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Complex Structures, T-duality and Worldsheet Instantons in Born Sigma Models

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Abstract. We study generalized (doubled) structures in $2D$-dimensional Born geometries in which T-duality symmetry is manifestly realized. We show that spacetime structures of Kähler, hyperkähler, bi-hermitian and bi-hypercomplex manifolds are implemented in Born geometries as generalized (doubled) structures. We find that the Born structures and the generalized Kähler (hyperkähler) structures appear as subalgebras of bi-quaternions $\mathbb{C} \times \mathbb{H}$ and split-tetra-quaternions $\mathbb{H} \times \mathrm{Sp}\mathbb{H}$. We investigate the nature of T-duality for the worldsheet instantons in Born sigma models. This manuscript is based on the original paper [1].

1. Introduction

It is well-known that the target space of two-dimensional $\mathcal{N} = (2,2)$ string sigma models composed of chiral multiplets is Kähler, admitting the complex structure J. On the other hand, the target space of $\mathcal{N} = (2,2)$ models by chiral and twisted chiral multiplets is the bi-hermitian manifold which is characterized by two commuting complex structures (J_+, J_-). Indeed, these geometries are related by the T-duality transformations [2, 3]. Similarly, there are T-duality relations between hyperkähler and bi-hypercomplex geometries in $\mathcal{N} = (4,4)$ models. The hyperkäher geometry has complex structures J^a $(a = 1,2,3)$ satisfying the $SU(2)$ algebra. The bi-hypercomplex geometry has a set of commuting complex structures J^a_+ and J^a_- $(a = 1,2,3)$. They satisfy the $SU(2)$ algebra independently.

The idea on the geometric realization of manifest T-duality has been studied in generalized geometry [4]. Given a D-dimensional spacetime manifold M, the generalized tangent bundle $\mathbb{T}M$ is defined by the sum of the tangent and the cotangent bundles $\mathbb{T}M = TM \oplus T^*M$. In this context, the complex structures of spacetime are realized on $\mathbb{T}M$ as generalized complex structures [5].

There is a deep connection between generalized geometry and the doubled formalism on a $2D$-dimensional doubled space $\mathcal{M}$. Double field theory (DFT) defined on $\mathcal{M}$ is a reformulation of supergravity that make T-duality be manifest [6]. T-duality symmetry is linearly realized as global $O(D,D)$ transformations in the doubled space $\mathcal{M}$. The general rules of T-duality transformation for (hyper)Kähler, bi-hermitian and bi-hypercomplex structures of spacetime M have been studied in the doubled formalism [7].

In the following, we discuss geometries of the doubled space and their relations to the T-duality transformation of complex structures. We also discuss a possible application of the T-duality covariantized complex structures by focusing on worldsheet instantons.

2. Double field theory and Born geometry

We first introduce double field theory (DFT) [6]. The fundamental fields of DFT are the generalized metric $\mathcal{H}_{MN}$ and the generalized dilaton d. They are defined in the $2D$-dimensional doubled space $\mathcal{M}$. The doubled coordinate $\mathbb{X}^M$, $(M = 1, \ldots, 2D)$ on $\mathcal{M}$ is decomposed as $\mathbb{X}^M = (X^\mu, \tilde{X}_\mu)$, $(\mu = 1, \ldots, D)$ where X^μ and $\tilde{X}_\mu$ may be the Kaluza-Klein and the winding coordinates, respectively. The generalized metric and the generalized dilaton are parameterized as

$$\mathcal{H}_{MN}(\mathbb{X}) = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\rho} B_{\rho\nu} & g^{\mu\nu} \end{pmatrix}, \qquad e^{-2d}(\mathbb{X}) = \sqrt{-g} e^{-2\phi}, \tag{1}$$

where $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ are D-dimensional (anti)symmetric matrices, ϕ is a real function on $\mathcal{M}$. We note that $\mathcal{H}_{MN}$ is an $O(D,D)$ element. On the other hand, d is invariant under the $O(D,D)$ transformation. The well-known Buscher's rule of T-duality transformations of fields are derived from an $O(D,D)$ rotation of $\mathcal{H}_{MN}$ and the invariance of d.

The DFT action is given by

$$S_{\mathrm{DFT}} = \int \mathrm{d}^{2D}\mathbb{X}\, e^{-2d} \mathscr{R}(\mathcal{H}, d), \tag{2}$$

where $\mathscr{R}(\mathcal{H}, d)$ is a function of $\mathcal{H}$ and d defined by

$$\begin{aligned}\mathscr{R}(\mathcal{H}, d) = {}& 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}.\end{aligned} \tag{3}$$

Here we have introduced the doubled derivative $\partial_M = \frac{\partial}{\partial\mathbb{X}^M}$. The doubled indices $M, N, \ldots$ are raised and lowered by the $O(D,D)$ invariant metric η_{MN} and its inverse η^{MN}. They are given by

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_\mu{}^\nu \\ \delta^\mu{}_\nu & 0 \end{pmatrix}, \qquad \eta^{MN} = \begin{pmatrix} 0 & \delta^\mu{}_\nu \\ \delta_\mu{}^\nu & 0 \end{pmatrix}. \tag{4}$$

The action defined in (2) is invariant under the global $O(D,D)$ transformation. In addition, the action is invariant under the DFT gauge transformations. They are the $O(D,D)$ covariantized diffeomorphism and the gauge transformation for the B-field. All the quantities in DFT, namely, the generalized metric, the generalized dilaton and the gauge parameters, satisfy the following constraints,

$$\partial_M\partial^M * = 0, \qquad \partial_M * \partial^M * = 0. \tag{5}$$

Here $*$ are any fields and gauge parameters in DFT. In (5), the first condition corresponds to the level-matching condition of closed strings. On the other hand, the second one is specific to DFT. This is called the strong constraint.

The D-dimensional physical spacetime M is defined by a hypersurface in the $2D$-dimensional doubled space $\mathcal{M}$. This is obtained as a solution to the constraints (5). It is obvious that a trivial solution is given by the DFT fields $\mathcal{H}_{MN}$, d and gauge parameters that depend only on X^μ. Therefore a hypersurface $\tilde{X}_\mu = \text{const.}$, parameterized by X^μ defines a D-dimensional

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physical spacetime. Then the matrices $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$ in (1) are identified with the spacetime metric, the NSNS B-field and the dilaton in the D-dimensional spacetime. It is easy to show that the DFT action (2) in this case reduces to that of the bosonic part of type II supergravities in D-dimensions,

$$S = \int \mathrm{d}^D X \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12}(H^{(3)})^2 \right]. \tag{6}$$

Here R is the D-dimensional Ricci scalar and $H^{(3)} = \mathrm{d}B$ is the field strength of the B-field. From this result we can say that DFT is, in a sense, a reformulation of supergravity in such a way that T-duality becomes a manifest symmetry.

We next introduce the geometry of the doubled space $\mathcal{M}$.

3. Born and generalized complex structures

The doubled space $\mathcal{M}$ has specific geometry characterized by Born structures [8–10].

3.1. Born geometry

An endomorphism on $T\mathcal{M}$ is called the doubled structure. The doubled space $\mathcal{M}$ in DFT has the doubled structures $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$. They are called the almost complex, the chiral, and the para-hermitian structures, respectively. The triple $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ satisfies and algebra given by

$$\begin{gathered} -\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = \mathbf{1}_{2D}, \qquad \mathcal{I}\mathcal{J}\mathcal{K} = -\mathbf{1}_{2D}, \\ \{\mathcal{I}, \mathcal{J}\} = \{\mathcal{J}, \mathcal{K}\} = \{\mathcal{K}, \mathcal{I}\} = 0. \end{gathered} \tag{7}$$

Here $\{\cdot, \cdot\}$ represents the anti-commutator. This is known as the algebra of split-quaternions $\mathrm{Sp}\mathbb{H}$. Given the structures $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$, we have the decompositions;

$$\mathcal{I} = \mathcal{H}^{-1}\omega_{\mathcal{K}} = -\omega_{\mathcal{K}}^{-1}\mathcal{H}, \qquad \mathcal{J} = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta, \qquad \mathcal{K} = \eta^{-1}\omega_{\mathcal{K}} = \omega_{\mathcal{K}}^{-1}\eta, \tag{8}$$

where $\mathcal{H}$ and η are the generalized and the $O(D, D)$ invariant metrices given above and $\omega_{\mathcal{K}}$ is an almost symplectic structure. The triple $(\eta, \omega_{\mathcal{K}}, \mathcal{H})$ here is known as the Born structure. They are explicitly given by $2D \times 2D$ matrices;

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_\mu{}^\nu \\ \delta^\mu{}_\nu & 0 \end{pmatrix}, \ (\omega_{\mathcal{K}})_{MN} = \begin{pmatrix} 2B_{\mu\nu} & -\delta_\mu{}^\nu \\ \delta^\mu{}_\nu & 0 \end{pmatrix}, \ \mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} & B_{\mu\rho}g^{\rho\nu} \\ -g^{\mu\rho}B_{\rho\nu} & g^{\mu\nu} \end{pmatrix}. \tag{9}$$

The other structures are given by

$$\begin{aligned} \mathcal{I}^M{}_N &= \mathcal{H}^{ML}(\omega_{\mathcal{K}})_{LN} = \begin{pmatrix} g^{\mu\rho}B_{\rho\nu} & -g^{\mu\nu} \\ g_{\mu\nu} + B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} & -B_{\mu\rho}g^{\rho\nu} \end{pmatrix}, \\ \mathcal{J}^M{}_N &= \eta^{ML}\mathcal{H}_{LN} = \begin{pmatrix} -g^{\mu\rho}B_{\rho\nu} & g^{\mu\nu} \\ g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} & B_{\mu\rho}g^{\rho\nu} \end{pmatrix}, \\ \mathcal{K}^M{}_N &= \eta^{ML}(\omega_{\mathcal{K}})_{LN} = \begin{pmatrix} \delta^\mu{}_\nu & 0 \\ 2B_{\mu\nu} & -\delta_\mu{}^\nu \end{pmatrix}. \end{aligned} \tag{10}$$

With these parametrization, we can show that $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ indeed satisfy the algebra (7). We call the expressions (9), (10) the standard representation.

The property of the para-hermitian structure $\mathcal{K}$ defines a D-dimensional physical spacetime M as follows. Since $\mathcal{K}^2 = \mathbf{1}_{2D}$ it defines $\mathcal{K} = \pm 1$ eigenbundles L, $\tilde{L}$ and $T\mathcal{M} = L \oplus \tilde{L}$. When $\mathcal{K}$

is an integrable structure on $T\mathcal{M}$, then the involutive bundle $L \subset T\mathcal{M}$ defines a foliation in $\mathcal{M}$. This introduces D-dimensional subspaces called leaves $\mathcal{F} \subset \mathcal{M}$ such that $L = T\mathcal{F}$. When the basis of L is given by $\partial_\mu = \frac{\partial}{\partial X^\mu}$, then the local coordinate of the leaves $\mathcal{F}$ is X^μ. The same is true for the $K = -1$-eigenbundle $\tilde{L} = T\tilde{\mathcal{F}}$. When $\tilde{L}$ is integrable, the coordinate of the corresponding base space $\tilde{\mathcal{F}}$ is $\tilde{X}_\mu$ and the decomposition of the doubled coordinate $\mathbb{X}^M = (X^\mu, \tilde{X}_\mu)$ that we have introduced in section 1 is now justified. Then a leaf in $\mathcal{F}$, specified by $\tilde{X}_\mu = \text{const.}$, is identified with a D-dimensional physical spacetime M. Note that after we solve the constraints (5), and make $g_{\mu\nu}$, B, ϕ be functions of X^μ, then they are the spacetime metric, the B-field and the dilaton on a leaf in $\mathcal{F}$.

In the following in this section, we set $B = 0$ in double structures for simplicity. The B-field is easily introduced through the B-transformation;

$$\mathcal{A} \to \mathcal{A}^B = e^B \mathcal{A} e^{-B}, \qquad e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \tag{11}$$

for any doubled structure $\mathcal{A}$.

3.2. Doubled space and generalized geometry

We here study the relation between the doubled structures on $T\mathcal{M}$ and endomorphisms on the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$. The $O(D,D)$ invariant metric η in $T\mathcal{M}$ enables us to define a map $T\mathcal{M} = L \oplus \tilde{L} \to L^* \oplus \tilde{L}^*$. Here $L^*(\tilde{L}^*)$ is the dual vector space of $L(\tilde{L})$. This induces the following maps;

$$\phi^+ : \tilde{L} \to L^*, \qquad \phi^- : L \to \tilde{L}^*. \tag{12}$$

With these relations, it is natural to identify $\tilde{L}$ with L^*. Therefore we now introduce maps called natural isomorphisms [8–10];

$$\Phi^+ : T\mathcal{M} \to L \oplus L^*, \qquad \Phi^- : T\mathcal{M} \to \tilde{L} \oplus \tilde{L}^*. \tag{13}$$

Then, after solving the strong constraint, a slice $\tilde{X}_\mu = \text{const.}$, $T\mathcal{M}$ is identified with the generalized tangent bundle $TM \oplus T^*M$ on the D-dimensional base space M by the natural isomorphisms. In the following, we consider a leaf $M \subset \mathcal{F}$ and always identify doubled structures on $T\mathcal{M}$ with generalized structures on $\mathbb{T}M$. Note that doubled and generalized vectors are explicitly identified as

$$V = V^M \partial_M = V^\mu \partial_\mu + \tilde{V}_\mu \tilde{\partial}^\mu \qquad \Longleftrightarrow \qquad V = V^\mu \partial_\mu + \tilde{V}_\mu \mathrm{d}X^\mu. \tag{14}$$

In the following, we assume that the D-dimensional spacetime M admits the integrable complex structure J and its associated closed two-form $\omega = -gJ$. Then M is a Kähler manifold. Given (J, ω) in M, we define the following doubled structures;

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{15}$$

where J^* is the adjoint of J. These are called the generalized complex structures on $\mathbb{T}M$ [5]. It is easy to show that $\mathcal{J}_J$ and $\mathcal{J}_\omega$ commute with each other. In addition, we find that their product

$$\mathcal{J} = \mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{16}$$

becomes the chiral structure given in (10). The structures $\mathcal{J}_J, \mathcal{J}_\omega$ and $\mathcal{J}$ obey the following algebra;

$$-\mathcal{J}_J^2 = -\mathcal{J}_\omega^2 = \mathcal{J}^2 = \mathbf{1}_{2D}, \qquad \mathcal{J}_J \mathcal{J}_\omega \mathcal{J} = \mathbf{1}_{2D},$$
$$[\mathcal{J}_J, \mathcal{J}_\omega] = [\mathcal{J}_\omega, \mathcal{J}] = [\mathcal{J}, \mathcal{J}_J] = 0. \tag{17}$$

Here $[\cdot, \cdot]$ is the commutator. This is known as the algebra of the bi-complex numbers $\mathbb{C}_2$.

3.3. Hypercomplex structures in doubled space

We next discuss the relation between the generalized complex and the Born structures in the doubled space. As we have shown, they satisfy the algebras of split-quaternions $\mathrm{Sp}\mathbb{H}$ in (7) and bi-complex numbers $\mathbb{C}_2$ in (17). We look for the algebra that contains (7) and (17) as subalgebras.

We find that the structures $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ and $(\mathcal{J}_J, \mathcal{J}_\omega)$ together with the extra doubled structures

$$\mathcal{Q} = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & -\omega^{-1} \\ -\omega & 0 \end{pmatrix}, \tag{18}$$

form the appropriate algebra known as bi-quaternions $\mathbb{C} \times \mathbb{H}$. The followings are the subalgebras of $\mathbb{C} \times \mathbb{H}$;

(i) $\mathbb{C}_2$: $(\mathbf{1}_{2D}, \mathcal{J}_J, \mathcal{J}_\omega, \mathcal{J})$, $\mathcal{J}_J^2 = \mathcal{J}_\omega^2 = -\mathbf{1}_{2D}$, $\mathcal{J}^2 = \mathbf{1}_{2D}$; commutative,
(ii) $\mathbb{C}_2$: $(\mathbf{1}_{2D}, \mathcal{J}_J, \mathcal{I}, \mathcal{P})$, $\mathcal{J}_J^2 = \mathcal{I}^2 = -\mathbf{1}_{2D}$, $\mathcal{P}^2 = \mathbf{1}_{2D}$; commutative,
(iii) $\mathbb{C}_2$: $(\mathbf{1}_{2D}, \mathcal{J}_J, \mathcal{K}, \mathcal{Q})$, $\mathcal{J}_J^2 = \mathcal{Q}^2 = -\mathbf{1}_{2D}$, $\mathcal{K}^2 = \mathbf{1}_{2D}$; commutative,
(iv) $\mathrm{Sp}\mathbb{H}$: $(\mathbf{1}_{2D}, \mathcal{I}, \mathcal{J}, \mathcal{K})$, $\mathcal{I}^2 = -\mathbf{1}_{2D}$, $\mathcal{J}^2 = \mathcal{K}^2 = \mathbf{1}_{2D}$; anti-commutative,
(v) $\mathrm{Sp}\mathbb{H}$: $(\mathbf{1}_{2D}, \mathcal{J}, \mathcal{P}, \mathcal{Q})$, $\mathcal{Q}^2 = -\mathbf{1}_{2D}$, $\mathcal{J}^2 = \mathcal{P}^2 = \mathbf{1}_{2D}$; anti-commutative,
(vi) $\mathrm{Sp}\mathbb{H}$: $(\mathbf{1}_{2D}, \mathcal{J}_\omega, \mathcal{K}, \mathcal{P})$, $\mathcal{J}_\omega^2 = -\mathbf{1}_{2D}$, $\mathcal{K}^2 = \mathcal{P}^2 = \mathbf{1}_{2D}$; anti-commutative,
(vii) $\mathbb{H}$: $(\mathbf{1}_{2D}, \mathcal{J}_\omega, \mathcal{I}, \mathcal{Q})$, $\mathcal{J}_\omega^2 = \mathcal{Q}^2 = \mathcal{I}^2 = -\mathbf{1}_{2D}$; anti-commutative.

Here "(anti-)commutative" means that any two of structures other than $\mathbf{1}_{2D}$ (anti-)commute. We note that the algebra of split-quaternions (iv), (v) and (vi) given above defines a Born structure. This algebra is isomorphic to Clifford algebras $\mathrm{Sp}\mathbb{H} \simeq Cl_{2,0}(\mathbb{R}) \simeq Cl_{1,1}(\mathbb{R})$. The algebra of quaternions (vii) defines a hypercomplex structure on the doubled space $\mathcal{M}$. It is $Cl_{0,2}(\mathbb{R})$ in the language of Clifford algebra. The algebra of bi-complex numbers (i), (ii) and (iii), which defines a generalized Kähler structure, is identified with the Clifford algebra $Cl_1(\mathbb{C})$.

Therefore geometries having the doubled structures of bi-quaternions represent a T-duality covariant realization of a spacetime admitting Kähler structures. Similar discussions hold in the cases of bi-hermitian, hyperkähler and bi-hypercomplex manifolds. The results are summarized in Table 1.

4. Born sigma models and instantons

In this section, we study the T-duality covariant doubled sigma model called the Born sigma model.

4.1. Born sigma models

The Born sigma model is defined as a two-dimensional sigma model whose target space is a $2D$-dimensional Born manifold $\mathcal{M}$. The action of the Born sigma model is given by

$$S = \frac{1}{4} \int_\Sigma \left[\mathcal{H}_{MN} \mathrm{d}\mathbb{X}^M \wedge * \mathrm{d}\mathbb{X}^N - \Omega_{MN} \mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N \right]. \tag{19}$$

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Structures on $T\mathcal{M} \simeq TM \oplus T^*M$	Algebras of hypercomplex numbers	Structures on TM
Generalized Kähler	bi-complex numbers (4)	Kähler (J, ω)
Generalized Kähler	bi-complex numbers over $\mathbb{C}$ (8)	bi-hermitian $(J_\pm, \omega_\pm)$
Generalized hyperkähler	split-bi-quaternions (8)	hyperkähler (J^a, ω^a)
Generalized hyperkähler	split-bi-quaternions over $\mathbb{H}$ (32)	bi-hypercomplex $(J^a_\pm, \omega^a_\pm)$
Born	split-quaternions (4)	
Born + generalized Kähler	bi-quaternions (8)	Kähler (J, ω)
Born + generalized Kähler	bi-quaternions over $\mathbb{C}$ (16)	bi-hermitian $(J_\pm, \omega_\pm)$
Born + generalized hyperkähler	split-tetra-quaternions (16)	hyperkähler (J^a, ω^a)
Born + generalized hyperkähler	split-tetra-quaternions over $\mathbb{H}$ (64)	bi-hypercomplex $(J^a_\pm, \omega^a_\pm)$

Table 1. The doubled structures on $T\mathcal{M} \simeq TM \oplus T^*M$ and their algebras and dimensions [1].

Here Σ stands for the two-dimensional worldsheet in the Minkowski signature, $\mathbb{X}^M = (X^\mu, \tilde{X}_\mu)$ is the local coordinate of $\mathcal{M}$, $*$ is the Hodge star operator in the worldsheet Σ, $\mathcal{H}_{MN}$ is the generalized metric in $\mathcal{M}$ and $\Omega_{MN} = -\Omega_{NM}$ is an anti-symmetric constant matrix. It is manifest that the action (19) is invariant under the following $O(D, D)$ transformations;

$$\mathrm{d}\mathbb{X}^M \to \mathcal{O}^M{}_N \mathrm{d}\mathbb{X}^N, \quad \mathcal{H}_{MN} \to (\mathcal{O}^t)_M{}^P \mathcal{H}_{PQ} \mathcal{O}^Q{}_N, \quad \Omega_{MN} \to (\mathcal{O}^t)_M{}^P \Omega_{PQ} \mathcal{O}^Q{}_N, \mathcal{O} \in O(D, D). \tag{20}$$

The topological term is given by $\Omega_{MN} \mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N = -2\mathrm{d}X^\mu \wedge \mathrm{d}\tilde{X}_\mu$ [11, 12].

The action (19) in the standard parametrization, is given by

$$\begin{aligned} S = \frac{1}{4} \int_\Sigma \Big[&(g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu}) \mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\rho} g^{\rho\nu} \mathrm{d}X^\mu \wedge *\mathrm{d}\tilde{X}_\nu \\ &- g^{\mu\rho} B_{\rho\nu} \mathrm{d}\tilde{X}_\mu \wedge *\mathrm{d}X^\nu + g^{\mu\nu} \mathrm{d}\tilde{X}_\mu \wedge *\mathrm{d}\tilde{X}_\nu \Big] \\ &+ \frac{1}{2} \int_\Sigma \mathrm{d}X^\mu \wedge d\tilde{X}_\mu. \end{aligned} \tag{21}$$

As in the case of DFT, the Born sigma model (19) contains extra degrees of freedom. In order to obtain a physical theory which contains degrees of freedom D, we impose constraints. First, the background field $\mathcal{H}_{MN}$ satisfies the constraints (5). Then all the background fields now depend only on X^μ. In addition, the ordinary string sigma model whose target space is a D-dimensional physical spacetime, is obtained by imposing the following chirality condition;

$$\mathrm{d}\mathbb{X}^M = \mathcal{J}^M{}_N * \mathrm{d}\mathbb{X}^N. \tag{22}$$

By using the explicit representation (10) together with the B-transformation (11) for the chiral structure $\mathcal{J}$, the above chirality condition (22) is expanded and it is possible to solve $\mathrm{d}\tilde{X}_\mu$ as

$$\mathrm{d}\tilde{X}_\mu = g_{\mu\nu} * \mathrm{d}X^\nu + B_{\mu\nu} \mathrm{d}X^\nu. \tag{23}$$

Using this result, we now remove the winding coordinate $\mathrm{d}\tilde{X}_\mu$ from the action (21). We find

$$S = \frac{1}{2} \int_\Sigma \left(g_{\mu\nu} \mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\nu} \mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \right). \tag{24}$$

This is nothing but the action for the ordinary string sigma model.

4.2. Doubled instantons

We then study the instantons in the Born sigma model. From now on, we assume the Euclidean spacetime and worldsheet and $*^2 = -1$. Since the complex structure is necessary for the worldsheet instantons in ordinary string sigma models, we focus on the spacetime of the bi-hermitian manifold. Since the generalized metric $\mathcal{H}_{MN}$ is positive-definite in the Euclidean space, we perform the Bogomol'nyi completion of the first term in the action,

$$\begin{aligned} S &= \frac{1}{8}\int \mathcal{H}_{MN}\Big(\mathrm{d}\mathbb{X}^M \pm \mathcal{J}_{\pm}{}^M{}_P * \mathrm{d}\mathbb{X}^P\Big) \wedge *\Big(\mathrm{d}\mathbb{X}^N \pm \mathcal{J}_{\pm}{}^N{}_Q * \mathrm{d}\mathbb{X}^Q\Big) \pm \frac{1}{4}\int (\omega_{\pm})_{MN}\mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N \\ &\geq \pm\frac{1}{4}\int (\omega_{\pm})_{MN}\mathrm{d}\mathbb{X}^M \wedge \mathrm{d}\mathbb{X}^N, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{J}_{\pm} &= \frac{1}{2}e^B \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix} e^{-B} \\ &= \frac{1}{2}e^B \Big(\mathcal{J}_{J_+} \pm \mathcal{J}_{J_-} + \mathcal{J}_{\omega_+} \mp \mathcal{J}_{\omega_-}\Big) e^{-B} \end{aligned} \tag{26}$$

are doubled structures in the Born manifold $\mathcal{M}$ satisfying $\mathcal{J}_{\pm}^2 = -\mathbf{1}_{2D}$. The fundamental two-forms associated with $\mathcal{J}_{\pm}$ are defined by $\omega_{\pm} = \mathcal{H}\mathcal{J}_{\pm}$. It is obvious that the bound is saturated when the map $\mathbb{X} : \Sigma \to \mathcal{M}$ satisfies the following doubled instanton equations;

$$\mathrm{d}\mathbb{X}^M \pm \mathcal{J}_{\pm}{}^M{}_N * \mathrm{d}\mathbb{X}^N = 0. \tag{27}$$

In the following, we only consider $\mathcal{J}_+$ without loss of generality. The doubled instanton equations (27) are decomposed as

$$\begin{aligned} &\frac{1}{2}\Big(\mathrm{d}\mathbb{X}^M \pm (\mathcal{J}^B_{J_+})^M{}_N * \mathrm{d}\mathbb{X}^N\Big) + \frac{1}{2}\Big(\mathrm{d}\mathbb{X}^M \pm (\mathcal{J}^B_{J_-})^M{}_N * \mathrm{d}\mathbb{X}^N\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{d}\mathbb{X}^M \pm (\mathcal{J}^B_{\omega_+})^M{}_N * \mathrm{d}\mathbb{X}^N\Big) - \frac{1}{2}\Big(\mathrm{d}\mathbb{X}^M \pm (\mathcal{J}^B_{\omega_-})^M{}_N * \mathrm{d}\mathbb{X}^N\Big) = 0. \end{aligned} \tag{28}$$

Using the chirality condition (22), we have the following equation from (28);

$$\mathrm{d}X^\mu - i * \mathrm{d}X^\mu = 0. \tag{29}$$

This means that the solutions to the instanton equations are given by the holomorphic functions as is well-known. In this case, we recover the bound in the ordinary worldsheet instantons that are defined by $J_{\pm}$;

$$\begin{aligned} S_{\mathrm{E}} &= \pm\frac{1}{2}\int (\omega_+)_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu + \frac{i}{2}\int B_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \\ &= \pm\frac{1}{2}\int (\omega_-)_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu + \frac{i}{2}\int B_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu. \end{aligned} \tag{30}$$

5. Conclusion

In this manuscript, we showed the T-duality covariant generalized (doubled) structures that contain (hyper)Kähler, bi-hermitian and bi-hypercomplex geometries of spacetime.

In DFT, the spacetime metric $g_{\mu\nu}$, the NSNS B-field and dilaton ϕ are packaged into the generalized metric $\mathcal{H}_{MN}$ and the generalized dilaton d. It is known that the natural $O(D,D)$ structures of DFT are introduced in the Born geometry on the $2D$-dimensional doubled space

$\mathcal{M}$. Assuming the strong constraint, there are natural isomorphisms that allow us to identify the doubled tangent bundle $T\mathcal{M}$ and the generalized tangent bundle $\mathbb{T}M$. Using the generalized Kähler structure $(\mathcal{J}_J, \mathcal{J}_\omega)$ on $T\mathcal{M}$, the Kähler structure on the physical spacetime M is organized into the T-duality covariant quantities. We showed that the generalized (doubled) structures satisfying the algebra of bi-quaternions $\mathbb{C} \times \mathbb{H}$ contain the subalgebras that define the Born and the generalized complex structures. These results are extended to the cases for bi-hermitian, hyperkähler and bi-hypercomplex manifolds.

We then introduced the doubled sigma model known as Born sigma model. The target space of the Born sigma model is the doubled space $\mathcal{M}$ admitting the Born and the generalized Kähler structures. The Born sigma model has manifest T-duality by definition. The action is given by the generalized metric $\mathcal{H}_{MN}$ and a topological term. The physical condition defined by the chiral structure $\mathcal{J}$ is imposed to obtain the ordinary string sigma model. We showed that the Bogomol'nyi completion is possible in Euclidean space and wrote down the doubled instanton equations. They are defined by the doubled complex structures $\mathcal{J}_\pm$. These equations reproduce the ordinary worldsheet instanton equations. In the original paper [1], we have discussed that the instantons in the bi-hermitian geometries. We have shown that they are represented by a linear combination of instanton equations defined by the bi-hermitian structures $(J_\pm, \omega_\pm)$. This analysis can be applied to the hyperkähler and the bi-hypercomplex cases. Details are found in the original paper.

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