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Summation of divergent field-theoretical series for exact and variable values of asymptotic parameters: numerical estimates for the ground-state energy of a cubic anharmonic oscillator

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Abstract. Using, as an example, the calculation of the ground-state energy of a cubic anharmonic oscillator, we demonstrate a new approach to summation of divergent series. Our approach based on the Borel-Leroy transformation in combination with a conformal mapping does not require the knowledge of exact values of asymptotic parameters that determine the large-order behaviour of the series. Resumming field-theoretical expansions by varying the asymptotic parameters in a wide range of their exact values, we postulate the independence of the result of numerical analysis from the asymptotic parameters and based on this criterion we give a numerical estimate of the ground state energy of the cubic anharmonic oscillator for different values of the parameters of expantion and anisotropy, taking into account various orders of perturbation theory. We demonstrate good agreement between the results of our numerical calculations and the estimates obtained in the framework of the resummation technique using exact values of the asymptotic parameters. The results we achieved for the simplest anisotropic model allow us to apply this approach to investigate more complicated field-theoretical models describing real phase transitions in condensed matter physics or elementary particle theory, where the perturbation theory used has no small parameter of expansion and the exact values of the asymptotic parameters of the model are unknown.

1. Introduction

In the paper we discuss the problem of summation of divergent series arising in various fields of physics. For example, when studying critical behavior of models describing phase transitions in real substances using the field-theoretical renormalization group (RG) approach, important physical quantities, such as fixed points (PT) of RG equations or critical exponents governing the anomalous behavior of the thermodynamic functions at the phase transition point, are represented by power series in a certain parameter. Unfortunately, the parameter of these expansions does not belong to the range of convergence of the perturbative series. A typical situation is when the radius of convergence is zero and the resulting series are at least asymptotic. On the other hand, these series can contain important physical information relevant to predicting the critical behavior of real systems. To extract reliable information from them, they should be processed by a proper resummation procedure [1].

At present, there are several resummation methods such as the simple Pade, Pade-Borel, or Pade-Borel-Leroy techniques, whose application is, however, limited to the series with coefficients alternating in sings. For the more sophisticated method, first proposed in [2] and based on the Borel transformation combined with a conformal mapping, this limitation is not crucial. This resummation technique has been elaborated and used systematically for various phase transition problems, and is now regarded to as a most sophisticated procedure [3-6]. However, it requires a knowledge of the exact asymptotic high-order behaviour of the series. As a rule, the coefficients of the series

$$F(g)=\sum_{k=0}^{\infty} f_k g^k$$

behave at large orders as follows

$$f_k=(-a)^k k!k^b c[1+O(\tfrac{1}{k})], \quad k\to\infty,$$

where the numbers a and b characterize the main divergent part. Nowadays these parameters are found only for the simplest models [6-11], and calculating them for the complex anisotropic field-theoretical models including several coupling constants in their Landau-Ginzburg-Wilson Hamiltonians is a most difficult problem as yet unsolved.

The main goal of this investigation is to suggest a new approach to summation of divergent series. Our method is based on the Borel-Leroy transformation in combination with a conformal mapping, and does not imply knowledge of the exact asymptotic parameters. The proposed resummation technique is tested on simple model functions expanded in their asymptotic power series and applied to estimating the ground state energy of simple quantum mechanical problems, including that of an isotropic anharmonic oscillator. Then, our method is used to estimate the ground state energy of anharmonic oscillator with cubic anisotropy for several values of the expansion parameter and the parameter of anisotropy. The numerical estimates obtained are compared with those obtained by processing the corresponding series, taking into account the exact values of the asymptotic parameters.

2. The method of summation, its testing and application

Below, we attempt to overcome the outlined above difficulties and suggest a new approach to summation of divergent field-theoretical series, that is based on the standard technique of Borel transformation combined with a conformal mapping [1-3] but which does not involve the exact values of the asymptotic parameters. We start from the Borel-Leroy transformation of some physical quantity $F(g)$ in the form [3]

$$F(g;a,b)=\sum_{k=0}^{\infty} A_k(\lambda)\int_0^{\infty} e^{-\frac{x}{ag}}\left(\frac{x}{ag}\right)^b d\left(\frac{x}{ag}\right)\frac{w^k(x)}{[1-w(x)]^{2\lambda}}. \qquad (1)$$

Conformal mapping

$$w=\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$$

transforms the complex cut-plane C\[-1,-∞) (x) onto the unity circle of the plane (w) so that the semi-axis [0,∞), the domain of integration, goes over into the interval [0,1). Obviously inside the unit circle the series converges.

The coefficients $A_k(\lambda)$ are determined from the equality

$$B(x(w))=\frac{A(\lambda,w)}{(1-w)^{2\lambda}}$$

and the Borel transform $B(x)$ is the analytical continuation of the series

$$\sum_k \frac{f_k}{a^k\Gamma(b+k+1)}x^k$$

absolutely convergent in the unit circle, f_k being the coefficients of the original series. An additional parameter λ is chosen from the condition of the most rapid convergence of the series (1), i.e., from minimizing the quantity

$$\left|1-\frac{F_L(g;a,b)}{F_{L-1}(g;a,b)}\right|,$$

where L is the step of truncation and $F_L(g;a,b)$ is the L-partial sum for $F(g;a,b)$. In the regular scheme [2-6], parameters a and b are related to the exact asymptotic values a_0 and b_0. Since we deal in practice only with a part of the series in which the asymptotic regime might not be established, we vary parameters a and b in the neighborhood of their exact values. Our principle observation is that the results of processing $F_L(g;a,b)$ exhibit very weak dependence on the transformation parameters a and b as they vary over a wide range. This dependence becomes weaker with the growth of the approximation order. Moreover, the smaller the parameter of expansion g, the better this property holds. So, we made the stability of the result of processing with respect to variations of a and b the foundation of our technique for the summation of divergent series [12]. Such an approach allows us to apply transformation (1) even if the exact asymptotic behavior of the series being processed is unknown.

Table 1. Numerical estimates for the ground state energy of isotropic anharmonic oscillator, g=1

L	8	9	10	11	12	Exact value
$E(L)$	1.392376	1.392357	1.392344	1.392349	1.392351	1.392352

A detailed analysis of the formulated resummation scheme applied to the simple model functions (zero-dimensional field theory and Euler function) expanded into their asymptotic series

$$\mathscr{F}(g)=\int_{-\infty}^{+\infty} e^{-x^2-gx^4}dx \sim \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(2k+\frac{1}{2}\right)}{k!}g^k,$$

$$\mathscr{E}(g)=\int_0^{\infty} e^{-x}\left(x\frac{\partial}{\partial x}\right)^{b_0}\frac{1}{1+gx}dx \sim \sum_{k=0}^{\infty}(-1)^k k!\,k^{b_0}g^k$$

was performed in [12]. So, let us demonstrate here how our approach to summation works to calculate the ground state energy E(g) for several simple quantum mechanical models. We consider first the isotropic anharmonic oscillator [13] with the Hamiltonian

$$H=x^2+gx^4.$$

We observe the same stability of the result of processing E(g) with respect to the parameters a and b as for the model functions. The estimates for the ground state energy at g=1 are listed in Table 1 according to the approximation order L. For L=8, our numerical result is by one order closer to the exact value than the number 1.391655±0.004562 found in [14] on the basis of Wynn's ε-algorithm.

We also studied the ground state energy E(g) of the Yukawa potential

$$V_g(x)=-\frac{1}{x}e^{-gx}.$$

The dependence E(g) presented in Figure 1 is in agreement with the known results [14,15]. The exact critical value of g when the bound state disappears is g_c = 1.190612…. Using Winn's ε-algorithm [14] gives us the value g_c = 1.1836. The best estimate in the frame of our approach yields g_c = 1.191.

Finally, the ground state energy E(g) of the cubic anharmonic oscillator with the Hamiltonian

$$H=\frac{1}{2}\left(x^2+y^2\right)+\frac{g}{4}\left[x^4+2(1-\delta)x^2y^2+y^4\right]$$

depending of the anisotropy parameter δ and the value of the coupling constant g was investigated in [9]. Those calculations were based on knowledge of the exact values of the asymptotic parameters. The ground state energy estimated on the basis of our approach proved to be very close to the values of [9]. The results given by the two different methods are listed in Table 2. In the framework of our resummation scheme we also observed the stability of the result of processing E(g) with respect to the variation of the asymptotic parameters a and b.

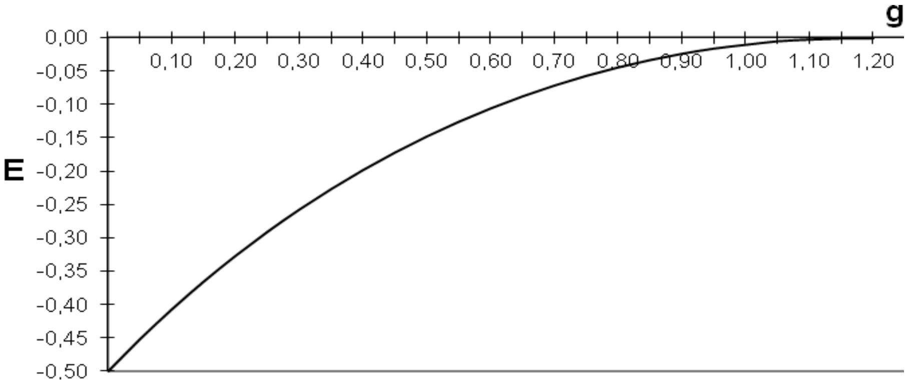

Figure 1. Dependence of the ground state energy E(g) for the Yukawa potential from the 8-th approximation order

Table 2. Numerical estimates of the ground state energy of the cubic anharmonic oscillator for various anisotropy parameters δ, coupling constant g and approximation order L

Results by Kleinert *et al* [9], $g/4$=0.1					
L/δ	-2.5	-1.5	-0.5	0.5	1.5
7	1.217107	1.192033	1.164803	1.134735	1.100604
9	1.217107	1.192034	1.164810	1.134736	1.100604
11	1.217107	1.192035	1.164810	1.134739	1.100604
Our estimates, $g/4$=0.1 ($0 \le b \le 60$, $0.5 \le a \le 1.5$)					
L/δ	-2.5	-1.5	-0.5	0.5	1.5
12	1.21705	1.19203	1.16480	1.13473	1.00600
Results by Kleinert *et al* [9], $g/4$=1.0					
L/δ	-2.5	-1.5	-0.5	0.5	1.5
7	1.941172	1.862806	1.733888	1.669172	1.535454
9	1.941172	1.862815	1.733909	1.669188	1.535425
11	1.941180	1.862823	1.733924	1.669199	1.535418
Our estimates, $g/4$=1.0 ($0 \le b \le 60$, $0.5 \le a \le 1.5$)					
L/δ	-2.5	-1.5	-0.5	0.5	1.5
12	1.9411	1.8627	1.7731	1.6691	1.5363

3. Conclusion

To sum up, let us formulate the results achieved in the present work. An approach to summation of divergent series has been suggested. The method employs the Borel transformation combined with a conformal mapping. It relies upon the stability of the result of processing on transformation parameters and therefore does not require knowing the exact asymptotic behavior of the series. The method has been tested on the functions expanded in their asymptotic series and applied to estimating the ground state energy of isotropic and cubic anharmonic oscillators. The principal observation is that within our approach, summation of the perturbative series of both simple and complex (anisotropic) models exhibits the same behaviour. This allows one to apply the developed technique to process divergent series arising in a number of anisotropic models describing phase transitions in real substances [6, 16-22]. It can be expected that the proposed summation method may be useful in other fields of physics, for example in QCD, where one deals with divergent series, but conventional resummation techniques are inapplicable.

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