## REMARKS ON THE SUPERSYMMETRIC WKB QUANTIZATION FORMULA

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The supersymmetric WKB quantization formula proposed by Comtet, Bandrauk and Campbell is shown to be seen as a supersymmetric counterpart of the WKB quantization formula.

### 1. Introduction

In recent years, supersymmetric quantum mechanics [1] has attracted considerable attention. In particular, the supersymmetric WKB formula proposed by Comtet, Bandrauk and Campbell [2] is a surprise and something worth attending to. The formula of Comtet, Bandrauk and Campbell (the CBC formula in short) provides exact energy quantization for a variety of nonrelativistic systems [2,3].

The standard semiclassical WKB quantization formula,

$$\int_{x_a}^{x_b} \{2M[E-V(x)]\}^{1/2}\, dx = (n+\tfrac{1}{2})\pi\hbar \qquad (n=0,1,2,\dots) \tag{1}$$

where $E = V(x_a) = V(x_b)$, is known to yield exact energy spectra for a certain class of systems provided that *ad hoc* modifications are made to the potential term [4]. For instance, the exact energy spectrum of the hydrogen atom can be obtained from the formula (1) if the well-known Langer replacement, $\ell(\ell+1) \to (\ell+\frac{1}{2})^2$, is applied. In contrast, the CBC formula [2],

$$\int_{\tilde{x}_a}^{\tilde{x}_b} \{2M[\tilde{E}-\phi^2(x)]\}^{1/2}\, dx = n\pi\hbar \qquad (n=0,1,2,\dots) \tag{2}$$

where $E=\tilde{E}+E_0$, $V(x)-E_0=\phi^2(x)-(\hbar^2/\sqrt{2M})d\phi/dx$, and $\tilde{E}=\phi^2(\tilde{x}_a)=\phi^2(\tilde{x}_b)$, can reproduce with no *ad hoc* modification all the exact results that have been obtained from the WKB formula (1) modified with the Langer-like term. Despite its surprising success, it is unclear why the CBC formula (2) works magically well. In fact, the CBC formula has never been unambiguously verified. Furthermore, there is no qualitative explanation of the observed exactness of the formula (2). It is certainly important to understand the real nature of the CBC formula in relation to the WKB formula. The aim of this paper is not to verify the CBC formula but to report our observations which might shed light on the understanding of the CBC formula.

## 2. Transformations of Variables

We start with the observed fact that the CBC formula (2) provides exact energy spectra at least for the following three systems;

a) the one-dimensional harmonic oscillator,

b) the three-dimensional harmonic oscillator, and

c) the Pöschl-Teller oscillator.

For convenience, let us call these three the elementary systems.

Our first observation is that any of the examples that have been reported exactly soluble via the CBC formula is reducible to one of the three elementary systems by a change of variable. The CBC action (2), if the variable x is changed into $y=f(x)$, is written as

$$W = \int_{y_a}^{y_b} \{2M[\tilde{E} - \phi^2(x)]/[f'(x)]^2\}^{1/2}\, dy. \qquad (3)$$

If the superpotential $\phi(x)$ has the form, $\phi^2(x)/f'^2(x)=C + \chi(y)$, where C is a constant and $\chi(y)$ is a well-behaved function of y, then the CBC formula can be put into the form

$$W = \int_{y_a}^{y_b} \{2M[\tilde{\mathscr{E}} - \varphi^2(y)]\}^{1/2}\, dy, \qquad (4)$$

where $\tilde{\mathscr{E}}=\tilde{C}-C$, $\varphi^2(y)=\chi(y)+\tilde{C}-E/f'^2(y)$ and $\tilde{C}$ is an adjustable constant. What we are claiming is that by an appropriate change of variable the transformed squared superpotential $\varphi^2(y)$ of (4) can be put into one of the following forms;

(a) $\varphi^2(y) = \alpha\, y^2$ $\quad (-\infty \le y < \infty)$

(b) $\varphi^2(y) = \alpha\, y^2 + \beta y^{-2} + \gamma$ $\quad (0 \le y < \infty)$

(c) $\varphi^2(y) = \kappa(\kappa-1)\sec^2(ay)+\lambda(\lambda-1)\csc^2(ay).$ $\quad (0 \le y \le \pi/a).$

For instance, the modified Morse potential, $V(x)=Ae^{-2ax} - Be^{-ax}$, has the superpotential of the form, $\phi^2(x)=Ae^{-2ax} - [B-a\hbar(A/2M)^{1/2}]e^{-ax} - E_0$, where $E_0$ is the ground state energy of the Morse system. If we let $e^{-ax} = y^2$ $(0 \le y < \infty)$, then we have $\tilde{\mathcal{E}} = \tilde{C} + 4a^{-2}[B - a\hbar(2m)^{-1/2}]$ and $\varphi^2=(4A/a^2)y^2 - (4E/a^2)y^{-2} + \tilde{C}$. Thus the CBC action for the Morse system is transformed into the CBC action for the radial harmonic oscillator. Since the solution of the radial harmonic oscillator is already known, the energy spectrum of the Morse system is determined by comparison of the constants involved. Similarly, the actions for the Kepler problem (in two as well as three dimensions), the charged particle in a uniform magnetic field, the Dirac-Coulomb problem among others can be reduced to the action of the radial harmonic oscillator. The actions for the Rosen-Morse potential, the Hulthén potential (including the Yukawa potential limit), the Kepler problem in a hypersphere and many others are reducible to that of the Pöschl-Teller potential. It is instructive to mention that these examples are soluble not only by the standard Schrödinger methods but also by the algebraic methods [5] and the path integral methods [6].

## 3. Conversion of WKB into CBC

Since the three elementary systems can be solved exactly and many of the integrable examples are reducible to the elementary systems, the exactness of the CBC formula for a wide class of examples is no longer a mystery. For the one-dimensional harmonic oscillator, in particular, the WKB formula can be reduced to the form of the CBC formula. Verification of this is rather simple and omitted here. What remains to be clarified is why the CBC formula is exact for the last two elementary systems.

Our next observation provides no solution for the remaining mystery, but presents a curious fact that the WKB formula with the Langer-like modification can be converted to the CBC formula by

(a) the replacement, $n \to n - 1/2$, in the case of the one-dimensional harmonic oscillator,

(b) the combined replacements, $n \to n - 1/2$ and $\ell \to \ell + 1/2$, in the case of the radial harmonic oscillator, and

(c) the replacement, $n \to n - 1/2$, combined with the changes of parameters, say $\kappa \to \kappa + 1/2$ and $\lambda \to \lambda + 1/2$, in the case of the Pöschl-Teller oscillator.

Let us examine consequences of the above replacements by rewriting the WKB formula in the form,

$$\int_{t_a}^{t_b} [(M/2)\dot{x}^2 - V_{eff}(x) + E_n]dt = (n + 1/2)\pi\hbar, \tag{5}$$

where $V_{eff}(x)$ is the effective potential modified with the Langer-like term. Upon replacement, the equation (5) changes into

$$\int_{t_a}^{t_b} [(M/2)\dot{x}^2 - \hat{V}_{eff}(x) + \hat{E}_n]dt = n\pi\hbar, \tag{6}$$

where $\hat{E}_n = E_{n-1/2}$ and $\hat{V}_{eff}(x)$ is the transformed effective potential. The time integral over a half period can be converted into the space integral over the range $(\tilde{x}_a, \tilde{x}_b)$ between the two turning points,

$$\int_{\tilde{x}_a}^{\tilde{x}_b} \{2M[\hat{E}_n - \hat{V}_{eff}(x)]\}^{1/2} dx = n\pi\hbar. \tag{7}$$

Now we shall show that for the three elementary systems, $\hat{E}_n = \tilde{E}_n + \hat{E}_0$ and $\hat{V}_{eff}(x) = \phi^2(x) + \hat{E}_0$, so that

$$\hat{E}_n - \hat{V}_{eff}(x) = \tilde{E}_n - \phi^2(x), \tag{8}$$

with which (7) coincides with the CBC formula (2).

**(a) The One-Dimensional Harmonic Oscillator:**

In this case, $V_{eff}(x) = V(x) = \phi^2(x) = M\omega^2/2$ and $E_n = (n+1/2)\hbar\omega$. By $n \to n - 1/2$, $\hat{E}_n = E_{n-1/2} = n\hbar\omega$ and $\hat{E}_0 = 0$. Consequently, we have $\hat{E}_n = \tilde{E}_n$ and $\hat{V}_{eff}(x) = \phi^2(x)$, arriving at the CBC formula for the one-dimensional oscillator.

**(b) The Radial Harmonic Oscillator:**

The radial potential of the harmonic oscillator in three dimensions is $V(r) = (M\omega^2/2)r^2 + \ell(\ell+1)\hbar^2/(2Mr^2)$. With the Langer modification, it becomes $V_{eff}(r) = (M\omega^2/2)r^2 + (\ell+1/2)^2\hbar^2/(2Mr^2)$. The superpotential corresponding to the radial potential $V(r)$ has the form $\phi^2(r) = (M\omega^2/2)r^2 + (\ell+1)^2\hbar^2/(2Mr^2) - (\ell+1)\hbar\omega$. The energy spectrum for this oscillator is $E_n = (2n+\ell+3/2)\hbar\omega$, and $\tilde{E}_n = 2n\hbar\omega$. In this case, the replacement $n \to n - 1/2$ alone does not lead us to any significant result, but if we associate with the above replacement another replacement, $\ell \to \ell+1/2$, then we get $\hat{E}_n = (2n + \ell + 1)\hbar\omega$, $\hat{E}_0 = (\ell + 1)\hbar\omega$, and hence $\tilde{E}_n = \hat{E}_n - \hat{E}_0$. Also we have $\hat{V}_{eff}(r) = (M\omega^2/2)r^2 + (\ell+1)^2\hbar^2/(2Mr^2) = \phi^2(r) + \hat{E}_0$. Thus, we obtain the desired relation (8).

**(c) The Pöschl-Teller Oscillator:**

The Pöschl-Teller potential, $V(x) = V_0\{\kappa(\kappa-1)\csc^2(ax) + \lambda(\lambda-1)\sec^2(ax)\}$ where $V_0=a^2\hbar^2/(2M)$, $\kappa>1$, $\lambda>1$, and $0\le x\le\pi/a$, when the Langer-like correction is made, becomes $V_{eff}(x) = V_0\{(\kappa-1/2)^2\csc^2(ax)+(\lambda-1/2)^2\sec^2(ax)\}$. With this modified potential, the WKB formula yields the exact energy spectrum $E_n=V_0(2n+\kappa+\lambda)^2$, from which we obtain $\tilde{E}_n=V_0(2n+\kappa+\lambda)^2-V_0(\kappa+\lambda)^2$. The superpotential $\phi(x)$ formed from the original potential $V(x)$ gives $\phi^2(x)=V_0\{\kappa^2\csc^2(ax)+\lambda^2\sec^2(ax)\}-V_0(\kappa+\lambda)^2$. By replacements, $n\to n-1/2$, $\kappa\to\kappa+1/2$, and $\lambda\to\lambda+1/2$, we get $\hat{E}_n=V_0(2n+\kappa+\lambda)^2=E_n$ and $\hat{E}_0= V_0(\kappa+\lambda)^2 =E_0$. Hence, $\tilde{E}_n=\hat{E}_n-\hat{E}_0$ and $\hat{V}_{eff}(x)=V_0\{\kappa^2\csc^2(ax)+\lambda^2\sec^2(ax)\}= \phi^2(x)+\hat{E}_0$. Thus, we see that this system satisfies the relation $\hat{E}_n-\hat{V}_{eff}(x) = \tilde{E}_n -\phi^2(x)$.

This observation does not immediately explain why the CBC formula is exact for the last two elementary systems, but is indeed interesting. The replacements apply not only to the WKB quantum number n but also to the parameters involved. Although the parameter quantization itself may be achieved within the scheme of dynamical symmetry [7], the 1/2 shift operations of quantum numbers cannot be accommodated within the conventional framework of quantum mechanics. Our observation seems to suggest that the CBC formula is a supersymmetric counterpart of the WKB formula in a broader framework where the 1/2 shift operations is well defined. Until a logical ground for the replacement procedure is provided, the exactness of the CBC formula would remain as a mystery.


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