Classification of cylindrically symmetric static Lorentzian manifolds according to their Petrov types and metrics

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The cylindrically symmetric static manifolds are classified for their Petrov types and metrics. This classification besides verifying the earlier result that such manifolds cannot be of petrov type **II**, **III** and **N,** gives a complete list of all static cylindrically symmetric metrics of Petrov type **O.** In the case of Petrov type **D** metrics, the results appear as three independent classes metrics.

Keywords: Petrov classification; static cylindrical symmetry; metrics.

The Petrov classification provides an invariant way of classifying the gravitational fields according to the number and the multiplicity of the eigenvalues of the Weyl tensor, C_{abcd} [1-2]. It has played a remarkable role in the investigation and the classification of the solutions of the Einstein's field equations besides helping in drawing certain conclusions about the physical nature of the gravitational fields belonging to a certain Petrov type [3 - 5]. It splits up gravitational fields into six different types, namely, algebraically general type, type **I** and the algebraically special types, types: **II, III, D, N,** and **O** [6-10].

The present attempt relates to classifying the cylindrically symmetric static Lorentzian manifolds according to their Petrov types and metrics. It is known [1] that these types of Lorentzian manifolds could be of Petrov type **I**, **D**, or **O**. It is yet to be known that for a given Petrov type, what kind of gravitational fields (g_{ab}) are possible. This work, besides independently verifying the already known result given in reference [1], finds: (i) explicitly all possible metrics of type **O**; (ii) all possible metrics of type **D** upto the first order differential constraints on their metrics coefficients; (iii) thus the residue of (i)-(ii) readily provide all type **I,** static cylindrically symmetric Lorentzian metrics. Earlier this work was undertaken in [11], where the type **O** metrics were identified as type **N**. This is also corrected here. In the Newman-Penrose formalism, the components of the Weyl tensor, C_{abcd} are expressed in terms of a complex null tetrad $\{e^a\} = (\underline{k}, \underline{l}, \underline{m}, \underline{n})$ [1]. These are:

$$\Psi_0 = C_{abcd}k^a m^b k^c m^d, \Psi_1 = C_{abcd}k^a l^b k^c m^d, \Psi_2 = \frac{1}{2}C_{abcd}k^a l^b (k^c l^d - m^c \overline{m}^d), \quad (1)$$

$$\Psi_3 = C_{abcd}l^a k^b l^c \overline{m}^d, \Psi_4 = C_{abcd}l^a \overline{m}^b l^c \overline{m}^d. \quad (2)$$

Then the invariants [1]:

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \tag{3}$$

$$J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}, \tag{4}$$

$$K = \Psi_1\Psi_4^2 - 3\Psi_4\Psi_3\Psi_2 + 2\Psi_3^3, \tag{5}$$

$$L = \Psi_2\Psi_4 - \Psi_3^2, \; N = 12L^2 - \Psi_4^2, \tag{6}$$

are constructed to give the invariant $I^3 - 27J^2$, to be used to find the Petrov classification [1]. All algebraically special gravitational fields, satisfy $I^3 - 27J^2 = 0$. In particular, for the types **III**, **N**, and **O**, both the invariants I and J vanish. If $I \neq 0 \neq J$ and $K = 0 = N$, the Petrov type is **D**. If $I = J = K = L = 0$, but $C_{abcd} \neq 0$, the metric is of Petrov type **N**. In case $C_{abcd} = 0$, the metric is of Petrov type **O**.

Here assuming that spacetime metric admits three commuting Killing vectors: $\partial/\partial t$, $\partial/\partial\theta$, and $\partial/\partial z$ in such a way that $\partial/\partial t$ is hyper surface orthogonal. Then the cylindrically symmetric static metric for a four dimensional Lorentzian manifold in isotropic coordinates, takes the form

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - a^2e^{\lambda(\rho)}d\theta^2 - e^{\mu(\rho)}dz^2. \tag{7}$$

The complex null tetrad basis for the metric (7) is given by

$$k^a = \frac{1}{\sqrt{2}}\left(e^{-\nu/2}\frac{\partial}{\partial t} - \frac{\partial}{\partial\rho}\right), l^a = \frac{1}{\sqrt{2}}\left(-e^{-\nu/2}\frac{\partial}{\partial t} + \frac{\partial}{\partial\rho}\right), \tag{8}$$

$$m^a = \frac{1}{\sqrt{2}}\left(\frac{e^{-\lambda/2}}{a}\frac{\partial}{\partial\theta} + ie^{-\mu/2}\frac{\partial}{\partial z}\right), \overline{m}^a = \frac{1}{\sqrt{2}}\left(\frac{e^{-\lambda/2}}{a}\frac{\partial}{\partial\theta} - ie^{-\mu/2}\frac{\partial}{\partial z}\right). \tag{9}$$

The null tetrad components of the Weyl tensor (1)-(2) for the metric (7) are therefore given by:

$$\Psi_0 = \tfrac{1}{16}\left[2\lambda'' - 2\mu'' + \lambda'^2 - \mu'^2 - \nu'\lambda' + \nu'\mu'\right] = \Psi_4, \; \Psi_1 = 0 = \Psi_3, \tag{10}$$

$$\Psi_2 = \tfrac{1}{48}[-4\nu'' + 2\lambda'' + 2\mu'' - 2\nu'^2 + \lambda'^2 + \mu'^2 + \nu'\lambda' + \nu'\mu' - 2\lambda'\mu']. \tag{11}$$

The invariants given by eqs.(3)-(6) are then reduced to:

$$I = \Psi_0^2 + 3\Psi_2^2, \; J = \Psi_2\left(\Psi_0^2 - \Psi_2^2\right), \; K = 0, \; L = \Psi_0\Psi_2, \; N = \Psi_0^2\left(3\Psi_2 + \Psi_0\right)\left(3\Psi_2 - \Psi_0\right). \tag{12}$$

This gives

$$I^3 - 27J^2 = \Psi_0^2\left(3\Psi_2 + \Psi_0\right)^2\left(3\Psi_2 - \Psi_0\right)^2. \tag{13}$$

Now to require that the metric be not of Petrov type **I,** we consider $I^3 - 27J^2 = 0$, which gives following four possibilities: (1) $\Psi_0 = 0, 3\Psi_2 + \Psi_0 \neq 0$, $3\Psi_2 - \Psi_0 \neq 0$; (2) $\Psi_0 \neq 0, 3\Psi_2 + \Psi_0 = 0$, $3\Psi_2 - \Psi_0 \neq 0$; (3) $\Psi_0 \neq 0, 3\Psi_2 + \Psi_0 \neq 0$, $3\Psi_2 - \Psi_0 = 0$; (4) $\Psi_0 = 0, 3\Psi_2 + \Psi_0 = 0$, $3\Psi_2 - \Psi_0 = 0$. Possibilities 1-3**,** due to eqs.(11)-(13), readily give $I \neq 0 \neq J$ and $K = 0 = N$. This concludes that in these cases the metrics (to be found) are of Petrov type **D.** Possibility 4 readily gives $I = J = K = L = 0$, thus in this case the metrics are of type **O.** All other metrics for which $I^3 - 27J^2 \neq 0$ are of type **I.** This proves that the static cylindrically symmetric Lorentzian manifolds cannot be of Petrov types **II**, **III** and **N.** For the Petrov type **O** metrics, eqs.(11)-(13), readily provide following differential constraints to be satisfied simultaneously:

$$2\left(\nu'' - \lambda''\right) + \nu'^2 - \lambda'^2 - \mu'\nu' + \mu'\lambda' = 0, \tag{14}$$

$$2\left(\lambda'' - \mu''\right) + \lambda'^2 - \mu'^2 - \nu'\lambda' + \nu'\mu' = 0, \tag{15}$$

$$2\left(\mu'' - \nu''\right) + \mu'^2 - \nu'^2 - \lambda'\mu' + \lambda'\nu' = 0. \tag{16}$$

For a complete solution of eqs.(14)-(16), all possible cases of ν', λ' and μ' are considered. These are: **Case I:** all ν', λ', μ' are identically zero; **Case II:** any one of ν', λ' and μ' is non zero at a point and hence in a coordinate neighbourhood of that point; **Case III:** any two of ν', λ' and μ' are non zero at a point and hence in a coordinate neighbourhood of that point; **Case IV:** all three ν', λ' and μ' are non zero at a point and hence in a coordinate neighbourhood of that point. We discuss all these cases one by one. **Case I:** If all ν', λ', μ' are zero then the differential constraints given by eqs.(14)-(16) are identically satisfied. In this case an infinite cylinder has a line cut at $\theta = 0, 2\pi$ and therefore, the circular coordinate can be straighten to write $ad\theta = dy$ with $d\rho = dx$. Thus the result is the "wrapped Minkowski" metric having a line singularity at $\theta = 0, 2\pi$. For this metric, since $C_{abcd} \equiv 0$, this is of the Petrov type **O**. **Case II:** In this case there arise three sub cases: **II**(a) $\nu' \neq 0, \lambda' = 0 = \mu'$; **II**(b) $\lambda' \neq 0$, $\nu' = 0 = \mu'$; **II**(c) $\mu' \neq 0$, $\nu' = 0 = \lambda'$. Then the differential constraints, eqs.(14)-(16), in each case reduce to $2\nu'' + \nu'^2 = 0$, $2\lambda'' + \lambda'^2 = 0$, $2\mu'' + \mu'^2 = 0$ respectively. The solutions of these equations readily give: $e^{\frac{\nu}{2}} = A\rho + B$; $e^{\frac{\lambda}{2}} = A\rho + B$; $e^{\frac{\mu}{2}} = A\rho + B$, where A and B are constants of integration. Thus for the **case II,** the solutions are: $e^{\frac{\nu}{2}} = A\rho + B$, $\lambda = 0 = \mu$; $e^{\frac{\lambda}{2}} = A\rho + B$, $\nu = 0 = \mu$; $e^{\frac{\mu}{2}} = A\rho + B$, $\nu = 0 = \lambda$. All these three solutions are isomorphic to the Minkowski metric. **Case III:** Any two of ν', $\lambda'$and $\mu'$are non zero e.g. we discuss the case where $\nu' \neq 0$, $\lambda' \neq 0$, $\mu' = 0$. The differential constraints given by eqs.(14)-(16) therefore reduce to

$$2\nu'' - 2\lambda'' + \nu'^2 - \lambda'^2 = 0, \tag{17}$$

$$2\lambda'' + \lambda'^2 - \nu'\lambda' = 0, \tag{18}$$

$$2\nu'' + \nu'^2 - \nu'\lambda' = 0. \tag{19}$$

Using eqs.(17)-(19), one gets $2\nu'\lambda'' + 2\nu''\lambda' = \left[\nu'\lambda'\right]' = 0$ or $\nu'\lambda' = 4k$, where $k \neq 0$ and 4 has been used for future convenience. Now for $\nu'\lambda' = 4k$, therefore eqs.(17)-(19) reduce to

$$\left(\nu'^2 + 4k\right)\left[\left(e^{\frac{\nu}{2}}\right)'' - ke^{\frac{\nu}{2}}\right] = 0, \tag{20}$$

$$\left(e^{\frac{\nu}{2}}\right)'' - k\, e^{\frac{\nu}{2}} = 0. \tag{21}$$

The solutions of eqs.(14)-(15) are: either $e^{\frac{\nu}{2}} = A\cosh(\alpha\rho+\beta)$, $e^{\frac{\lambda}{2}} = B\cosh(\alpha\rho+\beta)$, $\mu' = 0$ if $k = \alpha^2$; or $e^{\frac{\nu}{2}} = A\cos(\alpha\rho+\beta)$, $e^{\frac{\lambda}{2}} = B\cos(\alpha\rho+\beta)$, $\mu' = 0$, if $k = -\alpha^2$, where A, B and β are constants of integration. Analogously for the other two subcases of this type one gets: either $e^{\frac{\nu}{2}} = A\cosh(\alpha\rho+\beta)$, $\lambda' = 0$, $e^{\frac{\mu}{2}} = B\cosh(\alpha\rho+\beta)$; or $e^{\frac{\nu}{2}} = A\cos(\alpha\rho+\beta)$, $\lambda' = 0$, $e^{\frac{\mu}{2}} = B\cos(\alpha\rho+\beta)$; and either $\nu' = 0$, $e^{\frac{\lambda}{2}} = A\cosh(\alpha\rho+\beta)$, $e^{\frac{\mu}{2}} = B\cosh(\alpha\rho+\beta)$; or $\nu' = 0$, $e^{\frac{\lambda}{2}} = A\cos(\alpha\rho+\beta)$, $e^{\frac{\mu}{2}} = B\cos(\alpha\rho+\beta)$. **Case IV:** This is the case where all ν', λ' and μ' are nonzero. Eqs.(14)-(16) are integrated to give

$$(\nu-\lambda)'\, e^{\frac{1}{2}(\nu+\lambda-\mu)} = k_1, \tag{22}$$

$$(\lambda-\mu)'\, e^{\frac{1}{2}(\lambda+\mu-\nu)} = k_2, \tag{23}$$

$$(\mu-\nu)'\, e^{\frac{1}{2}(\mu+\nu-\lambda)} = k_3. \tag{24}$$

where k_i's $(i = 1,2,3)$ are constants of integration. We discuss all possible cases with respect to k_i as follows: **IV(a)** When all k_i's are zero; **IV(b)** any two of k_i's are zero; **IV(c)** any one of k_i's is zero; and **IV(d)** all k_i's are non zero.

Case **IV(a)** readily gives $\nu' = \lambda' = \mu'$. Case **IV (b)** is not possible because only one k_i can not be non zero as it is evident from the eqs.(22)-(24). Case **IV(c)** gives further three subcases, these are: **IVc(i)** $k_1 \neq 0$, $k_2 \neq 0$, $k_3 = 0$; **IVc(ii)** $k_1 \neq 0$, $k_2 = 0$, $k_3 \neq 0$; **IVc(iii)** $k_1 = 0$, $k_2 \neq 0$ and $k_1 \neq 0$. We discuss the Case **IVc(i)** and the results follow analogously for the other two sub cases. Case **IVc(i)** readily gives $\nu' = \mu'$ and eqs.(28)-(30) readily give $k_1 e^{\mu} + k_2 e^{\nu} = 0$. This implies that k_1 and k_2 have opposite signs $(k_1 k_2 < 0)$ and in fact ν and μ can be considered identical by absorbing the constant " $-\frac{k_1}{k_2}$ " in the definition of z. Therefore eqs.(22)-(23) reduce to

$$\mu' e^{\frac{1}{2}(\lambda)} - \lambda' e^{\frac{1}{2}(\lambda)} = k_1, \tag{25}$$

$$\lambda' e^{\frac{1}{2}(\lambda)} - \mu' e^{\frac{1}{2}(\lambda)} = k_2. \tag{26}$$

These equations readily give $k_1 + k_2 = 0$. Thus eqs.(22)-(23) simplify to $\left[e^{\frac{\lambda}{2}-\frac{\mu}{2}}\right]' = \frac{k_2}{2} e^{-\frac{\mu}{2}}$, which gives

$$e^{\frac{\lambda}{2}} = A e^{\frac{\mu}{2}} + \tfrac{1}{2} k_2 e^{\frac{\mu}{2}} \int e^{-\frac{\mu}{2}} d\rho, \tag{27}$$

where A is a constant of integration. Analogously the cases **IVc(ii)** and **IVc(iii)** can be solved to give:$\nu = \lambda$, $e^{\frac{\mu}{2}} = \frac{1}{2} k_2 e^{\frac{\nu}{2}} \int e^{-\frac{\nu}{2}} d\rho + A e^{\frac{\nu}{2}}$; $\lambda = \mu$, $e^{\frac{\nu}{2}} = \frac{1}{2} k_2 e^{\frac{\lambda}{2}} \int e^{-\frac{\lambda}{2}} d\rho + A e^{\frac{\lambda}{2}}$. This completes the solution of the case **IV(c).**

Case IV(d): Here all k_i's are non zero. Writing eqs.(22)-(24) in the form

$$(\nu-\lambda)'\, e^{\frac{1}{2}(\nu+\lambda+\mu)} = k_1 e^{\mu}, \tag{28}$$

$$(\lambda-\mu)'\, e^{\frac{1}{2}(\nu+\lambda+\mu)} = k_2 e^{\nu}, \tag{29}$$

$$(\mu-\nu)'\, e^{\frac{1}{2}(\nu+\lambda+\mu)} = k_3 e^{\lambda}. \tag{30}$$

By adding eqs.(28)-(29), one gets

$$k_1 e^{\mu} + k_2\, e^{\nu} + k_3\, e^{\lambda} = 0. \tag{31}$$

Again writing eqs.(28)-(29), in the form

$$\left(e^{\nu-\lambda}\right)' = k_1 e^{\frac{1}{2}(-3\nu+\lambda+\mu)}, \tag{32}$$

$$\left(e^{\lambda-\mu}\right)' = k_2 e^{\frac{1}{2}(\nu-3\lambda+\mu)}, \tag{33}$$

$$\left(e^{\mu-\nu}\right)' = k_3 e^{\frac{1}{2}(\nu+\lambda-3\mu)}. \tag{34}$$

Now eq.(31) can be written as

$$k_1 + k_2 e^{\nu-\mu} + k_3 e^{\lambda-\mu} = 0, \tag{35}$$

which on differentiation gives

$$k_3 \left(e^{\lambda-\mu}\right)' - k_2 \left(e^{\mu-\nu}\right)' = 0. \tag{36}$$

Using eqs.(32)-(34), eq.(36) reduces to

$$k_2\, k_3 \left[1 - e^{2(\lambda-\mu)}\right] = 0. \tag{37}$$

This implies $\lambda - \mu = 0$. Repeating the same argument one gets $\mu - \nu = 0$ and $\nu - \lambda = 0$. This concludes $\lambda = \mu = \nu$, but this contradicts the supposition that $k_i \neq 0$ for each i. So there does not exist any Petrov type **O** solution for the case **IVd.** This concludes the discussion Petrov type **O** solutions.

The conditions for the metric (7) to be of Petrov type **D** are $I^3 - 27J^2 = 0,\ I \neq 0 \neq J,\ K = 0 = N$. Thus eqs.(22)-(24) readily give that the metric will be of Petrov type **D** if either **(i)** $\Psi_0 = 0$ and $\Psi_2 \neq 0$ or **(ii)** $\Psi_0 \neq 0$, $\Psi_2 \neq 0$ and $\Psi_0 + 3\,\Psi_2 = 0$ or **(iii)** $\Psi_0 \neq 0$, $\Psi_2 \neq 0$ and $\Psi_0 - 3\Psi_2 = 0$. Thus each of eqs.(22)-(24) gives an independent condition to be satisfied for a metric to be of Petrov type **D.** Eqs.(22)-(24) with one of $k_i = 0$, readily give $\nu' = \lambda'$ or $\lambda' = \mu'$ or $\mu' = \nu'$. Thus all metrics with $\nu = \lambda$ or $\lambda = \mu$ or $\mu = \nu$ and not satisfying any of the conditions of the type **O** metrics obtained above, emerge as Petrov type **D** solutions. The other Petrov **D** solutions depends on the general solution of each of the eqs.(22)-(24).

The cylindrically symmetric static manifolds have been classified according to their Petrov types. It is shown that the Petrov types of these metrics can be determined by the nature of only two of the null tetrad components of the Weyl tensor, namely Ψ_0 and Ψ_2. It is verified that a cylindrically symmetric static metric cannot be of Petrov type **II** or **III** or **N**. Eqs.(22)-(24) with one of $k_i = 0$, readily give $\nu' = \lambda'$ or $\lambda' = \mu'$ or $\mu' = \nu'$. Thus all metrics with $\nu = \lambda$ or $\lambda = \mu$ or $\mu = \nu$ emerge as Petrov type **D** solutions. The cylindrically symmetric static Lorentzian manifolds given by the metric (7) are of Petrov type **O** if and only if one of the following conditions is satisfied:

(1) $e^{\frac{\nu}{2}} = A\rho + B,\ \lambda' = \mu' = 0;$
(2) $\nu' = 0,\ e^{\frac{\lambda}{2}} = A\rho + B,\ \mu' = 0;$

(3) $\nu' = \lambda' = 0,\ \ e^{\frac{\mu}{2}} = A\rho + B;$
(4) $e^{\frac{\nu}{2}} = A\cosh(\alpha\rho + B),\ \ e^{\frac{\lambda}{2}} = B\sinh(\alpha\rho + B),\ \ \mu' = 0;$
(5) $e^{\frac{\nu}{2}} = A\cos(\alpha\rho + B),\ \ e^{\frac{\lambda}{2}} = B\sin\alpha\rho + B,\ \ \mu' = 0;$
(6) $e^{\frac{\nu}{2}} = A\cosh(\alpha\rho + B),\ \ \lambda' = 0,\ \ e^{\frac{\mu}{2}} = B\sinh(\alpha\rho + B);$
(7) $e^{\frac{\nu}{2}} = A\cos(\alpha\rho + B),\ \ \lambda' = 0\ \ e^{\frac{\mu}{2}} = B\sin(\alpha\rho + B);$
(8) $\nu' = 0,\ \ e^{\frac{\lambda}{2}} = A\cosh(\alpha\rho + B),\ \ e^{\frac{\mu}{2}} = B\sinh(\alpha\rho + B);$
(9) $\nu' = 0,\ \ e^{\frac{\lambda}{2}} = A\cos(\alpha\rho + B),\ \ e^{\frac{\mu}{2}} = B\sin(\alpha\rho + B);$
(10) $\nu' = \lambda' = \mu';$
(11) $\nu = \lambda,\ \ e^{\frac{\mu}{2}} = \frac{1}{2}k_2 e^{\frac{\nu}{2}} \int e^{-\frac{\nu}{2}} d\rho + A e^{\frac{\nu}{2}};$
(12) $\lambda = \mu,\ \ e^{\frac{\nu}{2}} = \frac{1}{2}k_2 e^{\frac{\lambda}{2}} \int e^{-\frac{\lambda}{2}} d\rho + A e^{\frac{\lambda}{2}};$
(13) $\mu = \nu,\ \ e^{\frac{\lambda}{2}} = \frac{1}{2}k_2 e^{\frac{\mu}{2}} \int e^{-\frac{\mu}{2}} d\rho + A e^{\frac{\mu}{2}}.$

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