QED transition amplitudes in external fields

Dissertation
zur Erlangung des akademischen Grades
Doctor of Philosophy

vorgelegt von

Misha Arturo Lopez Lopez

Institut für Theoretische Physik
Fakultät Physik
Bereich Mathematik und Naturwissenschaften
Technische Universität Dresden
2024

Eingereicht am 06.06.2024

1. Gutachter: Prof. Dr. Ralf Schützhold
2. Gutachter: Prof. Dr. Holger Gies

Verteidigt am 17.10.2024

Abstract

The main purpose of this thesis is to study quantum-electrodynamics (QED) in the presence of external background fields. We address this purpose by computing the Delbrück scattering amplitudes in the low-energy limit, the low-energy N-photon amplitudes in the presence of a constant field, the low-energy four-photon amplitudes in the presence of a constant magnetic field, the forward Compton scattering amplitudes in a constant magnetic field and the one-loop vertex correction in an arbitrary plane-wave field. In most cases, except for the vertex correction, we employ the worldline formalism to perform all calculations simultaneously for both scalar and spinor QED.

We utilize the previously obtained result of the off-shell four-photon amplitude with two low-energy photons to calculate the circularly polarized amplitudes for the leading-order contributions to Delbrück scattering, assuming that the incoming and outgoing photons have low-energy.

We compute the one-loop N-photon amplitudes in a constant background field considering off-shell low-energy photons in various field configurations. Assuming parallel magnetic and electric components of the background field enables us to obtain compact representations for these amplitudes involving only simple algebra and a single global proper-time integral with trigonometric integrands. Similarly, assuming a constant crossed field, we derive compact expressions for these amplitudes, represented by a single proper-time integral. The outcome of this integral, for fixed parameters, takes the form of a factorial function. The latter case is further refined by employing the spinor helicity formalism, where the helicity components are expressed solely in terms of Bernoulli numbers and spinor products. Moreover, for an arbitrary constant field, we obtain another representation of these amplitudes as series expansions in the external field.

As an application, we compute the one-loop four-photon amplitudes in the presence of a pure magnetic field for off-shell low-energy photons. Using these results, we calculate the polarized amplitudes for linear and circular polarizations in two distinct scenarios: when the magnetic field is coplanar with the scattering plane and when it is orthogonal to it.

We study the polarization flip of a photon scattered by an off-shell particle in the presence of a magnetic field. Specifically, we compute the Compton scattering amplitudes in a magnetic background field for off-shell massive particles and on-shell photons under the assumption that the scattering occurs in the forward direction, aligned along the same axis as the magnetic field. Additionally, we consider the polarization of the external photons to be perpendicular to each other.

We apply the operator technique within the Furry picture (Volkov states) to compute the general expression of the one-loop vertex correction in an arbitrary plane-wave background field for the case of two on-shell external electrons and an off-shell external photon. We show that the ultraviolet divergence can be renormalized exactly as in vacuum while the infrared divergence is avoided by introducing a finite photon mass. This calculation completes the study of QED in a plane-wave background field at one-loop order.

In most cases, except for the Delbrück scattering amplitudes, we perform non-perturbative calculations, given that the external background fields are taken into account exactly.

Keywords: Quantum electrodynamics, one-loop, amplitude, worldline, operator technique, helicity, background field, scalar, spinor, low-energy, strong field, plane-wave.

Contents

Chapter 1

Introduction

In quantum-electrodynamics (QED), the classical theory described by Maxwell equations is modified by the appearance of nonlinear corrections that violate the superposition principle and are due to the purely quantum effects. Specifically, the prediction made by Dirac [1, 2] regarding the existence of positrons (the antiparticle of electrons, later discovered by C. D. Anderson [3, 4]) and the uncertainty principle (expressed as $\Delta E \Delta t \geq \hbar/2$ where $\hbar$ is the reduced Plank constant) allow the existence of virtual electron-positron pairs, i.e., electron-positron pairs that live for a very brief period of time. This argument can be extended to all existing particles, suggesting that the vacuum can be viewed as a non-trivial medium replete with quantum fluctuations such as virtual particle-antiparticle pairs. Therefore, exploring these nonlinear effects through perturbations of the vacuum with external fields could potentially lead to physics beyond the standard model.

The first quantum correction to the Maxwell Lagrangian that includes nonlinear corrections to QED was given by H. Euler and W. Heisenberg. They obtained the nonperturbative, renormalized, one-loop effective Lagrangian for spinor particles in a classical electromagnetic background of constant field strength [5]. This Euler-Heisenberg Lagrangian can be expressed as[1]

$$\mathcal{L}_{\text{EH}} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} \mathrm{e}^{-m^2 T} \left\{ \frac{(eaT)(ebT)}{\tan(eaT)\tanh(ebT)} - \frac{2}{3}(eT)^2 \mathcal{F} - 1 \right\}, \tag{1.1}$$

where

$$a = \left(\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F} \right)^{1/2}, \qquad b = \left(\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F} \right)^{1/2}, \tag{1.2}$$

are written in terms of the two invariants of the Maxwell field

$$-2\mathcal{F} = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \vec{E}^2 - \vec{B}^2, \qquad -\mathcal{G} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \vec{E} \cdot \vec{B}. \tag{1.3}$$

The same affective Lagrangian, for scalar particles, was later derived by Weisskopf [6]. Using the same conventions as for $\mathcal{L}_{\text{EH}}$, the Weisskopf Lagrangian is given by

$$\mathcal{L}_{\text{W}} = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} \mathrm{e}^{-m^2 T} \left\{ \frac{(eaT)(ebT)}{\sin(eaT)\sinh(ebT)} + \frac{1}{6}(eT)^2 \mathcal{F} - 1 \right\}. \tag{1.4}$$

The Euler-Heisenberg (1.1) and Weisskopf (1.4) Lagrangians give rise to two significant phenomena: light-by-light scattering and Schwinger pair production [7]. For instance, by expanding the real part of these effective Lagrangians up to the e^4 order, it is possible to obtain the low-energy limit of the four-photon amplitudes, which correspond to the leading contributions to photon-photon scattering for small photon energies compared to the electron mass. On the other hand, the imaginary part leads to the probability of electron-positron pair creation by the constant electric field, known as the Schwinger effect [7]. However, this probability is extremely small for typical field strengths, becoming more significant only when the electric field approaches the critical value. Here, the critical fields strengths of QED are $E_{cr} = m^2c^3/\hbar|e| \sim 1.3 \times 10^{18}$ V/m for the electric field and $B_{cr} = m^2c^2/\hbar|e| \sim 4.4 \times 10^{13}$ G for the magnetic, i. e. $F_{cr} = m^2/|e|$ within natural units. For a comprehensive review of the effective Lagrangians (1.1) and (1.4), their applications and generalizations, refer to [8].

[1]From now on, we use natural units with $\varepsilon_0 = c = \hbar = 1$ and m and e denote the electron mass and charge, respectively.

The first calculation of the cross section for photon-photon scattering was done by H. Euler and B. Kockel [9, 10], in the above mentioned low-energy limit. Shortly later, this was followed by a calculation of the opposite high-energy limit by A. Akhiezer et al. [11, 12].

The first treatment of the photon-photon scattering amplitude for arbitrary on-shell kinematics was done by Karplus and Neuman [13, 14]. They analyzed the tensor structure of the four-photon amplitude, showed its finiteness and gauge invariance. Later, De Tollis [15, 16] recalculated the amplitude using dispersion relation techniques, which led to a more compact form of the result.

The consideration of four-photon amplitudes with some legs off-shell arises from the realization that photons emitted or absorbed by an external field in general can not be assumed to obey on-shell conditions. An important example is Delbrück scattering, the elastic scattering of a photon by a nuclear electromagnetic field, where the scattered photon can be taken real but the interaction with the field is described by virtual photons (see Fig. 2.3). This process, and similar ones involving interactions with the Coulomb field, motivated V. Costantini et al. to study the four-photon amplitude with two photons on-shell and two off-shell [17]. Moreover, we have investigated the fully off-shell four-photon amplitudes using the worldline formalism [18, 19, 20]. In [18], they derived an optimized tensor decomposition for the integrands of these off-shell amplitudes in both scalar and spinor QED. The development of analytical expressions for these off-shell amplitudes remains an ongoing project, partly documented in this thesis. In [19, 20] the off-shell four-photon amplitudes involving two low-energy photons are obtained analytically. The utilization of these results to compute the Delbrück scattering amplitudes at low energies is a primary focus of this thesis, see Chapter 2.

The presence of external electromagnetic fields in vacuum can polarize its virtual pairs, causing the vacuum to exhibit medium-like behavior, such as birefringence and dichroism. In the weak-field limit, the study of vacuum birefringence and dichroism can be conducted using the aforementioned off-shell four-photon amplitudes or through the Euler-Heisenberg and Weisskopf Lagrangians [21, 22, 23]. Additionally, in the presence of a background field, it is anticipated that a photon may split into two or more, or conversely, multiple photons may merge into one [24, 25, 26], phenomena which can also be described by the off-shell four-photon amplitudes.

On the experimental front, the observation of Delbrück scattering [27, 28, 29] and photon splitting [30] in high energy experiments involving heavy ions represent an indirect confirmation of photon-photon scattering and consequently, the presence of virtual pair interactions. Moreover, light-by-light scattering has been observed in heavy-ion collisions [31, 32, 33], although this observation remains somewhat indirect as the experiment is not purely photonic.

The observability of vacuum birefringence has been extensively investigated experimentally (for instance see [34, 35, 36, 37] and references therein), yet it has proven elusive in laboratory settings. Currently, two experiments are being conducted whose aim is to observe this effect: *Biréfringence Magnétique du Vide* (BMV) in Toulouse, France [34] and Observing Vacuum with Laser (OVAL) based in Tokyo, Japan [35]. However, claims of its observation have been made in astrophysical measurements [38] and further discussed in [39, 40].

The pursuit of understanding photonic processes remains an ongoing focus in current experiments, particularly in heavy ion collisions, astrophysical observations and laser-assisted experiments. In these contexts, field strengths of the order of the critical fields E_{cr} and/or B_{cr} are often encountered, motivating the non-perturbative study of photonic phenomena, for instance, see [41, 42, 43, 44, 45]. The worldline formalism offers various technical advantages in the computation of N-photon amplitudes in the presence of constant fields [46, 47], plane-wave fields [48] or combined constant and plane-wave fields [49], for both scalar and spinor QED. Indeed, the advantages of the worldline formalism include treating scalar and spinor loops in a similar manner, along with the inclusion of all possible contributions from inequivalent Feynman diagrams in a single expression. Therefore, in the present thesis, we investigate the N-photon amplitudes in a constant background field for photon energies small compared to the electron mass by deriving compact expressions for various field configurations, as detailed in Chapter 3. These derivations closely follow the corresponding calculations in vacuum [50, 51, 52]. Subsequently, we utilize these results to compute explicit expressions for the low-energy limit of the four-photon amplitudes in the presence of a pure magnetic field.

The emergence of high intensity lasers and X-ray free-electron lasers (XFELs) have opened the possibility of testing QED in new regimes and promises the realization of more sensitive experiments that would provide direct observations of light-by-light scattering and vacuum birefringence. This has given rise to new proposals of laser experiments [53, 54, 55, 56, 57] for the direct measurement of light-by-light scattering. In particular, the Helmholtz International Beamline for Extreme Fields (HIBEF) has as one of its primary goals to test vacuum birefringence in experimental setups that combine XFELs and high-intensity lasers [58, 59], see also [60, 61, 62, 63, 64]. Alternatively, the possibility of measuring vacuum birefringence in Coulomb-assisted setups has been proposed in [65]. In these setups, the abundance of nuclei may imply the presence of electrons, which could interact with XFEL photons through Compton scattering. Consequently, it becomes essential to examine the birefringent effects arising from Compton scattering [66]. This aspect is addressed to some extent in the present thesis utilizing the worldline formalism approach [67, 68, 69, 70], see Chapter 4.

The continuous development and upgrade of optical high-intensity lasers have allowed to reach record intensity values on the order of 5.5×10^{22} W/cm^2 [71]. As a comparison, note that the critical field of QED ($F_{cr}=m^2/|e|$) correspond to laser intensities on the order of 10^{29} W/cm^2. This progress promises the observation of nonlinear phenomena in a controlled manner [72, 73], and potentially, the direct observation of Schwinger pair production in the future, which is one of the main motivations behind increasing the laser intensity.

The use of these high-intensity lasers with intensities $10^{18}-10^{19}$ W/cm^2, combined with relativistic electrons (of $46.6-49.1$ GeV), has already proven evidence of nonlinear effects in Compton scattering and Breit-Wheeler pair production, reported in [74] and [75, 76], respectively. Similar setups have also been employed to observe radiation reaction effects [77, 78]. These observation demonstrate the occurrence of nonlinear effects, yet further studies with higher accuracy are still required to validate QED in the presence of strong background fields or potentially to detect deviations from it.

In experiments involving intense laser beams and relativistic electrons, the controlling parameter is the so-called classical nonlinearity parameter [79], given by

$$\xi_0=\frac{|e|F_0}{m\omega_0}\,, \tag{1.5}$$

where F_0 is the amplitude of the laser field and ω_0 is its angular frequency. This parameter indicates that the effects of the laser field have to be taken into consideration exactly for $\xi_0\gtrsim 1$. Additionally, the so-called quantum nonlinearity parameter

$$\chi_0=\frac{\sqrt{-(p_\mu F_0^{\mu\nu})^2}}{m\,F_{cr}}\,, \tag{1.6}$$

where p^μ is the initial four-momentum of the electron, represents the amplitude of the laser field in units of the critical field of QED in the initial rest frame of the electron. The strong-field QED regime is entered when $\chi_0\gtrsim 1$, meaning that quantum corrections become relevant.

In upcoming facilities such as the Center for Relativistic Laser Science (CoReLS) [80], the Extreme Light Infrastructure (ELI) [81], the Exawatt Center for Extreme Light Studies (XCELS) [82] and Apollon [83], testing QED in the strong field regime is one of the primary goals. These facilities aim to achieve laser intensities on the order of $10^{23}-10^{24}$ W/cm^2. This has motivated the study of quantum processes in a non-perturbative manner, for instance, nonlinear Compton scattering [84, 85, 86, 87, 88], nonlinear Breit-Wheeler pair production [89, 90, 91, 92], nonlinear Bethe-Heitler pair production [93, 94] and nonlinear trident pair production [95, 96]. However, even at one-loop order the radiative corrections to the probabilities of these processes have never been computed.

The standard theoretical approach to describe a quantum process in a laser field typically involves working within the Furry picture [97], where the electron-positron field is quantized in the presence of the background field [98, 99]. However, solving the Dirac equation exactly in the presence of a laser field is not feasible due to the complexity of real lasers. Instead, a laser field can be approximated to the ideal case of a plane-wave field, for which analytical solutions of the Dirac equation have been obtained by Volkov [100]. Therefore, laser interactions are commonly studied within the framework of the Furry picture and using Volkov states [101, 102, 87, 103].

The study of loop corrections in background fields for the high-field limit has lead to the Ritus-Narozny conjecture [104, 105, 106, 107], which states that for $\chi_0\gg 1$ the effective coupling of QED in a constant crossed field scales as $\alpha\chi_0^{2/3}$, where $\alpha=e^2/4\pi$ is the fine structure constant.

In particular, the one-loop vertex correction in the presence of a constant crossed field has been derived [108, 109], showing agreement with the above mentioned Ritus-Narozny conjecture. In the present thesis, we extend this understanding by deriving a more general result: the one-loop vertex correction in the presence of a plane-wave field [110], employing a non-perturbative approach, the operator technique within the Furry picture, see Chapter 5. Here, we confirm the agreement with the Ritus-Narozny conjecture and discuss its applications in relation to nonlinear effects in, for instance, Compton scattering, Breit-Wheeler pair production, Bethe-Heitler pair production, and trident pair production.

This thesis can be divided into three parts whose main subjects are: light-by-light scattering in the presence of background fields, Compton scattering in the presence of a magnetic field, and the one-loop vertex correction in a plane-wave background field. In Chapter 2, we present a short introduction to the worldline formalism, we review the calculation of the N-photon amplitudes and the cross sections for light-by-light scattering at low energies. We present the results obtained in previous works [20, 19] for the off-shell four-photon amplitude with two low-energy photons, for scalar and spinor QED. Thus, we use these results to obtain the Delbrück scattering cross-sections at low energies.

In Chapter 3, we present the known worldline master formulas for the N-photon amplitude in an arbitrary constant background field for both scalar and spinor QED. We use these expressions to obtain compact formulas for the N-photon amplitudes for fixed field configurations and low-energy photons and we use these results to compute the four-photon amplitudes in a pure magnetic field for low energy photons.

In Chapter 4, we present the master formulas for the dressed scalar and spinor propagators in a constant field and we use these expressions to compute the amplitudes of the off-shell Compton scattering in a pure magnetic field, in the forward direction.

In Chapter 5, we present a short introduction to the operator technique and the Volkov states and propagators. The one-loop vertex correction in vacuum is discussed as well and we present the calculation of the renormalized one-loop vertex correction in a plane-wave field and discuss our results in relation to gauge invariance, infrared divergences, application as a building block, and strong fields.

The discussion of results is summarized in the conclusions, Chapter 6.

Supplementary information and details are provided in the appendices. In Appendix A, we present important conventions used in this thesis. In Appendix C, we present a list of integral results that supplement Section 3.8. In Appendix B, we present supplementary details for some calculations.

Part of the results obtained in this thesis have been published in [110, 18, 19, 66, 111, 112, 113].

Chapter 2

N-photon amplitudes within the worldline formalism

In this chapter, our primary aim is to introduce the worldline formalism and the spinor helicity formalism, both of which will be utilized in this and some subsequent chapters. Here, it is important to emphasize that for the worldline formalism, we use the metric tensor in Euclidean space $(g^{\mu\nu}) = \mathrm{diag}(+1, +1, +1, +1)$, whereas for the spinor helicity formalism, we employ Minkowski space convention $(\eta^{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$. We present the general expressions for the one-loop N-photon amplitudes for off-shell particles, see [18]. We review the low-energy limit of these N-photon amplitudes (see [51, 114, 52]) in preparation for the next chapter, wherein analogous results are obtained in the presence of a constant background field. As an application to processes in a background field, we compute the leading order correction to Delbrück scattering for external photons of low energies, see [19].

The worldline formalism or "string inspired" formalism is a first-quantized approach for amplitude calculations whose starting point is the path integral representations already obtained for QED by Feynman in [115, 116] but their computational advantages have been recognized only after the work of Z. Bern and D. Kosower who, inspired by string theory, developed the field theory limit at tree- and loop-level using various string models [117, 118]. In particular, this approach was first derived for calculations in quantum chromodynamics. Later, M. Strassler was able to rederive the results of Bern and Kosower from quantum field theory [119]. Furthermore, Strassler analyzed the QED photon amplitudes and effective action [120], and noted that the formalism allowed him, using certain integration by parts that homogeneized the integrand and led to the automatic appearance of photon field strength tensors, to arrive at an extremely compact integral representation for the four-photon amplitudes in scalar and spinor QED. The integration by parts procedure was improved in [121, 122]. For a pedagogical and more complete review of this formalism, refer to [123, 124, 125].

The worldline formalism has enabled the derivation of several general results that combine the contribution of every possible independent Feynman diagram in a single master formula. Particularly, it has allowed to obtain general representations of the N-photon amplitudes in vacuum for scalar and spinor QED at one-loop order and for off-shell photons [118, 119, 121]. Subsequently, these same results were extended to include the interaction of a background field exactly, for the cases where there is a constant background field [46, 47], a plane-wave background field [48] and a combined constant and plane-wave background fields [49]. This is one of the main subjects of this thesis, the computation of photon-amplitudes. Specifically, in this chapter, we review the vacuum results for the low-energy limit of the N-photon amplitudes as in [51, 50, 52] to later derive similar results in the presence of a constant background field, see Chapter 3 and [111, 112].

The above mentioned vacuum N-photon amplitudes [118, 119, 121] have been used, in previous works, to study the off-shell four-photon amplitudes [20, 18, 19] and it is continued in this chapter by applying these results to compute the leading order correction to the low-energy limit of the Delbrück scattering amplitudes [19].

The generalization to the open-line case in scalar QED, i.e. the scalar propagator dressed with N photons, was given in [126, 127], and extended to the constant-field case in [67], see Chapter 4. Recently, a computationally efficient worldline representation has also been constructed for the dressed electron propagator (spinor QED)[68] and also extended to the constant-field case in [69, 70], see Chapter 4. Moreover, for both scalar and spinor QED, the dressed propagators in the presence of a plane-wave field have been derived [128]. Multi-loop QED amplitudes have also been studied using this approach [129, 47, 130, 50, 114, 131, 132, 127, 133].

Although in the present thesis we focus on the QED amplitudes, it should be mentioned that Bern-Kosower type formulas have been derived also for many other cases. See [117, 118, 134] for the N-gluon amplitudes in quantum electrodynamics, [135, 136, 137] for amplitudes involving gravitons, [138, 139, 140, 141] for Yukawa and axial couplings, [142, 143] for worldline Monte Carlo and [144, 145] for pair creation from worldline instantons.

The spinor helicity formalism was originally developed for calculating scattering process at high energies, where particle masses can be neglected. This approach has been widely used in quantum electrodynamics and quantum chromodynamics. For historical notes and an extensive pedagogical review, see [146]. Notably, in the case of photons (usually considered massless), this formalism can be applied at any energy regime and offers a convenient way to express the polarization states of photons in terms of helicity projection operators for fermions, enabling efficient amplitude calculations. In this thesis, we use the advantages of this formalism to supplement the study of photonic amplitudes for on-shell states. Specifically, we use the results and conventions presented in [147, 51, 69] to study N-photon amplitudes.

2.1 N-photon amplitudes for scalar and spinor QED

In the worldline approach, the one-loop N-photon amplitude for scalar QED is expressed as the following path integral representation (see [124, 123, 18])

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = (-ie)^N \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(0)=x(T)} Dx\,\mathrm{e}^{-\int_0^T d\tau\frac{\dot{x}^2}{4}}\,V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]\,. \tag{2.1}$$

Here T is the proper-time of the scalar particle in the loop, and the path integral is performed over the space of all closed loops in (Euclidean) spacetime with periodicity T. Each photon is represented by the following photon vertex operator, integrated along the trajectory

$$V^\gamma_{\rm scal}[k,\varepsilon] = \int_0^T d\tau\,\varepsilon\cdot\dot{x}(\tau)\,\mathrm{e}^{ik\cdot x(\tau)}\,. \tag{2.2}$$

where ε and k are the polarization and momentum of the photon being absorbed or emitted. The path integral is of gaussian form, and thus can be performed by Wick contractions in the one-dimensional worldline field theory. Using a formal exponentiation of the factor $\varepsilon\cdot\dot{x} = \mathrm{e}^{\varepsilon\cdot\dot{x}}|_\varepsilon$, one straightforwardly arrives at the following "Bern-Kosower master formula"

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) &= (-ie)^N(2\pi)^d\delta^d\left(\sum_{i=1}^N k_i\right)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i\\ &\times \exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Big|_{\varepsilon_1\varepsilon_2\ldots\varepsilon_N}\,.\end{aligned} \tag{2.3}$$

Here the dependence on the proper-time parameters $T,\tau_1,\ldots,\tau_N$ is encoded in the "worldline Green's function"

$$G(\tau_1,\tau_2) = \mid \tau_1-\tau_2\mid - \frac{(\tau_1-\tau_2)^2}{T}\,, \tag{2.4}$$

as well as its first and second derivatives

$$\begin{aligned}\dot{G}(\tau_1,\tau_2) &= \mathrm{sgn}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}\,,\\ \ddot{G}(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}\,.\end{aligned} \tag{2.5}$$

Here, dots denote a derivative acting on the first variable, i. e., $\dot{G}(\tau_1,\tau_2) = \frac{\partial}{\partial\tau_1}G(\tau_1,\tau_2)$, and we abbreviate $G_{ij} = G(\tau_i,\tau_j)$ etc. The notation $\big|_{\varepsilon_1\varepsilon_2\ldots\varepsilon_N}$ means that, after the expansion of the exponential, the terms contributing to the amplitude are those that have each polarization vector $\varepsilon_1,\ldots,\varepsilon_N$ linearly.

In [119, 121], it is shown that by expanding the exponential in (2.3) and by performing integration by parts the scalar N-photon amplitude can always be expressed as (here we have re-scaled the τ-integrals to the unit circle $\tau_i = Tu_i$, see Appendix A.3)

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = (-ie)^N(2\pi)^d\delta^d\left(\sum_{i=1}^N k_i\right)\int_0^\infty \frac{dT}{T}T^N(4\pi T)^{-\frac{d}{2}}{\rm e}^{-m^2T}\\ \times\prod_{i=1}^N\int_0^1 du_i\, Q_{\rm scal}(\dot G_{ij})\exp\left\{\frac{1}{2}T\sum_{i,j=1}^N G_{ij}\,k_i\cdot k_j\right\}.\end{aligned} \tag{2.6}$$

where $Q_{\rm scal}$ is a polynomial that depends only on products of $(\dot G_{ij})$ and all possible traces of products of field strength tensors. These traces can be expressed as "Lorentz-cycles" $Z_n(i_1i_2\ldots i_n)$ as

$$\begin{aligned}Z_2(ij) &= \frac{1}{2}{\rm tr}\big(f_if_j\big) = \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_jk_i\cdot k_j\,,\\ Z_n(i_1i_2\ldots i_n) &= {\rm tr}\Big(\prod_{j=1}^n f_{i_j}\Big)\,,\quad (n\geq 3)\,.\end{aligned} \tag{2.7}$$

where, $f_i^{\mu\nu}$ is the field strength tensor for the external photon 'i' and has the following representation

$$f_i^{\mu\nu} = k_i^\mu\varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu\,. \tag{2.8}$$

In $Q_{\rm scal}$, each Lorentz-cycle $Z_n(i_1i_2\ldots i_n)$ appears multiplied by a corresponding "τ-cycle" [119, 121], which has the following form

$$\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}\,. \tag{2.9}$$

This motivated the definition of a "bicycle" [123] as the product of the two

$$\dot G(i_1i_2\cdots i_n) = \dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}Z_n(i_1i_2\cdots i_n)\,. \tag{2.10}$$

Moreover, the amplitude is not composed only by bicycles, starting from $N=3$ there will be leftovers, called "n-tails" [119], where n, the "length" of the tail, denotes the number of polarization vectors involved in the tail. In general Q_N will involve tails with length $n = 1, 2, \ldots, N-2$. Thus for our present purposes we will need to know only the one- and two-tails. Those are given by

$$\begin{aligned}T(i) &= \sum_{r\neq i}\dot G_{ir}\varepsilon_i\cdot k_r\,,\\ T(ij) &= \sum_{\substack{r\neq i,s\neq j\\(r,s)\neq(j,i)}}\dot G_{ir}\varepsilon_i\cdot k_r\dot G_{js}\varepsilon_j\cdot k_s + \frac{1}{2}\dot G_{ij}\varepsilon_i\cdot\varepsilon_j\Big[\sum_{r\neq i,j}\dot G_{ir}k_i\cdot k_r - \sum_{s\neq j,i}\dot G_{js}k_j\cdot k_s\Big]\,.\end{aligned} \tag{2.11}$$

Here it should be noted that $\dot G(\tau,\tau) = 0$, so that, for example, the term with $r = i$ drops out in the sum defining the one-tail. In [123], the polynomials $Q_{\rm scal}$ have been worked out explicitly up to $N = 6$. In [122], the general structure of the n-tails is examined by means of integration by parts. Specifically, a very compact and manifestly gauge-invariant representation of the 2-tail has been derived in [18] which is the representation that we will use in the case of the four-photon amplitudes.

The above presented representation of the scalar N-photon amplitude is called the "Q-representation". This representation has as advantage that it allows one to make the transition from scalar to spinor QED by a simple pattern-matching procedure, the "Bern-Kosower replacement rule" [117, 118]: after the removal of all $\ddot G_{ij}$, the integrand for the N-photon amplitude in spinor QED can be obtained from the scalar QED one by multiplying the whole amplitude by a global factor of "-2" (for statistics and degrees of freedom), and applying the following "cycle replacement rule",

$$\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}\to\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\,, \tag{2.12}$$

where

$$G_F(\tau,\tau') = {\rm sgn}(\tau-\tau') \tag{2.13}$$

is the fermionic Green's function. This replacement transforms the bicycle $\dot G(i_1i_2\cdots i_n)$ into the "super-bicycle"

$$\dot G_{\rm S}(i_1i_2\cdots i_n) = \Big(\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\Big)Z_n(i_1\cdots i_n)\,. \tag{2.14}$$

Then, the one-loop N-photon amplitude for spinor QED can be expressed as

$$\Gamma_{\rm spin}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = -2(-ie)^N(2\pi)^d\delta^d\left(\sum_{i=1}^N k_i\right)\int_0^\infty \frac{dT}{T}T^N(4\pi T)^{-\frac{d}{2}}\mathrm{e}^{-m^2T} \times \prod_{i=1}^N\int_0^1 du_i\, Q_{\rm spin}(\dot{G}_{ij},G_{Fij})\exp\left\{\frac{1}{2}T\sum_{i,j=1}^N G_{ij}\,k_i\cdot k_j\right\}. \tag{2.15}$$

It is important to mention that the N-photon amplitudes derived within the worldline formalism are valid for off-shell photons. For further insights into the various representations of these amplitudes, see [18, 123, 122].

2.2 Spinor helicity for photons

In this section rather than developing the technology of spinor helicity, we will recall some important results of such formalism that will be useful in the calculation of photon-amplitudes, adapted from [147, 51, 69] (see also [146, 148]). Here, the metric tensor in Minkowski space $(\eta^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$ is employed.

The helicity of a particle is defined as the component of the spin in the direction of the three-momentum. It is well known that for massless fermions the positive and negative energy solutions of the massless Dirac equation are identical such that for definite helicity, we have

$$u_\pm(k) = \frac{1}{2}(1\pm\gamma_5)u(k) \qquad \text{and} \qquad v_\mp(k) = \frac{1}{2}(1\pm\gamma_5)v(k). \tag{2.16}$$

These spinors with definite helicity are sometimes called twistors [148]. For the conjugate states, we have similar relations

$$\overline{u_\pm(k)} = \frac{1}{2}\overline{u(k)}(1\mp\gamma_5) \qquad \text{and} \qquad \overline{v_\mp(k)} = \frac{1}{2}\overline{v(k)}(1\mp\gamma_5). \tag{2.17}$$

In the present case and due to the fact that we are interested in amplitudes with a large number of momenta, we label them by $k_i,\ i=1,2...N$. Using a shorthand notation, the twistors are

$$|k_i^\pm\rangle = u_\pm(k_i) = v_\mp(k_i)\,, \qquad \langle k_i^\pm| = \overline{u_\pm(k_i)} = \overline{v_\mp(k_i)}\,, \tag{2.18}$$

with the basic spinor products

$$\langle ij\rangle_- = \langle ij\rangle = \langle k_i^-|k_j^+\rangle = \overline{u_-(k_i)}\,u_+(k_j)\,, \qquad \langle ij\rangle_+ = [ij] = \langle k_i^+|k_j^-\rangle = \overline{u_+(k_i)}\,u_-(k_j)\,. \tag{2.19}$$

Some useful identities are (the product of two four-vectors)

$$\langle ij\rangle[ji] = \langle k_i^-|k_j^+\rangle\langle k_j^+|k_i^-\rangle = 2k_i\cdot k_j\,, \tag{2.20}$$

the Gordon identity and the projection operator

$$\langle k^\pm|\gamma^\mu|k^\pm\rangle = 2k^\mu\,, \qquad |k^\pm\rangle\,\langle k^\pm| = \frac{1}{2}(\mathbb{1}\pm\gamma_5)\not{k}\,, \tag{2.21}$$

the complex conjugation and anti-symmetry

$$\langle ij\rangle_\pm^* = -\langle ij\rangle_\mp\,, \qquad \langle ij\rangle_\pm = -\langle ji\rangle_\pm\,, \qquad \langle ii\rangle_\pm = 0\,, \tag{2.22}$$

the Fierz rearrangement

$$\langle k_i^+|\gamma^\mu|k_j^+\rangle\langle k_r^+|\gamma_\mu|k_s^+\rangle = 2[ir]\langle sj\rangle\,, \tag{2.23}$$

the charge conjugation of current

$$\langle k_i^+|\gamma^\mu|k_j^+\rangle = \langle k_j^-|\gamma^\mu|k_i^-\rangle\,, \tag{2.24}$$

the Schouten identity

$$\langle ij\rangle\langle rs\rangle = \langle ir\rangle\langle js\rangle + \langle is\rangle\langle rj\rangle\,. \tag{2.25}$$

For an n-point amplitude additionally, we have

$$\sum_{i=1}^{n}[ji]\langle ir\rangle = 0 \tag{2.26}$$

due to momentum conservation (basically the same identity as the sum of Mandelstam variables).

The spinor representation of the polarization vector for a photon of definite helicity ± 1

$$\varepsilon_\mu^\pm(k) = \pm\frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\,\langle q^\mp|k^\pm\rangle}\,. \tag{2.27}$$

The Dirac-gamma matrices in the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{2.28}$$

where $\mathbb{1}$ is the 2×2 identity matrix and σ^i are Pauli sigma matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.29}$$

The use of spinor helicity technique allow us to obtain close expression in terms of twistor products for the Lorentz cycles (2.41) and for products of f_i as they appear in the tails of the N-photon amplitudes. The latter identities were already derived in [51, 69] and are listed below

The polarized field strength tensor for each photon is

$$f_i^{\pm\,\mu\nu} = k_i^\mu\varepsilon_i^{\pm\,\nu} - k_i^\nu\varepsilon_i^{\pm\,\mu} \tag{2.30}$$

which can be written in terms of twistors as

$$f_i^{\pm\,\mu\nu} = -\frac{1}{4\sqrt{2}}\,\langle k_i^\pm|[\gamma^\mu,\gamma^\nu]|k_j^\pm\rangle\,. \tag{2.31}$$

Commutators

$$[f_i^+, f_j^-]^{\mu\nu} = 0\,. \tag{2.32}$$

Anticommutators

$$\{f_i^\pm, f_j^\pm\}^{\mu\nu} = -\frac{1}{2}\langle ij\rangle_\pm^2\,\eta^{\mu\nu}\,. \tag{2.33}$$

Factorization of traces

$$\mathrm{tr}(f_{i_1}^+\cdots f_{i_M}^+ f_{j_1}^-\cdots f_{j_N}^-) = \frac{1}{4}\mathrm{tr}(f_{i_1}^+\cdots f_{i_M}^+)\,\mathrm{tr}(f_{j_1}^-\cdots f_{j_N}^-)\,. \tag{2.34}$$

Same-helicity traces

$$\mathrm{tr}(f_{i_1}^\pm\cdots f_{i_N}^\pm) = \frac{(-1)^N}{\sqrt{2^{N-2}}}\,\langle i_1 i_2\rangle_\pm\langle i_2 i_3\rangle_\pm\,\cdots\,\langle i_{N-1}i_N\rangle_\pm\langle i_N i_1\rangle_\pm\,. \tag{2.35}$$

Chain products (as in the one- and two-tail)

$$k_j\cdot f_i^\pm\cdot k_{j'} = \frac{-1}{2\sqrt{2}}\,\langle ji\rangle_\pm\,\langle ij'\rangle_\pm\,\langle jj'\rangle_\mp\,, \tag{2.36}$$

$$k_j\cdot f_{i_1}^+\cdot f_{i_2}^+\cdot k_{j'} = \frac{1}{4}\,\langle ji_1\rangle_+\,\langle i_1 i_2\rangle_+\,\langle i_2 j'\rangle_+\,\langle jj'\rangle_-\,, \tag{2.37}$$

$$k_j\cdot f_{i_1}^+\cdot f_{i_2}^-\cdot k_{j'} = \frac{1}{4}\,\langle ji_1\rangle_+\,\langle i_1 j'\rangle_+\,\langle j' i_2\rangle_-\,\langle i_2 j\rangle_-\,, \tag{2.38}$$

notice that the previous products exhibit a behavior similar to the traces.

2.3 Low-energy limit of the N - photon amplitudes in vacuum

In this section, we review the the low-energy limit of the N-photon amplitudes in vacuum, adapted from [50, 18, 51]. This is in preparation for the next chapter, in which similar results are obtained in the presence of a constant external field. The low-energy limit of the photon amplitudes is defined by the condition that all photon energies be small compared to the mass of the loop scalar or fermion,

$$\omega_i \ll m, \quad i = 1, \dots, N\,. \tag{2.39}$$

This condition then justifies truncating all the vertex operators to their terms linear in the momentum. Note that the leading, momentum-independent term in this expansion integrates to zero for a closed loop, so that the first non-vanishing contribution to the amplitude are the terms linear in the external momenta. By adding a suitable total-derivative term, we can write the vertex operator of a low-energy (LE) photon as

$$V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f] = \frac{i}{2}\int_0^T d\tau\, x(\tau)\cdot f\cdot \dot{x}(\tau) = \frac{i}{2}\int_0^T d\tau \mathrm{e}^{x(\tau)\cdot f\cdot \dot{x}(\tau)}\Big|_f\,, \tag{2.40}$$

where $f^{\mu\nu} = k^\mu\varepsilon^\nu - \varepsilon^\mu k^\nu$ is the photon field-strength tensor. The Wick contraction of a product of such objects produces products of "Lorentz cycles"

$$Z_n(i_1 i_2 \dots i_n) = \Big(\frac{1}{2}\Big)^{\delta_{n2}} \mathrm{tr}\Big(\prod_{j=1}^n f_{i_j}\Big)\,, \tag{2.41}$$

with coefficients that, by suitable partial integrations, can be written as integrals of the "τ - cycles" $\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots \dot{G}_{i_n i_1}$ introduced above. In this way, and with a rescaling $\tau_i = Tu_i$, we arrive at

$$\big\langle V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f_N]\big\rangle = (iT)^N \exp\Big\{\sum_{n=1}^\infty b_{2n} \sum_{\{i_1\dots i_{2n}\}} Z^{\mathrm{dist}}_{2n}(\{i_1 i_2 \dots i_{2n}\})\Big\}\Big|_{f_1\dots f_N}\,, \tag{2.42}$$

where $Z^{\mathrm{dist}}_k(\{i_1 i_2\dots i_k\})$ denotes the sum over all distinct Lorentz cycles which can be formed with a given subset of indices, e.g. $Z^{\mathrm{dist}}_4(\{ijkl\}) = Z_4(ijkl) + Z_4(ijlk) + Z_4(ikjl)$, and b_n denotes the basic "cycle integral"

$$b_n = \int_0^1 du_1 du_2 \dots du_n\, \dot{G}_{12}\dot{G}_{23}\cdots \dot{G}_{n1}\,. \tag{2.43}$$

This integral can be expressed in terms of the Bernoulli numbers $\mathcal{B}_n$ [149]

$$b_n = \begin{cases} -2^n \frac{\mathcal{B}_n}{n!} & n \quad \text{even}\,, \\ \quad 0 & n \quad \text{odd}\,. \end{cases} \tag{2.44}$$

Note that (2.42) can be further simplified using the combinatorial fact that

$$\mathrm{tr}\Big[(f_1 + \dots + f_N)^n\Big]\Big|_{\text{all different}} = 2n \sum_{\{i_1\dots i_n\}} Z^{\mathrm{dist}}_n(\{i_1 i_2 \dots i_n\})\,, \tag{2.45}$$

($n \neq 0$). Introducing $f_{\mathrm{tot}} = \sum_{i=1}^N f_i$, using all this in (2.1) and eliminating the T-integral, we arrive at the following formula for the low-energy limit of the one-loop N-photon ($N \geq 4$) amplitude for scalar QED is [50]

$$\Gamma^{(\mathrm{LE})}_{\mathrm{scal}}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = \frac{e^N\Gamma(N-2)}{(4\pi)^2 m^{2N-4}} \exp\Big\{\sum_{n=1}^\infty \frac{b_{2n}}{4n}\mathrm{tr}(f^{2n}_{\mathrm{tot}})\Big\}\Big|_{f_1\dots f_N}\,, \tag{2.46}$$

here the trivial case $N = 2$ have been exempted to be able to set $d = 4$ and we have omitted the Dirac-delta function of momentum conservation.

The low-energy limit of the one-loop N-photon amplitude for spinor QED is obtained by employing the above-mentioned Bern-Kosower replacement rule. This rule tell us to simply replace the cycle integral (2.44) by the "super - cycle integral"

$$\int_0^1 du_1 du_2 \dots du_n \Big(\dot{G}_{12}\dot{G}_{23}\cdots \dot{G}_{n1} - G_{F12}G_{F23}\cdots G_{Fn1}\Big) = (2-2^n)\, b_n \tag{2.47}$$

and add a global factor of (-2) for statistics and degrees of freedom. Then, the amplitude $(N \geq 4)$ become

$$\Gamma^{(\mathrm{LE})}_{\mathrm{spin}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = (-2)\frac{e^N\Gamma(N-2)}{(4\pi)^2m^{2N-4}}\,\exp\Big\{\sum_{n=1}^{\infty}(1-2^{2n-1})\frac{b_{2n}}{2n}\mathrm{tr}(f^{2n}_{\mathrm{tot}})\Big\}\Big|_{f_1\ldots f_N}. \tag{2.48}$$

Note that in the above derivation on-shell conditions have not yet been used.

2.3.1 Helicity components

In this section, we review the results obtained in [51], where it was shown how to obtain explicit expressions for all the helicity components of the on-shell N-photon amplitudes for low-energy photons, using the spinor helicity formalism. This is in preparation for the next chapter, in which similar results are obtained in the presence of a constant crossed field.

The starting point is to consider the one-loop effective Lagrangians in a constant background field: Euler-Heisenberg Lagrangian for spinor QED (1.1) and Weisskopf Lagrangian for scalar QED (1.4). In order to obtain the N-photon amplitude from the one-loop effective Lagrangian in a constant background field, the filed strength tensor is fixed as

$$F^{\mu\nu}_{\mathrm{tot}} = \sum_{i=0}^{N} f^{\mu\nu}_i\,, \tag{2.49}$$

where, as in (2.8), $f^{\mu\nu}_i = k^\mu_i\varepsilon^\nu_i - \varepsilon^\mu_i k^\nu_i$ represents the field strength tensor of an external photon with momentum k^μ_i and polarization ε^μ_i. The corresponding amplitude is then obtained by extracting the terms involving each $f_1, ..., f_N$ precisely once

$$\Gamma^{(\mathrm{LE})}_{\mathrm{spin/scal}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = \mathcal{L}_{\mathrm{EH/W}}(iF_{\mathrm{tot}})\Big|_{f_1\cdots f_N}. \tag{2.50}$$

Now, assuming fixed polarization for the photons, with L having the helicity '+' and $N-L$ the helicity '−', and using spinor helicity (see Section 2.2) , it is possible to compute the two Maxwell invariants. For each polarized photon, the field strength tensor is $f^{\pm,\mu\nu}_i = k^\mu_i\varepsilon^{\pm,\nu}_i - \varepsilon^{\pm,\mu}_i k^\nu_i$. Noticing that

$$\frac{1}{4}F_{\mathrm{tot}\,\mu\nu}F^{\mu\nu}_{\mathrm{tot}} = \chi_+ + \chi_-\,, \qquad \frac{1}{4}F_{\mathrm{tot}\,\mu\nu}\tilde{F}^{\mu\nu}_{\mathrm{tot}} = -i(\chi_+ - \chi_-)\,, \tag{2.51}$$

where

$$\chi_+ = \frac{1}{2}\sum_{1\leq i<j\leq N}[ij]^2\,, \qquad \chi_- = \frac{1}{2}\sum_{1\leq i<j\leq N}\langle ij\rangle^2\,, \tag{2.52}$$

it is possible to obtain

$$a = -i(\sqrt{\chi_+} - \sqrt{\chi_-})\,, \qquad b = \sqrt{\chi_+} + \sqrt{\chi_-}\,, \tag{2.53}$$

for the invariants in (1.2). Inserting these expressions into the Euler-Heisenberg Lagrangian (1.1) and after Taylor series expansion, this Lagrangian can be expressed as

$$\mathcal{L}_{\mathrm{EH}}(iF_{\mathrm{tot}}) = -2\frac{m^4}{(4\pi)^2}\sum_{N=4}^{\infty}\left(\frac{2e}{m^2}\right)^N\sum_{\substack{L=0\\ L\ \mathrm{even}}}^{N} C^{N,L}_{\mathrm{spin}}\,\chi_+^{\frac{L}{2}}\,\chi_-^{\frac{N-L}{2}}\,, \tag{2.54}$$

where

$$C^{N,L}_{\mathrm{spin}} = (-1)^{N/2}\,(N-3)!\sum_{r=0}^{L}\sum_{s=0}^{N-L}(-1)^{N-L-s}\,\frac{\mathcal{B}_{r+s}\,\mathcal{B}_{N-r-s}}{r!\,s!\,(L-r)!\,(N-L-s)!} \tag{2.55}$$

and $\mathcal{B}_n$ Bernoulli numbers. Similarly, the Weisskopf Lagrangian (see Eq. (1.4)) can be expressed as

$$\mathcal{L}_{\mathrm{W}}(iF_{\mathrm{tot}}) = \frac{m^4}{(4\pi)^2}\sum_{N=4}^{\infty}\left(\frac{2e}{m^2}\right)^N\sum_{\substack{L=0\\ L\ \mathrm{even}}}^{N} C^{N,L}_{\mathrm{scal}}\,\chi_+^{\frac{L}{2}}\,\chi_-^{\frac{N-L}{2}}\,, \tag{2.56}$$

where

$$C_{\rm scal}^{N,L} = (-1)^{N/2}\,(N-3)! \sum_{r=0}^{L}\sum_{s=0}^{N-L} (-1)^{N-L-s}\, \frac{\left(1-2^{1-r-s}\right)\left(1-2^{1-N+r+s}\right)\mathcal{B}_{r+s}\,\mathcal{B}_{N-r-s}}{r!\,s!\,(L-r)!\,(N-L-s)!}\,. \tag{2.57}$$

According to (2.50), the amplitudes with L '+' and $N-L$ '−' helicities are obtained from the corresponding term in the sum of (2.54) or (2.56) by picking out the terms multilinear in the f_i's. For L even, it is defined [51]

$$\chi_L^+ = (\chi_+)^{\frac{L}{2}}|_{\text{all different}} = \frac{(L/2)!}{2^{L/2}}\Big\{[12]^2[34]^2\cdots[(L-1)L]^2 + \text{all permutations}\Big\}\,, \tag{2.58}$$

$$\begin{aligned}\chi_{N-L}^- &= (\chi_-)^{\frac{N-L}{2}}|_{\text{all different}}\\ &= \frac{\left(\frac{N-L}{2}\right)!}{2^{\frac{N-L}{2}}}\Big\{\langle(L+1)(L+2)\rangle^2\langle(L+3)(L+4)\rangle^2\cdots\langle(N-1)N\rangle^2 + \text{all permutations}\Big\}\,.\end{aligned} \tag{2.59}$$

So that, the low-energy N-photon amplitudes with L external photons having helicity '+' and $N-L$ '−' are

$$\Gamma_{\rm spin}^{(LE)}(f_1^+;\ldots;f_L^+f_{L+1}^-;\ldots;f_N^-) = -2\,\frac{m^4}{(4\pi)^2}\left(\frac{2e}{m^2}\right)^N C_{\rm spin}^{N+n,L+\ell}\,\chi_L^+\,\chi_{N-L}^- \tag{2.60}$$

for spinor QED and

$$\Gamma_{\rm scal}^{(LE)}(f_1^+;\ldots;f_L^+f_{L+1}^-;\ldots;f_N^-) = \frac{m^4}{(4\pi)^2}\left(\frac{2e}{m^2}\right)^N C_{\rm scal}^{N+n,L+\ell}\,\chi_L^+\,\chi_{N-L}^- \tag{2.61}$$

for scalar QED. These amplitudes, in the low-energy regime, obey a "double Furry theorem", meaning that if there is an odd number of positive or negative helicities, the amplitude vanishes.

2.4 Four-photon amplitudes for scalar and spinor QED

In this section, we present the general expression for the four-photon amplitudes, see [18]. For scalar QED, we have

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4) = \frac{(-ie)^4}{(4\pi)^{\frac{d}{2}}}\,(2\pi)^d\delta^d(k_1+k_2+k_3+k_4)\int_0^\infty \frac{dT}{T}\,T^{4-\frac{d}{2}}\,{\rm e}^{-m^2T}\int_0^1\prod_{i=1}^4 du_i\,Q_{\rm scal}\,{\rm e}^{(\cdot)}\,. \tag{2.62}$$

Here, we have already done the usual rescaling $\tau_i = Tu_i$ such that the exponential part is

$${\rm e}^{(\cdot)} = \exp\Big\{T\sum_{i<j=1}^4 G_{ij}\,k_i\cdot k_j\Big\}\,, \tag{2.63}$$

the bosonic Green's functions are

$$G_{ij} = G(u_i,u_j) = |u_i-u_j| - (u_i-u_j)^2\,, \tag{2.64}$$

and the polynomial $Q_{\rm scal}$ is given by [1]

$$\begin{aligned}Q_{\rm scal} &= Q^4_{\rm scal} + Q^3_{\rm scal} + Q^2_{\rm scal} + Q^{22}_{\rm scal}\,,\\ Q^4_{\rm scal} &= \dot G(1234) + \dot G(2314) + \dot G(3124)\,,\\ Q^3_{\rm scal} &= \dot G(123)T(4) + \dot G(234)T(1) + \dot G(341)T(2) + \dot G(412)T(3)\,,\\ Q^2_{\rm scal} &= \dot G(12)T_{sh}(34) + \dot G(13)T_{sh}(24) + \dot G(14)T_{sh}(23) + \dot G(23)T_{sh}(14) + \dot G(24)T_{sh}(13) + \dot G(34)T_{sh}(12)\,,\\ Q^{22}_{\rm scal} &= \dot G(12)\dot G(34) + \dot G(13)\dot G(24) + \dot G(14)\dot G(23)\,.\end{aligned} \tag{2.65}$$

[1] When comparing with [123] note that there, a different basis was used for the four-cycle component Q^4. The two bases are related by cyclicity and inversion.

Let us remind that the expression for the bicycle is

$$\dot{G}(i_1 i_2 \cdots i_n) = \dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots \dot{G}_{i_n i_1} Z_n(i_1\cdots i_n)\,. \tag{2.66}$$

and for the Lorentz-cycle is

$$\begin{aligned} Z_2(ij) &= \frac{1}{2}\mathrm{tr}\big(f_i f_j\big) = \varepsilon_i\cdot k_j \varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\,,\\ Z_n(i_1 i_2\dots i_n) &= \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big)\,,\quad (n\ge 3)\,, \end{aligned} \tag{2.67}$$

where $f_i^{\mu\nu}$ is the field strength tensor of each photon (2.8). It is the "tails" that exist in various versions. For the present computation, we use the one-photon tail $T(i)$ of the original Q-representation [122] and the "short tail" $T_{sh}(ij)$ (this is the same 2-tail in (2.11) up to total derivatives), introduced in [18], as the two-photon tail:

$$\begin{aligned} T(i) &= \sum_{r\neq i}\dot{G}_{ir}\,\varepsilon_i\cdot k_r\,,\\ T_{sh}(ij) &= \sum_{r,s\neq i,j}\dot{G}_{ri}\dot{G}_{js}\,\frac{k_r\cdot f_i\cdot f_j\cdot k_s}{k_i\cdot k_j}\,. \end{aligned} \tag{2.68}$$

The spinor-loop result is obtained by employing the *Bern-Kosower replacement rule*, i.e. replacing simultaneously every closed (full) cycle $\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1}$ appearing in the integrand of the scalar-loop with

$$\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1} - G_{F i_1 i_2}G_{F i_2 i_3}\cdots G_{F i_n i_1}\,, \tag{2.69}$$

where $G_{Fij} = \mathrm{sgn}(u_i - u_j)$ is the fermionic Green's function. We write the spinor-loop amplitude as

$$\Gamma_{\rm spin}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4) = -2\frac{(-ie)^4}{(4\pi)^{\frac{d}{2}}}\,(2\pi)^d\delta^d(k_1{+}k_2{+}k_3{+}k_4)\int_0^\infty\frac{dT}{T}\,T^{4-\frac{D}{2}}\mathrm{e}^{-m^2T}\int_0^1\prod_{i=1}^{4}du_i\,Q_{\rm spin}\,\mathrm{e}^{(\cdot)}\,. \tag{2.70}$$

Thus, apart from a global factor of -2, the only difference to the scalar QED formula (2.62) is the replacement of $Q_{\rm scal}$ by $Q_{\rm spin}$ according to the rule (2.69). Let us also emphasize that equations (2.62), (2.70) are valid off-shell, and that the right-hand sides are manifestly finite term-by-term. The well-known spurious UV-divergences of the four-photon diagrams that usually cancel only in the sum of diagrams would show up here as logarithmic divergences of the T-integration at $T=0$, but have been eliminated already at the beginning by the integration-by-parts procedure that led to the Q-representation, see [122, 18].

To avoid carrying common prefactors, we have defined

$$\hat{\Gamma}_{\left\{\substack{\rm scal\\ \rm spin}\right\}} = \int_0^\infty\frac{dT}{T}\,T^{4-\frac{d}{2}}\mathrm{e}^{-m^2T}\int_0^1\prod_{i=1}^{4}du_i\,Q_{\left\{\substack{\rm scal\\ \rm spin}\right\}}\,\mathrm{e}^{(\cdot)}\,. \tag{2.71}$$

2.5 Low-energy limit of the four-photon amplitudes

In this section, our main interest is to compare three different methods of computing the cross section since it will be useful later for the calculation of the light-by-light polarized amplitudes in the presence of a constant field (Chapters 3) for some special cases. In order to do so, we explore the cross section of light-by-light scattering at low energies in vacuum for both scalar and spinor QED.

Here, a photon in the low-energy limit satisfies $\omega_i \ll m$, where ω_i is the energy of the photon and 'm' the mass in the loop. From the point of view of the amplitude we only consider the multi-linear terms in the momenta since this represent the first non-vanishing contribution to the four-photon amplitude (for more details in the present low-energy limit amplitudes, see Section 2.3 and Refs. [51, 50, 18]). The four-photon amplitudes at low energies (Fig. 2.1) is known for a long time. For spinor QED, it can be obtained from the well known Euler-Heisenberg effective Lagrangian [5] (see Eq. (1.1)) and for scalar QED the corresponding amplitude can be extracted from the Weisskopf effective Lagrangian [6] (see Eq. (1.4)).

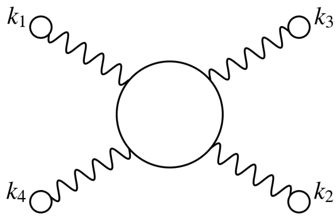

Figure 2.1: *Feynman diagram for the leading order contribution to light-by-light scattering with every photon having low-energy, indicated by empty bullets at their ends.*

The well known amplitudes for the leading order contribution to light-by-light scattering (within the worldline formalism conventions) are[2]

$$\begin{aligned}\Gamma_{\text{scal}}(k_1,\varepsilon_1;...;k_4,\varepsilon_4) = \frac{e^4}{(4\pi)^2m^4}\Big\{&b_4[Z_4(1234)+Z_4(2314)+Z_4(3124)]\\ &+b_2^2[Z_2(12)Z_2(34)+Z_2(23)Z_2(14)+Z_2(31)Z_2(24)]\Big\}\end{aligned} \tag{2.72}$$

for scalar QED, and

$$\begin{aligned}\Gamma_{\text{spin}}(k_1,\varepsilon_1;...;k_4,\varepsilon_4) = -\frac{2e^4}{(4\pi)^2m^4}\Big\{&-14b_4\,[Z_4(1234)+Z_4(2314)+Z_4(3124)]\\ &+(-2b_2)^2\,[Z_2(12)Z_2(34)+Z_2(23)Z_2(14)+Z_2(31)Z_2(24)]\Big\}\end{aligned} \tag{2.73}$$

for spinor QED.

The unpolarized differential cross section for the four-photon amplitude Γ_{4photon}, with photons of the same energy, is [17, 150, 151]

$$d\sigma = \frac{1}{64(2\pi)^2\omega^2}\,\overline{|\Gamma_{\text{4photon}}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4)|^2}\;\;d\Omega\,, \tag{2.74}$$

here the bar means the average of the amplitude over the polarizations, i.e.,

$$\overline{|\Gamma_{\text{4photon}}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4)|^2} = \frac{1}{4}\sum_{\varepsilon_i}|\Gamma_{\text{4photon}}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4)|^2 \tag{2.75}$$

and the differential of the solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$ with θ being the scattering angle and ϕ the azimuthal angle. For simplicity we define

$$d\hat{\sigma} := \sum_{\varepsilon_i}|\Gamma_{\text{4photon}}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4)|^2\,. \tag{2.76}$$

In the next subsections, we compute $d\hat{\sigma}$ using various methods.

2.5.1 Cross section from polarized amplitudes

Our initial approach to compute the unpolarized cross section involves collecting each polarized amplitude for fixed kinematics and then averaging over the polarized cross sections. The computation of the polarized amplitudes is carried out in Euclidean space where the convention of a four-vector is $v = (\boldsymbol{v}, v_4)$. We follow the kinematics set in [14, 15, 16] where the momenta k_1, k_2 are chosen to be

[2]Here, b_i are related to Bernoulli numbers by (2.44) and for this case $b_2 = \frac{-1}{3}$ and $b_4 = \frac{1}{45}$. The Lorentz cycles were defined in (2.41).

incoming while k_3, k_4 are outgoing

$$\begin{aligned} k_1 &= (0,0,-\omega,-i\omega)\ , \\ k_2 &= (0,0,\omega,-i\omega)\ , \\ k_3 &= (\omega\sin\theta,0,\omega\cos\theta,i\omega)\ , \\ k_4 &= (-\omega\sin\theta,0,-\omega\cos\theta,i\omega)\ . \end{aligned} \tag{2.77}$$

The linear polarizations $\varepsilon_i^{(\lambda_i)}$ are given by

$$\begin{aligned} \varepsilon_1^{(1)} = \varepsilon_2^{(1)} = \varepsilon_3^{(1)} = \varepsilon_4^{(1)} &= (0,1,0,0)\,, \\ -\varepsilon_1^{(2)} = \varepsilon_2^{(2)} &= (1,0,0,0)\,, \\ -\varepsilon_3^{(2)} = \varepsilon_4^{(2)} &= (\cos\theta,0,-\sin\theta,0)\ , \end{aligned} \tag{2.78}$$

as discussed in [14], the vectors $\varepsilon_j^{(1)}, \varepsilon_j^{(2)}$ are pointing in the perpendicular and parallel direction to the scattering plane, respectively. The set $(\varepsilon_j^{(1)}, \varepsilon_j^{(2)}, k_j)$ forms a right-handed system. The unit vectors for right $\varepsilon_j^{(+)}$ and left $\varepsilon_j^{(-)}$ handed circular polarization are given by

$$\varepsilon_j^{(\pm)} = \frac{1}{\sqrt{2}}\left[\varepsilon_j^{(1)} \pm i\varepsilon_j^{(2)}\right]\ . \tag{2.79}$$

Here, we use the following convention for the polarized amplitudes:

$$\Gamma^{\lambda_1\lambda_2\lambda_3\lambda_4}_{\text{scal/spin}} = \Gamma_{\text{scal/spin}}\left(k_1,\varepsilon_1^{*(\lambda_1)};k_2,\varepsilon_2^{*(\lambda_2)};k_3,\varepsilon_3^{(\lambda_3)};k_4,\varepsilon_4^{(\lambda_4)}\right)\ . \tag{2.80}$$

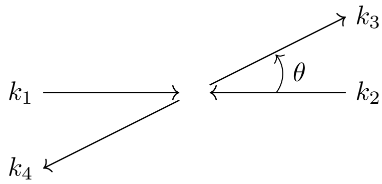

Figure 2.2: *Kinematics of the four-photon scattering in xz-plane.*

It is well known that, for identical photons, the four-photon amplitudes $\Gamma^{\lambda_1\lambda_2\lambda_3\lambda_4}$ satisfy the following relations [14, 16, 17]

$$\begin{aligned} &\Gamma^{1111}\,, \quad \Gamma^{2222}\,, \quad \Gamma^{1122} = \Gamma^{2211}\,, \quad \Gamma^{1212} = \Gamma^{2121}\,, \quad \Gamma^{1221} = \Gamma^{2112}\,, \\ &\Gamma^{1112} = \Gamma^{1121} = \Gamma^{1211} = \Gamma^{2111} = \Gamma^{2221} = \Gamma^{2212} = \Gamma^{2122} = \Gamma^{1222} = 0\,, \end{aligned} \tag{2.81}$$

for linear polarizations, and

$$\begin{aligned} &\Gamma^{++++} = \Gamma^{----}\,, \quad \Gamma^{++--} = \Gamma^{--++}\,, \quad \Gamma^{+-+-} = \Gamma^{-+-+}\,, \quad \Gamma^{+--+} = \Gamma^{-++-}\,, \\ &\Gamma^{+++-} = \Gamma^{++-+} = \Gamma^{---+} = \Gamma^{--+-} = \Gamma^{+-++} = \Gamma^{-+++} = \Gamma^{-+--} = \Gamma^{+---}\,, \end{aligned} \tag{2.82}$$

for circular polarizations. This is a consequence of the invariance under PT (parity and time-reversal) transformations. Additionally, for the present case of low-energy photons a "double Furry theorem" is satisfied by the amplitudes $\Gamma^{\lambda_1\lambda_2\lambda_3\lambda_4}$, which in this case implies $\Gamma^{+++-} = 0$ and $\Gamma^{1111} = \Gamma^{2222}$.

Therefore for the unpolarized cross section we have

$$\begin{aligned} \overline{|\Gamma_{\text{scal/spin}}(k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4)|^2} &= \frac{1}{2}\Big\{|\Gamma^{1111}_{\text{scal/spin}}|^2 + |\Gamma^{1122}_{\text{scal/spin}}|^2 + |\Gamma^{1212}_{\text{scal/spin}}|^2 + |\Gamma^{1221}_{\text{scal/spin}}|^2\Big\} \\ &= \frac{1}{2}\Big\{|\Gamma^{++++}_{\text{scal/spin}}|^2 + |\Gamma^{++--}_{\text{scal/spin}}|^2 + |\Gamma^{+-+-}_{\text{scal/spin}}|^2 + |\Gamma^{+--+}_{\text{scal/spin}}|^2\Big\}\,. \end{aligned} \tag{2.83}$$

The calculation of the non-vanishing contributions to the four-photon scalar and spinor amplitudes is a simple task. We computed these amplitudes with the aid of Mathematica [152] and the results are presented in Table 2.1. These results are in agreement with [14, 16][3].

[3] In [16] a misprint in [14] is pointed out where $\Gamma^{++++}_{\text{spin}}$ and $\Gamma^{++--}_{\text{spin}}$ seem to be interchanged.

Pol. amplitude	Scalar QED	Spinor QED
Γ^{1111}	$\frac{14(3+\cos^2\theta)\alpha^2\omega^4}{45m^4}$	$-\frac{32(3+\cos^2\theta)\alpha^2\omega^4}{45m^4}$
Γ^{1122}	$\frac{2(-13+\cos^2\theta)\alpha^2\omega^4}{45m^4}$	$\frac{8(1-7\cos^2\theta)\alpha^2\omega^4}{45m^4}$
Γ^{1212}	$-\frac{2[1+(-8+3\cos\theta)\cos\theta]\alpha^2\omega^4}{45m^4}$	$-\frac{4[31+(22+3\cos\theta)\cos\theta]\alpha^2\omega^4}{45m^4}$
Γ^{1221}	$-\frac{2[1+(8+3\cos\theta)\cos\theta]\alpha^2\omega^4}{45m^4}$	$-\frac{4[31+(-22+3\cos\theta)\cos\theta]\alpha^2\omega^4}{45m^4}$
Γ^{++++}	$\frac{32\alpha^2\omega^4}{45m^4}$	$-\frac{176\alpha^2\omega^4}{45m^4}$
Γ^{++--}	$\frac{4(3+\cos^2\theta)\alpha^2\omega^4}{15m^4}$	$\frac{8(3+\cos^2\theta)\alpha^2\omega^4}{15m^4}$
Γ^{+-+-}	$\frac{8(1+\cos\theta)^2\alpha^2\omega^4}{45m^4}$	$-\frac{44(1+\cos\theta)^2\alpha^2\omega^4}{45m^4}$
Γ^{+--+}	$\frac{8(1-\cos\theta)^2\alpha^2\omega^4}{45m^4}$	$-\frac{44(1-\cos\theta)^2\alpha^2\omega^4}{45m^4}$

Table 2.1: *Polarized amplitudes of light-by-light scattering for scalar and spinor QED.*

The differential cross section (2.74), after summing all components in Table 2.1 according to (2.83), is

$$d\sigma_{\rm scal} = \frac{34}{4(90)^2(2\pi)^2}\,\frac{\alpha^4}{m^8}\,\omega^6(3+\cos^2\theta)^2\,d\Omega \tag{2.84}$$

for scalar QED, and

$$d\sigma_{\rm spin} = \frac{139}{(90)^2\,(2\pi)^2}\,\frac{\alpha^4}{m^8}\,\omega^6(3+\cos^2\theta)^2\,d\Omega \tag{2.85}$$

for spinor QED.

The total cross section is obtained after integration[4]

$$\sigma_{\rm scal} = \frac{34}{4(90)^2(2\pi)}\,\frac{56}{5}\,\frac{\alpha^4}{m^8}\,\omega^6\,, \tag{2.86}$$

$$\sigma_{\rm spin} = \frac{139}{(90)^2\,(2\pi)}\,\frac{56}{5}\,\frac{\alpha^4}{m^8}\,\omega^6\,, \tag{2.87}$$

and are in complete agreement with the known results. The expression obtained in this section have been long studied by many scientists within which stand out Euler and Kockel who first studied the light-by-light cross section [9, 10] for spinor QED and later studied in more detail and for more general cases by [14, 16, 17]. This cross section nowadays is also found in several textbooks for instance see [98, 151, 150, 153][5], and has also been considered in the Born-Infeld theory [155, 154, 156]. The work [156] is of particular interest for this section since it contains both results the scalar and spinor QED cross sections and all elements in Table 2.1 can be compared with their results. Notice that in our case, the Mandelstam variables are

$$\begin{aligned} s &= 2k_1\cdot k_2 = -4\omega^2\,, \\ t &= 2k_1\cdot k_3 = 2\omega^2(1-\cos\theta)\,, \\ u &= 2k_1\cdot k_4 = 2\omega^2(1+\cos\theta)\,. \end{aligned} \tag{2.88}$$

However, in this chapter, we find more convenient to work with the momenta rather than the Mandelstam variables. For more details on the early stages of light-by-light scattering see [157].

2.5.2 Cross section from direct polarization sum

In the second approach, we compute the four-photon cross sections by performing the sum over polarizations according to the following prescription[6]

$$\sum_{\lambda=\pm}\varepsilon^{\mu}_{\lambda}(k_i)\,\varepsilon^{*\nu}_{\lambda}(k_i) = g^{\mu\nu} - (1-\xi)\frac{k_i^{\mu}k_i^{\nu}}{k^2}\,. \tag{2.89}$$

[4]Here, a factor of $\frac{1}{2}$ in needed to take into account the equivalence of the two final photons.
[5]In [154] a misprint in [150] is pointed out in the computation of the cross section.
[6]Here we clarify the equivalence between different conventions: $\varepsilon_i^{(\lambda)} = \varepsilon^{(\lambda)}(k_i) = \varepsilon_\lambda(k_i)$.

It is important to mention that the terms with '$k_i^\mu k_i^\nu$' will not contribute to the cross section due to the Ward identity (for simplicity, we use the Feynman gauge: $\xi = 1$).

Notice that we can compute the four-photon cross section simultaneously for both scalar and spinor QED since both amplitudes (2.72) and (2.73) have the following structure

$$\begin{aligned}\Gamma_{4\text{photon}}(k_1,\varepsilon_1;...;k_4,\varepsilon_4) = c_0 \Big\{ c_1 \Big[Z_4(1234) + Z_4(2314) + Z_4(3124) \Big] \\ + c_2 \Big[Z_2(12)Z_2(34) + Z_2(23)Z_2(14) + Z_2(31)Z_2(24) \Big] \Big\} .\end{aligned} \tag{2.90}$$

After squaring the amplitude, we notice that it is enough to compute the following seven contributions

$$\begin{aligned}
d\hat{\sigma}_1(1234) &= c_0^2\, c_1^2 \sum_{\varepsilon_i} Z(1234)^*\, Z(1234)\,, \\
d\hat{\sigma}_2(1234) &= c_0^2\, c_2^2 \sum_{\varepsilon_i} Z(12)^* Z(34)^*\, Z(12) Z(34)\,, \\
d\hat{\sigma}_3(1234) &= c_0^2\, c_1^2 \sum_{\varepsilon_i} Z(1234)^*\, Z(2314)\,, \\
d\hat{\sigma}_4(1234) &= c_0^2\, c_2^2 \sum_{\varepsilon_i} Z(12)^* Z(34)^*\, Z(23) Z(14)\,, \\
d\hat{\sigma}_5(1234) &= c_0^2\, c_1 c_2 \sum_{\varepsilon_i} Z(1234)^*\, Z(12) Z(34)\,, \\
d\hat{\sigma}_6(1234) &= c_0^2\, c_1 c_2 \sum_{\varepsilon_i} Z(1234)^*\, Z(23) Z(14)\,, \\
d\hat{\sigma}_7(1234) &= c_0^2\, c_1 c_2 \sum_{\varepsilon_i} Z(1234)^*\, Z(31) Z(24)\,,
\end{aligned} \tag{2.91}$$

every other contribution will be obtained through permutations, such that

$$d\hat{\sigma} = d\hat{\sigma}_1(1234) + d\hat{\sigma}_2(1234) + 2 \sum_{i=3}^{7} \Big[d\hat{\sigma}_i(1234)\; + (1 \to 2 \to 3 \to 1) + (1 \to 3 \to 2 \to 1) \Big] . \tag{2.92}$$

In the present case, the amplitudes depend exclusively on the field strength tensors of the photons. Then, the polarization sum can be carried out with [155]

$$\sum_{\lambda=\pm} f_i^{*\mu\mu'}\, f_i^{\nu\nu'} = g^{\mu\nu} k_i^{\mu'} k_i^{\nu'} + k_i^\mu k_i^\nu g^{\mu'\nu'} - g^{\mu\nu'} k_i^{\mu'} k_i^{\nu} - k_i^\mu k_i^{\nu'} g^{\mu'\nu} . \tag{2.93}$$

After summing over polarizations in (2.91) with (2.93), we perform all the contractions of the metric tensor $g^{\mu\nu}$ and the momenta k_i with the aid of Sympy [158] (a Python package), we obtain

$$\begin{aligned}
d\hat{\sigma}_1(1234) &= 2\, c_0^2\, c_1^2\, (k_{12}^4 + k_{13}^4 + k_{14}^4 + 4\, k_{12}^2 k_{14}^2)\,, \\
d\hat{\sigma}_2(1234) &= 4\, c_0^2\, c_2^2\, k_{12}^4\,, \\
d\hat{\sigma}_3(1234) &= 2\, c_0^2\, c_1^2\, (k_{12}^4 + k_{13}^4 + 3\, k_{14}^4 - 2\, k_{12}^2 k_{14}^2 - 2\, k_{13}^2 k_{14}^2)\,, \\
d\hat{\sigma}_4(1234) &= c_0^2\, c_2^2\, (k_{12}^4 + k_{13}^4 + k_{14}^4 - 2\, k_{12}^2 k_{13}^2 - 2\, k_{13}^2 k_{14}^2)\,, \\
d\hat{\sigma}_5(1234) &= 2\, c_0^2\, c_1 c_2\, (2\, k_{12}^4 - k_{12}^2 k_{13}^2 + k_{12}^2 k_{14}^2) \\
d\hat{\sigma}_6(1234) &= 2\, c_0^2\, c_1 c_2\, (2\, k_{14}^4 + k_{12}^2 k_{14}^2 - k_{13}^2 k_{14}^2)\,, \\
d\hat{\sigma}_7(1234) &= 2\, c_0^2\, c_1 c_2\, (k_{12}^4 + k_{13}^4 + k_{14}^4 - k_{12}^2 k_{13}^2 - 2\, k_{12}^2 k_{14}^2 - k_{13}^2 k_{14}^2)\,,
\end{aligned} \tag{2.94}$$

here, we have defined $k_{ij} = k_i \cdot k_j$.

The on-shell condition and momentum conservation imply

$$k_{12} = k_{34}, \quad k_{13} = k_{24}, \quad k_{14} = k_{23}, \quad k_{12}^2 k_{23}^2 + k_{12}^2 k_{31}^2 + k_{23}^2 k_{31}^2 = \frac{1}{2}(k_{12}^4 + k_{13}^4 + k_{14}^4) . \tag{2.95}$$

Using these identities and summing every contribution to the cross section, we obtain

$$d\hat{\sigma} = c_0^2\, (k_{12}^4 + k_{13}^4 + k_{14}^4)(22\, c_1^2 + 20\, c_1\, c_2 + 6\, c_2^2) . \tag{2.96}$$

Now the last step is to replace the constants c_0, c_1 and c_2 as their corresponding values in (2.72) and (2.73).

For scalar QED, we have that $c_0 = \frac{e^4}{(4\pi)^2 m^4} = \frac{\alpha^2}{m^4}$, $c_1 = b_4 = \frac{1}{45}$, $c_2 = b_2^2 = \frac{1}{9}$,

$$d\sigma_{\text{scal}} = \frac{1}{64(2\pi)^2\omega^2}\,\frac{1}{4}\,\frac{\alpha^4}{m^8}\,(k_{12}^4 + k_{13}^4 + k_{14}^4)\,\frac{(32)(34)}{(90)^2}\,d\Omega\,. \tag{2.97}$$

For spinor QED, we have that $c_0 = \frac{-2\alpha^2}{m^4}$, $c_1 = -14b_4 = \frac{-14}{45}$, $c_2 = 4\,b_2^2 = \frac{4}{9}$,

$$d\sigma_{\text{spin}} = \frac{1}{64(2\pi)^2\omega^2}\,\frac{1}{4}\,\frac{4\alpha^4}{m^8}\,(k_{12}^4 + k_{13}^4 + k_{14}^4)\,\frac{(32)(139)}{(90)^2}\,d\Omega\,. \tag{2.98}$$

As a final remark, notice that replacing the momenta in (2.77) into the expressions (2.97) and (2.98) will exactly reproduce the results in (2.84) and (2.85), respectively.

2.5.3 Cross section from spinor helicity

In the third approach, we adopt the conventions of [147, 51] with the tensor metric in Minkowski space such that $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. In the worldline formalism we work with the metric $(g^{\mu\nu}) = \text{diag}(+1, +1, +1, +1)$ in Euclidean space. Then in (2.72) and (2.73), we need to perform the replacement $g^{\mu\nu} \to -\eta^{\mu\nu}$ (see Appendix A) to compute the polarized amplitudes with the spinor helicity techniques, presented below.

The four-photon amplitude at low energies for two incoming and two outgoing photons can be expressed as

$$\begin{aligned}\Gamma_{\text{4photon}}(f_1^{\text{in}}; f_2^{\text{in}}; f_3^{\text{out}}; f_4^{\text{out}}) = c_0 \Big\{ & c_1 \Big[\text{tr}(f_1 f_2 f_3 f_4) + \text{tr}(f_2 f_3 f_1 f_4) + \text{tr}(f_3 f_1 f_2 f_4)\Big] \\ & + \frac{c_2}{4}\Big[\text{tr}(f_1 f_2)\text{tr}(f_3 f_4) + \text{tr}(f_2 f_3)\text{tr}(f_1 f_4) + \text{tr}(f_3 f_1)\text{tr}(f_2 f_4)\Big]\Big\}\,,\end{aligned} \tag{2.99}$$

which is exactly the same as (2.80) and here we make emphasis in the dependence on the field strength tensors of each photon f_i. We comment that in the previous section, in particular Eq. (2.90), was not necessary to fix the polarization propagation. Here, it is important to distinguish between incoming and outgoing polarizations since $(\varepsilon_i^{\pm})^* = \varepsilon_i^{\mp}$ which implies an effective change of the helicity of the photon. A similar analysis of the light-by-light amplitudes was carried out in [159] for the spinor part[7]. In the following we omit the super-indices 'in' and 'out'.

Using the spinor helicity identities introduced at the beginning of this section (more precisely equations (2.34), (2.35) and (2.25)), we can easily compute the polarized amplitudes for the four-photon case, for which we obtain

$$\Gamma_{\text{4photon}}(f_1^+; f_2^+; f_3^+; f_4^+) = \frac{c_0}{4}\Big(3c_1 + c_2\Big)\langle 12\rangle^2 [34]^2\,, \tag{2.100}$$

$$\Gamma_{\text{4photon}}(f_1^+; f_2^+; f_3^-; f_4^-) = \frac{c_0}{4}(c_1 + c_2)\Big(\langle 12\rangle^2\langle 34\rangle^2 + \langle 13\rangle^2\langle 24\rangle^2 + \langle 14\rangle^2\langle 23\rangle^2\Big)\,, \tag{2.101}$$

$$\Gamma_{\text{4photon}}(f_1^+; f_2^-; f_3^+; f_4^-) = \frac{c_0}{4}\Big(3c_1 + c_2\Big)[23]^2\langle 14\rangle^2\,, \tag{2.102}$$

$$\Gamma_{\text{4photon}}(f_1^+; f_2^-; f_3^-; f_4^+) = \frac{c_0}{4}\Big(3c_1 + c_2\Big)[24]^2\langle 13\rangle^2\,. \tag{2.103}$$

Notice that we can also derive these expressions from the results presented in Section 2.3, i.e., Ref. [51]. The contributions of these amplitudes to the cross section are

$$|\Gamma_{\text{4photon}}(f_1^+; f_2^+; f_3^+; f_4^+)|^2 = c_0^2\Big(3c_1 + c_2\Big)^2 k_{12}^4\,, \tag{2.104}$$

$$|\Gamma_{\text{4photon}}(f_1^+; f_2^+; f_3^-; f_4^-)|^2 = 2\,c_0^2 (c_1 + c_2)^2\Big(k_{12}^4 + k_{13}^4 + k_{14}^4\Big)\,, \tag{2.105}$$

[7]In [159] it seems that all photon polarizations are taken to be outgoing, that is why $\Gamma_{\text{spin}}^{++++}$ and $\Gamma_{\text{spin}}^{++--}$ appear interchanged.

$$|\Gamma_{4\text{photon}}(f_1^+; f_2^-; f_3^+; f_4^-)|^2 = c_0^2 \left(3c_1 + c_2\right)^2 k_{14}^4 \,, \tag{2.106}$$

$$|\Gamma_{4\text{photon}}(f_1^+; f_2^-; f_3^-; f_4^+)|^2 = c_0^2 \left(3c_1 + c_2\right)^2 k_{13}^4 \,. \tag{2.107}$$

From these expressions we can compute $d\hat{\sigma}$ and obtain exactly (2.96) or replace the four-momentum of each photon by (2.77) and obtain the results in Table 2.1.

2.6 Four-photon amplitudes with two low-energy photons

In previous works [19, 20], we have derived compact expressions for the off-shell four-photon amplitudes with two photons in their low-energy limit ($|\boldsymbol{k}_3|$, $|\boldsymbol{k}_4| \ll m$). In the case of d dimensions, we express the amplitudes entirely in terms of the hypergeometric function ${}_2F_1$ and its derivatives. However, for $d = 4$ (four space-time dimensions), it becomes possible to write the amplitudes completely in terms of elementary and trigonometric functions.

In this section, we present the explicit expressions for the off-shell four-photon amplitudes in $d = 4$ and with two photons in their low-energy limit, since we utilize these results in the next section to compute the Delbrück scattering amplitudes. For the sake of compactness, the following dimensionless variables have been introduced

$$\hat{k}_{12} = \frac{k_1 \cdot k_2}{m^2} \,, \qquad p_0 = \frac{\operatorname{arcsinh}\left(\frac{\sqrt{-\hat{k}_{12}}}{2}\right)}{\sqrt{(4 - \hat{k}_{12})(-\hat{k}_{12})}} \,. \tag{2.108}$$

Notice that in the following expressions the subindex '(34)' means that photon k_3 and k_4 are in their low-energy limit.

Scalar QED

For Q^4_{scal}

$$\begin{aligned}
\hat{\Gamma}^4_{\text{scal}(34)}(1234) &= -\frac{12 + 8(\hat{k}_{12} - 6)p_0}{3m^4\hat{k}_{12}^2} Z_4(1234)\,,\\
\hat{\Gamma}^4_{\text{scal}(34)}(2314) &= -\frac{2(\hat{k}_{12}^2 - 30\hat{k}_{12} + 108) - 8p_0(5\hat{k}_{12}^2 - 48\hat{k}_{12} + 108)}{9m^4(\hat{k}_{12} - 4)\hat{k}_{12}^2} Z_4(2314)\,,\\
\hat{\Gamma}^4_{\text{scal}(34)}(3124) &= -\frac{12 + 8(\hat{k}_{12} - 6)p_0}{3m^4\hat{k}_{12}^2} Z_4(3124)\,.
\end{aligned} \tag{2.109}$$

For Q^3_{scal}

$$\begin{aligned}
\hat{\Gamma}^3_{\text{scal}(34)}(123;4) &= -\frac{2(\hat{k}_{12}^2 - 48\hat{k}_{12} + 180) - 16p_0(4\hat{k}_{12}^2 - 39\hat{k}_{12} + 90)}{9m^6(\hat{k}_{12} - 4)\hat{k}_{12}^3} Z_3(123) k_2 \cdot f_4 \cdot k_1\,,\\
\hat{\Gamma}^3_{\text{scal}(34)}(412;3) &= -\frac{2(\hat{k}_{12}^2 - 48\hat{k}_{12} + 180) - 16p_0(4\hat{k}_{12}^2 - 39\hat{k}_{12} + 90)}{9m^6(\hat{k}_{12} - 4)\hat{k}_{12}^3} Z_3(412) k_2 \cdot f_3 \cdot k_1\,.
\end{aligned} \tag{2.110}$$

For $Q^2_{\rm scal}$

$$
\begin{aligned}
\hat\Gamma^2_{{\rm scal}(34)}(12;34) &= \frac{-1}{90} Z_2(12)\Biggl\{\Biggl[\frac{2(-\hat k_{12}+2)-16p_0}{m^6(\hat k_{12}-4)\hat k_{12}}\, k_1\cdot f_3\cdot f_4\cdot k_1 \\
&\quad + P_1\, k_1\cdot f_3\cdot f_4\cdot k_2 + (1\leftrightarrow 2)\Biggr] + 10P_2\; k_2\cdot f_4\cdot k_1 k_2\cdot f_3\cdot k_1\Biggr\}\,,\\
\hat\Gamma^2_{{\rm scal}(34)}(13;24) &= -\frac{2(\hat k_{12}-6)-16p_0(\hat k_{12}-3)}{9m^6(\hat k_{12}-4)\hat k_{12}^2} Z_2(13) k_1\cdot f_2\cdot f_4\cdot k_1\,,\\
\hat\Gamma^2_{{\rm scal}(34)}(23;14) &= -\frac{2(\hat k_{12}-6)-16p_0(\hat k_{12}-3)}{9m^6(\hat k_{12}-4)\hat k_{12}^2} Z_2(23) k_2\cdot f_1\cdot f_4\cdot k_2\,,\\
\hat\Gamma^2_{{\rm scal}(34)}(14;23) &= -\frac{2(\hat k_{12}-6)-16p_0(\hat k_{12}-3)}{9m^6(\hat k_{12}-4)\hat k_{12}^2} Z_2(14) k_1\cdot f_2\cdot f_3\cdot k_1\,,\\
\hat\Gamma^2_{{\rm scal}(34)}(24;31) &= -\frac{2(\hat k_{12}-6)-16p_0(\hat k_{12}-3)}{9m^6(\hat k_{12}-4)\hat k_{12}^2} Z_2(24) k_2\cdot f_1\cdot f_3\cdot k_2\,.
\end{aligned}
\tag{2.111}
$$

And finally, for $Q^{22}_{\rm scal}$

$$
\begin{aligned}
\hat\Gamma^{22}_{{\rm scal}(34)}(12,34) &= -\frac{2-8p_0}{3m^4\hat k_{12}} Z_2(12)Z_2(34)\,,\\
\hat\Gamma^{22}_{{\rm scal}(34)}(13,24) &= -\frac{2+8p_0}{9m^4(\hat k_{12}-4)} Z_2(13)Z_2(24)\,,\\
\hat\Gamma^{22}_{{\rm scal}(34)}(14,23) &= -\frac{2+8p_0}{9m^4(\hat k_{12}-4)} Z_2(14)Z_2(23)\,.
\end{aligned}
\tag{2.112}
$$

Here we have introduced two more functions, $P_i = P_i(\hat k_{12})$ with $i=1,2$, which are defined as

$$
P_1 = \frac{2(-\hat k_{12}^3+2\hat k_{12}^2-210\hat k_{12}+900)-32p_0(8\hat k_{12}^2-90\hat k_{12}+225)}{m^6(\hat k_{12}-4)\hat k_{12}^3}\,,
\tag{2.113}
$$

$$
P_2 = \frac{4(-\hat k_{12}^2+55\hat k_{12}-210)+48p_0(3\hat k_{12}^2-30\hat k_{12}+70)}{m^8(\hat k_{12}-4)\hat k_{12}^4}\,.
\tag{2.114}
$$

Spinor QED

For $Q^4_{\rm spin}$

$$
\begin{aligned}
\hat\Gamma^4_{{\rm spin}(34)}(1234) &= \frac{16[3+p_0(\hat k_{12}^2+2\hat k_{12}-12)]}{3m^4(\hat k_{12}-4)\hat k_{12}^2} Z_4(1234)\,,\\
\hat\Gamma^4_{{\rm spin}(34)}(2314) &= \frac{4[4\hat k_{12}^2-3\hat k_{12}-54-8p_0(\hat k_{12}^2+3\hat k_{12}-27)]}{9m^4(\hat k_{12}-4)\hat k_{12}^2} Z_4(2314)\,,\\
\hat\Gamma^4_{{\rm spin}(34)}(3124) &= \frac{16[3+p_0(\hat k_{12}^2+2\hat k_{12}-12)]}{3m^4(\hat k_{12}-4)\hat k_{12}^2} Z_4(3124)\,.
\end{aligned}
\tag{2.115}
$$

For $Q^3_{\rm spin}$

$$
\begin{aligned}
\hat\Gamma^3_{{\rm spin}(34)}(123;4) &= \frac{4(\hat k_{12}^2+15\hat k_{12}-90)+16p_0(\hat k_{12}^2-30\hat k_{12}+90)}{9m^6(\hat k_{12}-4)\hat k_{12}^3} Z_3(123) k_2\cdot f_4\cdot k_1\,,\\
\hat\Gamma^3_{{\rm spin}(34)}(412;3) &= \frac{4(\hat k_{12}^2+15\hat k_{12}-90)+16p_0(\hat k_{12}^2-30\hat k_{12}+90)}{9m^6(\hat k_{12}-4)\hat k_{12}^3} Z_3(412) k_2\cdot f_3\cdot k_1\,.
\end{aligned}
\tag{2.116}
$$

For $Q^2_{\rm spin}$

$$
\begin{aligned}
\hat{\Gamma}^2_{\text{spin}(34)}(12;34) &= \frac{1}{3}Z_2(12)\Bigg\{\Bigg[\frac{4(\hat{k}_{12}+2)+32p_0(\hat{k}_{12}-1)}{15m^6(4-\hat{k}_{12})^2\hat{k}_{12}}\,k_1\cdot f_3\cdot f_4\cdot k_1 \\
&\quad + \frac{2}{15}\tilde{P}_1\,k_1\cdot f_3\cdot f_4\cdot k_2 + (1\leftrightarrow 2)\Bigg] + \frac{4}{3}\tilde{P}_2\,k_2\cdot f_4\cdot k_1 k_2\cdot f_3\cdot k_1\Bigg\}, \\
\hat{\Gamma}^2_{\text{spin}(34)}(13;24) &= \frac{4(\hat{k}_{12}-6)-32p_0(\hat{k}_{12}-3)}{9m^6(\hat{k}_{12}-4)\hat{k}_{12}^2}Z_2(13)k_1\cdot f_2\cdot f_4\cdot k_1\,, \\
\hat{\Gamma}^2_{\text{spin}(34)}(23;14) &= \frac{4(\hat{k}_{12}-6)-32p_0(\hat{k}_{12}-3)}{9m^6(\hat{k}_{12}-4)\hat{k}_{12}^2}Z_2(23)k_2\cdot f_1\cdot f_4\cdot k_2\,, \\
\hat{\Gamma}^2_{\text{spin}(34)}(14;23) &= \frac{4(\hat{k}_{12}-6)-32p_0(\hat{k}_{12}-3)}{9m^6(\hat{k}_{12}-4)\hat{k}_{12}^2}Z_2(14)k_1\cdot f_2\cdot f_3\cdot k_1\,, \\
\hat{\Gamma}^2_{\text{spin}(34)}(24;31) &= \frac{4(\hat{k}_{12}-6)-32p_0(\hat{k}_{12}-3)}{9m^6(\hat{k}_{12}-4)\hat{k}_{12}^2}Z_2(24)k_2\cdot f_1\cdot f_3\cdot k_2\,.
\end{aligned}
\tag{2.117}
$$

And finally, for $Q^{22}_{\rm spin}$

$$
\begin{aligned}
\hat{\Gamma}^{22}_{\text{spin}(34)}(12,34) &= -\frac{16[1+2p_0(\hat{k}_{12}-2)]}{3m^4(\hat{k}_{12}-4)\hat{k}_{12}}Z_2(12)Z_2(34)\,, \\
\hat{\Gamma}^{22}_{\text{spin}(34)}(13,24) &= -\frac{8(1+4p_0)}{9m^4(\hat{k}_{12}-4)}Z_2(13)Z_2(24)\,, \\
\hat{\Gamma}^{22}_{\text{spin}(34)}(14,23) &= -\frac{8(1+4p_0)}{9m^4(\hat{k}_{12}-4)}Z_2(14)Z_2(23)\,,
\end{aligned}
\tag{2.118}
$$

Here, we have defined $\tilde{P}_i = \tilde{P}_i(\hat{k}_{12})$ as

$$
\tilde{P}_1 = \frac{2(\hat{k}_{12}^3+32\hat{k}_{12}^2-390\hat{k}_{12}+900)+16p_0(\hat{k}_{12}^3-46\hat{k}_{12}^2+270\hat{k}_{12}-450)}{m^6(\hat{k}_{12}-4)^2\hat{k}_{12}^3}, \tag{2.119}
$$

$$
\tilde{P}_2 = \frac{2(-23\hat{k}_{12}^2+200\hat{k}_{12}-420)-24p_0(\hat{k}_{12}^3-18\hat{k}_{12}^2+90\hat{k}_{12}-140)}{m^8(\hat{k}_{12}-4)^2\hat{k}_{12}^4}. \tag{2.120}
$$

2.7 Delbrück scattering for low-energy photons

In this section, we use the results in the previous section to compute the Delbrück scattering differential cross section for scalar and spinor QED under the assumption that the photon that interacts with the Coulomb field has low-energy (Fig. 2.3). This serves as an example of how to employ an off-shell amplitude to compute the amplitude of a specific process in the presence of an external background field. For the spinor QED case, this quantity was computed in detail in [17], therefore we will follow their conventions for easy comparison.

Here we use our above results for the four-photon amplitudes with two low-energy photons, and replace the two unrestricted legs 1 and 2 with two Coulomb photons. Furthermore, we now take the two low-energy photons (3 and 4) on-shell.

The vector potential for the photons from the Coulomb field is given by

$$
A_\mu(x) = \left(-\frac{Ze}{4\pi r},\,\mathbf{0}\right)\,, \tag{2.121}
$$

and has the following Fourier representation,

$$
A_\mu(x) = \varepsilon_\mu\int\frac{d^4k}{(2\pi)^4}\,\frac{Ze}{k^2}\,2\pi\,\delta(k_0)\,\mathrm{e}^{ik\cdot x}\,. \tag{2.122}
$$

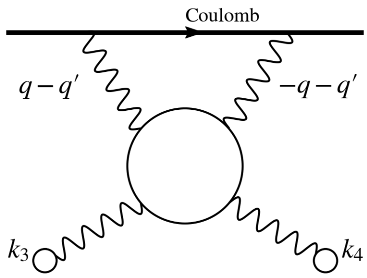

Figure 2.3: *Feynman diagram for the low-energy limit of Delbrück scattering. Empty bullets indicate low-energy photons.*

From here the new vertex operator for photons from the Coulomb field read as

$$\begin{aligned} V^{\gamma}_{\mathrm{Nuc}}[k,\varepsilon] &= \frac{Ze}{(2\pi)^3}\int d^4k\,\frac{\delta(k_0)}{k^2}\int_0^T d\tau\,\varepsilon\cdot\dot{x}\mathrm{e}^{ik\cdot x} \\ &= \frac{Ze}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{\mathbf{k}^2}\int_0^T d\tau\,\varepsilon\cdot\dot{x}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,. \end{aligned} \tag{2.123}$$

Thus in the present formalism, the Delbrück scattering amplitudes can be expressed as

$$\Gamma_{\left\{\substack{\mathrm{scal}(34)\\ \mathrm{spin}(34)}\right\}} = \left\{\begin{matrix}1\\-2\end{matrix}\right\}\frac{1}{2}\frac{e^4(Ze)^2}{(4\pi)^2(2\pi)^6}\int\frac{(d^3\mathbf{k}_1)(d^3\mathbf{k}_2)}{\mathbf{k}_1^2\mathbf{k}_2^2}(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\hat{\Gamma}_{\left\{\substack{\mathrm{scal}(34)\\ \mathrm{spin}(34)}\right\}}\,, \tag{2.124}$$

with $\hat{\Gamma}_{\left\{\substack{\mathrm{scal}(34)\\ \mathrm{spin}(34)}\right\}}$ as defined in (2.71) and with the results presented in the previous section. Note that the factor of $\frac{1}{2}$ that takes the symmetry between legs 1 and 2 into account.

In order to reproduce the result given in [17] for the differential cross section, we employ the following kinematics

$$\begin{aligned} k_1 &= (0,\mathbf{q}-\mathbf{q}')\,, \qquad k_2 = (0,-\mathbf{q}-\mathbf{q}'), \\ k_3 &= (i\omega,\mathbf{k}+\mathbf{q}')\,, \qquad k_4 = (-i\omega,\mathbf{q}'-\mathbf{k}), \end{aligned} \tag{2.125}$$

where

$$\begin{aligned} \mathbf{q}' &= (\omega\sin\theta/2,0,0), \\ \mathbf{k} &= (0,0,\omega\cos\theta/2), \\ \mathbf{q} &= (q_1,q_2,q_3), \end{aligned} \tag{2.126}$$

with ω as the energy and θ the scattering angle. The polarizations are chosen as

$$\begin{aligned} \varepsilon_1^{\mu} &= \varepsilon_2^{\mu} = (i,0,0,0), \\ \varepsilon_3^{\mu} &= \frac{1}{\sqrt{2}}(0,-i\lambda_3\cos\theta/2,1,+i\lambda_3\sin\theta/2), \\ \varepsilon_4^{\mu} &= \frac{1}{\sqrt{2}}(0,-i\lambda_4\cos\theta/2,1,-i\lambda_4\sin\theta/2), \end{aligned} \tag{2.127}$$

where $\lambda_i = \pm 1$ for right-and left-handed circular polarization, respectively. In the following it is understood that $\hat{\Gamma}^{+-}_{\mathrm{scal}(34)} = \hat{\Gamma}_{\mathrm{scal}(34)}|_{\lambda_3=1,\,\lambda_4=-1}$ etc. and we will also use the abbreviations

$$P_0 = \frac{\mathrm{arcsinh}\left(\frac{q}{2m}\right)}{q\sqrt{4m^2+q^2}}, \qquad S = \sin\frac{\theta}{2}, \qquad C = \cos\frac{\theta}{2}\,. \tag{2.128}$$

With the kinematics of (2.125) and using conservation of momentum we can write (2.124) as

$$\Gamma_{\left\{\substack{\mathrm{scal}(34)\\ \mathrm{spin}(34)}\right\}} = \left\{\begin{matrix}1\\-2\end{matrix}\right\}\frac{1}{2}\frac{e^4(Ze)^2}{(4\pi)^2(2\pi)^6}(2\pi)^4\delta(k_3^0+k_4^0)\int\frac{d^3\mathbf{q}}{|\mathbf{q}-\mathbf{q}'|^2|\mathbf{q}+\mathbf{q}'|^2}\hat{\Gamma}_{\left\{\substack{\mathrm{scal}(34)\\ \mathrm{spin}(34)}\right\}}\,. \tag{2.129}$$

For convenience, let us further define

$$\tilde{\Gamma}_{\left\{ \substack{\text{scal} \\ \text{spin}} \right\}} = \int \frac{d^3\mathbf{q}}{|\mathbf{q}-\mathbf{q}'|^2|\mathbf{q}+\mathbf{q}'|^2} \hat{\Gamma}_{\left\{ \substack{\text{scal}(34) \\ \text{spin}(34)} \right\}} . \tag{2.130}$$

Since we are considering the low-energy case, $\omega \ll m$, we neglect contributions of order superior to ω^2. We notice that

$$\begin{aligned} \int \frac{d^3\mathbf{q}}{|\mathbf{q}-\mathbf{q}'|^2|\mathbf{q}+\mathbf{q}'|^2} &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{q^2 \sin\theta' dq\, d\theta'\, d\phi'}{\left(q^2+\omega^2 \sin^2\frac{\theta}{2}\right)^2 - 4q_1^2\omega^2\sin^2\frac{\theta}{2}} \\ &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\theta' dq\, d\theta'\, d\phi'}{q^2} + \mathcal{O}(\omega) . \end{aligned} \tag{2.131}$$

Using the kinematics of (2.125), we find

$$\begin{aligned} \hat{\Gamma}^{+-}_{\text{scal}(34)} = \frac{4\omega^2}{3m^2q^4(4m^2+q^2)} \Big\{ & 3m^2[(6m^2+q^2) - 8m^2(3m^2+q^2)P_0](q_2^2-q_1^2) \\ & + [q^4(2m^2+q^2) - 3m^2(6m^2+q^2)q_2^2]S^2 - 8m^4[q^4 - 3(3m^2+q^2)q_2^2]S^2P_0 \Big\} + \mathcal{O}(\omega^3) , \end{aligned} \tag{2.132}$$

$$\begin{aligned} \hat{\Gamma}^{+-}_{\text{spin}(34)} = \frac{4\omega^2}{3q^4(4m^2+q^2)^2} \Big\{ & 3(4m^2+q^2)[(6m^2+q^2) - 8m^2(3m^2+q^2)P_0](q_2^2-q_1^2) \\ & + [q^4(2m^2-q^2) - 3(4m^2+q^2)(6m^2+q^2)q_2^2]S^2 \\ & - 8m^2[q^4(m^2+q^2) - 3(4m^2+q^2)(3m^2+q^2)q_2^2]S^2P_0 \Big\} + \mathcal{O}(\omega^3) , \end{aligned} \tag{2.133}$$

for the helicity non-conserving component, and

$$\begin{aligned} \hat{\Gamma}^{++}_{\text{scal}(34)} = \frac{4\omega^2}{3q^4(4m^2+q^2)} \Big\{ & 4[(6m^2+q^2) - 8m^2(3m^2+q^2)P_0](q_2^2-q_3^2) \\ & - [3q^2(2m^2+q^2) + 4(6m^2+q^2)q_2^2]C^2 \\ & + 4[q^2(6m^4+4m^2q^2+q^4) + 8m^2(3m^2+q^2)q_2^2]C^2P_0 \Big\} + \mathcal{O}(\omega^3) , \end{aligned} \tag{2.134}$$

$$\begin{aligned} \hat{\Gamma}^{++}_{\text{spin}(34)} = \frac{4\omega^2}{3q^4(4m^2+q^2)^2} \Big\{ & (6m^2+q^2)(16m^2+7q^2)(q_2^2-q_3^2) \\ & + 8[(3m^2+q^2)(4m^2+q^2)^2 - 3m^4q^2](q_3^2-q_2^2)P_0 \\ & - [3q^2(8m^4-2m^2q^2-q^4) + (6m^2+q^2)(16m^2+7q^2)q_2^2]C^2 \\ & + 8q^2(12m^6 - m^4q^2 - 5m^2q^4 - q^6)C^2P_0 \\ & + 8[(3m^2+q^2)(4m^2+q^2)^2 - 3m^4q^2]q_2^2C^2P_0 \Big\} + \mathcal{O}(\omega^3) , \end{aligned} \tag{2.135}$$

for the conserving ones. To perform the integral over $\mathbf{q}$ we use spherical coordinates:

$$q_1 = q\cos\theta' , \qquad q_2 = q\sin\theta'\cos\phi' , \qquad q_3 = q\sin\theta'\sin\phi' . \tag{2.136}$$

The integrals over θ' and ϕ' are trivial, and what remains to be calculated is

$$\tilde{\Gamma}^{+-}_{\text{scal}} = \frac{16\pi S^2\omega^2}{3} \int_0^\infty \frac{dq}{q^2} \frac{-6m^4 + m^2q^2 + q^4 + 24m^6P_0}{m^2q^2(4m^2+q^2)} , \tag{2.137}$$

$$\tilde{\Gamma}^{+-}_{\text{spin}} = \frac{32\pi S^2\omega^2}{3} \int_0^\infty \frac{dq}{q^2} \frac{-12m^4 - 4m^2q^2 - q^4 + 24m^4(2m^2+q^2)P_0}{q^2(4m^2+q^2)^2} , \tag{2.138}$$

$$\tilde{\Gamma}^{++}_{\text{scal}} = \frac{16\pi C^2\omega^2}{9} \int_0^\infty \frac{dq}{q^2} \frac{-42m^2 - 13q^2 + 4(42m^4 + 20m^2q^2 + 3q^4)P_0}{q^2(4m^2+q^2)} , \tag{2.139}$$

$$\tilde{\Gamma}^{++}_{\text{spin}} = \frac{32\pi C^2\omega^2}{9} \int_0^\infty \frac{dq}{q^2} \frac{-84m^4 - 20m^2q^2 + q^4 + 8(42m^6 + 17m^4q^2 - 2m^2q^4 - q^6)P_0}{q^2(4m^2+q^2)^2} . \tag{2.140}$$

Performing the integral over q, we get

$$\tilde{\Gamma}^{+-}_{\rm scal} = \frac{15\pi^3 S^2\omega^2}{32m^3}\,, \qquad \tilde{\Gamma}^{+-}_{\rm spin} = -\frac{5\pi^3 S^2\omega^2}{32m^3}\,, \tag{2.141}$$

$$\tilde{\Gamma}^{++}_{\rm scal} = \frac{3\pi^3 C^2\omega^2}{32m^3}\,, \qquad \tilde{\Gamma}^{++}_{\rm spin} = -\frac{73\pi^3 C^2\omega^2}{288m^3}\,. \tag{2.142}$$

Finally, the differential cross section is

$$d\sigma_{{\rm scal}(\lambda_3\lambda_4)} = \frac{(Z\alpha)^4\alpha^2}{4(2\pi)^6}\,|\tilde{\Gamma}^{\lambda_3\lambda_4}_{\rm scal}|^2\,d\Omega\,, \tag{2.143}$$

$$d\sigma_{{\rm spin}(\lambda_3\lambda_4)} = \frac{(Z\alpha)^4\alpha^2}{(2\pi)^6}\,|\tilde{\Gamma}^{\lambda_3\lambda_4}_{\rm spin}|^2\,d\Omega\,. \tag{2.144}$$

For scalar QED, we have

$$d\sigma_{{\rm scal}(++)} = d\sigma_{{\rm scal}(--)} = (Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{3}{16}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\cos^4\frac{\theta}{2}d\Omega\,, \tag{2.145}$$

$$d\sigma_{{\rm scal}(+-)} = d\sigma_{{\rm scal}(-+)} = (Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{15}{16}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\sin^4\frac{\theta}{2}d\Omega\,. \tag{2.146}$$

For spinor QED, we find

$$d\sigma_{{\rm spin}(++)} = d\sigma_{{\rm spin}(--)} = (Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{73}{72}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\cos^4\frac{\theta}{2}d\Omega\,, \tag{2.147}$$

$$d\sigma_{{\rm spin}(+-)} = d\sigma_{{\rm spin}(-+)} = (Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{5}{8}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\sin^4\frac{\theta}{2}d\Omega\,, \tag{2.148}$$

the spinor result is in agreement with [17] while the scalar is new, to the best of our knowledge.

Chapter 3

N-photon amplitudes in a constant background field

In the context of photon amplitudes in the presence of background fields, the worldline formalism [124, 123] have been used to derive master formulas for the N-photon amplitude in the presence of a constant [46, 47, 123, 122], plane-wave [48] and combined constant field & plane-wave [49] background fields. The previously mentioned master formulas have been derived for the cases in which the particle in the loop has spin zero or one-half. These master formulas can be used to study the polarization of the vacuum via, for instance, processes such as vacuum birefringence [65, 64, 160], photon splitting [24, 26, 44], photon merging [25, 161] and light-by-light scattering [18, 19, 45]. However, in order to properly study such effects we still need to calculate the multiple Schwinger-parameter integrals appearing in these master formulas.

Therefore, the approach in this chapter is to consider the simpler master formulas (the constant background field ones) and assume that all external photons are low-energy. In the next sections, we present the one-loop N-photon amplitude in the presence of a constant background field for scalar and spinor QED. In addition to the low-energy assumption, we consider different configurations for the external constant field. We first discuss the purely magnetic and later we generalized it to the case in which both electric and magnetic fields are pointing along the same direction. As a third case, we study the constant crossed field. And finally, we obtain alternative expressions for the case in which the field is considered arbitrarily constant.

3.1 N-photon amplitudes for scalar QED

Here, we make use of the results found in [46, 47] and later improved in [123, 122] for the scalar N-photon amplitude in a general constant field,

$$\Gamma_{\text{scal}}(k_1,\varepsilon_1;...;k_N,\varepsilon_N;F) = (2\pi)^d\delta^d\left(\sum_{i=1}^N k_i\right)\Gamma_{N,\text{scal}}(F)\,, \tag{3.1}$$

for convenience, we defined

$$\begin{aligned}\Gamma_{N,\text{scal}}(F) = (-ie)^N\int_0^\infty \frac{dT}{T}\,T^N(4\pi T)^{-d/2}\mathrm{e}^{-m^2T}\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\prod_{i=1}^N\int_0^1 du_i\\ \times\exp\left\{\sum_{i,j=1}^N\left(\frac{1}{2}k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j+\frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\right)\right\}\Big|_{\text{lin }\varepsilon_1\varepsilon_2...\varepsilon_N}.\end{aligned} \tag{3.2}$$

In the following, to avoid carrying the Dirac-delta function of momentum conservation, we work exclusively with the amplitudes $\Gamma_{N,\text{scal}}(F)$ for scalar and $\Gamma_{N,\text{spin}}(F)$ for spinor. In equation (3.2),

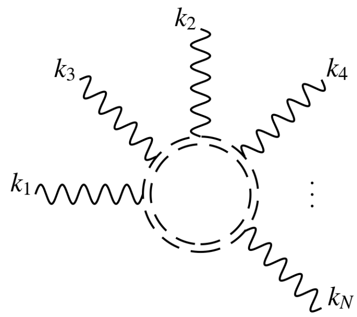

Figure 3.1: *N-photon one-loop Feynman diagram. The double dashed line indicate a particle of spin zero in a magnetic field.*

$\mathcal{Z} = eTF$ and the calligraphic Green's function and its derivatives are

$$\begin{aligned} \mathcal{G}_{Bij} &= \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{ij}} + i\mathcal{Z}\dot{G}_{ij} - 1\right) , \\ \dot{\mathcal{G}}_{Bij} &= \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{ij}} - 1\right) , \\ \ddot{\mathcal{G}}_{Bij} &= \frac{1}{T}\left(\ddot{G}_{ij} + 2 - \frac{2\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{ij}}\right) , \end{aligned} \tag{3.3}$$

derivatives are respect to the first parameter and G_{ij} is the vacuum Green's function, see (2.64). In this work, we mainly use re-scaled parameters $\dot{\mathcal{G}}_{Bij} = \dot{\mathcal{G}}_B(u_i, u_j)$, see Appendix A.3.

Note that, contrary to the vacuum case in which the coincidence limit vanishes i. e., $G(u,u) = 0$, the Green's function in the presence of a constant background field $\mathcal{G}_{Bii}$ and its derivatives have coincidence limit. Although $\ddot{\mathcal{G}}_{Bii}$ is not required for computing the N-photon amplitude, we present the coincidence limits of the calligraphic Green's functions[1]

$$\begin{aligned} \mathcal{G}_{Bii} &= \frac{T}{2\mathcal{Z}^2}\left(\mathcal{Z}\cot\mathcal{Z} - 1\right) , \\ \dot{\mathcal{G}}_{Bii} &= i\cot\mathcal{Z} - \frac{i}{\mathcal{Z}} , \\ \ddot{\mathcal{G}}_{Bii} &= -\frac{2}{T}\mathcal{Z}\cot\mathcal{Z} . \end{aligned} \tag{3.4}$$

As stated in [123, 122], the addition of a constant matrix to $\mathcal{G}_{Bij}$ and $\dot{\mathcal{G}}_{Bij}$ in (3.2) will have null effect due to momentum conservation. Then, we can use this fact to get rid of the coincidence limit of $\mathcal{G}_{Bij}$ and $\dot{\mathcal{G}}_{Bij}$, we define

$$\begin{aligned} \mathcal{G}_{ij} &= \mathcal{G}_{Bij} - \mathcal{G}_{Bii}, \\ \dot{\mathcal{G}}_{ij} &= \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii}, \end{aligned} \tag{3.5}$$

which allows to write the N-photon amplitude as

$$\begin{aligned} \Gamma_{N,\mathrm{scal}}(F) = (-ie)^N \int_0^\infty \frac{dT}{T} T^N (4\pi T)^{-d/2} \mathrm{e}^{-m^2 T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \\ \times \prod_{i=1}^N \int_0^1 du_i \, \exp\left\{\sum_{i,j=1}^N \left(\frac{1}{2}k_i \cdot \mathcal{G}_{ij} \cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{ij} \cdot k_j + \frac{1}{2}\varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j\right)\right\}\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2\ldots\varepsilon_N} , \end{aligned} \tag{3.6}$$

where now the calligraphic Green's functions $\mathcal{G}_{ij}$ and $\dot{\mathcal{G}}_{ij}$ do not have a coincidence limit. This allows us to work with $\Gamma_{N,\mathrm{scal}}(F)$ in an analogous way as in the vacuum case. After expansion of the

[1] See Appendix B.1.1 for the detailed calculation of one example of such coincidence limits.

amplitude to linear order in all $\varepsilon_1\varepsilon_2...\varepsilon_N$ we can remove all second derivatives $\ddot{\mathcal{G}}_{Bij}$ using integration by parts with the following choice

$$\frac{\partial}{\partial u_i}\dot{\mathcal{G}}_{ij} = \ddot{\mathcal{G}}_{Bij} \,, \tag{3.7}$$

which makes the calculation of the amplitude completely similar to the vacuum one, giving rise to the $\mathcal{Q}$-representation of the N-photon amplitude in a constant background field

$$\begin{aligned}\Gamma_{N,\text{scal}}(F) = (-ie)^N \int_0^\infty \frac{dT}{T}\, T^N (4\pi T)^{-d/2} \mathrm{e}^{-m^2T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \\ \times \prod_{i=1}^N \int_0^1 du_i\, \mathcal{Q}_{\text{scal}}(\dot{\mathcal{G}}_{ij}) \exp\left\{\frac{1}{2}\sum_{i,j=1}^N k_i\cdot\mathcal{G}_{ij}\cdot k_j\right\} ,\end{aligned} \tag{3.8}$$

where $\mathcal{Q}_{\text{scal}}(\dot{\mathcal{G}}_{ij})$ is polynomial where now the Lorentz-cycles and the so called τ-cycles appearing in the vacuum case [18] will mix. This motivates the definition of the "Lorentz trace"

$$\dot{\mathcal{G}}(i_1 i_2 \dots i_n) = \left(\frac{1}{2}\right)^{\delta_{n2}} \mathrm{tr}(f_{i_1}\cdot\dot{\mathcal{G}}_{i_1 i_2}\cdot f_{i_2}\cdot\dot{\mathcal{G}}_{i_2 i_3}\cdots f_{i_n}\cdot\dot{\mathcal{G}}_{i_n i_1})\,, \tag{3.9}$$

which will appear in $\mathcal{Q}_{\text{scal}}(\dot{\mathcal{G}}_{ij})$ together with the "n-tails". For instance, the one- and two-tails, in this case, are

$$\begin{aligned}\mathcal{T}(i) &= \sum_{r\neq i}\varepsilon_i\cdot\dot{\mathcal{G}}_{ir}\cdot k_r\,, \\ \mathcal{T}(ij) &= \sum_{\substack{r\neq i, s\neq j\\ (r,s)\neq(j,i)}} \varepsilon_i\cdot\dot{\mathcal{G}}_{ir}\cdot k_r \varepsilon_j\cdot\dot{\mathcal{G}}_{js}\cdot k_s + \frac{1}{2}\varepsilon_i\cdot\dot{\mathcal{G}}_{ij}\cdot\varepsilon_j\left[\sum_{r\neq i,j} k_i\cdot\dot{\mathcal{G}}_{ir}\cdot k_r - \sum_{s\neq j,i} k_j\cdot\dot{\mathcal{G}}_{js}\cdot k_s\right] .\end{aligned} \tag{3.10}$$

However, in this work, the inclusion of tails is unnecessary. Specifically, for low-energy photons, we solely consider terms linear in momenta, as those containing tails are of higher order and thus negligible. In Section 3.8.1, we use this master formula to compute the low-energy four-photon amplitudes in a pure magnetic field.

3.2 N-photon amplitudes for spinor QED

The procedure to obtain the spinor amplitude is similar to the one used in vacuum, Eq. (2.12). We start from the scalar amplitude and apply the *generalization of the Bern-Kosower replacement rule* [123, 122]. In order to apply this rule to the amplitude (3.8), we first need to replace $\dot{\mathcal{G}}_{ij}$ by

$$\hat{\dot{\mathcal{G}}}_{ij} = \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii} + \mathcal{G}_{Fii}\,, \tag{3.11}$$

in equation (3.6) to take into account the coincidence limit of the fermionic calligraphic Green's function $\mathcal{G}_{Fii} = -i\tan\mathcal{Z}$, see [122] and Appendix B.1.1. The replacement $\dot{\mathcal{G}}_{ij} \to \hat{\dot{\mathcal{G}}}_{ij}$ can be done directly in (3.8) but for the sake of clarity, we make the replacements $\mathcal{G}_{Bij}\to\mathcal{G}_{ij}$ and $\dot{\mathcal{G}}_{Bij}\to\hat{\dot{\mathcal{G}}}_{ij}$ in (3.2) to obtain

$$\begin{aligned}\Gamma_{N,\text{scal}}(F) = (-ie)^N \int_0^\infty \frac{dT}{T}\, T^N (4\pi T)^{-d/2} \mathrm{e}^{-m^2T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \\ \times \prod_{i=1}^N \int_0^1 du_i \exp\left\{\sum_{i,j=1}^N\left(\frac{1}{2}k_i\cdot\mathcal{G}_{ij}\cdot k_j - i\varepsilon_i\cdot\hat{\dot{\mathcal{G}}}_{ij}\cdot k_j + \frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{ij}\cdot\varepsilon_j\right)\right\}\Big|_{\text{lin }\varepsilon_1\varepsilon_2\dots\varepsilon_N} ,\end{aligned} \tag{3.12}$$

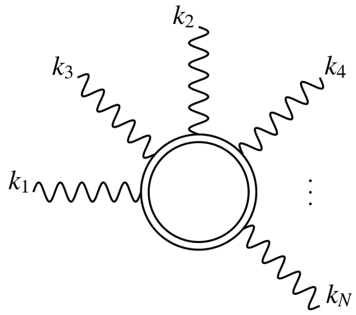

Figure 3.2: *N-photon one-loop Feynman diagram. The double solid line indicate a particle of spin one-half in a magnetic field.*

which again is justified by momentum conservation as in the previous section. After integration-by-parts, it can be expressed in a $\mathcal{Q}$-representation as

$$\begin{aligned}\Gamma_{N,\mathrm{scal}}(F) = (-ie)^N \int_0^\infty \frac{dT}{T}\, T^N (4\pi T)^{-d/2} \mathrm{e}^{-m^2 T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin \mathcal{Z}}\right] \\ \times \prod_{i=1}^{N} \int_0^1 du_i\, \mathcal{Q}_{\mathrm{scal}}(\dot{\hat{\mathcal{G}}}_{ij}) \exp\left\{\frac{1}{2}\sum_{i,j=1}^{N} k_i \cdot \mathcal{G}_{ij} \cdot k_j\right\},\end{aligned} \tag{3.13}$$

in which we can directly use the generalized Bern-Kosower replacement rule. The fermionic-Green's function in a constant field is

$$\mathcal{G}_{Fij} = G_{Fij}\, \frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{ij}}}{\cos \mathcal{Z}}\,. \tag{3.14}$$

In the following, we summarize the generalized Bern-Kosower replacement rule [122]:

1. Replace each Lorentz trace by the same trace minus the fermionic Lorentz trace

$$\dot{\hat{\mathcal{G}}}(i_1 i_2 \dots i_n) \quad \to \quad \dot{\hat{\mathcal{G}}}(i_1 i_2 \dots i_n) - \mathcal{G}_F(i_1 i_2 \dots i_n) = \dot{\mathcal{G}}_s(i_1 i_2 \dots i_n)\,, \tag{3.15}$$

   where we will refer to $\dot{\mathcal{G}}_s(i_1 i_2 \dots i_n)$ as the "Lorentz super-cycle", and

$$\mathcal{G}_F(i_1 i_2 \dots i_n) = \left(\frac{1}{2}\right)^{\delta_{n1}} \mathrm{tr}(f_{i_1} \cdot \mathcal{G}_{F i_1 i_2} \cdot f_{i_2} \cdot \mathcal{G}_{F i_2 i_3} \cdots f_{i_n} \cdot \mathcal{G}_{F i_n i_1})\,. \tag{3.16}$$

2. The scalar determinant must be replaced by the corresponding spinor determinant

$$\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin \mathcal{Z}}\right] \quad \to \quad \det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan \mathcal{Z}}\right]. \tag{3.17}$$

3. Multiply by the usual factor of -2 for statistics and degrees of freedom.

Therefore, the N-photon amplitude in a constant background field for spinor QED is

$$\begin{aligned}\Gamma_{N,\mathrm{spin}}(F) = -2(-ie)^N \int_0^\infty \frac{dT}{T}\, T^N (4\pi T)^{-d/2} \mathrm{e}^{-m^2 T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan \mathcal{Z}}\right] \\ \times \prod_{i=1}^{N} \int_0^1 du_i\, \mathcal{Q}_{\mathrm{spin}}(\dot{\hat{\mathcal{G}}}_{ij}, \mathcal{G}_{Fij}) \exp\left\{\frac{1}{2}\sum_{i,j=1}^{N} k_i \cdot \mathcal{G}_{ij} \cdot k_j\right\},\end{aligned} \tag{3.18}$$

where $\mathcal{Q}_{\mathrm{spin}}(\dot{\hat{\mathcal{G}}}_{ij}, \mathcal{G}_{Fij})$ will be determined by $\mathcal{Q}_{\mathrm{scal}}(\dot{\mathcal{G}}_{ij})$ through the replacement rule. For an example of it, see Section 3.8.2.

Notice that the generalized Bern-Kosower replacement rule presented in this section is different from the one in [123] due to the conventions of the calligraphic functions wherein the coincidence limits are subtracted, see [122].

3.3 Low-energy limit of the N-photon amplitudes in a constant field

In this section, we study the low-energy limit of N-photon amplitudes in a constant field. Similar as in the vacuum case (see Section 2.3 and [50, 18]), the N-photon amplitude for scalar QED, in terms of the vertex operator of each photon, is[2]

$$\Gamma^{(\mathrm{LE})}_{N,\mathrm{scal}}(F) = (-ie)^N \int_0^\infty \frac{dT}{T}\,(4\pi T)^{-d/2}\mathrm{e}^{-m^2T}\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\left\langle V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_N]\right\rangle_F , \quad (3.19)$$

for low-energy (LE) photons. In the presence of a constant field, the Wick contraction of the vertex operators generalizes to

$$\left\langle V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_N]\right\rangle_F = (iT)^N \exp\left\{\sum_{n=1}^{\infty}\sum_{i_1\ldots i_n}\prod_{k=1}^{n}\int_0^1 du_{i_k}\,\dot{\mathcal{G}}^{\mathrm{dist}}_B(\{i_1i_2...i_n\})\right\}\Bigg|_{f_1\ldots f_N} , \quad (3.20)$$

where $\dot{\mathcal{G}}^{\mathrm{dist}}_B(\{i_1i_2...i_n\})$ denotes the sum over all distinct Lorentz traces which can be formed with a given subset of indices, e.g. $\dot{\mathcal{G}}^{\mathrm{dist}}_B(\{i_1i_2i_3i_4\}) = \dot{\mathcal{G}}_B(i_1i_2i_3i_4) + \dot{\mathcal{G}}_B(i_1i_2i_4i_3) + \dot{\mathcal{G}}_B(i_1i_3i_2i_4)$.

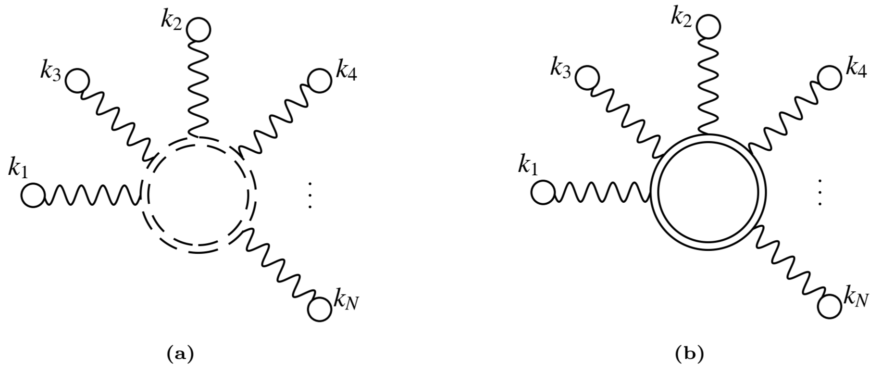

Figure 3.3: *N-photon one-loop Feynman diagrams with every leg low-energy indicated by empty bullets at their ends. (a): the double dashed line indicate a particle of spin zero in a magnetic field. (b): the double solid line indicate a particle of spin one-half in a magnetic field.*

It is important to mention that the representation (3.19) precedes (3.2), which is why the Lorentz trace is now expressed in terms of $\dot{\mathcal{G}}_{Bij}$'s

$$\dot{\mathcal{G}}_B(i_1i_2\ldots i_n) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \mathrm{tr}(f_{i_1}\cdot\dot{\mathcal{G}}_{Bi_1i_2}\cdot f_{i_2}\cdot\dot{\mathcal{G}}_{Bi_2i_3}\cdots f_{i_n}\cdot\dot{\mathcal{G}}_{Bi_ni_1}) . \quad (3.21)$$

The Wick contraction of the vertex operators, using combinatorics, can be expressed as

$$\left\langle V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_N]\right\rangle_F = (iT)^N \exp\left\{\sum_{n=1}^{\infty}\frac{1}{2n}\,I^{\mathrm{cyc}}_{\mathrm{scal}}(f_{\mathrm{tot}},\ldots,f_{\mathrm{tot}};F)\right\}\Bigg|_{f_1\ldots f_N} , \quad (3.22)$$

turning the problem of computing the amplitude into the calculation of the cyclic integral

$$I^{\mathrm{cyc}}_{\mathrm{scal}}(f_1,f_2,\ldots,f_n;F) = \int_0^1 du_1\cdots du_n\,\dot{\mathcal{G}}_B(12\ldots n) , \quad (3.23)$$

such that the N-photon amplitude in a constant background field for scalar QED under the assumption of low-energy photons (Fig. 3.3a) becomes

$$\Gamma^{(\mathrm{LE})}_{N,\mathrm{scal}}(F) = \frac{e^N}{(4\pi)^{\frac{d}{2}}}\int_0^\infty \frac{dT}{T}\,T^{N-\frac{d}{2}}\mathrm{e}^{-m^2T}\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\exp\left\{\sum_{n=1}^{\infty}\frac{1}{2n}\,I^{\mathrm{cyc}}_{\mathrm{scal}}(f_{\mathrm{tot}},\ldots,f_{\mathrm{tot}};F)\right\}\Bigg|_{f_1\ldots f_N} , \quad (3.24)$$

[2]Here, the subindex 'F' indicates that the Wick contraction includes the interaction with the external constant field and $V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_i]$ has been defined in (2.40).

since this amplitude is completely in terms of Lorentz traces, we can apply the generalized Bern-Kosower replacement rule, as stated in [123], transforming the scalar cyclic integral into a spinor cyclic integral

$$I^{\text{cyc}}_{\text{spin}}(f_1, f_2, \ldots, f_n; F) = \int_0^1 du_1 \cdots du_n \left[\dot{\mathcal{G}}_B(12\ldots n) - \mathcal{G}_F(12\ldots n)\right], \tag{3.25}$$

where now the fermionic Lorentz trace is

$$\mathcal{G}_F(i_1 i_2 \ldots i_n) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \operatorname{tr}(f_{i_1} \cdot \mathcal{G}_{Fi_1i_2} \cdot f_{i_2} \cdot \mathcal{G}_{Fi_2i_3} \cdots f_{i_n} \cdot \mathcal{G}_{Fi_ni_1}). \tag{3.26}$$

Therefore, the N-photon amplitude in a constant field at low energies for spinor QED (Fig. 3.3b) is

$$\Gamma^{(\text{LE})}_{N,\text{spin}}(F) = -2\frac{e^N}{(4\pi)^{\frac{d}{2}}} \int_0^\infty \frac{dT}{T} T^{N-\frac{d}{2}} \mathrm{e}^{-m^2T} \det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] \exp\left\{\sum_{n=1}^{\infty} \frac{1}{2n} I^{\text{cyc}}_{\text{spin}}(f_{\text{tot}}, \ldots, f_{\text{tot}}; F)\right\}\Bigg|_{f_1\ldots f_N}. \tag{3.27}$$

In the following sections, we calculate $I^{\text{cyc}}_{\text{scal}}$ and $I^{\text{cyc}}_{\text{spin}}$ for various field configurations, in four dimensions ($d = 4$).

3.4 Case 1: Pure magnetic or electric field

In this section, we focus in the case of a pure magnetic background field of constant strength for which the calligraphic Green's functions exhibit simplifications. Following [123], we choose the magnetic field pointing along the z axis, in Euclidean space,

$$F = \begin{pmatrix} 0 & B_z & 0 & 0 \\ -B_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

We define $z = eTB_z$, the matrices $g_\pm$ whose sum is the metric tensor

$$g_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.29}$$

and the matrices $r_\pm$

$$r_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.30}$$

which satisfy the following relations

$$\mathcal{Z} = eTF = z\, r_+, \qquad r_\pm^{2n} = (-1)^n\, g_\pm, \qquad r_\pm^{2n+1} = (-1)^n\, r_\pm. \tag{3.31}$$

Then, we see that

$$\mathcal{Z}^{2n} = (-1)^n z^{2n}\, g_+ \qquad \text{and} \qquad \mathcal{Z}^{2n+1} = (-1)^n z^{2n+1}\, r_+. \tag{3.32}$$

This allows us to write the scalar and spinor determinants as[3]

$$\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] = \frac{z}{\sinh z}, \qquad \det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] = \frac{z}{\tanh z} \tag{3.33}$$

[3]The calculation of the determinants is straightforward however we present an example of such calculation in Appendix B.1.3.

and the calligraphic Green's functions (3.3) and (3.14) as[4]

$$\begin{aligned}\mathcal{G}_{Bij} &= T\left[G_{Bij}\,g_- - \frac{1}{2z}A_{Bij}(z)g_+ + \frac{1}{2z}\left(S_{Bij}(z) - \dot{G}_{Bij}\right)ir_+\right],\\ \dot{\mathcal{G}}_{Bij} &= \dot{G}_{ij}\,g_- + S_{Bij}(z)g_+ - A_{Bij}(z)ir_+\,,\\ \ddot{\mathcal{G}}_{Bij} &= \frac{1}{T}\left(\ddot{G}_{ij}\,\mathbb{1} - 2zA_{Bij}(z)g_+ + 2zS_{Bij}(z)ir_+\right),\\ \mathcal{G}_{Fij} &= G_{Fij}g_- + S_{Fij}(z)g_+ - A_{Fij}(z)ir_+\,,\end{aligned}\tag{3.34}$$

where

$$\begin{aligned}A_{Bij}(z) &= \frac{\cosh(z\dot{G}_{ij})}{\sinh z} - \frac{1}{z}\,, & A_{Fij}(z) &= G_{Fij}\frac{\sinh(z\dot{G}_{ij})}{\cosh z}\,,\\ S_{Bij}(z) &= \frac{\sinh(z\dot{G}_{ij})}{\sinh z}\,, & S_{Fij}(z) &= G_{Fij}\frac{\cosh(z\dot{G}_{ij})}{\cosh z}\,.\end{aligned}\tag{3.35}$$

In order to efficiently perform integration over the u_i variables, we find convenient to introduce the following functions (here, we include their coincidence limits)

$$\begin{aligned}H^B_{ij}(z) &= \frac{\mathrm{e}^{z\dot{G}_{ij}}}{\sinh z} - \frac{1}{z}\,, & H^B_{ij}(0) &= \dot{G}_{ij}\,, & H^B_{ii}(z) &= \coth z - \frac{1}{z}\,,\\ H^F_{ij}(z) &= G_{Fij}\frac{\mathrm{e}^{z\dot{G}_{ij}}}{\cosh z}\,, & H^F_{ij}(0) &= G_{Fij}\,, & H^F_{ii}(z) &= \tanh z\,.\end{aligned}\tag{3.36}$$

Using these functions, A_{ij} and S_{ij} can now be written as

$$\begin{aligned}A_{Bij}(z) &= \frac{1}{2}\Big[H^B_{ij}(z) - H^B_{ij}(-z)\Big]\,, & A_{Fij}(z) &= \frac{1}{2}\Big[H^F_{ij}(z) - H^F_{ij}(-z)\Big]\,,\\ S_{Bij}(z) &= \frac{1}{2}\Big[H^B_{ij}(z) + H^B_{ij}(-z)\Big]\,, & S_{Fij}(z) &= \frac{1}{2}\Big[H^F_{ij}(z) + H^F_{ij}(-z)\Big]\,,\end{aligned}\tag{3.37}$$

consequently $\dot{\mathcal{G}}_{Bij}$, $\mathcal{G}_{Fij}$ and their coincidence limits can be expressed as well in terms of H_{ij}

$$\begin{aligned}\dot{\mathcal{G}}_{Bij} &= H^B_{ij}(0)g_- + \frac{1}{2}H^B_{ij}(z)(g_+ - ir_+) + \frac{1}{2}H^B_{ij}(-z)(g_+ + ir_+)\,,\\ \mathcal{G}_{Fij} &= H^F_{ij}(0)g_- + \frac{1}{2}H^F_{ij}(z)(g_+ - ir_+) + \frac{1}{2}H^F_{ij}(-z)(g_+ + ir_+)\,,\\ \dot{\mathcal{G}}_{Bii} &= -H^B_{ii}(z)\,ir_+\,,\\ \mathcal{G}_{Fii} &= -H^F_{ii}(z)\,ir_+\,.\end{aligned}\tag{3.38}$$

The pure electric case is obtained after replacing

$$g_+ \leftrightarrow g_-\,,\qquad r_+ \leftrightarrow r_-\,,\qquad z \to ieTE_z\,,\tag{3.39}$$

in (3.38).

The functions $H^B_{ij}(z)$ and $H^F_{ij}(z)$ have the property of reproducing themselves under integration. For both functions the following relations are satisfied[5]

$$H^{(2)}_{13}(z_1,z_2) = \int_0^1 du_2\,H_{12}(z_1)H_{23}(z_2) = \frac{H_{13}(z_1)}{z_2 - z_1} + \frac{H_{13}(z_2)}{z_1 - z_2}\,,\tag{3.40}$$

$$\begin{aligned}H^{(3)}_{14}(z_1,z_2,z_3) &= \int_0^1 du_2\,du_3\,H_{12}(z_1)H_{23}(z_2)H_{34}(z_3)\\ &= \frac{H_{14}(z_1)}{(z_2-z_1)(z_3-z_1)} + \frac{H_{14}(z_2)}{(z_1-z_2)(z_3-z_2)} + \frac{H_{14}(z_3)}{(z_1-z_3)(z_2-z_3)}\,,\end{aligned}\tag{3.41}$$

[4]See Appendix B.1.2 for the step by step calculation of $\dot{\mathcal{G}}_{Bij}$ in a pure magnetic field.

[5]In Appendix B.1.4, we show how to compute $H^{(2)}_{13}(z_1,z_2)$ for scalar and spinor QED.

$$
\begin{aligned}
H^{(4)}_{15}(z_1,z_2,z_3,z_4) &= \int_0^1 du_2\, du_3\, du_4\, H_{12}(z_1)H_{23}(z_2)H_{34}(z_3)H_{45}(z_4) \\
&= \frac{H_{15}(z_1)}{(z_2-z_1)(z_3-z_1)(z_4-z_1)} + \frac{H_{15}(z_2)}{(z_1-z_2)(z_3-z_2)(z_4-z_2)} \\
&+ \frac{H_{15}(z_3)}{(z_1-z_3)(z_2-z_3)(z_4-z_3)} + \frac{H_{15}(z_4)}{(z_1-z_4)(z_2-z_4)(z_3-z_4)}\,.
\end{aligned}
\tag{3.42}
$$

The above identities can be generalized to

$$
\begin{aligned}
H^{(n)}_{1(n+1)}(z_1,z_2,...,z_n) &= \int_0^1 du_2\, du_3\, \cdots du_n\, H_{12}(z_1)H_{23}(z_2)\cdots H_{n(n+1)}(z_n) \\
&= \sum_{\ell=1}^{n} \frac{H_{1(n+1)}(z_\ell)}{C_\ell}, \quad C_\ell = \prod_{j=1,j\neq\ell}^{n} (z_j - z_\ell)\,.
\end{aligned}
\tag{3.43}
$$

The coincidence limit of the $H^{(n)}_{ij}$ function is

$$
H^{(n)}_{11}(z_1,z_2,...,z_n) = \int_0^1 du_2\, du_3\, \cdots du_n\, H_{12}(z_1)H_{23}(z_2)\cdots H_{n1}(z_n) = \sum_{\ell=1}^{n}\frac{H_{11}(z_\ell)}{C_\ell}\,. \tag{3.44}
$$

This expression can be used for the calculation of the cyclic integrals (3.23) and (3.25). The outcome of these cyclic integrals will be outlined in the next section. When considering a magnetic and electric field aligned along the same axis, it is possible to derive a closed expression for the arbitrary n-point integral in a more symmetric form. This allows us to derive the scenarios of pure magnetic or electric fields as particular cases.

3.5 Case 2: Magnetic and electric fields parallel to each other

In this section, we focus on the case of a constant background field where both the magnetic and electric fields point along the z axis (here we follow [123] and the previous section)

$$
F = \begin{pmatrix} 0 & B_z & 0 & 0 \\ -B_z & 0 & 0 & 0 \\ 0 & 0 & 0 & iE_z \\ 0 & 0 & -iE_z & 0 \end{pmatrix}. \tag{3.45}
$$

We define $z_+ = eTB_z$ and $z_- = ieTE_z$. Similar to the previous section, now the following relations are satisfied

$$
\mathcal{Z}^{2n} = (-1)^n\left(z_+^{2n}\, g_+ + z_-^{2n}\, g_-\right) \qquad \text{and} \qquad \mathcal{Z}^{2n+1} = (-1)^n\left(z_+^{2n+1}\, r_+ + z_-^{2n+1}\, r_-\right), \tag{3.46}
$$

with $g_\pm$ and $r_\pm$ as defined in (3.29) and (3.30), respectively. In this case, the determinants become

$$
\det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] = \frac{z_+ z_-}{\tanh z_+\, \tanh z_-}\,, \qquad \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] = \frac{z_+ z_-}{\sinh z_+\, \sinh z_-} \tag{3.47}
$$

and the calligraphic Green's functions (3.3) and (3.14) in terms of A_{ij} and S_{ij} (3.35) are

$$
\dot{\mathcal{G}}_{Bij} = S_{Bij}(z_+)g_+ - A_{Bij}(z_+)ir_+ + S_{Bij}(z_-)g_- - A_{Bij}(z_-)ir_-\,, \tag{3.48}
$$

$$
\mathcal{G}_{Fij} = S_{Fij}(z_+)g_+ - A_{Fij}(z_+)ir_+ + S_{Fij}(z_-)g_- - A_{Fij}(z_-)ir_-\,, \tag{3.49}
$$

which can also be expressed in terms of H_{ij} as

$$
\begin{aligned}
\dot{\mathcal{G}}_{Bij} &= \frac{1}{2}\sum_{\alpha,\beta=\pm} H^B_{ij}(\alpha\, z_\beta)(g_\beta - \alpha\, ir_\beta)\,, \\
\mathcal{G}_{Fij} &= \frac{1}{2}\sum_{\alpha,\beta=\pm} H^F_{ij}(\alpha\, z_\beta)(g_\beta - \alpha\, ir_\beta)\,,
\end{aligned}
\tag{3.50}
$$

see equations (3.36) and (3.37).

The scalar cyclic-integral (3.23) is

$$I^{\text{cyc}}_{\text{scal}}(f_1, f_2, \dots, f_n; F) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \int_0^1 du_1 \cdots du_n \, \text{tr}(f_{i_1} \cdot \dot{\mathcal{G}}_{Bi_1i_2} \cdot f_{i_2} \cdot \dot{\mathcal{G}}_{Bi_2i_3} \cdots f_{i_n} \cdot \dot{\mathcal{G}}_{Bi_ni_1}) \,. \tag{3.51}$$

By expressing the calligraphic Green's functions as in (3.50) and employing (3.44) to integrate, we obtain

$$\begin{aligned} I^{\text{cyc}}_{\text{scal}}(f_1, f_2, \dots, f_n; F) = \left(\frac{1}{2}\right)^{n+\delta_{n1}+\delta_{n2}} & \sum_{\alpha_1,\beta_1=\pm} \cdots \sum_{\alpha_n,\beta_n=\pm} H^{B(n)}_{11}(\alpha_1 z_{\beta_1}, \alpha_2 z_{\beta_2}, ..., \alpha_n z_{\beta_n}) \\ & \times \text{tr}\left\{ f_1(g_{\beta_1} - \alpha_1\, ir_{\beta_1}) f_2(g_{\beta_2} - \alpha_2\, ir_{\beta_2}) \cdots f_n(g_{\beta_n} - \alpha_n\, ir_{\beta_n}) \right\} . \end{aligned} \tag{3.52}$$

Similarly for the spinor cyclic-integral (3.25), we obtain

$$\begin{aligned} I^{\text{cyc}}_{\text{spin}}(f_1, f_2, \dots, f_n; F) = \left(\frac{1}{2}\right)^{n+\delta_{n1}+\delta_{n2}} & \sum_{\alpha_1,\beta_1=\pm} \cdots \sum_{\alpha_n,\beta_n=\pm} \text{tr}\left\{ f_1(g_{\beta_1} - \alpha_1\, ir_{\beta_1}) \cdots f_n(g_{\beta_n} - \alpha_n\, ir_{\beta_n}) \right\} \\ & \times \left[H^{B(n)}_{11}(\alpha_1 z_{\beta_1}, ..., \alpha_n z_{\beta_n}) - H^{F(n)}_{11}(\alpha_1 z_{\beta_1}, ..., \alpha_n z_{\beta_n}) \right] . \end{aligned} \tag{3.53}$$

After substituting the determinants and considering the results (3.52) and (3.53), the N-photon amplitudes (3.24) and (3.27), within the field configuration of the present section, become

$$\Gamma^{(\text{LE})}_{N,\text{scal}}(F) = \frac{e^N}{(4\pi)^2} \int_0^\infty \frac{dT}{T} T^{N-2} \mathrm{e}^{-m^2 T} \frac{z_+ z_-}{\sinh z_+ \, \sinh z_-} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{2n} I^{\text{cyc}}_{\text{scal}}(f_{\text{tot}}, \dots, f_{\text{tot}}; F) \right\} \Bigg|_{f_1 \dots f_N} , \tag{3.54}$$

for scalar QED, and

$$\Gamma^{(\text{LE})}_{N,\text{spin}}(F) = -2 \frac{e^N}{(4\pi)^2} \int_0^\infty \frac{dT}{T} T^{N-2} \mathrm{e}^{-m^2 T} \frac{z_+ z_-}{\tanh z_+ \, \tanh z_-} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{2n} I^{\text{cyc}}_{\text{spin}}(f_{\text{tot}}, \dots, f_{\text{tot}}; F) \right\} \Bigg|_{f_1 \dots f_N} , \tag{3.55}$$

for spinor QED.

It is important to note some properties of these N-photon amplitudes: they are valid off-shell, they are expressed in a compact form that requires simple algebra to obtain the explicit amplitude expression and they have only one proper-time integral left. In Section 3.8, we specialize these results to the four-photon amplitudes in a pure constant magnetic field.

3.6 Case 3: Constant crossed field

In this section, we consider a constant crossed field defined by $\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$ where $E = |\boldsymbol{E}|$ and $B = |\boldsymbol{B}|$. Since the electric and magnetic fields are perpendicular and equal in magnitude both invariants $\boldsymbol{E} \cdot \boldsymbol{B}$ and $B^2 - E^2$ vanish. This implies, for the field strength tensor, $F^3 = 0$ so that the power series in the calligraphic Green's functions (3.3) and (3.14) terminates at the quadratic order

$$\begin{aligned} \mathcal{G}_{Bij} &= \frac{T}{2}\left[\frac{1}{6}\left(1 - 3\dot{G}^2_{ij}\right) - \frac{i}{6}\dot{G}_{ij}\left(1 - \dot{G}^2_{ij}\right)\mathcal{Z} + \frac{2}{3}\left(G^2_{ij} - \frac{1}{30}\right)\mathcal{Z}^2 \right] , \\ \dot{\mathcal{G}}_{Bij} &= \dot{G}_{ij} + \frac{i}{6}\left(1 - 3\dot{G}^2_{ij}\right)\mathcal{Z} + \frac{1}{6}\dot{G}_{ij}\left(1 - \dot{G}^2_{ij}\right)\mathcal{Z}^2 , \\ \ddot{\mathcal{G}}_{Bij} &= \frac{1}{T}\left[\ddot{G}_{ij} + 2i\dot{G}_{ij}\mathcal{Z} - \frac{2}{6}\left(1 - 3\dot{G}^2_{ij}\right)\mathcal{Z}^2 \right] , \\ \mathcal{G}_{Fij} &= G_{Fij}\left[1 - i\dot{G}_{ij}\mathcal{Z} + \frac{1}{2}\left(1 - \dot{G}^2_{ij}\right)\mathcal{Z}^2 \right] . \end{aligned} \tag{3.56}$$

The coincidence limits, in this case, are

$$\mathcal{G}_{Bii} = -T\left(\frac{1}{6}+\frac{1}{90}\mathcal{Z}^2\right)\,, \qquad \dot{\mathcal{G}}_{Bii} = -\frac{i}{3}\mathcal{Z}\,, \qquad \mathcal{G}_{Fii} = -i\mathcal{Z}\,. \tag{3.57}$$

And, the determinants now become

$$\det{}^{1/2}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] = \det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] = 1\,. \tag{3.58}$$

For instance, in four-dimensions, we could choose the field strength tensor as

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_x & 0 \\ 0 & -B_x & 0 & iE_z \\ 0 & 0 & -iE_z & 0 \end{pmatrix}\,, \qquad F^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -B_x^2 & 0 & iB_xE_z \\ 0 & 0 & 0 & 0 \\ 0 & iB_xE_z & 0 & E_z^2 \end{pmatrix}\,, \tag{3.59}$$

from which, it is clear that $F^3 = \mathcal{Z}^3 = 0$.

Notice that, the calligraphic Green's functions $\dot{\mathcal{G}}_{ij}$ and $\mathcal{G}_{Fij}$, as the Neumann series expansion [123, 162] of the Green's function operator, have the following representation[6]

$$\begin{aligned} \dot{\mathcal{G}}_{Bij} &= 2\sum_{\ell=0}^{2}\langle i|\partial_P^{-(\ell+1)}|j\rangle\,(2i\mathcal{Z})^\ell\,, \\ \mathcal{G}_{Fij} &= 2\sum_{\ell=0}^{2}\langle i|\partial_A^{-(\ell+1)}|j\rangle\,(2i\mathcal{Z})^\ell\,. \end{aligned} \tag{3.60}$$

Now, for the integral of 'n' bosonic calligraphic Green's functions $\dot{\mathcal{G}}_{Bij}$, we can define a generic bosonic cycle integral (as in the vacuum case [18]) by

$$b_{\ell_1+\ldots+\ell_n} = 2^{\ell_1+\ldots+\ell_n}\int_0^1 du_1\,du_2\cdots du_n\,\langle u_1|\partial_P^{-\ell_1}|u_2\rangle\langle u_2|\partial_P^{-\ell_2}|u_3\rangle\cdots\langle u_n|\partial_P^{-\ell_n}|u_1\rangle\,. \tag{3.61}$$

The calculation of the previous integral follows from the completeness relation $\int_0^1 du\,|u\rangle\langle u| = 1$ and it can be expressed in terms of the Bernoulli numbers [163, 149]

$$b_\ell = \begin{cases} -2^\ell\,\frac{\mathcal{B}_\ell}{\ell!} & \ell \quad \text{even}\,, \\ 0 & \ell \quad \text{odd}\,. \end{cases} \tag{3.62}$$

Similarly, for the integral of 'n' fermionic calligraphic Green's functions $\mathcal{G}_{Fij}$, we have

$$2^{\ell_1+\ldots+\ell_n}\int_0^1 du_1\cdots\int_0^1 du_n\,\langle 1|\partial_A^{-\ell_1}|2\rangle\langle 2|\partial_A^{-\ell_2}|3\rangle\cdots\langle n|\partial_A^{-\ell_n}|1\rangle = \left(1-2^{\ell_1+\ldots+\ell_n}\right)b_{\ell_1+\ldots+\ell_n}\,. \tag{3.63}$$

Therefore, we can use (3.61) and (3.63) to compute the cyclic integrals (3.23) and (3.25), and express the result in terms of Bernoulli numbers, for the present case. For the scalar cyclic-integral, we obtain

$$\begin{aligned} I^{\text{cyc}}_{\text{scal}}(f_1,f_2,\ldots,f_n;F) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} &\sum_{\ell_1=0}^{2}\cdots\sum_{\ell_n=0}^{2} i^{\ell_1+\ldots+\ell_n}\,b_{n+\ell_1+\ldots+\ell_n} \\ &\times \text{tr}\left(f_1\cdot\mathcal{Z}^{\ell_1}\cdot f_2\cdot\mathcal{Z}^{\ell_2}\cdots f_n\cdot\mathcal{Z}^{\ell_n}\right)\,. \end{aligned} \tag{3.64}$$

Similarly, by defining $h_\ell = \left(2-2^\ell\right)b_\ell$, the spinor cyclic-integral can be expressed as

$$\begin{aligned} I^{\text{cyc}}_{\text{spin}}(f_1,f_2,\ldots,f_n;F) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} &\sum_{\ell_1=0}^{2}\cdots\sum_{\ell_n=0}^{2} i^{\ell_1+\ldots+\ell_n}\,h_{n+\ell_1+\ldots+\ell_n} \\ &\times \text{tr}\left(f_1\cdot\mathcal{Z}^{\ell_1}\cdot f_2\cdot\mathcal{Z}^{\ell_2}\cdots f_n\cdot\mathcal{Z}^{\ell_n}\right)\,. \end{aligned} \tag{3.65}$$

[6] The subscript 'P' stands for periodic boundary conditions and 'A' for anti-periodic.

Note that, the Lorentz traces can be further simplified due to the symmetry $F^2 \cdot f_i \cdot F^2 = 0$, which has as consequence that the order of interactions in a cycle with the field cannot be greater than the number of photon in such cycle. Then, the cyclic integrals can be represented as

$$
\begin{aligned}
I^{\text{cyc}}_{\text{scal}}(f_1, f_2, \dots, f_n; F) &= \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \sum_{\ell=0}^{n} (ieT)^{\ell}\, b_{n+\ell}\, \text{tr}^{\text{dist}}\left(f_1 \cdot F \cdot f_2 \cdot F \cdots f_\ell \cdot F \cdot f_{\ell+1} \cdot f_{\ell+2} \cdots f_n\right), \\
I^{\text{cyc}}_{\text{spin}}(f_1, f_2, \dots, f_n; F) &= \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \sum_{\ell=0}^{n} (ieT)^{\ell}\, h_{n+\ell}\, \text{tr}^{\text{dist}}\left(f_1 \cdot F \cdot f_2 \cdot F \cdots f_\ell \cdot F \cdot f_{\ell+1} \cdot f_{\ell+2} \cdots f_n\right),
\end{aligned}
\tag{3.66}
$$

where ‘tr^{dist}’ denotes the sum of all different permutations of F for a fix set of f_i’s. For instance, the non-null contributions, for $n = 1$

$$
\text{tr}^{\text{dist}}(f_1 \cdot F) = \text{tr}(f_1 \cdot F)\,. \tag{3.67}
$$

For $n = 2$

$$
\text{tr}^{\text{dist}}\left(f_1 \cdot F \cdot f_2 \cdot F\right) = \text{tr}(F \cdot f_1 \cdot F \cdot f_2) + \text{tr}(F^2 \cdot f_1 \cdot f_2) + \text{tr}(f_1 \cdot F^2 \cdot f_2)\,. \tag{3.68}
$$

For $n = 3$

$$
\begin{aligned}
&\text{tr}^{\text{dist}}\left(f_1 \cdot F \cdot f_2 \cdot f_3\right) = \text{tr}(f_1 \cdot F \cdot f_2 \cdot f_3) + \text{tr}(f_1 \cdot f_2 \cdot F \cdot f_3) + \text{tr}(f_1 \cdot f_2 \cdot f_3 \cdot F)\,, \\
&\text{tr}^{\text{dist}}\left(f_1 \cdot F \cdot f_2 \cdot F \cdot f_3 \cdot F\right) = (f_1 F f_2 F f_3 F) + \text{tr}\left(f_1 f_2 F f_3 F^2 + 5\,\text{perm of } F,\ F^2\right)\,.
\end{aligned}
\tag{3.69}
$$

After substituting the determinants and considering the results in (3.66), the N-photon amplitudes (3.24) and (3.27), within the field configuration of the present section, become

$$
\Gamma^{(\text{LE})}_{N,\text{scal}}(F) = \frac{e^N}{(4\pi)^2} \int_0^\infty \frac{dT}{T}\, T^{N-2} \mathrm{e}^{-m^2 T} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{2n} I^{\text{cyc}}_{\text{scal}}(f_{\text{tot}}, \dots, f_{\text{tot}}; F) \right\} \Bigg|_{f_1 \dots f_N}, \tag{3.70}
$$

for scalar QED, and

$$
\Gamma^{(\text{LE})}_{N,\text{spin}}(F) = -2\frac{e^N}{(4\pi)^2} \int_0^\infty \frac{dT}{T}\, T^{N-2} \mathrm{e}^{-m^2 T} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{2n} I^{\text{cyc}}_{\text{spin}}(f_{\text{tot}}, \dots, f_{\text{tot}}; F) \right\} \Bigg|_{f_1 \dots f_N}, \tag{3.71}
$$

for spinor QED.

It is important to note that these N-photon amplitudes in a constant crossed field are valid off-shell and the proper-time integral left is straightforward to perform for a fixed number of photons. Moreover, the PT symmetry becomes manifest in this representation, facilitated by the Bernoulli numbers (3.62).

In the low-energy limit of the N-photon amplitudes within the worldline, it is well known that the leading contributions arise from the bicycles (2.10) or (2.14) in the vacuum case and from the Lorentz traces (3.9) or (3.15) in the constant field case, see [50, 51, 52, 111]. However, in the high-field and high energy limit of the N-photon amplitudes in a constant crossed field, the leading contributions may arise from the tails (3.10). In such case, the leading term of the amplitude would be the one containing the smallest Lorentz trace and the biggest tail. Since the external photons are off-shell, this presents the opportunity to systematically study the scaling with respect to the quantum nonlinearity parameter (1.6) for these N-photon amplitudes and even for multi-loop amplitudes, which is related to the Ritus-Narozhny conjecture [104, 164, 105, 106, 107]. This is a subject that is currently under development.

3.6.1 The plane-wave field limit

It is interesting to recall that a plane-wave field in the low-frequency approximation corresponds exactly to a constant crossed field. Then, in the light-cone coordinate system of a plane wave propagating along the $\mathbf{n}$ direction, the vector potential for a constant crossed field can be chosen as [165, 44]

$$
A(\phi) = E_0(\varepsilon_0^+ + \varepsilon_0^-)\phi\,, \tag{3.72}
$$

where $\phi = n_\mu x^\mu$ with Minkowski space metric $(\eta^{\mu\nu}) = \mathrm{diag}(+1,-1,-1,-1)$, E_0 a constant equal in magnitude to the electric and magnetic field strengths and $\varepsilon_0^\pm$ unitary four vectors orthogonal to n that can be regarded as the '+' and '−' helicity components of the plane wave field.

Notice that the field strength tensor of a constant crossed field can be seen as the sum of two 'polarized photon field strength tensors'

$$F = f_0^+ + f_0^- \,, \qquad f_0^{\pm,\mu\nu} = k_0^\mu\,\varepsilon_0^{\pm,\nu} - \varepsilon_0^{\pm,\mu}\,k_0^\nu \,, \qquad k_0^\mu = E_0\,n^\mu \,. \tag{3.73}$$

For instance, a good choice for the polarization four-vectors $\varepsilon_0^\pm = (0, \boldsymbol{\varepsilon}_0^\pm)$ in the light-cone basis [44] is

$$\boldsymbol{\varepsilon}_0^\pm = \frac{1}{\sqrt{2}}(\boldsymbol{a}_1 \pm i\boldsymbol{a}_2) \tag{3.74}$$

and assuming that, for the external photons, the scattering plane is formed by $(\boldsymbol{n}, \boldsymbol{a}_2)$, the polarization for each external photon will be

$$\boldsymbol{\varepsilon}_i^\pm = \frac{1}{\sqrt{2}}\left[\boldsymbol{a}_1 \pm i\boldsymbol{a}_1 \times \left(\frac{\boldsymbol{k}_i}{\omega_i}\right)\right] . \tag{3.75}$$

This imply that for the case of external polarized photons the effective interaction with the constant crossed field is through identical polarized photons which do not transfer energy.

3.6.2 Helicity amplitudes

The possibility of expressing the constant crossed field as (3.73) motivates the use of spinor helicity to compute the polarized amplitudes. Here, in order to use the spinor helicity formalism in Section 2.2, we change from the Euclidean $(g^{\mu\nu}) = \mathrm{diag}(+1,+1,+1,+1)$ to the Minkowski space convention $(\eta^{\mu\nu}) = \mathrm{diag}(+1,-1,-1,-1)$ as indicated in Appendix A.

For the present discussion we focus in the scalar N-photon amplitude since the spinor one follows analogously. It is convenient to use the representation in (3.20) to express the four dimensional amplitude (3.24) as

$$\Gamma^{(LE)}_{N,\mathrm{scal}}(F) = \frac{e^N}{(4\pi)^2}\int_0^\infty \frac{dT}{T}\,T^{N-2}\mathrm{e}^{-m^2T}\,\exp\left\{\sum_{n=1}^\infty \sum_{i_1\dots i_n} I^{\mathrm{cyc,dist}}_{\mathrm{scal}}(\{f_{i_1}, f_{i_2}, \dots, f_{i_n}\}; F)\right\}\Bigg|_{f_1\dots f_N} . \tag{3.76}$$

From this expression, we can see that the interaction with the external field in the cyclic integrals will be determined by the helicity of the external photons. For fixed helicity, the field strength tensor F is effectively replaced by one of the polarized field strength tensor $f_0^\pm$ as

$$I^{\mathrm{cyc}}_{\mathrm{scal}}(\{f_{i_1}, f_{i_2}, \dots, f_{i_n}\}; F) \quad \to \quad \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}} \sum_{\ell=1}^{n} (ieT)^\ell\, b_{n+\ell}\,\mathrm{tr}^{\mathrm{dist}}\left(f_{i_1}\cdot f_0^\pm \cdots f_{i_\ell}\cdot f_0^\pm \cdot f_{i_{\ell+1}}\cdots f_{i_n}\right) \tag{3.77}$$

and we have to take into account all the possible combinations in which $f_0^\pm$ can appear in the cyclic integral. Furthermore, the expansion of (3.76) to linear order in $f_1 ... f_N$ can be seen as the sum of 0 to N interactions with the field f_0, consequence of (3.66). Then, for fixed helicities, we can think of (3.76) as the sum of N vacuum N-photon amplitudes

$$\begin{aligned}\Gamma^{(LE)}_{\mathrm{scal}}(f_1^+; ...; f_L^+; \cdot f_{L+1}^-; ...; f_N^-; F) &= \frac{(-i)^N}{(4\pi)^2}\,(2\pi)^4\delta^4\left(\sum_{i=1}^N k_i\right)\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\sum_{n=0}^N\sum_{\ell=0}^n \\ &\times \exp\left\{\sum_{r=1}^\infty \sum_{i_1\dots i_r}\left(\frac{1}{2}\right)^{\delta_{r2}} (ieT)^r\, b_r\,\mathrm{tr}^{\mathrm{dist}}\left(f_{i_1}\cdot f_{i_2}\cdot\dots\cdot f_{i_r}\right)\right\}\Bigg|_{f_1^+\dots f_L^+ f_{L+1}^-\dots f_N^- f_{N+1}^+\dots f_{N+\ell}^+ f_{N+\ell+1}^-\dots f_{N+n}^-} ,\end{aligned} \tag{3.78}$$

for L external photons having helicity '+' and $N-L$ '−', and with $f_{N+1} = f_{N+2} = ... = f_{N+n} = f_0$. Here, we have written the momentum conservation explicitly to emphasize the fact that it is independent of the "effective photon" momentum k_0.

Now, we compare the results in [51] and [50] to obtain the following relation (see Section 2.3, specifically Eq. (2.46) and (2.61))

$$\exp\left\{\sum_{n=1}^{\infty}\sum_{i_1\dots i_n}\left(\frac{1}{2}\right)^{\delta_{n2}}(ieT)^n\, b_n\, \mathrm{tr}^{\mathrm{dist}}\left(f_{i_1}\cdot f_{i_2}\cdots f_{i_n}\right)\right\}\Bigg|_{f_1^+\dots f_L^+\cdot f_{L+1}^-\dots f_N^-} = \frac{(2ieT)^N\, C_{\mathrm{scal}}^{N,L}\, \chi_L^+\, \chi_{N-L}^-}{(N-3)!}\,, \tag{3.79}$$

where $C_{\mathrm{scal}}^{N,L}$ are scalar coefficients given in terms of Bernoulli numbers $\mathcal{B}_n$ by

$$C_{\mathrm{scal}}^{N,L} = (-1)^{N/2}\,(N-3)!\sum_{r=0}^{L}\sum_{s=0}^{N-L}(-1)^{N-L-s}\,\frac{\left(1-2^{1-r-s}\right)\left(1-2^{1-N+r+s}\right)\mathcal{B}_{r+s}\,\mathcal{B}_{N-r-s}}{r!\,s!\,(L-r)!\,(N-L-s)!} \tag{3.80}$$

and $\chi_L^{\pm}$ are twistor products that vanish for L odd. And, for L even we have

$$\chi_L^+ = (\chi_+)^{\frac{L}{2}}|_{\text{all different}} = \frac{(L/2)!}{2^{L/2}}\Big\{[12]^2[34]^2\cdots[(L-1)L]^2 + \text{all permutations}\Big\}\,, \tag{3.81}$$

$$\begin{aligned}\chi_{N-L}^- &= (\chi_-)^{\frac{N-L}{2}}|_{\text{all different}}\\ &= \frac{\left(\frac{N-L}{2}\right)!}{2^{\frac{N-L}{2}}}\Big\{\langle(L+1)(L+2)\rangle^2\langle(L+3)(L+4)\rangle^2\cdots\langle(N-1)N\rangle^2 + \text{all permutations}\Big\}\,.\end{aligned} \tag{3.82}$$

Here, we have provided the explicit expressions for $C_{\mathrm{scal}}^{N,L}$, χ_L^+ and χ_{N-L}^-, as introduced in Section 2.3, following the conventions outlined in Section 2.2. For further details, see [51].

In order to obtain a closed result for the N-photon amplitude in a constant crossed field with L external photons having helicity '+' and $N-L$ '−' for scalar QED we use equation (3.79) in (3.78). After integration over T, we obtain

$$\Gamma_{\mathrm{scal}}^{(LE)}(f_1^+;\dots;f_L^+ f_{L+1}^-;\dots;f_N^-;F) = \frac{m^4}{(4\pi)^2}\left(\frac{2e}{m^2}\right)^N\sum_{n=0}^{N}\sum_{\ell=0}^{n}\left(\frac{2ie}{m^2}\right)^n C_{\mathrm{scal}}^{N+n,L+\ell}\,\chi_{L+\ell}^+\,\chi_{N-L+n-\ell}^-\,. \tag{3.83}$$

Analogously, we obtain the N-photon amplitude in a constant crossed field with L external photons having helicity '+' and $N-L$ '−' for spinor QED

$$\Gamma_{\mathrm{spin}}^{(LE)}(f_1^+;\dots;f_L^+ f_{L+1}^-;\dots;f_N^-;F) = -2\,\frac{m^4}{(4\pi)^2}\left(\frac{2e}{m^2}\right)^N\sum_{n=0}^{N}\sum_{\ell=0}^{n}\left(\frac{2ie}{m^2}\right)^n C_{\mathrm{spin}}^{N+n,L+\ell}\,\chi_{L+\ell}^+\,\chi_{N-L+n-\ell}^-\,, \tag{3.84}$$

where $C_{\mathrm{spin}}^{N,L}$ are spinor coefficients given by

$$C_{\mathrm{spin}}^{N,L} = (-1)^{N/2}\,(N-3)!\sum_{r=0}^{L}\sum_{s=0}^{N-L}(-1)^{N-L-s}\,\frac{\mathcal{B}_{r+s}\,\mathcal{B}_{N-r-s}}{r!\,s!\,(L-r)!\,(N-L-s)!}\,. \tag{3.85}$$

In [51], it is shown that the N-photon amplitudes in vacuum obey a double Furry theorem for low-energy photons, i.e., the number of helicity components in an N-photon amplitudes should be even otherwise the latter vanishes. In the present case, the N-photon amplitudes in a constant crossed field do not obey the double Furry theorem, although each contribution to the amplitude does because $\chi_L^{\pm}$ are non-zero only for L even. Note that the polarized N-photon amplitudes in a constant crossed field (3.83) and (3.84) are valid for on-shell external photons, and keep in mind that $f_{N+n} = f_0$ for $n > 0$ and $[ii]_{\pm} = 0$.

3.7 Case 4: Arbitrary constant field

In this section, we notice that for the general case of an arbitrary constant background field the calligraphic Green's functions $\dot{\mathcal{G}}_{ij}$ and $\mathcal{G}_{Fij}$, as well as in (3.60), can be expressed as the Neumann series expansion [123, 162] of the Green's function operator[7]

$$\begin{aligned} \dot{\mathcal{G}}_{Bij} &= 2\sum_{\ell=0}^{\infty}\langle i|\partial_P^{-(\ell+1)}|j\rangle(2i\mathcal{Z})^{\ell}\,, \\ \mathcal{G}_{Fij} &= 2\sum_{\ell=0}^{\infty}\langle i|\partial_A^{-(\ell+1)}|j\rangle(2i\mathcal{Z})^{\ell}\,. \end{aligned} \tag{3.86}$$

Then, using (3.61) to compute scalar cyclic-integral (3.23), we obtain

$$I^{\rm cyc}_{\rm scal}(f_1,f_2,\dots,f_n;F) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_n=0}^{\infty} i^{\ell_1+\ldots+\ell_n}\,{\rm tr}\left(f_1\cdot\mathcal{Z}^{\ell_1}\cdot f_2\cdot\mathcal{Z}^{\ell_2}\cdots f_n\cdot\mathcal{Z}^{\ell_n}\right)\, b_{n+\ell_1+\ldots+\ell_n}\,, \tag{3.87}$$

here, b_ℓ is given by (3.62). Similarly, we can use (3.61) and (3.63) to compute spinor cyclic-integral (3.25), obtaining

$$I^{\rm cyc}_{\rm spin}(f_1,f_2,\dots,f_n;F) = \left(\frac{1}{2}\right)^{\delta_{n1}+\delta_{n2}}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_n=0}^{\infty} i^{\ell_1+\ldots+\ell_n}\,{\rm tr}\left(f_1\cdot\mathcal{Z}^{\ell_1}\cdot f_2\cdot\mathcal{Z}^{\ell_2}\cdots f_n\cdot\mathcal{Z}^{\ell_n}\right)\, h_{n+\ell_1+\ldots+\ell_n}\,, \tag{3.88}$$

with $h_\ell = \left(2-2^\ell\right)b_\ell\,.$

Therefore, assuming that external photons have low-energy, the N-photon amplitudes in the presence of an arbitrary constant background field can expressed as series expansions respect to the field strength tensor, as indicated by (3.87), (3.88),

$$\Gamma^{(LE)}_{N,{\rm scal}}(F) = \frac{(-ie)^N}{(4\pi)^{\frac{d}{2}}}\int_0^\infty \frac{dT}{T}\,T^{N-\frac{d}{2}}{\rm e}^{-m^2T}\,{\rm det}^{1/2}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\exp\left\{\sum_{n=1}^{\infty}\frac{1}{2n}\,I^{\rm cyc}_{\rm scal}(f_{\rm tot},\dots,f_{\rm tot};F)\right\}\Bigg|_{f_1\ldots f_N} \tag{3.89}$$

for scalar QED, and

$$\Gamma^{(LE)}_{N,{\rm spin}}(F) = -2\frac{(-ie)^N}{(4\pi)^{\frac{d}{2}}}\int_0^\infty \frac{dT}{T}\,T^{N-\frac{d}{2}}{\rm e}^{-m^2T}\,{\rm det}^{1/2}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\exp\left\{\sum_{n=1}^{\infty}\frac{1}{2n}\,I^{\rm cyc}_{\rm spin}(f_{\rm tot},\dots,f_{\rm tot};F)\right\}\Bigg|_{f_1\ldots f_N} \tag{3.90}$$

for spinor QED.

In this case, it is also evident that the PT symmetry becomes manifest in the above equations due to the properties of the Bernoulli numbers (3.62). Specifically, $b_{2\ell+1}=0$ corresponds to an odd power of the electric charge.

3.8 Low-energy limit of the four-photon amplitudes in a magnetic field

In this section, as an application of the formulas obtained above, we compute the four-photon amplitude at low energies in a pure magnetic field for both scalar and spinor QED. We point out that this amplitude for the spinor case have been studied in [45] from the Euler-Heisenberg Lagrangian [5]. The first step towards the study (within the worldline formalism) of light-by-light scattering in presence of a magnetic background field, in the following, we present analytical expressions for the corresponding scalar and spinor amplitudes as well as for the polarized amplitudes (see Appendix C for the results of the integrals).

[7] The subscript 'P' stands for periodic boundary conditions and 'A' for anti-periodic.

3.8.1 Four-photon amplitude in a magnetic field for scalar QED

In this section, we study the four-photon amplitude in a constant field for scalar QED (see Fig. 3.4) obtained from the N-photon amplitude (3.8). In four dimensions, this amplitude is

$$\begin{aligned}\Gamma_{4,\mathrm{scal}}(F) = (-ie)^4 \int_0^\infty \frac{dT}{T}\, T^4 (4\pi T)^{-2}\, \mathrm{e}^{-m^2 T} \det{}^{-1/2}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] \\ \times \int_0^1 du_1\, du_2\, du_3\, du_4\, \mathcal{Q}_{\mathrm{scal}}(\dot{\mathcal{G}}_{ij}) \exp\left\{\frac{1}{2}\sum_{i,j=1}^{4} k_i \cdot \mathcal{G}_{ij} \cdot k_j\right\}.\end{aligned} \tag{3.91}$$

In the case of $N = 4$, the polynomial $\mathcal{Q}_{\mathrm{scal}}(\dot{\mathcal{G}}_{ij})$ has the following representation

$$\begin{aligned}\mathcal{Q}_{\mathrm{scal}}(\dot{\mathcal{G}}_{ij}) &= \mathcal{Q}_{\mathrm{scal}}{}^4 + \mathcal{Q}_{\mathrm{scal}}{}^3 + \mathcal{Q}_{\mathrm{scal}}{}^2 + \mathcal{Q}_{\mathrm{scal}}{}^{22}\,,\\ \mathcal{Q}_{\mathrm{scal}}{}^4 &= \dot{\mathcal{G}}(1234) + \dot{\mathcal{G}}(2314) + \dot{\mathcal{G}}(3124)\,,\\ \mathcal{Q}_{\mathrm{scal}}{}^3 &= \dot{\mathcal{G}}(123)\mathcal{T}(4) + \dot{\mathcal{G}}(234)\mathcal{T}(1) + \dot{\mathcal{G}}(341)\mathcal{T}(2) + \dot{\mathcal{G}}(412)\mathcal{T}(3)\,,\\ \mathcal{Q}_{\mathrm{scal}}{}^2 &= \dot{\mathcal{G}}(12)\mathcal{T}(34) + \dot{\mathcal{G}}(13)\mathcal{T}(24) + \dot{\mathcal{G}}(14)\mathcal{T}(23) + \dot{\mathcal{G}}(23)\mathcal{T}(14) + \dot{\mathcal{G}}(24)\mathcal{T}(13) + \dot{\mathcal{G}}(34)\mathcal{T}(12)\,,\\ \mathcal{Q}_{\mathrm{scal}}{}^{22} &= \dot{\mathcal{G}}(12)\dot{\mathcal{G}}(34) + \dot{\mathcal{G}}(13)\dot{\mathcal{G}}(24) + \dot{\mathcal{G}}(14)\dot{\mathcal{G}}(23)\,,\end{aligned} \tag{3.92}$$

where the Lorentz cycles and tails are given by (3.9) and (3.10) respectively.

Remarkably, the latter representation of the leading light-by-light amplitude in a constant field follows the structure of the same amplitude in vacuum [18]. This is, in fact, an advantage since the Q-representation in vacuum has been studied in detail in [122, 123], even more, such representation is known for up to six external photons from which it is straightforward to obtain its extension to the case of a constant background field. Another advantage of this representation is its compactness which is due to the removal of the one-cycles $\dot{\mathcal{G}}(i)$.

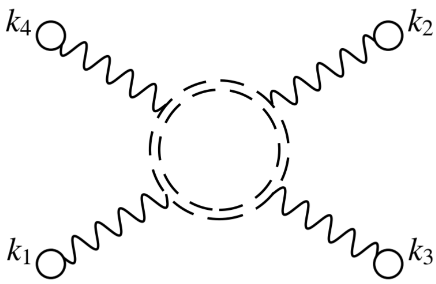

Figure 3.4: *Feynman diagram representing the four-photon amplitude with every incoming photon having low-energy, indicated by empty bullets at their ends. The double dashed line indicate a particle of spin zero in a magnetic field.*

In the following, we consider the fully low-energy case, i.e., we consider every photon to have much less energy than the mass in the loop. For such case, the leading non-vanishing contributions to the amplitude are those composed fully by Lorentz cycles. Then, the four-photon amplitude at low energies in a purely magnetic background field for scalar QED is

$$\Gamma^{(LE)}_{4,\mathrm{scal}}(F) = \alpha^2 \int_0^\infty \frac{dT}{T}\, T^2\, \mathrm{e}^{-m^2 T}\, \frac{z}{\sinh z} \int_0^1 du_1\, du_2\, du_3\, du_4\, \left(\mathcal{Q}_{\mathrm{scal}}{}^4 + \mathcal{Q}_{\mathrm{scal}}{}^{22}\right), \tag{3.93}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant and $z = eTB_z$, as in Section 3.4. Here, we have already used the determinant result (3.33). Notice that this amplitude involves only the following Lorentz cycles

$$\dot{\mathcal{G}}(12) = \frac{1}{2}\mathrm{tr}\left(f_1\dot{\mathcal{G}}_{12}f_2\dot{\mathcal{G}}_{21}\right)\,,\qquad \dot{\mathcal{G}}(1234) = \mathrm{tr}\left(f_1\dot{\mathcal{G}}_{12}f_2\dot{\mathcal{G}}_{23}f_3\dot{\mathcal{G}}_{34}f_4\dot{\mathcal{G}}_{41}\right)\,, \tag{3.94}$$

of course, they appear with different permutations. Notice that the u_i variables are removed after integration such that we can define

$$\begin{aligned}\mathcal{I}^{\mathrm{sc}}_{2f}(12) &= \frac{1}{2}\int_0^1 du_1\,du_2\;\mathrm{tr}\left(f_1\dot{\mathcal{G}}_{12}f_2\dot{\mathcal{G}}_{21}\right)\,,\\ \mathcal{I}^{\mathrm{sc}}_{4f}(1234) &= \int_0^1 du_1\,du_2\,du_3\,du_4\;\mathrm{tr}\left(f_1\dot{\mathcal{G}}_{12}f_2\dot{\mathcal{G}}_{23}f_3\dot{\mathcal{G}}_{34}f_4\dot{\mathcal{G}}_{41}\right)\,,\end{aligned} \tag{3.95}$$

allowing us to write the amplitude as

$$\Gamma^{(LE)}_{4,\mathrm{scal}}(F) = \alpha^2\int_0^\infty\frac{dT}{T}\,T^2\mathrm{e}^{-m^2T}\,\frac{z}{\sinh z}\Big[\mathcal{I}^{\mathrm{sc}}_{4f}(1234) + \mathcal{I}^{\mathrm{sc}}_{2f}(12)\,\mathcal{I}^{\mathrm{sc}}_{2f}(34) + 2\,\mathrm{perm}\Big]\,. \tag{3.96}$$

Notice that the integrals (3.95) can be expressed in terms of the cyclic integrals (3.52) as

$$\begin{aligned}\mathcal{I}^{\mathrm{sc}}_{2f}(12) &= I^{\mathrm{cyc}}_{\mathrm{scal}}(f_1,f_2;F) + \frac{1}{2}\,\mathrm{tr}\left(f_1\dot{\mathcal{G}}_{Bii}f_2\dot{\mathcal{G}}_{Bii}\right)\,,\\ \mathcal{I}^{\mathrm{sc}}_{4f}(1234) &= I^{\mathrm{cyc}}_{\mathrm{scal}}(f_1,f_2,f_3,f_4;F) + \mathrm{tr}\left(f_1\dot{\mathcal{G}}_{Bii}f_2\dot{\mathcal{G}}_{Bii}f_3\dot{\mathcal{G}}_{Bii}f_4\dot{\mathcal{G}}_{Bii}\right)\,.\end{aligned} \tag{3.97}$$

Now, to obtain the pure magnetic case we simply set $z_+ = z = eTB_z$ and $z_- = 0$ in (3.52). We must also recall the coincidence limit $\dot{\mathcal{G}}_{Bii}$ in a pure magnetic field (see 'Case 1' in Section 3.4). Then, in this way, we obtain

$$\mathcal{I}^{\mathrm{sc}}_{2f}(12) = I^{\mathrm{sc}}_{20}\,\mathrm{tr}(f_1g_-f_2g_-) + I^{\mathrm{sc}}_{21}\,\mathrm{tr}(f_1g_+f_2g_+) + I^{\mathrm{sc}}_{22}\,\mathrm{tr}(f_1r_+f_2r_+) + I^{\mathrm{sc}}_{23}\Big[\mathrm{tr}(f_1g_-f_2g_+) + \mathrm{tr}(f_1g_+f_2g_-)\Big]\,, \tag{3.98}$$

for the two-cycle integral. And

$$\begin{aligned}\mathcal{I}^{\mathrm{sc}}_{4f}(1234) = \Big\{&I^{\mathrm{sc}}_{0}\,\mathrm{tr}(f_1g_-f_2g_-f_3g_-f_4g_-) + I^{\mathrm{sc}}_{1}\,\mathrm{tr}(f_1g_+f_2g_+f_3g_+f_4g_+) + I^{\mathrm{sc}}_{2}\,\mathrm{tr}(f_1r_+f_2r_+f_3r_+f_4r_+)\\ &+ I^{\mathrm{sc}}_{3}\Big[\mathrm{tr}(f_1g_-f_2g_+f_3g_+f_4g_+) + 3\mathrm{perm}\Big] + I^{\mathrm{sc}}_{4}\Big[\mathrm{tr}(f_1g_-f_2g_-f_3g_-f_4g_+) + 3\mathrm{perm}\Big]\\ &+ I^{\mathrm{sc}}_{5}\Big[\mathrm{tr}(f_1g_-f_2g_-f_3g_+f_4g_+) + 5\mathrm{perm}\Big] + I^{\mathrm{sc}}_{6}\Big[\mathrm{tr}(f_1r_+f_2r_+f_3g_-f_4g_-) + 5\mathrm{perm}\Big]\\ &+ I^{\mathrm{sc}}_{7}\Big[\mathrm{tr}(f_1r_+f_2r_+f_3g_+f_4g_+) + 5\mathrm{perm}\Big] + I^{\mathrm{sc}}_{8}\Big[\mathrm{tr}(f_1r_+f_2r_+f_3g_-f_4g_+) + 11\mathrm{perm}\Big]\Big\}\,,\end{aligned} \tag{3.99}$$

for the four-cycle integral. Here, the functions I^{sc}_i's are trigonometric expressions and are explicitly written in terms of $H^{B(n)}_{11}$ (see Eq. (3.44) and Appendix C.1).

Finally, for the integration over T, we make the change of variables $z = eB_zT$ and set the following conventions

$$\begin{aligned}J^{\mathrm{sc}}_n &= \int_0^\infty dz\,\mathrm{e}^{-\beta_c z}\,\frac{z^2}{\sinh z}\,I^{\mathrm{sc}}_n\,,\\ \{J^{\mathrm{sc}}_{20},J^{\mathrm{sc}}_{21},J^{\mathrm{sc}}_{22},J^{\mathrm{sc}}_{23},J^{\mathrm{sc}}_{24}\} &= \int_0^\infty dz\,\mathrm{e}^{-\beta_c z}\,\frac{z^2}{\sinh z}\,\{(I^{\mathrm{sc}}_{20})^2, I^{\mathrm{sc}}_{20}I^{\mathrm{sc}}_{21}, I^{\mathrm{sc}}_{20}I^{\mathrm{sc}}_{22}, I^{\mathrm{sc}}_{20}I^{\mathrm{sc}}_{23}, I^{\mathrm{sc}}_{21}I^{\mathrm{sc}}_{22}\}\,,\\ \{J^{\mathrm{sc}}_{25},J^{\mathrm{sc}}_{26},J^{\mathrm{sc}}_{27},J^{\mathrm{sc}}_{28},J^{\mathrm{sc}}_{29}\} &= \int_0^\infty dz\,\mathrm{e}^{-\beta_c z}\,\frac{z^2}{\sinh z}\,\{I^{\mathrm{sc}}_{21}I^{\mathrm{sc}}_{23}, I^{\mathrm{sc}}_{22}I^{\mathrm{sc}}_{23}, (I^{\mathrm{sc}}_{21})^2, (I^{\mathrm{sc}}_{22})^2, (I^{\mathrm{sc}}_{23})^2\}\,,\end{aligned} \tag{3.100}$$

where $\beta_c = \frac{m^2}{eB_z} = \frac{B_{cr}}{B_z}$. Furthermore we use the following notation

$$\begin{aligned}\Gamma^{4,B}_{\mathrm{scal}}(1234) &= \frac{\alpha^2\beta_c^2}{m^4}\int_0^\infty dz\,\mathrm{e}^{-\beta_c z}\,\frac{z^2}{\sinh z}\,\mathcal{I}^{\mathrm{sc}}_{4f}(1234)\,,\\ \Gamma^{22,B}_{\mathrm{scal}}(12,34) &= \frac{\alpha^2\beta_c^2}{m^4}\int_0^\infty dz\,\mathrm{e}^{-\beta_c z}\,\frac{z^2}{\sinh z}\,\mathcal{I}^{\mathrm{sc}}_{2f}(12)\,\mathcal{I}^{\mathrm{sc}}_{2f}(34)\,.\end{aligned} \tag{3.101}$$

This allow us to express the four-photon amplitude in a pure magnetic field as

$$\Gamma^{(LE)}_{4,\text{scal}}(F) = \Gamma^{4,B}_{\text{scal}}(1234) + \Gamma^{4,B}_{\text{scal}}(2314) + \Gamma^{4,B}_{\text{scal}}(3124) + \Gamma^{22,B}_{\text{scal}}(12,34) + \Gamma^{22,B}_{\text{scal}}(23,14) + \Gamma^{22,B}_{\text{scal}}(31,24)\,, \tag{3.102}$$

for scalar QED. The contribution of a Lorentz four-cycle to this amplitude is

$$\begin{aligned}\Gamma^{4,B}_{\text{scal}}(1234) = \frac{\alpha^2\beta_c^2}{m^4}\Big\{& J_0^{\text{sc}}\,\text{tr}(f_1g_-f_2g_-f_3g_-f_4g_-) + J_1^{\text{sc}}\,\text{tr}(f_1g_+f_2g_+f_3g_+f_4g_+) + J_2^{\text{sc}}\,\text{tr}(f_1r_+f_2r_+f_3r_+f_4r_+)\\ &+ J_3^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_+f_3g_+f_4g_+) + 3\text{perm}\Big] + J_4^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_-f_3g_-f_4g_+) + 3\text{perm}\Big]\\ &+ J_5^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_-f_3g_+f_4g_+) + 5\text{perm}\Big] + J_6^{\text{sc}}\Big[\text{tr}(f_1r_+f_2r_+f_3g_-f_4g_-) + 5\text{perm}\Big]\\ &+ J_7^{\text{sc}}\Big[\text{tr}(f_1r_+f_2r_+f_3g_+f_4g_+) + 5\text{perm}\Big] + J_8^{\text{sc}}\Big[\text{tr}(f_1r_+f_2r_+f_3g_-f_4g_+) + 11\text{perm}\Big]\Big\}\end{aligned} \tag{3.103}$$

and of a Lorentz two two-cycle is

$$\begin{aligned}\Gamma^{22,B}_{\text{scal}}(12,34) = \frac{\alpha^2\beta_c^2}{m^4}\Big\{& J_{20}^{\text{sc}}\,\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_-f_4g_-) + J_{27}^{\text{sc}}\,\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_+f_4g_+)\\ &+ J_{21}^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_+f_4g_+) + \text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_-f_4g_-)\Big]\\ &+ J_{22}^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3r_+f_4r_+) + \text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_-f_4g_-)\Big]\\ &+ J_{23}^{\text{sc}}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_-f_4g_-)\Big]\\ &+ J_{24}^{\text{sc}}\Big[\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3r_+f_4r_+) + \text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_+f_4g_+)\Big]\\ &+ J_{25}^{\text{sc}}\Big[\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_+f_4g_+)\Big]\\ &+ J_{26}^{\text{sc}}\Big[\text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3r_+f_4r_+)\Big]\\ &+ J_{28}^{\text{sc}}\,\text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3r_+f_4r_+) + J_{29}^{\text{sc}}\,\text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-)\Big\}\,.\end{aligned} \tag{3.104}$$

The explicit expressions of J_i^{sc}'s can be found in Appendix C.5 as the integral of trigonometric functions and as combination of more general functions such as the Hurwitz-zeta and the polygamma functions (see [153, 166, 167, 168]).

3.8.2 Four-photon amplitude in a magnetic field for spinor QED

In this section, we study the four-photon amplitude in a constant field for spinor QED (see Fig. 3.5) obtained from the N-photon amplitude (3.18). In four dimensions, this amplitude is

$$\begin{aligned}\Gamma_{4,\text{spin}}(F) = -2(-ie)^4\int_0^\infty \frac{dT}{T}\,T^4(4\pi T)^{-2}\,\mathrm{e}^{-m^2T}\,\det{}^{-1/2}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right]\\ \times\int_0^1 du_1\,du_2\,du_3\,du_4\,\mathcal{Q}_{\text{spin}}(\dot{\hat{\mathcal{G}}}_{ij},\mathcal{G}_{Fij})\,\exp\left\{\frac{1}{2}\sum_{i,j=1}^4 k_i\cdot\mathcal{G}_{ij}\cdot k_j\right\}\end{aligned} \tag{3.105}$$

where, in the case of $N=4$, the polynomial $\mathcal{Q}_{\text{spin}}(\dot{\hat{\mathcal{G}}}_{ij},\mathcal{G}_{Fij})$ has the following representation

$$\begin{aligned}&\mathcal{Q}_{\text{spin}}(\dot{\hat{\mathcal{G}}}_{ij},\mathcal{G}_{Fij}) = {\mathcal{Q}_{\text{spin}}}^4 + {\mathcal{Q}_{\text{spin}}}^3 + {\mathcal{Q}_{\text{spin}}}^2 + {\mathcal{Q}_{\text{spin}}}^{22}\,,\\ &{\mathcal{Q}_{\text{spin}}}^4 = \dot{\mathcal{G}}_s(1234) + \dot{\mathcal{G}}_s(2314) + \dot{\mathcal{G}}_s(3124)\,,\\ &{\mathcal{Q}_{\text{spin}}}^3 = \dot{\mathcal{G}}_s(123)\mathcal{T}(4) + \dot{\mathcal{G}}_s(234)\mathcal{T}(1) + \dot{\mathcal{G}}_s(341)\mathcal{T}(2) + \dot{\mathcal{G}}_s(412)\mathcal{T}(3)\,,\\ &{\mathcal{Q}_{\text{spin}}}^2 = \dot{\mathcal{G}}_s(12)\mathcal{T}(34) + \dot{\mathcal{G}}_s(13)\mathcal{T}(24) + \dot{\mathcal{G}}_s(14)\mathcal{T}(23) + \dot{\mathcal{G}}_s(23)\mathcal{T}(14) + \dot{\mathcal{G}}_s(24)\mathcal{T}(13) + \dot{\mathcal{G}}_s(34)\mathcal{T}(12)\,,\\ &{\mathcal{Q}_{\text{spin}}}^{22} = \dot{\mathcal{G}}_s(12)\dot{\mathcal{G}}_s(34) + \dot{\mathcal{G}}_s(13)\dot{\mathcal{G}}_s(24) + \dot{\mathcal{G}}_s(14)\dot{\mathcal{G}}_s(23)\,,\end{aligned} \tag{3.106}$$

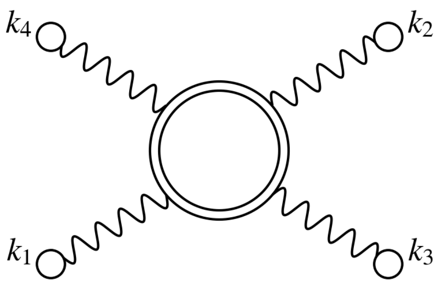

Figure 3.5: *Feynman diagram representing the four-photon amplitude with every incoming photon having low-energy, indicated by empty bullets at their ends. The double solid line indicate a particle of spin one-half in a magnetic field.*

obtained after the scalar one through the replacement rule described in Section 3.2. Here, the tails are given by (3.10) and the Lorentz super-cycles are given by

$$\begin{aligned}\dot{\mathcal{G}}_s(1234) &= \mathrm{tr}\left(f_1\hat{\dot{\mathcal{G}}}_{12}f_2\hat{\dot{\mathcal{G}}}_{23}f_3\hat{\dot{\mathcal{G}}}_{34}f_4\hat{\dot{\mathcal{G}}}_{41}\right) - \mathrm{tr}\Big(f_1\mathcal{G}_{F12}f_2\mathcal{G}_{F23}f_3\mathcal{G}_{F34}f_4\mathcal{G}_{F41}\Big)\,,\\ \dot{\mathcal{G}}_s(123) &= \mathrm{tr}\left(f_1\hat{\dot{\mathcal{G}}}_{12}f_2\hat{\dot{\mathcal{G}}}_{23}f_3\hat{\dot{\mathcal{G}}}_{31}\right) - \mathrm{tr}\Big(f_1\mathcal{G}_{F12}f_2\mathcal{G}_{F23}f_3\mathcal{G}_{F31}\Big)\,,\\ \dot{\mathcal{G}}_s(12) &= \frac{1}{2}\left[\mathrm{tr}\left(f_1\hat{\dot{\mathcal{G}}}_{12}f_2\hat{\dot{\mathcal{G}}}_{21}\right) - \mathrm{tr}\Big(f_1\mathcal{G}_{F12}f_2\mathcal{G}_{F21}\Big)\right]\,.\end{aligned} \tag{3.107}$$

It is convenient to recall that

$$\hat{\dot{\mathcal{G}}}_{ij} = \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii} + \mathcal{G}_{Fii}\,. \tag{3.108}$$

In the following, we consider the fully low-energy case. The next steps are completely analogous to the scalar case. The leading non-vanishing contributions to the amplitude are those composed fully by Lorentz super-cycles. Then, the four-photon amplitude at low energies in a purely magnetic background field for spinor QED is

$$\Gamma^{(LE)}_{4,\mathrm{spin}}(F) = -2\alpha^4\int_0^\infty \frac{dT}{T}\,T^2\,\mathrm{e}^{-m^2T}\,\frac{z}{\tanh z}\int_0^1 du_1\,du_2\,du_3\,du_4\,\left({\mathcal{Q}_{\mathrm{spin}}}^4 + {\mathcal{Q}_{\mathrm{spin}}}^{22}\right)\,. \tag{3.109}$$

Here, we have already used the determinant result (3.33). We define

$$\begin{aligned}\mathcal{I}^{\mathrm{sp}}_{2f}(12) &= \frac{1}{2}\int_0^1 du_1\,du_2\left[\mathrm{tr}\left(f_1\hat{\dot{\mathcal{G}}}_{12}f_2\hat{\dot{\mathcal{G}}}_{21}\right) - \mathrm{tr}\left(f_1\mathcal{G}_{F12}f_2\mathcal{G}_{F21}\right)\right]\,,\\ \mathcal{I}^{\mathrm{sp}}_{4f}(1234) &= \int_0^1 du_1\,du_2\,du_3\,du_4\,\left[\mathrm{tr}\left(f_1\hat{\dot{\mathcal{G}}}_{12}f_2\hat{\dot{\mathcal{G}}}_{23}f_3\hat{\dot{\mathcal{G}}}_{34}f_4\hat{\dot{\mathcal{G}}}_{41}\right) - \mathrm{tr}\left(f_1\mathcal{G}_{F12}f_2\mathcal{G}_{F23}f_3\mathcal{G}_{F34}f_4\mathcal{G}_{F41}\right)\right]\,,\end{aligned} \tag{3.110}$$

which in terms of the cyclic integrals (3.53) can be expressed as

$$\begin{aligned}\mathcal{I}^{\mathrm{sp}}_{2f}(12) &= I^{\mathrm{cyc}}_{\mathrm{spin}}(f_1,f_2;F) + \frac{1}{2}\,\mathrm{tr}\left[f_1(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})f_2(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})\right]\,,\\ \mathcal{I}^{\mathrm{sp}}_{4f}(1234) &= I^{\mathrm{cyc}}_{\mathrm{spin}}(f_1,f_2,f_3,f_4;F) + \mathrm{tr}\left[f_1(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})f_2(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})f_3(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})f_4(\dot{\mathcal{G}}_{Bii}-\mathcal{G}_{Fii})\right]\,.\end{aligned} \tag{3.111}$$

We set $z_+ = z = eTB_z$ and $z_- = 0$ in (3.53) and recall the coincidence limit of the calligraphic Green's functions in a pure magnetic field, Section 3.4. Then, in this way, we obtain

$$\mathcal{I}^{\mathrm{sp}}_{2f}(12) = I^{\mathrm{sp}}_{20}\,\mathrm{tr}(f_1g_-f_2g_-) + I^{\mathrm{sp}}_{21}\,\mathrm{tr}(f_1g_+f_2g_+) + I^{\mathrm{sp}}_{22}\,\mathrm{tr}(f_1r_+f_2r_+) + I^{\mathrm{sp}}_{23}\left[\mathrm{tr}(f_1g_-f_2g_+) + \mathrm{tr}(f_1g_+f_2g_-)\right]\,, \tag{3.112}$$

for the two-cycle integral. And

$$\mathcal{I}^{\rm sp}_{4f}(1234) = \Big\{ I^{\rm sp}_0\,{\rm tr}(f_1g_-f_2g_-f_3g_-f_4g_-) + I^{\rm sp}_1\,{\rm tr}(f_1g_+f_2g_+f_3g_+f_4g_+) + I^{\rm sp}_2\,{\rm tr}(f_1r_+f_2r_+f_3r_+f_4r_+) \\ + I^{\rm sp}_3\Big[{\rm tr}(f_1g_-f_2g_+f_3g_+f_4g_+) + 3{\rm perm}\Big] + I^{\rm sp}_4\Big[{\rm tr}(f_1g_-f_2g_-f_3g_-f_4g_+) + 3{\rm perm}\Big] \\ + I^{\rm sp}_5\Big[{\rm tr}(f_1g_-f_2g_-f_3g_+f_4g_+) + 5{\rm perm}\Big] + I^{\rm sp}_6\Big[{\rm tr}(f_1r_+f_2r_+f_3g_-f_4g_-) + 5{\rm perm}\Big] \\ + I^{\rm sp}_7\Big[{\rm tr}(f_1r_+f_2r_+f_3g_+f_4g_+) + 5{\rm perm}\Big] + I^{\rm sp}_8\Big[{\rm tr}(f_1r_+f_2r_+f_3g_-f_4g_+) + 11{\rm perm}\Big]\Big\}\,, \tag{3.113}$$

for the four-cycle integral. Here, the functions $I_i^{\rm sp}$'s are trigonometric expressions and are explicitly written in terms of $H_{11}^{B(n)}$ and $H_{11}^{F(n)}$ (see Eq. (3.44) and Appendix C.1).

Finally, for the integration over T, we make the change of variables $z = eB_zT$ and set the following definitions

$$\begin{aligned} J^{\rm sp}_n &= \int_0^\infty dz\,{\rm e}^{-\beta_c z}\,\frac{z^2}{\tanh z}\,I^{\rm sp}_n\,, \\ \{J^{\rm sp}_{20}, J^{\rm sp}_{21}, J^{\rm sp}_{22}, J^{\rm sp}_{23}, J^{\rm sp}_{24}\} &= \int_0^\infty dz\,{\rm e}^{-\beta_c z}\,\frac{z^2}{\tanh z}\,\{(I^{\rm sp}_{20})^2, I^{\rm sp}_{20}I^{\rm sp}_{21}, I^{\rm sp}_{20}I^{\rm sp}_{22}, I^{\rm sp}_{20}I^{\rm sp}_{22}, I^{\rm sp}_{21}I^{\rm sp}_{22}\}\,, \\ \{J^{\rm sp}_{25}, J^{\rm sp}_{26}, J^{\rm sp}_{27}, J^{\rm sp}_{28}, J^{\rm sp}_{29}\} &= \int_0^\infty dz\,{\rm e}^{-\beta_c z}\,\frac{z^2}{\tanh z}\,\{I^{\rm sp}_{21}I^{\rm sp}_{23}, I^{\rm sp}_{22}I^{\rm sp}_{23}, (I^{\rm sp}_{21})^2, (I^{\rm sp}_{22})^2, (I^{\rm sp}_{23})^2\} \end{aligned} \tag{3.114}$$

and

$$\begin{aligned} \Gamma^{4,B}_{\rm scal}(1234) &= -2\,\frac{\alpha^2\beta_c^2}{m^4}\int_0^\infty dz\,{\rm e}^{-\beta_c z}\,\frac{z^2}{\tanh z}\,\mathcal{I}^{\rm sp}_{4f}(1234)\,, \\ \Gamma^{22,B}_{\rm scal}(12,34) &= -2\,\frac{\alpha^2\beta_c^2}{m^4}\int_0^\infty dz\,{\rm e}^{-\beta_c z}\,\frac{z^2}{\tanh z}\,\mathcal{I}^{\rm sp}_{2f}(12)\,\mathcal{I}^{\rm sp}_{2f}(34)\,. \end{aligned} \tag{3.115}$$

This allows us to express the four-photon amplitude in a pure magnetic field as

$$\Gamma^{(LE)}_{4,{\rm spin}}(F) = \Gamma^{4,B}_{\rm spin}(1234) + \Gamma^{4,B}_{\rm spin}(2314) + \Gamma^{4,B}_{\rm spin}(3124) + \Gamma^{22,B}_{\rm spin}(12,34) + \Gamma^{22,B}_{\rm spin}(23,14) + \Gamma^{22,B}_{\rm spin}(31,24)\,, \tag{3.116}$$

for spinor QED. The contribution of a Lorentz four-cycle to this amplitude is

$$\Gamma^{4,B}_{\rm spin}(1234) = -2\,\frac{\alpha^2\beta_c^2}{m^4}\,{\rm tr}\Big[J^{\rm sp}_0\,f_1g_-f_2g_-f_3g_-f_4g_- + J^{\rm sp}_1\,f_1g_+f_2g_+f_3g_+f_4g_+ + J^{\rm sp}_2\,f_1r_+f_2r_+f_3r_+f_4r_+ \\ + J^{\rm sp}_3\Big(f_1g_-f_2g_+f_3g_+f_4g_+ + 3{\rm perm}\Big) + J^{\rm sp}_4\Big(f_1g_-f_2g_-f_3g_-f_4g_+ + 3{\rm perm}\Big) \\ + J^{\rm sp}_5\Big(f_1g_-f_2g_-f_3g_+f_4g_+ + 5{\rm perm}\Big) + J^{\rm sp}_6\Big(f_1r_+f_2r_+f_3g_-f_4g_- + 5{\rm perm}\Big) \\ + J^{\rm sp}_7\Big(f_1r_+f_2r_+f_3g_+f_4g_+ + 5{\rm perm}\Big) + J^{\rm sp}_8\Big(f_1r_+f_2r_+f_3g_-f_4g_+ + 11{\rm perm}\Big)\Big] \tag{3.117}$$

and of a Lorentz two two-cycle is

$$\Gamma^{22,B}_{\text{spin}}(12,34) = -2\,\frac{\alpha^2\beta_c^2}{m^4}\Big\{J^{\text{sp}}_{20}\,\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_-f_4g_-) + J^{\text{sp}}_{27}\,\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_+f_4g_+)$$
$$+ J^{\text{sp}}_{21}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_+f_4g_+) + \text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_-f_4g_-)\Big]$$
$$+ J^{\text{sp}}_{22}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3r_+f_4r_+) + \text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_-f_4g_-)\Big]$$
$$+ J^{\text{sp}}_{23}\Big[\text{tr}(f_1g_-f_2g_-)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_-f_4g_-)\Big]$$
$$+ J^{\text{sp}}_{24}\Big[\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3r_+f_4r_+) + \text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_+f_4g_+)\Big]$$
$$+ J^{\text{sp}}_{25}\Big[\text{tr}(f_1g_+f_2g_+)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_+f_4g_+)\Big]$$
$$+ J^{\text{sp}}_{26}\Big[\text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-) + \text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3r_+f_4r_+)\Big]$$
$$+ J^{\text{sp}}_{28}\,\text{tr}(f_1r_+f_2r_+)\,\text{tr}(f_3r_+f_4r_+) + J^{\text{sp}}_{29}\,\text{tr}(f_1g_-f_2g_+ + f_1g_+f_2g_-)\,\text{tr}(f_3g_-f_4g_+ + f_3g_+f_4g_-)\Big\}\,. \tag{3.118}$$

The explicit expression of J^{sp}_i can be found in the Appendix C.6 as the integral of a trigonometric function and as combination of more general functions such as the Hurwitz-zeta and the polygamma functions (see [153, 166, 167, 168]).

3.9 Low-energy limit of the polarized four-photon amplitudes in a magnetic field

In this section, we present explicit expressions for the polarized four-photon amplitudes following the conventions set in [14, 16].

3.9.1 Magnetic field parallel to the scattering plane: $B\,||\,k_i$

In this section, we present the four-photon polarized amplitudes in the presence of a pure magnetic background field pointing in the z axis. The scattering plane for the external photons is chosen as the xz-plane. For convenience, we define

$$\Gamma^{(LE)}_{\text{scal}}(k_1,\varepsilon_1^{*(\lambda_1)};k_2,\varepsilon_2^{*(\lambda_2)};k_3,\varepsilon_3^{(\lambda_3)};k_4,\varepsilon_4^{(\lambda_4)};F) = \alpha^2\,(2\pi)^4\delta^4\,(k_1+k_2+k_3+k_4)\;\hat{\Gamma}^{B,\lambda_1\lambda_2\lambda_3\lambda_4}_{\text{scal}}$$
$$\Gamma^{(LE)}_{\text{spin}}(k_1,\varepsilon_1^{*(\lambda_1)};k_2,\varepsilon_2^{*(\lambda_2)};k_3,\varepsilon_3^{(\lambda_3)};k_4,\varepsilon_4^{(\lambda_4)};F) = -2\alpha^2\,(2\pi)^4\delta^4\,(k_1+k_2+k_3+k_4)\;\hat{\Gamma}^{B,\lambda_1\lambda_2\lambda_3\lambda_4}_{\text{spin}} \tag{3.119}$$

since this allows us to compute simultaneously both polarized amplitudes. Then, we express the general amplitude $\hat{\Gamma}^B$ in terms of the J_i's functions which must be identified with J^{sc}_i's (listed in Appendix C.5) for the scalar amplitudes or J^{sp}_i's (listed in Appendix C.6) for the spinor amplitudes.

Here, the momenta and polarizations are chosen as in Section 2.5.1 (we follow the conventions of [14, 16]) with k_1, k_2 as incoming and k_3, k_4 as outgoing

$$\begin{aligned} k_1 &= (0,0,-\omega,-i\omega)\\ k_2 &= (0,0,\omega,-i\omega)\\ k_3 &= (\omega\sin\theta,0,\omega\cos\theta,i\omega)\\ k_4 &= (-\omega\sin\theta,0,-\omega\cos\theta,i\omega) \end{aligned} \tag{3.120}$$

The linear polarizations $\varepsilon_i^{(\lambda_i)}$ are given by

$$\begin{aligned} \varepsilon_1^{(1)} = \varepsilon_2^{(1)} = \varepsilon_3^{(1)} = \varepsilon_4^{(1)} &= (0,1,0,0)\\ -\varepsilon_1^{(2)} = \varepsilon_2^{(2)} &= (1,0,0,0)\\ -\varepsilon_3^{(2)} = \varepsilon_4^{(2)} &= (\cos\theta,0,-\sin\theta,0) \end{aligned} \tag{3.121}$$

and for circularly polarized, we have

$$\varepsilon_j^{(\pm)} = \frac{1}{\sqrt{2}}\left[\varepsilon_j^{(1)} \pm i\varepsilon_j^{(2)}\right] \tag{3.122}$$

where $\varepsilon_j^{(+)}$ and $\varepsilon_j^{(-)}$ are for right and left handed circular polarizations respectively.

For linear polarizations, we find that the polarized amplitudes satisfy the following relations

$$\begin{aligned}
&\hat{\Gamma}^{B,1111}, \quad \hat{\Gamma}^{B,2222}, \quad \hat{\Gamma}^{B,1122}, \quad \hat{\Gamma}^{B,2211}, \quad \hat{\Gamma}^{B,1212} = \hat{\Gamma}^{B,2121}, \quad \hat{\Gamma}^{B,1221} = \hat{\Gamma}^{B,2112}\\
&\hat{\Gamma}^{B,1112} = \hat{\Gamma}^{B,1121} = \hat{\Gamma}^{B,1211} = \hat{\Gamma}^{B,2111} = \hat{\Gamma}^{B,2221} = \hat{\Gamma}^{B,2212} = \hat{\Gamma}^{B,2122} = \hat{\Gamma}^{B,1222} = 0
\end{aligned} \tag{3.123}$$

and the result of these linearly polarized amplitudes are

$$\begin{aligned}
\hat{\Gamma}^{B,1111} &= 2\frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{25} - J_{26} + 2J_3 - 6J_8) + 2(1+\cos^2\theta)(J_{29} + 2J_5)\Big]\\
\hat{\Gamma}^{B,2222} &= 2\frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{23} + 2J_4) + 2(1+\cos^2\theta)(J_{29} + 2J_5)\Big]\\
\hat{\Gamma}^{B,1122} &= 2\frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{23} + 2J_4) + (1+\cos^2\theta)(J_{29} + J_5 - J_6)\Big]\\
\hat{\Gamma}^{B,2211} &= 2\frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{25} - J_{26} + 2J_3 - 6J_8) + (1+\cos^2\theta)(J_{29} + J_5 - J_6)\Big]\\
\hat{\Gamma}^{B,1212} &= \frac{\omega^4}{(eB_z)^2}\Big[(1-\cos\theta)^2 J_{29} + [-3 + (-6+\cos\theta)\cos\theta]J_5 - [1 + (2+5\cos\theta)\cos\theta]J_6\Big]\\
\hat{\Gamma}^{B,1221} &= \frac{\omega^4}{(eB_z)^2}\Big[(1+\cos\theta)^2 J_{29} + [-3 + (6+\cos\theta)\cos\theta]J_5 - [1 + (-2+5\cos\theta)\cos\theta]J_6\Big]
\end{aligned} \tag{3.124}$$

For circular polarizations, we find that the polarized amplitudes satisfy the following relations

$$\begin{aligned}
&\hat{\Gamma}^{B,++++} = \hat{\Gamma}^{B,----}, \qquad \hat{\Gamma}^{B,++--} = \hat{\Gamma}^{B,--++},\\
&\hat{\Gamma}^{B,+-+-} = \hat{\Gamma}^{B,-+-+}, \qquad \hat{\Gamma}^{B,+--+} = \hat{\Gamma}^{B,-++-},\\
&\hat{\Gamma}^{B,+++-} = \hat{\Gamma}^{B,++-+} = \hat{\Gamma}^{B,---+} = \hat{\Gamma}^{B,--+-},\\
&\hat{\Gamma}^{B,+-++} = \hat{\Gamma}^{B,-+++} = \hat{\Gamma}^{B,-+--} = \hat{\Gamma}^{B,+---} = 0,
\end{aligned} \tag{3.125}$$

where the non-vanishing circularly polarized amplitudes are

$$\begin{aligned}
\hat{\Gamma}^{B,++++} &= \frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{23} + J_{25} - J_{26} + 2J_3 + 2J_4 - 6J_8)\\
&\qquad + 4(1+\cos^2\theta)J_{29} + 2(1+3\cos^2\theta)(J_5 - J_6)\Big]\\
\hat{\Gamma}^{B,++--} &= \frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)(J_{23} + J_{25} - J_{26} + 2J_3 + 2J_4 - 6J_8)\\
&\qquad + 2(1+\cos^2\theta)(J_{29} + 4J_5) - 4(J_5 - J_6)\cos^2\theta\Big]\\
\hat{\Gamma}^{B,+-+-} &= \frac{\omega^4}{(eB_z)^2}(1-\cos\theta)^2(J_{29} + 3J_5 + J_6)\\
\hat{\Gamma}^{B,+--+} &= \frac{\omega^4}{(eB_z)^2}(1+\cos\theta)^2(J_{29} + 3J_5 + J_6)\\
\hat{\Gamma}^{B,+++-} &= \frac{\omega^4}{(eB_z)^2}(1-\cos^2\theta)(-J_{23} + J_{25} - J_{26} + 2J_3 - 2J_4 - 6J_8)
\end{aligned} \tag{3.126}$$

In the next section we look at the case in which the magnetic field is perpendicular to the scattering plane.

3.9.2 Magnetic field orthogonal to the scattering plane: $B \perp k_i$

In this section, we present the four-photon polarized amplitudes in the presence of a pure magnetic background field pointing in the z axis. The scattering plane for the external photons is chosen as the xy-plane and we use the same conventions as in the previous section (see Eq. (3.119)). Here we choose the momenta k_1, k_2 as incoming and k_3, k_4 as outgoing

$$\begin{aligned} k_1 &= (0, -\omega, 0, -i\omega) \\ k_2 &= (0, \omega, 0, -i\omega) \\ k_3 &= (\omega \sin\theta, \omega\cos\theta, 0, i\omega) \\ k_4 &= (-\omega\sin\theta, -\omega\cos\theta, 0, i\omega) \end{aligned} \tag{3.127}$$

The polarizations $\varepsilon_i^{(\lambda_i)}$ are given by, for linear polarizations,

$$\begin{aligned} \varepsilon_1^{(1)} = \varepsilon_2^{(1)} = \varepsilon_3^{(1)} = \varepsilon_4^{(1)} &= (0,0,1,0) \\ -\varepsilon_1^{(2)} = \varepsilon_2^{(2)} &= (1,0,0,0) \\ -\varepsilon_3^{(2)} = \varepsilon_4^{(2)} &= (\cos\theta, -\sin\theta, 0, 0) \end{aligned} \tag{3.128}$$

and, for circular polarizations,

$$\varepsilon_j^{(\pm)} = \frac{1}{\sqrt{2}}\left[\varepsilon_j^{(1)} \pm i\varepsilon_j^{(2)}\right] \tag{3.129}$$

For linear polarizations, the following relations are satisfied

$$\begin{aligned} &\hat{\Gamma}^{B,1111}, \quad \hat{\Gamma}^{B,2222}, \quad \hat{\Gamma}^{B,1122} = \hat{\Gamma}^{B,2211}, \quad \hat{\Gamma}^{B,1212} = \hat{\Gamma}^{B,2121}, \quad \hat{\Gamma}^{B,1221} = \hat{\Gamma}^{B,2112} \\ &\hat{\Gamma}^{B,1112} = \hat{\Gamma}^{B,1121} = \hat{\Gamma}^{B,1211} = \hat{\Gamma}^{B,2111} = \hat{\Gamma}^{B,2221} = \hat{\Gamma}^{B,2212} = \hat{\Gamma}^{B,2122} = \hat{\Gamma}^{B,1222} = 0 \end{aligned} \tag{3.130}$$

and the result of these linearly polarized amplitudes are

$$\begin{aligned} \hat{\Gamma}^{B,1111} &= \frac{\omega^4}{(eB_z)^2}\Big[(1+2\cos^2\theta)(J_{29}+2J_5) + 3(J_{20}+2J_0) + 2(J_{23}+2J_4)\Big] \\ \hat{\Gamma}^{B,2222} &= \frac{\omega^4}{(eB_z)^2}\Big[(1+2\cos^2\theta)(J_{29}+2J_5) + 6(J_1+J_2-6J_7-2J_8-J_{24}) \\ &\qquad + 2(J_{25}-J_{26}+2J_3) + 3(J_{27}+J_{28})\Big] \\ \hat{\Gamma}^{B,1122} &= \frac{\omega^4}{(eB_z)^2}\Big[(1-\cos^2\theta)J_5 - J_6\cos^2\theta + 2(J_3+J_4-3J_8) \\ &\qquad + J_{21}-J_{22}+J_{23}+J_{25}-J_{26}+J_{29}\Big] \\ \hat{\Gamma}^{B,1212} &= \frac{\omega^4}{(eB_z)^2}\Big[(J_{29}+J_5-J_6)\cos^2\theta - 2(J_3+J_4+J_5-J_6-3J_8)\cos\theta \\ &\qquad - (J_{23}+J_{25}-J_{26})\cos\theta + J_{21}-J_{22}-3J_5+J_6\Big] \\ \hat{\Gamma}^{B,1221} &= \frac{\omega^4}{(eB_z)^2}\Big[(J_{29}+J_5-J_6)\cos^2\theta + 2(J_3+J_4+J_5-J_6-3J_8)\cos\theta \\ &\qquad + (J_{23}+J_{25}-J_{26})\cos\theta + J_{21}-J_{22}-3J_5+J_6\Big] \end{aligned} \tag{3.131}$$

For circular polarizations, the following relations are satisfied

$$\begin{aligned} &\hat{\Gamma}^{B,++++} = \hat{\Gamma}^{B,----}, \quad \hat{\Gamma}^{B,++--} = \hat{\Gamma}^{B,--++}, \quad \hat{\Gamma}^{B,+-+-} = \hat{\Gamma}^{B,-+-+}, \quad \hat{\Gamma}^{B,+--+} = \hat{\Gamma}^{B,-++-}, \\ &\hat{\Gamma}^{B,+++-} = \hat{\Gamma}^{B,++-+} = \hat{\Gamma}^{B,---+} = \hat{\Gamma}^{B,--+-} = \hat{\Gamma}^{B,+-++} = \hat{\Gamma}^{B,-+++} = \hat{\Gamma}^{B,-+--} = \hat{\Gamma}^{B,+---}, \end{aligned} \tag{3.132}$$

where these circularly polarized amplitudes are given by

$$\begin{aligned}
\hat{\Gamma}^{B,++++} &= \frac{\omega^4}{4(eB_z)^2}\Big[-4(1-2\cos^2\theta)(J_5-J_6)+4(1+2\cos^2\theta)J_{29}+6(J_0+J_1+J_2) \\
&\quad +8(J_3+J_4)-12(3J_7+2J_8)+3(J_{20}+J_{27}+J_{28}) \\
&\quad +6(J_{21}-J_{22}-J_{24})+4(J_{23}+J_{25}-J_{26})\Big] \\
\hat{\Gamma}^{B,++--} &= \frac{\omega^4}{4(eB_z)^2}\Big[6(J_0+J_1+J_2)+8(J_3+J_4)+4(5J_5-J_6)-12(3J_7+2J_8) \\
&\quad +3(J_{20}+J_{27}+J_{28})-2(J_{21}-J_{22}+3J_{24})+4(J_{23}+J_{25}-J_{26}+J_{29})\Big] \\
\hat{\Gamma}^{B,+-+-} &= \frac{\omega^4}{4(eB_z)^2}\Big[4(J_{29}+3J_5-J_6)\cos^2\theta-4(J_{23}+J_{25}-J_{26})\cos\theta \\
&\quad -8(J_3+J_4+J_5-J_6-3J_8)\cos\theta+6(J_0+J_1+J_2-6J_7) \\
&\quad +3(J_{20}+J_{27}+J_{28})-2(J_{21}-J_{22}+3J_{24})+4(J_{23}+J_{25}-J_{26})\Big] \\
\hat{\Gamma}^{B,+--+} &= \frac{\omega^4}{4(eB_z)^2}\Big[4(J_{29}+3J_5-J_6)\cos^2\theta+4(J_{23}+J_{25}-J_{26})\cos\theta \\
&\quad +8(J_3+J_4+J_5-J_6-3J_8)\cos\theta+6(J_0+J_1+J_2-6J_7) \\
&\quad +3(J_{20}+J_{27}+J_{28})-2(J_{21}-J_{22}+3J_{24})+4(J_{23}+J_{25}-J_{26})\Big] \\
\hat{\Gamma}^{B,+++-} &= \frac{\omega^4}{4(eB_z)^2}\Big[6(J_0-J_1-J_2+6J_7+2J_8)-4(J_3-J_4) \\
&\quad +3(J_{20}-J_{27}-J_{28})+2(J_{23}+3J_{24}-J_{25}+J_{26})\Big]
\end{aligned} \tag{3.133}$$

In this Sections 3.8 and 3.9 (together with Appendix C), we presented analytic results for the unpolarized and polarized amplitudes for scalar and spinor QED of the leading contribution to light-by-light scattering in the presence of a magnetic background field. Notice that these results need to be studied further in order to compare with observations made in heavy ion collisions such as [31, 32, 33] (see also [45]) or in astrophysical observations [38, 169] in which it is possible to have very strong magnetic fields. Further details of the presented results in this chapter can be found in [113].

Chapter 4

Dressed propagators in a constant background field

In this chapter, we provide a concise overview of the derivation of the dressed scalar and fermion propagators in the presence of a constant background field, adapted from [67, 68, 69, 70, 170, 124]. This is done with the aim of investigating the polarization effects resulting from Compton scattering [66], which could be relevant in the observation of Coulomb-assisted birefringence [65].

Theoretical advancements regarding the propagation of both spin-zero (scalar) and spin one-half (fermion) particles within the worldline formalism have led to the development of master formulas for various configurations. The master formula for the propagator of a scalar particle moving from x to x' and interacting with N photons is derived in [126, 127] and later generalized to the spinor case in [68, 69] (see also [124]). Additionally, the propagator of a scalar particle moving in a constant background field while interacting with N photons is derived in [67]. The extension to the spinor case of the latter is presented in [70, 170], formulated in terms of a Grassman path integral (as discussed in this chapter). Furthermore, the scenario involving a particle moving from x to x' dressed with N photons and in the presence of a plane-wave field is obtained in [128], for both scalar and spinor QED.

4.1 Dressed propagators in a external field

The propagators of an off-shell scalar and spinor particles propagating from x to x' in a background field in d dimensions are [67, 68, 170]

$$
\begin{aligned}
D^{xx'}_{\rm sc}(A) &= \langle x'|\frac{1}{m^2+\Pi^2}|x\rangle\,, \\
D^{xx'}_{\rm sp}(A) &= \langle x'|\frac{1}{m-\not{\Pi}}|x\rangle = \left(m+\not{\Pi}_{x'}\right)\langle x'|\frac{1}{m^2+\Pi^2+\frac{ie}{4}F_{\mu\nu}[\gamma^\mu,\gamma^\nu]}|x\rangle\,,
\end{aligned}
\tag{4.1}
$$

respectively. Here, we use the subscripts ‘sc’ and ‘sp’ for scalar and spinor respectively. The spinor propagator is valid only for even dimensions (see [68]). The four-momentum of the particle in the presence of a background field is

$$
\Pi^\mu = -p^\mu - eA^\mu = i\partial^\mu - eA^\mu \tag{4.2}
$$

and $\not{\Pi} = \gamma_\mu \Pi^\mu$ with γ_μ the Dirac-gamma matrices in Euclidean space, see Appendix A.

The use of Schwinger parameters allow us to express the previous propagators as

$$
\begin{aligned}
D^{xx'}_{\rm sc}(A) &= \int_0^\infty dT\, {\rm e}^{-Tm^2} K^{xx'}_{\rm sc}(A,T)\,, \\
D^{xx'}_{\rm sp}(A) &= \left(m+\not{\Pi}_{x'}\right)\int_0^\infty dT\, {\rm e}^{-Tm^2} K^{xx'}_{\rm sp}(A,T)\,.
\end{aligned}
\tag{4.3}
$$

The kernels $K_{\rm sc}^{xx'}$ and $K_{\rm sp}^{xx'}$ are defined as

$$K_{\rm sc}^{xx'}(A,T) = \langle x'|{\rm e}^{-T\Pi^2}|x\rangle = \int\limits_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\,{\rm e}^{-S[x(\tau)]}\,,$$
$$K_{\rm sp}^{xx'}(A,T) = \langle x'|{\rm e}^{-T\Pi^2-\frac{ie}{4}TF_{\mu\nu}[\gamma^\mu,\gamma^\nu]}|x\rangle = \int\limits_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\,{\rm e}^{-S[x(\tau)]}\;\mathcal{P}{\rm e}^{-\frac{ie}{4}\int_0^T d\tau F_{\mu\nu}[\gamma^\mu,\gamma^\nu]}\,, \tag{4.4}$$

where $\mathcal{P}$ is the "path-ordering" operator and $S[x(\tau)]$ is the "worldline action"

$$S[x(\tau)] = \int\limits_0^T d\tau\left[\frac{\dot{x}_\mu\dot{x}^\mu}{4} + ie\dot{x}^\mu A_\mu\big(x(\tau)\big)\right]. \tag{4.5}$$

Here, the spin term can be expressed as a path-integral over Grassman variables

$$\mathcal{P}{\rm e}^{-\frac{ie}{4}\int_0^T d\tau F_{\mu\nu}[\gamma^\mu,\gamma^\nu]} = 2^{-d/2}{\rm symb}^{-1}\left[\int_C \mathcal{D}\psi(\tau)\,{\rm e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi^\mu\dot{\psi}_\mu - ie\left(\psi+\frac{1}{2}\eta\right)^\mu F_{\mu\nu}\left(\psi+\frac{1}{2}\eta\right)^\nu\right]}\right], \tag{4.6}$$

with the boundary condition $C = \{\psi^\mu(0)+\psi^\mu(T)=0\}$. The 'symb' function is

$$\left(i\sqrt{2}\right)^n {\rm symb}\left(\gamma^{\alpha_1\alpha_2\ldots\alpha_n}\right) = \eta^{\alpha_1}\eta^{\alpha_2}\cdots\eta^{\alpha_n}\,, \tag{4.7}$$

where $\gamma^{\alpha_1\alpha_2\ldots\alpha_n}$ is the totally antisymmetric product of gamma matrices

$$\gamma^{\alpha_1\alpha_2\ldots\alpha_n} = \frac{1}{n!}\,\varepsilon^{a_1a_2\ldots a_n}\,\gamma^{\alpha_{a_1}}\gamma^{\alpha_{a_2}}\cdots\gamma^{\alpha_{a_n}}\,,\quad a_i = 1,2,...,n \tag{4.8}$$

and $\varepsilon^{a_1a_2\ldots a_n}$ the Levi-Civita symbol satisfying the convention $\varepsilon^{12}=\varepsilon^{1234}=+1$. For instance,

$$\gamma^{[\alpha\beta]} = \frac{1}{2!}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right), \tag{4.9}$$

which correspond to the case $n=2$.

It is important to point out that these expressions for the propagators are valid for any vector potential $A_\mu(x)$ and that for their representation in momentum space the Fourier transform is employed. The position space propagators are expressed as

$$D_{\rm sc/sp}^{pp'}(A) = \int d^dx\int d^dx'\,{\rm e}^{i(px+p'x')}\,D_{\rm sc/sp}^{xx'}(A)\,, \tag{4.10}$$

for initial momentum p and final $-p'$, as shown in Fig. 4.1.

4.2 Dressed propagators in a constant field

Now we consider the propagators interacting with N external photons (propagator dressed with N photons) in the presence of a constant background field. In order to produce the external photons we choose one background field A_N^μ as a sum of plane waves. And for the constant field $A_{\rm ct}^\mu$ we consider the Fock-Schwinger gauge in which the gauge condition is $(x-x_c)_\mu A^\mu(x)=0$, centered in x_c. Then the total vector potential A^μ is

$$A^\mu\big(x(\tau)\big) = A_N^\mu\big(x(\tau)\big) + A_{\rm ct}^\mu\big(x(\tau)\big) = \sum_{i=1}^N {\varepsilon_i}^\mu\,{\rm e}^{ik_i\cdot x(\tau)} - \frac{1}{2}F_{\rm ct}^{\mu\nu}\big(x(\tau)-x\big)_\nu\,, \tag{4.11}$$

with this choice, the field strength tensor is

$$F^{\mu\nu}\big(x(\tau)\big) = i\sum_{i=1}^N f_i^{\mu\nu}\,{\rm e}^{ik_i\cdot x(\tau)} + F_{\rm ct}^{\mu\nu}\,, \tag{4.12}$$

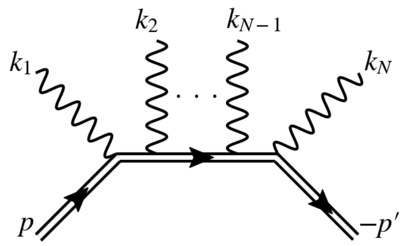

Figure 4.1: *Feynman for the scalar or spinor propagator dressed with N-photons. Double lines indicate that the particles is in the presence of a constant background field.*

where

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\mu \tag{4.13}$$

is the field strength tensor of each photon.

The use of the vector potential (4.11), and since the terms contributing to the propagators are those that are multilinear in all polarizations [67, 70, 124], the QED propagators in this case become

$$D^{xx'}_{\text{sc},N}(A_{\text{ct}}) = \int_0^\infty dT\, \mathrm{e}^{-Tm^2} K^{xx'}_{\text{sc},N}(A_{\text{ct}},T) \tag{4.14}$$

and

$$D^{xx'}_{\text{sp},N}(A_{\text{ct}}) = \int_0^\infty dT\, \mathrm{e}^{-Tm^2} \left[\left(m + \not{\Pi}_{\text{ct},x'}\right) K^{xx'}_{\text{sp},N}(A_{\text{ct}},T) - e\sum_{i=1}^N \not{\varepsilon}_i\, \mathrm{e}^{ik_i\cdot x'} K^{xx'}_{\text{sp},N-1}(A_{\text{ct}},T)\right], \tag{4.15}$$

for an off-shell scalar and spinor particle dressed with N photons, respectively. Here, the external photons can be taken as off-shell or on-shell right away. Here we have used the following convention $\Pi^\mu_{\text{ct},x'} = i\partial^\mu_{x'} + \frac{e}{2}F^{\mu\nu}_{\text{ct}}(x'-x)_\nu$ and the short hand notation

$$D^{xx'}_{\text{sc},N}(A_{\text{ct}}) = D_{\text{sc}}(A_{\text{ct}}|x,x';k_1,\varepsilon_1;k_2,\varepsilon_2;...;k_N,\varepsilon_N)\,. \tag{4.16}$$

The kernel for scalar particles can be written as [67]

$$\begin{aligned} K^{xx'}_{\text{sc},N}(A_{\text{ct}},T) = (-ie)^N(4\pi T)^{-d/2}\left[\det\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right)\right]^{1/2} \mathrm{e}^{-\frac{1}{4T}(x'-x)\mathcal{Z}\cot\mathcal{Z}(x'-x)+\, i\sum_{i=1}^N k_i x} \\ \times \prod_{i=1}^N \int_0^T d\tau_i\; \mathrm{e}^{\frac{i(x'-x)^\mu}{T}\sum_{i=1}^N\left[(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})_{\mu\nu}k_i^\nu - i\left(1-i\mathcal{Z}\dot{\mathcal{G}}_{B0i}\right)_{\mu\nu}\varepsilon_i^\nu\right]} \\ \times\, \mathrm{e}^{\sum_{i,j=1}^N\left(k_i\underline{\Delta}_{ij}k_j - 2i\,\varepsilon_i{}^\bullet\underline{\Delta}_{ij}k_j - \varepsilon_i{}^\bullet\underline{\Delta}^\bullet_{ij}\varepsilon_j\right)}\Big|_{\text{lin }\varepsilon_1\cdots\varepsilon_N}\,. \end{aligned} \tag{4.17}$$

The kernel for spinor particles is [70]

$$\begin{aligned} K^{xx'}_{\text{sp},N}(A_{\text{ct}},T) = (-ie)^N(4\pi T)^{-d/2}\left[\det\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right)\right]^{1/2} \mathrm{e}^{-\frac{1}{4T}(x'-x)\mathcal{Z}\cot\mathcal{Z}(x'-x)+\, i\sum_{i=1}^N k_i x} \\ \times \prod_{i=1}^N \int_0^T d\tau_i\; \mathrm{e}^{\frac{i(x'-x)^\mu}{T}\sum_{i=1}^N\left[(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})_{\mu\nu}k_i^\nu - i\left(1-i\mathcal{Z}\dot{\mathcal{G}}_{B0i}\right)_{\mu\nu}\varepsilon_i^\nu\right]} \\ \times\, \mathrm{e}^{\sum_{i,j=1}^N\left(k_i\underline{\Delta}_{ij}k_j - 2i\,\varepsilon_i{}^\bullet\underline{\Delta}_{ij}k_j - \varepsilon_i{}^\bullet\underline{\Delta}^\bullet_{ij}\varepsilon_j\right)} S^\gamma_N(F_{\text{ct}},T)\Big|_{\text{lin }\varepsilon_1\cdots\varepsilon_N}\,. \end{aligned} \tag{4.18}$$

where the "spin-term" is[1]

$$S^\gamma_N(F_{\text{ct}},T) = 2^{-d/2}\text{symb}^{-1}\left[\int_C \mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)\left(eF_{\text{ct}} - \sum_{i=1}^N\delta(\tau-\tau_i)f_i\right)\left(\psi+\frac{1}{2}\eta\right)\right]}\right]. \tag{4.19}$$

[1] Here, 'γ' is not a Lorentz index. It indicates that the structure of the Dirac-gamma matrices originates from this spin-term.

For the moment, we do not perform the Grassman path integral, we do it later using Wick contractions for specific cases, see [124] for path integrals and Wick contractions.

Here, $\underline{\Delta}_{ij} = \underline{\Delta}(\tau_i, \tau_j)$ is the Green's function in a constant background with Dirichlet boundary conditions which can be expressed in terms of the calligraphic Green's functions as

$$
\begin{aligned}
\underline{\Delta}(\tau,\tau') &= \frac{1}{2}\left[\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\right] , \\
{}^\bullet\underline{\Delta}(\tau,\tau') &= \frac{1}{2}\Big(\dot{\mathcal{G}}_B(\tau,\tau') - \dot{\mathcal{G}}_B(\tau,0)\Big) , \\
{}^\bullet\underline{\Delta}^\bullet(\tau,\tau') &= -\frac{1}{2}\ddot{\mathcal{G}}_B(\tau,\tau') ,
\end{aligned}
\tag{4.20}
$$

with ${}^\bullet\underline{\Delta}_{ij} = \frac{\partial}{\partial\tau_i}\underline{\Delta}(\tau_i,\tau_j)$, $\underline{\Delta}^\bullet_{ij} = \frac{\partial}{\partial\tau_j}\underline{\Delta}(\tau_i,\tau_j)$ as the first derivative with respect to the first and second parameter, respectively. Recall that $\mathcal{Z} = eF_{\text{ct}}T$ and $\mathcal{G}_{0i} = \mathcal{G}(0,\tau_i)$, $\mathcal{G}_{ij} = \mathcal{G}_{Bij} - \mathcal{G}_{Bii}$, $\dot{\mathcal{G}}_{ij} = \frac{\partial}{\partial\tau_i}\mathcal{G}(\tau_i,\tau_j)$ are the calligraphic Green's functions that follow the conventions of Chapter 3 (see also Appendix A.3).

4.2.1 Off-shell amplitudes in momentum space

In this section, we present the propagators in momentum space for which the Fourier transform have been taken according to (4.10). In the following expressions the momenta of the external photons can be considered as off-shell or on-shell while momentum of the mass particle can only be off-shell. In order to take the on-shell limit, it is necessary to first remove all spurious poles at $p^2 = p'^2 = m^2$. Note that the on-shell calculation is not covered in this thesis, for such case, it is advise to follow the procedure presented in [128] and references therein.

For the momentum space, we use the short hand notation

$$
D_{\text{sc/sp}}(A_{\text{ct}}|p,p';k_1,\varepsilon_1;k_2,\varepsilon_2;...;k_N,\varepsilon_N) = (2\pi)^d\delta^d\left(p+p'+\sum_{i=1}^N k_i\right) D^{pp'}_{\text{sc/sp},N} ,
\tag{4.21}
$$

for the propagator. And similarly,

$$
K_{\text{sc/sp}}(A_{\text{ct}},T|p,p';k_1,\varepsilon_1;k_2,\varepsilon_2;...;k_N,\varepsilon_N) = (2\pi)^d\delta^d\left(p+p'+\sum_{i=1}^N k_i\right) K^{pp'}_{\text{sc/sp},N} ,
\tag{4.22}
$$

for the kernel.

The dressed scalar propagator, in momentum space, with N external photons in a constant background field is given by [67]

$$
D^{pp'}_{\text{sc},N} = \int_0^\infty dT \mathrm{e}^{-Tm^2} K^{pp'}_{\text{sc},N} ,
\tag{4.23}
$$

with the kernel

$$
\begin{aligned}
K^{pp'}_{\text{sc},N} = (-ie)^N \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \prod_{i=1}^N \int_0^T d\tau_i \; \mathrm{e}^{-T\, b^{\mathrm{T}}(\mathcal{Z}\cot\mathcal{Z})^{-1} b} \\
\times\, \mathrm{e}^{\sum_{i,j=1}^N\left(k_i\underline{\Delta}_{ij}k_j - 2i\,\varepsilon_i{}^\bullet\underline{\Delta}_{ij}k_j - \varepsilon_i{}^\bullet\underline{\Delta}^\bullet_{ij}\varepsilon_j\right)}\Big|_{\text{lin }\varepsilon_1\cdots\varepsilon_N} ,
\end{aligned}
\tag{4.24}
$$

where

$$
b^\mu = p'^\mu + \frac{1}{T}\sum_{i=1}^N\left[(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})^{\mu\nu}k_{i\nu} - i\left(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B0i}\right)^{\mu\nu}\varepsilon_{i\nu}\right] .
\tag{4.25}
$$

The dressed fermion propagator, in momentum space, with N external photons in a constant background field is given by [70]

$$
D^{pp'}_{\text{sp},N} = \int_0^\infty dT\mathrm{e}^{-Tm^2}\left[\left(m + \not{p}'\right)K^{pp'}_{\text{sp},N} + K^{pp'}_{\text{ct},N} - e\sum_{i=1}^N \not{\varepsilon}_i\, K^{p(p'+k_i)}_{\text{sp},N-1}(\text{without photon } <i>)\right] .
\tag{4.26}
$$

Here, we have used

$$(2\pi)^d \delta^d \left(p + p' + \sum_{i=1}^{N} k_i\right) K^{pp'}_{\mathrm{ct},N} = \frac{e}{2}\,\gamma_\mu F^{\mu\nu}_{\mathrm{ct}} \int d^d x \int d^d x' \, \mathrm{e}^{i(px+p'x')}\,(x'-x)_\nu K^{xx'}_{\mathrm{sp},N}(A_{\mathrm{ct}}, T)\,. \tag{4.27}$$

After performing x and x' integrals, $K^{pp'}_{\mathrm{ct},N}$ becomes

$$K^{pp'}_{\mathrm{ct},N} = i\gamma_\mu(\tan\mathcal{Z})^{\mu\nu}\,(-ie)^N \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \prod_{i=1}^{N}\int_0^T d\tau_i\, b_\nu\, \mathrm{e}^{\sum_{i,j=1}^{N}\left(k_i \Delta_{ij} k_j - 2i\,\varepsilon_i {}^{\bullet}\Delta_{ij} k_j - \varepsilon_i {}^{\bullet}\Delta^{\bullet}_{ij}\varepsilon_j\right)} \times \mathrm{e}^{-T\, b^{\mathrm{T}}(\mathcal{Z}\cot\mathcal{Z})^{-1} b}\, S^{\gamma}_N(F_{\mathrm{ct}}, T)\Big|_{\mathrm{lin}\ \varepsilon_1\cdots\varepsilon_N}\,. \tag{4.28}$$

Finally, the spinor kernel in momentum space is

$$K^{pp'}_{\mathrm{sp},N} = (-ie)^N \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \prod_{i=1}^{N}\int_0^T d\tau_i \;\; \mathrm{e}^{\sum_{i,j=1}^{N}\left(k_i \Delta_{ij} k_j - 2i\,\varepsilon_i {}^{\bullet}\Delta_{ij} k_j - \varepsilon_i {}^{\bullet}\Delta^{\bullet}_{ij}\varepsilon_j\right)} \times \mathrm{e}^{-T\, b^{\mathrm{T}}(\mathcal{Z}\cot\mathcal{Z})^{-1} b}\, S^{\gamma}_N(F_{\mathrm{ct}}, T)\Big|_{\mathrm{lin}\ \varepsilon_1\cdots\varepsilon_N}\,. \tag{4.29}$$

In the following sections, we specialize the propagator expressions (4.23) and (4.26) to the case of $N = 2$ i. e., the off-shell amplitude of Compton scattering in a constant background field. Subsequently, in order to simplify the obtained expression, we consider a pure magnetic background field of constant strength and special kinematics.

4.3 Off-shell Compton scattering in a constant field for scalar QED

In this section, we consider the scalar propagator (4.23) for $N = 2$ and expand it to linear order in the photon polarizations ε_1 and ε_2. The amplitude of the scalar Compton scattering in a constant background field (see Section 4.2.1 and Fig. 4.2) is

$$D_{\mathrm{sc}}(A_{\mathrm{ct}}|p, p'; k_1, \varepsilon_1; k_2, \varepsilon_2) = (2\pi)^d \delta^d\left(p + p' + k_1 + k_2\right) \int_0^\infty dT\, \mathrm{e}^{-Tm^2}\, K^{pp'}_{\mathrm{sc},2}\,. \tag{4.30}$$

For this case, the kernel of an off-shell scalar particle in a constant background interacting with two photons is

$$K^{pp'}_{\mathrm{sc},2} = (-ie)^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 \int_0^T d\tau_2\; \mathrm{e}^{-Tb^{\mathrm{T}}(\mathcal{Z}\cot\mathcal{Z})^{-1} b} \times \mathrm{e}^{\sum_{i,j=1}^{2}\left(k_i\, \Delta_{ij} k_j - 2i\varepsilon_i\, {}^{\bullet}\Delta_{ij} k_j - \varepsilon_i\, {}^{\bullet}\Delta^{\bullet}_{ij}\varepsilon_j\right)}\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2}\,, \tag{4.31}$$

where

$$b^\mu = p'^\mu + \frac{1}{T}\sum_{i=1}^{2}\left[\left(\tau_i + i\mathcal{Z}\mathcal{G}_{0i}\right)^{\mu\nu} k_{i\nu} - i\left(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B0i}\right)^{\mu\nu}\varepsilon_{i\nu}\right]\,. \tag{4.32}$$

For the explicit expressions of the Green's functions, see the previous chapter or Appendix A.3.

Expanding b^μ in (4.31) such that the dependence on the polarizations is explicit, we can rewrite the kernel as

$$\begin{aligned} K^{pp'}_{\mathrm{sc},2} = (-ie)^2 &\left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 d\tau_2\; \mathrm{e}^{-Tb_0^{\mathrm{T}}(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0}\, \mathrm{e}^{2k_1\Delta_{12}k_2 + k_1\Delta_{11}k_1 + k_2\Delta_{22}k_2} \\ \times\ &\mathrm{e}^{-2i\varepsilon_1\left[{}^{\bullet}\Delta_{11}k_1 + {}^{\bullet}\Delta_{12}k_2 - \left(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\right)(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\right]}\, \mathrm{e}^{-2i\varepsilon_2\left[{}^{\bullet}\Delta_{22}k_2 + {}^{\bullet}\Delta_{21}k_1 - \left(1 - i\dot{\mathcal{G}}_{B20}\mathcal{Z}\right)(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\right]} \\ \times\ &\mathrm{e}^{-2\varepsilon_1\left[{}^{\bullet}\Delta^{\bullet}_{12} - \frac{1}{T}\left(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\right)(\mathcal{Z}\cot\mathcal{Z})^{-1}\left(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B02}\right)\right]\varepsilon_2}\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2}\,, \end{aligned} \tag{4.33}$$

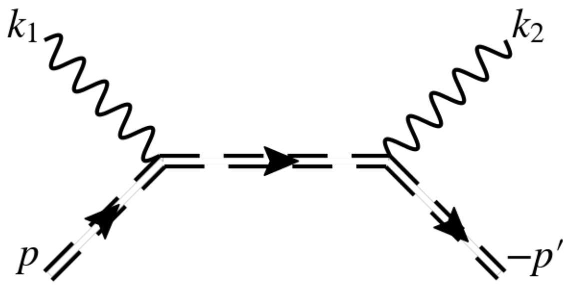

Figure 4.2: *Compton scattering diagram for a scalar particle. The double dashed line indicates that a scalar particle propagates in a constant field.*

where b_0^μ is the polarization independent part of b^μ. Notice that this kernel can be written as

$$K^{pp'}_{\mathrm{sc},2} = (-ie)^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 d\tau_2\, \mathrm{e}^{-T\,h_{12}}\, \mathrm{e}^{-2i\varepsilon_1 b_1}\, \mathrm{e}^{-2i\varepsilon_2 b_2}\, \mathrm{e}^{-\frac{2}{T}\varepsilon_1 b_{12}\varepsilon_2}\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2}\,, \tag{4.34}$$

by defining the vectors b_i^μ, the matrix $b_{12}^{\mu\nu}$ and the function h_{12} as

$$\begin{aligned} b_0 &= p' + \frac{1}{T}\Big[(\tau_1 + i\mathcal{Z}\mathcal{G}_{01})\,k_1 + (\tau_2 + i\mathcal{Z}\mathcal{G}_{B02})\,k_2\Big]\,,\\ b_1 &= {}^\bullet\underline{\Delta}_{11}k_1 + {}^\bullet\underline{\Delta}_{12}k_2 - \big(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,,\\ b_2 &= {}^\bullet\underline{\Delta}_{22}k_2 + {}^\bullet\underline{\Delta}_{21}k_1 - \big(1 - i\dot{\mathcal{G}}_{B20}\mathcal{Z}\big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,,\\ \frac{1}{T}b_{12} &= {}^\bullet\underline{\Delta}^\bullet_{12} - \frac{1}{T}\big(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}\big(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B02}\big)\,,\\ -Th_{12} &= 2k_1\underline{\Delta}_{12}k_2 + k_1\underline{\Delta}_{11}k_1 + k_2\underline{\Delta}_{22}k_2 - Tb_0^T(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,. \end{aligned} \tag{4.35}$$

Expanding the exponential in (4.34) at linear order in each polarization ε_1 and ε_2, we get

$$K^{pp'}_{\mathrm{sc},2} = -e^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 d\tau_2\, \mathrm{e}^{-T\,h_{12}} \left[-\frac{2}{T}\varepsilon_1 b_{12}\varepsilon_2 + (-2i)^2\varepsilon_1 b_1\,\varepsilon_2 b_2\right]. \tag{4.36}$$

Finally we re-scale to the unit circle $\tau_i = Tu_i$

$$K^{pp'}_{\mathrm{sc},2} = 2e^2T^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^1 du_1 du_2\, \mathrm{e}^{-T\,h_{12}} \left(\frac{1}{T}\varepsilon_1 b_{12}\varepsilon_2 + 2\,\varepsilon_1 b_1\,\varepsilon_2 b_2\right). \tag{4.37}$$

The Green's function $\underline{\Delta}_{ij}$ and its derivatives can be written in terms of $\mathcal{G}_{Bij}$, $\dot{\mathcal{G}}_{Bij}$ and $\ddot{\mathcal{G}}_{Bij}$

$$\begin{aligned} 2k_1\underline{\Delta}_{12}k_2 &= k_1\left[\mathcal{G}_{B12} - \mathcal{G}_{B10} - \mathcal{G}_{B02} + \mathcal{G}_{B00}\right]k_2\,,\\ k_1\underline{\Delta}_{11}k_1 &= k_1\left[\mathcal{G}_{B00} - \mathcal{G}_{B10}\right]k_1\,,\\ k_2\underline{\Delta}_{22}k_2 &= k_2\left[\mathcal{G}_{B00} - \mathcal{G}_{B20}\right]k_2\,,\\ {}^\bullet\underline{\Delta}_{12} &= \frac{1}{2}\Big(\dot{\mathcal{G}}_{B12} - \dot{\mathcal{G}}_{B10}\Big)\,,\\ {}^\bullet\underline{\Delta}_{21} &= \frac{1}{2}\Big(\dot{\mathcal{G}}_{B21} - \dot{\mathcal{G}}_{B20}\Big)\,,\\ {}^\bullet\underline{\Delta}_{11} &= \frac{1}{2}\Big(\dot{\mathcal{G}}_{B00} - \dot{\mathcal{G}}_{B10}\Big)\,,\\ {}^\bullet\underline{\Delta}_{22} &= \frac{1}{2}\Big(\dot{\mathcal{G}}_{B00} - \dot{\mathcal{G}}_{B20}\Big)\,,\\ {}^\bullet\underline{\Delta}^\bullet_{12} &= -\frac{1}{2}\ddot{\mathcal{G}}_{B12}\,. \end{aligned} \tag{4.38}$$

Here, the calligraphic Green's functions are thought to be written in terms of u_i variables, see Appendix A.3.

4.4 Off-shell Compton scattering in a constant field for spinor QED

In this section, we consider the fermion propagator (4.26) for $N = 2$ and expand it to linear order in the photon polarizations ε_1 and ε_2. The amplitude of Compton scattering in a constant background field (see Section 4.2.1 and Fig. 4.3) is

$$D^{pp'}_{\mathrm{sp},2} = \int_0^\infty dT\, \mathrm{e}^{-Tm^2} \Big[\big(m + \not{p}'\big)\, K^{pp'}_{\mathrm{sp},2} + K^{pp'}_{\mathrm{ct},2} - e\not{\varepsilon}_1\, K^{p(p'+k_1)}_{\mathrm{sp},1}(k_2, \varepsilon_2) - e\not{\varepsilon}_2\, K^{p(p'+k_2)}_{\mathrm{sp},1}(k_1, \varepsilon_1) \Big] \,. \tag{4.39}$$

Following the scalar calculation, the different kernels of an off-shell spinor particle in constant background interacting with two photons are

$$\begin{aligned}
K^{pp'}_{\mathrm{sp},2} &= (-ie)^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 d\tau_2\ \mathrm{e}^{-Th_{12}}\, \mathrm{e}^{-2i\varepsilon_1 b_1} \mathrm{e}^{-2i\varepsilon_2 b_2}\ \mathrm{e}^{-\frac{2}{T}\varepsilon_1 b_{12}\varepsilon_2}\, S^{\gamma}_2(T, F_{\mathrm{ct}})\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2} \,,\\
K^{pp'}_{\mathrm{ct},2} &= i\gamma_\mu (\tan\mathcal{Z})^{\mu\nu} (-ie)^2 \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_1 d\tau_2\ b_\nu\, \mathrm{e}^{-Th_{12}}\, \mathrm{e}^{-2i\varepsilon_1 b_1} \mathrm{e}^{-2i\varepsilon_2 b_2}\ \mathrm{e}^{-\frac{2}{T}\varepsilon_1 b_{12}\varepsilon_2}\, S^{\gamma}_2(T, F_{\mathrm{ct}})\Big|_{\mathrm{lin}\ \varepsilon_1\varepsilon_2} \,,\\
S^{\gamma}_2(T, F_{\mathrm{ct}}) &= 2^{-2}\, \mathrm{symb}^{-1} \left[\int_C \mathcal{D}\psi(\tau) \mathrm{e}^{-\int_0^T d\tau \left\{ \frac{1}{2}\psi\dot{\psi} - i\left(\psi + \frac{1}{2}\eta\right)[eF - \delta(\tau-\tau_1) f_1 - \delta(\tau-\tau_2) f_2]\left(\psi + \frac{1}{2}\eta\right) \right\}} \right] \,,
\end{aligned} \tag{4.40}$$

where the vectors b^μ_i, $b^{\mu\nu}_{12}$ and the function h_{12} were computed in the scalar case, Eq. (4.35).

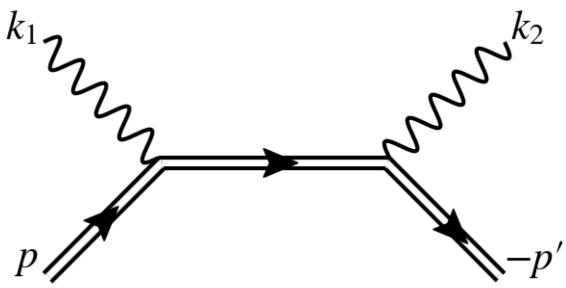

Figure 4.3: *Compton scattering diagram for a spinor particle. The double solid line indicates that a spinor particle propagates in a constant field.*

For the one-photon kernel,

$$K_{\mathrm{sp}}(A_{\mathrm{ct}}|p, p'; k_i, \varepsilon_i) = (2\pi)^4 \delta^4(p + p' + k_i)\, K^{pp'}_{\mathrm{sp},1}(k_i, \varepsilon_i) \,, \tag{4.41}$$

we have

$$\begin{aligned}
K^{pp'}_{\mathrm{sp},1}(k_i, \varepsilon_i) &= (-ie) \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} \int_0^T d\tau_i\ \mathrm{e}^{-Th_{ii}(p')}\, \mathrm{e}^{-2i\varepsilon_i h_i(p')}\, S^{\gamma}_1(T, F_{\mathrm{ct}}, f_i)\Big|_{\mathrm{lin}\ \varepsilon_i} \,,\\
S^{\gamma}_1(T, F_{\mathrm{ct}}, f_i) &= 2^{-2}\, \mathrm{symb}^{-1} \left[\int_C \mathcal{D}\psi(\tau)\, \mathrm{e}^{-\int_0^T d\tau \left\{ \frac{1}{2}\psi\dot{\psi} - i\left(\psi + \frac{1}{2}\eta\right)[eF - \delta(\tau-\tau_i) f_i]\left(\psi + \frac{1}{2}\eta\right) \right\}} \right] \,,
\end{aligned} \tag{4.42}$$

where

$$\begin{aligned}
h_i(p') &= {}^\bullet\underline{\Delta}_{ii} k_i - \big(1 - i\dot{\mathcal{G}}_{Bi0}\mathcal{Z}\big)\, (\mathcal{Z}\cot\mathcal{Z})^{-1} \left[p' + \frac{1}{T}(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})k_i\right] \,,\\
-Th_{ii}(p') &= k_i \underline{\Delta}_{ii} k_i - T\left[p' + \frac{1}{T}k_i(\tau_i - i\mathcal{G}_{i0}\mathcal{Z})\right] (\mathcal{Z}\cot\mathcal{Z})^{-1} \left[p' + \frac{1}{T}(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})k_i\right] \,.
\end{aligned} \tag{4.43}$$

At this point, it is necessary to perform the fermionic path integral.

4.4.1 Fermionic path integral and Wick contractions

For the path integral over Grassman variables in equation (4.42), we define

$$I_{\psi,1} = \int_C \mathcal{D}\psi(\tau) \mathrm{e}^{-\int_0^T d\tau \left\{\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)\left[eF-\delta(\tau-\tau_1)f_1\right]\left(\psi+\frac{1}{2}\eta\right)\right\}}\Big|_{\mathcal{O}(\varepsilon_1)} \tag{4.44}$$

and expand up to linear order with respect to ε_1

$$I_{\psi,1} = \int_C \mathcal{D}\psi(\tau) \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]} \left[1 - i\left(\psi(\tau_1)+\frac{1}{2}\eta\right) f_1 \left(\psi(\tau_1)+\frac{1}{2}\eta\right)\right]. \tag{4.45}$$

The previous path integral will be non-zero only for an even number of $\psi(\tau_i)$ in the integrand. Here the Gaussian path integral is well known [124] to be

$$I_{\psi,0} = \int_C \mathcal{D}\psi(\tau)\, \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]} = 2^{\frac{d}{2}} \det{}^{\frac{1}{2}}(\cos\mathcal{Z})\, \mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}. \tag{4.46}$$

In this case, we have only the following Wick contraction

$$\frac{\int_C \mathcal{D}\psi(\tau)\, \psi(\tau_i)\psi(\tau_j)\, \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]}}{\int_C \mathcal{D}\psi(\tau)\, \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]}} = \langle\psi(\tau_i)\psi(\tau_j)\rangle. \tag{4.47}$$

This Wick contraction is proportional to the fermion Green's function in a constant field [123]

$$\langle\psi(\tau_i)\psi(\tau_j)\rangle = \langle\tau_i|\left(\frac{d}{d\tau} - 2ieF\right)^{-1}|\tau_j\rangle = \frac{1}{2}\mathcal{G}^{\mu\nu}_{Fij}. \tag{4.48}$$

Writing $I_{\psi,1}$ in terms of Wick contractions, we get

$$I_{\psi,1} = I_{\psi_0}\left[1 - i\left(\langle\psi_1^\mu\psi_1^\nu\rangle + \frac{1}{4}\eta^\mu\eta^\nu\right) f_1^{\mu\nu}\right]. \tag{4.49}$$

Then, in terms of calligraphic Green's functions

$$S_1^\gamma(T, F_{\rm ct}) = \det{}^{\frac{1}{2}}(\cos\mathcal{Z})\,\mathrm{symb}^{-1}\left\{\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\left[1 - \frac{i}{2}\left(\mathcal{G}^{\mu\nu}_{F11} + \frac{1}{2}\eta^\mu\eta^\nu\right) f_1^{\mu\nu}\right]\right\}, \tag{4.50}$$

valid for arbitrary even dimension.

Notice that, in $d = 4$, the exponential in (4.50) has the following Taylor expansion

$$\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta} = \mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta + \frac{1}{2}\left(\frac{i}{4}\right)^2 \eta(\tan\mathcal{Z})\eta\eta(\tan\mathcal{Z})\eta, \tag{4.51}$$

due to the fact that η^i is a Grassman variable and the square of it is equal to zero. Let's define

$$\begin{aligned} S_0(\mathcal{Z}) &= \mathrm{symb}^{-1}\left[\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\right], \\ S_1(\mathcal{Z}, f_1) &= \frac{1}{2}\mathrm{symb}^{-1}\left[\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\mathcal{G}^{\mu\nu}_{F11} + \frac{1}{2}\left(\mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta\right)\eta^\mu\eta^\nu\right] f_1^{\mu\nu}. \end{aligned} \tag{4.52}$$

For the present case, we require the following symbol functions, from (4.7),

$$\begin{aligned} \mathrm{symb}^{-1}(\eta^\mu\eta^\nu) &= -[\gamma^\mu, \gamma^\nu] = 2\sigma^{\mu\nu}, \qquad \sigma^{\mu\nu} = -\frac{1}{2}[\gamma^\mu, \gamma^\nu], \\ \mathrm{symb}^{-1}(\eta^\mu\eta^\nu\eta^\sigma\eta^\rho) &= \varepsilon^{\mu\nu\sigma\rho}\ \mathrm{symb}^{-1}(\eta^1\eta^2\eta^3\eta^4) = \varepsilon^{\mu\nu\sigma\rho}(i\sqrt{2})^4\,\gamma^1\gamma^2\gamma^3\gamma^4 = 4\,\varepsilon^{\mu\nu\sigma\rho}\,\gamma^5, \end{aligned} \tag{4.53}$$

where $\varepsilon^{\mu\nu\sigma\rho}$ is the fully antisymmetric Levi-Civita tensor with $\varepsilon^{1234} = 1$. This allows us to express S_0 and S_1 in terms of gamma matrices, we have

$$\begin{aligned} S_0(\mathcal{Z}) &= \mathbb{1} + \frac{i}{2}\sigma^{\mu\nu}(\tan\mathcal{Z})^{\mu\nu} + 2\left(\frac{i}{4}\right)^2 \varepsilon^{\mu\nu\sigma\rho}\,\gamma^5(\tan\mathcal{Z})^{\mu\nu}(\tan\mathcal{Z})^{\sigma\rho}, \\ S_1(\mathcal{Z}, f_1) &= \frac{1}{2}\left[S_0(\mathcal{Z})\,\mathcal{G}^{\mu\nu}_{F11} + \sigma^{\mu\nu} + \frac{i}{2}\varepsilon^{\sigma\rho\mu\nu}\,\gamma^5(\tan\mathcal{Z})^{\sigma\rho}\right] f_1^{\mu\nu}. \end{aligned} \tag{4.54}$$

Therefore, the spin contribution in equation (4.42) is

$$S_1^{\gamma}(T,F_{\rm ct},f_i) = \det{}^{\frac{1}{2}}(\cos\mathcal{Z})\Big[S_0(\mathcal{Z}) - iS_1(\mathcal{Z},f_i)\Big]\,. \tag{4.55}$$

Similarly, the spin contribution in equation (4.40) is

$$S_2^{\gamma}(T,F_{\rm ct}) = \det{}^{\frac{1}{2}}(\cos\mathcal{Z})\Big[S_0(\mathcal{Z}) - iS_1(\mathcal{Z},f_1) - iS_1(\mathcal{Z},f_2) - S_2(\mathcal{Z},f_1f_2)\Big]\,, \tag{4.56}$$

for the step-by-step calculation, see Appendix B.2.1.

4.4.2 One- and two-photon kernels

In this section, we summarize the kernel results at linear order in each polarization ε_1 and ε_2 (here we have made the re-scaling $\tau_i = Tu_i$). For the two-photon kernel, we have

$$K^{pp'}_{\rm sp,2} = 2e^2T^2\int_0^1 du_1du_2\;\;{\rm e}^{-Th_{12}}\;\left[\left(\frac{1}{T}\varepsilon_1 b_{12}\varepsilon_2 + 2\,\varepsilon_1 b_1\,\varepsilon_2 b_2\right)S_0 + \varepsilon_2 b_2\,S_1(f_1) + \varepsilon_1 b_1\,S_1(f_2) + \frac{1}{2}S_2(f_1f_2)\right]\,. \tag{4.57}$$

The two-photon kernel mixed with the external field is

$$\begin{aligned}K^{pp'}_{\rm ct,2} = 2ie^2T^2\,\gamma^\mu(\tan\mathcal{Z})_{\mu\nu}\int_0^1 du_1du_2\;\;{\rm e}^{-Th_{12}}\Bigg\{\Bigg[&(a_1\varepsilon_1)^\nu\varepsilon_2 b_2 + (a_2\varepsilon_2)^\nu\varepsilon_1 b_1 + b_0^\nu\left(\frac{1}{T}\varepsilon_1 b_{12}\varepsilon_2 + 2\,\varepsilon_1 b_1\,\varepsilon_2 b_2\right)\Bigg]S_0\\ + b_0^\nu\left[\varepsilon_2 b_2\,S_1(f_1) + \varepsilon_1 b_1\,S_1(f_2) + \frac{1}{2}S_2(f_1f_2)\right] + \frac{1}{2}&\left[(a_1\varepsilon_1)^\nu S_1(f_2) + (a_2\varepsilon_2)^\nu S_1(f_1)\right]\Bigg\}\,,\end{aligned} \tag{4.58}$$

with $a_i = \frac{1}{T}(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B0i})$. And finally, the one-photon kernels are

$$\begin{aligned}K^{p,p'+k_1}_{\rm sp,1}(k_2,\varepsilon_2) &= -eT\int_0^1 du_2\;\;{\rm e}^{-T\,h_{22}(p'+k_1)}\Big[2\,\varepsilon_2\,h_2(p'+k_1)\,S_0 + S_1(f_2)\Big]\,,\\ K^{p,p'+k_2}_{\rm sp,1}(k_1,\varepsilon_1) &= -eT\int_0^1 du_1\;\;{\rm e}^{-T\,h_{11}(p'+k_2)}\Big[2\,\varepsilon_1\,h_1(p'+k_2)\,S_0 + S_1(f_1)\Big]\,.\end{aligned} \tag{4.59}$$

Here, we collect the expression of every term in the one- and two-photon kernels. First, the spin terms are

$$\begin{aligned}S_0 &= \mathbb{1} + \frac{i}{2}\sigma^{\mu\nu}(\tan\mathcal{Z})_{\mu\nu} + 2\left(\frac{i}{4}\right)^2\varepsilon^{\mu\nu\sigma\rho}\,\gamma^5(\tan\mathcal{Z})_{\mu\nu}(\tan\mathcal{Z})_{\sigma\rho}\,,\\ S_1(f_i) &= \frac{1}{2}\left[S_0\,\mathcal{G}^{\mu\nu}_{Fii} + \sigma^{\mu\nu} + \frac{i}{2}\varepsilon^{\sigma\rho\mu\nu}\,\gamma^5(\tan\mathcal{Z})_{\sigma\rho}\right]f_{i,\mu\nu}\,,\\ S_2(f_1f_2) &= \frac{1}{4}\Bigg[S_0\Big(\mathcal{G}^{\mu\nu}_{F11}\mathcal{G}^{\sigma\rho}_{F22} + 2\,\mathcal{G}^{\mu\rho}_{F12}\mathcal{G}^{\nu\sigma}_{F12}\Big) + \Big(\mathcal{G}^{\mu\nu}_{F11}\sigma^{\sigma\rho} + \sigma^{\mu\nu}\mathcal{G}^{\sigma\rho}_{F22} + 4\,\mathcal{G}^{\mu\rho}_{F12}\sigma^{\nu\sigma}\Big)\\ &\quad + \varepsilon^{\mu\nu\sigma\rho}\,\gamma^5 + \frac{i}{2}(\tan\mathcal{Z})_{\alpha\beta}\Big(\mathcal{G}^{\mu\nu}_{F11}\varepsilon^{\alpha\beta\sigma\rho} + \varepsilon^{\alpha\beta\mu\nu}\mathcal{G}^{\sigma\rho}_{F22} + 4\,\mathcal{G}^{\mu\rho}_{F12}\varepsilon^{\alpha\beta\nu\sigma}\Big)\gamma^5\Bigg]f_{1,\mu\nu}f_{2,\sigma\rho}\,.\end{aligned} \tag{4.60}$$

Second, the bosonic terms (which do not contain gamma matrices)

$$\begin{aligned}b_0 &= p' + \frac{1}{T}\Big[(\tau_1 + i\mathcal{Z}\mathcal{G}_{01})\,k_1 + (\tau_2 + i\mathcal{Z}\mathcal{G}_{B02})\,k_2\Big]\,,\\ b_1 &= {}^\bullet\!\underline{\Delta}_{11}k_1 + {}^\bullet\!\underline{\Delta}_{12}k_2 - \Big(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\Big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,,\\ b_2 &= {}^\bullet\!\underline{\Delta}_{22}k_2 + {}^\bullet\!\underline{\Delta}_{21}k_1 - \Big(1 - i\dot{\mathcal{G}}_{B20}\mathcal{Z}\Big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,,\\ \frac{1}{T}b_{12} &= {}^\bullet\!\underline{\Delta}^\bullet_{12} - \frac{1}{T}\,\Big(1 - i\dot{\mathcal{G}}_{B10}\mathcal{Z}\Big)\,(\mathcal{Z}\cot\mathcal{Z})^{-1}\,\Big(1 - i\mathcal{Z}\dot{\mathcal{G}}_{B02}\Big)\,,\\ -Th_{12} &= 2k_1\underline{\Delta}_{12}k_2 + k_1\underline{\Delta}_{11}k_1 + k_2\underline{\Delta}_{22}k_2 - Tb_0^T(\mathcal{Z}\cot\mathcal{Z})^{-1}b_0\,.\end{aligned} \tag{4.61}$$

And third, the bosonic terms for the one-photon kernel

$$\begin{aligned} h_i(p') &= {}^{\bullet}\underline{\Delta}_{ii} k_i - \left(1 - i\dot{\mathcal{G}}_{Bi0}\mathcal{Z}\right)(\mathcal{Z}\cot\mathcal{Z})^{-1}\left[p' + \frac{1}{T}(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})k_i\right], \\ -Th_{ii}(p') &= k_i\underline{\Delta}_{ii}k_i - T\left[p' + \frac{1}{T}k_i(\tau_i - i\mathcal{G}_{i0}\mathcal{Z})\right](\mathcal{Z}\cot\mathcal{Z})^{-1}\left[p' + \frac{1}{T}(\tau_i + i\mathcal{Z}\mathcal{G}_{0i})k_i\right]. \end{aligned} \tag{4.62}$$

In the next sections, the above expressions are simplified by specializing to a pure magnetic field and forward scattering.

4.5 Pure magnetic field

In this section, we present the calligraphic Green's functions in the case of a pure magnetic background field of constant strength. Following [123], we choose the magnetic field pointing along the x axis, in Euclidean space,

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_x & 0 \\ 0 & -B_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.63}$$

For convenience, we define the following projectors

$$\hat{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_\perp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_{||} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.64}$$

such that for $z = eTB_x$, we have that

$$\mathcal{Z}^{2n} = (-1)^n g_\perp z^{2n}, \qquad \mathcal{Z}^{2n+1} = (-1)^n \hat{F} z^{2n+1} \qquad \text{and} \qquad g_{||}^{\mu\nu} + g_\perp^{\mu\nu} = \mathbb{1}. \tag{4.65}$$

Then the calligraphic Green's function $\mathcal{G}_{ij} = \mathcal{G}(u_i, u_j)$ can be expressed as

$$\begin{aligned} \mathcal{G}_{Bij} &= T\left[\left(G_{ij} - \frac{1}{6}\right) g_{||} - \frac{1}{2z}\left(C_{ij} - \frac{1}{z}\right) g_\perp + \frac{1}{2z}\left[S_{ij} - \dot{G}_{ij}\right] i\hat{F}\right], \\ \dot{\mathcal{G}}_{Bij} &= \dot{G}_{ij}\, g_{||} + S_{ij}\, g_\perp - \left(C_{ij} - \frac{1}{z}\right) i\hat{F}, \\ \ddot{\mathcal{G}}_{Bij} &= \frac{2}{T}\left[\delta(u_i - u_j)\,\mathbb{1} - g_{||} - zC_{ij}\, g_\perp + zS_{ij}\, i\hat{F}\right], \\ \mathcal{G}_{Fij} &= G_{Fij}\left(g_{||} + C_{Fij}\, g_\perp - S_{Fij}\, i\hat{F}\right), \end{aligned} \tag{4.66}$$

where, we have defined[2]

$$\begin{aligned} S_{ij}(z) &= \frac{\sinh(z\,\dot{G}_{ij})}{\sinh(z)}, \qquad & S_{Fij}(z) &= \frac{\sinh(z\,\dot{G}_{ij})}{\cosh(z)}, \\ C_{ij}(z) &= \frac{\cosh(z\,\dot{G}_{ij})}{\sinh(z)}, \qquad & C_{Fij}(z) &= \frac{\cosh(z\,\dot{G}_{ij})}{\cosh(z)}. \end{aligned} \tag{4.67}$$

The coincidence limits of the calligraphic functions for this case are

$$\begin{aligned} \mathcal{G}_{Bii} &= -T\left[\frac{1}{6} g_{||} + \frac{1}{2z}\left(\coth z - \frac{1}{z}\right) g_\perp\right], \\ \dot{\mathcal{G}}_{Bii} &= -\left(\coth z - \frac{1}{z}\right) i\hat{F}, \\ \mathcal{G}_{Fii} &= -i\hat{F}\tanh z. \end{aligned} \tag{4.68}$$

[2]These definitions are independent from those used in Chapter 3.

The calligraphic Green's function with no coincidence limit ($\mathcal{G}_{ij} = \mathcal{G}_{Bij} - \mathcal{G}_{Bii}$) can be written as

$$\begin{aligned} \mathcal{G}_{ij} &= T\left[G_{ij}\, g_{||} - \frac{1}{2z}\left(C_{ij} - \coth z\right) g_{\perp} + \frac{1}{2z}\left(S_{ij} - \dot{G}_{ij}\right) i\hat{F}\right], \\ \dot{\mathcal{G}}_{ij} &= \dot{G}_{ij}\, g_{||} + S_{ij}\, g_{\perp} - \left(C_{ij} - \coth z\right) i\hat{F}\,. \end{aligned} \tag{4.69}$$

In a pure magnetic field, the determinant appearing in the propagators (4.23) and (4.26) becomes

$$\det{}^{-1/2}\left(\cos \mathcal{Z}\right) = \frac{1}{\cosh z}\,. \tag{4.70}$$

The pure electric case is obtained after replacing

$$\hat{F} \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad g_{\perp} \leftrightarrow g_{||}\,, \qquad z \to ieTE_z\,, \tag{4.71}$$

in (4.66).

In the following sections, we apply the field configuration described earlier to compute the scalar and spinor QED amplitudes for Compton scattering. Our focus is on the forward direction, aligned with the magnetic background field. In this calculation, we consider the external photons as on-shell, with linear polarizations assumed to be orthogonal to each other. Our objective is to explore the polarization effects arising from Compton scattering [66], which could have implications for the observation of Coulomb-assisted birefringence [65].

4.6 Compton scattering in a magnetic field for scalar QED

In this section, we present the off-shell amplitude of Compton scattering in a pure magnetic background field of constant strength for scalar QED and specialize it to the forward scattering, aligned with the direction of the magnetic field. The Green's function $\underline{\Delta}_{ij}$ and its derivatives appearing in (4.30) for a magnetic field in $d = 4$, after replacing $\mathcal{G}_{Bij}$, $\dot{\mathcal{G}}_{Bij}$ and $\ddot{\mathcal{G}}_{Bij}$ by the expressions in Section 4.5, become

$$\begin{aligned} 2k_1\underline{\Delta}_{12}k_2 &= T\, k_1 \cdot \Big[\left(G_{12} - G_{10} - G_{02}\right) g_{||} - \frac{1}{2z}\left(C_{12} - C_{10} - C_{02} - \coth z\right) g_{\perp} \\ &\qquad + \frac{1}{2z}\left(S_{12} - S_{10} - S_{02} - \dot{G}_{12} + \dot{G}_{10} + \dot{G}_{02}\right) i\hat{F}\Big] \cdot k_2\,, \\ k_1\underline{\Delta}_{11}k_1 &= -T\, k_1 \cdot \left[G_{10} g_{||} + \frac{1}{2z}\left(\coth z - C_{10}\right) g_{\perp}\right] \cdot k_1\,, \\ k_2\underline{\Delta}_{22}k_2 &= -T\, k_2 \cdot \left[G_{20} g_{||} + \frac{1}{2z}\left(\coth z - C_{20}\right) g_{\perp}\right] \cdot k_2\,, \\ {}^{\bullet}\underline{\Delta}_{12} &= \frac{1}{2}\left[\left(\dot{G}_{12} - \dot{G}_{10}\right) g_{||} + \left(S_{12} - S_{10}\right) g_{\perp} - \left(C_{12} - C_{10}\right) i\hat{F}\right], \\ {}^{\bullet}\underline{\Delta}_{21} &= \frac{1}{2}\left[\left(\dot{G}_{21} - \dot{G}_{20}\right) g_{||} + \left(S_{21} - S_{20}\right) g_{\perp} - \left(C_{21} - C_{20}\right) i\hat{F}\right], \\ {}^{\bullet}\underline{\Delta}_{11} &= -\frac{1}{2}\left[G_{10} g_{||} + S_{10} g_{\perp} + \left(\coth z - C_{10}\right) i\hat{F}\right], \\ {}^{\bullet}\underline{\Delta}_{22} &= -\frac{1}{2}\left[G_{20} g_{||} + S_{20} g_{\perp} + \left(\coth z - C_{20}\right) i\hat{F}\right], \\ {}^{\bullet}\underline{\Delta}^{\bullet}_{12} &= -\frac{1}{T}\left[\delta(u_i - u_j)\,\mathbb{1} - g_{||} - zC_{ij}\, g_{\perp} + zS_{ij}\, i\hat{F}\right]. \end{aligned} \tag{4.72}$$

4.6.1 Forward scattering aligned with the magnetic field

In order to simplify the exponent h_{12} and therefore be able to perform the integral analytically, we make the following assumptions:

1. We consider the forward direction such that all momenta are parallel to the B-field, implying that

$$k_i \cdot g_\perp = 0, \qquad k_i \cdot \hat{F} = 0, \qquad p' \cdot g_\perp = 0, \qquad p' \cdot \hat{F} = 0\,. \tag{4.73}$$

2. Both photons are on-shell and their polarizations are perpendicular to each other. Which implies

$$k_i \cdot \varepsilon_j = 0, \qquad \varepsilon_i \cdot g_{||} = 0, \qquad \varepsilon_1 \cdot \varepsilon_2 = 0\,. \tag{4.74}$$

These assumptions allow the exponent in (4.37) to become

$$h_{12} = (p' + u_1 k_1 + u_2 k_2)^2 = p'^2 + 2u_1\, p' \cdot k_1 + 2u_2\, p' \cdot k_2 \tag{4.75}$$

and the terms linear in the polarizations simplify to (remember $z = eB_x T$)

$$\begin{aligned} &\varepsilon_1 \cdot b_1 = 0, \qquad \varepsilon_2 \cdot b_2 = 0\,, \\ &\varepsilon_1 \cdot b_{12} \cdot \varepsilon_2 = z\,[-S_{12} + \tanh z\,(S_{10}C_{02} + S_{02}C_{10})]\, i\,\varepsilon_1 \cdot \hat{F} \cdot \varepsilon_2\,. \end{aligned} \tag{4.76}$$

Then, the kernel in (4.37) becomes

$$K^{pp'}_{\mathrm{sc},2} = 2ie^2 T \left(\frac{z}{\cosh z}\right) \int_0^1 du_1 du_2\, \mathrm{e}^{-T\,h_{12}}\, [-S_{12} + \tanh z\,(S_{10}C_{02} + S_{02}C_{10})]\; \varepsilon_1 \cdot \hat{F} \cdot \varepsilon_2\,. \tag{4.77}$$

This means that for the propagator we must perform the following integral

$$I_{\mathrm{sc}} = \int_0^\infty \mathrm{e}^{-Tm^2}\, T \left(\frac{z}{\cosh z}\right) \int_0^1 du_1 du_2\;\; \mathrm{e}^{-Th_{12}}\, [-S_{12} + \tanh z\,(S_{10}C_{20} - S_{20}C_{10})]\,. \tag{4.78}$$

All integrals are performed with Mathematica [152], such that the propagator becomes

$$D^{pp'}_{\mathrm{sc},2} = 2ie^2\, \varepsilon_1 \cdot \hat{F} \cdot \varepsilon_2\, \frac{(p' \cdot k_2 - p' \cdot k_1)eB_x}{4[(eB_x)^2 - (p' \cdot k_1)^2][(eB_x)^2 - (p' \cdot k_2)^2]}\, (I_{\mathrm{sc},1} + I_{\mathrm{sc},2} + I_{\mathrm{sc},3} + I_{\mathrm{sc},4})\,. \tag{4.79}$$

Here, we have set

$$I_{\mathrm{sc}} = \frac{(p' \cdot k_2 - p' \cdot k_1)eB_x}{4[(eB_x)^2 - (p' \cdot k_1)^2][(eB_x)^2 - (p' \cdot k_2)^2]}\, (I_{\mathrm{sc},1} + I_{\mathrm{sc},2} + I_{\mathrm{sc},3} + I_{\mathrm{sc},4})\,, \tag{4.80}$$

where $I_{\mathrm{sc},i}$ can be written in terms of di-gamma functions $\psi(x)$

$$\begin{aligned} I_{\mathrm{sc},1} &= 1 - \left[m^2 + p^2 - \frac{(eB_x)^2 + p' \cdot k_2\, p' \cdot k_1}{p' \cdot k_2 + p' \cdot k_1}\right] \frac{1}{eB_x} \beta\left(\frac{m^2 + p^2 + eB_x}{2eB_x}\right), \\ I_{\mathrm{sc},2} &= 1 - \left[m^2 + {p'}^2 + \frac{(eB_x)^2 + p' \cdot k_2\, p' \cdot k_1}{p' \cdot k_2 + p' \cdot k_1}\right] \frac{1}{eB_x} \beta\left(\frac{m^2 + {p'}^2 + eB_x}{2eB_x}\right), \\ I_{\mathrm{sc},3} &= 1 - \left[m^2 + (p' + k_2)^2 - \frac{(eB_x)^2 - p' \cdot k_2\, p' \cdot k_1}{p' \cdot k_2 - p' \cdot k_1}\right] \frac{1}{eB_x} \beta\left(\frac{m^2 + (p' + k_2)^2 + eB_x}{2eB_x}\right), \\ I_{\mathrm{sc},4} &= 1 - \left[m^2 + (p' + k_1)^2 + \frac{(eB_x)^2 - p' \cdot k_2\, p' \cdot k_1}{p' \cdot k_2 - p' \cdot k_1}\right] \frac{1}{eB_x} \beta\left(\frac{m^2 + (p' + k_1)^2 + eB_x}{2eB_x}\right), \end{aligned} \tag{4.81}$$

with $\beta(x)$ defined as (see [166, 167, 168])

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right], \qquad \psi(x) = \frac{d}{dx} \ln \Gamma(x)\,, \tag{4.82}$$

which also have the following integral representation

$$\beta\left(\frac{x+1}{2}\right) = \int_0^\infty dt\, \frac{\mathrm{e}^{-xt}}{\cosh t} \tag{4.83}$$

and series expansion

$$\frac{1}{b}\,\beta\left(\frac{x+b}{2\,b}\right) = \int_0^\infty dt\, \frac{\mathrm{e}^{-xt}}{\cosh(bt)} = \sum_{n=0}^\infty \frac{E_{2n}}{(2n)!} \int_0^\infty dt\, \mathrm{e}^{-xt}(bt)^{2n} = \sum_{n=0}^\infty \frac{E_{2n}}{(2n)!} \frac{b^{2n}}{x^{2n+1}}\, \Gamma(2n+1)\,, \tag{4.84}$$

E_{2n} are the Euler numbers.

4.6.2 Minkowski space representation

In the previous sections, we have considered all particles to be ingoing. Now we consider a realistic set up in which k_1 is the momentum of the incoming photon while k_2 is the outgoing photon. Furthermore, we change to Minkowski space by applying the following replacements[3]

$$\begin{aligned} g_{\mu\nu} &\longrightarrow -(\eta_{\mu\nu}) = -\mathrm{diag}(1,-1,-1,-1)\,, \\ k^4 &\longrightarrow -ik^0\,, \\ T &\longrightarrow -is\,, \\ \gamma^4 &\longrightarrow \gamma^0\,. \end{aligned} \tag{4.85}$$

In particular, for the external photons in the forward direction, we have

$$\begin{aligned} k_{\rm in} &= \omega_{\rm in}(1,1,0,0)\,, \\ k_{\rm out} &= \omega_{\rm out}(1,1,0,0)\,, \\ \varepsilon_{\rm in} &= (0,0,1,0)\,, \\ \varepsilon_{\rm out} &= (0,0,0,1)\,, \end{aligned} \tag{4.86}$$

with $p_{\rm in}$, $p_{\rm out}$ parallel to $k_{\rm in}$, $k_{\rm out}$ and the magnetic field $\mathbf{B} = (B,0,0)$. We use the following definition

$$D_{\rm scal}(p_{\rm in},p_{\rm out};k_{\rm in},\varepsilon_{\rm in};k_{\rm out},\varepsilon_{\rm out};B) = (2\pi)^4\,\delta^4(p_{\rm in}-p_{\rm out}+k_{\rm in}-k_{\rm out})\,D_{\rm scal}\,. \tag{4.87}$$

Then, the scalar Compton scattering in the forward direction can be expressed as

$$D_{\rm scal} = 2e^2\,\varepsilon_{\rm in}\cdot eF\cdot\varepsilon_{\rm out}\int_0^\infty ds\, s^2\int_0^1 du_1du_2\,{\rm e}^{+is\,m^2}\,{\rm e}^{-is(p_{\rm out}-u_1k_{\rm in}+u_2k_{\rm out})^2}\,\frac{K(u_1,u_2,eBs)}{\cos(eBs)}, \tag{4.88}$$

where

$$K(u_1,u_2,z) = -\frac{\sin(z\,\dot G_{12})}{\sin z} + \frac{\sin(z\,\dot G_{10})}{\sin z}\frac{\cos(z\,\dot G_{20})}{\cos z} - \frac{\sin(z\,\dot G_{20})}{\sin z}\frac{\cos(z\,\dot G_{10})}{\cos z}\,. \tag{4.89}$$

Notice that the field strength tensor $F^{\mu\nu}$ and Minkowski metric $\eta^{\mu\nu} = \eta_\perp^{\mu\nu}+\eta_{||}^{\mu\nu}$ now are expressed as

$$F = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&B\\ 0&0&-B&0 \end{pmatrix},\qquad \eta_\perp = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix},\qquad \eta_{||} = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}. \tag{4.90}$$

The scalar propagator (4.88), after integration, can be written as

$$D_{\rm scal} = -ie^2\,\frac{\varepsilon_{\rm in}\cdot(eF)\cdot\varepsilon_{\rm out}\,(p_{\rm out}\cdot k_{\rm out}+p_{\rm out}\cdot k_{\rm in})}{2[(eB)^2-(p_{\rm out}\cdot k_{\rm in})^2][(eB)^2-(p_{\rm out}\cdot k_{\rm out})^2]}\,(I_{\rm sc,1}+I_{\rm sc,2}+I_{\rm sc,3}+I_{\rm sc,4})\,, \tag{4.91}$$

where[4]

$$\begin{aligned} I_{\rm sc,1} &= 1-\left[p_{\rm in}^2-m^2-\lambda_-\right]\frac{1}{eB}\,\beta\left(\frac{p_{\rm in}^2-m^2+eB}{2eB}\right)\,, \\ I_{\rm sc,2} &= 1-\left[p_{\rm out}^2-m^2+\lambda_-\right]\frac{1}{eB}\,\beta\left(\frac{p_{\rm out}^2-m^2+eB}{2eB}\right)\,, \\ I_{\rm sc,3} &= 1-\left[(p_{\rm out}+k_{out})^2-m^2-\lambda_+\right]\frac{1}{eB}\,\beta\left(\frac{(p_{\rm out}+k_{out})^2-m^2+eB}{2eB}\right)\,, \\ I_{\rm sc,4} &= 1-\left[(p_{\rm out}-k_{in})^2-m^2+\lambda_+\right]\frac{1}{eB}\,\beta\left(\frac{(p_{\rm out}-k_{in})^2-m^2+eB}{2eB}\right)\,. \end{aligned} \tag{4.92}$$

For simplicity, we have defined $\lambda_\pm$ as

$$\lambda_\pm = \frac{(eB)^2\pm p_{\rm out}\cdot k_{\rm in}p_{\rm out}\cdot k_{\rm out}}{p_{\rm out}\cdot k_{\rm out}\pm p_{\rm out}\cdot k_{\rm in}}\,. \tag{4.93}$$

[3]Notice that in Appendix A and in this section the Wick rotations are different. This is due the fact the here we consider that $m^2 < p^2$ for the square of the external off-shell momenta.

[4]In deriving these expression, we have considered that $p^2 > m^2$ and used that $\beta(1-x) = -\beta(x)$ for the function $\beta(x)$.

4.7 Compton scattering in a magnetic field for spinor QED

In this section, we present the off-shell amplitude of Compton scattering in a pure magnetic background field of constant strength for spinor QED and specialize it to the forward scattering, aligned with the direction of the magnetic field. The fermionic contributions S_i of (4.39) for the pure magnetic field in $d=4$, after replacing the calligraphic Green's functions by the expressions in Section 4.5, are

$$\begin{aligned} S_0 &= \mathbb{1} + \frac{i}{2}\sigma^{\mu\nu}\hat{F}_{\mu\nu}\tanh z\,, \\ S_1(f_i) &= \frac{1}{2}\left[\sigma_{\mu\nu} - i\left(S_0\,\hat{F}_{\mu\nu} - \frac{1}{2}\varepsilon_{\sigma\rho\mu\nu}\,\gamma^5\hat{F}^{\sigma\rho}\right)\tanh z\right] f_i^{\mu\nu}\,, \end{aligned} \tag{4.94}$$

$$\begin{aligned} S_2(f_1f_2) = \frac{1}{4}\Bigg\{ & S_0\left[2\left(g_{||} + C_{F12}g_\perp - S_{F12}i\hat{F}\right)_{\mu\rho}\left(g_{||} + C_{F12}g_\perp - S_{F12}i\hat{F}\right)_{\nu\sigma} - \hat{F}_{\mu\nu}\hat{F}_{\sigma\rho}\tanh^2 z\right] \\ & + \frac{1}{2}\hat{F}^{\alpha\beta}\left[\left(\hat{F}_{\mu\nu}\varepsilon_{\alpha\beta\sigma\rho} + \hat{F}_{\sigma\rho}\varepsilon_{\alpha\beta\mu\nu}\right)\tanh z + 4iG_{F12}\left(g_{||} + C_{F12}g_\perp - S_{F12}i\hat{F}\right)_{\mu\rho}\varepsilon_{\alpha\beta\nu\sigma}\right]\gamma^5\tanh z \\ & + \varepsilon_{\mu\nu\sigma\rho}\,\gamma^5 - i\left(\hat{F}_{\mu\nu}\sigma_{\sigma\rho} + \sigma_{\mu\nu}\hat{F}_{\sigma\rho}\right)\tanh z + 4G_{F12}\left(g_{||} + C_{F12}g_\perp - S_{F12}i\hat{F}\right)_{\mu\rho}\sigma_{\nu\sigma}\Bigg\} f_1^{\mu\nu}f_2^{\sigma\rho} \end{aligned} \tag{4.95}$$

and the bosonic contributions are given by (4.72).

4.7.1 Forward scattering aligned with the magnetic field

In order to simplify the exponents in the kernels and therefore be able to perform the integral analytically, we make the following assumptions:

1. We consider the forward direction such that all momenta are parallel to the B-field, implying that

$$k_i \cdot g_\perp = 0, \qquad k_i \cdot \hat{F} = 0, \qquad p' \cdot g_\perp = 0, \qquad p' \cdot \hat{F} = 0\,. \tag{4.96}$$

2. Both photons are on-shell and their polarizations are perpendicular to each other. Which implies

$$k_i \cdot \varepsilon_j = 0, \qquad \varepsilon_i \cdot g_{||} = 0, \qquad \varepsilon_1 \cdot \varepsilon_2 = 0\,. \tag{4.97}$$

With such assumptions, the exponents become

$$\begin{aligned} h_{12} &= (p' + u_1k_1 + u_2k_2)^2 = p'^2 + 2u_1\,p'\cdot k_1 + 2u_2\,p'\cdot k_2\,, \\ h_{11}(p'+k_2) &= (p' + k_2 + u_1k_1)^2 = p'^2 + 2\,p'\cdot k_2 + 2u_1\,p'\cdot k_1\,, \\ h_{22}(p'+k_1) &= (p' + k_1 + u_2k_2)^2 = p'^2 + 2\,p'\cdot k_1 + 2u_2\,p'\cdot k_2\,, \end{aligned} \tag{4.98}$$

and the terms linear in the polarizations simplify to (remember $z = eB_xT$)

$$\begin{aligned} \varepsilon_1.b_1 = \varepsilon_1.h_1 = \varepsilon_2.b_2 = \varepsilon_2.h_2 &= \gamma^\mu(\tan\mathcal{Z})_{\mu\nu}b_0^\nu = 0\,, \\ \varepsilon_1.b_{12}.\varepsilon_2 &= z\left[-S_{12} + (S_{10}C_{02} + S_{02}C_{10})\tanh z\right] i\,\varepsilon_1.\hat{F}.\varepsilon_2\,, \\ \gamma^\mu(\tan\mathcal{Z})_{\mu\nu}\,(a_i\varepsilon_i)^\nu &= -\frac{i}{T}\,z\tanh z\,\left(S_{i0}\,\not{\varepsilon}_i + i\,C_{i0}\,\gamma.\hat{F}.\varepsilon_i\right)\,, \\ S_0 &= \mathbb{1} + \frac{i}{2}\sigma^{\mu\nu}\hat{F}_{\mu\nu}\tanh z\,, \\ S_1(f_i) &= \frac{1}{2}\sigma_{\mu\nu}f_i^{\mu\nu} = -\not{k}_i\not{\varepsilon}_i\,, \\ S_2(f_1f_2) &= \varepsilon_{\mu\nu\sigma\rho}\,f_1^{\mu\nu}f_2^{\sigma\rho}\,\gamma^5\,. \end{aligned} \tag{4.99}$$

Then, the kernels become

$$
\begin{aligned}
K^{pp'}_{\mathrm{sp},2} &= e^2 T \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}} \Big\{ 2\, S_0\, z \Big[-S_{12} + (S_{10}C_{20} - S_{20}C_{10}) \tanh z \Big] i\, \varepsilon_1 \cdot \hat{F} \cdot \varepsilon_2 + T\, S_2(f_1 f_2) \Big\} \,, \\
K^{pp'}_{\mathrm{ct},2} &= -e^2 T\, z \tanh z \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}} \Big[\Big(S_{10}\, \not{\varepsilon}_1 + iC_{10}\, \gamma.\hat{F}.\varepsilon_1 \Big) \not{k}_2 \not{\varepsilon}_2 + \Big(S_{20}\, \not{\varepsilon}_2 + iC_{20}\, \gamma.\hat{F}.\varepsilon_2 \Big) \not{k}_1 \not{\varepsilon}_1 \Big] \,, \\
K^{p,p'+k_1}_{\mathrm{sp},1}(k_2, \varepsilon_2) &= eT \int_0^1 du_2\ \mathrm{e}^{-T(p'+k_1+u_2k_2)^2} \not{k}_2 \not{\varepsilon}_2 \,, \\
K^{p,p'+k_2}_{\mathrm{sp},1}(k_1, \varepsilon_1) &= eT \int_0^1 du_1\ \mathrm{e}^{-T(p'+k_2+u_1k_1)^2} \not{k}_1 \not{\varepsilon}_1 \,,
\end{aligned}
\tag{4.100}
$$

We recall that the spinor propagator interacting with two photons in a background field is

$$
D^{pp'}_{\mathrm{sp},2} = \int_0^\infty dT\, \mathrm{e}^{-Tm^2} \Big[(m + \not{p}')\, K^{pp'}_{\mathrm{sp},2} + K^{pp'}_{\mathrm{ct},2} - e\not{\varepsilon}_1\, K^{p(p'+k_1)}_{\mathrm{sp},1}(k_2, \varepsilon_2) - e\not{\varepsilon}_2\, K^{p(p'+k_2)}_{\mathrm{sp},1}(k_1, \varepsilon_1) \Big] \,.
\tag{4.101}
$$

Notice that for the pure magnetic case in the forward direction, we need to compute the following integrals

$$
\begin{aligned}
I_{\mathrm{sp},1} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}}\, z \left[-S_{12} + \tanh z\, (S_{10}C_{20} - S_{20}C_{10}) \right] \,, \\
I_{\mathrm{sp},2} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}}\, z \tanh z \left[-S_{12} + \tanh z\, (S_{10}C_{20} - S_{20}C_{10}) \right] \,, \\
I^{\mu\nu}_{\mathrm{sp},3} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}}\, z \tanh z \left(S_{10}\, g^{\mu\nu}_\perp + iC_{10}\, \hat{F}^{\mu\nu} \right) \,, \\
I^{\mu\nu}_{\mathrm{sp},4} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}}\, z \tanh z \left(S_{20}\, g^{\mu\nu}_\perp + iC_{20}\, \hat{F}^{\mu\nu} \right) \,, \\
I_{\mathrm{sp},5} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T^2 \int_0^1 du_1 du_2\ \mathrm{e}^{-Th_{12}} = \frac{2(m^2 + p'^2 + p' \cdot k_1 + p' \cdot k_2)}{(m^2 + p^2)(m^2 + p'^2)[m^2 + (p' + k_2)^2][m^2 + (p' + k_1)^2]} \,, \\
I_{\mathrm{sp},6} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_2\ \mathrm{e}^{-T(p'+k_1+u_2k_2)^2} = \frac{1}{(m^2 + p^2)[m^2 + (p' + k_1)^2]} \,, \\
I_{\mathrm{sp},7} &= \int_0^\infty \mathrm{e}^{-Tm^2}\, T \int_0^1 du_1\ \mathrm{e}^{-T(p'+k_2+u_1k_1)^2} = \frac{1}{(m^2 + p^2)[m^2 + (p' + k_2)^2]} \,.
\end{aligned}
\tag{4.102}
$$

These integrals can be performed with Mathematica [152] or using the integral identities in [123]. Similar as the scalar calculation, we see that the propagator (4.101) can be written as

$$
\begin{aligned}
D^{pp'}_{\mathrm{sp},2} = {}& e^2\, (m + \not{p}') \left(2i\, \varepsilon_1.\hat{F}.\varepsilon_2\, I_{\mathrm{sp},1} - \varepsilon_1.\hat{F}.\varepsilon_2\, \sigma^{\mu\nu} \hat{F}_{\mu\nu}\, I_{\mathrm{sp},2} + \varepsilon_{\mu\nu\sigma\rho}\, f_1^{\mu\nu} f_2^{\sigma\rho}\, \gamma^5\, I_{\mathrm{sp},5} \right) \\
& - e^2 \left(\gamma_\mu I^{\mu\nu}_{\mathrm{sp},3}\, \varepsilon_{1\nu}\, \not{k}_2 \not{\varepsilon}_2 + \gamma_\mu I^{\mu\nu}_{\mathrm{sp},4}\, \varepsilon_{2\nu}\, \not{k}_1 \not{\varepsilon}_1 + I_{\mathrm{sp},6}\, \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2 + I_{\mathrm{sp},7}\, \not{\varepsilon}_2 \not{k}_1 \not{\varepsilon}_1 \right) .
\end{aligned}
\tag{4.103}
$$

Here $I_{\mathrm{sp},1}$ to $I_{\mathrm{sp},4}$ contain information about the magnetic field. And $I_{\mathrm{sp},5}$ to $I_{\mathrm{sp},7}$ have no interaction with the magnetic field therefore such terms can not contribute to the on-shell amplitude.

In order to have compact expressions for the integrals (4.102), we define

$$
\begin{aligned}
D_{p'} &= m^2 + p'^2, \qquad & D_{p'1} &= m^2 + (p' + k_1)^2 \,, \\
D_p &= m^2 + p^2, \qquad & D_{p'2} &= m^2 + (p' + k_2)^2 \,,
\end{aligned}
\tag{4.104}
$$

and decompose the integrals into four parts as

$$\begin{aligned} I_{\mathrm{sp},1} &= I_0\left(I_{\mathrm{sp},11}+I_{\mathrm{sp},12}+I_{\mathrm{sp},13}+I_{\mathrm{sp},14}\right)\,, \\ I_{\mathrm{sp},2} &= I_0\left(I_{\mathrm{sp},21}+I_{\mathrm{sp},22}+I_{\mathrm{sp},23}+I_{\mathrm{sp},24}\right)\,, \\ I^{\mu\nu}_{\mathrm{sp},3} &= I_{30}\left(I_{\mathrm{sp},31}+I_{\mathrm{sp},32}+I_{\mathrm{sp},33}+I_{\mathrm{sp},34}\right)^{\mu\nu}\,, \\ I^{\mu\nu}_{\mathrm{sp},4} &= I_{40}\left(I_{\mathrm{sp},41}+I_{\mathrm{sp},42}+I_{\mathrm{sp},43}+I_{\mathrm{sp},44}\right)^{\mu\nu}\,. \end{aligned} \tag{4.105}$$

Now, we simply present the results for each integral. For $I_{\mathrm{sp},1}$

$$\begin{aligned} I_0 &= \frac{(p'\cdot k_2 - p'\cdot k_1)eB_x}{4[(eB_x)^2-(p'\cdot k_1)^2][(eB_x)^2-(p'\cdot k_2)^2]}\,, \\ I_{\mathrm{sp},11} &= \frac{(eB_x-p'\cdot k_1)(eB_x-p'\cdot k_2)}{(p'\cdot k_2+p'\cdot k_1)D_p}+\beta\left(\frac{D_p}{2eB_x}\right)\,, \\ I_{\mathrm{sp},12} &= -\frac{(eB_x+p'\cdot k_1)(eB_x+p'\cdot k_2)}{(p'\cdot k_2+p'\cdot k_1)D_{p'}}+\beta\left(\frac{D_{p'}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},13} &= \frac{(eB_x+p'\cdot k_1)(eB_x-p'\cdot k_2)}{(p'\cdot k_2-p'\cdot k_1)D_{p'2}}+\beta\left(\frac{D_{p'2}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},14} &= -\frac{(eB_x-p'\cdot k_1)(eB_x+p'\cdot k_2)}{(p'\cdot k_2-p'\cdot k_1)D_{p'1}}+\beta\left(\frac{D_{p'1}}{2eB_x}\right)\,. \end{aligned} \tag{4.106}$$

For $I_{\mathrm{sp},2}$

$$\begin{aligned} I_{\mathrm{sp},21} &= \left[1-\frac{(eB_x-p'\cdot k_1)(eB_x-p'\cdot k_2)}{(p'\cdot k_2+p'\cdot k_1)D_p}\right]-\left[D_p-\frac{(eBx)^2+p'\cdot k_1 p'\cdot k_2}{p'\cdot k_2+p'\cdot k_1}\right]\frac{1}{eB_x}\,\beta\left(\frac{D_p}{2eB_x}\right)\,, \\ I_{\mathrm{sp},22} &= \left[1+\frac{(eB_x+p'\cdot k_1)(eB_x+p'\cdot k_2)}{(p'\cdot k_2+p'\cdot k_1)D_{p'}}\right]-\left[D_{p'}+\frac{(eBx)^2+p'\cdot k_1 p'\cdot k_2}{p'\cdot k_2+p'\cdot k_1}\right]\frac{1}{eB_x}\,\beta\left(\frac{D_{p'}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},23} &= \left[1-\frac{(eB_x+p'\cdot k_1)(eB_x-p'\cdot k_2)}{(p'\cdot k_2-p'\cdot k_1)D_{p'2}}\right]-\left[D_{p'2}-\frac{(eBx)^2-p'\cdot k_1 p'\cdot k_2}{p'\cdot k_2-p'\cdot k_1}\right]\frac{1}{eB_x}\,\beta\left(\frac{D_{p'2}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},24} &= \left[1+\frac{(eB_x-p'\cdot k_1)(eB_x+p'\cdot k_2)}{(p'\cdot k_2-p'\cdot k_1)D_{p'1}}\right]-\left[D_{p'1}+\frac{(eBx)^2-p'\cdot k_1 p'\cdot k_2}{p'\cdot k_2-p'\cdot k_1}\right]\frac{1}{eB_x}\,\beta\left(\frac{D_{p'1}}{2eB_x}\right)\,. \end{aligned} \tag{4.107}$$

For $I_{\mathrm{sp},3}$

$$\begin{aligned} I_{\mathrm{sp},30} &= \frac{1}{4[(eB_x)^2-(p'\cdot k_1)^2]p'.k_2}\,, \\ I_{\mathrm{sp},31} &= \frac{eB_x}{D_p}(i\,\hat{F}+g_\perp)(eB_x-p'.k_1)-(ieB_x\,\hat{F}-p'.k_1\,g_\perp)\beta\left(\frac{D_p}{2eB_x}\right)\,, \\ I_{\mathrm{sp},32} &= -\frac{eB_x}{D_{p'}}(i\,\hat{F}-g_\perp)(eB_x+p'.k_1)+(ieB_x\,\hat{F}-p'.k_1\,g_\perp)\beta\left(\frac{D_{p'}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},33} &= \frac{eB_x}{D_{p'2}}(i\,\hat{F}-g_\perp)(eB_x+p'.k_1)-(ieB_x\,\hat{F}-p'.k_1\,g_\perp)\beta\left(\frac{D_{p'2}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},34} &= -\frac{eB_x}{D_{p'1}}(i\,\hat{F}+g_\perp)(eB_x-p'.k_1)+(ieB_x\,\hat{F}-p'.k_1\,g_\perp)\beta\left(\frac{D_{p'1}}{2eB_x}\right)\,. \end{aligned} \tag{4.108}$$

And for $I_{\mathrm{sp},4}$

$$\begin{aligned} I_{\mathrm{sp},40} &= \frac{1}{4[(eB_x)^2-(p'\cdot k_2)^2]p'.k_1}\,, \\ I_{\mathrm{sp},41} &= \frac{eB_x}{D_p}(i\,\hat{F}+g_\perp)(eB_x-p'.k_2)-(ieB_x\,\hat{F}-p'.k_2\,g_\perp)\beta\left(\frac{D_p}{2eB_x}\right)\,, \\ I_{\mathrm{sp},42} &= -\frac{eB_x}{D_{p'}}(i\,\hat{F}-g_\perp)(eB_x+p'.k_2)+(ieB_x\,\hat{F}-p'.k_2\,g_\perp)\beta\left(\frac{D_{p'}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},43} &= -\frac{eB_x}{D_{p'2}}(i\,\hat{F}+g_\perp)(eB_x-p'.k_2)+(ieB_x\,\hat{F}-p'.k_2\,g_\perp)\beta\left(\frac{D_{p'2}}{2eB_x}\right)\,, \\ I_{\mathrm{sp},44} &= \frac{eB_x}{D_{p'1}}(i\,\hat{F}-g_\perp)(eB_x+p'.k_2)-(ieB_x\,\hat{F}-p'.k_2\,g_\perp)\beta\left(\frac{D_{p'1}}{2eB_x}\right)\,. \end{aligned} \tag{4.109}$$

4.7.2 Minkowski space representation

Here, we change to Minkowski space, as we did for the scalar amplitude, by performing the replacements in (4.85), we assume scattering in the forward direction parallel to the magnetic field and photon kinematics given by (4.86), and we use the following convention

$$D_{\text{spin}}(p_{\text{in}}, p_{\text{out}}; k_{\text{in}}, \varepsilon_{\text{in}}; k_{\text{out}}, \varepsilon_{\text{out}}; B) = (2\pi)^4\,\delta^4(p_{\text{in}} - p_{\text{out}} + k_{\text{in}} - k_{\text{out}})\,D_{\text{spin}}\,. \tag{4.110}$$

Then, the spinor Compton scattering in the forward direction can be expressed as

$$\begin{aligned} D_{\text{spin}} = -e^2 \int_0^\infty ds\, s \int_0^1 du_1 du_2\, \mathrm{e}^{+is\,m^2} \Big\{ &\gamma^\mu J_{\mu\nu}(u_1, eBs)\varepsilon^\nu_{\text{in}}\, \not{k}_{\text{out}}\not{\varepsilon}_{\text{out}} - \gamma^\mu J_{\mu\nu}(u_2, eBs)\varepsilon^\nu_{\text{out}}\, \not{k}_{\text{in}}\not{\varepsilon}_{\text{in}} \\ &+ 2\left(m + \not{p}_{\text{out}}\right) \varepsilon_{\text{in}}\cdot(eFs)\cdot\varepsilon_{\text{out}}\Big[1 + \frac{i}{B}\Sigma^{\mu\nu}F_{\mu\nu}\,\tan(eBs)\Big]K(u_1, u_2, eBs)\Big\}\,\mathrm{e}^{-is(p_{\text{out}} - u_1 k_{\text{in}} + u_2 k_{\text{out}})^2}\,, \end{aligned} \tag{4.111}$$

where $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, $K(u_1, u_2, z)$ is given by (4.89) and

$$J(u_i, z) = z\left(\frac{\sin(z\,\dot{G}_{i0})}{\cos z}\,\eta_\perp + \frac{1}{B}\,\frac{\cos(z\,\dot{G}_{i0})}{\cos z}\,F\right)\,. \tag{4.112}$$

The spinor propagator (4.111), after integration, can be expressed as

$$\begin{aligned} D_{\text{spin}} = ie^2\Big[2\left(m + \not{p}_{\text{out}}\right)\varepsilon_{\text{in}}\cdot(eF)\cdot\varepsilon_{\text{out}}\Big(I_{\text{sp},1} - \frac{1}{B}\Sigma^{\mu\nu}F_{\mu\nu}\,I_{\text{sp},2}\Big) \\ - i\gamma_\mu I^{\mu\nu}_{\text{sp},3}\varepsilon_{\text{in},\nu}\,\not{k}_{\text{out}}\not{\varepsilon}_{\text{out}} - i\gamma_\mu I^{\mu\nu}_{\text{sp},4}\varepsilon_{\text{out},\nu}\,\not{k}_{\text{in}}\not{\varepsilon}_{\text{in}}\Big]\,. \end{aligned} \tag{4.113}$$

For simplicity we define

$$\lambda'_{\pm\pm} = \frac{(eB \pm p_{\text{out}}\cdot k_{\text{in}})(eB \pm p_{\text{out}}\cdot k_{\text{out}})}{p_{\text{out}}\cdot k_{\text{out}} + (\pm)(\pm)p_{\text{out}}\cdot k_{\text{in}}}\,. \tag{4.114}$$

Now, let us compute (4.105). For $I_{\text{sp},1}$

$$\begin{aligned} I_0 &= -\frac{p_{\text{out}}\cdot k_{\text{out}} + p_{\text{out}}\cdot k_{\text{in}}}{4[(eB)^2 - (p_{\text{out}}\cdot k_{\text{in}})^2][(eB)^2 - (p_{\text{out}}\cdot k_{\text{out}})^2]}\,, \\ I_{\text{sp},11} &= \frac{\lambda'_{-+}}{p^2_{\text{in}} - m^2} - \beta\left(\frac{p^2_{\text{in}} - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},12} &= -\frac{\lambda'_{+-}}{p^2_{\text{out}} - m^2} - \beta\left(\frac{p^2_{\text{out}} - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},13} &= \frac{\lambda'_{++}}{(p_{\text{out}} + k_{out})^2 - m^2} - \beta\left(\frac{(p_{\text{out}} + k_{out})^2 - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},14} &= -\frac{\lambda'_{--}}{(p_{\text{out}} - k_{in})^2 - m^2} - \beta\left(\frac{(p_{\text{out}} - k_{in})^2 - m^2}{2eB} + 1\right)\,. \end{aligned} \tag{4.115}$$

For $I_{\text{sp},2}$ (see (4.93) for $\lambda_\pm$ definition)

$$\begin{aligned} I_{\text{sp},21} &= \left[1 - \frac{\lambda'_{-+}}{p^2_{\text{in}} - m^2}\right] - \left[p^2_{\text{in}} - m^2 - \lambda_-\right]\frac{1}{eB}\,\beta\left(\frac{p^2_{\text{in}} - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},22} &= \left[1 + \frac{\lambda'_{+-}}{p^2_{\text{out}} - m^2}\right] - \left[p^2_{\text{out}} - m^2 + \lambda_-\right]\frac{1}{eB}\,\beta\left(\frac{p^2_{\text{out}} - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},23} &= \left[1 - \frac{\lambda'_{++}}{(p_{\text{out}} + k_{out})^2 - m^2}\right] - \left[(p_{\text{out}} + k_{out})^2 - m^2 - \lambda_+\right]\frac{1}{eB}\,\beta\left(\frac{(p_{\text{out}} + k_{out})^2 - m^2}{2eB} + 1\right)\,, \\ I_{\text{sp},24} &= \left[1 + \frac{\lambda'_{--}}{(p_{\text{out}} - k_{in})^2 - m^2}\right] - \left[(p_{\text{out}} - k_{in})^2 - m^2 + \lambda_+\right]\frac{1}{eB}\,\beta\left(\frac{(p_{\text{out}} - k_{in})^2 - m^2}{2eB} + 1\right)\,. \end{aligned} \tag{4.116}$$

For $I_{\rm sp,3}$

$$
\begin{aligned}
I_{\rm sp,30} &= \frac{1}{4[(eB)^2-(p_{\rm out}\cdot k_{\rm in})^2]p_{\rm out}\cdot k_{\rm out}}\,,\\
I^{\mu}_{\rm sp,31} &= \frac{eB(eB-p_{\rm out}\cdot k_{\rm in})}{p_{\rm in}^2-m^2}\left(\frac{i}{B}F-\eta_\perp\right)-(ieF+p_{\rm out}\cdot k_{\rm in}\,\eta_\perp)\beta\left(\frac{p_{\rm in}^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,32} &= -\frac{eB(eB+p_{\rm out}\cdot k_{\rm in})}{p_{\rm out}^2-m^2}\left(\frac{i}{B}F+\eta_\perp\right)+(ieF+p_{\rm out}\cdot k_{\rm in}\,\eta_\perp)\beta\left(\frac{p_{\rm out}^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,33} &= \frac{eB(eB+p_{\rm out}\cdot k_{\rm in})}{(p_{\rm out}+k_{out})^2-m^2}\left(\frac{i}{B}F+\eta_\perp\right)-(ieF+p_{\rm out}\cdot k_{\rm in}\,\eta_\perp)\beta\left(\frac{(p_{\rm out}+k_{out})^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,34} &= -\frac{eB(eB-p_{\rm out}\cdot k_{\rm in})}{(p_{\rm out}-k_{in})^2-m^2}\left(\frac{i}{B}F-\eta_\perp\right)+(ieF+p_{\rm out}\cdot k_{\rm in}\,\eta_\perp)\beta\left(\frac{(p_{\rm out}-k_{in})^2-m^2}{2eB}+1\right)\,.
\end{aligned}
\tag{4.117}
$$

And for $I_{\rm sp,4}$

$$
\begin{aligned}
I_{\rm sp,40} &= \frac{1}{4[(eB)^2-(p_{\rm out}\cdot k_{\rm out})^2]p_{\rm out}\cdot k_{\rm in}}\,,\\
I^{\mu}_{\rm sp,41} &= \frac{eB(eB+p_{\rm out}\cdot k_{\rm out})}{p_{\rm in}^2-m^2}\left(\frac{i}{B}F-\eta_\perp\right)-(ieF-p_{\rm out}\cdot k_{\rm out}\,\eta_\perp)\beta\left(\frac{p_{\rm in}^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,42} &= -\frac{eB(eB-p_{\rm out}\cdot k_{\rm out})}{p_{\rm out}^2-m^2}\left(\frac{i}{B}F+\eta_\perp\right)+(ieF-p_{\rm out}\cdot k_{\rm out}\,\eta_\perp)\beta\left(\frac{p_{\rm out}^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,43} &= -\frac{eB(eB+p_{\rm out}\cdot k_{\rm out})}{(p_{\rm out}+k_{out})^2-m^2}\left(\frac{i}{B}F-\eta_\perp\right)+(ieF-p_{\rm out}\cdot k_{\rm out}\,\eta_\perp)\beta\left(\frac{(p_{\rm out}+k_{out})^2-m^2}{2eB}+1\right)\,,\\
I^{\mu}_{\rm sp,44} &= \frac{eB(eB-p_{\rm out}\cdot k_{\rm out})}{(p_{\rm out}-k_{in})^2-m^2}\left(\frac{i}{B}F+\eta_\perp\right)-(ieF-p_{\rm out}\cdot k_{\rm out}\,\eta_\perp)\beta\left(\frac{(p_{\rm out}-k_{in})^2-m^2}{2eB}+1\right)\,.
\end{aligned}
\tag{4.118}
$$

In this chapter, we have presented the scalar and spinor amplitude of Compton scattering in a pure magnetic field for off-shell massive particles and on-shell photons. The integration of such amplitudes is carried out analytically for the forward scattering in which all particles are parallel to the direction of the magnetic field. We have checked that the final results (4.91) and (4.113) in the weak-field expansion (at first order with respect to the B-filed) and on-shell limit reproduce the expected results [66]. This shows that Compton scattering in a magnetic background field could lead to the observation of polarization changes that do not correspond to the vacuum birefringence assisted with Coulomb fields [65].

It is important to mention that our main results (4.91) and (4.113) do not correspond to the experimental Compton scattering in a magnetic field since we are considering the incoming and outgoing (scalar or spinor) particles to be off-shell and taking the on-shell limit is not straightforward when we consider the external field exactly. This is because we use the worldline formalism to derive such results in which the methods to obtain on-shell amplitudes for external massive particles are still under development [171, 128]. However, these results give us an idea of the important parameters involved in this scattering process, for instance, our formulas depend mainly on the parameter eB. We can also notice that the denominators of (4.91) and (4.113): $(eB)^2-(p_{\rm out}\cdot k_{\rm out})^2$ and $(eB)^2-(p_{\rm out}\cdot k_{\rm in})^2$ have poles related to the Landau levels of the corresponding particles, for which, the cyclotron frequency is $\omega_c = eB/m$, see [66] for a more detailed review of the birefringent Compton scattering.

Chapter 5

Electron propagator in a plane-wave field: One-loop vertex correction

In this chapter, we provide a brief introduction to the operator technique, the Furry picture, and the Volkov states and propagators. We discuss the one-loop vertex correction in vacuum, emphasizing its renormalization and gauge invariance. This correction is significant in QED due to its connection with the anomalous magnetic moment of the electron, which has enabled high-precision measurements of fundamental constants such as the fine structure constant and the anomalous magnetic moment of the electron [172, 173, 174].

We then present the calculation of the renormalized one-loop vertex correction in an arbitrary plane-wave field, discussing its relevance to gauge invariance, infrared divergences, application as a building block, and behavior in strong fields. This is a non-perturbative calculation, given that the plane-wave field is taken into account exactly.

It is important to note that, unlike in vacuum, extracting a correction to the anomalous magnetic moment of the electron from the one-loop vertex correction in a plane-wave field is not well defined. However, this amplitude completes the study of QED in a plane-wave field at one-loop order. This completion is significant for future experimental comparisons, as the closer the laser intensities approach the critical field, the more significant loop corrections become [73].

It is important to point out some conventions used in this chapter that differ from the rest of the thesis. Here, we use the metric tensor in Minkowski space $(\eta^{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$, the product of two four-vectors is denoted by $(xy) = x_\mu y^\mu$, the energy component of an on-shell electron is $\varepsilon = \sqrt{\boldsymbol{p}^2 + m^2}$ (where $\boldsymbol{p}$ is its kinetic momentum and m its mass) and the polarization of a photon with momentum q is written as $e_l^\mu(q)$.

5.1 Plane waves as a laser approximation

The match between theory and experiment is crucial in the understanding of physical laws. Therefore to compare experimental data from laser-particle collisions with theory, it is desirable to have a mathematical description of laser fields. Due to the complexity of lasers, the exact mathematical description of lasers is not possible. However, a plane wave field is a reasonably good approximation of a laser, if the radius of the minimal focusing area is much larger than the central wavelength of the laser pulse [72].

A general plane wave is characterized mainly by the direction of propagation which is represented by the unit vector $\mathbf{n}$. Then, the plane-wave field is described by the four-vector potential $A^\mu(\phi)$, which only depends on the "light-cone time" $\phi = t - \boldsymbol{n} \cdot \boldsymbol{x}$, with $x^\mu = (t, \boldsymbol{x})$ being the space-time Cartesian coordinates. Assuming that the vector potential fulfills the Lorenz gauge condition $\partial_\mu A^\mu = 0$, $A^\mu(\phi) \to 0$ in the limit $\phi \to \pm\infty$ i.e., it is a localized field, and imposing the restriction $A^0(\phi) = 0$ which implies $A^\mu(\phi) = (0, \boldsymbol{A}(\phi))$ we have as a consequence that

$$\mathbf{n} \cdot \boldsymbol{A}'(\phi) = 0 \qquad \text{and} \qquad \mathbf{n} \cdot \boldsymbol{A}(\phi) = 0\,, \tag{5.1}$$

where $\boldsymbol{A}'(\phi) = \frac{d}{d\phi}\boldsymbol{A}(\phi)$. This motivates the use of light-cone coordinates

$$n^\mu = (1, \mathbf{n}), \qquad \tilde{n}^\mu = (1, -\mathbf{n})/2, \qquad a_1^\mu = (0, \boldsymbol{a}_1), \qquad a_2^\mu = (0, \boldsymbol{a}_2). \tag{5.2}$$

The previous coordinates satisfy the following properties

$$n^2 = \tilde{n}^2 = (na_j) = (\tilde{n}a_j) = 0\,, \qquad (n\tilde{n}) = 1\,, \qquad (a_i a_j) = -\boldsymbol{a}_i \cdot \boldsymbol{a}_j = -\delta_{ij}\,, \tag{5.3}$$

and fulfill the completeness relation

$$\eta^{\mu\nu} = n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu\,, \tag{5.4}$$

which indicates that the light-cone coordinates form a complete basis for Minkowski space. Then, the "light-cone components" of an arbitrary four-vector $v^\mu = (v^0, \boldsymbol{v})$ can be expressed as

$$\begin{aligned} &v_- = (nv) = v^0 - v_n\,, \qquad v_+ = (\tilde{n}v) = (v^0 + v_n)/2\,, \qquad \text{with} \qquad v_n = \boldsymbol{n}\cdot\boldsymbol{v}\,, \\ &\boldsymbol{v}_\perp = (v_{\perp 1}, v_{\perp 2}) = -((va_1), (va_2)) = (\boldsymbol{v}\cdot\boldsymbol{a}_1, \boldsymbol{v}\cdot\boldsymbol{a}_2)\,. \end{aligned} \tag{5.5}$$

Notice that the most general form of the vector potential $\boldsymbol{A}(\phi)$ is expressed as

$$\boldsymbol{A}(\phi) = \psi_1(\phi)\boldsymbol{a}_1 + \psi_2(\phi)\boldsymbol{a}_2\,, \tag{5.6}$$

where $\psi_1(\phi)$ and $\psi_2(\phi)$ are arbitrary functions that vanish for $\phi \to \pm\infty$ and they satisfy the same differential properties of the four-vector potential $A^\mu(\phi)$, i.e. $\partial_\mu \psi_i(\phi)\boldsymbol{a}_i = 0$.

5.2 Dirac equation within the light-cone coordinates

In this section, we review the simpler case of the equation of motion for a spin-one-half particle propagating in a vacuum, i.e., the Dirac equation. We will first examine this in the usual Cartesian coordinate system and then in the light-cone system. We will focus on some important aspects that will be useful for our main calculation later (for more details, see [150, 98]). The Dirac equation is

$$\left(\not{P} - m\right)\psi = 0\,, \tag{5.7}$$

in which $P^\mu = i\partial^\mu$ is the four-momentum operator. We use the convention $\not{v} = \gamma^\mu v_\mu$ for a generic four-vector v^μ, with γ^μ being the Dirac-gamma matrices, which satisfy the anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Assuming that the four vectors x^μ and $p^\mu = (\varepsilon, \boldsymbol{p})$ are the eigenvalues of the position and momentum operators X^μ and P^μ with their respective eigenstates $|x\rangle$ and $|p\rangle$ that are normalized as

$$\langle x|y\rangle = \delta^{(4)}(x - y)\,, \qquad \langle p|q\rangle = (2\pi)^4\delta^{(4)}(p - q)\,, \tag{5.8}$$

and satisfying the completeness relations

$$\int d^4x\, |x\rangle\langle x| = 1\,, \qquad \int \frac{d^4p}{(2\pi)^4}\, |p\rangle\langle p| = 1. \tag{5.9}$$

The positive solution of Dirac equation is given by

$$\psi_s(p) = u_s(p)\, \mathrm{e}^{-i(px)}\,, \tag{5.10}$$

where $u_s(p)$ are the free, positive-energy spinors normalized as $u_s^\dagger(p)u_{s'}(p) = 2\varepsilon\, \delta_{ss'}$. This solution satisfies the on-shell condition $p^2 = m^2$. Here, we can go to the light-cone basis by using (5.4) so that

$$\langle x|p\rangle = \exp(-i(px)) = \exp[-i(p_+\phi + p_- T - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp)]\,. \tag{5.11}$$

Notice that, the eigenvalue equation for the position operator $X^\mu|x\rangle = x^\mu|x\rangle$, in the light-cone coordinate system, can be re-expressed as

$$\Phi|x\rangle = (nX)|x\rangle = \phi|x\rangle, \qquad T|x\rangle = (\tilde{n}X)|x\rangle = T|x\rangle\,, \qquad \boldsymbol{X}_\perp|x\rangle = \boldsymbol{x}_\perp|x\rangle\,. \tag{5.12}$$

Here, we have defined

$$\phi = (nx), \qquad T = (\tilde{n}x), \qquad \boldsymbol{x}_\perp = (x_{\perp 1}, x_{\perp 2}) = -((xa_1), (xa_2)). \tag{5.13}$$

On the other hand, for the eigenvalue equation of the momentum operator $P^\mu|p\rangle = p^\mu|p\rangle$, it is convenient to write the momenta operators in the light-cone basis

$$P_\phi = -i\partial_\phi = -(\tilde{n}P) = -(i\partial_t - i\partial_{x_n})/2\,, \tag{5.14}$$

$$P_T = -i\partial_T = -(nP) = -(i\partial_t + i\partial_{x_n})\,, \tag{5.15}$$

$$\boldsymbol{P}_\perp = (P_{\perp,1}, P_{\perp,2}) = -i(\boldsymbol{a}_1\cdot\boldsymbol{\nabla}, \boldsymbol{a}_2\cdot\boldsymbol{\nabla})\,, \tag{5.16}$$

for which, the non-null commutators are

$$[\Phi, P_\phi] = [T, P_T] = i\,, \qquad [X_{\perp,j}, P_{\perp,k}] = i\delta_{jk}\,, \tag{5.17}$$

and with the eigenvalues

$$P_\phi|p\rangle = -p_+|p\rangle, \qquad P_T|p\rangle = -p_-|p\rangle, \qquad \text{and} \qquad \boldsymbol{P}_\perp|p\rangle = \boldsymbol{p}_\perp|p\rangle\,. \tag{5.18}$$

We can obtain useful shifting identities by recalling the commutation relations $[X^\mu, P^\nu] = -i\eta^{\mu\nu}$, which imply

$$[P^\mu, f(X)] = i\partial^\mu_X f(X)\,, \tag{5.19}$$

where $f(X)$ is an arbitrary function of the four-position operator that can be expanded in Taylor series and $\partial^\mu_X = \partial/\partial X_\mu$. Analogously, it can easily be shown that

$$\mathrm{e}^{if(X)}\, P^\mu\, \mathrm{e}^{-if(X)} = P^\mu + \partial^\mu f(X) \tag{5.20}$$

and then formally that

$$\mathrm{e}^{if(X)}\, g(P)\, \mathrm{e}^{-if(X)} = g(P + \partial f(X))\,, \tag{5.21}$$

where $g(P)$ is a function of the four-momentum that can be expanded in Taylor series. The same commutation relations imply that

$$\mathrm{e}^{ig(P)}\, X^\mu\, \mathrm{e}^{-ig(P)} = X^\mu - \partial^\mu_P g(P) \qquad \text{and} \qquad \mathrm{e}^{ig(P)}\, f(X)\, \mathrm{e}^{-ig(P)} = f(X - \partial_P g(P))\,, \tag{5.22}$$

where $\partial^\mu_P = \partial/\partial P_\mu$, and $f(X)$ can be represented by a Taylor series expansion. In particular, we will consider the case where the functions in the exponents are linear either in X^μ or in P^μ

$$\mathrm{e}^{i(Xq)}\, g(P)\, \mathrm{e}^{-i(Xq)} = g(P+q)\,, \tag{5.23}$$

$$\mathrm{e}^{i(Py)}\, f(X)\, \mathrm{e}^{-i(Py)} = f(X-y)\,, \tag{5.24}$$

where q^μ and y^μ are constant four-vectors.

Then, in the light-cone basis, the commutation relations $[\phi, P_\phi] = [T, P_T] = i$ will imply, in particular, the identities

$$\mathrm{e}^{ia\phi}\, \tilde{g}(P_\phi)\, \mathrm{e}^{-ia\phi} = \tilde{g}(P_\phi - a), \tag{5.25}$$

$$\mathrm{e}^{ibP_T}\, \tilde{f}(T)\, \mathrm{e}^{-ibP_T} = \tilde{f}(T+b), \tag{5.26}$$

with a and b being two constants and $\tilde{f}(T)$ and $\tilde{g}(P_\phi)$ being two arbitrary functions.

5.3 Vertex function in vacuum

In this section, since the main goal of this chapter is to compute the one-loop vertex correction in a plane wave field, it is convenient to review the known result in vacuum (adapted from [150]), with a particular focus on renormalization and gauge invariance. The complete irreducible[1] vertex function is

$$-ie\Lambda_{s,s',l}(p,p',q) = -ie\delta(p-p'-q)\,\bar{u}_{s'}(p')\,\Lambda^\mu(p,p')\,e^*_{l,\mu}(q)\,u_s(p)\,, \tag{5.27}$$

where, at one-loop order,

$$\Lambda^\mu(p,p') = Z_1\,\gamma^\mu + \Lambda^{(1)\mu}(p,p') + \delta\Lambda^{(\xi)\mu}(p,p')\,, \tag{5.28}$$

[1]We recall that an irreducible diagram is such that it cannot be split into two pieces by removing a single line [175].

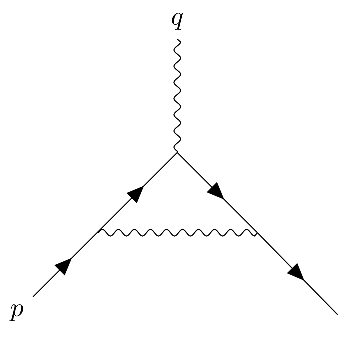

Figure 5.1: *Feynman diagram for the electron one-loop vertex correction.*

where Z_1 is a renormalization constant [148]. For the sake of convenience, we have defined $\Lambda^{(1)\mu}(p,p')$ as the one-loop vertex correction in Feynman-gauge and $\delta\Lambda^{(\xi)\mu}(p,p')$ the additional contribution to the one-loop vertex function when considering the internal photon propagator in an arbitrary gauge as in (5.38) and with the inclusion of a positive photon mass 'κ' to avoid the infrared divergence. From Feynman rules, we see that the complete one-loop correction (see Fig. 5.1) in an arbitrary gauge $\Lambda^{(1)\mu}(p,p') + \delta\Lambda^{(\xi)\mu}(p,p')$ is given by

$$\Lambda^{(1)\mu}(p,p') = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\kappa^2+i0}\gamma^\lambda \frac{1}{\not{p}'+\not{k}-m+i0}\gamma^\mu \frac{1}{\not{p}+\not{k}-m+i0}\gamma_\lambda \,, \tag{5.29}$$

$$\delta\Lambda^{(\xi)\mu}(p,p') = -ie^2\left(1-\frac{1}{\xi}\right) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\kappa^2+i0)^2}\,\hat{k}\,\frac{1}{\not{p}'+\not{k}-m+i0}\gamma^\mu \frac{1}{\not{p}+\not{k}-m+i0}\,\hat{k}\,. \tag{5.30}$$

The four-momentum integral in (5.29) can be performed by using Schwinger parameters such that, we obtain[2]

$$\Lambda^{(1)\mu}(p,p') = -i\frac{\alpha}{2\pi}\int_0^\infty \frac{ds\,du\,dt}{S^3}\Bigg\{ [2S(pp')+i]\,\gamma^\mu - \not{p}(\not{p}'s+\not{p}u)\gamma^\mu - \gamma^\mu(\not{p}'s+\not{p}u)\not{p}' - \frac{1}{S}(\not{p}'s+\not{p}u)\gamma^\mu(\not{p}'s+\not{p}u)\Bigg\}\, e^{-i\kappa^2 t - i\frac{(p's+pu)^2}{S}} \,, \tag{5.31}$$

where $S = s+t+u$. In the case of (5.30), we can simplify it by noticing that the external spinors in the expression $\bar{u}_{s'}(p')\,\delta\Lambda^{(\xi)\mu}(p,p')u_s(p)$ satisfy the Dirac equation and after straightforward manipulations, we obtain

$$\delta\Lambda^{(\xi)\mu}(p,p') = -ie^2\left(1-\frac{1}{\xi}\right)\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\kappa^2+i0)^2}\,\gamma^\mu = Z^{(\xi)}\,\gamma^\mu \,. \tag{5.32}$$

Then, the irreducible vertex function can be decomposed as

$$\Lambda^\mu(p,p') = (Z_1 + Z^{(\xi)})\,\gamma^\mu + \Lambda^{(1)\mu}(p,p') \,, \tag{5.33}$$

note that $Z^{(\xi)}$ is a logarithmically divergent constant that depends on the gauge parameter ξ.

In order to regularize the vertex function, we define the electric charge via

$$\bar{u}_{s'}(p')\,e\Lambda^\mu_R(p,p')\,u_s(p)\big|_{\not{p}=\not{p}'=m} = \bar{u}_{s'}(p')\,e\gamma^\mu\,u_s(p) \,. \tag{5.34}$$

This condition will fix the renormalization constant as $Z_1 = 1 - Z^{(\xi)}$ and the renormalization of the one loop vertex correction as

$$\Lambda^{(1)\mu}_R(p,p') = \Lambda^{(1)\mu}(p,p') - \Lambda^{(1)\mu}(p,p)\big|_{\not{p}=m} \,, \tag{5.35}$$

[2] Here, for the present discussion, it is more useful to have a symmetric expression for the one loop vertex function. However, we can further simplify it by taking into account the on-shell condition, Dirac equation and the Gordon identity, see [150, 148, 175].

where, from (5.31), is obvious that

$$\Lambda^{(1)\mu}(p,p)|_{\not p=m} = -i\frac{\alpha}{2\pi}\gamma^\mu \int_0^\infty \frac{dsdudt}{S^3}\, e^{-i\kappa^2 t - i\frac{(s+u)^2}{S}m^2}\left\{m^2\left[2t - \frac{(s+u)^2}{S}\right] + i\right\}. \tag{5.36}$$

It is well known that the renormalized one-loop vertex correction can be expressed as [150, 175]

$$\Lambda_R^{(1)\mu}(p,p') = \gamma^\mu\, F_1(q^2) + \frac{i}{2m}\,\sigma^{\mu\nu}q_\nu F_2(q^2)\,, \tag{5.37}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the well-known QED form factors, derived in detail in [150], and whose explicit expressions are not presented in this thesis but only discussed.

In this context, $F_2(q^2)$ is usually called the magnetic form factor since it provides the leading quantum correction to the magnetic moment of the electron. Notably, $F_2(q^2)$ has been computed at higher loop orders and subsequently compared with experimental results. These comparisons have enabled precise measurements of fundamental constants such as the fine structure constant and the anomalous magnetic moment of the electron [172, 173, 174], marking one of the significant triumphs of QED. Additionally, it is noteworthy that this form factor exhibits no infrared divergence in the limit of a massless photon: $\kappa \to 0$.

On the other hand, the form factor $F_1(q^2)$ has an infrared divergence in the limit $\kappa \to 0$ which, in order to be removed, it is necessary to apply the Bloch-Nordsieck method [176, 175]. In this section we saw that the gauge-dependent constant $Z^{(\xi)}$ can be absorbed in the renormalization constant Z_1 nevertheless the complete gauge dependent one loop vertex correction $\Lambda^{(1)\mu}(p,p') + \delta\Lambda^{(\xi)\mu}(p,p')$ have been considered in [177], where it is discussed the relation between the infrared diverge and the gauge choice, in particular, with relation to the Yennie-Fried gauge choice $1 - \frac{1}{\xi} = 2(1-2\epsilon)$ where ϵ is a small positive constant.

In the following sections, we generalize the result of the one-loop vertex correction in vacuum to the one in the presence of a plane-wave field.

5.4 Operator technique

Schwinger first proposed the operator technique [7] as a method to compute transition amplitudes in the presence of external fields. This method does not require the explicit solution of the Dirac equation in the external field, as it is sufficient to know the spectrum of certain operators.

We work within the Furry picture [97], an interaction picture used to analyze the interactions between fermions and bosons in the presence of an external field. This picture assumes that the external field is strong enough such that it remains unchanged by the interaction with and between fermions and bosons, the external field satisfy the Lorenz gauge condition $\partial_\mu A^\mu = 0$, and the action vanishes on the boundaries [84].

These assumptions imply, in particular for the spinor QED Lagrangian in the presence of a background field, that the electron equation of motion will be the Dirac equation in the presence of the external field and for the photon field the external field will have no effect in the equation of motion, i.e. the photon propagator remains as in vacuum, which in a general gauge is

$$D^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{e}^{-i(kx)}}{k^2+i0}\left[\eta^{\mu\nu} + \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2+i0}\right], \tag{5.38}$$

with ξ as the gauge parameter. To compute transition amplitudes within the Furry picture, we use the same Feynman rules as in vacuum except that now we use the Dirac operator in the presence of the external field (see [87] for a short review of the Feynman rules for the case of a plane-wave background field).

The Furry picture and the operator technique have been widely applied to study quantum corrections in various background fields. For a constant background field, these applications include the mass operator [178], N-photon amplitude and polarization operator [179]. For a constant crossed background field, studies include the polarization operator [41], photon splitting [180, 164], the one-loop vertex correction [108, 109] and nonlinear double Compton scattering [181]. For a plane-wave background field, they include the mass operator [182], the polarization operator [183, 179, 184], nonlinear Compton scattering [84, 85, 86, 87, 88], nonlinear double Compton scattering [185, 186], Breit-Wheeler pair production [90, 91, 92], radiation reaction [77, 78], trident pair production [95, 96], nonlinear Bethe-Heitler pair production [94], photon splitting [44] and photon merging [161, 187, 26]. This list provides some examples of applications; see also [101, 72, 73] and references therein.

In the following, within the Furry picture, we consider the exact solution of the Dirac equation in a plane-wave field (Volkov states) and the Dirac propagator in a plane-wave field (Volkov propagator) to obtain the spectrum of the necessary set of operators required to compute the one-loop vertex correction in a plane-wave field.

5.4.1 Dirac equation in a plane-wave background field

On the theoretical side, a laser field is often approximated as a plane wave. Therefore, to describe the propagation of an electron immersed in a laser field, it is important to have the solution of the Dirac equation in a plane wave field. This solution was first obtained by Volkov, even before the invention of lasers [100] and it can also be found in text-books such as [150, 98]. The Dirac equation in a plane-wave background field is

$$\Big[\not{\Pi}(\Phi) - m\Big]U = 0\,, \tag{5.39}$$

where $\Pi^\mu = P^\mu - eA^\mu(\Phi)$ is the four-momentum operator in the presence of a plane-wave field. The positive solution is given by the Volkov state $U_s(p,x) = E(p,x)u_s(p)$ [100], where

$$E(p,x) = \left[1 + \frac{e\not{n}\not{A}(\phi)}{2p_-}\right]\mathrm{e}^{iS_p(x)}, \tag{5.40}$$

with

$$S_p(x) \equiv -(px) - \int_{-\infty}^{\phi} d\varphi\left[\frac{e(pA(\varphi))}{p_-} - \frac{e^2A^2(\varphi)}{2p_-}\right] \tag{5.41}$$

and $u_s(p)$ are the free spinors introduced in Section 5.2 (see Appendix B.3.1 for a short derivation of this result). It is convenient to explicitly write the conjugate Volkov solution $\bar{U}_s(p,x) = \bar{u}_s(p)\bar{E}(p,x)$, where

$$\bar{E}(p,x) = \left[1 + \frac{e\not{A}(\phi)\not{n}}{2p_-}\right]\mathrm{e}^{-iS_p(x)}\,. \tag{5.42}$$

Here, $\bar{E}(p,x)$ and $E(p,x)$ are known as the Ritus matrices [101] and satisfy the following orthogonality and completeness relations

$$\int d^4x\bar{E}(p,x)E(p',x) = (2\pi)^4\delta^4(p-p')\,,\qquad \int\frac{d^4p}{(2\pi)^4}\bar{E}(p,x)E(p,x') = \delta^4(x-x')\,. \tag{5.43}$$

The negative Volkov states are $V_s(p,x) = E(-p,x)v_s(p)$, where $v_s(p)$ are the free, negative-energy spinors normalized as $v_s^\dagger(p)v_{s'}(p) = 2\varepsilon\,\delta_{ss'}$.

It is important to know the electron propagator in a plane-wave background field which is known as the Volkov propagator and, it is the Green's function of the Dirac operator in a plane-wave background field, defined by

$$\Big[\not{\Pi}(\Phi) - m\Big]G(x,x') = \delta^{(4)}(x-x'). \tag{5.44}$$

Notice that the operator G is such that $G(x,x') = \langle x|G|x'\rangle$ or in other words

$$G = \frac{1}{\not{\Pi} - m + i0}\,. \tag{5.45}$$

Here, we have assumed the Feynman prescription corresponding to the shift $m \to m - i0$ [98]. In [103] is shown that the operator G can be written in the form (see Appendix B.3.2 for a short derivation of this result)

$$\begin{aligned} G = (\not{\Pi} + m)\frac{1}{\not{\Pi}^2 - m^2 + i0} &= (\not{\Pi} + m)(-i)\int_0^\infty ds\, e^{-im^2s}e^{2isP_TP_\phi} \\ &\times e^{-i\int_0^s ds'[\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi - 2s'P_T)]^2}\left\{1 - \frac{e}{2P_T}\not{n}[\not{A}(\Phi - 2sP_T) - \not{A}(\Phi)]\right\}. \end{aligned} \tag{5.46}$$

Equivalently, G can be written in the form

$$G = \frac{1}{\slashed{\Pi}^2 - m^2 + i0}(\slashed{\Pi} + m) = (-i)\int_0^\infty ds\, e^{-im^2 s}\Big\{1 + \frac{e}{2P_T}\slashed{n}[\slashed{A}(\Phi + 2sP_T) - \slashed{A}(\Phi)]\Big\} \\ \times e^{-i\int_0^s ds' [\boldsymbol{P}_\perp - e\boldsymbol{A}_\perp(\Phi + 2s'P_T)]^2} e^{2isP_T P_\phi}(\slashed{\Pi} + m)\,. \tag{5.47}$$

Alternatively, we can write the Volkov propagator in the Ritus representation [101] as

$$G(x, x') = \int \frac{d^4p}{(2\pi)^4}\, E(p, x)\, \frac{\slashed{p} + m}{p^2 - m^2 + i0}\, \bar{E}(p, x')\,. \tag{5.48}$$

For the present case of a plane-wave background field, we require only the eigenvalues of the position and momenta operators, which can be used to show that the momentum operator in presence of the external field satisfies (see Eq. (5.40))

$$\Pi^\lambda(\phi)U_s(p, x) = \left[\pi_p^\lambda(\phi) + i\frac{e\slashed{n}\slashed{A}'(\phi)}{2p_-}n^\lambda\right]U_s(p, x), \tag{5.49}$$

where

$$\pi_p^\lambda(\phi) = p^\lambda - eA^\lambda(\phi) + \frac{e(pA(\phi))}{p_-}n^\lambda - \frac{e^2A^2(\phi)}{2p_-}n^\lambda \tag{5.50}$$

is the classical kinetic four-momentum of an electron in the plane-wave $A^\mu(\phi)$, and due to the boundary conditions it satisfy $\lim_{\phi\to\pm\infty}\pi_p^\lambda(\phi) = p^\lambda$. The shifting relation (5.25) imply that

$$\mathrm{e}^{i\alpha P_\phi}\, f(\phi)U_s(p, x) = \mathrm{e}^{-i\alpha p_+}\, f(\phi + \alpha)\left[1 + \frac{e\slashed{n}[\slashed{A}(\phi + \alpha) - \slashed{A}(\phi)]}{2p_-}\right]\mathrm{e}^{-\int_\phi^{\phi+\alpha} d\varphi\left[\frac{e(pA(\varphi))}{p_-} - \frac{e^2A^2(\varphi)}{2p_-}\right]}U_s(p, x)\,. \tag{5.51}$$

In addition, it is possible to obtain the Gordon identity for the Volkov states (see Appendix B.3.3)

$$\bar{U}_{s'}(p', x)\gamma^\mu U_s(p, x) = \bar{U}_{s'}(p', x)\left[\frac{\pi_{p'}^\mu(\phi) + \pi_p^\mu(\phi)}{2m} + i\,\frac{\sigma^{\mu\nu}\,[\pi_{p'}(\phi) - \pi_p(\phi)]_\nu}{2m}\right]U_s(p, x)\,. \tag{5.52}$$

Note that, some quantities can be expressed as manifestly gauge-invariant by writing them in terms of the field strength tensor of the plane-wave

$$F^{\mu\nu}(\phi) = \partial^\mu A^\nu(\phi) - \partial^\nu A^\mu(\phi) = n^\mu A'^\nu(\phi) - n^\nu A'^\mu(\phi) \tag{5.53}$$

and its integral

$$\mathscr{F}^{\mu\nu}(\phi) = \int_{-\infty}^{\phi} d\phi' F^{\mu\nu}(\phi') = n^\mu A^\nu(\phi) - A^\mu(\phi)\, n^\nu\,, \tag{5.54}$$

which is gauge invariant as well. For instance, the kinetic four-momentum $\pi_p^\lambda(\phi)$ can be written in the manifestly gauge-invariant form as

$$\pi_p^\lambda(\phi) = p^\lambda - \frac{ep_\mu\mathscr{F}^{\mu\lambda}(\phi)}{p_-} + \frac{e^2p_\mu\mathscr{F}^{\mu\rho}(\phi)\mathscr{F}_{\rho\nu}(\phi)p^\nu}{2p_-^3}n^\lambda. \tag{5.55}$$

5.4.2 Feynman rules for Volkov states

In this section, we present the Feynman rules used to compute transition amplitudes in the presence of a plane-wave field within the Furry picture [87]. These rules, in the coordinate space, follow the same structure as in vacuum and they include the interaction with the plane-wave field exactly through the Volkov propagator and states.

The value of a diagram consist of the following factors:

- For each internal fermion propagator, $iG(x, x')$.
- For each internal photon propagator, $-iD^{\mu\nu}(x - x')$.
- For each fermion-fermion-photon vertex, $-ie\gamma^\mu$.

- For each incoming photon, $e_l^{\mu}(q)\,\mathrm{e}^{-i(qx)}$.
- For each outgoing photon, $e_l^{*\mu}(q)\,\mathrm{e}^{i(qx)}$.
- For each incoming fermion, $U_s(p,x)$.
- For each outgoing fermion, $\bar{U}_s(p,x)$.
- For each incoming anti-fermion, $\bar{V}_s(p,x)$.
- For each outgoing anti-fermion, $V_s(p,x)$.

Additionally, for each vertex, an integral over the space-time coordinates $\int d^4x$ must be included. These rules reduce to the vacuum Feynman rules for $A^{\mu}=0$.

5.5 One-loop vertex correction in a plane-wave field

In this section, we present the calculation of the one-loop vertex correction corresponding to the Feynman diagram in Fig. 5.2. For this amplitude, we assume that the external photon with four-momentum q^{μ} is outgoing and off-shell ($q^2 \neq 0$), the electron at the initial and final position is real, i.e., the four-momenta p and p' are on-shell ($p^2 = p'^{\,2} = m^2$). We denote by s (s') the spin quantum number of the incoming (outgoing) electron and by l the polarization quantum number of the outgoing photon. Then, using Feynman rules (see Section 5.4.2), the amplitude for the one-loop vertex correction in a plane-wave field can be expressed as

$$-ie\Gamma_{s,s',l}(p,p',q) = \int d^4x\, d^4y\, d^4z\, \bar{U}_{s'}(p',y)\; (-ie\gamma^{\lambda})\; iG(y,z)\; (-ie)\not{e}_l^{*}(q)\mathrm{e}^{i(qz)}\; iG(z,x) \\ \times (-ie\gamma^{\nu})\; U_s(p,x)\; (-i)D_{\lambda\nu}(x-y), \tag{5.56}$$

where, $e_l^{\mu}(q)$ is the polarization four-vector of the outgoing photon and, for the photon propagator (5.38), we choose the Feynman gauge ($\xi = 1$) such that

$$D^{\lambda\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\frac{\eta^{\lambda\nu}}{k^2-\kappa^2+i0}e^{-i(kx)}, \tag{5.57}$$

with κ^2 the square of a fictitious photon mass, which has been introduced to avoid infrared divergences.

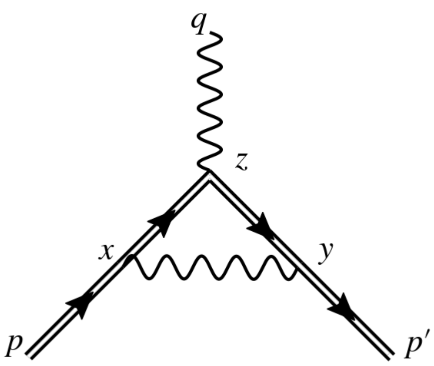

Figure 5.2: *Feynman diagram for the one-loop vertex correction in a plane-wave field. The double lines indicates that the electron/positron states and propagators include the exact interaction with plane-wave field.*

Now, we can choose to write the internal propagators as operators (see equations (5.46) and (5.47)) or as Green's functions (see equation (5.48)). We choose to use operators since with the use of the completeness relation in (5.9), we can immediately remove two of the integrals such that

$$-ie\Gamma_{s,s',l}(p,p',q) = -e^3\int d^4x \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-\kappa^2+i0}\,\bar{U}_{s'}(p',x)\mathrm{e}^{i(kx)}\gamma^{\lambda}G\mathrm{e}^{i(qx)}\not{e}_l^{*}(q)G\mathrm{e}^{-i(kx)}\gamma_{\lambda}U_s(p,x)\,. \tag{5.58}$$

Now the internal propagator can be expressed as in (5.46) and (5.47) to obtain a semi-operator representation of the amplitude

$$-ie\Gamma_{s,s',l}(p,p',q) = -e^3 \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\kappa^2+i0} \bar{U}_{s'}(p',x)\gamma^\lambda[\not{\Pi}(\phi)+\not{k}+m] \\ \times \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} \mathrm{e}^{i(qx)} \not{e}^*_l(q) \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} [\not{\Pi}(\phi)+\not{k}+m]\gamma_\lambda U_s(p,x)\,. \tag{5.59}$$

Here, we have used the translation properties in (5.23). Using the fact that the operator $\Pi^\mu(\phi)$ satisfy Dirac equation $[\not{\Pi}(\phi)-m]U_s(p,x) = [\not{\Pi}(\phi)-m]U_{s'}(p',x) = 0$, we obtain[3]

$$-ie\Gamma_{s,s',l}(p,p',q) = -e^3 \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\kappa^2+i0} \bar{U}_{s'}(p',x)[2\Pi^\lambda(\phi)+\gamma^\lambda\not{k}] \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} \\ \times \mathrm{e}^{i(qx)} \not{e}^*_l(q) \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} [2\Pi_\lambda(\phi)+\not{k}\gamma_\lambda]U_s(p,x). \tag{5.60}$$

We use the eigenvalue relation (5.49) for $\Pi^\mu(\phi)$ to obtain

$$-ie\Gamma_{s,s',l}(p,p',q) = -e^3 \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\kappa^2+i0} \bar{U}_{s'}(p',x) \left[2\pi^\lambda_{p'}(\phi) + i\frac{e\not{n}\not{A}'(\phi)}{p'_-}n^\lambda + \gamma^\lambda\not{k}\right] \\ \times \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} \mathrm{e}^{i(qx)} \not{e}^*_l(q) \frac{1}{[\not{\Pi}(\phi)+\not{k}]^2-m^2+i0} \\ \times \left[2\pi_{p,\lambda}(\phi) + i\frac{e\not{n}\not{A}'(\phi)}{p_-}n_\lambda + \not{k}\gamma_\lambda\right] U_s(p,x). \tag{5.61}$$

At this point, it is convenient to use the representation in (5.46) for the first square Volkov propagator and (5.47) for the second. Then, we obtain

$$-ie\Gamma_{s,s',l}(p,p',q) = e^3 \int d^4x \int \frac{d^4k}{(2\pi)^4} \int_0^\infty ds \int_0^\infty du \, \frac{e^{i(qx)}}{k^2-\kappa^2+i0} \\ \times \bar{U}_{s'}(p',x) \left[2\pi^\lambda_{p'}(\phi) + i\frac{e\not{n}\not{A}'(\phi)}{p'_-}n^\lambda + \gamma^\lambda\not{k}\right] e^{-im^2 s} e^{-2is(p_- - q_- + k_-)(P_\phi - k_+ + q_+)} \\ \times e^{-i\int_0^s ds' [\boldsymbol{p}_\perp - \boldsymbol{q}_\perp + \boldsymbol{k}_\perp - e\boldsymbol{A}_\perp(\phi + 2s'(p_- - q_- + k_-))]^2} \left\{1 + \frac{e\not{n}[\not{A}(\phi+2s(p_- - q_- + k_-)) - \not{A}(\phi)]}{2(p_- - q_- + k_-)}\right\} \\ \times \not{e}^*_l(q) e^{-im^2 u} \left\{1 - \frac{e\not{n}[\not{A}(\phi - 2u(p_- + k_-)) - \not{A}(\phi)]}{2(p_- + k_-)}\right\} e^{-i\int_0^u du' [\boldsymbol{p}_\perp + \boldsymbol{k}_\perp - e\boldsymbol{A}_\perp(\phi - 2u'(p_- + k_-))]^2} \\ \times e^{-2iu(p_- + k_-)(P_\phi - k_+)} \left[2\pi_{p,\lambda}(\phi) + i\frac{e\not{n}\not{A}'(\phi)}{p_-}n_\lambda + \not{k}\gamma_\lambda\right] U_s(p,x), \tag{5.62}$$

where we have exploited the fact that Volkov states are eigenstates of the operators P_T and $\boldsymbol{P}_\perp$ as in (5.18). Indeed, the only operator remaining in this equation is P_ϕ. Now, we use the translation property in (5.25) which imply (5.51) and, analogously to the vacuum case, we write the amplitude $-ie\Gamma_{s,s',l}(p,p',q)$ in the form

$$-ie\Gamma_{s,s',l}(p,p',q) = -ie \int d^4x \, e^{i(qx)} \bar{U}_{s'}(p',x)\Gamma^\mu(p,p',q;\phi)U_s(p,x)e^*_{l,\mu}(q), \tag{5.63}$$

[3] Here, we use the commutator $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$.

where

$$
\begin{aligned}
-ie\Gamma^\mu(p,p',q;\phi) = e^3 \int \frac{d^4k}{(2\pi)^4} \int_0^\infty ds \int_0^\infty du\, \frac{e^{-im^2(s+u)}}{k^2-\kappa^2+i0} \\
\times\, e^{2ik_+[s(p'_-+k_-)+u(p_-+k_-)]} e^{i\left\{p'_+(\phi_s-\phi)+\int_\phi^{\phi_s} d\phi' \left[-\frac{e\boldsymbol{p}'_\perp\cdot\boldsymbol{A}_\perp(\phi')}{p'_-}+\frac{e^2\boldsymbol{A}^2_\perp(\phi')}{2p'_-}\right]\right\}} \\
\times\, e^{-i\int_0^s ds'[\boldsymbol{p}'_\perp+\boldsymbol{k}_\perp-e\boldsymbol{A}_\perp(\phi_{s'})]^2} e^{-i\int_0^u du'[\boldsymbol{p}_\perp+\boldsymbol{k}_\perp-e\boldsymbol{A}_\perp(\phi_{u'})]^2} \\
\times\, e^{i\left\{-p_+(\phi_u-\phi)-\int_\phi^{\phi_u} d\phi' \left[-\frac{e\boldsymbol{p}_\perp\cdot\boldsymbol{A}_\perp(\phi')}{p_-}+\frac{e^2\boldsymbol{A}^2_\perp(\phi')}{2p_-}\right]\right\}} M^\mu(\phi,k,s,u).
\end{aligned}
\tag{5.64}
$$

Here, we have introduced the quantities

$$\phi_s = \phi + 2s(p'_- + k_-), \tag{5.65}$$

$$\phi_u = \phi - 2u(p_- + k_-), \tag{5.66}$$

and the matrix

$$
\begin{aligned}
M^\mu(k,s,u;\phi) = \left\{1 - \frac{e\not{n}[\not{A}(\phi_s)-\not{A}(\phi)]}{2p'_-}\right\} \left[2\pi^\lambda_{p'}(\phi_s) + i\frac{e\not{n}\not{A}'(\phi_s)}{p'_-}n^\lambda + \gamma^\lambda\not{k}\right] \\
\times \left\{1 + \frac{e\not{n}[\not{A}(\phi_s)-\not{A}(\phi)]}{2(p'_-+k_-)}\right\} \gamma^\mu \left\{1 - \frac{e\not{n}[\not{A}(\phi_u)-\not{A}(\phi)]}{2(p_-+k_-)}\right\} \\
\times \left[2\pi_{p,\lambda}(\phi_u) + i\frac{e\not{n}\not{A}'(\phi_u)}{p_-}n_\lambda + \not{k}\gamma_\lambda\right] \left\{1 + \frac{e\not{n}[\not{A}(\phi_u)-\not{A}(\phi)]}{2p_-}\right\}.
\end{aligned}
\tag{5.67}
$$

The phase in (5.64) can be written in a compact form by turning the integral from ϕ to ϕ_s (from ϕ to ϕ_u) into an integral in s' (u') like that in the third line of (5.64). Using Schwinger parameters, we can exponentiate the denominator $k^2 - \kappa^2 + i0$ in the photon propagator and then the quantity $\Gamma^\mu(p,p',q;\phi)$ can be written as

$$\Gamma^\mu(p,p',q;\phi) = e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^\infty ds \int_0^\infty du \int_0^\infty dt\, e^{iSk^2 - i\kappa^2 t + 2i(k\tilde{F})} M^\mu(k,s,u;\phi), \tag{5.68}$$

where $S = u + s + t$ and

$$\tilde{F}^\mu = \int_0^s ds' \pi^\mu_{p'}(\phi_{s'}) + \int_0^u du' \pi^\mu_p(\phi_{u'}). \tag{5.69}$$

As next step, we can perform the integrals in d^4k analytically by shifting the four-momentum k^μ by setting $k'^\mu = k^\mu + \tilde{F}^\mu/S$, which, since all components of $\tilde{F}^\mu$ except $\tilde{F}_-$ depend on k_-, implies that $k^\mu = k'^\mu - \tilde{G}^\mu/S$, where

$$\tilde{G}^\mu = \int_0^s ds' \pi^\mu_{p'}(\tilde{\psi}_{s'}) + \int_0^u du' \pi^\mu_p(\tilde{\psi}_{u'}), \tag{5.70}$$

such that $\tilde{G}_- = \tilde{F}_- = sp'_- + up_-$. Here, we have introduced the two shifted phases

$$\tilde{\psi}_s = \phi + 2s\tau'_- + 2sk_-, \tag{5.71}$$

$$\tilde{\psi}_u = \phi - 2u\tau_- - 2uk_-, \tag{5.72}$$

where

$$\tau'_- = p'_- - \frac{\tilde{G}_-}{S} = \frac{tp'_- - uq_-}{S}, \tag{5.73}$$

$$\tau_- = p_- - \frac{\tilde{G}_-}{S} = \frac{tp_- + sq_-}{S}. \tag{5.74}$$

After the shift of the four-momentum k^μ, we can write $\Gamma^\mu(p,p',q;\phi)$ in the form

$$\Gamma^\mu(p,p',q;\phi) = e^2 \int_0^\infty dsdudt \int \frac{d^4k}{(2\pi)^4} e^{-i\kappa^2 t - i\frac{\tilde{G}^2}{S} + iSk^2} \tilde{L}(\tilde{Q}'^\lambda + \gamma^\lambda\not{k})\tilde{C}^\mu(\tilde{Q}_\lambda + \not{k}\gamma_\lambda)\tilde{R}, \tag{5.75}$$

where

$$\tilde{L} = 1 - \frac{e\not{n}[\not{A}(\tilde{\psi}_s) - \not{A}(\phi)]}{2p'_-}, \tag{5.76}$$

$$\tilde{Q}^\lambda = 2\pi_p^\lambda(\tilde{\psi}_u) + i\frac{e\not{n}\not{A}'(\tilde{\psi}_u)}{p_-}n^\lambda - \frac{\tilde{\not{G}}}{S}\gamma^\lambda, \tag{5.77}$$

$$\tilde{C}^\mu = \left\{1 + \frac{e\not{n}[\not{A}(\tilde{\psi}_s) - \not{A}(\phi)]}{2(\tau'_- + k_-)}\right\}\gamma^\mu\left\{1 - \frac{e\not{n}[\not{A}(\tilde{\psi}_u) - \not{A}(\phi)]}{2(\tau_- + k_-)}\right\}, \tag{5.78}$$

$$\tilde{Q}'^\lambda = 2\pi_{p'}^\lambda(\tilde{\psi}_s) + i\frac{e\not{n}\not{A}'(\tilde{\psi}_s)}{p'_-}n^\lambda - \gamma^\lambda\frac{\tilde{\not{G}}}{S}, \tag{5.79}$$

$$\tilde{R} = 1 + \frac{e\not{n}[\not{A}(\tilde{\psi}_u) - \not{A}(\phi)]}{2p_-}. \tag{5.80}$$

The integral in d^4k in $\Gamma^\mu(p,p',q;\phi)$ is complicated by the fact that the variable k_- is contained in the argument of the four-vector potential of the plane-wave. Thus, we first compute the integral in $d^2\boldsymbol{k}_\perp$, which is Gaussian, so that we obtain

$$\Gamma^\mu(p,p',q;\phi) = -i\alpha\int_0^\infty \frac{dsdudt}{S}\int\frac{dk_-dk_+}{(2\pi)^2}e^{-i\kappa^2 t - i\frac{\tilde{G}^2}{S} + 2iSk_-k_+}\tilde{M}^\mu(k_-,k_+,s,u,t;\phi), \tag{5.81}$$

where

$$\begin{aligned}\tilde{M}^\mu(k_-,k_+,s,u,t;\phi) =& \tilde{L}\left[(\tilde{Q}'^\lambda + k_-\gamma^\lambda\not{n})\tilde{C}^\mu(\tilde{Q}_\lambda + k_-\not{n}\gamma_\lambda) - \frac{i}{2S}\gamma^\lambda\gamma_{\perp,i}\tilde{C}^\mu\gamma_{\perp,i}\gamma_\lambda\right]\tilde{R}\\ &+ k_+\tilde{L}[\gamma^\lambda\not{n}\tilde{C}^\mu(\tilde{Q}_\lambda + k_-\not{n}\gamma_\lambda) + (\tilde{Q}'^\lambda + k_-\gamma^\lambda\not{n})\tilde{C}^\mu\not{n}\gamma_\lambda]\tilde{R} + k_+^2\tilde{L}\gamma^\lambda\not{n}\gamma^\mu\not{n}\gamma_\lambda\tilde{R}\,.\end{aligned} \tag{5.82}$$

Finally, the integral in dk_+ results in a delta function and its first and second derivatives all evaluated at $2Sk_-$. This allows us to then compute the integral in dk_- and, after straightforward manipulations, the resulting expression of $\Gamma^\mu(p,p',q;\phi)$ can be written as

$$\begin{aligned}\Gamma^\mu(p,p',q;\phi) = -\frac{i\alpha}{4\pi}\int_0^\infty \frac{dsdudt}{S^3}\,e^{-i\kappa^2 t}\Bigg\{& e^{-i\frac{\tilde{G}^2}{S}}\tilde{L}(S\tilde{Q}'^\lambda\tilde{C}^\mu\tilde{Q}_\lambda + 2i\tilde{C}^\mu)\tilde{R}\\ &+\frac{i}{2}\frac{d}{dk_-}\left[e^{-i\frac{\tilde{G}^2}{S}}\tilde{L}(\gamma^\lambda\not{n}\tilde{C}^\mu\tilde{Q}_\lambda + \tilde{Q}'^\lambda\tilde{C}^\mu\not{n}\gamma_\lambda)\tilde{R}\right] + \frac{\not{n}}{S}n^\mu\frac{d^2}{dk_-^2}\left(e^{-i\frac{\tilde{G}^2}{S}}\right)\Bigg\}\Bigg|_{k_-=0},\end{aligned} \tag{5.83}$$

and after taking the derivative of the exponential, we get

$$\begin{aligned}\Gamma^\mu(p,p',q;\phi) =& -\frac{i\alpha}{4\pi}\int_0^\infty\frac{dsdudt}{S^3}\,e^{-i\kappa^2 t - i\frac{\tilde{G}^2}{S}}\\ &\times\Bigg\{\tilde{L}\left[S\tilde{Q}'^\lambda\tilde{C}^\mu\tilde{Q}_\lambda + 2i\tilde{C}^\mu + \frac{1}{2S}\frac{d\tilde{G}^2}{dk_-}(\gamma^\lambda\not{n}\tilde{C}^\mu\tilde{Q}_\lambda + \tilde{Q}'^\lambda\tilde{C}^\mu\not{n}\gamma_\lambda)\right]\tilde{R}\\ &+\frac{i}{2}\frac{d}{dk_-}\left[\tilde{L}(\gamma^\lambda\not{n}\tilde{C}^\mu\tilde{Q}_\lambda + \tilde{Q}'^\lambda\tilde{C}^\mu\not{n}\gamma_\lambda)\tilde{R}\right] - \frac{\not{n}}{S^2}n^\mu\left[\frac{1}{S}\left(\frac{d\tilde{G}^2}{dk_-}\right)^2 + i\frac{d^2\tilde{G}^2}{dk_-^2}\right]\Bigg\}\Bigg|_{k_-=0}.\end{aligned} \tag{5.84}$$

This expression can be further manipulated especially to simplify its matrix structure. However, it is first convenient to make the following considerations related to the Ward identity to be fulfilled by $\Gamma^\mu(p,p',q;\phi)$ [150]. From now on we assume that $q_- > 0$. Thus, by using the three four-vectors

$$N^\mu = q^\mu - \frac{q^2 n^\mu}{2q_-}, \tag{5.85}$$

$$\Lambda_i^\mu = a_i^\mu + \frac{q_{\perp,i}n^\mu}{q_-}, \tag{5.86}$$

with $i = 1,2$ together with n^μ, one can build a light-cone basis such that

$$\eta^{\mu\nu} = \frac{N^\mu n^\nu + n^\mu N^\nu}{q_-} - \Lambda_1^\mu \Lambda_1^\nu - \Lambda_2^\mu \Lambda_2^\nu. \tag{5.87}$$

Then, taking into account that we work in the Lorenz gauge where $(qe_l^*(q)) = 0$, the quantity $\Gamma^\mu(p,p',q;\phi)e^*_{l,\mu}(q)$ can be written as

$$\begin{aligned}\Gamma^\mu(p,p',q;\phi)e^*_{l,\mu}(q) = &\frac{e^*_{l,-}(q)}{q_-}\Gamma_q(p,p',q;\phi) - \frac{q^2 e^*_{l,-}(q)}{q_-^2}\Gamma_-(p,p',q;\phi)\\ &- (\Gamma(p,p',q;\phi)\Lambda_1)(\Lambda_1 e_l^*(q)) - (\Gamma(p,p',q;\phi)\Lambda_2)(\Lambda_2 e_l^*(q)),\end{aligned} \tag{5.88}$$

where $\Gamma_q(p,p',q;\phi) = (\Gamma^\mu(p,p',q;\phi)q_\mu)$ and $\Gamma_-(p,p',q;\phi) = (\Gamma^\mu(p,p',q;\phi)n_\mu)$. Notice that due to the gauge invariance of QED the term $\Gamma_q(p,p',q;\phi)$ does not contribute to any transition amplitude since, due to the Ward identity, this term cancels out exactly (see [110] for a detailed proof). Then, from now on we can forget about the term $\Gamma_q(p,p',q;\phi)$ and express the amplitude as

$$\begin{aligned}\Gamma^\mu(p,p',q;\phi)e^*_{l,\mu}(q) = &-\left[\frac{q^2 e^*_{l,-}(q)}{q_-^2} + \frac{q_{\perp,1}}{q_-}(\Lambda_1 e_l^*(q)) + \frac{q_{\perp,2}}{q_-}(\Lambda_2 e_l^*(q))\right]\Gamma_-(p,p',q;\phi)\\ &+ \Gamma_{\perp,1}(p,p',q;\phi)(\Lambda_1 e_l^*(q)) + \Gamma_{\perp,2}(p,p',q;\phi)(\Lambda_2 e_l^*(q)),\end{aligned} \tag{5.89}$$

with $\Gamma_{\perp,j}(p,p',q;\phi) = -(\Gamma^\mu(p,p',q;\phi)a_{j,\mu})$. Now, we should notice that it is easier to work with the terms $\Gamma_-(p,p',q;\phi)$ and $\Gamma_{\perp,j}(p,p',q;\phi)$ since in these components many terms will cancel due to the quantities n^μ and a_i^μ.

First, we consider the component $\Gamma_-(p,p',q;\phi)$, whose structure is particularly easy. In fact, starting from (5.83), we have that[4]

$$\begin{aligned}\Gamma_-(p,p',q;\phi) &= -\frac{i\alpha}{4\pi}\int_0^\infty \frac{dsdudt}{S^3}\, e^{-i\kappa^2 t - i\frac{G^2}{S}}\left(SLQ'^\lambda \not{n} Q_\lambda R + 2i\not{n}\right)\\ &= -\frac{i\alpha}{2\pi}\int_0^\infty \frac{dsdudt}{S^3}\, e^{-i\kappa^2 t - i\frac{G^2}{S}}\left[\left(2S(\pi_s\pi_u) + \frac{G^2}{S} + i\right)\not{n} - 2G_-(\not{\pi}_s R + L\not{\pi}_u)\right.\\ &\left. -\not{G}\not{\pi}_s\not{n} - \not{n}\not{\pi}_u\not{G} + 2\tau_- L\not{G} + 2\tau'_-\not{G}R + 2\frac{G_-}{S}\not{G} - \frac{G_-^2}{S}\frac{\not{\Delta}_s\not{n}\not{\Delta}_u}{p_-p'_-}\right],\end{aligned} \tag{5.90}$$

where

$$\pi_s^\mu = \pi_{p'}^\mu(\psi_s), \quad \Delta_s^\mu = e[A^\mu(\psi_s) - A^\mu(\phi)], \qquad L = 1 - \frac{\not{n}\not{\Delta}_s}{2p'_-}, \quad \psi_s = \phi + 2s\tau'_-, \tag{5.91}$$

$$\pi_u^\mu = \pi_p^\mu(\psi_u), \quad \Delta_u^\mu = e[A^\mu(\psi_u) - A^\mu(\phi)], \qquad R = 1 + \frac{\not{n}\not{\Delta}_u}{2p_-}, \quad \psi_u = \phi - 2u\tau_-, \tag{5.92}$$

$$C^\mu = \left(1 + \frac{\not{n}\not{\Delta}_s}{2\tau'_-}\right)\gamma^\mu\left(1 - \frac{\not{n}\not{\Delta}_u}{2\tau_-}\right), \qquad G^\mu = \int_0^s ds'\pi_{p'}^\mu(\psi_{s'}) + \int_0^u du'\pi_p^\mu(\psi_{u'}), \tag{5.93}$$

$$Q^\lambda = 2\pi_u^\lambda + i\frac{e\not{n}\not{A}'(\psi_u)}{p_-}n^\lambda - \frac{\not{G}}{S}\gamma^\lambda, \qquad Q'^\lambda = 2\pi_s^\lambda + i\frac{e\not{n}\not{A}'(\psi_s)}{p'_-}n^\lambda - \gamma^\lambda\frac{\not{G}}{S}. \tag{5.94}$$

Finally, for the calculation of the components $\Gamma_{\perp,j}(p,p',q;\phi)$ we can effectively assume that the matrix $\not{n}$ anticommutes with γ^μ. In the following four equations, with an abuse of notation, we use the equal symbol also for two matrices that are equal to each other up to terms proportional to n^μ, which can anyway be ignored in the computation of $\Gamma_{\perp,j}(p,p',q;\phi)$. Going through the terms in Eq.

[4]The quantities with and without 'tilde' are related after setting $k_- = 0$, for instance, $\tilde{G}|_{k_-=0} = G$

(5.84) in order of complexity, one can easily show that

$$LC^\mu R = \gamma^\mu + \frac{G_-}{2p'_-\tau'_- S}\not{p}\not{\Delta}_s\gamma^\mu - \frac{G_-}{2p_-\tau_- S}\gamma^\mu\not{p}\not{\Delta}_u\,, \tag{5.95}$$

$$\begin{aligned} L(\gamma^\lambda\not{p}C^\mu Q_\lambda + Q'^\lambda C^\mu\not{p}\gamma_\lambda)R = &-\frac{2\tau_-}{p'_-}\not{p}\not{\Delta}_s\gamma^\mu + \frac{2\tau'_-}{p_-}\gamma^\mu\not{p}\not{\Delta}_u - \frac{4G_-}{S}\gamma^\mu + \frac{4G^\mu}{S}\not{p} \\ &- 2\not{p}\gamma^\mu\not{\pi}_s - 2\not{\pi}_u\gamma^\mu\not{p}\,, \end{aligned} \tag{5.96}$$

$$\begin{aligned} \frac{d}{dk_-}\left[\tilde{L}(\gamma^\lambda\not{p}\tilde{C}^\mu\tilde{Q}_\lambda + \tilde{Q}'^\lambda\tilde{C}^\mu\not{p}\gamma_\lambda)\tilde{R}\right]_{k_-=0} = &\, 8\left(\frac{G_1^\mu}{S} - \frac{s\tau_-}{p'_-}\mathcal{A}'^\mu_s + \frac{u\tau'_-}{p_-}\mathcal{A}'^\mu_u\right)\not{p} \\ &+ 4s\left(1+\frac{\tau_-}{p'_-}\right)\not{p}\gamma^\mu\not{\mathcal{A}}'_s - 4u\left(1+\frac{\tau'_-}{p_-}\right)\not{\mathcal{A}}'_u\gamma^\mu\not{p} \end{aligned} \tag{5.97}$$

and

$$\begin{aligned} LQ'^\lambda C^\mu Q_\lambda R = &\, 4(\pi_s\pi_u)\left(\gamma^\mu + \frac{G_-}{2p'_-\tau'_- S}\not{p}\not{\Delta}_s\gamma^\mu - \frac{G_-}{2p_-\tau_- S}\gamma^\mu\not{p}\not{\Delta}_u\right) \\ &+ 2i\frac{\tau_-}{p'_-}\not{p}\not{\mathcal{A}}'_s\gamma^\mu + 2i\frac{\tau'_-}{p_-}\gamma^\mu\not{p}\not{\mathcal{A}}'_u - \frac{2}{S}LC^\mu\not{G}\not{\pi}_s R - \frac{2}{S}L\not{\pi}_u\not{G}C^\mu R \\ &- \frac{2}{S^2}L\not{G}\left(\gamma^\mu + \frac{\gamma^\mu\not{\Delta}_s\not{p}}{2\tau'_-} - \frac{\not{\Delta}_u\not{p}\gamma^\mu}{2\tau_-}\right)\not{G}R\,. \end{aligned} \tag{5.98}$$

Here, we have used integration by parts to re-express $\left.\frac{d\tilde{G}}{dk_-}\right|_{k_-=0}$ as twice G_1:

$$\begin{aligned} G_1^\mu &= \frac{d}{d\phi}\left[\int_0^s ds'\, s'\pi^\mu_{p'}(\psi_{s'}) - \int_0^u du'\, u'\pi^\mu_p(\psi_{u'})\right] \\ &= \frac{1}{2\tau'_-}\left[s\pi^\mu_{p'}(\psi_s) - \int_0^s ds'\pi^\mu_{p'}(\psi_{s'})\right] + \frac{1}{2\tau_-}\left[u\pi^\mu_p(\psi_u) - \int_0^u du'\pi^\mu_p(\psi_{u'})\right]\,, \end{aligned} \tag{5.99}$$

we have further introduced

$$\mathcal{A}^\mu_{s/u} = eA^\mu(\psi_{s/u})\,, \tag{5.100}$$

for which the prime on these quantities indicates the derivative with respect to ϕ.

Then, putting all together, we obtain the following expressions of the transverse components $\Gamma_{\perp,i}(p,p',q;\phi)$:

$$\begin{aligned} \Gamma_{\perp,j}(p,p',q;\phi) = &\frac{i\alpha}{2\pi}\int_0^\infty \frac{dsdudt}{S^3}\, e^{-i\kappa^2 t - i\frac{G^2}{S}} \\ &\times\Bigg\{(2S(\pi_s\pi_u)+i)\left(\not{a}_j + \frac{G_-}{2p'_-\tau'_- S}\not{p}\not{\Delta}_s\not{a}_j - \frac{G_-}{2p_-\tau_- S}\not{a}_j\not{p}\not{\Delta}_u\right) \\ &- L(Ca_j)\not{G}\not{\pi}_s R - L\not{\pi}_u\not{G}(Ca_j)R - \frac{1}{S}L\not{G}\left(\not{a}_j + \frac{\not{a}_j\not{\Delta}_s\not{p}}{2\tau'_-} - \frac{\not{\Delta}_u\not{p}\not{a}_j}{2\tau_-}\right)\not{G}R \\ &- \frac{2(GG_1)}{S}\left(\frac{\tau_-}{p'_-}\not{p}\not{\Delta}_s\not{a}_j - \frac{\tau'_-}{p_-}\not{a}_j\not{p}\not{\Delta}_u + \frac{2G_-}{S}\not{a}_j - \frac{2(Ga_j)}{S}\not{p} + \not{p}\not{a}_j\not{\pi}_s + \not{\pi}_u\not{a}_j\not{p}\right) \\ &+ 2i\left(\frac{(G_1a_j)}{S} + s(\mathcal{A}'_s a_j) - u(\mathcal{A}'_u a_j)\right)\not{p} \\ &+ i\left[s - (u+t)\frac{\tau_-}{p'_-}\right]\not{\mathcal{A}}'_s\not{p}\not{a}_j - i\left[u - (s+t)\frac{\tau'_-}{p_-}\right]\not{a}_j\not{p}\not{\mathcal{A}}'_u\Bigg\}\,. \end{aligned} \tag{5.101}$$

Finally, we observe that all the terms in equations (5.89), (5.90), and (5.101) have at most three gamma matrices except the three terms on the third line of (5.101). These three terms can be easily reduced to expressions containing at most five gamma matrices. For the first, we have

$$\begin{aligned} L(Ca_j)\not{C}\not{\pi}_s R &= \left(\not{a}_j + \frac{G_-}{2Sp'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j - \frac{G_-}{2Sp_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)\not{C}\not{\pi}_s \\ &+ \left(\not{a}_j + \frac{G_-}{2Sp'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j\right)(p'_-\not{C} - G_-\not{\pi}_s)\frac{\not{\Delta}_u}{p_-} \\ &+ \frac{\not{a}_j\not{n}}{2p_-\tau_-}[2(G_-(\Delta_u\pi_s) - p'_-(\Delta_u G))\not{\Delta}_u - (G_-\Delta_u^2 + 2\tau_-(\Delta_u G))\not{\pi}_s + (p'_-\Delta_u^2 + 2\tau_-(\Delta_u\pi_s))\not{C}]\,. \end{aligned} \tag{5.102}$$

For the second, we have

$$\begin{aligned} L\not{\pi}_u\not{G}(Ca_j)R &= \not{\pi}_u\not{G}\left(\not{a}_j + \frac{G_-}{2Sp'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j - \frac{G_-}{2Sp_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right) \\ &+ \frac{\not{\Delta}_s}{p'_-}(p_-\not{C} - G_-\not{\pi}_u)\left(\not{a}_j - \frac{G_-}{2Sp_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right) \\ &+ [2(G_-(\Delta_s\pi_u) - p_-(\Delta_s G))\not{\Delta}_s - (G_-\Delta_s^2 + 2\tau'_-(\Delta_s G))\not{\pi}_u + (p_-\Delta_s^2 + 2\tau'_-(\Delta_s\pi_u))\not{C}]\frac{\not{n}\not{a}_j}{2p'_-\tau'_-}\,. \end{aligned} \tag{5.103}$$

And for the third, we have

$$\begin{aligned} &L\not{G}\left(\not{a}_j + \frac{\not{a}_j\not{\Delta}_s\not{n}}{2\tau'_-} - \frac{\not{\Delta}_u\not{n}\not{a}_j}{2\tau_-}\right)\not{G}R = \frac{G_-}{2}\left(\frac{\not{G}\not{a}_j\not{\Delta}_s\not{n}\not{\Delta}_u}{p_-\tau'_-} + \frac{\not{\Delta}_s\not{n}\not{\Delta}_u\not{a}_j\not{G}}{p'_-\tau_-}\right) + G_-\left(\frac{\not{a}_j\not{\Delta}_s\not{G}}{\tau'_-} + \frac{\not{G}\not{\Delta}_u\not{a}_j}{\tau_-}\right) \\ &+ \left[(p'_- + \tau'_-)(Ga_j) + G_-(\Delta_s a_j)\right]\frac{\not{\Delta}_s\not{n}\not{G}}{p'_-\tau'_-} + [(p_- + \tau_-)(Ga_j) + G_-(\Delta_u a_j)]\frac{\not{G}\not{n}\not{\Delta}_u}{p_-\tau_-} - G^2\not{a}_j \\ &- \left[(p'_- + \tau'_-)G^2 + \frac{\tau'_-G_-^2}{p_-\tau_-}\Delta_u^2\right]\frac{\not{\Delta}_s\not{n}\not{a}_j}{2p'_-\tau'_-} - \left[(p_- + \tau_-)G^2 + \frac{\tau_-G_-^2}{p'_-\tau'_-}\Delta_s^2\right]\frac{\not{a}_j\not{n}\not{\Delta}_u}{2p_-\tau_-} + 2(Ga_j)\not{G} \\ &- [G_-\Delta_s^2 + 2p'_-(G\Delta_s)]\frac{\not{a}_j\not{n}\not{G}}{2p'_-\tau'_-} - [G_-\Delta_u^2 + 2p_-(G\Delta_u)]\frac{\not{G}\not{n}\not{a}_j}{2p_-\tau_-} \\ &+ \frac{G_-}{p_-p'_-}\left[(Ga_j) + \frac{G_-}{\tau'_-}(\Delta_s a_j) + \frac{G_-}{\tau_-}(\Delta_u a_j)\right]\not{\Delta}_s\not{n}\not{\Delta}_u - G^2\left[\frac{(\Delta_s a_j)}{\tau'_-} + \frac{(\Delta_u a_j)}{\tau_-}\right]\not{n}\,. \end{aligned} \tag{5.104}$$

In order to compute, for example, the probability of nonlinear Compton scattering at second order in α with the above obtained amplitude, it is required to square the sum of the leading order correction in Fig. 5.3 and all the radiative corrections at one-loop order (corresponding to the diagrams in Figs. 5.2 and 5.6). And, to compute the square of such quantities it is necessary to calculate the trace of gamma matrices along with the remaining integrals. However, to simplify this task and avoid lengthy calculations during the final stage, it is important to simplify the matrix structure of the amplitude as much as possible.

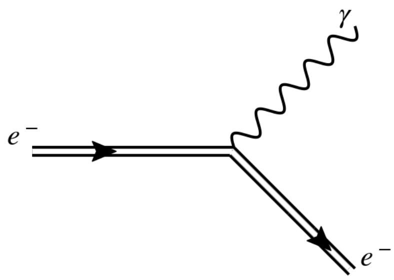

Figure 5.3: *Feynman diagram for the leading contribution to nonlinear Compton scattering. The double lines represent the exact electron states in a plane-wave field (Volkov states).*

Note that, we can further reduce the number of gamma matrices, in the terms with three or more, by using following the identity

$$\begin{aligned} \not{A}\not{B}\not{C} &= \frac{1}{4}\mathrm{tr}(\gamma_\mu \not{A}\not{B}\not{C})\gamma^\mu - \frac{1}{4}\mathrm{tr}(\gamma^5\gamma_\mu \not{A}\not{B}\not{C})\gamma^5\gamma^\mu \\ &= \not{A}(BC) - \not{B}(AC) + \not{C}(AB) + i\varepsilon_{\mu\nu\lambda\rho}\gamma^5\gamma^\mu A^\nu B^\lambda C^\rho, \end{aligned} \tag{5.105}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon^{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor with $\varepsilon^{0123} = +1$, which is valid for three arbitrary four-vectors A^μ, B^μ, and C^μ. However, this reduces the number of gamma matrices but increases the number of terms.

In the next section, we will further investigate the structure of the vertex correction and discuss its divergences.

5.6 Renormalization of the one-loop vertex correction

We can see that if we expand the vertex correction $\Gamma_{s,s',l}(p,p',q)$ respect to the vector potential, the only ultraviolet divergence will be the same present in vacuum. Then, the regularization of the vertex correction can be carried out in exactly the same way as in vacuum. Since the Volkov states for the limit of zero fields reduce to the vacuum states, we can perform the regularization in the quantity $\Gamma^\mu(p,p',q;\phi)$ by subtracting the vacuum expression $\Lambda^{(1)\mu}(p,p')$ in (5.31) evaluated for $q^\mu = 0$ and for $\not{p} = \not{p}' = m$ (see [150] for the vacuum case and [108, 109] for the constant crossed field case). Therefore the regularized amplitude for the vertex correction is obtained via

$$\Gamma^\mu_R(p,p',q;\phi) = \Gamma^\mu(p,p',q;\phi) - \Lambda^{(1)\mu}(p,p)|_{\not{p}=m}\,, \tag{5.106}$$

where $\Lambda^{(1)\mu}(p,p)|_{\not{p}=m}$ is given by (5.36). From the expression (5.36) and since $q^\mu = 0$, it is clear that $\Lambda^{(1)\mu}(p,p)|_{\not{p}=m}$ has only components $\Lambda^{(1)\mu}_-(p,p)|_{\not{p}=m}$ and $\Lambda^{(1)\mu}_{\perp,j}(p,p)|_{\not{p}=m}$, and then that $\Gamma_{R,q}(p,p',q;\phi) = \Gamma_q(p,p',q;\phi)$, which can be shown to vanish for $A^\mu(\phi) = 0$.

Now, we would like to investigate the convergence properties of the proper time integrals in $\Gamma_{R,-}(p,p',q;\phi)$ and $\Gamma_{R,\perp,j}(p,p',q;\phi)$. It is first convenient to use the following identity [123]

$$\begin{aligned} \int_0^\infty ds \int_0^\infty du \int_0^\infty dt &= \int_0^\infty ds \int_0^\infty du \int_0^\infty dt \int_0^\infty dS\, \delta(S-s-u-t) \\ &= \int_0^\infty dS \int_0^S ds \int_0^S du \int_0^S dt\, \delta(S-s-u-t) \\ &= \int_0^\infty dS\, S^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z), \end{aligned} \tag{5.107}$$

where in the last line we performed the changes of variables $s = xS$, $u = yS$, and $t = zS$. By setting

$$\int_\delta dxdydz = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z), \tag{5.108}$$

it is instructive to report the expression of $\Lambda^{(1)\mu}(p,p)|_{\not{p}=m}$ in terms of the new variables:

$$\Lambda^{(1)\mu}(p,p)|_{\not{p}=m} = -i\frac{\alpha}{2\pi}\gamma^\mu \int_0^\infty dS \int_\delta dxdydz\, e^{-i\kappa^2 zS - im^2(x+y)^2 S}\left\{m^2[2z-(x+y)^2] + \frac{i}{S}\right\}, \tag{5.109}$$

because it clearly shows that only the term whose integrand is proportional to i is (logarithmically) divergent (in the limit $S \to 0$). This divergence is related with the ultraviolet logarithmic divergence of the vertex-correction function. Keeping in mind that $z = 1-x-y$, see (5.108), another divergence for $x+y \to 0$ arises for a massless photon ($\kappa^2 = 0$), which corresponds to the infrared divergence of the vertex-correction function. By means of the above change of variables, we obtain

$$
\begin{aligned}
\Gamma_{R,-}(p,p',q;\phi) &= \frac{\alpha}{2\pi}\not{n}\int_0^\infty \frac{dS}{S}\int_\delta dxdydz\, e^{-i\kappa^2 zS}\left[e^{-ig^2S}-e^{-im^2(x+y)^2S}\right]\\
&-\frac{i\alpha}{2\pi}\int_0^\infty dS\int_\delta dxdydz\, e^{-i\kappa^2 zS}\Bigg\{e^{-ig^2S}\Bigg[\left(2(\pi_s\pi_u)+g^2\right)\not{n}-2g_-(\not{\pi}_sR+L\not{\pi}_u)-\not{g}\not{\pi}_s\not{n}-\not{n}\not{\pi}_u\not{g}\\
&+2\tau_-L\not{g}+2\tau'_-\not{g}R+2g_-\not{g}-g_-^2\frac{\not{\Delta}_s\not{n}\not{\Delta}_u}{p_-p'_-}\Bigg]-m^2\not{n}e^{-im^2(x+y)^2S}[2z-(x+y)^2]\Bigg\},
\end{aligned}
\tag{5.110}
$$

and

$$
\begin{aligned}
\Gamma_{R,\perp,j}(p,p',q;\phi) &= -\frac{\alpha}{2\pi}\not{a}_j\int_0^\infty\frac{dS}{S}\int_\delta dxdydz\, e^{-i\kappa^2 zS}\left[e^{-ig^2S}-e^{-im^2(x+y)^2S}\right]\\
&+\frac{i\alpha}{2\pi}\int_0^\infty dS\int_\delta dxdydz\, e^{-i\kappa^2 zS}\Big\langle e^{-ig^2S}\Big\{2(\pi_s\pi_u)\left(\not{a}_j+\frac{g_-}{2p'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j-\frac{g_-}{2p_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)\\
&+\frac{i}{S}\left(\frac{g_-}{2p'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j-\frac{g_-}{2p_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)-L(Ca_j)\not{g}\not{\pi}_sR-L\not{\pi}_u\not{g}(Ca_j)R\\
&-L\not{g}\left(\not{a}_j+\frac{\not{a}_j\not{\Delta}_s\not{n}}{2\tau'_-}-\frac{\not{\Delta}_u\not{n}\not{a}_j}{2\tau_-}\right)\not{g}R-2S(gg_1)\left(\frac{\tau_-}{p'_-}\not{n}\not{\Delta}_s\not{a}_j-\frac{\tau'_-}{p_-}\not{a}_j\not{n}\not{\Delta}_u+2g_-\not{a}_j-2(ga_j)\not{n}\right.\\
&\left.+\not{n}\not{a}_j\not{\pi}_s+\not{\pi}_u\not{a}_j\not{n}\right)+2i\left((g_1a_j)+x(\mathcal{A}'_sa_j)-y(\mathcal{A}'_ua_j)\right)\not{n}+i\left[x-(y+z)\frac{\tau_-}{p'_-}\right]\not{\mathcal{A}}'_s\not{n}\not{a}_j\\
&-i\left[y-(x+z)\frac{\tau'_-}{p_-}\right]\not{a}_j\not{n}\not{\mathcal{A}}'_u\Big\}-m^2\not{a}_je^{-im^2(x+y)^2S}[2z-(x+y)^2]\Big\rangle,
\end{aligned}
\tag{5.111}
$$

where

$$
\begin{aligned}
L(Ca_j)\not{g}\not{\pi}_sR &= \left(\not{a}_j+\frac{g_-}{2p'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j-\frac{g_-}{2p_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)\not{g}\not{\pi}_s\\
&+\left(\not{a}_j+\frac{g_-}{2p'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j\right)\left(p'_-\not{g}-g_-\not{\pi}_s\right)\frac{\not{\Delta}_u}{p_-}\\
&+\frac{\not{a}_j\not{n}}{2p_-\tau_-}[2(g_-(\Delta_u\pi_s)-p'_-(\Delta_ug))\not{\Delta}_u-(g_-\Delta_u^2+2\tau_-(\Delta_ug))\not{\pi}_s+(p'_-\Delta_u^2+2\tau_-(\Delta_u\pi_s))\not{g}],
\end{aligned}
\tag{5.112}
$$

$$
\begin{aligned}
L\not{\pi}_u\not{g}(Ca_j)R &= \not{\pi}_u\not{g}\left(\not{a}_j+\frac{g_-}{2p'_-\tau'_-}\not{n}\not{\Delta}_s\not{a}_j-\frac{g_-}{2p_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)\\
&+\frac{\not{\Delta}_s}{p'_-}\left(p_-\not{g}-g_-\not{\pi}_u\right)\left(\not{a}_j-\frac{g_-}{2p_-\tau_-}\not{a}_j\not{n}\not{\Delta}_u\right)\\
&+[2(g_-(\Delta_s\pi_u)-p_-(\Delta_sg))\not{\Delta}_s-(g_-\Delta_s^2+2\tau'_-(\Delta_sg))\not{\pi}_u+(p_-\Delta_s^2+2\tau'_-(\Delta_s\pi_u))\not{g}]\frac{\not{n}\not{a}_j}{2p'_-\tau'_-},
\end{aligned}
\tag{5.113}
$$

$$
\begin{aligned}
L\not{g}\left(\not{a}_j+\frac{\not{a}_j\not{\Delta}_s\not{n}}{2\tau'_-}-\frac{\not{\Delta}_u\not{n}\not{a}_j}{2\tau_-}\right)\not{g}R &= \frac{g_-}{2}\left(\frac{\not{g}\not{a}_j\not{\Delta}_s\not{n}\not{\Delta}_u}{p_-\tau'_-}+\frac{\not{\Delta}_s\not{n}\not{\Delta}_u\not{a}_j\not{g}}{p'_-\tau_-}\right)+g_-\left(\frac{\not{a}_j\not{\Delta}_s\not{g}}{\tau'_-}+\frac{\not{g}\not{\Delta}_u\not{a}_j}{\tau_-}\right)\\
&+\left[(p'_-+\tau'_-)(ga_j)+g_-(\Delta_sa_j)\right]\frac{\not{\Delta}_s\not{n}\not{g}}{p'_-\tau'_-}+\left[(p_-+\tau_-)(ga_j)+g_-(\Delta_ua_j)\right]\frac{\not{g}\not{n}\not{\Delta}_u}{p_-\tau_-}-g^2\not{a}_j\\
&-\left[(p'_-+\tau'_-)g^2+\frac{\tau'_-g_-^2}{p_-\tau_-}\Delta_u^2\right]\frac{\not{\Delta}_s\not{n}\not{a}_j}{2p'_-\tau'_-}-\left[(p_-+\tau_-)g^2+\frac{\tau_-g_-^2}{p'_-\tau'_-}\Delta_s^2\right]\frac{\not{a}_j\not{n}\not{\Delta}_u}{2p_-\tau_-}+2(ga_j)\not{g}\\
&-[g_-\Delta_s^2+2p'_-(g\Delta_s)]\frac{\not{a}_j\not{n}\not{g}}{2p'_-\tau'_-}-[g_-\Delta_u^2+2p_-(g\Delta_u)]\frac{\not{g}\not{n}\not{a}_j}{2p_-\tau_-}\\
&+\frac{g_-}{p_-p'_-}\left[(ga_j)+\frac{g_-}{\tau'_-}(\Delta_sa_j)+\frac{g_-}{\tau_-}(\Delta_ua_j)\right]\not{\Delta}_s\not{n}\not{\Delta}_u-g^2\left[\frac{(\Delta_sa_j)}{\tau'_-}+\frac{(\Delta_ua_j)}{\tau_-}\right]\not{n},
\end{aligned}
\tag{5.114}
$$

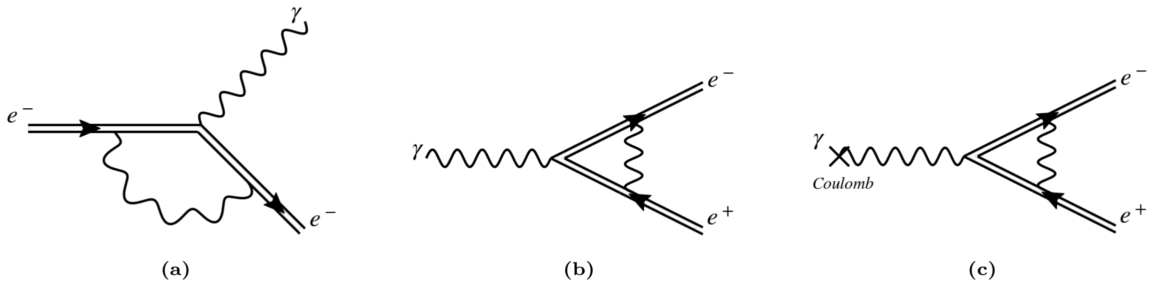

Figure 5.4: *One-loop vertex corrections to (a) nonlinear Compton scattering, (b) nonlinear Breit-Wheeler pair production, and (c) nonlinear Bethe-Heitler pair production. The double lines indicates that the electron/positron states and propagators include the exact interaction with plane-wave field.*

and where it is clear that also in the case of $\Gamma_{R,\perp,j}(p,p',q;\phi)$ the only term requiring regularization is the one analogous to that in the first line of equation (5.110). Due to the above change of variables, the various quantities appearing in $\Gamma_{R,-}(p,p',q;\phi)$, and $\Gamma_{R,\perp,j}(p,p',q;\phi)$ have to be interpreted as

$$\tau'_- = zp'_- - yq_- = (1-x-y)p'_- - yq_-, \qquad \pi^\mu_s = \pi^\mu_{p'}(\theta'_S), \qquad \Delta^\mu_s = \mathcal{A}^\mu(\theta'_S) - \mathcal{A}^\mu(\phi), \tag{5.115}$$
$$\tau_- = zp_- + xq_- = (1-x-y)p_- + xq_-, \qquad \pi^\mu_u = \pi^\mu_p(\theta_S), \qquad \Delta^\mu_u = \mathcal{A}^\mu(\theta_S) - \mathcal{A}^\mu(\phi), \tag{5.116}$$

where

$$\theta'_S = \phi + 2x\tau'_- S = \phi + 2x[(1-x-y)p'_- - yq_-]S, \tag{5.117}$$
$$\theta_S = \phi - 2y\tau_- S = \phi - 2y[(1-x-y)p_- + xq_-]S. \tag{5.118}$$

The formal definitions of the other quantities like L, R, C^μ, Q^λ, and Q'^λ remain unchanged and the additional quantities

$$g^\mu = \frac{G^\mu}{S} = x\int_0^1 d\eta\,\pi^\mu_{p'}(\theta'_{\eta S}) + y\int_0^1 d\eta\,\pi^\mu_p(\theta_{\eta S}) \tag{5.119}$$

and

$$\begin{aligned} g_1^\mu = \frac{G_1^\mu}{S^2} &= \frac{d}{d\phi}\left[x^2\int_0^1 d\eta\,\eta\pi^\mu_{p'}(\theta'_{\eta S}) - y^2\int_0^1 d\eta\,\eta\pi^\mu_p(\theta'_{\eta S})\right] \\ &= \frac{x}{2\tau'_- S}\left[\pi^\mu_{p'}(\theta'_S) - \int_0^1 d\eta\,\pi^\mu_{p'}(\theta'_{\eta S})\right] + \frac{y}{2\tau_- S}\left[\pi^\mu_p(\theta_S) - \int_0^1 d\eta\,\pi^\mu_p(\theta_{\eta S})\right], \end{aligned} \tag{5.120}$$

which is regular in the limit $S\to 0$ (and also in the limits $\tau_-\to 0$ and $\tau'_-\to 0$), have been also introduced.

5.7 Properties of the one-loop vertex correction

Here, the main result for one-loop vertex correction in a plane-wave field $-ie\Gamma_{s,s',l}(p,p',q)$ in (5.63) is given by the equations (5.89), (5.110), and (5.111). These expression, when the background field is turned off, reduce to the known result in vacuum, in particular, the one presented in Section 5.3 and in [150]. These results are free of ultraviolet divergences and the infrared divergences have been avoided by assigning a positive mass κ to the photon. The infrared divergence in the case of a massless photon must be removed to properly evaluate the components $\Gamma_{R,-}(p,p',q;\phi)$ and $\Gamma_{R,\perp,j}(p,p',q;\phi)$ numerically. For this purpose, this divergence must be further studied along the lines of [176, 188].

Notice that the integrals with respect to $d^4x = d\phi\,dT\,d^2\boldsymbol{x}_\perp$ in $-ie\Gamma_{s,s',l}(p,p',q)$ in (5.63) can be performed only for T and $\boldsymbol{x}_\perp$ which lead to Dirac-delta functions for the conservations laws $p_- = p'_- + q_-$ and $p_\perp = p'_\perp + q_\perp$. The ϕ integral can not be performed at this point since we are assuming that the vector potential $A^\mu(\phi)$ is arbitrary and depends on the ϕ coordinate.

In Section 5.3, we mentioned that from the physical consequence of the vacuum vertex correction $\Lambda_R^{(1)\mu}(p,p')$ in (5.37) is the possible extraction of the anomalous magnetic moment of the electron.

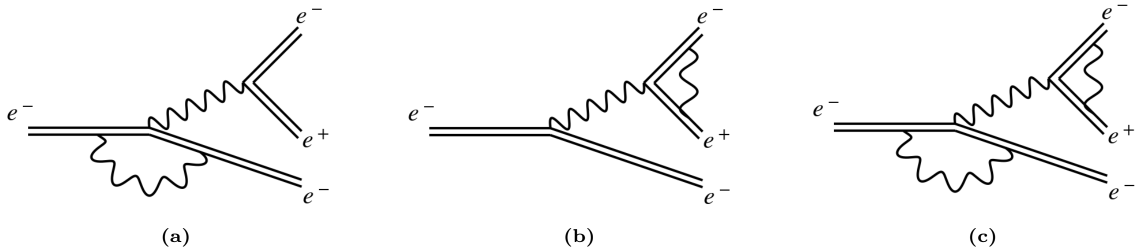

Figure 5.5: *Loop corrections the nonlinear trident pair production involving the one-loop vertex correction: (a) and (b) one-loop, and (c) two-loop. The double lines indicates that the electron/positron states and propagators include the exact interaction with plane-wave field.*

This could suggest to compute the correction to the magnetic moment in the present case of the vertex correction in a plane-wave field, nevertheless it is difficult because the electron interacts with the total magnetic field which contains the magnetic field from the external photon and from the plane-wave. And since it is assumed that the magnetic field from the plane wave is much stronger, we find the calculation of the anomalous moment to be more suitable when considering only the presence of the plane wave, as presented in [104].

The fact that the external photon, in our main result, is off-shell allows to use the amplitude $-ie\Gamma_{s,s',l}(p,p',q)$ as a building block to compute loop-corrections of nonlinear quantum processes in the presence of an arbitrary plane-wave. For instance, in QED, we can use it to compute the nonlinear one-loop corrections to Compton scattering, Breit-Wheeler and Bethe-Heitler pair production as shown in Fig. 5.4. Furthermore, it can be used to obtain the one- and two-loop corrections to the trident pair production, as in Fig. 5.5, although such calculation is already challenging at tree level [95, 96].

The amplitude $-ie\Gamma_{s,s',l}(p,p',q)$ is invariant under gauge transformations of the plane-wave field and of the interaction field, in particular, referring to the internal photon propagator. From equation (5.75) and due to the fact that the kinetic four-momentum of an electron in a plane-wave $\pi^\mu_p(\phi)$ is gauge invariant (see (5.55)), it is easy to see that $-ie\Gamma_{s,s',l}(p,p',q)$ is invariant under a gauge transformation of the external plane-wave field, or in other words, by the replacement of the four-vector potential $A^\mu(\phi)$ with $A^\mu(\phi)+\partial^\mu f(\phi)=A^\mu(\phi)+n^\mu f'(\phi)$, for an arbitrary function $f(\phi)$ that depends only on the parameter ϕ. Furthermore, it is straightforward to show that by choosing the photon propagator in an arbitrary gauge, as in equation (5.38), the additional term coming from the gauge-parameter dependence has the same structure as the tree-level vertex amplitude. At this point, we can follow the same procedure as in vacuum (see Section 5.3 and Ref. [150]) to absorb such term in the renormalization of the electric charge.

The amplitude $-ie\Gamma_{s,s',l}(p,p',q)$ is not by itself gauge invariant respect to the external photon, in other words, this amplitude alone does not satisfy the Ward identity. However, the gauge invariance of QED ensures that the sum of all the one-loop corrections to the nonlinear Compton scattering with an external off-shell photon (corresponding to the diagrams of Figs. 5.2 and 5.6) fulfill the Ward identity.

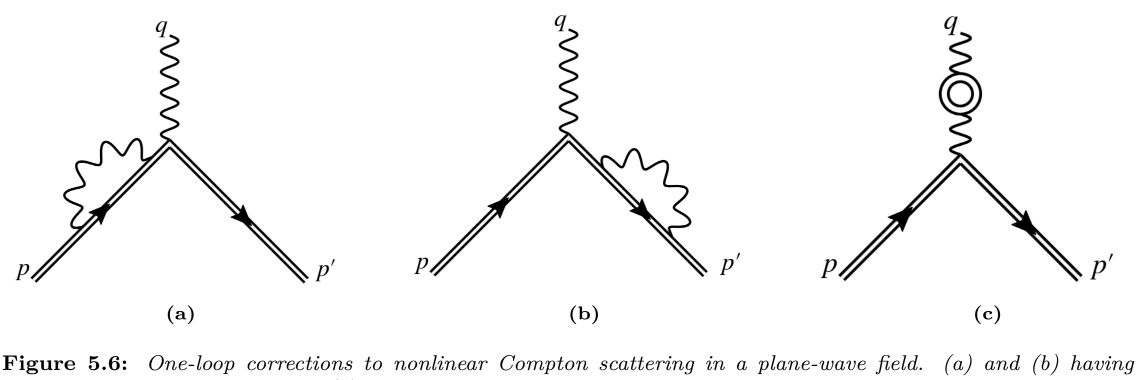

Figure 5.6: *One-loop corrections to nonlinear Compton scattering in a plane-wave field. (a) and (b) having as sub-diagram the mass operator. (c) the contribution with the polarization operator as sub-diagram. The double lines indicates that the electron/positron states and propagators include the exact interaction with plane-wave field.*

In [183, 160], it is shown that the correction in Fig. 5.6c is by itself gauge invariant. Then, if we denote by $i\mathcal{M}^{(1)}_{s,s',\mu}(p,p',q)\, e_l^{*\mu}(q)$ the amplitude resulting from the sum of the corrections corresponding to the diagrams in Figs. 5.2, 5.6a and 5.6b, we can easily show that $\mathcal{M}^{(1)}_{s,s',\mu}(p,p',q)\, q^\mu = 0$. In such a calculation, we can see the component $\Gamma_q(p,p',q;\phi)$ is exactly canceled by the vertex corrections corresponding to Figs. 5.6a and 5.6b, see [110].

In [108, 109], it is shown that the ratio between the one-loop vertex correction and the vertex function (tree level) in a constant crossed field for strong fields ($\xi_0 \gg 1$, $\chi_0 \gg 1$) scales as $\alpha\chi_0^{2/3}$ in agreement with the Ritus-Narozny conjecture [104, 105, 106, 107]. The same fact can be confirmed, using the expressions in this chapter, for the case of the one-loop vertex correction in a plane-wave field. However, the analysis carried out in [110] shows that the terms scaling as $\alpha\chi_0^{2/3}$ do not contribute to any transition amplitudes since those come from the quantity $\Gamma_q(p,p',q;\phi)$ (the same argument apply for the scaling in [108, 109]). Then, the dominant scaling for transition amplitude is provided by the components $\Gamma_{R,-}(p,p',q;\phi)$ and $\Gamma_{R,\perp,j}(p,p',q;\phi)$ which scale as $\alpha\chi_0^{1/3}$.

Chapter 6

Conclusions

In this thesis, we studied various nonlinear processes in the presence of background fields, at the amplitude level. We have explicitly demonstrated the importance of off-shell amplitudes in the computation of quantum corrections for phenomena involving the interaction with background fields. Specifically, we have employed the fully off-shell four-photon amplitudes within the worldline formalism to obtain the circularly polarized amplitudes and cross sections for the scattering of low-energy photons by a Coulomb field (Delbrück scattering) [18, 19]. In this application of the four-photon amplitudes, the results obtained are in agreement with known results for spinor QED [17] and represent novel results for the scalar QED case.

The low-energy limit of the off-shell one-loop N-photon amplitudes in the presence of a constant background field was calculated for two different field configurations: parallel magnetic and electric fields and a constant crossed field. In both cases compact expressions were obtained for scalar and spinor QED, leaving only one proper-time integral left.

In the case of parallel magnetic and electric fields, we specialize our expressions to the case of the four-photon amplitudes in a pure magnetic field at low energies, expressing the results in terms of one proper-time integral, for which we provide a list of analytical results in Appendix C. However, a more detailed analysis of the polarized and total cross sections is still in preparation. In future work, the analysis of the helicity components for these N-photon amplitudes within the framework of Section 2.2 is under consideration, given the advantages of the helicity formalism in calculating photonic amplitudes.

For a constant crossed field, we also obtain the helicity components by applying the techniques presented in [50, 51] (Sections 2.2 and 2.3). In this case, every integral can be carried out analytically, and it is clear that a double Furry theorem is no longer valid due to the interaction with the background field.

We have observed that in the high-field and high-energy limit of the N-photon amplitudes in a constant crossed field, the leading contributions may arise from the tails (3.10). Then, it is interesting to study the scaling with respect to the quantum nonlinearity parameter χ_0 (1.6) for such amplitudes in future work. Given that the photons are off-shell, this approach can also be used to study the scaling with respect to χ_0 of multi-loop amplitudes, which is related to the Ritus-Narozhny conjecture [104, 164, 105, 106, 107].

In addition, we obtained the N-photon amplitudes in an arbitrary constant field and expressed them as the product of infinite sums of traces. These results seem to have no practical applications, and the only obvious property that appears is the manifest charge parity indicated by the Bernoulli numbers.

We made use of the worldline formalism to study the amplitude of Compton scattering in the presence of a purely magnetic background field for off-shell scalar and fermion particles and on-shell external photons. Here, due to the complex structure of the integrand in the amplitudes, we have assumed that the scattering occurs in the forward direction pointing along the same axis as the magnetic field and with the polarization of the external photons being perpendicular to each other. This allowed us to perform every integral and obtain compact expressions for the amplitudes. The outcome of our calculations shows marks of polarization changes in the forward scattering of photons with scalar or spinor particles in the presence of a magnetic field. However, to properly study the birefringent Compton scattering, further analysis of the on-shell limit should be performed.

We applied the operator technique within the Furry picture and the Volkov states to compute the regularized amplitude of the irreducible one-loop vertex correction to nonlinear Compton scattering in the presence of a plane-wave field background assuming that the incoming and outgoing electrons are on-shell, and the external photon is off-shell. The final result is decomposed into the light-cone components, which allows to obtain compact expressions. We have shown that this vertex amplitude is invariant under gauge transformations with respect to the plane-wave background field and the photon propagator. Given that the external photon is off-shell, this result can be used as a building block to compute loop-corrections of nonlinear quantum processes.

We studied the strong field ($\xi_0 \gg 1$, $\chi_0 \gg 1$) behavior of this vertex amplitude and found that its scaling is in agreement with the Ritus-Narozhny conjecture. We checked the Ward identity and found that by fulfilling this identity, the component pointing along the external photon direction does not contribute to transition amplitudes of on-shell states. In the high-field limit, this component exhibits a dominant scaling behavior with respect to χ_0, scaling as $\alpha\chi_0^{2/3}$, while the remaining amplitude scaling is $\alpha\chi_0^{1/3}$.

The infrared divergences in this vertex amplitude are cured by including a positive photon mass in the photon propagator. In future work, we plan to remove these divergences using the Bloch-Nordsieck method [176, 188], to include this result in the cross-section of nonlinear Compton scattering at the next to leading order.

Appendix A

Conventions

In this thesis, we have used natural units $\epsilon_0 = c = \hbar = 1$ and employed both Euclidean and Minkowski space conventions. In this section, we present the main differences between these conventions and point out the sections or chapters in which have been employed. In adition, we present a list of the worldline Green's functions used in this thesis.

A.1 Euclidean space

In Chapters 2 to 4, we work within the worldline formalism in Euclidean space with metric tensor $(g^{\mu\nu}) = \text{diag}(+1, +1, +1, +1)$. However, in Sections 2.5.3, 3.6.2, 4.6.2 and 4.7.2 we use Minkowski space conventions. In the case of Sections 2.5.3 and 3.6.2, in order to use the spinor helicity formalism it is more convenient to work in Minkowski space. In Sections 4.6.2 and 4.7.2, we express our results within the Minkowski space conventions in order to compare with known results as well as to discuss the physical scenario.

We use the absolute value of the electron charge $e = |e|$, corresponding to a covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. Momenta of external photons in the master formulas are ingoing. The field strength tensor for a constant field is

$$F = \begin{pmatrix} 0 & B_z & -B_y & iE_x \\ -B_z & 0 & B_x & iE_y \\ B_y & -B_x & 0 & iE_z \\ -iE_x & -iE_y & -iE_z & 0 \end{pmatrix} . \tag{A.1}$$

The Dirac-gamma matrices in the Chiral representation (conventions for Chapters 9 and 10)

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \qquad \gamma^4 = i \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} . \tag{A.2}$$

Minkowski space amplitudes with metric $(g_M^{\mu\nu}) = \text{diag}(-1, +1, +1, +1)$ are obtained by analytically continuing

$$\begin{aligned} (g^{\mu\nu}) = \mathbb{1} \quad &\longrightarrow \quad (g_M^{\mu\nu}) = \text{diag}(-1, +1, +1, +1) \, , \\ k^4 \quad &\longrightarrow \quad -ik^0 \, , \\ T \quad &\longrightarrow \quad is \, . \end{aligned} \tag{A.3}$$

However, to obtain the convention used in this thesis an extra step should be done. This is, we replace the metric $g_M^{\mu\nu}$ by $\eta^{\mu\nu}$ through

$$(g_M^{\mu\nu}) = \text{diag}(-1, +1, +1, +1) = -(\eta^{\mu\nu}) \, . \tag{A.4}$$

A.2 Minkowski space

In Chapters 5, we work within the operator technique in Minkowski space with metric tensor $(\eta^{\mu\nu}) = \mathrm{diag}(+1,-1,-1,-1)$.

We use $e < 0$ as value of the electron charge, corresponding to a covariant derivative $D^\mu = \partial^\mu + ieA^\mu$.

The field strength tensor for a constant field is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{A.5}$$

The Dirac-gamma matrices in the Chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} , \tag{A.6}$$

where $\mathbb{1}$ is the 2×2 identity matrix and σ^i are Pauli sigma matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.7}$$

The Dirac-gamma matrices in the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{A.8}$$

A.3 Green's functions in vacuum

In this thesis, we mainly use re-scaled parameters 'u' for the worldline Green's functions. For this reason, we clarify the relation between the vacuum Green's functions with 'τ' parameters and the re-scaled ones, see table A.1. For more details see [124, 123].

In table A.1, the expressions for the Green's function follow the convention of [18, 19]. Note that, the constant factor of '$T/6$' is irrelevant for flat space calculations [124], for this reason, it is omitted in the calculation of the N-photon amplitudes.

τ representation	u representation: $\tau_i = T\,u_i$	Comparison of reps.
$G_{Bij} = \lvert\tau_i-\tau_j\rvert - \frac{1}{T}(\tau_i-\tau_j)^2 - \frac{T}{6}$	$G_{Bij} = \lvert u_i-u_j\rvert - (u_i-u_j)^2 - \frac{1}{6}$	$G_B(\tau_i,\tau_j) = TG_B(u_i,u_j)$
$G_{ij} = \lvert\tau_i-\tau_j\rvert - \frac{1}{T}(\tau_i-\tau_j)^2$	$G_{ij} = \lvert u_i-u_j\rvert - (u_i-u_j)^2$	$G(\tau_i,\tau_j) = TG(u_i,u_j)$
$\dot G_{ij} = \mathrm{sign}(\tau_i-\tau_j) - \frac{2}{T}(\tau_i-\tau_j)$	$\dot G_{ij} = \mathrm{sign}(u_i-u_j) - 2(u_i-u_j)$	$\dot G(\tau_i,\tau_j) = \dot G(u_i,u_j)$
$\ddot G_{ij} = 2\delta(\tau_i-\tau_j) - \frac{2}{T}$	$\ddot G_{ij} = 2\delta(u_i-u_j) - 2$	$\ddot G(\tau_i,\tau_j) = \frac{1}{T}\ddot G(u_i,u_j)$
$G_{Fij} = \mathrm{sign}(\tau_i-\tau_j)$	$G_{Fij} = \mathrm{sign}(u_i-u_j)$	$G_F(\tau_i,\tau_j) = G_F(u_i,u_j)$

Table A.1: *Vacuum Green's functions with string inspired boundary conditions.*

Some obvious but useful identities, that we often use, are

$$\dot G_{ij}^2 = 1 - 4G_{ij} , \tag{A.9}$$

$$\dot G_{F31}^2 = 1 , \tag{A.10}$$

$$-G_{F12}G_{F23}G_{F31} = \dot G_{12} + \dot G_{23} + \dot G_{31} . \tag{A.11}$$

A.4 Green's functions in a constant field

It is convenient to remember that in the case of a constant field we use the following convention for the field strength tensor

$$\mathcal{Z} = eFT\,. \tag{A.12}$$

The Green's function in a constant background with string-inspired boundary conditions and its derivatives are [123]

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau') &= \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')} + i\mathcal{Z}\dot{G}_B(\tau,\tau') - 1\right)\,,\\
\dot{\mathcal{G}}_B(\tau,\tau') &= \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')} - 1\right)\,,\\
\ddot{\mathcal{G}}_B(\tau,\tau') &= 2\delta(\tau,\tau') - \frac{2}{T}\frac{\mathcal{Z}}{\sin\mathcal{Z}}\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')}\,,\\
\mathcal{G}_F(\tau,\tau') &= G_F(\tau,\tau')\,\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')}}{\cos\mathcal{Z}}\,.
\end{aligned} \tag{A.13}$$

The coincidence limit of the previous functions are

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau) &= \frac{T}{2}\left(\frac{\cot\mathcal{Z}}{\mathcal{Z}} - \frac{1}{\mathcal{Z}^2}\right)\,,\\
\dot{\mathcal{G}}_B(\tau,\tau) &= i\cot\mathcal{Z} - \frac{i}{\mathcal{Z}}\,,\\
\mathcal{G}_F(\tau,\tau) &= -i\tan\mathcal{Z}\,.
\end{aligned} \tag{A.14}$$

The Green's function in a constant background with Dirichlet boundary conditions and its derivatives are [67]

$$\begin{aligned}
\underline{\Delta}(\tau,\tau') &= \frac{1}{2}\left[\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\right]\,,\\
{}^\bullet\underline{\Delta}(\tau,\tau') &= \frac{1}{2}\Big(\dot{\mathcal{G}}_B(\tau,\tau') - \dot{\mathcal{G}}_B(\tau,0)\Big)\,,\\
{}^\bullet\underline{\Delta}^\bullet(\tau,\tau') &= -\frac{1}{2}\ddot{\mathcal{G}}_B(\tau,\tau')\,.
\end{aligned} \tag{A.15}$$

Here, we have presented a list of the worldline Green's functions used in this thesis for more details, see Chapters 3, 4 and references [124, 123].

Appendix B

Talacha: Complementary details

Talacha: Spanish word defined by the dictionary as "long and tiring mechanical work". Here is used to refer to the intermediate steps in a calculation to obtain the desired results.

B.1 Light-by-light scattering

B.1.1 Coincidence limit

As an example, we present the step by step calculation (adapted from [123, 170]) of the coincidence limit of (3.14)

$$\mathcal{G}_{Fij} = G_{Fij}\,\frac{\cos(\mathcal{Z}\dot{G}_{ij}) - i\sin(\mathcal{Z}\dot{G}_{ij})}{\cos\mathcal{Z}}\,, \tag{B.1}$$

and by coincidence limit, we mean $\mathcal{G}_F(u_i, u_j) \to \mathcal{G}_F(u_i, u_i)$. Here, in order to compute this limit, we need to keep in mind that G_{Fij} and $\dot{G}_{ij}$ contain one sign-function σ_{ij}, which has the property that $\sigma_{ij}^2 = 1$. Then, the first step is to expand[1]

$$\mathcal{G}_{Fij} = \frac{G_{Fij}}{\cos\mathcal{Z}}\left[\sum_{n=0}^{\infty} c_{2n}(\mathcal{Z}\dot{G}_{ij})^{2n} - i\sum_{n=0}^{\infty} c_{2n+1}(\mathcal{Z}\dot{G}_{ij})^{2n+1}\right]\,, \tag{B.2}$$

use the σ_{ij} property mentioned before, or equivalently the identity $\dot{G}_{ij}^2 = 1 - 4G_{ij}$, to obtain

$$\mathcal{G}_{Fij} = \frac{G_{Fij}}{\cos\mathcal{Z}}\left[\sum_{n=0}^{\infty} c_{2n}\mathcal{Z}^{2n}(1-4G_{ij})^n - i\sum_{n=0}^{\infty} c_{2n+1}\mathcal{Z}^{2n+1}(1-4G_{ij})^n\dot{G}_{ij}\right]\,. \tag{B.3}$$

Now, notice that $G_{Fij}\dot{G}_{ij} = 1 - 2|u_i - u_j|$, such that

$$\mathcal{G}_{Fij} = \frac{1}{\cos\mathcal{Z}}\left[G_{Fij}\sum_{n=0}^{\infty} c_{2n}\mathcal{Z}^{2n}(1-4G_{ij})^n - i\sum_{n=0}^{\infty} c_{2n+1}\mathcal{Z}^{2n+1}(1-4G_{ij})^n(1-2|u_i-u_j|)\right]\,, \tag{B.4}$$

and finally, since we have removed all σ_{ij}^2, we can take the limit $u_j \to u_i$ and consider $G_{Fii} = G_{ii} = \dot{G}_{ii} = 0$. Therefore, we obtain

$$\mathcal{G}_{Fij} = -i\frac{1}{\cos\mathcal{Z}}\sum_{n=0}^{\infty} c_{2n+1}\mathcal{Z}^{2n+1} = -i\tan\mathcal{Z}\,. \tag{B.5}$$

B.1.2 Calligraphic Green's functions in a pure magnetic field

Here, we present the step by step calculation (adapted from [123, 170]) of $\dot{\mathcal{G}}_{Bij}$ for a magnetic field in z direction, as in (3.34). First, we look at the series expansion[2] of $\dot{\mathcal{G}}_{Bij}$ respect to $\mathcal{Z}$

$$\dot{\mathcal{G}}_{Bij} = \frac{\sin(\mathcal{Z}\dot{G}_{ij})}{\sin\mathcal{Z}} + i\left[\frac{\cos(\mathcal{Z}\dot{G}_{ij})}{\sin\mathcal{Z}} - \frac{1}{\mathcal{Z}}\right] = \dot{G}_{ij}\mathbb{1} + \sum_{n=1}^{\infty} c_{2n}\mathcal{Z}^{2n} + i\sum_{n=1}^{\infty} c_{2n+1}\mathcal{Z}^{2n+1}\,. \tag{B.6}$$

[1] Here c_n are constant coefficients whose exact expression in not required but can be found in terms of Bernoulli numbers [166, 167].

[2] Now, the coefficients c_n depend of the scalar Green's functions $\dot{G}_{ij}$.

Here, we should look at the odd and even functions in $\mathcal{Z}$ such that using equation (3.32), we obtain

$$\dot{\mathcal{G}}_{Bij} = \dot{G}_{ij}(g_- + g_+) + \sum_{n=1}^{\infty} c_{2n}(-1)^n z^{2n} g_+ + i \sum_{n=0}^{\infty} c_{2n+1}(-1)^n z^{2n+1} r_+ \tag{B.7}$$

or

$$\dot{\mathcal{G}}_{Bij} = \dot{G}_{ij} g_- + \left(\dot{G}_{ij} + \sum_{n=1}^{\infty} c_{2n}(iz)^{2n} \right) g_+ + \sum_{n=0}^{\infty} c_{2n+1}(iz)^{2n+1} r_+ \,, \tag{B.8}$$

now comparing the even and odd parts, we can see that

$$\dot{\mathcal{G}}_{Bij} = \dot{G}_{ij} g_- + \frac{\sin(iz\dot{G}_{ij})}{\sin(iz)} g_+ + \left[\frac{\cos(iz\dot{G}_{ij})}{\sin(iz)} - \frac{1}{iz} \right] r_+ \,. \tag{B.9}$$

At this point, it is obvious that

$$\dot{\mathcal{G}}_{Bij} = \dot{G}_{ij} g_- + \frac{\sinh(z\dot{G}_{ij})}{\sinh(z)} g_+ - \left[\frac{\cosh(z\dot{G}_{ij})}{\sinh(z)} - \frac{1}{z} \right] i r_+ \,. \tag{B.10}$$

All other calligraphic Green's functions in a magnetic field and for the case of Section 3.5 are computed analogously.

B.1.3 Determinants in a pure magnetic field

As an example, we present the step by step calculation (adapted from [123, 170]) of one of the determinants in (3.33). First, it is important to notice the following series expansion

$$\frac{\mathcal{Z}}{\sin \mathcal{Z}} = \mathbb{1} + \sum_{n=1}^{\infty} c_n \mathcal{Z}^{2n} \,, \tag{B.11}$$

where c_n are constant coefficients that are not important for this calculation. Since $\mathcal{Z}^{2n} = (-1)^n z^{2n}\, g_+$,

$$\frac{\mathcal{Z}}{\sin \mathcal{Z}} = g_- + g_+ + \sum_{n=1}^{\infty} c_n (-1)^n z^{2n}\, g_+ = g_- + \frac{iz}{\sin(iz)} g_+ \,. \tag{B.12}$$

Then,

$$\det\left[\frac{\mathcal{Z}}{\sin \mathcal{Z}}\right] = \det\left[g_- + \frac{z}{\sinh z} g_+\right] = \begin{vmatrix} \frac{z}{\sinh z} & 0 & 0 & 0 \\ 0 & \frac{z}{\sinh z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \,, \tag{B.13}$$

making obvious that

$$\det{}^{1/2}\left[\frac{\mathcal{Z}}{\sin \mathcal{Z}}\right] = \frac{z}{\sinh z} \,. \tag{B.14}$$

The calculation of the spinor determinant and the determinants in Section 3.5 are similar.

B.1.4 Integration in a pure magnetic field

As an example, we compute (3.40) step by step for both bosonic and fermionic cases.

Scalar QED: First, we write the bosonic functions H^B_{ij} explicitly

$$H^{B(2)}_{13}(z_1, z_2) = \int_0^1 du_2\, H^B_{12}(z_1) H^B_{23}(z_2) = \int_0^1 du_2 \left(\frac{\mathrm{e}^{z_1 \dot{G}_{12}}}{\sinh z_1} - \frac{1}{z_1} \right) \left(\frac{\mathrm{e}^{z_2 \dot{G}_{23}}}{\sinh z_2} - \frac{1}{z_2} \right) \,, \tag{B.15}$$

for which, we use the following identity [123]

$$\int_0^1 du\, \mathrm{e}^{\sum_{i=1}^n c_i \dot{G}(u,u_i)} = \frac{\sum_{i=1}^n \sinh(c_i) \mathrm{e}^{\sum_{j=1}^n c_j \dot{G}_{ij}}}{\sum_{i=1}^n c_i} \,, \tag{B.16}$$

obtaining

$$H_{13}^{B(2)}(z_1,z_2) = \frac{1}{(z_2-z_1)\sinh z_1\ \sinh z_2}\left(\mathrm{e}^{-z_1\dot{G}_{31}}\ \sinh z_2 + \mathrm{e}^{z_2\dot{G}_{13}}\ \sinh(-z_1)\right) - \frac{1}{z_1 z_2}\,, \tag{B.17}$$

that simplifies to

$$H_{13}^{B(2)}(z_1,z_2) = \frac{1}{(z_2-z_1)}\left(\frac{\mathrm{e}^{z_1\dot{G}_{13}}}{\sinh z_1} - \frac{1}{z_1}\right) + \frac{1}{(z_1-z_2)}\left(\frac{\mathrm{e}^{z_2\dot{G}_{13}}}{\sinh z_2} - \frac{1}{z_2}\right) = \frac{H_{13}^B(z_1)}{z_2-z_1} + \frac{H_{13}^B(z_2)}{z_1-z_2}\,. \tag{B.18}$$

Spinor QED: First, we write the fermionic functions H_{ij}^F explicitly

$$H_{13}^{F(2)}(z_1,z_2) = \int_0^1 du_2\, H_{12}^F(z_1)H_{23}^F(z_2) = \int_0^1 du_2\left(G_{F12}\frac{\mathrm{e}^{z_1\dot{G}_{12}}}{\cosh z_1}\right)\left(G_{F23}\frac{\mathrm{e}^{z_2\dot{G}_{23}}}{\cosh z_2}\right)\,, \tag{B.19}$$

for which, we use the following identities [123]

$$-G_{F12}G_{F23}G_{F31} = \dot{G}_{12} + \dot{G}_{23} + \dot{G}_{31}\,, \qquad \dot{G}_{F31}^2 = 1\,, \tag{B.20}$$

to get

$$H_{13}^{F(2)}(z_1,z_2) = \frac{1}{\cosh z_1\ \cosh z_2}\int_0^1 du_2\ \left(\dot{G}_{12} + \dot{G}_{23} + \dot{G}_{31}\right) G_{F13}\,\mathrm{e}^{z_1\dot{G}_{12}+z_2\dot{G}_{23}}\,. \tag{B.21}$$

Here, we can replace $\dot{G}_{12}, \dot{G}_{23}$ by derivatives of their respective coefficients

$$H_{13}^{F(2)}(z_1,z_2) = \frac{1}{\cosh z_1\ \cosh z_2}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \dot{G}_{31}\right) G_{F13}\int_0^1 du_2\,\mathrm{e}^{z_1\dot{G}_{12}+z_2\dot{G}_{23}} \tag{B.22}$$

and using the identity (B.16), it becomes

$$H_{13}^{F(2)}(z_1,z_2) = \frac{1}{\cosh z_1\ \cosh z_2} G_{F13}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \dot{G}_{31}\right)\left(\frac{\mathrm{e}^{z_1\dot{G}_{13}}\ \sinh z_2}{z_1-z_2} + \frac{\mathrm{e}^{z_2\dot{G}_{13}}\ \sinh z_1}{z_2-z_1}\right)\,. \tag{B.23}$$

Finally, we perform the derivatives and simplify, obtaining

$$H_{13}^{F(2)}(z_1,z_2) = G_{F13}\left(\frac{1}{z_2-z_1}\frac{\mathrm{e}^{z_1\dot{G}_{13}}}{\cosh z_1} + \frac{1}{z_1-z_2}\frac{\mathrm{e}^{z_2\dot{G}_{13}}}{\cosh z_2}\right) = \frac{H_{13}^F(z_1)}{z_2-z_1} + \frac{H_{13}^F(z_2)}{z_1-z_2}\,. \tag{B.24}$$

B.2 Compton scattering in magnetic field

B.2.1 Spinor path integral for $N=2$

In order to compute the path integral in the following expression

$$S_2^\gamma(T,F_{\rm ct}) = 2^{-2}\,\mathrm{symb}^{-1}\left[\int_C \mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^T d\tau\left\{\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)\left[eF-\delta(\tau-\tau_1)f_1-\delta(\tau-\tau_2)f_2\right]\left(\psi+\frac{1}{2}\eta\right)\right\}}\right] \tag{B.25}$$

and re-expressed as (Eq. 4.56), we define

$$I_{\psi,2} \equiv \int_C \mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^T d\tau\left\{\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)\left[eF-\delta(\tau-\tau_1)f_1-\delta(\tau-\tau_2)f_2\right]\left(\psi+\frac{1}{2}\eta\right)\right\}}\Bigg|_{\mathcal{O}(\varepsilon_1\varepsilon_2)}\,. \tag{B.26}$$

Expanding up to linear order in $\varepsilon_1\varepsilon_2$ we get

$$\begin{aligned} I_{\psi,2} = \int_C \mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi} - i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]}\Bigg\{ 1 - i\left(\psi(\tau_1)+\frac{1}{2}\eta\right) f_1 \left(\psi(\tau_1)+\frac{1}{2}\eta\right) \\ - i\left(\psi(\tau_2)+\frac{1}{2}\eta\right) f_2 \left(\psi(\tau_2)+\frac{1}{2}\eta\right) - \left(\psi(\tau_1)+\frac{1}{2}\eta\right) f_1 \left(\psi(\tau_1)+\frac{1}{2}\eta\right)\left(\psi(\tau_2)+\frac{1}{2}\eta\right) f_2 \left(\psi(\tau_2)+\frac{1}{2}\eta\right)\Bigg\}\,. \end{aligned} \tag{B.27}$$

Here, as well as in the calculation of S_1^γ, the previous path integral will be non-zero only for an even number of $\psi(\tau_i)$ in the pre-exponent. In this case, additionally to (4.47), we have the following Wick contraction

$$\frac{\int_C \mathcal{D}\psi(\tau)\,\psi(\tau_i)\psi(\tau_j)\psi(\tau_k)\psi(\tau_l)\,\mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi}-i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]}}{\int_C \mathcal{D}\psi(\tau)\,\mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi\dot{\psi}-i\left(\psi+\frac{1}{2}\eta\right)eF\left(\psi+\frac{1}{2}\eta\right)\right]}} = \langle\psi(\tau_i)\psi(\tau_j)\psi(\tau_k)\psi(\tau_l)\rangle\,. \tag{B.28}$$

It is well know that [124, 123]

$$\begin{aligned}\langle\psi(\tau_i)\psi(\tau_j)\psi(\tau_k)\psi(\tau_l)\rangle &= \langle\psi(\tau_i)\psi(\tau_j)\rangle\langle\psi(\tau_k)\psi(\tau_l)\rangle - \langle\psi(\tau_i)\psi(\tau_k)\rangle\langle\psi(\tau_j)\psi(\tau_l)\rangle \\ &\quad + \langle\psi(\tau_i)\psi(\tau_l)\rangle\langle\psi(\tau_j)\psi(\tau_k)\rangle\,.\end{aligned} \tag{B.29}$$

Writing the previous integral in terms of Wick contractions

$$\begin{aligned}I_{\psi,2} = I_{\psi,0}\Bigg\{&1 - i\left(\langle\psi_1^\mu\psi_1^\nu\rangle + \frac{1}{4}\eta^\mu\eta^\nu\right)f_1^{\mu\nu} - i\left(\langle\psi_2^\mu\psi_2^\nu\rangle + \frac{1}{4}\eta^\mu\eta^\nu\right)f_2^{\mu\nu} - \Bigg[\langle\psi_1^\mu\psi_1^\nu\rangle\langle\psi_2^\sigma\psi_2^\rho\rangle \\ &- \langle\psi_1^\mu\psi_2^\sigma\rangle\langle\psi_1^\nu\psi_2^\rho\rangle + \langle\psi_1^\mu\psi_2^\rho\rangle\langle\psi_1^\nu\psi_2^\sigma\rangle + \frac{1}{4}\langle\psi_1^\mu\psi_1^\nu\rangle\eta^\sigma\eta^\rho + \frac{1}{4}\langle\psi_2^\sigma\psi_2^\rho\rangle\eta^\mu\eta^\nu - \frac{1}{4}\langle\psi_1^\mu\psi_2^\sigma\rangle\eta^\nu\eta^\rho \\ &+ \frac{1}{4}\langle\psi_1^\mu\psi_2^\rho\rangle\eta^\nu\eta^\sigma + \frac{1}{4}\langle\psi_1^\nu\psi_2^\sigma\rangle\eta^\mu\eta^\rho - \frac{1}{4}\langle\psi_1^\nu\psi_2^\rho\rangle\eta^\mu\eta^\sigma + \frac{1}{16}\eta^\mu\eta^\nu\eta^\sigma\eta^\rho\Bigg]f_1^{\mu\nu}f_2^{\sigma\rho}\Bigg\}\,.\end{aligned} \tag{B.30}$$

Then, in terms of calligraphic Green's functions for $d = 4$ (see Eq. (4.48)), we have

$$\begin{aligned}I_{\psi,2} = 2^2{\det}^{\frac{1}{2}}(\cos\mathcal{Z})\,\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\Bigg\{&1 - \frac{i}{2}\left(\mathcal{G}_{F11}^{\mu\nu} + \frac{1}{2}\eta^\mu\eta^\nu\right)f_1^{\mu\nu} - \frac{i}{2}\left(\mathcal{G}_{F22}^{\mu\nu} + \frac{1}{2}\eta^\mu\eta^\nu\right)f_2^{\mu\nu} \\ - \frac{1}{4}\Bigg[\mathcal{G}_{F11}^{\mu\nu}\mathcal{G}_{F22}^{\sigma\rho} + 2\,\mathcal{G}_{F12}^{\mu\rho}\mathcal{G}_{F12}^{\nu\sigma} + \frac{1}{2}\mathcal{G}_{F11}^{\mu\nu}\eta^\sigma\eta^\rho &+ \frac{1}{2}\eta^\mu\eta^\nu\mathcal{G}_{F22}^{\sigma\rho} + 2\,\mathcal{G}_{F12}^{\mu\rho}\eta^\nu\eta^\sigma + \frac{1}{4}\eta^\mu\eta^\nu\eta^\sigma\eta^\rho\Bigg]f_1^{\mu\nu}f_2^{\sigma\rho}\Bigg\}\,.\end{aligned} \tag{B.31}$$

This implies that

$$\begin{aligned}S_2^\gamma(T,F_{\rm ct}) = {\det}^{\frac{1}{2}}(\cos\mathcal{Z})\,{\rm symb}^{-1}\Bigg\langle\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\Bigg\{&1 - \frac{i}{2}\left(\mathcal{G}_{F11}^{\mu\nu} + \frac{1}{2}\eta^\mu\eta^\nu\right)f_1^{\mu\nu} - \frac{i}{2}\left(\mathcal{G}_{F22}^{\mu\nu} + \frac{1}{2}\eta^\mu\eta^\nu\right)f_2^{\mu\nu} \\ - \frac{1}{4}\Bigg[\mathcal{G}_{F11}^{\mu\nu}\mathcal{G}_{F22}^{\sigma\rho} + 2\,\mathcal{G}_{F12}^{\mu\rho}\mathcal{G}_{F12}^{\nu\sigma} + \frac{1}{2}\mathcal{G}_{F11}^{\mu\nu}\eta^\sigma\eta^\rho &+ \frac{1}{2}\eta^\mu\eta^\nu\mathcal{G}_{F22}^{\sigma\rho} + 2\,\mathcal{G}_{F12}^{\mu\rho}\eta^\nu\eta^\sigma + \frac{1}{4}\eta^\mu\eta^\nu\eta^\sigma\eta^\rho\Bigg]f_1^{\mu\nu}f_2^{\sigma\rho}\Bigg\}\Bigg\rangle\,.\end{aligned} \tag{B.32}$$

Notice that, we can re-write the previous expression as

$$S_2^\gamma(T,F_{\rm ct}) = {\det}^{\frac{1}{2}}(\cos\mathcal{Z})\Big[S_0(\mathcal{Z}) - iS_1(\mathcal{Z},f_1) - iS_1(\mathcal{Z},f_2) - S_2(\mathcal{Z},f_1f_2)\Big]\,. \tag{B.33}$$

Due to the fact that η^i are Grassmann variables whose square are equal to zero, in $d = 4$, we have

$$S_0(\mathcal{Z}) = {\rm symb}^{-1}\left[\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\right] = {\rm symb}^{-1}\left[\mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta + \frac{1}{2}\left(\frac{i}{4}\right)^2\eta(\tan\mathcal{Z})\eta\eta(\tan\mathcal{Z})\eta\right]\,. \tag{B.34}$$

Similarly, we have

$$\begin{aligned}S_1(\mathcal{Z},f_1) &= \frac{1}{2}{\rm symb}^{-1}\left[\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\mathcal{G}_{F11}^{\mu\nu} + \frac{1}{2}\left(\mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta\right)\eta^\mu\eta^\nu\right]f_1^{\mu\nu}\,, \\ S_1(\mathcal{Z},f_2) &= \frac{1}{2}{\rm symb}^{-1}\left[\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\mathcal{G}_{F22}^{\mu\nu} + \frac{1}{2}\left(\mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta\right)\eta^\mu\eta^\nu\right]f_2^{\mu\nu}\,, \\ S_2(\mathcal{Z},f_1f_2) &= \frac{1}{8}{\rm symb}^{-1}\Bigg[\left(\mathbb{1} + \frac{i}{4}\eta(\tan\mathcal{Z})\eta\right)\left(\mathcal{G}_{F11}^{\mu\nu}\eta^\sigma\eta^\rho + \eta^\mu\eta^\nu\mathcal{G}_{F22}^{\sigma\rho} + 4\,\mathcal{G}_{F12}^{\mu\rho}\eta^\nu\eta^\sigma\right) \\ &\quad + 2\,\mathrm{e}^{\frac{i}{4}\eta(\tan\mathcal{Z})\eta}\left(\mathcal{G}_{F11}^{\mu\nu}\mathcal{G}_{F22}^{\sigma\rho} + 2\,\mathcal{G}_{F12}^{\mu\rho}\mathcal{G}_{F12}^{\nu\sigma}\right) + \frac{1}{2}\eta^\mu\eta^\nu\eta^\sigma\eta^\rho\Bigg]f_1^{\mu\nu}f_2^{\sigma\rho}\,.\end{aligned} \tag{B.35}$$

On the other hand, the symb functions (see (4.7)) that we require, in this section, are

$$\begin{aligned} \mathrm{symb}^{-1}(\eta^\mu\eta^\nu) = -[\gamma^\mu,\gamma^\nu] = 2\sigma^{\mu\nu}\,, \qquad \sigma^{\mu\nu} = -\frac{1}{2}[\gamma^\mu,\gamma^\nu]\,,\\ \mathrm{symb}^{-1}(\eta^\mu\eta^\nu\eta^\sigma\eta^\rho) = \varepsilon^{\mu\nu\sigma\rho}\;\mathrm{symb}^{-1}(\eta^1\eta^2\eta^3\eta^4) = 4\,\varepsilon^{\mu\nu\sigma\rho}\,\gamma^5\,, \end{aligned} \tag{B.36}$$

where $\varepsilon^{\mu\nu\sigma\rho}$ is the fully antisymmetric Levi-Civita tensor with $\varepsilon^{1234}=1$. Then, in terms of gamma matrices we have

$$\begin{aligned} S_0(\mathcal{Z}) &= \mathbb{1} + \frac{i}{2}\sigma^{\mu\nu}(\tan\mathcal{Z})^{\mu\nu} + 2\left(\frac{i}{4}\right)^2 \varepsilon^{\mu\nu\sigma\rho}\,\gamma^5(\tan\mathcal{Z})^{\mu\nu}(\tan\mathcal{Z})^{\sigma\rho}\,,\\ S_1(\mathcal{Z},f_1) &= \frac{1}{2}\left[S_0\,\mathcal{G}^{\mu\nu}_{F11} + \sigma^{\mu\nu} + \frac{i}{2}\varepsilon^{\sigma\rho\mu\nu}\,\gamma^5(\tan\mathcal{Z})^{\sigma\rho}\right] f_1^{\mu\nu}\,,\\ S_1(\mathcal{Z},f_2) &= \frac{1}{2}\left[S_0\,\mathcal{G}^{\mu\nu}_{F22} + \sigma^{\mu\nu} + \frac{i}{2}\varepsilon^{\sigma\rho\mu\nu}\,\gamma^5(\tan\mathcal{Z})^{\sigma\rho}\right] f_2^{\mu\nu}\,,\\ S_2(\mathcal{Z},f_1f_2) &= \frac{1}{4}\Bigg[S_0\left(\mathcal{G}^{\mu\nu}_{F11}\mathcal{G}^{\sigma\rho}_{F22} + 2\,\mathcal{G}^{\mu\rho}_{F12}\mathcal{G}^{\nu\sigma}_{F12}\right) + \left(\mathcal{G}^{\mu\nu}_{F11}\sigma^{\sigma\rho} + \sigma^{\mu\nu}\mathcal{G}^{\sigma\rho}_{F22} + 4\,\mathcal{G}^{\mu\rho}_{F12}\sigma^{\nu\sigma}\right)\\ &\quad + \varepsilon^{\mu\nu\sigma\rho}\,\gamma^5 + \frac{i}{2}(\tan\mathcal{Z})^{\alpha\beta}\left(\mathcal{G}^{\mu\nu}_{F11}\varepsilon^{\alpha\beta\sigma\rho} + \varepsilon^{\alpha\beta\mu\nu}\mathcal{G}^{\sigma\rho}_{F22} + 4\,\mathcal{G}^{\mu\rho}_{F12}\varepsilon^{\alpha\beta\nu\sigma}\right)\gamma^5\Bigg] f_1^{\mu\nu} f_2^{\sigma\rho}\,. \end{aligned} \tag{B.37}$$

B.3 Strong field QED

B.3.1 Dirac equation in a plane-wave field

In this section, we briefly review the steps to obtain the Volkov states solution (5.40) of the Dirac equation in a plane-wave field background (adapted from [98]). Notice that the Dirac equation in a plane-wave field can be written as

$$\left(\Pi^2 - m^2 - \frac{1}{2}ieF_{\mu\nu}\sigma^{\mu\nu}\right)U = \left[(i\partial - eA)^2 - m^2 - \frac{1}{2}ieF_{\mu\nu}\sigma^{\mu\nu}\right]U = 0\,. \tag{B.38}$$

In which, we have

$$(i\partial - eA)^2\,U = \left(-\partial_\mu\partial^\mu - 2ieA_\mu\partial^\mu + e^2A_\mu A^\mu\right)U\,, \tag{B.39}$$

$$F_{\mu\nu}\sigma^{\mu\nu} = 2\hat{n}\hat{A}'\,, \tag{B.40}$$

here we have used the Lorenz gauge condition $\partial_\mu A^\mu = 0$. Then, the second order Dirac equation become

$$\left(-\partial_\mu\partial^\mu - 2ieA_\mu\partial^\mu + e^2A_\mu A^\mu - m^2 - ie\hat{n}\hat{A}'\right)U = 0\,. \tag{B.41}$$

Since $A = A(\phi)$, we seek a solution as

$$U = \mathrm{e}^{-i(px)}H(\phi)\,. \tag{B.42}$$

Substituting into Dirac equation, we get

$$2i(np)\frac{d}{d\phi}H + \left[(eA-p)^2 - m^2 - ie\hat{n}\hat{A}'\right]H = 0\,. \tag{B.43}$$

The solution of the previous equation is well known to be

$$H(\phi) = \mathrm{e}^{i\int_0^\phi d\phi'\,\frac{[eA(\phi')-p]^2 - m^2 - ie\hat{n}\hat{A}'(\phi')}{2(np)}}\,\frac{u}{\sqrt{2p_0}}\,. \tag{B.44}$$

Then, the solution of the Dirac equation is[3]

$$U(p,x) = \left[1 + \frac{e\hat{n}\hat{A}}{2(np)}\right]\mathrm{e}^{-i(px)}\mathrm{e}^{-i\int_0^\phi d\phi'\,\frac{2e(pA(\phi')) - e^2A^2(\phi')}{2(np)}}\,\frac{u}{\sqrt{2p_0}}\,. \tag{B.45}$$

Similarly for the conjugate Volkov states, we have

$$\bar{U}(p,x) = \frac{\bar{u}}{\sqrt{2p_0}}\left[1 - \frac{e\hat{n}\hat{A}}{2(np)}\right]\mathrm{e}^{i(px)}\mathrm{e}^{i\int_0^\phi d\phi'\,\frac{2e(pA(\phi')) - e^2A^2(\phi')}{2(np)}}\,. \tag{B.46}$$

[3] Here, we have used that $\mathrm{e}^{\hat{n}\hat{A}} = 1 + \hat{n}\hat{A}$ since $n^2 = 0$.

B.3.2 Green's function for Dirac operator

In this section, we briefly review the steps to obtain the Green's function (5.40) of the Dirac operator in a plane-wave field background (adapted from [98]). The Green's function of Dirac operator satisfy

$$\left(\hat{\Pi} - m\right) G(x,y) = \delta(x,y)\,, \tag{B.47}$$

i.e, G is the inverse Dirac operator that can be expressed as

$$G = \frac{1}{\hat{\Pi} - m + i0} = \left(\hat{\Pi} + m\right) \frac{1}{\hat{\Pi}^2 - m^2 + i0} = -i\left(\hat{\Pi} + m\right) \int_0^\infty ds\, \mathrm{e}^{is(\hat{\Pi}^2 - m^2)}\,. \tag{B.48}$$

It is equivalent to have the operator $\hat{\Pi}+m$ acting from the left or right. In the last equality, Schwinger parameter have been used. Notice that

$$\begin{aligned}\hat{\Pi}^2 - m^2 &= p_0^2 - p_1^2 - [p_\perp - eA_\perp(\phi)]^2 - m^2 - \frac{1}{2} ieF_{\mu\nu}\sigma^{\mu\nu} \\ &= 2p_\phi p_T - [p_\perp - eA_\perp(\phi)]^2 - m^2 - ie\hat{n}\hat{A}'(\phi)\,.\end{aligned} \tag{B.49}$$

To disentangle the exponential operator, we write it as

$$\mathrm{e}^{is\left\{2p_\phi p_T - [p_\perp - eA_\perp(\phi)]^2 - m^2 - ie\hat{n}\hat{A}'(\phi)\right\}} = L(s)\, \mathrm{e}^{2isp_\phi p_T}\,. \tag{B.50}$$

Taking the derivative respect to 's', we obtain the following differential equation for $L(s)$

$$i\mathrm{e}^{is(\Pi^2 - m^2)} \left\{2p_\phi p_T - [p_\perp - eA_\perp(\phi)]^2 - m^2 - ie\hat{n}\hat{A}'(\phi)\right\} = \frac{dL(s)}{ds}\, \mathrm{e}^{2isp_\phi p_T} + 2iL(s) p_\phi p_T\, \mathrm{e}^{2isp_\phi p_T}\,, \tag{B.51}$$

which can be written as

$$-iL(s)\, \mathrm{e}^{2isp_\phi p_T} \left\{[p_\perp - eA_\perp(\phi)]^2 + m^2 + ie\hat{n}\hat{A}'(\phi)\right\} \mathrm{e}^{-2isp_\phi p_T} = \frac{dL(s)}{ds}\,, \tag{B.52}$$

shifting operators, $\mathrm{e}^{2isp_\phi p_T} f(\phi) \mathrm{e}^{-2isp_\phi p_T} = f(\phi + 2sp_T)$, we get

$$\frac{dL(s)}{ds} = -iL(s) \left\{[p_\perp - eA_\perp(\phi + 2sp_T)]^2 + m^2 + ie\hat{n}\hat{A}'(\phi + 2sp_T)\right\}\,, \tag{B.53}$$

where the solution with initial condition $L(0) = 1$ is

$$L(s) = \mathrm{e}^{-i\int_0^s ds' \left\{\left[p_\perp - eA_\perp(\phi + 2s'p_T)\right]^2 + m^2 + ie\hat{n}\hat{A}'(\phi + 2s'p_T)\right\}}\,. \tag{B.54}$$

Then, the Green's function of Dirac operator in a plane-wave field is[4]

$$G = -i(\hat{\Pi}+m) \int_0^\infty ds \left\{1 + \frac{e}{2p_T}\hat{n}\left[\hat{A}(\phi + 2sp_T) - \hat{A}(\phi)\right]\right\} \mathrm{e}^{-i\int_0^s ds' \left\{\left[p_\perp - eA_\perp(\phi + 2s'p_T)\right]^2 + m^2\right\}} \mathrm{e}^{2isp_\phi p_T}\,. \tag{B.55}$$

Similarly, if we consider $\hat{\Pi} + m$ acting from the right, we obtain

$$G = -i \int_0^\infty ds\, \mathrm{e}^{2isp_\phi p_T} \left\{1 - \frac{e}{2p_T}\hat{n}\left[\hat{A}(\phi - 2sp_T) - \hat{A}(\phi)\right]\right\} \mathrm{e}^{-i\int_0^s ds' \left\{\left[p_\perp - eA_\perp(\phi - 2s'p_T)\right]^2 + m^2\right\}} (\hat{\Pi}+m)\,. \tag{B.56}$$

B.3.3 Gordon identity

In this section, we derive the Gordon identity in plane-wave field (5.52). First, remember that for the Volkov states we have

$$\hat{\Pi}(\phi)\, U_s(p,x) = \hat{\pi}_p(\phi)\, U_s(p,x) = m\, U_s(p,x)\,. \tag{B.57}$$

[4] Here, we have used that $\mathrm{e}^{\hat{n}\hat{A}} = 1 + \hat{n}\hat{A}$ since $n^2 = 0$.

We can make use of the previous identity to show that

$$\bar{U}_{s'}(p',x)\gamma^\mu U_s(p,x) = \frac{1}{2m}\Big[\bar{U}_{s'}(p',x)\hat{\pi}_{p'}(\phi)\gamma^\mu U_s(p,x) + \bar{U}_{s'}(p',x)\gamma^\mu\hat{\pi}_p(\phi)U_s(p,x)\Big]\,, \tag{B.58}$$

which, using the anti-commutator relation $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$, we can express as

$$\begin{aligned}\bar{U}_{s'}(p',x)\gamma^\mu U_s(p,x) = \;&\frac{1}{4m}\bar{U}_{s'}(p',x)\Big[2\pi^\mu_{p'}(\phi) - \gamma^\mu\hat{\pi}_{p'}(\phi) + \hat{\pi}_{p'}(\phi)\gamma^\mu\Big]U_s(p,x)\\ &+\frac{1}{4m}\bar{U}_{s'}(p',x)\Big[2\pi^\mu_p(\phi) - \hat{\pi}_p(\phi)\gamma^\mu + \gamma^\mu\hat{\pi}_p(\phi)\Big]U_s(p,x)\,.\end{aligned} \tag{B.59}$$

Simplifying in terms of $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, we finally get

$$\bar{U}_{s'}(p',x)\gamma^\mu U_s(p,x) = \bar{U}_{s'}(p',x)\left\{\frac{\pi^\mu_{p'}(\phi) + \pi^\mu_p(\phi)}{2m} + i\,\frac{\sigma^{\mu\nu}\left[\pi_{p'}(\phi) - \pi_p(\phi)\right]_\nu}{2m}\right\}U_s(p,x)\,. \tag{B.60}$$

Appendix C

Proper-time integrals for the four-photon amplitude in a B-field

C.1 Relations for the two- and four-cycle integrals

For 2-cycle integral in (3.97), we have used the following identities (see Eq. (3.44))

$$\begin{aligned}
&H_{11}^{B(2)}(0,0) = -\frac{1}{3}\,,\\
&H_{11}^{B(2)}(z,0) = H_{11}^{B(2)}(z,-z) = \frac{1-z\coth(z)}{z^2}\,,\\
&H_{11}^{B(2)}(z,z) = H_{11}^{B(2)}(-z,-z) = \mathrm{csch}^2(z) - \frac{1}{z^2}\,,
\end{aligned} \tag{C.1}$$

to simplify such integral result as much as possible. Then, the trigonometric functions I_i^{sc} in (3.98) are given by

$$\begin{aligned}
&I_{20}^{\mathrm{sc}} = \frac{1}{2}H_{11}^{B(2)}(0,0)\,, && I_{21}^{\mathrm{sc}} = \frac{1}{4}\left[H_{11}^{B(2)}(z,0) + H_{11}^{B(2)}(z,z)\right]\,,\\
&I_{23}^{\mathrm{sc}} = \frac{1}{2}H_{11}^{B(2)}(z,0)\,, && I_{22}^{\mathrm{sc}} = \frac{1}{4}\left\{H_{11}^{B(2)}(z,0) - H_{11}^{B(2)}(z,z) - 2[H_{11}^{B}(z)]^2\right\}\,.
\end{aligned} \tag{C.2}$$

For the 4-cycle integral in (3.97), we have used the following identities

$$\begin{aligned}
&H_{11}^{B(4)}(z,0,0,0) = H_{11}^{B(4)}(-z,0,0,0) = H_{11}^{B(4)}(-z,z,0,0)\,,\\
&H_{11}^{B(4)}(z,z,0,0) = H_{11}^{B(4)}(-z,-z,0,0)\,,\\
&H_{11}^{B(4)}(z,z,z,0) = H_{11}^{B(4)}(-z,-z,-z,0)\,,\\
&H_{11}^{B(4)}(z,z,z,z) = H_{11}^{B(4)}(-z,-z,-z,-z)\,,\\
&H_{11}^{B(4)}(-z,z,z,0) = H_{11}^{B(4)}(-z,-z,z,0) = H_{11}^{B(4)}(-z,-z,z,z)\,,\\
&H_{11}^{B(4)}(-z,z,z,z) = H_{11}^{B(4)}(-z,-z,-z,z)\,,
\end{aligned} \tag{C.3}$$

to simplify such integral result as much as possible. The explicit trigonometric expression for every

$H_{11}^{B(4)}$ is

$$\begin{aligned}
H_{11}^{B(4)}(0,0,0,0) &= \frac{1}{45}\,,\\
H_{11}^{B(4)}(z,0,0,0) &= \frac{z^2-3z\coth(z)+3}{3z^4}\,,\\
H_{11}^{B(4)}(z,z,0,0) &= \frac{-4z^2+3z\coth(z)(z\coth(z)+2)-9}{3z^4}\,,\\
H_{11}^{B(4)}(z,z,z,0) &= \frac{3-z\left(\coth(z)+z(z\coth(z)+1)\mathrm{csch}^2(z)\right)}{z^4}\,,\\
H_{11}^{B(4)}(z,z,z,z) &= -\frac{1}{z^4}+\mathrm{csch}^4(z)+\frac{2\mathrm{csch}^2(z)}{3}\,,\\
H_{11}^{B(4)}(-z,z,z,0) &= \frac{z\left(\coth(z)+z\mathrm{csch}^2(z)\right)-2}{2z^4}\,,\\
H_{11}^{B(4)}(-z,z,z,z) &= \frac{4-z\left(\coth(z)+z(2z\coth(z)+1)\mathrm{csch}^2(z)\right)}{4z^4}\,.
\end{aligned} \tag{C.4}$$

Then, the functions I_i^{sc} in (3.99) are given by

$$\begin{aligned}
I_0^{\mathrm{sc}} &= H_{11}^{B(4)}(0,0,0,0)\,,\\
I_1^{\mathrm{sc}} &= \frac{1}{8}H_{11}^{B(4)}(z,z,z,z)+\frac{3}{8}H_{11}^{B(4)}(-z,z,z,0)+\frac{1}{2}H_{11}^{B(4)}(-z,z,z,z)\,,\\
I_2^{\mathrm{sc}} &= \frac{1}{8}H_{11}^{B(4)}(z,z,z,z)+\frac{3}{8}H_{11}^{B(4)}(-z,z,z,0)-\frac{1}{2}H_{11}^{B(4)}(-z,z,z,z)+[H_{11}^{B}(z)]^4\,,\\
I_3^{\mathrm{sc}} &= \frac{1}{4}H_{11}^{B(4)}(z,z,z,0)+\frac{3}{4}H_{11}^{B(4)}(-z,z,z,0),\\
I_4^{\mathrm{sc}} &= H_{11}^{B(4)}(z,0,0,0)\,,\\
I_5^{\mathrm{sc}} &= \frac{1}{2}H_{11}^{B(4)}(z,0,0,0)+\frac{1}{2}H_{11}^{B(4)}(z,z,0,0)\,,\\
I_6^{\mathrm{sc}} &= \frac{1}{2}H_{11}^{B(4)}(z,0,0,0)-\frac{1}{2}H_{11}^{B(4)}(z,z,0,0)\,,\\
I_7^{\mathrm{sc}} &= -\frac{1}{8}H_{11}^{B(4)}(z,z,z,z)+\frac{1}{8}H_{11}^{B(4)}(-z,z,z,0)\,,\\
I_8^{\mathrm{sc}} &= -\frac{1}{4}H_{11}^{B(4)}(z,z,z,0)+\frac{1}{4}H_{11}^{B(4)}(-z,z,z,0)\,.
\end{aligned} \tag{C.5}$$

In the case of fermionic variables, we obtain similar expressions for the integrals (3.44). For 2-cycle integral in (3.111), we have that

$$\begin{aligned}
H_{11}^{F(2)}(0,0) &= -1\,,\\
H_{11}^{F(2)}(z,0) = H_{11}^{F(2)}(z,-z) &= -\frac{\tanh(z)}{z}\,,\\
H_{11}^{F(2)}(z,z) = H_{11}^{F(2)}(-z,-z) &= \tanh^2(z)-1\,,
\end{aligned} \tag{C.6}$$

Then, the trigonometric functions I_i^{sp} in (3.112) are given by

$$\begin{aligned}
I_{20}^{\mathrm{sp}} &= \frac{1}{2}\left[H_{11}^{B(2)}(0,0)-H_{11}^{F(2)}(0,0)\right]\,,\\
I_{21}^{\mathrm{sp}} &= \frac{1}{4}\left[H_{11}^{B(2)}(z,0)+H_{11}^{B(2)}(z,z)\right]-\frac{1}{4}\left[H_{11}^{F(2)}(z,0)+H_{11}^{F(2)}(z,z)\right]\,,\\
I_{22}^{\mathrm{sp}} &= \frac{1}{4}\left[H_{11}^{B(2)}(z,0)-H_{11}^{B(2)}(z,z)\right]-\frac{1}{4}\left[H_{11}^{F(2)}(z,0)-H_{11}^{F(2)}(z,z)\right]-\frac{1}{2}[H_{11}^{B}(z)-H_{11}^{F}(z)]^2\,,\\
I_{23}^{\mathrm{sp}} &= \frac{1}{2}\left[H_{11}^{B(2)}(z,0)-H_{11}^{F(2)}(z,0)\right]\,.
\end{aligned} \tag{C.7}$$

For the 4-cycle integral in (3.111), we have used the following identities

$$\begin{aligned}
&H_{11}^{F(4)}(z,0,0,0) = H_{11}^{F(4)}(-z,0,0,0) = H_{11}^{F(4)}(-z,z,0,0)\,,\\
&H_{11}^{F(4)}(z,z,0,0) = H_{11}^{F(4)}(-z,-z,0,0)\,,\\
&H_{11}^{F(4)}(z,z,z,0) = H_{11}^{F(4)}(-z,-z,-z,0)\,,\\
&H_{11}^{F(4)}(z,z,z,z) = H_{11}^{F(4)}(-z,-z,-z,-z)\,,\\
&H_{11}^{F(4)}(-z,z,z,0) = H_{11}^{F(4)}(-z,-z,z,0) = H_{11}^{F(4)}(-z,-z,z,z)\,,\\
&H_{11}^{F(4)}(-z,z,z,z) = H_{11}^{F(4)}(-z,-z,-z,z)\,,
\end{aligned} \tag{C.8}$$

to simplify such integral result as much as possible. The explicit trigonometric expression for every $H_{11}^{F(4)}$ is

$$\begin{aligned}
&H_{11}^{F(4)}(0,0,0,0) = \frac{1}{3}\,,\\
&H_{11}^{F(4)}(z,0,0,0) = \frac{z-\tanh(z)}{z^3}\,,\\
&H_{11}^{F(4)}(z,z,0,0) = \frac{\tanh(z)(z\tanh(z)+2)-2z}{z^3}\,,\\
&H_{11}^{F(4)}(z,z,z,0) = \frac{z(z\tanh(z)+1)\mathrm{sech}^2(z)-\tanh(z)}{z^3}\,,\\
&H_{11}^{F(4)}(z,z,z,z) = \mathrm{sech}^4(z) - \frac{2\mathrm{sech}^2(z)}{3}\,,\\
&H_{11}^{F(4)}(-z,z,z,0) = \frac{\tanh(z)-z\,\mathrm{sech}^2(z)}{2z^3}\,,\\
&H_{11}^{F(4)}(-z,z,z,z) = \frac{z(2z\tanh(z)+1)\mathrm{sech}^2(z)-\tanh(z)}{4z^3}\,.
\end{aligned} \tag{C.9}$$

Then, the functions $I_i^{\rm sp}$ in (3.113) are given by

$$\begin{aligned}
I_0^{\rm sp} &= H_{11}^{B(4)}(0,0,0,0) - H_{11}^{F(4)}(0,0,0,0)\,,\\
I_1^{\rm sp} &= \frac{1}{8}\left[H_{11}^{B(4)}(z,z,z,z) - H_{11}^{F(4)}(z,z,z,z)\right] + \frac{3}{8}\left[H_{11}^{B(4)}(-z,z,z,0) - H_{11}^{F(4)}(-z,z,z,0)\right]\\
&\quad + \frac{1}{2}\left[H_{11}^{B(4)}(-z,z,z,z) - H_{11}^{F(4)}(-z,z,z,z)\right]\,,\\
I_2^{\rm sp} &= \frac{1}{8}\left[H_{11}^{B(4)}(z,z,z,z) - H_{11}^{F(4)}(z,z,z,z)\right] + \frac{3}{8}\left[H_{11}^{B(4)}(-z,z,z,0) - H_{11}^{F(4)}(-z,z,z,0)\right]\\
&\quad - \frac{1}{2}\left[H_{11}^{B(4)}(-z,z,z,z) - H_{11}^{F(4)}(-z,z,z,z)\right] + [H_{11}^{B}(z) - H_{11}^{F}(z)]^4\,,\\
I_3^{\rm sp} &= \frac{1}{4}\left[H_{11}^{B(4)}(z,z,z,0) - H_{11}^{F(4)}(z,z,z,0)\right] + \frac{3}{4}\left[H_{11}^{B(4)}(-z,z,z,0) - H_{11}^{F(4)}(-z,z,z,0)\right]\,,\\
I_4^{\rm sp} &= H_{11}^{B(4)}(z,0,0,0) - H_{11}^{F(4)}(z,0,0,0)\,,\\
I_5^{\rm sp} &= \frac{1}{2}\left[H_{11}^{B(4)}(z,0,0,0) - H_{11}^{F(4)}(z,0,0,0)\right] + \frac{1}{2}\left[H_{11}^{B(4)}(z,z,0,0) - H_{11}^{F(4)}(z,z,0,0)\right]\,,\\
I_6^{\rm sp} &= \frac{1}{2}\left[H_{11}^{B(4)}(z,0,0,0) - H_{11}^{F(4)}(z,0,0,0)\right] - \frac{1}{2}\left[H_{11}^{B(4)}(z,z,0,0) - H_{11}^{F(4)}(z,z,0,0)\right]\,,\\
I_7^{\rm sp} &= -\frac{1}{8}\left[H_{11}^{B(4)}(z,z,z,z) - H_{11}^{F(4)}(z,z,z,z)\right] + \frac{1}{8}\left[H_{11}^{B(4)}(-z,z,z,0) - H_{11}^{F(4)}(-z,z,z,0)\right]\,,\\
I_8^{\rm sp} &= -\frac{1}{4}\left[H_{11}^{B(4)}(z,z,z,0) - H_{11}^{F(4)}(z,z,z,0)\right] + \frac{1}{4}\left[H_{11}^{B(4)}(-z,z,z,0) - H_{11}^{F(4)}(-z,z,z,0)\right]\,.
\end{aligned} \tag{C.10}$$

C.2 Proper time 'T' integrals for the pure magnetic case

In this section, we briefly discuss the procedure to analytically compute the integrals in (3.100) and (3.114). For such purpose, it is required the following three basic expressions[1]

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh(z)} = 2^{-n}\Gamma(n+1)\zeta\left(n+1, \frac{\beta+1}{2}\right), \tag{C.11}$$

$$\int dz\, e^{-\beta z}\, z^n \coth(z) = 2^{-n-1}\Gamma(n+1)\left[\zeta\left(n+1, \frac{\beta}{2}\right) + \zeta\left(n+1, \frac{\beta+2}{2}\right)\right], \tag{C.12}$$

$$\int dz\, e^{-\beta z}\, z^n \tanh(z) = 2^{-2(n+1)}\Gamma(n+1)\left[\zeta\left(n+1, \frac{\beta}{4}\right) - 2\zeta\left(n+1, \frac{\beta+2}{4}\right) + \zeta\left(n+1, \frac{\beta}{4}+1\right)\right], \tag{C.13}$$

where $\zeta(x, y)$ is the Hurwitz-zeta function defined by

$$\zeta(x, y) = \sum_{k=0}^{\infty}(k+y)^{-x} \tag{C.14}$$

and $\Gamma(n)$ the usual gamma function (see [168]).

Now, we follow the steps presented in [153] for the calculation of integrals of the form

$$\int_0^\infty dz\, e^{-\beta z}\, f(z) = \int dz\, e^{-\beta z}\, f(z), \tag{C.15}$$

for β a positive constant and $f(z)$ a trigonometric function.

In order to perform integrals of this kind, we first rewrite them as the derivative respect to an 'artificial' parameter 'α' of one of our basic integrals (C.11), (C.12) or (C.13). For instance,

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^2(z)} = -\left\{\partial_\alpha \int dz\, e^{-\beta z}\, \frac{z^{n-1}}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1}. \tag{C.16}$$

Finally, we make the change of variables $z' = \alpha z$ and perform the derivative

$$\left\{\partial_\alpha \int dz\, e^{-\beta z}\, \frac{z^{n-1}}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1} = \left\{\partial_\alpha \int dz'\, e^{-\frac{\beta}{\alpha}z'}\, \frac{\alpha^{-n}\,(z')^{n-1}}{\sinh(z')}\right\}\Bigg|_{\alpha=1} = \int dz\, e^{-\beta z}\, \frac{z^{n-1}(\beta z - n)}{\sinh(z)}, \tag{C.17}$$

which returns an expression of the form of (C.11).

We list the different integrals that appear in the four-photon amplitudes in a magnetic field: For scalar QED

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^2(z)} = -\left\{\partial_\alpha \int dz\, e^{-\beta z}\, \frac{z^{n-1}}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1}, \tag{C.18}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^3(z)} = \frac{1}{2}\left\{\partial_\alpha^2 \int dz\, e^{-\beta z}\, \frac{z^{n-2}}{\sinh(\alpha z)} - \int dz\, e^{-\beta z}\, \frac{z^n}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1}, \tag{C.19}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh^2(z)}{\sinh^3(z)} = \frac{1}{2}\left\{\partial_\alpha^2 \int dz\, e^{-\beta z}\, \frac{z^{n-2}}{\sinh(\alpha z)} + \int dz\, e^{-\beta z}\, \frac{z^n}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1}, \tag{C.20}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^4(z)} = -\frac{1}{3!}\left\{\partial_\alpha^3 \int dz\, e^{-\beta z}\, \frac{z^{n-3}}{\sinh(\alpha z)} - \partial_\alpha \int dz\, e^{-\beta z}\, \frac{z^{n-1}}{\sinh(\alpha z)},\right\}\Bigg|_{\alpha=1} \tag{C.21}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^5(z)} = \frac{1}{4!}\left\{\partial_\alpha^4 \int dz\, e^{-\beta z}\, \frac{z^{n-4}}{\sinh(\alpha z)} - 10\,\partial_\alpha^2 \int dz\, e^{-\beta z}\, \frac{z^{n-2}}{\sinh(\alpha z)} + 9\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh(\alpha z)}\right\}\Bigg|_{\alpha=1}. \tag{C.22}$$

[1] In this appendix, the integral symbol without limits is understood to be form 0 to ∞.

And for spinor QED

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^2(z)} = -\left\{\partial_\alpha \int dz\, e^{-\beta z}\, z^{n-1} \coth(\alpha z)\right\}\Bigg|_{\alpha=1}, \tag{C.23}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^3(z)} = \frac{1}{2}\left\{\partial^2_\alpha \int dz\, e^{-\beta z}\, z^{n-2} \coth(\alpha z)\right\}\Bigg|_{\alpha=1}, \tag{C.24}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^4(z)} = -\frac{1}{3!}\left\{\partial^3_\alpha \int dz\, e^{-\beta z}\, z^{n-3} \coth(\alpha z) - 4\,\partial_\alpha \int dz\, e^{-\beta z}\, z^{n-1} \coth(\alpha z)\right\}\Bigg|_{\alpha=1}, \tag{C.25}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^5(z)} = \frac{1}{4!}\left\{\partial^4_\alpha \int dz\, e^{-\beta z}\, z^{n-4} \coth(\alpha z) - 4\,\partial^2_\alpha \int dz\, e^{-\beta z}\, z^{n-2} \coth(\alpha z)\right\}\Bigg|_{\alpha=1}, \tag{C.26}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\cosh^2(z)} = \left\{\partial_\alpha \int dz\, e^{-\beta z}\, z^{n-1} \tanh(\alpha z)\right\}\Bigg|_{\alpha=1}, \tag{C.27}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \sinh(z)}{\cosh^3(z)} = -\frac{1}{2}\left\{\partial^2_\alpha \int dz\, e^{-\beta z}\, z^{n-2} \tanh(\alpha z)\right\}\Bigg|_{\alpha=1}. \tag{C.28}$$

C.3 Basic integral results for the scalar amplitude

Here, we use (C.11) to write the analytic result of all the integrals appearing in the scalar amplitudes in terms of the Hurwitz-zeta function $\zeta(x, y)$. So we have

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^2(z)} = 2^{-n}\Gamma(n+1)\left[2\,\zeta\left(n, \frac{\beta+1}{2}\right) - \beta\,\zeta\left(n+1, \frac{\beta+1}{2}\right)\right], \tag{C.29}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^3(z)} = 2^{-n-1}\Gamma(n+1)\left[\left(\beta^2-1\right)\zeta\left(n+1, \frac{\beta+1}{2}\right) + 4\,\zeta\left(n-1, \frac{\beta+1}{2}\right) - 4\beta\,\zeta\left(n, \frac{\beta+1}{2}\right)\right], \tag{C.30}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n \cosh^2(z)}{\sinh^3(z)} = 2^{-n-1}\Gamma(n+1)\left[\left(\beta^2+1\right)\zeta\left(n+1, \frac{\beta+1}{2}\right) + 4\,\zeta\left(n-1, \frac{\beta+1}{2}\right) - 4\beta\,\zeta\left(n, \frac{\beta+1}{2}\right)\right], \tag{C.31}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n \cosh(z)}{\sinh^4(z)} = \frac{2^{-n-1}}{3}\Gamma(n+1)\Bigg[&-\beta^3\,\zeta\left(n+1, \frac{\beta+1}{2}\right) + 6\beta^2\,\zeta\left(n, \frac{\beta+1}{2}\right)\\ &- 12\beta\,\zeta\left(n-1, \frac{\beta+1}{2}\right) + \beta\,\zeta\left(n+1, \frac{\beta+1}{2}\right) + 8\,\zeta\left(n-2, \frac{\beta+1}{2}\right) - 2\,\zeta\left(n, \frac{\beta+1}{2}\right)\Bigg],\end{aligned} \tag{C.32}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^5(z)} = \frac{2^{-n-3}}{3}\Gamma(n+1)\Bigg[&\beta^4\,\zeta\left(n+1, \frac{\beta+1}{2}\right) - 8\beta^3\,\zeta\left(n, \frac{\beta+1}{2}\right) + 24\beta^2\,\zeta\left(n-1, \frac{\beta+1}{2}\right)\\ &- 10\beta^2\,\zeta\left(n+1, \frac{\beta+1}{2}\right) - 32\beta\,\zeta\left(n-2, \frac{\beta+1}{2}\right) + 40\beta\,\zeta\left(n, \frac{\beta+1}{2}\right)\\ &+ 16\,\zeta\left(n-3, \frac{\beta+1}{2}\right) - 40\,\zeta\left(n-1, \frac{\beta+1}{2}\right) + 9\,\zeta\left(n+1, \frac{\beta+1}{2}\right)\Bigg].\end{aligned} \tag{C.33}$$

It is important to mention that J_i^{sc} contain spurious poles that we remove with the aid of dimensional regularization. For such we replace $z^n \to z^{n+\epsilon}$ and take the limit of $\epsilon \to 0$. Notice that n is different in every J_i^{sc}.

C.4 Basic integral results for the spinor amplitude

Here we use (C.12) and (C.13) to write the analytic result of all the integrals appearing in the scalar amplitudes in terms of the Hurwitz-zeta function $\zeta(x,y)$. So we have

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^2(z)} = 2^{-n-1}\Gamma(n+1)\left\{2\left[\zeta\left(n,\frac{\beta}{2}\right)+\zeta\left(n,\frac{\beta+2}{2}\right)\right]-\beta\left[\zeta\left(n+1,\frac{\beta}{2}\right)+\zeta\left(n+1,\frac{\beta+2}{2}\right)\right]\right\}, \tag{C.34}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n\cosh(z)}{\sinh^3(z)} = 2^{-n-2}\Gamma(n+1)\Bigg\{&4\left[\zeta\left(n-1,\frac{\beta}{2}\right)+\zeta\left(n-1,\frac{\beta+2}{2}\right)\right]-4\beta\left[\zeta\left(n,\frac{\beta}{2}\right)+\zeta\left(n,\frac{\beta+2}{2}\right)\right]\\ &+\beta^2\left[\zeta\left(n+1,\frac{\beta}{2}\right)+\zeta\left(n+1,\frac{\beta+2}{2}\right)\right]\Bigg\},\end{aligned} \tag{C.35}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n}{\sinh^4(z)} = \frac{1}{3}2^{-n-2}\Gamma(n+1)\Bigg\{&-\beta^3\left[\zeta\left(n+1,\frac{\beta}{2}\right)+\zeta\left(n+1,\frac{\beta+2}{2}\right)\right]\\ &+6\beta^2\left[\zeta\left(n,\frac{\beta}{2}\right)+\zeta\left(n,\frac{\beta+2}{2}\right)\right]-12\beta\left[\zeta\left(n-1,\frac{\beta}{2}\right)+\zeta\left(n-1,\frac{\beta+2}{2}\right)\right]\\ &+4\beta\left[\zeta\left(n+1,\frac{\beta}{2}\right)+\zeta\left(n+1,\frac{\beta+2}{2}\right)\right]-8\left[\zeta\left(n,\frac{\beta}{2}\right)+\zeta\left(n,\frac{\beta+2}{2}\right)\right]\\ &+8\left[\zeta\left(n-2,\frac{\beta}{2}\right)+\zeta\left(n-2,\frac{\beta+2}{2}\right)\right]\Bigg\},\end{aligned} \tag{C.36}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n\cosh(z)}{\sinh^5(z)} = \frac{1}{3}2^{-n-4}\Gamma(n+1)\Bigg\{&\left(\beta^2-4\right)\beta^2\left[\zeta\left(n+1,\frac{\beta}{2}\right)+\zeta\left(n+1,\frac{\beta+2}{2}\right)\right]\\ -8\left(\beta^2-2\right)\beta\left[\zeta\left(n,\frac{\beta}{2}\right)+\zeta\left(n,\frac{\beta+2}{2}\right)\right]&+8\left(3\beta^2-2\right)\left[\zeta\left(n-1,\frac{\beta}{2}\right)+\zeta\left(n-1,\frac{\beta+2}{2}\right)\right]\\ -32\beta\left[\zeta\left(n-2,\frac{\beta}{2}\right)+\zeta\left(n-2,\frac{\beta+2}{2}\right)\right]&+16\left[\zeta\left(n-3,\frac{\beta}{2}\right)+\zeta\left(n-3,\frac{\beta+2}{2}\right)\right]\Bigg\},\end{aligned} \tag{C.37}$$

$$\int dz\, e^{-\beta z}\, \frac{z^n}{\cosh^2(z)} = 2^{-2n-1}\Gamma(n+1)\left\{-4\left[\zeta\left(n,\frac{\beta}{4}\right)-\zeta\left(n,\frac{\beta+2}{4}\right)\right]+\beta\left[\zeta\left(n+1,\frac{\beta}{4}\right)-\zeta\left(n+1,\frac{\beta+2}{4}\right)\right]\right\}, \tag{C.38}$$

$$\begin{aligned}\int dz\, e^{-\beta z}\, \frac{z^n\sinh(z)}{\cosh^3(z)} = 4^{-n-1}\Gamma(n+1)\Bigg\{&-16\left[\zeta\left(n-1,\frac{\beta}{4}\right)-\zeta\left(n-1,\frac{\beta+2}{4}\right)\right]\\ &+8\beta\left[\zeta\left(n,\frac{\beta}{4}\right)-\zeta\left(n,\frac{\beta+2}{4}\right)\right]-\beta^2\left[\zeta\left(n+1,\frac{\beta}{4}\right)-\zeta\left(n+1,\frac{\beta+2}{4}\right)\right]\Bigg\}.\end{aligned} \tag{C.39}$$

It is important to mention that J_i^{sp} contain spurious poles that we remove with the aid of dimensional regularization. For such we replace $z^n \to z^{n+\epsilon}$ and take the limit of $\epsilon \to 0$. Notice that n is different in every J_i^{sp}.

C.5 Integral results for the scalar QED: $J_i^{\rm sc}$

Here, we present the explicit expression for each integral in (3.100) as the proper time integral of a trigonometric function

$$\begin{aligned}
\frac{J_0^{\rm sc}}{b_4} &= \frac{J_{20}^{\rm sc}}{b_2^2} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\, \frac{z^2}{\sinh z}\,,\\
J_{21}^{\rm sc} &= \frac{b_2}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{z^2}{\sinh^3 z} - \frac{z\cosh z}{\sinh^2 z}\right),\\
J_{22}^{\rm sc} &= -b_2\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{3}{2}\left(\frac{z^2}{\sinh^3 z} - \frac{z\cosh z}{\sinh^2 z}\right) + \frac{z^2}{\sinh z}\right],\\
J_{23}^{\rm sc} &= b_2\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{1}{\sinh z} - \frac{z\cosh z}{\sinh^2 z}\right),
\end{aligned} \tag{C.40}$$

$$\begin{aligned}
J_{24}^{\rm sc} &= -\frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{3}{2}\left(\frac{z^2}{\sinh^5 z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{2\,z\cosh z}{\sinh^4 z}\right) - \frac{z\cosh z}{\sinh^2 z} + \frac{z^2}{\sinh^3 z}\right],\\
J_{25}^{\rm sc} &= \frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{1}{\sinh^3 z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{\cosh z}{z\sinh^2 z} - \frac{z\cosh z}{\sinh^4 z}\right),\\
J_{26}^{\rm sc} &= -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{3}{2}\left(\frac{1}{\sinh^3 z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{\cosh z}{z\sinh^2 z} - \frac{z\cosh z}{\sinh^4 z}\right) + \frac{1}{\sinh z} - \frac{z\cosh z}{\sinh^2 z}\right],
\end{aligned} \tag{C.41}$$

$$\begin{aligned}
J_{27}^{\rm sc} &= \frac{1}{4}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{z^2}{\sinh^5 z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{2\,z\cosh z}{\sinh^4 z}\right),\\
J_{28}^{\rm sc} &= \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{9}{4}\left(\frac{z^2}{\sinh^5 z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{2\,z\cosh z}{\sinh^4 z}\right) + 3\left(\frac{z^2}{\sinh^3 z} + \frac{z\cosh z}{\sinh^2 z}\right) + \frac{z^2}{\sinh z}\right],\\
J_{29}^{\rm sc} &= \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{1}{z^2\sinh z} + \frac{\cosh^2 z}{\sinh^3 z} - \frac{2\,\cosh z}{z\sinh^2 z}\right),
\end{aligned} \tag{C.42}$$

$$\begin{aligned}
J_1^{\rm sc} &= \frac{1}{16}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{z\sinh^2 z} + \frac{1}{\sinh^3 z} + \frac{2z^2}{\sinh^5 z} + \frac{4z^2}{3\sinh^3 z} - \frac{2\,z\cosh z}{\sinh^4 z}\right),\\
J_2^{\rm sc} &= \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{101}{16\,\sinh^3 z} - \frac{59\,\cosh z}{16\,z\,\sinh^2 z} - \frac{15\,z\cosh z}{4\,\sinh^4 z} - \frac{4\,z\cosh z}{\sinh^2 z} + \frac{9\,z^2}{8\,\sinh^5 z} + \frac{25\,z^2}{12\sinh^3 z} + \frac{6+z^2}{\sinh z}\right),
\end{aligned} \tag{C.43}$$

$$\begin{aligned}
J_3^{\rm sc} &= \frac{1}{8}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{z\sinh^2 z} + \frac{1}{\sinh^3 z} - \frac{2\,z\cosh z}{\sinh^4 z}\right),\\
J_4^{\rm sc} &= \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{1}{3\sinh z} + \frac{1}{z^2\sinh z} - \frac{\cosh z}{z\sinh^2 z}\right),\\
J_5^{\rm sc} &= \frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{z\sinh^2 z} + \frac{1}{\sinh^3 z} - \frac{2}{z^2\sinh z}\right),
\end{aligned} \tag{C.44}$$

$$\begin{aligned}
J_6^{\rm sc} &= -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{1}{2\,\sinh^3 z} - \frac{1}{3\,\sinh z} - \frac{2}{z^2\,\sinh z} + \frac{3\,\cosh z}{2\,z\,\sinh^2 z}\right),\\
J_7^{\rm sc} &= \frac{1}{16}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{z\sinh^2 z} + \frac{1}{\sinh^3 z} - \frac{2z^2}{\sinh^5 z} - \frac{4z^2}{3\sinh^3 z}\right),\\
J_8^{\rm sc} &= -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[-\frac{3}{8}\left(\frac{\cosh z}{z\sinh^2 z} + \frac{1}{\sinh^3 z}\right) + \frac{1}{z^2\sinh z} - \frac{z\,\cosh z}{4\,\sinh^4 z}\right].
\end{aligned} \tag{C.45}$$

After using the expressions in Section C.3 and implementing the dimensional regularization, we obtain the following expressions for each $J_i^{\rm sc}$ in terms of the Hurwitz-zeta function, its first derivative (respect to the first parameter),the digamma function $\psi(z) = \psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and the polygamma function $\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z)$[2]

$$
\begin{aligned}
J_{20}^{\rm sc} &= \frac{1}{18}\zeta\left(3, \frac{\beta+1}{2}\right) , \\
J_{21}^{\rm sc} &= \frac{1}{48}\left[\left(\beta^2-1\right)\psi^{(2)}\left(\frac{\beta+1}{2}\right) + 4\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) - 4\right] , \\
J_{22}^{\rm sc} &= -\frac{1}{24}\left[6\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) - \left(3\beta^2+1\right)\zeta\left(3, \frac{\beta+1}{2}\right) - 6\right] , \\
J_{23}^{\rm sc} &= \frac{1}{6}\left[2 - \beta\psi^{(1)}\left(\frac{\beta+1}{2}\right)\right] ,
\end{aligned}
\tag{C.46}
$$

$$
J_{24}^{\rm sc} = -\frac{1}{128}\left[-4\left(\beta^2+13\right) - \left(\beta^2-1\right)^2\psi^{(2)}\left(\frac{\beta+1}{2}\right) + 32\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right)\right] ,
\tag{C.47}
$$

$$
\begin{aligned}
J_{25}^{\rm sc} = \frac{1}{72}\Bigg[&- 144\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + 21\beta^2 + 3\left(\beta^2-1\right)\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) - 6\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) \\
&- 36\beta\log(8\pi) + 72\beta\log(\beta-1) + 72\beta\log\Gamma\left(\frac{\beta-1}{2}\right) + 11\Bigg] ,
\end{aligned}
\tag{C.48}
$$

$$
\begin{aligned}
J_{26}^{\rm sc} = -\frac{1}{24}\Bigg[&21\beta^2 + 3\left(\beta^2+3\right)\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) - 6\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) - 36\beta\log(8\pi) \\
&+ 72\beta\log(\beta-1) + 72\beta\log\Gamma\left(\frac{\beta-1}{2}\right) - 13\Bigg] + 6\,\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) ,
\end{aligned}
\tag{C.49}
$$

$$
\begin{aligned}
J_{27}^{\rm sc} &= \frac{1}{384}\left[-4\left(\beta^2+21\right) - \left(\beta^4-10\beta^2+9\right)\psi^{(2)}\left(\frac{\beta+1}{2}\right) + 64\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right)\right] , \\
J_{28}^{\rm sc} &= \frac{1}{128}\left[-12\left(\beta^2+5\right) - \left(3\left(\beta^2+6\right)\beta^2+11\right)\psi^{(2)}\left(\frac{\beta+1}{2}\right)\right] , \\
J_{29}^{\rm sc} &= -\frac{1}{12} - 12\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + \frac{5\beta^2}{4} - \frac{1}{2}\left(\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) \\
&\quad - 2\beta\log(8\pi) + 4\beta\log(\beta-1) + 4\beta\log\Gamma\left(\frac{\beta-1}{2}\right) ,
\end{aligned}
\tag{C.50}
$$

$$
\begin{aligned}
J_{0}^{\rm sc} &= \frac{1}{90}\zeta\left(3, \frac{\beta+1}{2}\right) , \\
J_{1}^{\rm sc} &= \frac{1}{768}\Bigg[192\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) - \left(\beta^2-1\right)^2\psi^{(2)}\left(\frac{\beta+1}{2}\right) + 8\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) \\
&\quad + 8\beta\Big(6\log(8\pi) - 5\beta\Big) - 96\beta\log(\beta-1) - 96\beta\log\Gamma\left(\frac{\beta-1}{2}\right) + 8\Bigg] ,
\end{aligned}
\tag{C.51}
$$

$$
\begin{aligned}
J_{2}^{\rm sc} = \frac{1}{768}\Bigg[&- 11328\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + 1896\beta^2 + 32\left(3\beta^2+13\right)\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) \\
&- 472\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) - \left(9\beta^4+110\beta^2+73\right)\psi^{(2)}\left(\frac{\beta+1}{2}\right) \\
&- 8496\beta\log(2) - 2832\beta\log(\pi) + 5664\beta\log(\beta-1) + 5664\beta\log\Gamma\left(\frac{\beta-1}{2}\right) - 936\Bigg] , \\
J_{3}^{\rm sc} = \frac{1}{288}\Bigg[&144\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) - 39\beta^2 + 6\left(\beta^2-1\right)\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) + 6\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) \\
&+ 36\beta\log(8\pi) - 72\beta\log(\beta-1) - 72\beta\log\Gamma\left(\frac{\beta-1}{2}\right) + 13\Bigg] ,
\end{aligned}
\tag{C.52}
$$

[2]We clarify that $\log(x)$ is the natural logarithm and $\Gamma(x)$ is the gamma-function.

$$
\begin{aligned}
J_4^{\mathrm{sc}} &= -\frac{1}{6} - 8\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + \frac{1}{2}\beta(\beta - 2\log(8\pi)) + 2\beta\log(\beta-1) \\
&\quad - \frac{1}{3}\psi^{(0)}\left(\frac{\beta+1}{2}\right) + 2\beta\log\Gamma\left(\frac{\beta-1}{2}\right), \\
J_5^{\mathrm{sc}} &= \frac{5}{24} + 6\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) - \frac{1}{4}\left(\beta^2-1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) \\
&\quad - \frac{1}{8}\beta(\beta - 4\log(8\pi)) - \beta\log(\beta-1) - \beta\log\Gamma\left(\frac{\beta-1}{2}\right), \\
J_6^{\mathrm{sc}} &= -\frac{1}{24}\left[-15\beta^2 + 2\left(7-3\beta^2\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) + 36\beta\log(8\pi) - 72\beta\log(\beta-1) + 9\right] \\
&\quad - 14\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + 3\beta\log\Gamma\left(\frac{\beta-1}{2}\right),
\end{aligned}
\tag{C.53}
$$

$$
\begin{aligned}
J_7^{\mathrm{sc}} = -\frac{1}{2304}\Bigg[&-80 - 576\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + 192\beta^2 - 48\left(\beta^2-1\right)\beta\psi^{(1)}\left(\frac{\beta+1}{2}\right) \\
&- 24\left(3\beta^2+1\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) - 3\left(\beta^2-1\right)^2\psi^{(2)}\left(\frac{\beta+1}{2}\right) - 432\beta\log(2) \\
&- 144\beta\log(\pi) + 288\beta\log(\beta-1) + 288\beta\log\Gamma\left(\frac{\beta-1}{2}\right)\Bigg],
\end{aligned}
\tag{C.54}
$$

$$
\begin{aligned}
J_8^{\mathrm{sc}} = -\frac{1}{288}\Bigg[&-47 - 1584\zeta^{(1,0)}\left(-1, \frac{\beta+1}{2}\right) + \left(90\beta^2-66\right)\psi^{(0)}\left(\frac{\beta+1}{2}\right) + 6\beta\left(\beta^2-1\right)\psi^{(1)}\left(\frac{\beta+1}{2}\right) \\
&- 3\beta(\beta + 36\log(8\pi)) + 216\beta\log(\beta-1) + 216\beta\log\Gamma\left(\frac{\beta-1}{2}\right)\Bigg].
\end{aligned}
\tag{C.55}
$$

C.6 Integral results for the spinor QED: J_i^{sp}

Here, we present the explicit expression for each integral in (3.114) as the proper time integral of a trigonometric function

$$
\begin{aligned}
\frac{J_0^{\mathrm{sp}}}{h_4} &= \frac{J_{20}^{\mathrm{sp}}}{h_2^2} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\, z^2\, \coth z\,, \\
J_{21}^{\mathrm{sp}} &= \frac{h_2}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{z^2\cosh z}{\sinh^3 z} - \frac{z}{\sinh^2 z} + z^2\left(\coth z - \tanh z\right)\right], \\
J_{22}^{\mathrm{sp}} &= -\frac{h_2}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[3\left(\frac{z^2\cosh z}{\sinh^3 z} - \frac{z}{\sinh^2 z}\right) - z^2\left(\coth z - \tanh z\right)\right], \\
J_{23}^{\mathrm{sp}} &= h_2\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\coth z - \frac{z}{\sinh^2 z}\right),
\end{aligned}
\tag{C.56}
$$

$$
\begin{aligned}
J_{24}^{\mathrm{sp}} &= -\frac{1}{4}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\Bigg[3\left(\frac{\cosh z}{\sinh^3 z} - \frac{2\,z}{\sinh^4 z} + \frac{z^2\cosh z}{\sinh^5 z} - \coth z + \tanh z\right) - 3z^2\left(\coth z - \tanh z\right) \\
&\qquad + 2\left(\frac{z^2\cosh z}{\sinh^3 z} - \frac{z}{\sinh^2 z} + \frac{z}{\cosh^2 z}\right) + \frac{z^2\sinh z}{\cosh^3 z}\Bigg], \\
J_{25}^{\mathrm{sp}} &= \frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{2\cosh z}{\sinh^3 z} - \frac{1}{z\,\sinh^2 z} + \frac{z}{\cosh^2 z} - \frac{z}{\sinh^2 z} - \frac{z}{\sinh^4 z}\right), \\
J_{26}^{\mathrm{sp}} &= -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{3}{2}\left(\frac{2\cosh z}{\sinh^3 z} - \frac{1}{z\,\sinh^2 z} - \frac{z}{\sinh^4 z}\right) - \frac{1}{2}\left(\frac{z}{\cosh^2 z} - \frac{z}{\sinh^2 z}\right) - 2\left(\coth z - \tanh z\right)\right],
\end{aligned}
\tag{C.57}
$$

$$J^{\rm sp}_{27} = \frac{1}{4}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{\cosh z}{\sinh^3 z} - \frac{2\,z}{\sinh^4 z} + \frac{z^2\cosh z}{\sinh^5 z} - \coth z + \tanh z - z^2\left(\coth z - \tanh z\right) + 2\left(\frac{z^2\cosh z}{\sinh^3 z} - \frac{z}{\sinh^2 z} + \frac{z}{\cosh^2 z}\right) - \frac{z^2\sinh z}{\cosh^3 z}\right],$$

$$J^{\rm sp}_{28} = \frac{1}{4}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[9\left(\frac{\cosh z}{\sinh^3 z} - \frac{2z}{\sinh^4 z} + \frac{z^2\cosh z}{\sinh^5 z} - \coth z + \tanh z\right) + 7z^2\left(\coth z - \tanh z\right) - 6\left(\frac{z^2\cosh z}{\sinh^3 z} - \frac{z}{\sinh^2 z} + \frac{z}{\cosh^2 z}\right) - \frac{z^2\sinh z}{\cosh^3 z}\right], \tag{C.58}$$

$$J^{\rm sp}_{29} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{\sinh^3 z} - \frac{2}{z\,\sinh^2 z} - \coth z + \tanh z + \frac{\coth z}{z^2}\right),$$

$$J^{\rm sp}_{1} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{1}{16}\left(\frac{\cosh z}{\sinh^3 z} + \frac{1}{z\sinh^2 z} + \coth z - \tanh z\right) + \frac{1}{24}\left(\frac{2z^2\cosh z}{\sinh^3 z} - z^2\coth z + z^2\tanh z\right) - \frac{1}{4}\left(\frac{z}{\cosh^2 z} + \frac{z}{\sinh^4 z} + \frac{z}{\sinh^2 z}\right) + \frac{1}{8}\left(\frac{z^2\cosh z}{\sinh^5 z} + \frac{z^2\sinh z}{\cosh^3 z}\right)\right], \tag{C.59}$$

$$J^{\rm sp}_{2} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{1}{16}\left(\frac{101\,\cosh z}{\sinh^3 z} - \frac{59}{z\sinh^2 z} - 91\,\coth z + 91\,\tanh z\right) + \frac{1}{8}\left(\frac{9\,z^2\cosh z}{\sinh^5 z} - \frac{7\,z^2\sinh z}{\cosh^3 z}\right) - \frac{1}{24}\left(\frac{46\,z^2\cosh z}{\sinh^3 z} - 71\,z^2\coth z + 71\,z^2\tanh z\right) - \frac{1}{4}\left(\frac{15\,z}{\cosh^2 z} + \frac{15\,z}{\sinh^4 z} - \frac{17\,z}{\sinh^2 z}\right)\right], \tag{C.60}$$

$$J^{\rm sp}_{3} = \frac{1}{8}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{\cosh z}{\sinh^3 z} + \frac{1}{z\sinh^2 z} + \coth z - \tanh z - 2\left(\frac{z}{\cosh^2 z} + \frac{z}{\sinh^4 z} + \frac{z}{\sinh^2 z}\right)\right],$$
$$J^{\rm sp}_{4} = \int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\coth z}{z^2} - \frac{2}{3}\coth z - \frac{1}{z\sinh^2 z}\right), \tag{C.61}$$

$$J^{\rm sp}_{5} = \frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{\sinh^3 z} + \frac{1}{z\sinh^2 z} + \coth z - \tanh z - \frac{2\coth z}{z^2}\right),$$
$$J^{\rm sp}_{6} = -\frac{1}{2}\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left(\frac{\cosh z}{\sinh^3 z} + \frac{3}{z\,\sinh^2 z} + \frac{7}{3}\coth z - \tanh z - \frac{4\coth z}{z^2}\right), \tag{C.62}$$

$$J^{\rm sp}_{7} = -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{-1}{16}\left(\frac{\cosh z}{\sinh^3 z} + \frac{1}{z\sinh^2 z} + \coth z - \tanh z\right) + \frac{1}{8}\left(\frac{z^2\cosh z}{\sinh^5 z} + \frac{z^2\sinh z}{\cosh^3 z}\right) + \frac{1}{24}\left(\frac{2\,z^2\cosh z}{\sinh^3 z} - z^2\coth z + z^2\tanh z\right)\right],$$
$$J^{\rm sp}_{8} = -\int_0^\infty dz\, \mathrm{e}^{-\beta z}\left[\frac{-3}{8}\left(\frac{\cosh z}{\sinh^3 z} + \frac{1}{z\sinh^2 z} + \coth z - \tanh z\right) - \frac{1}{4}\left(\frac{z}{\cosh^2 z} + \frac{z}{\sinh^4 z} + \frac{z}{\sinh^2 z}\right) + \frac{\coth z}{z^2}\right]. \tag{C.63}$$

After using the expressions in Section C.4 and implementing the dimensional regularization, we obtain the following expressions for each $J^{\rm sp}_i$ in terms of the Hurwitz-zeta function, its first derivative (with respect to the first parameter), the digamma function and the polygamma function

$$\begin{aligned}
J_{20}^{\rm sp} &= \frac{4}{9}\left[-\frac{2}{\beta^3}-\frac{1}{4}\psi^{(2)}\left(\frac{\beta}{2}\right)\right],\\
J_{21}^{\rm sp} &= \frac{1}{96}\left[-4\left(\beta^2+2\right)\psi^{(2)}\left(\frac{\beta}{2}\right)-16\beta\psi^{(1)}\left(\frac{\beta}{2}\right)+\psi^{(2)}\left(\frac{\beta}{4}\right)-\psi^{(2)}\left(\frac{\beta+2}{4}\right)+16\right],\\
J_{22}^{\rm sp} &= \frac{-1}{96}\left[\left(8-12\beta^2\right)\psi^{(2)}\left(\frac{\beta}{2}\right)-48\beta\psi^{(1)}\left(\frac{\beta}{2}\right)-\psi^{(2)}\left(\frac{\beta}{4}\right)+\psi^{(2)}\left(\frac{\beta+2}{4}\right)+48\right],\\
J_{23}^{\rm sp} &= \frac{1}{3}\left[-\frac{2}{\beta}+\beta\psi^{(1)}\left(\frac{\beta}{2}\right)-2\right],
\end{aligned} \tag{C.64}$$

$$\begin{aligned}
J_{24}^{\rm sp} = \frac{-1}{256}\Bigg[&-8\beta^2-2\left(\beta^4+4\beta^2-24\right)\psi^{(2)}\left(\frac{\beta}{2}\right)+\left(\beta^2-6\right)\psi^{(2)}\left(\frac{\beta}{4}\right)-\left(\beta^2-6\right)\psi^{(2)}\left(\frac{\beta+2}{4}\right)\\
&-16\beta-128\beta\psi^{(1)}\left(\frac{\beta}{2}\right)+32\beta\psi^{(1)}\left(\frac{\beta}{4}\right)-32\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right)+208\Bigg],\\
J_{25}^{\rm sp} = \frac{1}{144}\Bigg[&-144\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)-144\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+\left(24-36\beta^2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)\\
&+6\beta\left(\beta^2+2\right)\psi^{(1)}\left(\frac{\beta}{2}\right)+6\beta\{7\beta-4[2+\log(64)+3\log(\pi)]\}+72\beta\log(\beta)+36\psi^{(0)}\left(\frac{\beta}{4}\right)\\
&-36\psi^{(0)}\left(\frac{\beta+2}{4}\right)+9\beta\psi^{(1)}\left(\frac{\beta}{4}\right)-9\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right)+144\beta\log\Gamma\left(\frac{\beta}{2}\right)-44\Bigg],
\end{aligned} \tag{C.65}$$

$$\begin{aligned}
J_{26}^{\rm sp} = \frac{-1}{48}\Bigg\{&-144\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)-144\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+\left(24-36\beta^2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)\\
&+3\beta\left[2\left(\beta^2-6\right)\psi^{(1)}\left(\frac{\beta}{2}\right)-\psi^{(1)}\left(\frac{\beta}{4}\right)+\psi^{(1)}\left(\frac{\beta+2}{4}\right)\right]+6\beta\{7\beta-4[2+\log(64)+3\log(\pi)]\}\\
&+72\beta\log(\beta)-60\psi^{(0)}\left(\frac{\beta}{4}\right)+60\psi^{(0)}\left(\frac{\beta+2}{4}\right)+144\beta\log\Gamma\left(\frac{\beta}{2}\right)+52\Bigg\},\\
J_{27}^{\rm sp} = \frac{1}{768}\Bigg[&-8\beta^2-2\left(\beta^4+20\beta^2-24\right)\psi^{(2)}\left(\frac{\beta}{2}\right)-3\left(\beta^2+2\right)\psi^{(2)}\left(\frac{\beta}{4}\right)\\
&+3\left(\beta^2+2\right)\psi^{(2)}\left(\frac{\beta+2}{4}\right)-16\beta-256\beta\psi^{(1)}\left(\frac{\beta}{2}\right)+336\Bigg],
\end{aligned} \tag{C.66}$$

$$\begin{aligned}
J_{28}^{\rm sp} &= \frac{1}{256}\Bigg\{24(10-2\beta-\beta^2)-2\left(3\beta^4-36\beta^2+56\right)\psi^{(2)}\left(\frac{\beta}{2}\right)-\left(\beta^2-14\right)\left[\psi^{(2)}\left(\frac{\beta}{4}\right)-\psi^{(2)}\left(\frac{\beta+2}{4}\right)\right]\\
&-64\beta\left[\psi^{(1)}\left(\frac{\beta}{4}\right)-\psi^{(1)}\left(\frac{\beta+2}{4}\right)\right]-512\left[\psi^{(0)}\left(\frac{\beta}{4}\right)-\psi^{(0)}\left(\frac{\beta+2}{4}\right)\right]\Bigg\},\\
J_{29}^{\rm sp} &= -6\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)-6\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+\frac{1}{12}\Bigg[-6\left(\beta^2-2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)+3\beta(5\beta-2-12\log(4)-8\log(\pi))\\
&+24\beta(\log(\beta-2)+\log(\beta))-6\psi^{(0)}\left(\frac{\beta}{4}\right)+6\psi^{(0)}\left(\frac{\beta+2}{4}\right)+24\beta\log\Gamma\left(\frac{\beta}{2}\right)+2\Bigg]+2\beta\log\Gamma\left(\frac{\beta}{2}-1\right),
\end{aligned} \tag{C.67}$$

$$
\begin{aligned}
J_0^{\mathrm{sp}} &= \frac{7\left(\beta^3\psi^{(2)}\left(\frac{\beta}{2}\right)+8\right)}{90\beta^3}\,,\\
J_1^{\mathrm{sp}} &= \frac{1}{1536}\Bigg[192\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)+192\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)-80\beta^2+16\left(3\beta^2-2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)\\
&\quad-2\left(\beta^4+4\beta^2-8\right)\psi^{(2)}\left(\frac{\beta}{2}\right)+\left(3\beta^2-2\right)\psi^{(2)}\left(\frac{\beta}{4}\right)+\left(2-3\beta^2\right)\psi^{(2)}\left(\frac{\beta+2}{4}\right)+32\beta+192\beta\log(2)\\
&\quad+96\beta\log(\pi)-96\beta\log(\beta)-48\psi^{(0)}\left(\frac{\beta}{4}\right)+48\psi^{(0)}\left(\frac{\beta+2}{4}\right)-192\beta\log\Gamma\left(\frac{\beta}{2}\right)-32\Bigg]\,,
\end{aligned}
\tag{C.68}
$$

$$
\begin{aligned}
J_2^{\mathrm{sp}} &= \frac{1}{768}\Bigg[-5664\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)-5664\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+1896\beta^2+32\left(3\beta^2-26\right)\beta\psi^{(1)}\left(\frac{\beta}{2}\right)\\
&\quad+472\left(2-3\beta^2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)+\left(-9\beta^4+220\beta^2-568\right)\psi^{(2)}\left(\frac{\beta}{2}\right)+\left(71-\frac{21\beta^2}{2}\right)\psi^{(2)}\left(\frac{\beta}{4}\right)\\
&\quad+\left(\frac{21\beta^2}{2}-71\right)\psi^{(2)}\left(\frac{\beta+2}{4}\right)-1680\beta-5664\beta\log(2)-2832\beta\log(\pi)+2832\beta\log(\beta)-528\beta\psi^{(1)}\left(\frac{\beta}{4}\right)\\
&\quad+528\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right)-3960\psi^{(0)}\left(\frac{\beta}{4}\right)+3960\psi^{(0)}\left(\frac{\beta+2}{4}\right)+5664\beta\log\Gamma\left(\frac{\beta}{2}\right)+1872\Bigg]\,,
\end{aligned}
\tag{C.69}
$$

$$
\begin{aligned}
J_3^{\mathrm{sp}} &= \frac{1}{288}\Bigg[72\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)+72\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+6\left(3\beta^2-2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)\\
&\quad+6\beta\left(\beta^2+2\right)\psi^{(1)}\left(\frac{\beta}{2}\right)+3\beta(-13\beta+2+24\log(2)+12\log(\pi))-36\beta\log(\beta)-18\psi^{(0)}\left(\frac{\beta}{4}\right)\\
&\quad+18\psi^{(0)}\left(\frac{\beta+2}{4}\right)-9\beta\psi^{(1)}\left(\frac{\beta}{4}\right)+9\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right)-72\beta\log\Gamma\left(\frac{\beta}{2}\right)-26\Bigg]\,,
\end{aligned}
\tag{C.70}
$$

$$
\begin{aligned}
J_4^{\mathrm{sp}} &= \frac{1}{6}\Bigg[-24\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)-24\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)+3\beta^2+\frac{4}{\beta}-\beta\log\left(4096\pi^6\right)\\
&\quad+6\beta\log(\beta)+4\psi^{(0)}\left(\frac{\beta}{2}\right)+12\beta\log\Gamma\left(\frac{\beta}{2}\right)+2\Bigg]\,,
\end{aligned}
\tag{C.71}
$$

$$
\begin{aligned}
J_5^{\mathrm{sp}} &= \frac{1}{24}\Bigg[72\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)+72\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)-6\left(\beta^2+2\right)\psi^{(0)}\left(\frac{\beta}{2}\right)-3\beta(\beta+2-12\log(2)-4\log(\pi))\\
&\quad-12\beta(\log(\beta-2)+\log(\beta))+6\psi^{(0)}\left(\frac{\beta}{4}\right)-6\psi^{(0)}\left(\frac{\beta+2}{4}\right)-12\beta\log\Gamma\left(\frac{\beta}{2}\right)-12\beta\log\Gamma\left(\frac{\beta}{2}-1\right)-10\Bigg]\,,
\end{aligned}
\tag{C.72}
$$

$$
\begin{aligned}
J_6^{\mathrm{sp}} &= \frac{1}{24\beta}\Bigg[-168\beta\left(\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right)+\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right)\right)+36\beta^2(\log(\beta-2)+\log(\beta))\\
&\quad+2\left(3\beta^2+14\right)\beta\psi^{(0)}\left(\frac{\beta}{2}\right)+3\beta(\beta(5\beta+2-36\log(2)-12\log(\pi))+6)\\
&\quad+6\beta\left(-\psi^{(0)}\left(\frac{\beta}{4}\right)+\psi^{(0)}\left(\frac{\beta+2}{4}\right)+6\beta\left(\log\Gamma\left(\frac{\beta}{2}\right)+\log\Gamma\left(\frac{\beta}{2}-1\right)\right)\right)+16\Bigg]\,,
\end{aligned}
\tag{C.73}
$$

$$J_7^{\mathrm{sp}} = \frac{-1}{4608}\Bigg[-576\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right) - 576\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right) + 384\beta^2 - 96\left(\beta^2+2\right)\beta\psi^{(1)}\left(\frac{\beta}{2}\right)$$
$$+ 48\left(2-3\beta^2\right)\psi^{(0)}\left(\frac{\beta}{2}\right) - 6\left(\beta^4+4\beta^2-8\right)\psi^{(2)}\left(\frac{\beta}{2}\right) + \left(9\beta^2-6\right)\psi^{(2)}\left(\frac{\beta}{4}\right)$$
$$+ \left(6-9\beta^2\right)\psi^{(2)}\left(\frac{\beta+2}{4}\right) - 864\beta\log(2) - 288\beta\log(\pi) + 288\beta\log(\beta-2) + 288\beta\log(\beta) + 144\beta\psi^{(1)}\left(\frac{\beta}{4}\right)$$
$$- 144\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right) + 144\psi^{(0)}\left(\frac{\beta}{4}\right) - 144\psi^{(0)}\left(\frac{\beta+2}{4}\right) + 288\beta\log\Gamma\left(\frac{\beta}{2}\right) + 288\beta\log\Gamma\left(\frac{\beta}{2}-1\right) + 320\Bigg],$$

$$J_8^{\mathrm{sp}} = \frac{-1}{288}\Bigg[-792\zeta^{(1,0)}\left(-1,\frac{\beta}{2}\right) - 792\zeta^{(1,0)}\left(-1,\frac{\beta+2}{2}\right) + 6\left(15\beta^2+22\right)\psi^{(0)}\left(\frac{\beta}{2}\right)$$
$$+ 6\beta\left(\beta^2+2\right)\psi^{(1)}\left(\frac{\beta}{2}\right) - 3\beta(\beta - 26 + 72\log(2) + 36\log(\pi)) + 108\beta\log(\beta) - 90\psi^{(0)}\left(\frac{\beta}{4}\right)$$
$$+ 90\psi^{(0)}\left(\frac{\beta+2}{4}\right) - 9\beta\psi^{(1)}\left(\frac{\beta}{4}\right) + 9\beta\psi^{(1)}\left(\frac{\beta+2}{4}\right) + 216\beta\log\Gamma\left(\frac{\beta}{2}\right) + 94\Bigg].$$

(C.74)

List of publications

The present thesis is partially based in the the following publications:

i) A. Di Piazza and M. A. Lopez-Lopez, *One-loop vertex correction in a plane wave*, Phys. Rev. D **102** (2020) 076018, [110].
A. Di Piazza and I performed all calculations in parallel for this article. I contributed in discussions and proof reading of the paper.

ii) N. Ahmadiniaz, C. Lopez-Arcos, M. A. Lopez-Lopez and C. Schubert, *The QED four-photon amplitudes off-shell: Part 1*, Nucl. Phys. B **991** (2023) 116216, [18].
I performed calculations for the four-photon amplitudes in scalar and spinor QED, and contributed in writing the paper.

iii) N. Ahmadiniaz, C. Lopez-Arcos, M. A. Lopez-Lopez and C. Schubert, *The QED four-photon amplitudes off-shell: Part 2*, Nucl. Phys. B **991** (2023) 116217, [19].
I performed calculations for the four-photon amplitudes in scalar and spinor QED, and contributed in writing the paper.

iv) N. Ahmadiniaz, T. E. Cowan, M. Ding, M. A. Lopez-Lopez, R. Sauerbrey, R. Shaisultanov et al., *Field-assisted birefringent Compton scattering*, in preparation [66].
I performed calculations for the forward Compton scattering in scalar QED.

v) N. Ahmadiniaz, M. A. Lopez-Lopez and C. Schubert, *Low-energy limit of N-photon amplitudes in a constant field*, Phys. Lett. B (2024) 138610, [111].
I performed calculations for the N-photon amplitudes in scalar and spinor QED, and contributed in discussions and proof reading of the paper.

vi) M. A. Lopez-Lopez, *Low-energy limit of N-photon amplitudes in a constant field: part II*, in preparation [112].
I performed calculations for the N-photon amplitudes in scalar and spinor QED, and wrote the paper.

vii) N. Ahmadiniaz, M. A. Lopez-Lopez, C. Schubert, R. Schützhold and M. A. Trejo, *Low-energy limit of the four-photon amplitude in a constant field*, in preparation [113].
I performed calculations for the four-photon amplitudes in scalar and spinor QED, and contributed in writing the paper.

List of Tables

List of Figures

Bibliography

[1] P. A. M. Dirac and R. H. Fowler, *The quantum theory of the electron*, Proc. R. Soc. Lond. A **117** (1928) 610.

[2] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. R. Soc. Lond. A **133** (1931) 60.

[3] C. D. Anderson, *The Apparent Existence of Easily Deflectable Positives*, Science **76** (1932) 238.

[4] C. D. Anderson, *The Positive Electron*, Phys. Rev. **43** (1933) 491.

[5] W. Heisenberg and H. Euler, *Consequences of Dirac's theory of positrons*, Z. Phys. **98** (1936) 714 [physics/0605038].

[6] V. Weisskopf, *The electrodynamics of the vacuum based on the quantum theory of the electron*, Kong. Dan. Vid. Sel. Mat. Fys. Med. **14N6** (1936) 1.

[7] J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. **82** (1951) 664.

[8] G. V. Dunne, *Heisenberg-Euler effective Lagrangians: Basics and extensions*, in *From Fields to Strings: Circumnavigating Theoretical Physics*, pp. 445–522. 2005. hep-th/0406216. DOI.

[9] H. Euler and B. Kockel, *The scattering of light by light in Dirac's theory*, Naturwiss. **23** (1935) 246.

[10] H. Euler, *On the scattering of light by light according to Dirac's theory*, Annalen der Physik **418** (1936) 398.

[11] A. Akhiezer, L. Landau and I. Pomeranchuk, *Scattering of light by light*, Nature **138** (1936) 206.

[12] A. Akhiezer and V. Berestetsky, *Quantum electrodynamics*. Nauka, Moscow, 1981 (in Russian).

[13] R. Karplus and M. Neuman, *Non-linear interactions between electromagnetic fields*, Phys. Rev. **80** (1950) 380.

[14] R. Karplus and M. Neuman, *The scattering of light by light*, Phys. Rev. **83** (1951) 776.

[15] B. De Tollis, *Dispersive approach to photon-photon scattering*, Nuovo Cim. **32** (1964) 757.

[16] B. De Tollis, *The scattering of photons by photons*, Nuovo Cim. **35** (1965) 1182.

[17] V. Costantini, B. De Tollis and G. Pistoni, *Nonlinear effects in quantum electrodynamics*, Nuovo Cim. A **2** (1971) 733.

[18] N. Ahmadiniaz, C. Lopez-Arcos, M. A. Lopez-Lopez and C. Schubert, *The QED four-photon amplitudes off-shell: Part 1*, Nucl. Phys. B **991** (2023) 116216.

[19] N. Ahmadiniaz, C. Lopez-Arcos, M. A. Lopez-Lopez and C. Schubert, *The QED four-photon amplitudes off-shell: Part 2*, Nucl. Phys. B **991** (2023) 116217.

[20] M. A. Lopez-Lopez, *Off-shell four photon amplitudes in the worldline formalism*, Master's thesis, IFM-UMSNH, Michoacan, 2018.

[21] R. Baier and P. Breitenlohner, *The Vacuum refraction Index in the presence of External Fields*, Nuovo Cim. B **47** (1967) 117.

[22] W. Dittrich and H. Gies, *Vacuum birefringence in strong magnetic fields*, in *Workshop on Frontier Tests of Quantum Electrodynamics and Physics of the Vacuum*, pp. 29–43, 6, 1998, hep-ph/9806417.

[23] J. S. Heyl and L. Hernquist, *Birefringence and dichroism of the QED vacuum*, J. Phys. A: Math. Gen. **30** (1997) 6485 [hep-ph/9705367].

[24] S. L. Adler, *Photon splitting and photon dispersion in a strong magnetic field*, Annals Phys. **67** (1971) 599 .

[25] H. Gies, F. Karbstein and R. Shaisultanov, *Laser photon merging in an electromagnetic field inhomogeneity*, Phys. Rev. D **90** (2014) 033007 [1406.2972].

[26] H. Gies, F. Karbstein and N. Seegert, *Photon merging and splitting in electromagnetic field inhomogeneities*, Phys. Rev. D **93** (2016) 085034 [1603.00314].

[27] R. R. Wilson, *Scattering of* 1.3^3 *MeV Gamma-Rays by an Electric Field*, Phys. Rev. **90** (1953) 720.

[28] M. Schumacher, I. Borchert, F. Smend and P. Rullhusen, *Delbrück scattering of* 2.75 *MeV photons by lead*, Phys. Lett. B **59** (1975) 134 .

[29] T. Bar-Noy and S. Kahane, *Numerical calculations of Delbrück scattering amplitudes*, Nucl. Phys. A **288** (1977) 132 .

[30] S. Z. Akhmadaliev, G. Y. Kezerashvili, S. Klimenko, R. Lee, V. Malyshev, A. Maslennikov et al., *Experimental investigation of high-energy photon splitting in atomic fields*, *Phys. Rev. Lett.* **89** (2002) 061802.

[31] ATLAS collaboration, M. Aaboud et al., *Evidence for light-by-light scattering in heavy-ion collisions with the ATLAS detector at the LHC*, Nature Phys. **13** (2017) 852–858 [1702.01625].

[32] ATLAS collaboration, G. Aad et al., *Observation of light-by-light scattering in ultraperipheral Pb+Pb collisions with the ATLAS detector*, Phys. Rev. Lett. **123** (2019) 052001 [1904.03536].

[33] CMS collaboration, A. M. Sirunyan et al., *Evidence for light-by-light scattering and searches for axion-like particles in ultraperipheral PbPb collisions at* $\sqrt{s_{\rm NN}} = 5.02$ *TeV*, Phys. Lett. B **797** (2019) 134826 [1810.04602].

[34] A. Cadène, P. Berceau, M. Fouché, R. Battesti and C. Rizzo, *Vacuum magnetic linear birefringence using pulsed fields: status of the BMV experiment*, Eur. Phys. J. D **68** (2014) 16 [1302.5389].

[35] X. Fan et al., *The OVAL experiment: A new experiment to measure vacuum magnetic birefringence using high repetition pulsed magnets*, Eur. Phys. J. D **71** (2017) 308 [1705.00495].

[36] A. Ejlli, F. Della Valle, U. Gastaldi, G. Messineo, R. Pengo, G. Ruoso et al., *The PVLAS experiment: A 25 year effort to measure vacuum magnetic birefringence*, Phys. Rept. **871** (2020) 1 [2005.12913].

[37] R. Battesti et al., *High magnetic fields for fundamental physics*, Phys. Rept. **765-766** (2018) 1 [1803.07547].

[38] R. P. Mignani, V. Testa, D. G. Caniulef, R. Taverna, R. Turolla, S. Zane et al., *Evidence for vacuum birefringence from the first optical-polarimetry measurement of the isolated neutron star RX J1856.5−3754*, Mon. Not. Roy. Astron. Soc. **465** (2017) 492 [1610.08323].

[39] L. M. Capparelli, A. Damiano, L. Maiani and A. D. Polosa, *A note on polarized light from Magnetars*, Eur. Phys. J. C **77** (2017) 754 [1705.01540].

[40] R. Turolla, S. Zane, R. Taverna, D. G. Caniulef, R. P. Mignani, V. Testa et al., *A Comment on "A note on polarized light from Magnetars: QED effects and axion-like particles" by L.M. Capparelli, L. Maiani and A.D. Polosa*, 1706.02505.

[41] N. B. Narozhny, *Propagation of plane electromagnetic waves in a constant field*, *Sov. Phys. JETP* **28** (1969) 371.

[42] A. Di Piazza, K. Z. Hatsagortsyan and C. H. Keitel, *Non-perturbative vacuum-polarization effects in proton-laser collisions*, *Phys. Rev. Lett.* **100** (2008) 010403 [0708.0475].

[43] A. Di Piazza, K. Z. Hatsagortsyan and C. H. Keitel, *Laser photon merging in proton-laser collisions*, *Phys. Rev.* **A78** (2008) 062109 [0906.5576].

[44] A. Di Piazza, A. I. Milstein and C. H. Keitel, *Photon splitting in a laser field*, *Phys. Rev. A* **76** (2007) 032103.

[45] R. Baier, A. Rebhan and M. Wödlinger, *Light-by-light scattering in the presence of magnetic fields*, *Phys. Rev. D* **98** (2018) 056001.

[46] R. Shaisultanov, *On the string-inspired approach to QED in external field*, *Phys. Lett. B* **378** (1996) 354.

[47] M. Reuter, M. G. Schmidt and C. Schubert, *Constant external fields in gauge theory and the spin 0,* 1/2*, 1 path integrals*, *Annals Phys.* **259** (1997) 313 [hep-th/9610191].

[48] J. P. Edwards and C. Schubert, *N-photon amplitudes in a plane-wave background*, *Phys. Lett. B* **822** (2021) 136696 [2105.08173].

[49] C. Schubert and R. Shaisultanov, *Master formulas for photon amplitudes in a combined constant and plane-wave background field*, *Phys. Lett. B* **843** (2023) 137969 [2303.08907].

[50] G. V. Dunne and C. Schubert, *Two-loop self-dual Euler-Heisenberg Lagrangians (I): Real part and helicity amplitudes*, *JHEP* **08** (2002) 053 [hep-th/0205004].

[51] L. C. Martin, C. Schubert and V. M. Villanueva, *On the low-energy limit of the QED N-photon amplitudes*, *Nucl. Phys. B* **668** (2003) 335 [hep-th/0301022].

[52] J. P. Edwards, A. Huet and C. Schubert, *On the low-energy limit of the QED N-photon amplitudes: part 2*, *Nucl. Phys. B* **935** (2018) 198 [1807.10697].

[53] J. Lundin, M. Marklund, E. Lundström, G. Brodin, J. Collier, R. Bingham et al., *Analysis of four-wave mixing of high-power lasers for the detection of elastic photon-photon scattering*, *Phys. Rev. A* **74** (2006) 043821 [hep-ph/0606136].

[54] D. Tommasini, A. Ferrando, H. Michinel and M. Seco, *Precision tests of QED and non-standard models by searching photon-photon scattering in vacuum with high power lasers*, *JHEP* **11** (2009) 043 [0909.4663].

[55] H. Gies, F. Karbstein, C. Kohlfürst and N. Seegert, *Photon-photon scattering at the high-intensity frontier*, *Phys. Rev. D* **97** (2018) 076002 [1712.06450].

[56] H. Gies, F. Karbstein and L. Klar, *Quantum vacuum signatures in multicolor laser pulse collisions*, *Phys. Rev. D* **103** (2021) 076009.

[57] F. Karbstein, D. Ullmann, E. A. Mosman and M. Zepf, *Direct Accessibility of the Fundamental Constants Governing Light-by-Light Scattering*, *Phys. Rev. Lett.* **129** (2022) 061802 [2207.09866].

[58] "Helmholtz international beamline for extreme fields (HIBEF)." www.hibef.eu.

[59] T. Heinzl, B. Liesfeld, K.-U. Amthor, H. Schwoerer, R. Sauerbrey and A. Wipf, *On the observation of vacuum birefringence*, *Opt. Commun.* **267** (2006) 318 [hep-ph/0601076].

[60] A. Di Piazza, K. Z. Hatsagortsyan and C. H. Keitel, *Light diffraction by a strong standing electromagnetic wave*, *Phys. Rev. Lett.* **97** (2006) 083603 [hep-ph/0602039].

[61] H.-P. Schlenvoigt, T. Heinzl, U. Schramm, T. E. Cowan and R. Sauerbrey, *Detecting vacuum birefringence with x-ray free electron lasers and high-power optical lasers: a feasibility study*, *Phys. Scripta* **91** (2016) 023010.

[62] S. Bragin, S. Meuren, C. H. Keitel and A. Di Piazza, *High-Energy Vacuum Birefringence and Dichroism in an Ultrastrong Laser Field*, *Phys. Rev. Lett.* **119** (2017) 250403 [1704.05234].

[63] F. Karbstein, *Vacuum birefringence in the head-on collision of x-ray free-electron laser and optical high-intensity laser pulses*, *Phys. Rev. D* **98** (2018) 056010 [1807.03302].

[64] F. Karbstein, C. Sundqvist, K. S. Schulze, I. Uschmann, H. Gies and G. G. Paulus, *Vacuum birefringence at x-ray free-electron lasers*, *New J. Phys.* **23** (2021) 095001 [2105.13869].

[65] N. Ahmadiniaz, M. Bussmann, T. E. Cowan, A. Debus, T. Kluge and R. Schützhold, *Observability of Coulomb-assisted quantum vacuum birefringence*, *Phys. Rev. D* **104** (2021) L011902 [2012.04484].

[66] N. Ahmadiniaz, T. E. Cowan, M. Ding, M. A. Lopez-Lopez, R. Sauerbrey, R. Shaisultanov et al., *Field-assisted birefringent Compton scattering*, 2212.03350.

[67] A. Ahmad, N. Ahmadiniaz, O. Corradini, S. P. Kim and C. Schubert, *Master formulas for the dressed scalar propagator in a constant field*, *Nucl. Phys. B* **919** (2017) 9 [1612.02944].

[68] N. Ahmadiniaz, V. M. Banda Guzmán, F. Bastianelli, O. Corradini, J. P. Edwards and C. Schubert, *Worldline master formulas for the dressed electron propagator. Part I. Off-shell amplitudes*, *JHEP* **08** (2020) 049 [2004.01391].

[69] N. Ahmadiniaz, V. M. B. Guzman, F. Bastianelli, O. Corradini, J. P. Edwards and C. Schubert, *Worldline master formulas for the dressed electron propagator. Part 2. On-shell amplitudes*, *JHEP* **01** (2022) 050 [2107.00199].

[70] N. Ahmadiniaz, V. M. B. Guzman, F. Bastianelli, O. Corradini, J. P. Edwards and C. Schubert, *Worldline master formulas for the dressed electron propagator. Part III*, *Unpublished* .

[71] J. W. Yoon, C. Jeon, J. Shin, S. K. Lee, H. W. Lee, I. W. Choi et al., *Achieving the laser intensity of* 5.5×10^{22} W/cm^2 *with a wavefront-corrected multi-pw laser*, *Opt. Express* **27** (2019) 20412.

[72] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan and C. H. Keitel, *Extremely high-intensity laser interactions with fundamental quantum systems*, *Rev. Mod. Phys.* **84** (2012) 1177.

[73] A. Fedotov, A. Ilderton, F. Karbstein, B. King, D. Seipt, H. Taya et al., *Advances in QED with intense background fields*, *Phys. Rept.* **1010** (2023) 1 [2203.00019].

[74] C. Bula, K. T. McDonald, E. J. Prebys, C. Bamber, S. Boege, T. Kotseroglou et al., *Observation of nonlinear effects in Compton scattering*, *Phys. Rev. Lett.* **76** (1996) 3116.

[75] D. L. Burke, R. C. Field, G. Horton-Smith, J. E. Spencer, D. Walz, S. C. Berridge et al., *Positron production in multiphoton light-by-light scattering*, *Phys. Rev. Lett.* **79** (1997) 1626.

[76] C. Bamber, S. J. Boege, T. Koffas, T. Kotseroglou, A. C. Melissinos, D. D. Meyerhofer et al., *Studies of nonlinear QED in collisions of 46.6 GeV electrons with intense laser pulses*, *Phys. Rev. D* **60** (1999) 092004.

[77] J. M. Cole, K. T. Behm, E. Gerstmayr, T. G. Blackburn, J. C. Wood, C. D. Baird et al., *Experimental evidence of radiation reaction in the collision of a high-intensity laser pulse with a laser-wakefield accelerated electron beam*, *Phys. Rev. X* **8** (2018) 011020.

[78] K. Poder, M. Tamburini, G. Sarri, A. Di Piazza, S. Kuschel, C. D. Baird et al., *Experimental signatures of the quantum nature of radiation reaction in the field of an ultraintense laser*, *Phys. Rev. X* **8** (2018) 031004.

[79] A. Di Piazza, *Analytical tools for investigating strong-field QED processes in tightly focused laser fields*, *Phys. Rev. A* **91** (2015) 042118 [1501.06475].

[80] "Center for relativistic laser science (CoReLS)." https://corels.ibs.re.kr/html/corels_en/.

[81] "Extreme light infrastructure (ELI)." https://eli-laser.eu/.

[82] "Exawatt center for extreme light studies (XCELS)." http://www.xcels.iapras.ru/.

[83] "Apollon." https://apollonlaserfacility.cnrs.fr/.

[84] A. Hartin and G. Moortgat-Pick, *High Intensity Compton Scattering in a strong plane wave field of general form*, *Eur. Phys. J. C* **71** (2011) 1729 [1106.1671].

[85] B. King and S. Tang, *Nonlinear Compton scattering of polarized photons in plane-wave backgrounds*, *Phys. Rev. A* **102** (2020) 022809.

[86] D. Seipt and B. Kämpfer, *Nonlinear Compton scattering of ultrashort intense laser pulses*, *Phys. Rev. A* **83** (2011) 022101.

[87] D. Seipt, *Volkov States and Non-linear Compton Scattering in Short and Intense Laser Pulses*, in *Quantum Field Theory at the Limits: from Strong Fields to Heavy Quarks*, pp. 24–43, 2017, 1701.03692, DOI.

[88] A. Di Piazza, M. Tamburini, S. Meuren and C. H. Keitel, *Implementing nonlinear Compton scattering beyond the local-constant-field approximation*, *Phys. Rev. A* **98** (2018) 012134.

[89] G. Breit and J. A. Wheeler, *Collision of two light quanta*, *Phys. Rev.* **46** (1934) 1087.

[90] H. R. Reiss, *Absorption of Light by Light*, *J. Math. Phys.* **3** (2004) 59.

[91] T. G. Blackburn and B. King, *Higher fidelity simulations of nonlinear Breit–Wheeler pair creation in intense laser pulses*, *Eur. Phys. J. C* **82** (2022) 44 [2108.10883].

[92] A. Di Piazza, M. Tamburini, S. Meuren and C. H. Keitel, *Improved local-constant-field approximation for strong-field QED codes*, *Phys. Rev. A* **99** (2019) 022125.

[93] H. Bethe and W. Heitler, *On the Stopping of fast particles and on the creation of positive electrons*, *Proc. Roy. Soc. Lond. A* **146** (1934) 83.

[94] C. Muller, *Nonlinear Bethe-Heitler pair creation with attosecond laser pulses at the LHC*, *Phys. Lett. B* **672** (2009) 56 [0811.0976].

[95] U. Hernandez Acosta and B. Kämpfer, *Laser pulse-length effects in trident pair production*, *Plasma Phys. Control. Fusion* **61** (2019) 084011 [1901.08860].

[96] A. I. Titov, U. H. Acosta and B. Kämpfer, *Positron energy distribution in a factorized trident process*, *Phys. Rev. A* **104** (2021) 062811.

[97] W. H. Furry, *On bound states and scattering in positron theory*, *Phys. Rev.* **81** (1951) 115.

[98] V. Berestetskii, E. Lifshitz and L. Pitaevskii, *Quantum Electrodynamics*. Butterworth-Heinemann, Oxford, second ed., 1982.

[99] E. Fradkin, D. Gitman and S. Shvartsman, *Quantum electrodynamics with unstable vacuum*. 12, 1991.

[100] D. M. Volkov, *Uber eine Klasse von Losungen der Diracschen Gleichung*, *Z. Phys.* **94** (1935) 250.

[101] V. I. Ritus, *Quantum effects of the interaction of elementary particles with an intense electromagnetic field*, *J. Sov. Laser Res.* **6** (1985) 497.

[102] A. Hartin, *Fierz relations for Volkov spinors and the simplification of Furry picture traces*, *Phys. Rev. D* **94** (2016) 073002.

[103] A. Di Piazza, *Completeness and orthonormality of the Volkov states and the Volkov propagator in configuration space*, *Phys. Rev. D* **97** (2018) 056028.

[104] V. Ritus, *Radiative effects and their enhancement in an intense electromagnetic field*, *Sov. Phys. JETP* **30** (1970) 052805.

[105] N. B. Narozhny, *Radiation corrections to quantum processes in an intense electromagnetic field*, *Phys. Rev. D* **20** (1979) 1313.

[106] N. B. Narozhny, *Expansion parameter of perturbation theory in intense-field quantum electrodynamics*, *Phys. Rev. D* **21** (1980) 1176.

[107] A. A. Mironov, S. Meuren and A. M. Fedotov, *Resummation of QED radiative corrections in a strong constant crossed field*, *Phys. Rev. D* **102** (2020) 053005.

[108] D. A. Morozov, N. B. Narozhny and V. I. Ritus, *Vertex function of electron in a constant electromagnetic field*, *Sov. Phys. JETP* **53** (1981) 1103.

[109] D. A. Morozov, N. B. Narozhny and V. I. Ritus, *Vertex function of electron in a constant electromagnetic field*, *Fiz. Inst. Akad. Nauk* **84** (1981) 1.

[110] A. Di Piazza and M. A. Lopez-Lopez, *One-loop vertex correction in a plane wave*, *Phys. Rev. D* **102** (2020) 076018.

[111] N. Ahmadiniaz, M. A. Lopez-Lopez and C. Schubert, *Low-energy limit of N-photon amplitudes in a constant field*, *Phys. Lett. B* (2024) 138610.

[112] M. A. Lopez-Lopez, *Low-energy limit of N-photon amplitudes in a constant field: Part II*, *In preparation* (2024) [2408.16474].

[113] N. Ahmadiniaz, M. A. Lopez-Lopez, C. Schubert, R. Schützhold and M. A. Trejo, *Low-energy limit of the four-photon amplitude in a constant field*, *In preparation* .

[114] G. V. Dunne and C. Schubert, *Two loop selfdual Euler-Heisenberg Lagrangians. 2. Imaginary part and Borel analysis*, *JHEP* **06** (2002) 042 [hep-th/0205005].

[115] R. P. Feynman, *Mathematical formulation of the quantum theory of electromagnetic interaction*, *Phys. Rev.* **80** (1950) 440.

[116] R. P. Feynman, *An Operator calculus having applications in quantum electrodynamics*, *Phys. Rev.* **84** (1951) 108.

[117] Z. Bern and D. A. Kosower, *Efficient calculation of one-loop QCD amplitudes*, *Phys. Rev. Lett.* **66** (1991) 1669.

[118] Z. Bern and D. A. Kosower, *The computation of loop amplitudes in gauge theories*, *Nucl. Phys. B* **379** (1992) 451 .

[119] M. J. Strassler, *Field theory without feynman diagrams: One-loop effective actions*, *Nucl. Phys. B* **385** (1992) 145.

[120] M. J. Strassler, *Field theory without Feynman diagrams: A Demonstration using actions induced by heavy particles*, *SLAC-PUB-5978, Unpublished* (1992) .

[121] C. Schubert, *The Structure of the Bern-Kosower integrand for the N gluon amplitude*, *Eur. Phys. J.* **C5** (1998) 693 [hep-th/9710067].

[122] N. Ahmadiniaz, C. Schubert and V. M. Villanueva, *String-inspired representations of photon/gluon amplitudes*, *JHEP* **01** (2013) 132 [1211.1821].

[123] C. Schubert, *Perturbative quantum field theory in the string-inspired formalism*, *Phys. Rep.* **355** (2001) 73 [hep-th/0101036].

[124] O. Corradini, C. Schubert, J. P. Edwards and N. Ahmadiniaz, *Spinning Particles in Quantum Mechanics and Quantum Field Theory*, 12, 2015, 1512.08694.

[125] F. Bastianelli and C. Schubert, *Worldline Path Integrals and Quantum Field Theory*. Cambridge University Press, to appear.

[126] K. Daikouji, M. Shino and Y. Sumino, *Bern-Kosower rule for scalar QED*, *Phys. Rev. D* **53** (1996) 4598.

[127] N. Ahmadiniaz, A. Bashir and C. Schubert, *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, *Phys. Rev. D* **93** (2016) 045023 [1511.05087].

[128] P. Copinger, J. P. Edwards, A. Ilderton and K. Rajeev, *Master Formulae for N-photon tree level amplitudes in plane wave backgrounds*, 2311.14638.

[129] M. G. Schmidt and C. Schubert, *Multiloop calculations in the string-inspired formalism: The single spinor loop in QED*, *Phys. Rev. D* **53** (1996) 2150.

[130] D. Fliegner, M. Reuter, M. Schmidt and C. Schubert, *The Two loop Euler-Heisenberg Lagrangian in dimensional renormalization*, *Theor. Math. Phys.* **113** (1997) 1442 [hep-th/9704194].

[131] I. Huet, D. McKeon and C. Schubert, *Euler-Heisenberg lagrangians and asymptotic analysis in 1+1 QED, part 1: Two-loop*, *JHEP* **12** (2010) 036 [1010.5315].

[132] I. Huet, M. Rausch De Traubenberg and C. Schubert, *Three-loop Euler-Heisenberg Lagrangian in 1+1 QED, part 1: single fermion-loop part*, *JHEP* **03** (2019) 167 [1812.08380].

[133] J. Nicasio, J. P. Edwards, C. Schubert and N. Ahmadiniaz, *Non-perturbative gauge transformations of arbitrary fermion correlation functions in quantum electrodynamics*, 2010.04160.

[134] Z. Bern, L. Dixon and D. A. Kosower, *One-loop corrections to five-gluon amplitudes*, *Phys. Rev. Lett.* **70** (1993) 2677.

[135] F. Bastianelli and A. Zirotti, *Worldline formalism in a gravitational background*, *Nucl. Phys. B* **642** (2002) 372 .

[136] F. Bastianelli, O. Corradini and A. Zirotti, *Dimensional regularization for $\mathcal{N} = 1$ supersymmetric sigma models and the worldline formalism*, *Phys. Rev. D* **67** (2003) 104009 [hep-th/0211134].

[137] N. Ahmadiniaz, F. M. Balli, O. Corradini, J. M. Dávila and C. Schubert, *Compton-like scattering of a scalar particle with N photons and one graviton*, *Nucl. Phys. B* **950** (2020) 114877.

[138] E. D'Hoker and D. G. Gagne, *Worldline path integrals for fermions with scalar, pseudoscalar and vector couplings*, *Nucl. Phys. B* **467** (1996) 272 [hep-th/9508131].

[139] E. D'Hoker and D. G. Gagne, *Worldline path integrals for fermions with general couplings*, *Nucl. Phys. B* **467** (1996) 297 [hep-th/9512080].

[140] M. Mondragon, L. Nellen, M. G. Schmidt and C. Schubert, *Axial couplings on the worldline*, *Phys. Lett. B* **366** (1996) 212 [hep-th/9510036].

[141] M. Mondragon, L. Nellen, M. G. Schmidt and C. Schubert, *Yukawa couplings for the spinning particle and the worldline formalism*, *Phys. Lett. B* **351** (1995) 200 [hep-th/9502125].

[142] H. Gies and K. Langfeld, *Loops and loop clouds: A Numerical approach to the worldline formalism in QED*, *Int. J. Mod. Phys. A* **17** (2002) 966 [hep-ph/0112198].

[143] H. Gies and K. Langfeld, *Quantum diffusion of magnetic fields in a numerical worldline approach*, *Nucl. Phys. B* **613** (2001) 353 [hep-ph/0102185].

[144] G. V. Dunne and C. Schubert, *Worldline instantons and pair production in inhomogeneous fields*, *Phys. Rev. D* **72** (2005) 105004 [hep-th/0507174].

[145] G. V. Dunne, Q.-h. Wang, H. Gies and C. Schubert, *Worldline instantons. II. The Fluctuation prefactor*, *Phys. Rev. D* **73** (2006) 065028 [hep-th/0602176].

[146] R. Gastmans and T. T. Wu, *The Ubiquitous photon: Helicity method for QED and QCD*, vol. 80. 1990.

[147] L. J. Dixon, *Calculating scattering amplitudes efficiently*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond*, pp. 539–584, 1, 1996, hep-ph/9601359.

[148] M. Srednicki, *Quantum Field Theory*. Cambridge University Press, 2007, 10.1017/CBO9780511813917.

[149] M. G. Schmidt and C. Schubert, *On the calculation of effective actions by string methods*, *Phys. Lett. B* **318** (1993) 438 .

[150] C. Itzykson and J. Zuber, *Quantum Field Theory*, International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

[151] J. M. Jauch and F. Rohrlich, *The theory of photons and electrons. The relativistic quantum field theory of charged particles with spin one-half*, Texts and Monographs in Physics. Springer, Berlin, 2nd ed. ed., 1976, 10.1007/978-3-642-80951-4.

[152] W. R. Inc., "Mathematica, Version 12.0."

[153] W. Dittrich and H. Gies, *Probing the quantum vacuum. Perturbative effective action approach in quantum electrodynamics and its application*, vol. 166. 2000, 10.1007/3-540-45585-X.

[154] J. M. Davila, C. Schubert and M. A. Trejo, *Photonic processes in Born-Infeld theory*, *Int. J. Mod. Phys. A* **29** (2014) 1450174 [1310.8410].

[155] M. A. Trejo, *Cotas fenomenológicas para el parámetro libre de la teoría Born-Infeld*, Master's thesis, IFM, UMSNH, 2011.

[156] A. Rebhan and G. Turk, *Polarization effects in light-by-light scattering: Euler–Heisenberg versus Born–Infeld*, *Int. J. Mod. Phys. A* **32** (2017) 1750053 [1701.07375].

[157] K. Scharnhorst, *Photon-photon scattering and related phenomena. Experimental and theoretical approaches: The early period*, 1711.05194.

[158] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin et al., *Sympy: symbolic computing in python*, *PeerJ Comp. Sci.* **3** (2017) e103.

[159] Y. Liang and A. Czarnecki, *Photon-photon scattering: A Tutorial*, *Can. J. Phys.* **90** (2012) 11 [1111.6126].

[160] S. Meuren, C. H. Keitel and A. Di Piazza, *Polarization operator for plane-wave background fields*, *Phys. Rev. D* **88** (2013) 013007.

[161] A. Di Piazza, K. Z. Hatsagortsyan and C. H. Keitel, *Laser-photon merging in proton-laser collisions*, *Phys. Rev. A* **78** (2008) 062109.

[162] S. Hassani, *Mathematical physics: a modern introduction to its foundations*. Springer Science & Business Media, New York, 2013.

[163] J. P. Edwards, C. M. Mata, U. Müller and C. Schubert, *New Techniques for Worldline Integration*, *SIGMA* **17** (2021) 065 [2106.12071].

[164] V. O. Papanyan and V. I. Ritus, *Three-photon interaction in an intense field and scaling invariance*, *Zh. Eksp. Teor. Fiz.* **65** (1973) 1756.

[165] A. Barducci, F. Bordi and R. Casalbuoni, *Path integral quantization of spinning particles interacting with crossed external electromagnetic fields*, *Nuov. Cim. B* **64** (1981) 287.

[166] H. Bateman and B. M. Project, *Higher Transcendental Functions [Volumes I-III]*. McGraw-Hill Book Company, 1953.

[167] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh ed., 2007.

[168] G. B. Arfken, H. J. Weber and F. E. Harris, *Mathematical Methods for Physicists*. Academic Press, Boston, seventh ed., 2013.

[169] C. M. Kim and S. P. Kim, *Schwinger Pair Production and Vacuum Birefringence around Highly Magnetized Neutron Stars*, *Astron. Rep.* **67** (2023) S122 [2308.15830].

[170] C. Schubert, *Lecture notes on the worldline approach to QED*, *Unpublished* (2020) .

[171] G. Mogull, J. Plefka and J. Steinhoff, *Classical black hole scattering from a worldline quantum field theory*, *JHEP* **02** (2021) 048 [2010.02865].

[172] D. Hanneke, S. Fogwell and G. Gabrielse, *New measurement of the electron magnetic moment and the fine structure constant*, *Phys. Rev. Lett.* **100** (2008) 120801 [0801.1134].

[173] L. Morel, Z. Yao, P. Cladé and S. Guellati-Khélifa, *Determination of the fine-structure constant with an accuracy of 81 parts per trillion*, *Nature* **588** (2020) 61.

[174] NA64 Collaboration collaboration, Y. M. Andreev, D. Banerjee, J. Bernhard, V. E. Burtsev, A. G. Chumakov, D. Cooke et al., *Constraints on new physics in electron $g-2$ from a search for invisible decays of a scalar, pseudoscalar, vector, and axial vector*, *Phys. Rev. Lett.* **126** (2021) 211802.

[175] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[176] F. Bloch and A. Nordsieck, *Note on the radiation field of the electron*, *Phys. Rev.* **52** (1937) 54.

[177] G. S. Adkins, M. Lymberopoulos and D. D. Velkov, *One-loop vertex function in Yennie gauge QED*, *Phys. Rev. D* **50** (1994) 4194.

[178] V. N. Baier, V. M. Katkov and V. M. Strakhovenko, *Operator Approach to Quantum Electrodynamics in External Field: Mass Operator*, *Zh. Eksp. Teor. Fiz.* **67** (1974) 453.

[179] V. N. Baier, V. M. Katkov and V. M. Strakhovenko, *Operator Approach to Quantum Electrodynamics in External Field. Electron Loops*, *Zh. Eksp. Teor. Fiz.* **68** (1975) 405.

[180] V. O. Papanyan and V. I. Ritus, *Vacuum polarization and photon splitting in an intense field*, *Zh. Eksp. Teor. Fiz.* **61** (1971) 2231.

[181] B. King, *Double Compton scattering in a constant crossed field*, *Phys. Rev. A* **91** (2015) 033415.

[182] V. N. Baier, V. M. Katkov, A. I. Milstein and V. M. Strakhovenko, *The theory of quantum processes in the field of a strong electromagnetic wave*, *Sov. Phys. JETP* **42** (1976) 400.

[183] V. N. Baier, A. I. Milstein and V. Strakhovenko, *Interaction between a photon and an intense electromagnetic wave*, *Sov. Phys. JETP* **42** (1976) 961.

[184] A. J. Macleod, J. P. Edwards, T. Heinzl, B. King and S. V. Bulanov, *Strong-field vacuum polarisation with high energy lasers*, *New J. Phys.* **25** (2023) 093002 [2304.02114].

[185] F. Mackenroth and A. Di Piazza, *Nonlinear Double Compton Scattering in the Ultrarelativistic Quantum Regime*, *Phys. Rev. Lett.* **110** (2013) 070402.

[186] D. Seipt and B. Kämpfer, *Two-photon Compton process in pulsed intense laser fields*, *Phys. Rev. D* **85** (2012) 101701.

[187] C. Sundqvist and F. Karbstein, *Two-beam laser photon merging*, *Phys. Rev. D* **108** (2023) 056028 [2303.11904].

[188] A. Ilderton and G. Torgrimsson, *Scattering in plane-wave backgrounds: Infrared effects and pole structure*, *Phys. Rev. D* **87** (2013) 085040.

Acknowledgments

I would like to thank my family and friends for being part of my life.

I am particularly grateful to Ralf Schützhold, Naser Ahmadiniaz, Christian Schubert and Antonino Di Piazza for supervise me during this thesis.

I would like to thank Anabel Trejo, Michael G. Schmidt, Felix Karbstein, James Edwards, Cesar Mata, Victor Banda, A. M. Fedotov, S. Meuren, A. A. Mironov for a fruitful discussions. I wish to thank K. Scharnhorst, G. S. Adkins for correspondence.

I would like to express my sincere gratitude to *Consejo Nacional de Ciencia y Tecnología* for financial support during the first period of this project, as well as, to Helmholtz-Zentrum Dresden-Rossendorf for financial support during the second period of this project.

I gratefully acknowledge the hospitality of C. H. Keitel and of the Max Planck Institute for Nuclear Physics. I gratefully acknowledge the hospitality of F. Karbstein and of Helmholtz-Institut Jena.

Erklärung zum Antrag auf Eröffnung des Promotionsverfahrens

1. Die folgende Promotionsordnung in ihrer gültigen Fassung erkenne ich an:

   Bereich Mathematik und Naturwissenschaften - Promotionsordnung vom 23.02.2011.

2. Die Promotion wurde an folgendem Institut/an folgender Professur durchgeführt:

   Helmholtz-Zentrum Dresden-Rossendorf.

3. Folgende Personen haben die Promotion wissenschaftlich betreut und/oder mich bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts unterstützt:

   Prof. Dr. Ralf Schützhold,

   Prof. Dr. Christian Schubert,

   Prof. Dr. Antonino Di Piazza,

   Dr. Naser Ahmadiniaz.

4. Ich bestätige, dass für meine Person bisher keine früheren, erfolglosen Promotionsverfahren stattgefunden haben. Andernfalls habe ich diesem Antrag eine Erklärung bzw. Nachweise beigelegt, in dem ersichtlich ist, wo, wann, mit welchem Thema und mit welchem Bescheid diese Promotionsversuche stattgefunden haben

5. Ich versichere weiterhin, dass

   (a) ich die vorliegende Arbeit mit dem Titel “QED transition amplitudes in external fields” ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel selbst angefertigt habe. Hilfe Dritter wurde nur in wissenschaftlich vertretbarem und prüfungsrechtlich zulässigem Ausmaß in Anspruch genommen. Es sind keine unzulässigen geldwerten Leistungen, weder unmittelbar noch mittelbar, im Zusammenhang mit dem Inhalt der vorliegenden Dissertation an Dritte erfolgt. Insbesondere habe ich nicht die Hilfe einer kommerziellen Promotionsberatung in Anspruch genommen.

   (b) die aus fremden Quellen direkt oder indirekt übernommenen Gedanken als solche kenntlich gemacht sind.

   (c) ich die vorliegende Arbeit bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde zum Zwecke einer Promotion oder eines anderen Prüfungsverfahrens vorgelegt habe und diese auch noch nicht veröffentlicht worden ist.

6. Mir ist bekannt, dass die Nichteinhaltung dieser Erklärung oder unrichtige Angaben zum Verfahrensabbruch oder zum nachträglichen Entzug des Doktortitels führen können.

Ort, Datum

Unterschrift Antragsteller:in