High Energy Cosmology

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Settore Scientifico: Astronomia e Astrofisica FIS 05

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Contents

1 Introduction **1**

2 The Big-Bang Model **4**
2.1 The Friedmann equations 4
2.2 The Early Universe 7
2.3 The Standard Big Bang Problems 10

3 The Inflationary Universe **16**
3.1 Inflationary Solution to the Standard Big Bang Problems 16
3.2 The Inflaton 18
3.3 The Slow-roll 19
3.4 Inflation and Reheating 22
3.5 The Inflationary models 23

4 The Cosmological Perturbations **25**
4.1 Quantum fluctuations of a scalar field during inflation 27
4.2 The Power Spectrum 31
4.3 The Metric Fluctuations 35
4.4 The comoving curvature perturbation 46
4.5 Adiabatic and isocurvature perturbations 48
4.6 Gravitational waves 58
4.7 The post-inflationary evolution of the cosmological perturbations 60

5 High Energy Cosmological Models **68**
5.1 Transplanckian physics 77
5.2 String Theory Inflation 80
5.3 String Cosmology 81
5.4 Brane cosmology 87
5.5 Strings in de Sitter space 95
5.6 Holography 97

6 Trans-Planckian Effects on CMB **103**
6.1 The sub-Planckian effective theory 103
6.2 Minimum uncertainty principle 104
6.3 Power-law inflation 105

7 Baryogenesis **109**
7.1 The Spontaneous Baryogenesis Mechanism 110
7.2 Radion Induced Spontaneous Baryogenesis 111
7.3 Applications 113

8 Brane Cosmology **115**
8.1 Einstein equations 116
8.2 The Low Density Expansion 119
8.3 First order results 120

8.4 Second order results 124
8.5 Second order cosmology 125
8.6 RS I 126
8.7 RS II 128
8.8 Approximation analysis 130
8.9 Summary 132

9 Gauss-Bonnet Brane Inflation **134**
9.1 Einstein equations and Static Solutions 134
9.2 The perturbed solutions 136
9.3 Summary 139

10 Conclusions **140**

A Evolution of the curvature perturbation on superhorizon scales **141**

References **142**

1 Introduction

Recent years have provided a wealth of observational data about the cosmos. We have high resolution maps of the anisotropies in the temperature of the cosmic microwave background (CMB) [1], surveys of the large-scale structure (LSS) - the distribution of galaxies in three-dimensional space - are increasing in size and in accuracy (see e.g. [2] and [3]), and new techniques which will allow us to measure the distribution of the dark matter are being pioneered. All of this data involves small deviations of the cosmos from homogeneity and isotropy. The cosmological observations reveal that the Universe has non-random fluctuations on all scales smaller than the present Hubble radius.

Parallel to this spectacular progress in observational cosmology, new cosmological scenarios have emerged within which it is possible to explain the origin of non-random inhomogeneities by means of causal physics. The scenario which has attracted most attention is inflationary cosmology [4, 5], according to which there was a period in the early Universe in which space was expanding at an accelerated rate. One of its basic ideas is that there was an epoch early in the history of the universe when potential, or vacuum, energy dominated other forms of energy density such as matter or radiation. During the vacuum-dominated era the scale factor grew exponentially (or nearly exponentially) in time. In this phase, known as inflation, a small, smooth spatial region of size less than the Hubble radius at that time can grow so large as to easily encompass the comoving volume of the entire presently observable universe. If the early universe underwent this period of rapid expansion, then one can understand why the observed universe is so homogeneous and isotropic to high accuracy. All these virtues of inflation were noted when it was first proposed by Guth in 1981 [35]. A more dramatic consequence of the inflationary paradigm was noticed soon after [38, 72, 36]. Starting with a universe which is absolutely homogeneous and isotropic at the classical level, the inflationary expansion of the universe will 'freeze in' the vacuum fluctuation of the inflaton field so that it becomes an essentially classical quantity. On each comoving scale, this happens soon after horizon exit. Associated with this vacuum fluctuation is a primordial energy density perturbation, which survives after inflation and may be the origin of all structure in the universe. In particular, it may be responsible for the observed cosmic microwave background (CMB) anisotropy and for the large-scale distribution of galaxies and dark matter. Inflation also generates primordial gravitational waves as a vacuum fluctuation, which may contribute to the low multipoles of the CMB anisotropy. Therefore, a prediction of inflation is that all of the structure we see in the universe is a result of quantum-mechanical fluctuations during the inflationary epoch.

However, there are also alternative proposals [7, 8] in which our current stage of cosmological expansion is preceded by a phase of contraction. These scenarios have in common the fact that for scales of cosmological interest today, although their physical wavelength is larger than the Hubble length during most of the history of the universe, it is smaller than the Hubble radius at very early times, thus in principle allowing for a causal origin of the cosmological fluctuations.

In order to connect theories of fundamental physics providing an origin of perturbations with the data on the late time universe, one must be able to evolve cosmological fluctuations from earliest times to today. Since on large scales (scales larger than about 10 Mpc - 1 Mpc being roughly three million light years) the relative density fluctuations are smaller than one today, and since these relative fluctuations grow in time as a consequence of gravitational instability, they were smaller than one throughout their history - at least in a universe which is always expanding. Thus, it is reasonable to expect that a linearized analysis of the fluctuations will give reliable results. Inflationary cosmology is at the present time the most successful framework of connecting physics of the very early universe with the present structure (although alternatives such as the Pre-Big-Bang [7] and Ekpyrotic [8] scenarios have been proposed and may turn out to be successful as well).

Within this framework one is face with the so called trans-planckian problem, whose name is due to the fact that inflation magnifies all quantum fluctuations and, therefore, red-shifts originally trans-Planckian frequencies down to the range of low energy physics. This causes two main concerns: first of all, there is currently no universally accepted (if at all) theory of quantum gravity which allows us to describe the original quantum fluctuations in such an high energy regime; further, it is not clear whether the red-shifted trans-Planckian frequencies can indeed be observed with the precision of present and future experiments.

Regarding the first problem, one can take the pragmatic approach of modern renormalization theory and assume that quantum fluctuations are effectively described by quantum field theory after they have been red-shifted below the scale of quantum gravity, henceforth called Λ, and forget about their previous dynamics. The second problem is instead more of a phenomenological interest and needs actual investigation to find the size of corrections to the CMBR. It then seems that the answer depends on the details of the model that one considers and no general consensus has been reached so far. In Ref. [157], a principle of least uncertainty on the quantum fluctuations at the time of emergence from the Planckian domain (when the physical momentum $p \sim \Lambda$) was imposed. Without a good understanding of physics at the Planck scale, this can be regarded as an empirical way of accounting for new physics. In the present thesis, we apply the same approach to power-law inflation, where some new interesting feature emerge. This will allow us to check the final result against an inflationary model with time-dependent Hubble parameter.

In the cosmological context theories with extra dimensions have become of increasing importance. The main role of such theories, originally introduced in the 20's by Kaluza and Klein [186, 187], is to provide a connection between particle physics and gravity at some level. At a deeper level, string theory unifies all the interactions by means of some n-dimensional manifold (with $n > 4$) where the fundamental objects are supposedly living; at a more phenomenological level, models which assume the existence of extra dimensions, no matter their origin, are considered in order to solve some puzzles of particle physics, cosmology and astrophysics, giving rise to many possible observable consequences.

Originally proposed in order to solve the problem of the large hierarchy between Gravity and Standard Model scales, the Randall-Sundrum model of Ref. [188] (RS I) has acquired considerable relevance due to its stringy inspiration. It represents the prototype of the so-called brane-world and differs from previous models in that it constrains standard matter on a four-dimensional manifold (the brane) just letting gravity (and exotic matter) propagate everywhere. The RS I solution to the hierarchy problem needs one additional compactified (orbifolded) spatial dimension with two branes located at its fixed points, plus a negative cosmological constant filling the space between such branes (the bulk). The bulk cosmological constant Λ warps the extra dimension and generates the effective four-dimensional physical constants we measure. It was soon realized that the modifications to four-dimensional gravity induced by the fifth dimension may be reduced to such a short distance effect to be unobservable even in the presence of just one brane and infinite compactification radius.

The cosmological features of the RS models are nowadays being investigated even more than its particle physics consequences, due to the refined results lately obtained and to the major problems recent astrophysical data have revealed: the possible late time acceleration from supernovae, CMBR spectrum, dark matter and dark energy quests suggest either a full revision of the modern theoretical physics approach or the possibility of the existence of further, up to now ignored, ingredients such as the extra dimensions. We will examine these models with the intent to give a possible solution to the observed current cosmological acceleration and to the problem of dynamically generating the baryonic asymmetry.

One of the most peculiar features of our Universe is, in fact, the observed baryonic asymmetry. This can be conveniently characterized by the dimensionless number $n_B/s \equiv \eta \simeq 10^{-10}$ where $n_B \equiv n_b - n_{\bar{b}}$ is the difference between the baryon and anti-baryon densities and s is the density of entropy. The consistency of primordial nucleosynthesis, which yields some of the most precise results in the standard model of cosmology, requires that η took the above value at the time when the light elements (*i.e.*, ^{3}He, ^{4}He, and ^{7}Li) were produced, and it is believed to have then remained the same up to the present epoch. The necessary conditions for generating the baryonic asymmetry in quantum field theory were formulated by Sacharov in 1967 [162] (see also Ref. [163]). The so called mechanism of spontaneous baryogenesis [166, 167] uses the natural (strong) CPT non-invariance of the Universe during its early history to bypass this third condition. We know that an expanding Universe at finite temperature violates both Lorentz invariance and time reversal, and this can lead to effective CPT violating interactions [164, 165]. Thus the cosmological expansion of the early Universe leads us naturally to examine the possibility of generating the baryon asymmetry in thermal equilibrium. The main ingredient for implementing this mechanism is a scalar field ϕ with a derivative coupling to the baryonic current. The brane-world model with two branes proposed by Randall and Sundrum (RS) in Ref. [168] contains a metric degree of freedom called the *radion* which determines the distance between the two branes and appears as a scalar field ϕ on the branes. Cosmological solutions have also been examined rather extensively in this context. In particular, it has been shown that, when matter is added on one (or both) of the two branes, the standard Friedmann equation for the scale factor of the Universe is recovered (with possible corrections) provided the radion is suitably stabilized. In this brane-world model, we therefore have both a scalar field (the radion) and the cosmological evolution as required by spontaneous baryogenesis, and we shall show that the radion field does in fact couple differently with baryons and anti-baryons. This scenario might therefore naturally reproduce the observed baryonic asymmetry In section 2 we give a brief review of the Big-Bang theory. In section 3 we describe the idea of inflation as solution to the shortcomings of the Big-Bang theory and in section 4 we present the modern theory of the Cosmological perturbations as seeds of the large scale structures and of CMB fluctuations. In section 5 we introduce the basic concepts of high energy cosmology which are applied in to the cosmological transplanckian problem (section 6), to a model possibily generating the observed baryonic asymmetry (section 7) and to the problem of the observed present cosmological acceleration (section 8 and 9).

2 The Big-Bang Model

The standard cosmology is based upon the maximally spatially symmetric Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]; \tag{1}$$

where $a(t)$ is the cosmic-scale factor, $R_{\rm curv} \equiv a(t)|k|^{-1/2}$ is the curvature radius, and $k = -1, 0, 1$ is the curvature signature. All three models are without boundary: the positively curved model is finite and "curves" back on itself; the negatively curved and flat models are infinite in extent. The Robertson-Walker metric embodies the observed isotropy and homogeneity of the universe. It is interesting to note that this form of the line element was originally introduced for sake of mathematical simplicity; we now know that it is well justified at early times or today on large scales ($\gg 10\,{\rm Mpc}$), at least within our visible patch.

The coordinates, r, θ, and ϕ, are referred to as *comoving* coordinates: A particle at rest in these coordinates remains at rest, *i.e.*, constant r, θ, and ϕ. A freely moving particle eventually comes to rest these coordinates, as its momentum is red shifted by the expansion, $p \propto a^{-1}$. Motion with respect to the comoving coordinates (or cosmic rest frame) is referred to as peculiar velocity; unless "supported" by the inhomogeneous distribution of matter peculiar velocities decay away as a^{-1}. Thus the measurement of peculiar velocities, which is not easy as it requires independent measures of both the distance and velocity of an object, can be used to probe the distribution of mass in the universe.

Physical separations between freely moving particles scale as $a(t)$; or said another way the physical separation between two points is simply $a(t)$ times the coordinate separation. The momenta of freely propagating particles decrease, or "red shift," as $a(t)^{-1}$, and thus the wavelength of a photon stretches as $a(t)$, which is the origin of the cosmological red shift. The red shift suffered by a photon emitted from a distant galaxy $1 + z = a_0/a(t)$; that is, a galaxy whose light is red shifted by $1+z$, emitted that light when the universe was a factor of $(1+z)^{-1}$ smaller. Thus, when the light from the most distant quasar yet seen ($z = 4.9$) was emitted the universe was a factor of almost six smaller; when CMB photons last scattered the universe was about 1100 times smaller.

2.1 The Friedmann equations

The evolution of the scale factor $a(t)$ is governed by Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} \equiv G_{\mu\nu} = 8\pi G \, T_{\mu\nu} \tag{2}$$

where $R_{\mu\nu}$ $(\mu, \nu = 0, \cdots 3)$ is the Riemann tensor and R is the Ricci scalar constructed via the metric (525) [45] and $T_{\mu\nu}$ is the energy-momentum tensor. Under the hypothesis of homogeneity and isotropy, we can always write the energy-momentum tensor under the form $T_{\mu\nu} = {\rm diag}\,(\rho, p, p, p)$ where ρ is the energy density of the system and p its pressure. They are functions of time.

The evolution of the cosmic-scale factor is governed by the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}; \tag{3}$$

where ρ is the total energy density of the universe, matter, radiation, vacuum energy, and so on.

Differentiating wrt to time both members of Eq. (177) and using the the mass conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{4}$$

we find the equation for the acceleration of the scale-factor

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{5}$$

Combining Eqs. (3) and (5) we find

$$\dot{H} = -4\pi G\left(\rho + p\right). \tag{6}$$

The evolution of the energy density of the universe is governed by

$$d(\rho a^3) = -pd\left(a^3\right); \tag{7}$$

which is the First Law of Thermodynamics for a fluid in the expanding universe. (In the case that the stress energy of the universe is comprised of several, noninteracting components, this relation applies to each separately; *e.g.*, to the matter and radiation separately today.) For $p = \rho/3$, ultra-relativistic matter, $\rho \propto a^{-4}$ and $a \sim t^{\frac{1}{2}}$; for $p = 0$, very nonrelativistic matter, $\rho \propto a^{-3}$ and $a \sim t^{\frac{2}{3}}$; and for $p = -\rho$, vacuum energy, $\rho =$const. If the rhs of the Friedmann equation is dominated by a fluid with equation of state $p = \gamma\rho$, it follows that $\rho \propto a^{-3(1+\gamma)}$ and $a \propto t^{2/3(1+\gamma)}$.

We can use the Friedmann equation to relate the curvature of the universe to the energy density and expansion rate:

$$\Omega - 1 = \frac{k}{a^2H^2}; \qquad \Omega = \frac{\rho}{\rho_{\rm crit}}; \tag{8}$$

and the critical density today $\rho_{\rm crit} = 3H^2/8\pi G = 1.88h^2\,{\rm g\,cm^{-3}} \simeq 1.05 \times 10^4\,{\rm eV\,cm^{-3}}$. There is a one to one correspondence between Ω and the spatial curvature of the universe: positively curved, $\Omega_0 > 1$; negatively curved, $\Omega_0 < 1$; and flat ($\Omega_0 = 1$). Further, the "fate of the universe" is determined by the curvature: model universes with $k \leq 0$ expand forever, while those with $k > 0$ necessarily recollapse. The curvature radius of the universe is related to the Hubble radius and Ω by

$$R_{\rm curv} = \frac{H^{-1}}{|\Omega - 1|^{1/2}}. \tag{9}$$

In physical terms, the curvature radius sets the scale for the size of spatial separations where the effects of curved space become "pronounced." And in the case of the positively curved model it is just the radius of the 3-sphere.

The energy content of the universe consists of matter and radiation (today, photons and neutrinos). Since the photon temperature is accurately known, $T_0 = 2.73 \pm 0.01\,\mathrm{K}$, the fraction of critical density contributed by radiation is also accurately known: $\Omega_R h^2 = 4.2 \times 10^{-5}$, where $h = 0.72 \pm 0.07$ is the present Hubble rate in units of 100 km sec^{-1} Mpc^{-1} [34]. The remaining content of the universe is another matter. Rapid progress has been made recently toward the measurement of cosmological parameters [15]. Over the past three years the basic features of our universe have been determined. The universe is spatially flat; accelerating; comprised of one third of dark matter and two third a new form of dark energy. The measurements of the cosmic microwave background anisotropies at different angular scales performed by Boomerang, Maxima, DASI, CBI and VSA have recently significantly increase the case for accelerated expansion in the early universe (the inflationary paradigm) and at the current epoch (dark energy dominance), especially when combined with data on high redshift supernovae (SN1) and large scale structure (LSS) [15]. A recent analysis [23] shows that the CMB+LSS+SN1 data give

$$\Omega_0 = 1.00^{+0.07}_{-0.03},$$

meaning tha the present universe is spatially flat (or at least very close to being flat). Restricting to $\Omega_0 = 1$, the dark matter density is given by [23]

$$\Omega_{\rm DM} h^2 = 0.12^{+0.01}_{-0.01},$$

and a baryon density

$$\Omega_B h^2 = 0.022^{+0.003}_{-0.002},$$

while the Big Bang nucleosynthesis estimate is $\Omega_B h^2 = 0.019 \pm 0.002$. Substantial dark (unclustered) energy is inferred,

$$\Omega_Q \approx 0.68 \pm 0.05,$$

compatible with the independent SN1 estimates! What is most relevant for us, this universe is apparently born from a burst of rapid expansion, inflation, during which quantum noise was stretched to astrophysical size seeding cosmic structure. This is exactly the phenomena we want to address.

Before launching ourselves into the description of the early universe, we would like to introduce the concept of conformal time which will be useful in the next sections. The conformal time τ is defined through the following relation

$$d\tau = \frac{dt}{a}. \tag{10}$$

The metric (525) then becomes

$$ds^2 = -a^2(\tau)\left[d\tau^2 - \frac{dr^2}{1-kr^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{11}$$

The reason why τ is called conformal is manifest from Eq. (11): the corresponding FRW line element is conformal to the Minkowski line element describing a static four dimensional hypersurface.

Any function $f(t)$ satisfies the rule

$$\dot{f}(t) = \frac{f'(\tau)}{a(\tau)}, \tag{12}$$

$$\ddot{f}(t) = \frac{f''(\tau)}{a^2(\tau)} - \mathcal{H}\frac{f'(\tau)}{a^2(\tau)}, \tag{13}$$

where a prime now indicates differentation wrt to the conformal time τ and

$$\mathcal{H} = \frac{a'}{a}. \tag{14}$$

In particular we can set the following rules

$$\begin{aligned}
H &= \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\mathcal{H}}{a}, \\
\ddot{a} &= \frac{a''}{a^2} - \frac{\mathcal{H}^2}{a}, \\
\dot{H} &= \frac{\mathcal{H}'}{a^2} - \frac{\mathcal{H}^2}{a^2}, \\
H^2 &= \frac{8\pi G\rho}{3} - \frac{k}{a^2} \Longrightarrow \mathcal{H}^2 = \frac{8\pi G\rho a^2}{3} - k \\
\dot{H} &= -4\pi G\,(\rho + p) \Longrightarrow \mathcal{H}' = -\frac{4\pi G}{3}\,(\rho + 3p)\,a^2, \\
\dot{\rho} &+ 3H(\rho + p) = 0 \Longrightarrow \rho' + 3\,\mathcal{H}(\rho + p) = 0
\end{aligned}$$

Finally, if the scale factor $a(t)$ scales like $a \sim t^n$, solving the relation (10) we find

$$a \sim t^n \Longrightarrow a(\tau) \sim \tau^{\frac{n}{1-n}}. \tag{15}$$

2.2 The Early Universe

In any case, at present, matter outweighs radiation by a wide margin. However, since the energy density in matter decreases as a^{-3}, and that in radiation as a^{-4} (the extra factor due to the red shifting of the energy of relativistic particles), at early times the universe was radiation dominated—indeed the calculations of primordial nucleosynthesis provide excellent evidence for this. Denoting the epoch of matter-radiation equality by subscript 'EQ,' and using $T_0 = 2.73\,\mathrm{K}$, it follows that

$$a_{\rm EQ} = 4.18 \times 10^{-5}\,(\Omega_0 h^2)^{-1}; \qquad T_{\rm EQ} = 5.62(\Omega_0 h^2)\,\mathrm{eV}; \tag{16}$$

$$t_{\rm EQ} = 4.17 \times 10^{10} (\Omega_0 h^2)^{-2}\,\mathrm{sec} \tag{17}$$

At early times the expansion rate and age of the universe were determined by the temperature of the universe and the number of relativistic degrees of freedom:

$$\rho_{\rm rad} = g_*(T)\frac{\pi^2 T^4}{30}; \qquad H \simeq 1.67 g_*^{1/2} T^2/m_{\rm Pl}; \tag{18}$$

$$\Rightarrow a \propto t^{1/2}; \qquad t \simeq 2.42 \times 10^{-6} g_*^{-1/2} (T/\,{\rm GeV})^{-2}\ {\rm sec}; \tag{19}$$

where $g_*(T)$ counts the number of ultra-relativistic degrees of freedom ($\approx$ the sum of the internal degrees of freedom of particle species much less massive than the temperature) and $m_{\rm Pl} \equiv G^{-1/2} = 1.22 \times 10^{19}\,{\rm GeV}$ is the Planck mass. For example, at the epoch of nucleosynthesis, $g_* = 10.75$ assuming three, light ($\ll$ MeV) neutrino species; taking into account all the species in the standard model, $g_* = 106.75$ at temperatures much greater than 300 GeV.

A quantity of importance related to g_* is the entropy density in relativistic particles,

$$s = \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_* T^3,$$

and the entropy per comoving volume,

$$S \quad \propto \quad a^3 s \quad \propto \quad g_* a^3 T^3.$$

By a wide margin most of the entropy in the universe exists in the radiation bath. The entropy density is proportional to the number density of relativistic particles. At present, the relativistic particle species are the photons and neutrinos, and the entropy density is a factor of 7.04 times the photon-number density: $n_\gamma = 413\,{\rm cm}^{-3}$ and $s = 2905\,{\rm cm}^{-3}$.

In thermal equilibrium—which provides a good description of most of the history of the universe—the entropy per comoving volume S remains constant. This fact is very useful. First, it implies that the temperature and scale factor are related by

$$T \propto g_*^{-1/3} a^{-1}, \tag{20}$$

which for $g_* =$const leads to the familiar $T \propto a^{-1}$.

Second, it provides a way of quantifying the net baryon number (or any other particle number) per comoving volume:

$$N_B \equiv R^3 n_B = \frac{n_B}{s} \simeq (4-7) \times 10^{-11}. \tag{21}$$

The baryon number of the universe tells us two things: (1) the entropy per particle in the universe is extremely high, about 10^{10} or so compared to about 10^{-2} in the sun and a few in the core of a newly formed neutron star. (2) The asymmetry between matter and antimatter is very small, about 10^{-10}, since at early times quarks and antiquarks were roughly as abundant as photons. One of the great successes of particle cosmology is baryogenesis, the idea that B, C, and CP violating interactions occurring out-of-equilibrium early on allow the universe to develop a net baryon number of this magnitude [68, 69].

Finally, the constancy of the entropy per comoving volume allows us to characterize the size of comoving volume corresponding to our present Hubble volume in a very physical way: by the entropy it contains,

$$S_U = \frac{4\pi}{3} H_0^{-3} s \simeq 10^{90}. \tag{22}$$

The standard cosmology is tested back to times as early as about 0.01 sec; it is only natural to ask how far back one can sensibly extrapolate. Since the fundamental particles of Nature are point-like quarks and leptons whose interactions are perturbatively weak at energies much greater than 1 GeV, one can imagine extrapolating as far back as the epoch where general relativity becomes suspect, i.e., where quantum gravitational effects are likely to be important: the Planck epoch, $t \sim 10^{-43}$ sec and $T \sim 10^{19}$ GeV. Of course, at present, our firm understanding of the elementary particles and their interactions only extends to energies of the order of 100 GeV, which corresponds to a time of the order of 10^{-11} sec or so. We can be relatively certain that at a temperature of 100 MeV − 200 MeV ($t \sim 10^{-5}$ sec) there was a transition (likely a second-order phase transition) from quark/gluon plasma to very hot hadronic matter, and that some kind of phase transition associated with the symmetry breakdown of the electroweak theory took place at a temperature of the order of 300 GeV ($t \sim 10^{-11}$ sec).

In spite of the fact that the universe was vanishingly small at early times, the rapid expansion precluded causal contact from being established throughout. Photons travel on null paths characterized by $dr = dt/a(t)$; the physical distance that a photon could have traveled since the bang until time t, the distance to the particle horizon, is

$$
\begin{aligned}
R_H(t) &= a(t) \int_0^t \frac{dt'}{a(t')} \\
&= \frac{t}{(1-n)} = n\,\frac{H^{-1}}{(1-n)} \sim H^{-1} \qquad \text{for } a(t) \propto t^n, \quad n<1. \qquad (23)
\end{aligned}
$$

Using the conformal time, the particle horizon becomes

$$
R_H(t) = a(\tau) \int_{\tau_0}^{\tau} d\tau, \qquad (24)
$$

where τ_0 indicates the conformal time corresponding to $t = 0$. Note, in the standard cosmology the distance to the horizon is finite, and up to numerical factors, equal to the age of the universe or the Hubble radius, H^{-1}. For this reason, we will use horizon and Hubble radius interchangeably.[1]

Note also that a physical length scale λ is within the horizon if $\lambda < R_H \sim H^{-1}$. Since we can identify the length scale λ with its wavenumber k, $\lambda = 2\pi a/k$, we will have the following rule

$$
\begin{aligned}
\frac{k}{aH} &\ll 1 \Longrightarrow \text{SCALE } \lambda \text{ OUTSIDE THE HORIZON} \\
\frac{k}{aH} &\gg 1 \Longrightarrow \text{SCALE } \lambda \text{ WITHIN THE HORIZON}
\end{aligned}
$$

[1] As we shall see, in inflationary models the horizon and Hubble radius are not roughly equal as the horizon distance grows exponentially relative to the Hubble radius; in fact, at the end of inflation they differ by e^N, where N is the number of e-folds of inflation. However, we will slip and use "horizon" and "Hubble radius" interchangeably, though we will always mean Hubble radius.

An important quantity is the entropy within a horizon volume: $S_{\rm HOR} \sim H^{-3}T^3$; during the radiation-dominated epoch $H \sim T^2/m_{\rm Pl}$, so that

$$S_{\rm HOR} \sim \left(\frac{m_{\rm Pl}}{T}\right)^3; \tag{25}$$

from this we will shortly conclude that at early times the comoving volume that encompasses all that we can see today (characterized by an entropy of about 10^{90}) was comprised of a very large number of causally disconnected regions.

By now the shortcomings of the standard cosmology are well appreciated: the horizon or large-scale smoothness problem; the small-scale inhomogeneity problem (origin of density perturbations); and the flatness or oldness problem. We will only briefly review them here. They do not indicate any logical inconsistencies of the standard cosmology; rather, that very special initial data seem to be required for evolution to a universe that is qualitatively similar to ours today. Nor is inflation the first attempt to address these shortcomings: over the past two decades cosmologists have pondered this question and proposed alternative solutions. Inflation is a solution based upon well-defined, albeit speculative, early universe microphysics describing the post-Planck epoch.

2.3 The Standard Big Bang Problems

Let us make a tremendous extrapolation and assume that Einstein equations are valid until the Plank era, when the temperature of the universe is $T_{\rm Pl} \sim m_{\rm Pl} \sim 10^{19}$ GeV. From Eq. (8), we read that if the universe is perfectly flat, then $(\Omega = 1)$ at all times. On the other hand, if there is even a small curvature term, the time dependence of $(\Omega - 1)$ is quite different.

During a radiation-dominated period, we have that $H^2 \propto \rho_R \propto a^{-4}$ and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-4}} \propto a^2. \tag{26}$$

During Matter Domination, $\rho_M \propto a^{-3}$ and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-3}} \propto a. \tag{27}$$

In both cases $(\Omega - 1)$ decreases going backwards with time. Since we know that today $(\Omega_0 - 1)$ is of order unity at present, we can deduce its value at $t_{\rm Pl}$ (the time at which the temperature of the universe is $T_{\rm Pl} \sim 10^{19}$ GeV)

$$\frac{\mid \Omega - 1 \mid_{T=T_{\rm Pl}}}{\mid \Omega - 1 \mid_{T=T_0}} \approx \left(\frac{a^2_{\rm Pl}}{a^2_0}\right) \approx \left(\frac{T^2_0}{T^2_{\rm Pl}}\right) \approx \mathcal{O}(10^{-64}). \tag{28}$$

where 0 stands for the present epoch, and $T_0 \sim 10^{-13}$ GeV is the present-day temperature of the CMB radiation. If we are not so brave and go back simply to the epoch of nucleosynthesis when light elements abundances were formed, at $T_N \sim 1$ MeV, we get

$$\frac{\mid \Omega - 1 \mid_{T=T_N}}{\mid \Omega - 1 \mid_{T=T_0}} \approx \left(\frac{a^2_N}{a^2_0}\right) \approx \left(\frac{T^2_0}{T^2_N}\right) \approx \mathcal{O}(10^{-16}). \tag{29}$$

In order to get the correct value of $(\Omega_0 - 1) \sim 1$ at present, the value of $(\Omega - 1)$ at early times have to be fine-tuned to values amazingly close to zero, but without being exactly zero. This is the reason why the flatness problem is also dubbed the 'fine-tuning problem'.

Let us now see how the hypothesis of adiabatic expansion of the universe is connected with the flatness problem. From the Friedman equation (177) we know that during a radiation-dominated period

$$H^2 \simeq \rho_R \simeq \frac{T^4}{{m_{\rm Pl}}^2}, \tag{30}$$

from which we deduce

$$\Omega - 1 = \frac{k{m_{\rm Pl}}^2}{a^4 T^4} = \frac{k{m_{\rm Pl}}^2}{S^{\frac{2}{3}} T^2}. \tag{31}$$

Under the hypothesis of adiabaticity, S is constant over the evolution of the universe and therefore

$$|\Omega - 1|_{t=t_{\rm Pl}} = \frac{{m_{\rm Pl}}^2}{T_{\rm Pl}^2} \frac{1}{S_U^{2/3}} = \frac{1}{S_U^{2/3}} \approx 10^{-60}. \tag{32}$$

We have discovered that $(\Omega - 1)$ is so close to zero at early epochs because the total entropy of our universe is so incredibly large. The flatness problem is therefore a problem of understanding why the (classical) initial conditions corresponded to a universe that was so close to spatial flatness. In a sense, the problem is one of fine–tuning and although such a balance is possible in principle, one nevertheless feels that it is unlikely. On the other hand, the flatness problem arises because the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be resolved if the cosmic expansion was non–adiabatic for some finite time interval during the early history of the universe.

According to the standard cosmology, photons decoupled from the rest of the components (electrons and baryons) at a temperature of the order of 0.3 eV. This corresponds to the so-called surface of 'last-scattering' at a red shift of about 1100 and an age of about $180,000\,(\Omega_0 h^2)^{-1/2}$ yrs. From the epoch of last-scattering onwards, photons free-stream and reach us basically untouched. Detecting primordial photons is therefore equivalent to take a picture of the universe when the latter was about 300,000 yrs old. The spectrum of the cosmic background radiation (CBR) is consistent that of a black body at temperature 2.73 K over more than three decades in wavelength; see Fig. 1.

The most accurate measurement of the temperature and spectrum is that by the FIRAS instrument on the COBE satellite which determined its temperature to be $2.726 \pm 0.01\,{\rm K}$ [58]. The length corresponding to our present Hubble radius (which is approximately the radius of our observable universe) at the time of last-scattering was

$$\lambda_H(t_{\rm LS}) = R_H(t_0)\left(\frac{a_{\rm LS}}{a_0}\right) = R_H(t_0)\left(\frac{T_0}{T_{\rm LS}}\right).$$

On the other hand, during the matter-dominated period, the Hubble length has decreased with a different law

$$H^2 \propto \rho_M \propto a^{-3} \propto T^3.$$

At last-scattering

$$H_{LS}^{-1} = R_H(t_0)\left(\frac{T_{LS}}{T_0}\right)^{-3/2} \ll R_H(t_0).$$

The length corresponding to our present Hubble radius was much larger that the horizon at that time. This can be shown comparing the volumes corresponding to these two scales

$$\frac{\lambda_H^3(T_{LS})}{H_{LS}^{-3}} = \left(\frac{T_0}{T_{LS}}\right)^{-\frac{3}{2}} \approx 10^6. \tag{33}$$

There were $\sim 10^6$ casually disconnected regions within the volume that now corresponds to our horizon! It is difficult to come up with a process other than an early hot and dense phase in the history of the universe that would lead to a precise black body [64] for a bath of photons which were causally disconnected the last time they interacted with the surrounding plasma.

The horizon problem is well represented by Fig. 2 where the green line indicates the horizon scale and the red line any generic physical length scale λ. Suppose, indeed that λ indicates the distance between two photons we detect today. From Eq. (33) we discover that at the time of emission (last-scattering) the two photons could not talk to each other, the red line is above the green line. There is another aspect of the horizon problem which is related to the problem of initial conditions for the cosmological perturbations. We have every indication that the universe at early times, say $t \ll 300,000$ yrs, was very homogeneous; however, today inhomogeneity (or structure) is ubiquitous: stars ($\delta\rho/\rho \sim 10^{30}$), galaxies ($\delta\rho/\rho \sim 10^5$), clusters of galaxies ($\delta\rho/\rho \sim 10 - 10^3$), superclusters, or "clusters of clusters" ($\delta\rho/\rho \sim 1$), voids ($\delta\rho/\rho \sim -1$), great walls, and so on. For some twenty-five years the standard cosmology has provided a general framework for understanding this picture. Once the universe becomes matter dominated (around 1000 yrs after the bang) primeval density inhomogeneities ($\delta\rho/\rho \sim 10^{-5}$) are amplified by gravity and grow into the structure we see today [63]. The existence of density inhomogeneities has another important consequence: fluctuations in the temperature of the CMB radiation of a similar amplitude. The temperature difference measured between two points separated by a large angle ($\gtrsim 1^\circ$) arises due to a very simple physical effect: the difference in the gravitational potential between the two points on the last-scattering surface, which in turn is related to the density perturbation, determines the temperature anisotropy on the angular scale subtended by that length scale,

$$\left(\frac{\delta T}{T}\right)_\theta \approx \left(\frac{\delta\rho}{\rho}\right)_\lambda, \tag{34}$$

where the scale $\lambda \sim 100h^{-1}\,\mathrm{Mpc}(\theta/\mathrm{deg})$ subtends an angle θ on the last-scattering surface. This is known as the Sachs-Wolfe effect [70]. The CMB experiments looking for the tiny anisotropies are of three kinds: satellite experiments, balloon experiments, and ground based experiments. The technical and economical advantages of ground based experiments are evident, but their main problem is atmospheric fluctuations. The problem can be limited choosing a very high and cold site, or working on small scales (as the Dasi experiment [27]). Balloon based experiments limit the atmospheric problems, but have to face the following problems: they must be limited in weight, they can not be manipulated during the flight, they have a rather short duration (and they have to be recovered intact). Maxima [60], and

Boomerang [24] are experiments of this kind.
At present, there is a satellite experiment – MAP (Microwave Anisotropy Probe) sponsored by NASA mission, which is taking data [59]. Finally, a satellite mission PLANCK is planned by ESA to be launched in 2007 [65]. The temperature anisotropy is commonly expanded in spherical harmonics

$$\frac{\Delta T}{T}(x_0, \tau_0, \mathbf{n}) = \sum_{\ell m} a_{\ell,m}(x_0) Y_{\ell m}(\mathbf{n}), \tag{35}$$

where x_0 and τ_0 are our position and the preset time, respectively, $\mathbf{n}$ is the direction of observation, ℓ's are the different multipoles and[2]

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell,\ell'} \delta_{m,m'} C_\ell, \tag{36}$$

where the deltas are due to the fact that the process that created the anisotropy is statistically isotropic. The C_ℓ are the so-called CMB power spectrum. For homogeneity and isotropy, the C_ℓ's are neither a function of x_0, nor of m. The two-point-correlation function is related to the C_l's in the following way

$$\begin{aligned} \left\langle \frac{\delta T(\mathbf{n})}{T} \frac{\delta T(\mathbf{n}')}{T} \right\rangle &= \sum_{\ell \ell' m m'} \langle a_{\ell m} a^*_{\ell' m'} \rangle Y_{\ell m}(\mathbf{n}) Y^*_{\ell' m'}(\mathbf{n}') \\ &= \sum_\ell C_\ell \sum_m Y_{\ell m}(\mathbf{n}) Y^*_{\ell m}(\mathbf{n}') = \frac{1}{4\pi} \sum_\ell (2\ell+1) C_\ell P_\ell(\mu = \mathbf{n} \cdot \mathbf{n}') \end{aligned} \tag{37}$$

where we have used the addition theorem for the spherical harmonics, and P_ℓ is the Legendre polynom of order ℓ. In expression (37) the expectation value is an ensamble average. It can be regarded as an average over the possible observer positions, but not in general as an average over the single sky we observe, because of the cosmic variance[3].

Let us now consider the last-scattering surface. In comoving coordinates the latter is 'far' from us a distance equal to

$$\int_{t_{\rm LS}}^{t_0} \frac{dt}{a} = \int_{\tau_{\rm LS}}^{\tau_0} d\tau = (\tau_0 - \tau_{\rm LS}) \,. \tag{38}$$

A given comoving scale λ is therefore projected on the last-scattering surface sky on an angular scale

$$\theta \simeq \frac{\lambda}{(\tau_0 - \tau_{\rm LS})}, \tag{39}$$

[2] An alternative definition is $C_\ell = \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$.

[3] The usual hypothesis is that we observe a typical realization of the ensamble. This means that we expect the difference between the observed values $|a_{\ell m}|^2$ and the ensamble averages C_ℓ to be of the order of the mean-square deviation of $|a_{\ell m}|^2$ from C_ℓ. The latter is called cosmic variance and, because we are dealing with a gaussian distribution, it is equal to $2C_\ell$ for each multipole ℓ. For a single ℓ, averaging over the $(2\ell+1)$ values of m reduces the cosmic variance by a factor $(2\ell+1)$, but it remains a serious limitation for low multipoles.

where we have neglected tiny curvature effects. Consider now that the scale λ is of the order of the comoving sound horizon at the time of last-scattering, $\lambda \sim c_s \tau_{\rm LS}$, where $c_s \simeq 1/\sqrt{3}$ is the sound velocity at which photons propagate in the plasma at the last-scattering. This corresponds to an angle

$$\theta \simeq c_s \frac{\tau_{\rm LS}}{(\tau_0 - \tau_{\rm LS})} \simeq c_s \frac{\tau_{\rm LS}}{\tau_0}, \tag{40}$$

where the last passage has been performed knowing that $\tau_0 \gg \tau_{\rm LS}$. Since the universe is matter-dominated from the time of last-scattering onwards, the scale factor has the following behaviour: $a \sim T^{-1} \sim t^{2/3} \sim \tau^2$, where we have made use of the relation (15). The angle $\theta_{\rm HOR}$ subtended by the sound horizon on the last-scattering surface then becomes

$$\theta_{\rm HOR} \simeq c_s \left(\frac{T_0}{T_{\rm LS}} \right)^{1/2} \sim 1^\circ, \tag{41}$$

where we have used $T_{\rm LS} \simeq 0.3$ eV and $T_0 \sim 10^{-13}$ GeV. This corresponds to a multipole $\ell_{\rm HOR}$

$$\ell_{\rm HOR} = \frac{\pi}{\theta_{\rm HOR}} \simeq 200. \tag{42}$$

From these estimates we conclude that two photons which on the last-scattering surface were separated by an angle larger than $\theta_{\rm HOR}$, corresponding to multipoles smaller than $\ell_{\rm HOR} \sim 200$ were not in causal contact. On the other hand, from Fig. (3) it is clear that small anisotropies, of the *same* order of magnitude $\delta T/T \sim 10^{-5}$ are present at $\ell \ll 200$. We conclude that one of the striking features of the CMB fluctuations is that they appear to be noncausal. Photons at the last-scattering surface which were causally disconnected have the same small anisotropies! The existence of particle horizons in the standard cosmology precludes explaining the smoothness as a result of microphysical events: the horizon at decoupling, the last time one could imagine temperature fluctuations being smoothed by particle interactions, corresponds to an angular scale on the sky of about 1°, which precludes temperature variations on larger scales from being erased.

To account for the small-scale lumpiness of the universe today, density perturbations with horizon-crossing amplitudes of 10^{-5} on scales of 1 Mpc to 10^4 Mpc or so are required. As can be seen in Fig. 2, in the standard cosmology the physical size of a perturbation, which grows as the scale factor, begins larger than the horizon and relatively late in the history of the universe crosses inside the horizon. This precludes a causal microphysical explanation for the origin of the required density perturbations.

From the considerations made so far, it appears that solving the shortcomings of the standard Big Bang theory requires two basic modifications of the assumptions made so far:

- The universe has to go through a non-adiabatic period. This is necessary to solve the entropy and the flatness problem. A non-adiabatic phase may give rise to the large entropy S_U we observe today.

- The universe has to go through a primordial period during which the physical scales λ evolve faster than the horizon scale H^{-1}.

The second condition is obvious from Fig. 4. If there is period during which physical length scales grow faster than H^{-1}, length scales λ which are within the horizon today, $\lambda < H^{-1}$ (such as the distance between two detected photons) and were outside the horizon for some period, $\lambda > H^{-1}$ (for istance at the time of last-scattering when the two photons were emitted), had a chance to be within the horizon at some primordial epoch, $\lambda < H^{-1}$ again. If this happens, the homogeneity and the isotropy of the CMB can be easily explained: photons that we receive today and were emitted from the last-scattering surface from causally disconnected regions have the same temperature because they had a chance to 'talk' to each other at some primordial stage of the evolution of the universe.

The second condition can be easily expressed as a condition on the scale factor a. Since a given scale λ scales like $\lambda \sim a$ and $H^{-1} = a/\dot{a}$, we need to impose that there is a period during which

$$\left(\frac{\lambda}{H^{-1}}\right)^{\cdot} = \ddot{a} > 0.$$

We can therefore introduced the following rigorous definition: an inflationary stage [35] is a period of the universe during which the latter accelerates

$$\text{INFLATION} \quad \Longleftrightarrow \quad \ddot{a} > 0.$$

*Comment:*Let us stress that during such a accelerating phase the universe expands *adiabatically*. This means that during inflation one can exploit the usual FRW equations (3) and (5). It must be clear therefore that the non-adiabaticity condition is satisfied not during inflation, but during the phase transition between the end of inflation and the beginning of the radiation-dominated phase. At this transition phase a large entropy is generated under the form of relativistic degrees of freedom: the Big Bange has taken place.

3 The Inflationary Universe

From the previous section we have learned that an accelerating stage during the primordial phases of the evolution of the universe might be able to solve the horizon problem. From Eq. (5) we learn that

$$\ddot{a} > 0 \Longleftrightarrow (\rho + 3p) < 0.$$

An accelerating period is obtainable only if the overall pressure p of the universe is negative: $p < -\rho/3$. Neither a radiation-dominated phase nor a matter-dominated phase (for which $p = \rho/3$ and $p = 0$, respectively) satisfy such a condition. Let us postpone for the time being the problem of finding a 'candidate' able to provide the condition $p < -\rho/3$. For sure, inflation is a phase of the history of the universe occurring before the era of nucleosynthesis ($t \approx 1$ sec, $T \approx 1$ MeV) during which the light elements abundances were formed. This is because nucleosynthesis is the earliest epoch we have experimental data from and they are in agreement with the predictions of the standard Big-Bang theory. However, the thermal history of the universe before the epoch of nucleosynthesis is unknown.

In order to study the properties of the period of inflation, we assume the extreme condition $p = -\rho$ which considerably simplifies the analysis. A period of the universe during which $p = -\rho$ is called *de Sitter* stage. By inspecting Eqs. (3) and (4), we learn that during the de Sitter phase

$$\begin{aligned} \rho &= \text{constant}, \\ H_I &= \text{constant}, \end{aligned}$$

where we have indicated by H_I the value of the Hubble rate during inflation. Correspondingly, solving Eq. (3) gives

$$a = a_i \, e^{H_I(t-t_i)}, \tag{43}$$

where t_i denotes the time at which inflation starts. Let us now see how such a period of exponential expansion takes care of the shortcomings of the standard Big Bang Theory.[4]

3.1 Inflationary Solution to the Standard Big Bang Problems

During the inflationary (de Sitter) epoch the horizon scale H^{-1} is constant. If inflation lasts long enough, all the physical scales that have left the horizon during the radiation-dominated or matter-dominated phase can re-enter the horizon in the past: this is because such scales are exponentially reduced. As we have seen in the previous section, this explains both the problem of the homogeneity of CMB and the initial condition problem of small cosmological perturbations. Once the physical length is within the horizon, microphysics can act, the universe can be made approximately homogeneous and the primaeval inhomogeneities can be created.

[4] Despite the fact that the growth of the scale factor is exponential and the expansion is *superluminal*, this is not in contradiction with what dictated by relativity. Indeed, it is the spacetime itself which is progating so fast and not a light signal in it.

Let us see how long inflation must be sustained in order to solve the horizon problem. Let t_i and t_f be, respectively, the time of beginning and end of inflation. We can define the corresponding number of e-foldings N

$$N = \ln\left[H_I(t_e - t_i)\right]. \tag{44}$$

A necessary condition to solve the horizon problem is that the largest scale we observe today, the present horizon H_0^{-1}, was reduced during inflation to a value $\lambda_{H_0}(t_i)$ smaller than the value of horizon length H_I^{-1} during inflation. This gives

$$\lambda_{H_0}(t_i) = H_0^{-1}\left(\frac{a_{t_f}}{a_{t_0}}\right)\left(\frac{a_{t_i}}{a_{t_f}}\right) = H_0^{-1}\left(\frac{T_0}{T_f}\right)e^{-N} \lesssim H_I^{-1},$$

where we have neglected for simplicity the short period of matter-domination and we have called T_f the temperature at the end of inflation (to be indentified with the reheating temperature T_{RH} at the beginning of the radiation-dominated phase after inflation, see later). We get

$$N \gtrsim \ln\left(\frac{T_0}{H_0}\right) - \ln\left(\frac{T_f}{H_I}\right) \approx 67 + \ln\left(\frac{T_f}{H_I}\right).$$

Apart from the logarithmic dependence, we obtain $N \gtrsim 70$.

Inflation solves elegantly the flatness problem. Since during inflation the Hubble rate is constant

$$\Omega - 1 = \frac{k}{a^2H^2} \propto \frac{1}{a^2}.$$

On the other end the condition (32) tells us that to reproduce a value of $(\Omega_0 - 1)$ of order of unity today the initial value of $(\Omega - 1)$ at the beginning of the radiation-dominated phase must be $|\Omega - 1| \sim 10^{-60}$. Since we identify the beginning of the radiation-dominated phase with the beginning of inflation, we require

$$\left.|\Omega - 1|\right|_{t=t_f} \sim 10^{-60}.$$

During inflation

$$\frac{\left.|\Omega - 1|\right|_{t=t_f}}{\left.|\Omega - 1|\right|_{t=t_i}} = \left(\frac{a_i}{a_f}\right)^2 = e^{-2N}. \tag{45}$$

Taking $\left.|\Omega - 1|\right|_{t=t_i}$ of order unity, it is enough to require that $N \approx 70$ to solve the flatness problem.

1. Comment: In the previous section we have written that the flateness problem can be also seen as a fine-tuning problem of one part over 10^{60}. Inflation ameliorates this fine-tuning problem, by explaining a tiny number $\sim 10^{-60}$ with a number N of the order 70.

2. Comment: The number $N \simeq 70$ has been obtained requiring that the present-day value of $(\Omega_0 - 1)$ is of order unity. For the expression (45), it is clear that –if the period of inflation lasts longer than 70 e-foldings the present-day value of Ω_0 will be equal to unity with a great precision. One can say that a generic prediction of inflation is that

$$\text{INFLATION} \quad \Longrightarrow \quad \Omega_0 = 1.$$

This statement, however, must be taken *cum grano salis* and properly specified. Inflation does not change the global geometric properties of the spacetime. If the universe is open or closed, it will always remain flat or closed, independently from inflation. What inflation does is to magnify the radius of curvature $R_{\rm curv}$ defined in Eq. (9) so that locally the universe is flat with a great precision. As we have seen in section 2, the current data on the CMB anisotropies confirm this prediction!

In the previous section, we have seen that the flatness problem arises because the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be resolved if the cosmic expansion was non-adiabatic for some finite time interval during the early history of the universe. We need to produce a large amount of entropy $S_U \sim 10^{90}$. Let us postulate that the entropy changed by an amount

$$S_f = Z^3\, S_i \tag{46}$$

from the beginning to the end of the inflationary period, where Z is a numerical factor. It is very natural to assume that the total entropy of the universe at the beginning of inflation was of order unity, one particle per horizon. Since, from the end of inflation onwards, the universe expands adiabatically, we have $S_f = S_U$. This gives $Z \sim 10^{30}$. On the other hand, since $S_f \sim \left(a_f T_f\right)^3$ and $S_i \sim \left(a_i T_i\right)^3$, where T_f and T_i are the temperatures of the universe at the end and at the beginning of inflation, we get

$$\left(\frac{a_f}{a_i}\right) = e^N \approx 10^{30}\left(\frac{T_i}{T_f}\right), \tag{47}$$

which gives again $N \sim 70$ up to the logarithmic factor $\ln\left(\frac{T_i}{T_f}\right)$. We stress again that such a large amount of entopy is not produced during inflation, but during the non-adiabatic phase transition which gives rise to the usual radiation-dominated phase.

3.2 The Inflaton

In the previous subsections we have described the various adavantages of having a period of accelerating phase. The latter required $p < -\rho/3$. Now, we would like to show that this condition can be attained by means of a simple scalar field. We shall call this field the *inflaton* ϕ.

The action of the inflaton field reads

$$S = \int d^4x\, \sqrt{-g}\, \mathcal{L} = \int\, d^4x\, \sqrt{-g}\, \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right], \tag{48}$$

where $\sqrt{-g} = a^3$ for the FRW metric (525). From the Eulero-Lagrange equations

$$\partial^\mu \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\,\partial^\mu\phi} - \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\phi} = 0, \tag{49}$$

we obtain

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0, \tag{50}$$

where $V'(\phi) = (dV(\phi)/d\phi)$. Note, in particular, the appearance of the friction term $3H\dot{\phi}$: a scalar field rolling down its potential suffers a friction due to the expansion of the universe.

We can write the energy-momentum tensor of the scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\,\mathcal{L}.$$

The corresponding energy density ρ_ϕ and pressure density p_ϕ are

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}, \tag{51}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{(\nabla\phi)^2}{6a^2}. \tag{52}$$

Notice that, if the gradient term were dominant, we would obtain $p_\phi = -\frac{\rho_\phi}{3}$, not enough to drive inflation. We can now split the inflaton field in

$$\phi(t) = \phi_0(t) + \delta\phi(\mathbf{x}, t),$$

where ϕ_0 is the 'classical' (infinite wavelength) field, that is the expectation value of the inflaton field on the initial isotropic and homogeneous state, while $\delta\phi(\mathbf{x}, t)$ represents the quantum fluctuations around ϕ_0. In this section, we will be only concerned with the evolution of the classical field ϕ_0. The next section will be devoted to the crucial issue of the evolution of quantum perturbations during inflation. This separation is justified by the fact that quantum fluctuations are much smaller than the classical value and therefore negligible when looking at the classical evolution. To not be overwhelmed by the notation, we will keep indicating from now on the classical value of the inflaton field by ϕ. The energy-momentum tensor becomes

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{53}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{54}$$

If

$$V(\phi) \gg \dot{\phi}^2$$

we obtain the following condition

$$p_\phi \simeq -\rho_\phi$$

From this simple calculation, we realize that a scalar field whose energy is dominant in the universe and whose potential energy dominates over the kinetic term gives inflation! Inflation is driven by the vacuum energy of the inflaton field.

3.3 The Slow-roll

Let us now quantify better under which circumstances a scalar field may give rise to a period of inflation. The equation of motion of the field is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{55}$$

If we require that $\dot{\phi}^2 \ll V(\phi)$, the scalar field is slowly rolling down its potential. This is the reason why such a period is called *slow-roll*. We may also expect that – being the potential flat – $\ddot{\phi}$ is negligible as well. We will assume that this is true and we will quantify this condition soon. The FRW equation (3) becomes

$$H^2 \simeq \frac{8\pi G}{3} V(\phi), \tag{56}$$

where we have assumed that the inflaton field dominates the energy density of the universe. The new equation of motion becomes

$$3H\dot{\phi} = -V'(\phi) \tag{57}$$

which gives $\dot{\phi}$ as a function of $V'(\phi)$. Using Eq. (57) slow-roll conditions then require

$$\dot{\phi}^2 \ll V(\phi) \Longrightarrow \frac{(V')^2}{V} \ll H^2$$

and

$$\ddot{\phi} \ll 3H\dot{\phi} \Longrightarrow V'' \ll H^2.$$

It is now useful to define the slow-roll parameters, ϵ and η in the following way

$$\begin{aligned} \epsilon &= -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \\ \eta &= \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2}, \\ \delta &= \eta - \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}}. \end{aligned}$$

It might be useful to have the same parameters expressed in terms of conformal time

$$\begin{aligned} \epsilon &= 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = 4\pi G \frac{\phi'^2}{\mathcal{H}^2} \\ \delta &= \eta - \epsilon = 1 - \frac{\phi''}{\mathcal{H}\phi'}. \end{aligned}$$

The parameter ϵ quantifies how much the Hubble rate H changes with time during inflation. Notice that, since

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)\, H^2,$$

inflation can be attained only if $\epsilon < 1$:

$$\text{INFLATION} \quad \Longleftrightarrow \quad \epsilon < 1.$$

As soon as this condition fails, inflation ends. In general, slow-roll inflation is attained if $\epsilon \ll 1$ and $|\eta| \ll 1$. During inflation the slow-roll parameters ϵ and η can be considered to be approximately constant since the potential $V(\phi)$ is very flat.

<u>Comment</u>: In the following, we will work at *first-order* perturbation in the slow-roll parameters, that is we will take only the first power of them. Since, using their definition, it is easy to see that $\dot{\epsilon}, \dot{\eta} = \mathcal{O}\left(\epsilon^2, \eta^2\right)$, this amounts to saying that we will trat the slow-roll parameters as constant in time.

Within these approximations, it is easy to compute the number of e-foldings between the beginning and the end of inflation. If we indicate by ϕ_i and ϕ_f the values of the inflaton field at the beginning and at the end of inflation, respectively, we have that the *total* number of e-foldings is

$$\begin{aligned} N &\equiv \int_{t_i}^{t_f} H\, dt \\ &\simeq H \int_{\phi_i}^{\phi_f} \frac{d\phi}{\dot{\phi}} \\ &\simeq -3H^2 \int_{\phi_i}^{\phi_f} \frac{d\phi}{V'} \\ &\simeq -8\pi G \int_{\phi_i}^{\phi_f} \frac{V}{V'}\, d\phi. \end{aligned} \tag{58}$$

We may also compute the number of e-foldings ΔN which are left to go to the end of inflation

$$\Delta N \simeq 8\pi G \int_{\phi_f}^{\phi_{\Delta N}} \frac{V}{V'}\, d\phi, \tag{59}$$

where $\phi_{\Delta N}$ is the value of the inflaton field when there are ΔN e-foldings to the end of inflation.

1. <u>*Comment*</u>*:* According to the criterion given in subsection 2.4, a given scale length $\lambda = a/k$ leaves the horizon when $k = aH_k$ where H_k is the the value of the Hubble rate at that time. One can compute easily the rate of change of H_k^2 as a function of k

$$\frac{d\ln H_k^2}{d\ln k} = \left(\frac{d\ln H_k^2}{dt}\right)\left(\frac{dt}{d\ln a}\right)\left(\frac{d\ln a}{d\ln k}\right) = 2\frac{\dot{H}}{H} \times \frac{1}{H} \times 1 = 2\frac{\dot{H}}{H^2} = -2\epsilon. \tag{60}$$

2. <u>*Comment*</u>*:* Take a given physical scale λ today which crossed the horizon scale during inflation. This happened when

$$\lambda\left(\frac{a_f}{a_0}\right) e^{-\Delta N_\lambda} = \lambda\left(\frac{T_0}{T_f}\right) e^{-\Delta N_\lambda} = H_I^{-1}$$

where ΔN_λ indicates the number of e-foldings from the time the scale crossed the horizon during inflation and the end of inflation. This relation gives a way to determine the number of e-foldings to the end of inflation corresponding to a given scale

$$\Delta N_\lambda \simeq 65 + \ln\left(\frac{\lambda}{3000\ \mathrm{Mpc}}\right) + 2\ln\left(\frac{V^{1/4}}{10^{14}\ \mathrm{GeV}}\right) - \ln\left(\frac{T_f}{10^{10}\ \mathrm{GeV}}\right).$$

Scales relevant for the CMB anisotropies correspond to $\Delta N \sim 60$.

3.4 Inflation and Reheating

Inflation ended when the potential energy associated with the inflaton field became smaller than the kinetic energy of the field. By that time, any pre-inflation entropy in the universe had been inflated away, and the energy of the universe was entirely in the form of coherent oscillations of the inflaton condensate around the minimum of its potential. The universe may be said to be frozen after the end of inflation. We know that somehow the low-entropy cold universe dominated by the energy of coherent motion of the ϕ field must be transformed into a high-entropy hot universe dominated by radiation. The process by which the energy of the inflaton field is transferred from the inflaton field to radiation has been dubbed *reheating*. In the old theory of reheating [30, 10], the simplest way to envision this process is if the comoving energy density in the zero mode of the inflaton decays into normal particles, which then scatter and thermalize to form a thermal background. It is usually assumed that the decay width of this process is the same as the decay width of a free inflaton field.

Of particular interest is a quantity known usually as the reheat temperature, denoted as T_{RH}[5]. The reheat temperature is calculated by assuming an instantaneous conversion of the energy density in the inflaton field into radiation when the decay width of the inflaton energy, Γ_ϕ, is equal to H, the expansion rate of the universe.

The reheat temperature is calculated quite easily. After inflation the inflaton field executes coherent oscillations about the minimum of the potential. Averaged over several oscillations, the coherent oscillation energy density redshifts as matter: $\rho_\phi \propto a^{-3}$, where a is the Robertson–Walker scale factor. If we denote as ρ_I and a_I the total inflaton energy density and the scale factor at the initiation of coherent oscillations, then the Hubble expansion rate as a function of a is

$$H^2(a) = \frac{8\pi}{3}\frac{\rho_I}{{m_{\mathrm{Pl}}}^2}\left(\frac{a_I}{a}\right)^3. \tag{61}$$

Equating $H(a)$ and Γ_ϕ leads to an expression for a_I/a. Now if we assume that all available coherent energy density is instantaneously converted into radiation at this value of a_I/a, we can find the reheat temperature by setting the coherent energy density, $\rho_\phi = \rho_I(a_I/a)^3$, equal to the radiation energy density, $\rho_R = (\pi^2/30)g_* T_{RH}^4$, where g_* is the effective number of relativistic degrees of freedom at temperature T_{RH}. The result is

$$T_{RH} = \left(\frac{90}{8\pi^3 g_*}\right)^{1/4}\sqrt{\Gamma_\phi m_{\mathrm{Pl}}} = 0.2\left(\frac{200}{g_*}\right)^{1/4}\sqrt{\Gamma_\phi m_{\mathrm{Pl}}}\ . \tag{62}$$

[5] So far, we have indicated it with T_f.

In some models of inflation reheating can be anticipated by a period of preheating [44] when the the classical inflaton field very rapidly (explosively) decays into ϕ-particles or into other bosons due to broad parametric resonance. This stage cannot be described by the standard elementary approach to reheating based on perturbation theory. The bosons produced at this stage further decay into other particles, which eventually become thermalized.

3.5 The Inflationary models

Even restricting ourselves to a simple single-field inflation scenario, the number of models available to choose from is large [57]. It is convenient to define a general classification scheme, or "zoology" for models of inflation. We divide models into three general types [29]: *large-field*, *small-field*, and *hybrid*, with a fourth classification. A generic single-field potential can be characterized by two independent mass scales: a "height" Λ^4, corresponding to the vacuum energy density during inflation, and a "width" μ, corresponding to the change in the field value $\Delta\phi$ during inflation:

$$V(\phi) = \Lambda^4 f\left(\frac{\phi}{\mu}\right). \tag{63}$$

Different models have different forms for the function f. Let us now briefly describe the different class of models.

Large-field models are potentials typical of the "chaotic" inflation scenario[52], in which the scalar field is displaced from the minimum of the potential by an amount usually of order the Planck mass. Such models are characterized by $V''(\phi) > 0$, and $-\epsilon < \delta \leq \epsilon$. The generic large-field potentials we consider are polynomial potentials $V(\phi) = \Lambda^4 (\phi/\mu)^p$, and exponential potentials, $V(\phi) = \Lambda^4 \exp(\phi/\mu)$. In the chaotic inflation scenario, it is assumed that the universe emerged from a quantum gravitational state with an energy density comparable to that of the Planck density. This implies that $V(\phi) \approx {m_{\rm Pl}}^4$ and results in a large friction term in the Friedmann equation (180). Consequently, the inflaton will slowly roll down its potential. The condition for inflation is therefore satisfied and the scale factor grows as

$$a(t) = a_i e^{\left(\int_{t_i}^{t} dt' H(t')\right)}. \tag{64}$$

The simplest chaotic inflation model is that of a free field with a quadratic potential, $V(\phi) = m^2\phi^2/2$, where m represents the mass of the inflaton. During inflation the scale factor grows as

$$a(t) = a_i e^{2\pi(\phi_i^2 - \phi^2(t))} \tag{65}$$

and inflation ends when $\phi = \mathcal{O}(1)\ m_{\rm Pl}$. If inflation begins when $V(\phi_{\rm i}) \approx {m_{\rm Pl}}^4$, the scale factor grows by a factor $\exp(4\pi {m_{\rm Pl}}^2/m^2)$ before the inflaton reaches the minimum of its potential. We will later show that the mass of the field should be $m \approx 10^{-6} m_{\rm Pl}$ if the microwave background constraints are to be satisfied. This implies that the volume of the universe will increase by a factor of $Z^3 \approx 10^{3\times 10^{12}}$ and this is more than enough inflation to solve the problems of the hot big bang model.

In the chaotic inflationary scenarios, the present-day universe is only a small portion of the universe which suffered inflation! Notice also that the typical values of the inflaton field during inflation are of the order of m_{Pl}, giving rise to the possibility of testing planckian physics [26].

Small-field models are the type of potentials that arise naturally from spontaneous symmetry breaking (such as the original models of "new" inflation [51, 12]) and from pseudo Nambu-Goldstone modes (natural inflation[33]). The field starts from near an unstable equilibrium (taken to be at the origin) and rolls down the potential to a stable minimum. Small-field models are characterized by $V''(\phi) < 0$ and $\eta < -\epsilon$. Typically ϵ is close to zero. The generic small-field potentials we consider are of the form $V(\phi) = \Lambda^4 \left[1 - (\phi/\mu)^p\right]$, which can be viewed as a lowest-order Taylor expansion of an arbitrary potential about the origin. See, for instance, Ref. [28].

The hybrid scenario[53, 54, 25] frequently appears in models which incorporate inflation into supersymmetry [66] and supergravity [55]. In a typical hybrid inflation model, the scalar field responsible for inflation evolves toward a minimum with nonzero vacuum energy. The end of inflation arises as a result of instability in a second field. Such models are characterized by $V''(\phi) > 0$ and $0 < \epsilon < \delta$. We consider generic potentials for hybrid inflation of the form $V(\phi) = \Lambda^4 \left[1 + (\phi/\mu)^p\right]$. The field value at the end of inflation is determined by some other physics, so there is a second free parameter characterizing the models. This enumeration of models is certainly not exhaustive. There are a number of single-field models that do not fit well into this scheme, for example logarithmic potentials $V(\phi) \propto \ln(\phi)$ typical of supersymmetry [57, 37, 22, 31, 56, 67, 32, 43]. Another example is potentials with negative powers of the scalar field $V(\phi) \propto \phi^{-p}$ used in intermediate inflation [17] and dynamical supersymmetric inflation [40, 41]. Both of these cases require and auxilliary field to end inflation and are more properly categorized as hybrid models, but fall into the small-field class. However, the three classes categorized by the relationship between the slow-roll parameters as $-\epsilon < \delta \leq \epsilon$ (large-field), $\delta \leq -\epsilon$ (small-field) and $0 < \epsilon < \delta$ (hybrid) seems to be good enough for comparing theoretical expectations with experimental data.

4 The Cosmological Perturbations

As we have seen in the previous section, the early universe was made very nearly uniform by a primordial inflationary stage. However, the important caveat in that statement is the word 'nearly'. Our current understanding of the origin of structure in the universe is that it originated from small 'seed' perturbations, which over time grew to become all of the structure we observe. Once the universe becomes matter dominated (around 1000 yrs after the bang) primeval density inhomogeneities ($\delta\rho/\rho \sim 10^{-5}$) are amplified by gravity and grow into the structure we see today [63]. The fact that a fluid of self-gravitating particles is unstable to the growth of small inhomogeneities was first pointed out by Jeans and is known as the Jeans instability. Furthermore, the existence of these inhomogeneities was confirmed by the COBE discovery of CMB anisotropies; the temperature anisotropies detected almost certainly owe their existence to primeval density inhomogeneities, since, as we have seen, causality precludes microphysical processes from producing anisotropies on angular scales larger than about 1°, the angular size of the horizon at last-scattering.

The growth of small matter inhomogeneities of wavelength smaller than the Hubble scale ($\lambda \lesssim H^{-1}$) is governed by a Newtonian equation:

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + v_s^2 \frac{k^2}{a^2}\delta_{\mathbf{k}} = 4\pi G\rho_M \delta_{\mathbf{k}}, \tag{66}$$

where $v_s^2 = \partial p/\partial\rho_M$ is the square of the sound speed and we have expanded the perturbation to the matter density in plane waves

$$\frac{\delta\rho_M(\mathbf{x},t)}{\rho_M} = \frac{1}{(2\pi)^3}\int d^3k\, \delta_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{67}$$

Competition between the pressure term and the gravity term on the rhs of Eq. (66) determines whether or not pressure can counteract gravity: perturbations with wavenumber larger than the Jeans wavenumber, $k_J^2 = 4\pi G a^2 \rho_M / v_s^2$, are Jeans stable and just oscillate; perturbations with smaller wavenumber are Jeans unstable and can grow.

Let us discuss solutions to this equation under different circumstances. First, consider the Jeans problem, evolution of perturbations in a static fluid, *i.e.*, $H = 0$. In this case Jeans unstable perturbations grow exponentially, $\delta_{\mathbf{k}} \propto \exp(t/\tau)$ where $\tau = 1/\sqrt{4G\pi\rho_M}$. Next, consider the growth of Jeans unstable perturbations in a matter-dominated universe, *i.e.*, $H^2 = 8\pi G\rho_M/3$ and $a \propto t^{2/3}$. Because the expansion tends to "pull particles away from one another," the growth is only power law, $\delta_{\mathbf{k}} \propto t^{2/3}$; *i.e.*, at the same rate as the scale factor. Finally, consider a radiation-dominated universe. In this case, the expansion is so rapid that matter perturbations grow very slowly, as $\ln a$ in radiation-dominated epoch. Therefore, perturbations may grow only in a matter-dominated period. Once a perturbation reaches an overdensity of order unity or larger it "separates" from the expansion –*i.e.*, becomes its own self-gravitating system and ceases to expand any further. In the process of virial relaxation, its size decreases by a factor of two—density increases by a factor of 8; thereafter, its density contrast grows as a^3 since the average matter density is decreasing as a^{-3}, though smaller scales could become Jeans unstable and collapse further to form smaller objects of higher density.

In order for structure formation to occur via gravitational instability, there must have been small preexisting fluctuations on physical length scales when they crossed the Hubble radius in the radiation-dominated and matter-dominated eras. In the standard Big-Bang model these small perturbations have to be put in by hand, because it is impossible to produce fluctuations on any length scale while it is larger than the horizon. Since the goal of cosmology is to understand the universe on the basis of physical laws, this appeal to initial conditions is unsatisfactory. The challenge is therefore to give an explanation to the small seed perturbations which allow the gravitational growth of the matter perturbations.

Our best guess for the origin of these perturbations is quantum fluctuations during an inflationary era in the early universe. Although originally introduced as a possible solution to the cosmological conundrums such as the horizon, flatness and entopy problems, by far the most useful property of inflation is that it generates spectra of both density perturbations and gravitational waves. These perturbations extend from extremely short scales to scales considerably in excess of the size of the observable universe.

During inflation the scale factor grows quasi-exponentially, while the Hubble radius remains almost constant. Consequently the wavelength of a quantum fluctuation – either in the scalar field whose potential energy drives inflation or in the graviton field – soon exceeds the Hubble radius. The amplitude of the fluctuation therefore becomes 'frozen in'. This is quantum mechanics in action at macroscopic scales!

According to quantum field theory, empty space is not entirely empty. It is filled with quantum fluctuations of all types of physical fields. The fluctuations can be regarded as waves of physical fields with all possible wavelenghts, moving in all possible directions. If the values of these fields, averaged over some macroscopically large time, vanish then the space filled with these fields seems to us empty and can be called the vacuum.

In the exponentially expanding universe the vacuum structure is much more complicated. The wavelenghts of all vacuum fluctuations of the inflaton field ϕ grow exponentially in the expnading universe. When the wavelength of any particular fluctuation becomes greater than H^{-1}, this fluctuation stops propagating, and its amplitude freezes at some nonzero value $\delta\phi$ because of the large friction term $3H\dot{\phi}$ i the equation of motion of the field ϕ. The amplitude of this fluctuation then remains almost unchanged for a very long time, whereas its wavelength grows exponentially. Therefore, the appearance of such frozen fluctuation is equivalent to the appearance of a classical field $\delta\phi$ that does not vanish after having averaged over some macroscopic interval of time. Because the vacuum contains fluctuations of all possible wavelength, inflation leads to the creation of more and more new perturbations of the classical field with wavelength larger than the horizon scale.

Once inflation has ended, however, the Hubble radius increases faster than the scale factor, so the fluctuations eventually reenter the Hubble radius during the radiation- or matter-dominated eras. The fluctuations that exit around 60 e-foldings or so before reheating reenter with physical wavelengths in the range accessible to cosmological observations. These spectra provide a distinctive signature of inflation. They can be measured in a variety of different ways including the analysis of microwave background anisotropies.

The physical processes which give rise to the structures we observe today are well-explained in Fig. 8. Quantum fluctuations of the inflaton field are generated during inflation. Since gravity talks to any component of the universe, small fluctuations of the inflaton field are intimately related to fluctuations of the spacetime metric, giving rise to perturbations of the curvature $\mathcal{R}$ (which will be defined in the following; the reader may loosely think of it as a gravitational potential). The wavelenghts λ of these perturbations grow exponentially and leave soon the horizon when $\lambda > R_H$. On superhorizon scales, curvature fluctuations are frozen in and may be considered as classical. Finally, when the wavelength of these fluctuations reenters the horizon, at some radiation- or matter-dominated epoch, the curvature (gravitational potential) perturbations of the spacetime give rise to matter (and temperature) perturbations $\delta\rho$ via the Poisson equation. These fluctuations will then start growing giving rise to the structures we observe today.

In summary, two are the key ingredients for understanding the observed structures in the universe within the inflationary scenario:

- Quantum fluctuations of the inflaton field are excited during inflation and stretched to cosmological scales. At the same time, being the inflaton fluctuations connected to the metric perturbations through Einstein's equations, ripples on the metric are also excited and stretched to cosmological scales.

- Gravity acts a messanger since it communicates to baryons and photons the small seed perturbations once a given wavelength becomes smaller than the horizon scale after inflation.

Let us know see how quantum fluctuations are generated during inflation. We will proceed by steps. First, we will consider the simplest problem of studying the quantum fluctuations of a generic scalar field during inflation: we will learn how perturbations evolve as a function of time and compute their spectrum. Then – since a satisfactory description of the generation of quantum fluctuations have to take both the inflaton and the metric perturbations into account – we will study the system composed by quantum fluctuations of the inflaton field and quantum fluctuations of the metric.

4.1 Quantum fluctuations of a scalar field during inflation

Let us first see how the fluctuations of a generic scalar field χ, which is *not* the inflaton field, behave during inflation. To warm up we first consider a de Sitter epoch during which the Hubble rate is constant.

We assume this field to be massless. The massive case will be analyzed in the next subsection.

Expanding the scalar field χ in Fourier modes

$$\delta\chi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \delta\chi_{\mathbf{k}}(t),$$

we can write the equation for the fluctuations as

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\,\delta\dot{\chi}_{\mathbf{k}} + \frac{k^2}{a^2}\,\delta\chi_{\mathbf{k}} = 0. \tag{68}$$

Let us study the qualitative behaviour of the solution to Eq. (68).

- For wavelengths within the horizon, $\lambda \ll H^{-1}$, the corresponding wavenumber satisfies the relation $k \gg a\,H$. In this regime, we can neglect the friction term $3H\,\delta\dot{\chi}_{\mathbf{k}}$ and Eq. (68) reduces to

$$\delta\ddot{\chi}_{\mathbf{k}} + \frac{k^2}{a^2}\,\delta\chi_{\mathbf{k}} = 0, \tag{69}$$

which is – basically – the equation of motion of an harmonic oscillator. Of course, the frequency term k^2/a^2 depends upon time because the scale factor a grows exponentially. On the qualitative level, however, one expects that when the wavelength of the fluctuation is within the horizon, the fluctuation oscillates.

- For wavelengths above the horizon, $\lambda \gg H^{-1}$, the corresponding wavenumber satisfies the relation $k \ll aH$ and the term k^2/a^2 can be safely neglected. Eq. (68) reduces to

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\,\delta\dot{\chi}_{\mathbf{k}} = 0, \tag{70}$$

which tells us that on superhorizon scales $\delta\chi_{\mathbf{k}}$ remains constant.

We have therefore the following picture: take a given fluctuation whose initial wavelength $\lambda \sim a/k$ is within the horizon. The fluctuations oscillates till the wavelength becomes of the order of the horizon scale. When the wavelength crosses the horizon, the fluctuation ceases to oscillate and gets frozen in.

Let us know study the evolution of the fluctuation is a more quantitative way. To do so, we perform the following redefinition

$$\delta\chi_{\mathbf{k}} = \frac{\delta\sigma_{\mathbf{k}}}{a}$$

and we work in conformal time $d\tau = dt/a$. For the time being, we solve the problem for a pure de Sitter expansion and we take the scale factor exponentially growing as $a \sim e^{Ht}$; the corresponding conformal factor reads (after choosing properly the integration constants)

$$a(\tau) = -\frac{1}{H\tau} \quad (\tau < 0).$$

In the following we will also solve the problem in the case of quasi de Sitter expansion. The beginning of inflation coincides with some initial time $\tau_i \ll 0$. Using the set of rules (15), we find that Eq. (68) becomes

$$\delta\sigma''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_{\mathbf{k}} = 0. \tag{71}$$

We obtain an equation which is very 'close' to the equation for a Klein-Gordon scalar field in flat spacetime, the only difference being a negative time-dependent mass term $-a''/a = -2/\tau^2$. Eq. (71) can be obtained from an action of the type

$$\delta S_{\mathbf{k}} = \int d\tau \left[\frac{1}{2}\delta\sigma'^2_{\mathbf{k}} - \frac{1}{2}\left(k^2 - \frac{a''}{a}\right)\delta\sigma^2_{\mathbf{k}}\right], \tag{72}$$

which is the canonical action for a simple harmonic oscillator with canonical commutation relations $\delta\sigma^*_{\mathbf{k}}\delta\sigma'_{\mathbf{k}} - \delta\sigma_{\mathbf{k}}\delta\sigma^{*\prime}_{\mathbf{k}} = -i$.

Let us study the behaviour of this equation on subhorizon and superhorizon scales. Since

$$\frac{k}{aH} = -k\,\tau,$$

on subhorizon scales $k^2 \gg a''/a$ Eq. (71) reduces to

$$\delta\sigma''_{\mathbf{k}} + k^2\,\delta\sigma_{\mathbf{k}} = 0,$$

whose solution is a plane wave

$$\delta\sigma_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2k}} \quad (k \gg aH). \tag{73}$$

We find again that fluctuations with wavelength within the horizon oscillate exactly like in flat spacetime. This does not come as a surprise. In the ultraviolet regime, that is for wavelengths much smaller than the horizon scale, one expects that approximating the spacetime as flat is a good approximation.

On superhorizon scales, $k^2 \ll a''/a$ Eq. (71) reduces to

$$\delta\sigma''_{\mathbf{k}} - \frac{a''}{a}\delta\sigma_{\mathbf{k}} = 0,$$

which is satisfied by

$$\delta\sigma_{\mathbf{k}} = B(k)\,a \quad (k \ll aH). \tag{74}$$

where $B(k)$ is a constant of integration. Roughly matching the (absolute values of the) solutions (73) and (74) at $k = aH$ $(-k\tau = 1)$, we can determine the (absolute value of the) constant $B(k)$

$$|B(k)|\,a = \frac{1}{\sqrt{2k}} \Longrightarrow |B(k)| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}.$$

Going back to the original variable $\delta\chi_{\mathbf{k}}$, we obtain that the quantum fluctuation of the χ field on superhorizon scales is constant and approximately equal to

$$|\delta\chi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}} \quad \text{(ON SUPERHORIZON SCALES)}$$

In fact we can do much better, since Eq. (71) has an *exact* solution:

$$\delta\sigma_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{75}$$

This solution reproduces all what we have found by qualitative arguments in the two extreme regimes $k \ll aH$ and $k \gg aH$. The reason why we have performed the matching procedure is to show that the latter can be very useful to determine the behaviour of the solution on superhorizon scales when the exact solution is not known.

So far, we have solved the equation for the quantum perturbations of a generic massless field, that is neglecting the mass squared term m_χ^2. Let us know discuss the solution when such a mass term is present. Eq. (71) becomes

$$\delta\sigma''_{\mathbf{k}} + \left[k^2 + M^2(\tau)\right]\delta\sigma_{\mathbf{k}} = 0, \tag{76}$$

where

$$M^2(\tau) = \left(m_\chi^2 - 2H^2\right) a^2(\tau) = \frac{1}{\tau^2}\left(\frac{m^2}{H^2} - 2\right).$$

Eq. (76) can be recast in the form

$$\delta\sigma_{\mathbf{k}}'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu_\chi^2 - \frac{1}{4}\right)\right]\delta\sigma_{\mathbf{k}} = 0, \tag{77}$$

where

$$\nu_\chi^2 = \left(\frac{9}{4} - \frac{m_\chi^2}{H^2}\right). \tag{78}$$

The generic solution to Eq. (76) for ν_χ *real* is

$$\delta\sigma_{\mathbf{k}} = \sqrt{-\tau}\left[c_1(k)\, H_{\nu_\chi}^{(1)}(-k\tau) + c_2(k)\, H_{\nu_\chi}^{(2)}(-k\tau)\right],$$

where $H_{\nu_\chi}^{(1)}$ and $H_{\nu_\chi}^{(2)}$ are the Hankel's functions of the first and second kind, respectively. If we impose that in the ultraviolet regime $k \gg aH$ ($-k\tau \gg 1$) the solution matches the plane-wave solution $e^{-ik\tau}/\sqrt{2k}$ that we expect in flat spacetime and knowing that

$$H_{\nu_\chi}^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4}\right)} \quad , H_{\nu_\chi}^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4}\right)},$$

we set $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2}\, e^{i\left(\nu_\chi + \frac{1}{2}\right)\frac{\pi}{2}}$. The exact solution becomes

$$\delta\sigma_{\mathbf{k}} = \frac{\sqrt{\pi}}{2}\, e^{i\left(\nu_\chi + \frac{1}{2}\right)\frac{\pi}{2}}\, \sqrt{-\tau}\, H_{\nu_\chi}^{(1)}(-k\tau). \tag{79}$$

On superhorizon scales, since $H_{\nu_\chi}^{(1)}(x \ll 1) \sim \sqrt{2/\pi}\, e^{-i\frac{\pi}{2}}\, 2^{\nu_\chi - \frac{3}{2}}\left(\Gamma(\nu_\chi)/\Gamma(3/2)\right) x^{-\nu_\chi}$, the fluctuation (79) becomes

$$\delta\sigma_{\mathbf{k}} = e^{i\left(\nu_\chi - \frac{1}{2}\right)\frac{\pi}{2}}\, 2^{\left(\nu_\chi - \frac{3}{2}\right)}\, \frac{\Gamma(\nu_\chi)}{\Gamma(3/2)}\, \frac{1}{\sqrt{2k}}\, (-k\tau)^{\frac{1}{2} - \nu_\chi}.$$

Going back to the old variable $\delta\chi_{\mathbf{k}}$, we find that on superhorizon scales, the fluctuation with nonvanishing mass is not exactly constant, but it acquires a tiny dependence upon the time

$$|\delta\chi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2} - \nu_\chi} \quad \text{(ON SUPERHORIZON SCALES)}$$

If we now define, in analogy with the definition of the slow roll parameters η and ϵ for the inflaton field, the parameter $\eta_\chi = (m_\chi^2/3H^2) \ll 1$, one finds

$$\frac{3}{2} - \nu_\chi \simeq \eta_\chi. \tag{80}$$

It is instructive to analyze the case in which ν_χ is *imaginary*, that is $m_\chi/H > 3/2$. In such a case, we define $\widetilde{\nu} = i\nu$. In superhorizon scales, performing the same steps we have done for the case of ν_χ real, we find

$$|\delta\chi_{\mathbf{k}}|^2 = \frac{\pi}{4}\frac{e^{-\pi\widetilde{\nu}}}{a^2}\frac{1}{aH}\Big[\frac{(1+\cot(\pi\widetilde{\nu})^2)\sinh(\pi\widetilde{\nu})}{\pi\widetilde{\nu}} + \frac{\widetilde{\nu}}{\pi\sinh(\pi\widetilde{\nu})} + \qquad (81)$$
$$2\,\mathrm{Re}\Big(i\left(\cos\left(2\,\widetilde{\nu}\ln(\frac{k\eta}{2})\right) + i\sin\left(2\,\widetilde{\nu}\ln(\frac{k\tau}{2})\right)\right)\frac{1-\cot(\pi\widetilde{\nu})}{\Gamma(1+i\widetilde{\nu})}\frac{\Gamma(-i\widetilde{\nu})}{\pi}\Big)\Big]\left(\frac{k}{aH}\right)^{n-1}.$$

In the limit of long wavelengths, the highly oscillating term which appears in the real part can be neglected because its average on k is 0. The resulting power spectrum is the following

$$\mathcal{P}_{\delta\chi}(k) = \tfrac{\pi}{4}e^{-\pi\widetilde{\nu}}\left(\tfrac{H}{2\pi}\right)^2\left(\tfrac{(1+\cot(\pi\widetilde{\nu})^2)\sinh(\pi\widetilde{\nu})}{\pi\widetilde{\nu}} + \tfrac{\widetilde{\nu}}{\pi\sinh(\pi\widetilde{\nu})}\right) \qquad (82)$$
$$\times\left(\tfrac{k}{aH}\right)^{n-1} \simeq \left(\tfrac{H}{2\pi}\right)^2\left(\tfrac{H}{m_\chi}\right)\left(\tfrac{k}{aH}\right)^3.$$

Therefore, for very massive scalar fields, $m_\chi > 3H/2$, the power spectrum has an amplitude which is suppressed by the ratio (H/m_χ) and the spectrum falls down rapidly al large wavelengths k^{-1} as k^3.

We have previously said that the quantum flactuations can be regarded as classical when their corresponding wavelengths cross the horizon. To better motivate this statement, we should compute the number of particles $n_{\mathbf{k}}$ per wavenumber $\mathbf{k}$ on superhorizon scales and check that it is indeed much larger than unity, $n_{\mathbf{k}} \gg 1$ (in this limit one can neglect the "quantum" factor 1/2 in the Hamiltonian $H_{\mathbf{k}} = \omega_{\mathbf{k}}\left(n_{\mathbf{k}} + \frac{1}{2}\right)$ where $\omega_{\mathbf{k}}$ is the energy eigenvalue). If so, the fluctuation can be regarded as classical. The number of particles $n_{\mathbf{k}}$ can be estimated to be of the order of $H_{\mathbf{k}}/\omega_{\mathbf{k}}$, where $H_{\mathbf{k}}$ is the Hamiltonian corresponding to the action

$$\delta S_{\mathbf{k}} = \int d\tau\,\left[\frac{1}{2}\delta\sigma_{\mathbf{k}}'^2 + \frac{1}{2}\left(k^2 - M^2(\tau)\right)\delta\sigma_{\mathbf{k}}^2\right], \qquad (83)$$

One obtains on superhorizon scales

$$n_{\mathbf{k}} \simeq \frac{M^2(\tau)\,|\delta\chi_{\mathbf{k}}|^2}{\omega_{\mathbf{k}}} \sim \left(\frac{k}{aH}\right)^{-3} \gg 1,$$

which confirms that fluctuations on superhorizon scales may be indeed considered as classical.

4.2 The Power Spectrum

Let us define now the power spectrum, a useful quantity to characterize the properties of the perturbations. For a generic quantity $g(\mathbf{x}, t)$, which can expanded in Fourier space as

$$g(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,g_{\mathbf{k}}(t),$$

the power spectrum can be defined as

$$\langle 0|g^*_{\mathbf{k}_1} g_{\mathbf{k}_2}|0\rangle \equiv \delta^{(3)}\left(\mathbf{k}_1 - \mathbf{k}_2\right) \frac{2\pi^2}{k^3} \mathcal{P}_g(k), \tag{84}$$

where $|0\rangle$ is the vacuum quantum state of the system. This definition leads to the usual relation

$$\langle 0|g^2(\mathbf{x},t)|0\rangle = \int \frac{dk}{k} \mathcal{P}_g(k). \tag{85}$$

So far, we have computed the time evolution and the spectrum of the quantum flutuations of a generic scalar field χ supposing that the scale factor evolves like in a pure de Sitter expansion, $a(\tau) = -1/(H\tau)$. However, during inflation the Hubble rate is not exactly constant, but changes with time as $\dot{H} = -\epsilon\, H^2$ (quasi de Sitter expansion), In this subsection, we will solve for the perturbations in a quasi de Sitter expansion. Using the definition of the conformal time, one can show that the scale factor for small values of ϵ becomes

$$a(\tau) = -\frac{1}{H}\frac{1}{\tau(1-\epsilon)}.$$

Eq. (76) has now a squared mass term

$$M^2(\tau) = m_\chi^2 a^2 - \frac{a''}{a},$$

where

$$\begin{aligned} \frac{a''}{a} &= a^2\left(\frac{\ddot{a}}{a} + H^2\right) = a^2\left(\dot{H} + 2\,H^2\right) \\ &= a^2\,(2-\epsilon)\,H^2 = \frac{(2-\epsilon)}{\tau^2\,(1-\epsilon)^2} \\ &\simeq \frac{1}{\tau^2}\,(2+3\epsilon)\,. \end{aligned} \tag{86}$$

Taking $m_\chi^2/H^2 = 3\eta_\chi$ and expanding for small values of ϵ and η we get Eq. (77) with

$$\nu_\chi \simeq \frac{3}{2} + \epsilon - \eta_\chi. \tag{87}$$

Armed with these results, we may compute the variance of the perturbations of the generic χ field

$$\begin{aligned} \langle 0|\left(\delta\chi(\mathbf{x},t)\right)^2|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\,|\delta\chi_{\mathbf{k}}|^2 \\ &= \int \frac{dk}{k}\,\frac{k^3}{2\pi^2}\,|\delta\chi_{\mathbf{k}}|^2 \\ &= \int \frac{dk}{k}\,\mathcal{P}_{\delta\chi}(k), \end{aligned} \tag{88}$$

which defines the power spectrum of the fluctuations of the scalar field χ

$$\mathcal{P}_{\delta\chi}(k) \equiv \frac{k^3}{2\pi^2}\,|\delta\chi_{\mathbf{k}}|^2\,. \tag{89}$$

Since we have seen that fluctuations are (nearly) frozen in on superhorizon scales, a way of characterizing the perturbations is to compute the spectrum on scales larger than the horizon. For a massive scalar field, we obtain

$$\mathcal{P}_{\delta\chi}(k) = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_\chi}. \tag{90}$$

We may also define the *spectral index* $n_{\delta\chi}$ of the fluctuations as

$$n_{\delta\chi} - 1 = \frac{d\ln \mathcal{P}_{\delta\phi}}{d\ln k} = 3 - 2\nu_\chi = 2\eta_\chi - 2\epsilon.$$

The power spectrum of fluctuations of the scalar field χ is therefore *nearly flat*, that is is nearly independent from the wavelength $\lambda = \pi/k$: the amplitude of the fluctuation on superhorizon scales does not (almost) depend upon the time at which the fluctuations crosses the horizon and becomes frozen in. The small tilt of the power spectrum arises from the fact that the scalar field χ is massive and because during inflation the Hubble rate is not exactly constant, but nearly constant, where 'nearly' is quantified by the slow-roll parameters ϵ. Adopting the traditional terminology, we may say that the spectrum of perturbations is blue if $n_{\delta\chi} > 1$ (more power in the ultraviolet) and red if $n_{\delta\chi} < 1$ (more power in the infrared). The power spectrum of the perturbations of a generic scalar field χ generated during a period of slow roll inflation may be either blue or red. This depends upon the relative magnitude between η_χ and ϵ. For instance, in chaotic inflation with a quadric potential $V(\phi) = \frac{m_\phi^2\phi^2}{2}$, one can easily compute

$$n_{\delta\chi} - 1 = 2\eta_\chi - 2\epsilon = \frac{2}{3H^2}\left(m_\chi^2 - m_\phi^2\right),$$

which tells us that the spectrum is blue (red) if $m_\chi^2 > m_\phi^2$ ($m_\chi^2 > m_\phi^2$).

<u>*Comment*</u>*:* We might have computed the spectral index of the spectrum $\mathcal{P}_{\delta\chi}(k)$ by first solving the equation for the perturbations of the field χ in a di Sitter stage, with H = constant and therefore $\epsilon = 0$, and then taking into account the time-evolution of the Hubble rate introducing the subscript in H_k whose time variation is determined by Eq. (60). Correspondingly, H_k is the value of the Hubble rate when a given wavelength $\sim k^{-1}$ crosses the horizon (from that point on the fluctuations remains frozen in). The power spectrum in such an approach would read

$$\mathcal{P}_{\delta\chi}(k) = \left(\frac{H_k}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_\chi} \tag{91}$$

with $3 - 2\nu_\chi \simeq \eta_\chi$. Using Eq. (60), one finds

$$n_{\delta\chi} - 1 = \frac{d\ln \mathcal{P}_{\delta\phi}}{d\ln k} = \frac{d\ln H_k^2}{d\ln k} + 3 - 2\nu_\chi = 2\eta_\chi - 2\epsilon$$

which reproduces our previous findings.

<u>Comment:</u> Since on superhorizon scales

$$\delta\chi_{\mathbf{k}} \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta_\chi - \epsilon} \simeq \frac{H}{\sqrt{2k^3}}\left[1 + (\eta_\chi - \epsilon)\ln\left(\frac{k}{aH}\right)\right],$$

we discover that

$$|\delta\dot{\chi}_{\mathbf{k}}| \simeq |H\,(\eta_\chi - \epsilon)\;\delta\chi_{\mathbf{k}}| \ll |H\,\delta\chi_{\mathbf{k}}|\,, \tag{92}$$

that is on superhorizon scales the time variation of the perturbations can be safely neglected.

As we have mentioned in the previous section, the linear theory of the cosmological perturbations represent a cornerstone of modern cosmology and is used to describe the formation and evolution of structures in the universe as well as the anisotrpies of the CMB. The seeds for these inhomegeneities were generated during inflation and stretched over astronomical scales because of the rapid superluminal expansion of the universe during the (quasi) de Sitter epoch.

In the previous section we have already seen that pertubations of a generic scalar field χ are generated during a (quasi) de Sitter expansion. The inflaton field is a scalar field and, as such, we conclude that inflaton fluctuations will be generated as well. However, the inflaton is special from the point of view of perturbations. The reason is very simple. By assumption, the inflaton field dominates the energy density of the universe during inflation. Any perturbation in the inflaton field means a perturbation of the stress energy-momentum tensor

$$\delta\phi \Longrightarrow \delta T_{\mu\nu}.$$

A perturbation in the stress energy-momentum tensor implies, through Einstein's equations of motion, a perturbation of the metric

$$\delta T_{\mu\nu} \Longrightarrow \left[\delta R_{\mu\nu} - \frac{1}{2}\delta\left(g_{\mu\nu}R\right)\right] = 8\pi G\delta T_{\mu\nu} \Longrightarrow \delta g_{\mu\nu}.$$

On the other hand, a pertubation of the metric induces a backreaction on the evolution of the inflaton perturbation through the perturbed Klein-Gordon equation of the inflaton field

$$\delta g_{\mu\nu} \Longrightarrow \delta\left(\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi}\right) = 0 \Longrightarrow \delta\phi.$$

This logic chain makes us conclude that the perturbations of the inflaton field and of the metric are tightly coupled to each other and have to be studied together

$$\boxed{\delta\phi \Longleftrightarrow \delta g_{\mu\nu}}$$

As we will see shortly, this relation is stronger than one might thought because of the issue of gauge invariance.

Before launching ourselves into the problem of finding the evolution of the quantum perturbations of the inflaton field when they are coupled to gravity, let us give a heuristic explanation of why we expect that during inflation such fluctuations are indeed present.

If we take Eq. (50) and split the inflaton field as its classical value ϕ_0 plus the quantum flucutation $\delta\phi$, $\phi(\mathbf{x},t)=\phi_0(t)+\delta\phi(\mathbf{x},t)$, the quantum perturbation $\delta\phi$ satisfies the equation of motion

$$\delta\ddot{\phi}+3H\,\delta\dot{\phi}-\frac{\nabla^2\delta\phi}{a^2}+V''\,\delta\phi=0. \tag{93}$$

Differentiating Eq. (55) wrt time and taking H constant (de Sitter expansion) we find

$$(\phi_0)^{\dddot{}}+3H\ddot{\phi}_0+V''\,\dot{\phi}_0=0. \tag{94}$$

Let us consider for simplicity the limit $\mathbf{k}^2/a^2\ll 1$ and let us disregard the gradient term. Under this condition we see that $\dot{\phi}_0$ and $\delta\phi$ solve the same equation. The solutions have therefore to be related to each other by a constant of proportionality which depends upon time , that is

$$\delta\phi=-\dot{\phi}_0\,\delta t(\mathbf{x}). \tag{95}$$

This tells us that $\phi(\mathbf{x},t)$ will have the form

$$\phi(\mathbf{x},t)=\phi_0\left(\mathbf{x},t-\delta t(\mathbf{x})\right).$$

This equation indicates that the inflaton field does not acquire the same value at a given time t in all the space. On the contrary, when the inflaton field is rolling down its potential, it acquires different values from one spatial point $\mathbf{x}$ to the other. The inflaton field is not homogeneous and fluctuations are present. These fluctuations, in turn, will induce fluctuations in the metric.

4.3 The Metric Fluctuations

The mathematical tool do describe the linear evolution of the cosmological perturbations is obtained by perturbing at the first-order the FRW metric $g_{\mu\nu}^{(0)}$, see Eq. (525)

$$g_{\mu\nu} \;=\; g_{\mu\nu}^{(0)}(t)\,+\,g_{\mu\nu}(\mathbf{x},t)\,;\qquad g_{\mu\nu}\ll g_{\mu\nu}^{(0)}\,. \tag{96}$$

The metric perturbations can be decomposed according to their spin with respect to a local rotation of the spatial coordinates on hypersurfaces of constant time. This leads to

- *scalar perturbations*
- *vector perturbations*
- *tensor perturbations*

Tensor perturbations or gravitational waves have spin 2 and are the "true" degrees of freedom of the gravitational fields in the sense that they can exist even in the vacuum. Vector perturbations are spin 1 modes arising from rotational velocity fields and are also called vorticity modes. Finally, scalar perturbations have spin 0.

Let us make a simple exercise to count how many scalar degrees of freedom are present. Take a spacetime of dimensions $D = n + 1$, of which n coordinates are spatial coordinates. The symmetric metric tensor $g_{\mu\nu}$ has $\frac{1}{2}(n+2)(n+1)$ degrees of freedom. We can perform $(n+1)$ coordinate transformations in order to eliminate $(n+1)$ degrees of freedom, this leaves us with $\frac{1}{2}n(n+1)$ degrees of freedom. These $\frac{1}{2}n(n+1)$ degrees of freedom contain scalar, vector and tensor modes. According to Helmholtz's theorem we can always decompose a vector u_i $(i = 1, \cdots, n)$ as $u_i = \partial_i v + v_i$, where v is a scalar (usually called potential flow) which is curl-free, $v_{[i,j]} = 0$, and v_i is a real vector (usually called vorticity) which is divergence-free, $\nabla \cdot v = 0$. This means that the real vector (vorticity) modes are $(n-1)$. Furthermore, a generic traceless tensor Π_{ij} can always be decomposed as $\Pi_{ij} = \Pi^S_{ij} + \Pi^V_{ij} + \Pi^T_{ij}$, where $\Pi^S_{ij} = \left(-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right)\Pi$, $\Pi^V_{ij} = (-i/2k)\left(k_i \Pi_j + k_j \Pi_i\right)$ $(k_i \Pi_i = 0)$ and $k_i \Pi^T_{ij} = 0$. This means that the true symmetric, traceless and transverse tensor degreees of freedom are $\frac{1}{2}(n-2)(n+1)$.

The number of scalar degrees of freedom are therefore

$$\frac{1}{2}n(n+1) - (n-1) - \frac{1}{2}(n-2)(n+1) = 2,$$

while the degrees of freedom of true vector modes are $(n-1)$ and the number of degrees of freedom of true tensor modes (gravitational waves) are $\frac{1}{2}(n-2)(n+1)$. In four dimensions $n = 3$, meaning that one expects 2 scalar degrees of freedom, 2 vector degrees of freedom and 2 tensor degrees of freedom. As we shall see, to the 2 scalar degrees of freedom from the metric, one has to add an another one, the inflaton field perturbation $\delta\phi$. However, since Einstein's equations will tell us that the two scalar degrees of freedom from the metric are equal during inflation, we expect a total number of scalar degrees of freedom equal to 2.

At the linear order, the scalar, vector and tensor perturbations evolve independently (they decouple) and it is therefore possible to analyze them separately. Vector perturbations are not excited during inflation because there are no rotational velocity fields during the inflationary stage. We will analyze the generation of tensor modes (gravitational waves) in the following. For the time being we want to focus on the scalar degrees of freedom of the metric.

Considering only the scalar degrees of freedom of the perturbed metric, the most generic perturbed metric reads

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2A & \partial_i B \\ \partial_i B & (1 - 2\psi)\,\delta_{ij} + D_{ij}E \end{pmatrix}, \tag{97}$$

while the line-element can be written as

$$ds^2 = a^2\big((-1 - 2A)d\tau^2 + 2\,\partial_i B\, d\tau\, dx^i + ((1-2\psi)\delta_{ij} + D_{ij}E)\, dx^i\, dx^j\big). \tag{98}$$

Here $D_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\,\delta_{ij}\,\nabla^2\right)$.

We now want to determine the inverse $g^{\mu\nu}$ of the metric at the linear order

$$g^{\mu\alpha}\, g_{\alpha\nu} \,=\, \delta^\mu_\nu. \tag{99}$$

We have therefore to solve the equations

$$\left(g^{\mu\alpha}_{(0)} \,+\, g^{\mu\alpha}\right)\left(g^{(0)}_{\alpha\nu} \,+\, g_{\alpha\nu}\right) \,=\, \delta^\mu_\nu\,, \tag{100}$$

where $g^{\mu\alpha}_{(0)}$ is simply the unperturbed FRW metric (525). Since

$$g^{\mu\nu}_{(0)} \,=\, \frac{1}{a^2}\left(\begin{array}{cc} -1 & 0 \\ 0 & \delta^{ij} \end{array}\right), \tag{101}$$

we can write in general

$$\begin{aligned} g^{00} &= \frac{1}{a^2}\,(\,-1 \,+\, X)\;; \\ g^{0i} &= \frac{1}{a^2}\,\partial^i Y\;; \\ g^{ij} &= \frac{1}{a^2}\,\left((1 \,+\, 2\,Z)\,\delta^{ij} \,+\, D^{ij}K\right). \end{aligned} \tag{102}$$

Plugging these expressions into Eq. (100) we find for $\mu = \nu = 0$

$$(-1 \,+\, X)(-1 \,-\, 2\,A) \,+\, \partial^i Y\, \partial_i B \,=\, 1. \tag{103}$$

Neglecting the terms $-\,2\,A{\cdot}X$ e $\partial^i Y \cdot \partial_i B$ because they are second-order in the perturbations, we find

$$1 \,-\, X \,+\, 2\,A \,=\, 1 \qquad \Rightarrow \qquad X \,=\, 2\,A\,. \tag{104}$$

Analogously, the components $\mu = 0$, $\nu = i$ of Eq. (100) give

$$(-1 \,+\, 2\,A)(\partial_i B) \,+\, \partial^j Y\,[(1 \,-\, 2\,\psi)\delta_{ji} \,+\, D_{ji}E] \,=\, 0. \tag{105}$$

At the first-order, we obtain

$$-\partial_i B \,+\, \partial_i Y \,=\, 0 \qquad \Rightarrow \qquad Y = B\,. \tag{106}$$

Finally, the components $\mu = i, \;\; \nu = j$ give

$$\partial^i B\, \partial_j B \,+\, \left((1 \,+\, 2\,Z)\delta^{ik} \,+\, D^{ik}K\right)\,((1 - 2\,\psi)\delta_{kj} \,+\, D_{kj}E) \,=\, \delta^i_j. \tag{107}$$

Neglecting the second-order terms, we obtain

$$(1 \,-\, 2\,\psi \,+\, 2\,Z)\delta^i_j \,+\, D^i_j E \,+\, D^i_j K \,=\, \delta^i_j \Rightarrow Z \,=\, \psi\,; \qquad K \,=\, -\,E\,. \tag{108}$$

The metric $g^{\mu\nu}$ finally reads

$$g^{\mu\nu} \,=\, \frac{1}{a^2}\left(\begin{array}{cc} -1 \,+\, 2\,A & \partial^i B \\ \partial^i B & (1 \,+\, 2\,\psi)\delta^{ij} \,-\, D^{ij}E \end{array}\right). \tag{109}$$

In this subsection we provide the reader with the perturbed affine connections and Einstein's tensor.

First, let us list the unperturbed affine connections

$$\Gamma^0_{00} = \frac{a'}{a}\,; \qquad \Gamma^i_{0j} = \frac{a'}{a}\,\delta^i_j\,; \qquad \Gamma^0_{ij} = \frac{a'}{a}\,\delta_{ij}\,; \tag{110}$$

$$\Gamma^i_{00} = \Gamma^0_{0i} = \Gamma^i_{jk} = 0\,. \tag{111}$$

The expression for the affine connections in terms of the metric is

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\rho}\left(\frac{\partial g_{\rho\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho}\right) \tag{112}$$

which implies

$$\begin{aligned}\delta\Gamma^\alpha_{\beta\gamma} &= \frac{1}{2}\,\delta g^{\alpha\rho}\left(\frac{\partial g_{\rho\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho}\right)\\ &+ \frac{1}{2}\,g^{\alpha\rho}\left(\frac{\partial \delta g_{\rho\gamma}}{\partial x^\beta} + \frac{\partial \delta g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial \delta g_{\beta\gamma}}{\partial x^\rho}\right),\end{aligned} \tag{113}$$

or in components

$$\delta\Gamma^0_{00} = A'\,; \tag{114}$$

$$\delta\Gamma^0_{0i} = \partial_i\,A + \frac{a'}{a}\partial_i\,B\,; \tag{115}$$

$$\delta\Gamma^i_{00} = \frac{a'}{a}\,\partial^i B + \partial^i B' + \partial^i A\,; \tag{116}$$

$$\begin{aligned}\delta\Gamma^0_{ij} &= -2\,\frac{a'}{a}\,A\,\delta_{ij} - \partial_i\partial_j B - 2\,\frac{a'}{a}\,\psi\,\delta_{ij}\\ &-\psi'\,\delta_{ij} - \tfrac{a'}{a}\,D_{ij}E + \tfrac{1}{2}\,D_{ij}E'\,;\end{aligned} \tag{117}$$

$$\delta\Gamma^i_{0j} = -\,\psi'\delta_{ij} + \frac{1}{2}\,D_{ij}E'\,; \tag{118}$$

$$\begin{aligned}\delta\Gamma^i_{jk} &= \partial_j\psi\,\delta^i_k - \partial_k\psi\,\delta^i_j + \partial^i\psi\,\delta_{jk} - \frac{a'}{a}\,\partial^i B\,\delta_{jk}\\ &+ \frac{1}{2}\,\partial_j D^i_k E + \frac{1}{2}\,\partial_k D^i_j E - \frac{1}{2}\,\partial^i D_{jk}E\,.\end{aligned} \tag{119}$$

We may now compite the Ricci scalar defines as

$$R_{\mu\nu} = \partial_\alpha\,\Gamma^\alpha_{\mu\nu} - \partial_\mu\,\Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha}\,. \tag{120}$$

Its variation at the first-order reads

$$\begin{aligned}\delta R_{\mu\nu} &= \partial_\alpha\,\delta\Gamma^\alpha_{\mu\nu} - \partial_\mu\,\delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{\mu\nu}\\ &- \delta\Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\,\delta\Gamma^\sigma_{\mu\alpha}\,.\end{aligned} \tag{121}$$

The background values are given by

$$R_{00} \,=\, -\,3\,\frac{a''}{a}\,+\,3\left(\frac{a'}{a}\right)^2; \qquad R_{0i}\,=\,0\,; \tag{122}$$

$$R_{ij}\,=\,\left(\frac{a''}{a}\,+\,\left(\frac{a'}{a}\right)^2\right)\delta_{ij} \tag{123}$$

which give

$$\delta R_{00} = \frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A' \ ; \tag{124}$$

$$\delta R_{0i} = \frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2\partial_i B + 2\partial_i\psi' + 2\frac{a'}{a}\partial_i A + \frac{1}{2}\partial_k D_i^k E' \ ; \tag{125}$$

$$\begin{aligned}\delta R_{ij} \ &= \ \Big(-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A \\ &- \ 2\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi - \frac{a'}{a}\partial_k\partial^k B\Big)\delta_{ij} \\ &- \ \partial_i\partial_j B' + \frac{a'}{a}D_{ij}E' + \frac{a''}{a}D_{ij}E + \left(\frac{a'}{a}\right)^2 D_{ij}E \\ &+ \ \frac{1}{2}D_{ij}E'' + \partial_i\partial_j\psi - \partial_i\partial_j A - 2\frac{a'}{a}\partial_i\partial_j B \\ &+ \ \frac{1}{2}\partial_k\partial_i D_j^k E + \frac{1}{2}\partial_k\partial_j D_i^k E - \frac{1}{2}\partial_k\partial^k D_{ij}E \ ;\end{aligned} \tag{126}$$

The perturbation of the scalar curvature

$$R\,=\,g^{\mu\alpha}\,R_{\alpha\mu}\,, \tag{127}$$

for which the first-order perturbation is

$$\delta R\,=\,\delta g^{\mu\alpha}\,R_{\alpha\mu}\,+\,g^{\mu\alpha}\,\delta R_{\alpha\mu}\,. \tag{128}$$

The background value is

$$R\,=\,\frac{6}{a^2}\,\frac{a''}{a} \tag{129}$$

while from Eq. (128) one finds

$$\begin{aligned}\delta R \ &= \ \frac{1}{a^2}\Big(-6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i A - 6\psi'' \\ &- \ 6\frac{a'}{a}A' - 18\frac{a'}{a}\psi' - 12\frac{a''}{a}A + 4\partial_i\partial^i\psi + \partial_k\partial^i D_i^k E\Big).\end{aligned} \tag{130}$$

Finally, we may compute the perturbations of the Einstein tensor

$$G_{\mu\nu}\,=\,R_{\mu\nu}\,-\,\frac{1}{2}\,g_{\mu\nu}\,R\,, \tag{131}$$

whose background components are

$$G_{00} \;=\; 3\left(\frac{a'}{a}\right)^2; \qquad G_{0i} \;=\; 0\,; \qquad G_{ij} \;=\; \left(-\,2\,\frac{a''}{a} \;+\; \left(\frac{a'}{a}\right)^2\right)\delta_{ij}\,. \tag{132}$$

At first-order, one finds

$$\delta G_{\mu\nu} \;=\; \delta R_{\mu\nu} \;-\; \frac{1}{2}\,\delta g_{\mu\nu}\,R \;-\; \frac{1}{2}\,g_{\mu\nu}\,\delta R\,, \tag{133}$$

or in components

$$\delta G_{00} = -2\frac{a'}{a}\,\partial_i\partial^i\,B - 6\frac{a'}{a}\,\psi' + 2\,\partial_i\partial^i\,\psi + \frac{1}{2}\,\partial_k\partial^i D_i^k E\,; \tag{134}$$

$$\delta G_{0i} = -2\frac{a''}{a}\,\partial_i B + \left(\frac{a'}{a}\right)^2\,\partial_i B + 2\partial_i\,\psi' + \frac{1}{2}\partial_k\,D_i^k E' + 2\,\frac{a'}{a}\,\partial_i A\,; \tag{135}$$

$$\begin{aligned} \delta G_{ij} \;\;=\;\; & \Bigg(2\frac{a'}{a}\,A' + 4\frac{a'}{a}\,\psi' + 4\frac{a''}{a}\,A - 2\left(\frac{a'}{a}\right)^2 A \\ +\;\; & 4\frac{a''}{a}\,\psi - 2\left(\frac{a'}{a}\right)^2\psi + 2\psi'' - \partial_k\partial^k\,\psi \\ +\;\; & 2\frac{a'}{a}\,\partial_k\partial^k B + \partial_k\partial^k B' + \partial_k\partial^k A + \frac{1}{2}\partial_k\partial^m\,D_m^k\,E\Bigg)\delta_{ij} \\ -\;\; & \partial_i\partial_j B' + \partial_i\partial_j\psi - \partial_i\partial_j A + \frac{a'}{a}\,D_{ij}E' - 2\,\frac{a''}{a}\,D_{ij}E \\ +\;\; & \left(\frac{a'}{a}\right)^2 D_{ij}E + \frac{1}{2}D_{ij}E'' + \frac{1}{2}\partial_k\partial_i D_j^k E \\ +\;\; & \frac{1}{2}\partial^k\partial_j D_{ik}E - \frac{1}{2}\partial_k\partial^k D_{ij}E - 2\,\frac{a'}{a}\partial_i\partial_j B\,. \end{aligned} \tag{136}$$

For convenience, we also give the expressions for the pertubations with one index up and one index down

$$\begin{aligned} \delta G^\mu_\nu \;&=\; \delta(g^{\mu\alpha\,G_{\alpha\nu}}) \\ &=\; \delta g^{\mu\alpha}\,G_{\alpha\nu} \;+\; g^{\mu\alpha}\,\delta G_{\alpha\nu}\,, \end{aligned} \tag{137}$$

or in components

$$\delta G^0_0 \;=\; 6\left(\frac{a'}{a}\right)^2 A \;+\; 6\,\frac{a'}{a}\psi' \;+\; 2\,\frac{a'}{a}\partial_i\,\partial^i B \;-\; 2\,\partial_i\,\partial^i\psi \;-\; \frac{1}{2}\,\partial_k\partial^i\,D_i^k E\,. \tag{138}$$

$$\delta G^0_i \;=\; -\,2\,\frac{a'}{a}\partial_i A \;-\; 2\,\partial_i\psi' \;-\; \frac{1}{2}\,\partial_k D_i^k E'\,. \tag{139}$$

$$
\begin{aligned}
\delta G^i_j \;&=\; \left(2\,\frac{a'}{a}A' \,+\, 4\,\frac{a''}{a}A \,-\, 2\left(\frac{a'}{a}\right)^2 A \,+\, \partial_i\,\partial^i A \,+\, 4\,\frac{a'}{a}\psi' \,+\, 2\,\psi'' \right.\\
&-\; \left. \partial_i\,\partial^i\psi \,+\, 2\,\frac{a'}{a}\partial_i\,\partial^i B \,+\, \partial_i\,\partial^i B' \,+\, \frac{1}{2}\partial_k\partial^m D^k_m E \right)\delta^i_j \\
&-\; \partial^i\partial_j A \,+\, \partial^i\partial_j\psi \,-\, 2\,\frac{a'}{a}\partial^i\partial_j B \,-\, \partial^i\partial_j B' \,+\, \frac{a'}{a}D^i_j E' \,+\, \frac{1}{2}\,D^i_j E'' \\
&+\; \frac{1}{2}\,\partial_k\partial^i\,D^k_j E \,+\, \frac{1}{2}\,\partial_k\partial_j\,D^{ik}E \,-\, \frac{1}{2}\,\partial_k\partial^k\,D^i_j E\,. \qquad (140)
\end{aligned}
$$

As we have seen previously, the perturbations of the metric are induced by the perturbations of the stress energy-momentum tensor of the inflaton field

$$
T_{\mu\nu} \,=\, \partial_\mu\phi\,\partial_\nu\phi \,-\, g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,-\, V(\phi)\right)\,, \qquad (141)
$$

whose background values are

$$
\begin{aligned}
T_{00} \;&=\; \frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2 \\
T_{0i} \;&=\; 0\,; \\
T_{ij} \;&=\; \left(\frac{1}{2}\,\phi'^2 \,-\, V(\phi)\,a^2\right)\,\delta_{ij}\,. \qquad (142)
\end{aligned}
$$

The perturbed stress energy-momentum tensor reads

$$
\begin{aligned}
\delta T_{\mu\nu} \;&=\; \partial_\mu\delta\phi\,\partial_\nu\phi \,+\, \partial_\mu\phi\,\partial_\nu\delta\phi \,-\, \delta g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
&-\; g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi \,+\, \frac{\partial V}{\partial\phi}\,\delta\phi \,+\, \frac{\partial V}{\partial\phi}\delta\phi\right)\,. \qquad (143)
\end{aligned}
$$

In components we have

$$
\delta T_{00} \,=\, \delta\phi'\,\phi' \,+\, 2\,A\,V(\phi)\,a^2 \,+\, a^2\,\frac{\partial V}{\partial\phi}\,\delta\phi\,; \qquad (144)
$$

$$
\delta T_{\,0i} \,=\, \partial_i\,\delta\phi\,\phi' \,+\, \frac{1}{2}\,\partial_i B\,\phi'^2 \,-\, \partial_i B\,V(\phi)\,a^2\,; \qquad (145)
$$

$$
\begin{aligned}
\delta T_{\,ij} \;&=\; \left(\delta\phi'\,\phi' \,-\, A\,\phi'^2 \,-\, a^2\,\frac{\partial V}{\partial\phi}\,\delta^{(1)}\phi \,-\, \psi\,\phi'^2 \,+\, 2\,\psi\,V(\phi)\,a^2\right)\delta_{ij} \\
&+\; \frac{1}{2}\,D_{ij}E\,\phi'^2 \,-\, D_{ij}E\,V(\phi)\,a^2\;. \qquad (146)
\end{aligned}
$$

For covenience, we list the mixed components

$$
\begin{aligned}
\delta T^\mu_\nu \;&=\; \delta(g^{\mu\alpha}\,T_{\alpha\nu}) \\
&=\; \delta g^{\mu\alpha}\,T_{\alpha\nu} \,+\, g^{\mu\alpha}\,\delta T_{\alpha\nu} \qquad (147)
\end{aligned}
$$

or

$$\begin{aligned}
\delta T^0_0 &= A\,\phi'^2 \,-\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\,;\\
\delta T^i_0 &= \partial^i B\,\phi'^2 \,+\, \partial^i\delta\phi\,\phi'\,;\\
\delta T^0_i &= -\,\partial^i\delta\phi\,\phi'\,;\\
\delta T^i_j &= \left(-\,A\,\phi'^2 \,+\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\right)\delta^i_j\,.
\end{aligned} \tag{148}$$

The inflaton equation of motion is the Klein-Gordon equation of a scalar field under the action of its potential $V(\phi)$. The equation to perturb is therefore

$$\begin{aligned}
\partial^\mu\partial_\mu\,\phi &= \frac{\partial V}{\partial\phi}\,;\\
\partial_\mu\partial^\mu\phi &= \frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\phi\right)\,;
\end{aligned} \tag{149}$$

which at the zero-th order gives the inflaton equation of motion

$$\phi'' \,+\, 2\,\frac{a'}{a}\,\phi' \,=\, -\,\frac{\partial V}{\partial\phi}\,a^2\,. \tag{150}$$

The variation of Eq. (381) is the sum of four different contributions corresponding to the variations of $\frac{1}{\sqrt{-g}}$, $\sqrt{-g}$, $g^{\mu\nu}$ and ϕ. For the variation of g we have

$$\delta g \,=\, g\,g^{\mu\nu}\delta g_{\nu\mu} \,=\, dg \,=\, g^{\mu\nu}dg_{\mu\nu} \tag{151}$$

which give at the linear order

$$\begin{aligned}
\delta\sqrt{-g} &= -\,\frac{\delta g}{2\,\sqrt{-g}}\,;\\
\delta\frac{1}{\sqrt{-g}} &= \frac{\delta\sqrt{-g}}{g}\,.
\end{aligned} \tag{152}$$

Plugging these results into the expression for the variation of Eq. (150)

$$\begin{aligned}
\delta\partial_\mu\partial^\mu\,\phi &= -\,\delta\phi'' \,-\, 2\,\frac{a'}{a}\delta\phi' \,+\, \partial_i\,\partial^i\delta\phi \,+\, 2\,A\,\phi'' \,+\, 4\,\frac{a'}{a}A\,\phi' \,+\, A'\phi'\\
&+\; 3\,\psi'\phi' \,+\, \partial_i\,\partial^i B\,\phi'\\
&= \delta\phi\,\frac{\partial^2 V}{\partial\phi^2}\,a^2\,.
\end{aligned} \tag{153}$$

Using Eq. (150) to write

$$2\,A\,\phi'' \,+\, 4\,\frac{a'}{a}\phi' \,=\, 2\,A\,\frac{\partial V}{\partial\phi}\,, \tag{154}$$

Eq. (153) becomes

$$\delta\phi'' + 2\,\frac{a'}{a}\delta\phi' \;-\; \partial_i\,\partial^i\delta\phi \,-\, A'\phi' \,-\, 3\,\psi'\phi' \,-\, \partial_i\,\partial^i B\,\phi'$$
$$= \; -\delta\phi\,\frac{\partial^2 V}{\partial\phi^2}\,a^2 \,-\, 2\,A\,\frac{\partial V}{\partial\phi}. \qquad (155)$$

After having computed the perturbations at the linear order of the Einstein's tensor and of the stress energy-momentum tensor, we are ready to solve the perturbed Einstein's equations in order to quantify the inflaton and the metric fluctuations. We pause, however, for a moment in order to deal with the problem of gauge invariance.

When studying the cosmological density perturbations, what we are interested in is following the evolution of a spacetime which is neither homogeneous nor isotropic. This is done by following the evolution of the differences between the actual spacetime and a well understood reference spacetime. So we will consider small perturbations away from the homogeneous, isotropic spacetime (see Fig. 9). The reference system in our case is the spatially flat Friedmann–Robertson–Walker (FRW) spacetime, with line element $ds^2 = a^2(\tau)\,\{d\tau^2 - \delta_{ij}dx^i dx^j\}$. Now, the key issue is that general relativity is a gauge theory where the gauge transformations are the generic coordinate transformations from a local reference frame to another.

When we compute the perturbation of a given quantity, this is defined to be the difference between the value that this quantity assumes on the real physical spacetime and the value it assumes on the unperturbed background. Nonetheless, to perform a comparison between these two values, it is necessary to compute the at the same spacetime point. Since the two values "live" on two different geometries, it is necessary to specify a map which allows to link univocally the same point on the two different spacetimes. This correspondance is called a gauge choice and changing the map means performing a gauge transformation.

Fixing a gauge in general relativity implies choosing a coordinate system. A choice of coordinates defines a *threading* of spacetime into lines (corresponding to fixed spatial coordinates $\mathbf{x}$) and a *slicing* into hypersurfaces (corresponding to fixed time τ). A choice of coordinates is is called a *gauge* and there is no unique preferred gauge

GAUGE CHOICE $\Longleftrightarrow$ SLICING AND THREADING

From a more formal point of view, operating an infinitesimal gauge tranformation on the coordinates

$$\widetilde{x^\mu} \,=\, x^\mu \,+\, \delta x^\mu \qquad (156)$$

implies on a generic quantity Q a tranformation on its perturbation

$$\widetilde{\delta Q} \,=\, \delta Q \,+\, \pounds_{\delta x}\,Q_0 \qquad (157)$$

where Q_0 is the value assumed by the quantity Q on the background and $\pounds_{\delta x}$ is the Lie-derivative of Q along the vector δx^μ.

Decomposing in the usual manner the vector δx^μ

$$\begin{aligned} \delta x^0 &= \xi^0(x^\mu)\,; \\ \delta x^i &= \partial^i\beta(x^\mu) + v^i(x^\mu)\,; \qquad \partial_i v^i = 0\,, \end{aligned} \tag{158}$$

we can easily deduce the transformation law of a scalar quantity f (like the inflaton scalar field ϕ and energy density ρ). Instead of applying the formal definition (157), we find the transformation law in an alternative (and more pedagogical) way. We first write $\delta f(x) = f(x) - f_0(x)$, where $f_0(x)$ is the background value. Under a gauge transformation we have $\widetilde{\delta f}(\widetilde{x^\mu}) = \widetilde{f}(\widetilde{x^\mu}) - \widetilde{f}_0(\widetilde{x^\mu})$. Since f is a scalar we can write $f(\widetilde{x^\mu}) = f(x^\mu)$ (the value of the scalar function in a given physical point is the same in all the coordinate system). On the other side, on the unperturbed background hypersurface $\widetilde{f}_0 = f_0$. We have therefore

$$\begin{aligned} \widetilde{\delta f}(\widetilde{x^\mu}) &= \widetilde{f}(\widetilde{x^\mu}) - \widetilde{f}_0(\widetilde{x^\mu}) \\ &= f(x^\mu) - f_0(\widetilde{x^\mu}) \\ &= f\left(\widetilde{x^\mu}\right) - f_0(\widetilde{x^\mu}) \\ &= f(\widetilde{x^\mu}) - \delta x^\mu \frac{\partial f}{\partial x^\mu}(\widetilde{x}) - f_0(\widetilde{x^\mu}), \end{aligned} \tag{159}$$

from which we finally deduce, being $f_0 = f_0(x^0)$,

$$\widetilde{\delta f} = \delta f - f'\,\xi^0$$

For the spin zero perturbations of the metric, we can proceed analogously. We use the following trick. Upon a coordinate transformation $x^\mu \to \widetilde{x^\mu} = x^\mu + \delta x^\mu$, the line element is left invariant, $ds^2 = \widetilde{ds^2}$. This implies, for instance, that $a^2(\widetilde{x^0})\left(1+\widetilde{A}\right)\left(d\widetilde{x^0}\right)^2 = a^2(x^0)\,(1+A)\,(dx^0)^2$. Since $a^2(\widetilde{x^0}) \simeq a^2(x^0) + 2a\,a'\,\xi^0$ and $d\widetilde{x^0} = (1+\xi^{0\prime})\,dx^0 + \frac{\partial x^0}{\partial x^i}\,dx^i$, we obtain $1 + 2A = 1 + 2\widetilde{A} + 2\,\mathcal{H}\xi^0 + 2\xi^{0\prime}$. A similar procedure leads to the following transformation laws

$$\begin{aligned} \widetilde{A} &= A - \xi^{0\prime} - \frac{a'}{a}\xi^0\,; \\ \widetilde{B} &= B + \xi^0 + \beta' \\ \widetilde{\psi} &= \psi - \frac{1}{3}\nabla^2\beta + \frac{a'}{a}\xi^0\,; \\ \widetilde{E} &= E + 2\beta\,. \end{aligned}$$

The gauge problem stems from the fact that a change of the map (a change of the coordinate system) implies the variation of the perturbation of a given quantity which may therefore assume different values (all of them on a equal footing!) according to the gauge choice. To eliminate this ambiguity, one has therefore a double choice:

- Indentify those combinations representing gauge invariant quantities;
- choose a given gauge and perform the calculations in that gauge.

Both options have advantages and drawbacks. Choosing a gauge may render the computation technically simpler with the danger, however, of including gauge artifacts, *i.e.* gauge freedoms which are not physical. Performing a gauge-invariant computation may be technically more involved, but has the advantage of treating only physical quantities.

Let us first indicate some gauge-invariant quantities which have been introduced first in Ref. [16]. They are the so-called gauge invariant potentials or Bardeen's potentials

$$\Phi = -A + \frac{1}{a}\left[\left(-B + \frac{E'}{2}\right) a\right]' , \tag{160}$$

$$\Psi = -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left(B - \frac{E'}{2}\right) . \tag{161}$$

Analogously, one can define a gauge invariant quantity for the perturbation of the inflaton field. Since ϕ is a scalar field $\widetilde{\delta\phi} = (\delta\phi - \phi'\,\xi^0)$ and therefore

$$\boxed{\delta\phi^{(\mathrm{GI})} = -\delta\phi + \phi'\left(\frac{E'}{2} - B\right)}$$

is gauge-invariant.

Analogously, one can define a gauge-invariant energy-density perturbation

$$\boxed{\delta\rho^{(\mathrm{GI})} = -\delta\rho + \rho'\left(\frac{E'}{2} - B\right)}$$

We now want to pause to introduce in details some gauge-invariant quantities which play a major role in the computation of the density perturbations. In the following we will be interested only in the coordinate transformations on constant time hypersurfaces and therefore gauge invariance will be equivalent to independent of the slicing.

4.4 The comoving curvature perturbation

The intrinsic spatial curvature on hypersurfaces on constant conformal time τ and for a flat universe is given by

$$^{(3)}R = \frac{4}{a^2}\nabla^2\,\psi.$$

The quantity ψ is usually referred to as the *curvature perturbation.* We have seen, however, that the the curvature potential ψ is *not* gauge invariant, but is defined only on a given slicing. Under a transformation on constant time hypersurfaces $t \to t + \delta\tau$ (change of the slicing)

$$\psi \to \psi + \mathcal{H}\,\delta\tau.$$

We now consider the *comoving slicing* which is defined to be the slicing orthogonal to the worldlines of comoving observers. The latter are are free-falling and the expansion defined by them is isotropic. In practice, what this means is that there is no flux of energy measured by these observers, that is $T_{0i} = 0$. During inflation this means that these observers measure $\delta\phi_{\rm com} = 0$ since T_{0i} goes like $\partial_i\delta\phi(\mathbf{x},\tau)\phi'(\tau)$.

Since $\delta\phi \to \delta\phi - \phi'\delta\tau$ for a transformation on constant time hypersurfaces, this means that

$$\delta\phi \to \delta\phi_{\rm com} = \delta\phi - \phi'\,\delta\tau = 0 \Longrightarrow \delta\tau = \frac{\delta\phi}{\phi'},$$

that is $\delta\tau = \frac{\delta\phi}{\phi'}$ is the time-displacement needed to go from a generic slicing with generic $\delta\phi$ to the comoving slicing where $\delta\phi_{\rm com} = 0$. At the same time the curvature pertubation ψ transforms into

$$\psi \to \psi_{\rm com} = \psi + \mathcal{H}\,\delta\tau = \psi + \mathcal{H}\frac{\delta\phi}{\phi'}.$$

The quantity

$$\boxed{\mathcal{R} = \psi + \mathcal{H}\frac{\delta\phi}{\phi'} = \psi + H\frac{\delta\phi}{\dot\phi}}$$

is the *comoving curvature perturbation.* This quantity is gauge invariant by construction and is related to the gauge-dependent curvature perturbation ψ on a generic slicing to the inflaton perturbation $\delta\phi$ in that gauge. By construction, the meaning of $\mathcal{R}$ is that it represents the gravitational potential on comoving hypersurfaces where $\delta\phi = 0$

$$\mathcal{R} = \psi|_{\delta\phi=0}\,.$$

We now consider the *slicing of uniform energy density* which is defined to be the the slicing where there is no perturbation in the energy density, $\delta\rho = 0$.

Since $\delta\rho \rightarrow \delta\rho - \rho'\,\delta\tau$ for a transformation on constant time hypersurfaces, this means that

$$\delta\rho \rightarrow \delta\rho_{\rm unif} = \delta\rho - \rho'\,\delta\tau = 0 \Longrightarrow \delta\tau = \frac{\delta\rho}{\rho'},$$

that is $\delta\tau = \frac{\delta\rho}{\rho'}$ is the time-displacement needed to go from a generic slicing with generic $\delta\rho$ to the slicing of uniform energy density where $\delta\rho_{\rm unif} = 0$. At the same time the curvature pertubation ψ transforms into

$$\psi \rightarrow \psi_{\rm unif} = \psi + \mathcal{H}\,\delta\tau = \psi + \mathcal{H}\frac{\delta\rho}{\rho'}.$$

The quantity

$$\boxed{\zeta = \psi + \mathcal{H}\frac{\delta\rho}{\rho'} = \psi + H\frac{\delta\rho}{\dot\rho}}$$

is the *curvature perturbation on slices of uniform energy density.* This quantity is gauge invariant by construction and is related to the gauge-dependent curvature perturbation ψ on a generic slicing and to the energy density perturbation $\delta\rho$ in that gauge. By construction, the meaning of ζ is that it represents the gravitational potential on slices of uniform energy density

$$\zeta = \psi|_{\delta\rho=0}\,.$$

Notice that, using the energy-conservation equation $\rho' + 3\,\mathcal{H}(\rho + p) = 0$, the curvature perturbation on slices of uniform energy density can be also written as

$$\zeta = \psi - \frac{\delta\rho}{3(\rho + p)}.$$

During inflation $\rho + p = \dot\phi^2$. Furthermore, on superhorizon scales from what we have learned in the previous section (and will be rigously shown in the following) the inflaton fluctuation $\delta\phi$ is frozen in and $\delta\dot\phi =$ (slow roll parameters) $\times\, H\,\delta\phi$. This implies that $\delta\rho = \dot\phi\delta\dot\phi + V'\delta\phi \simeq V'\delta\phi \simeq -3H\dot\phi$, leading to

$$\zeta \simeq \psi + \frac{3H\dot\phi}{3\dot\phi^2} = \psi + H\frac{\delta\phi}{\dot\phi}\mathcal{R} \qquad \text{(ON SUPERHORIZON SCALES)}$$

The comoving curvature pertubation and the curvature perturbation on uniform energy density slices are equal on superhorizon scales.

We now consider the *spatially flat gauge* which is defined to be the the slicing where there is no curvature $\psi_{\rm flat} = 0$.

Since $\psi \rightarrow \psi + \mathcal{H}\,\delta\tau$ for a transformation on constant time hypersurfaces, this means that

$$\psi \to \psi_{\rm flat} = \psi + \mathcal{H}\,\delta\tau = 0 \Longrightarrow \delta\tau = -\frac{\psi}{\mathcal{H}},$$

that is $\delta\tau = -\psi/\mathcal{H}$ is the time-displacement needed to go from a generic slicing with generic ψ to the spatially flat gauge where $\psi_{\rm flat} = 0$. At the same time the fluctuation of the inflaton field transforms a

$$\delta\phi \to \delta\phi - \phi'\,\delta\tau = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\psi.$$

The quantity

$$\boxed{Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\psi = \delta\phi + \frac{\dot\phi}{H}\psi \equiv \frac{\dot\phi}{H}\,\mathcal{R}}$$

is the inflaton perturbation on spatially flat gauges. This quantity is gauge invariant by construction and is related to the inflaton perturbation $\delta\phi$ on a generic slicing and to to the curvature perturbation ψ in that gauge. By construction, the meaning of Q is that it represents the inflaton potential on spatially flat slices

$$Q = \left.\delta\phi\right|_{\delta\psi=0}.$$

This quantity is often referred to as the Sasaki or Mukhanov variable [71, 61].

Notice that $\delta\phi = -\phi'\delta\tau = -\dot\phi\delta t$ on flat slices, where δt is the time displacement going from flat to comoving slices. This relation makes somehow rigorous the expression (95). Analogously, going from flat to comoving slices one has $\mathcal{R} = H\,\delta t$.

4.5 Adiabatic and isocurvature perturbations

Arbitrary cosmological perturbations can be decomposed into:

- *adiabatic or curvature perturbations* which perturb the solution along the same trajectory in phase-space as the as the background solution. The perturbations in any scalar quantity X can be described by a unique perturbation in expansion with respect to the background

  $$H\,\delta t = H\,\frac{\delta X}{\dot X} \qquad \text{FOR EVERY } X.$$

  In particular, this holds for the energy density and the pressure

  $$\frac{\delta\rho}{\dot\rho} = \frac{\delta p}{\dot p}$$

  which implies that $p = p(\rho)$. This explains why they are called adiabatic. They are called curvature perturbations because a given time displacement δt causes the same relative change $\delta X/\dot{X}$ for all quantities. In other words the perturbations is democratically shared by all components of the universe.

- *isocurvature perturbations* which perturb the solution off the background solution

$$\frac{\delta X}{\dot{X}} \neq \frac{\delta Y}{\dot{Y}} \quad \text{FOR SOME } X \text{ AND } Y.$$

One way of specifying a generic isocurvature perturbation δX is to give its value on uniform-density slices, related to its value on a different slicing by the gauge-invariant definition

$$H \left. \frac{\delta X}{\dot{X}} \right|_{\delta\rho=0} = H \left(\frac{\delta X}{\dot{X}} - \frac{\delta\rho}{\dot{\rho}} \right).$$

For a set of fluids with energy density ρ_i, the isocurvature perturbations are conventionally defined by the gauge invariant quantities

$$S_{ij} = 3H \left(\frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j} \right) = 3\left(\zeta_i - \zeta_j\right).$$

One simple example of isocurvature perturbations is the baryon-to-photon ratio $S = \delta(n_B/n_\gamma) = (\delta n_B/n_B) - (\delta n_\gamma/n_\gamma)$.

1. Comment:

From the definitions above, it follows that the cosmological perturbations generated during inflation are of the adiabatic type *if* the inflaton field is the only fiels driving inflation. However, if inflation is driven by more than one field, isocurvature perturbations are expected to be generated (and they might even be cross-correlated to the adiabatic ones [18, 19, 21]). In the following, however, we will keep focussing – as done so far – on the one-single field case, that is we will be dealing only with adiabatic/curvature perturbations.

2. Comment: The perturbations generated during inflation are *gaussian*, *i.e.* the two-point correlation functions (like the power spectrum) suffice to define all the higher-order even correlation fucntions, while the odd correlation functions (such as the three-point correlation function) vanish. This conclusion is drawn by the very same fact that cosmological perturbations are studied *linearizing* Einstein's and Klein-Gordon equations. This turns out to be a good approximation because we know that the inflaton potential needs to be very flat in order to drive inflation and the interaction terms in the inflaton potential might be present, but they are small. Non-gaussian features are therefore suppressed since the non-linearities of the inflaton potential are suppressed too. The same argument applies to the metric perturbations; non-linearities appear only at the second-order in deviations from the homogeneous background solution and are therefore small. This expectation has been recently confirmed by the first computation of the cosmological perturbations generated during inflation up to second-order in deviations from the homogeneous background solution which fully account for the inflaton self-interactions as well as for the second-order fluctuations of the background metric [11].

After all these technicalities, it is useful to rest for a moment and to go back to physics. Up to now we have learned that during inflation quantum fluctuations of the inflaton field are generated and their wavelengths are stretched on large scales by the rapid expansion of the universe. We have also seen that the quantum fluctuations of the inflaton field are in fact impossible to disantagle from the metric perturbations. This happens not only because they are tightly coupled to each other through Einstein's equations, but also because of the issue of gauge invariance. Take, for instance, the gauge invariant quantity $Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\psi$. We can always go to a gauge where the fluctuation is entirely in the curvature potential ψ, $Q = \frac{\phi'}{\mathcal{H}}\,\psi$, or entirely in the inflaton field, $Q = \delta\phi$. However, as we have stressed at the end of the previous section, once ripples in the curvature are frozen in on superhorizon scales during inflation, it is in fact gravity that acts as a messanger communicating to baryons and photons the small seeds of perturbations once a given scale reenters the horizon after inflation. This happens thanks to Newtonian physics; a small perturbation in the gravitational potential ψ induces a small perturbation of the energy density ρ through Poisson's equiation $\nabla^2\psi = 4\pi G\delta\rho$. Similarly, if perturbations are adiabatic/curvature perturbations and, as such, treat democratically all the components, a ripple in the curvature is communicated to photons as well, giving rise to a nonvanishing $\delta T/T$.

These considerations make it clear that the next steps will be

- Compute the curvature perturbation generated during inflation on superhorizon scales. As we have seen we can either compute the comoving curvature perturbation $\mathcal{R}$ or the curvature on uniform energy density hypersurfaces ζ. They will tell us about the fluctuations of the gravitational potential.

- See how the fluctuations of the gravitational potential are transmitted to baryons and photons.

We now intend to address the first point. As stressed previously, we are free to follow two alternative roads: either pick up a gauge and compute the gauge-invariant curvature in that gauge or perform a gauge-invariant calculation. We take both options.

The longitudinal (or conformal newtonian) gauge is a convenient gauge to compute the cosmological perturbations. It is defined by performing a coordinate transformation such that $B = E = 0$. This leaves behind two degrees of freedom in the scalar perturbations, A and ψ. As we have previously seen in subsection 7.1, these two degrees of freedom fully account for the scalar perturbations in the metric.

First of all, we take the non-diagonal part ($i \neq j$) of Eq. (140). Since the stress energy-momentum tensor does not have any non-diagonal component (no stress), we have

$$\partial_i\partial_j\left(\psi - A\right) = 0 \Longrightarrow \psi = A$$

and we can now work only with one variable, let it be ψ.

Eq. (139) gives (in cosmic time)

$$\dot{\psi} + H\,\psi = 4\pi G\dot{\phi}\,\delta\phi = \epsilon\,H\frac{\delta\phi}{\dot{\phi}}, \tag{162}$$

while Eq. (138) and the diagonal part of (140) ($i = j$) give respectively

$$
\begin{aligned}
-3H\left(\dot{\psi}+H\psi\right)+\frac{\nabla^2\psi}{a^2} &= 4\pi G\left(\dot{\phi}\delta\dot{\phi}-\dot{\phi}^2\psi+V'\delta\phi\right), && (163)\\
-\left(2\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right)\psi-3H\dot{\psi}-\ddot{\psi} &= -\left(\dot{\phi}\delta\dot{\phi}-\dot{\phi}^2\psi-V'\delta\phi\right), && (164)
\end{aligned}
$$

If we now use the fact that $\dot{H}=4\pi G\dot{\phi}^2$, sum Eqs. (163) and 164) and use the background Klein-Gordon equation to eliminate V', we arrive at the equation for the gravitational potential

$$
\ddot{\psi}_{\mathbf{k}}+\left(H-2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\psi}_{\mathbf{k}}+2\left(\dot{H}-H\frac{\ddot{\phi}}{\dot{\phi}}\right)\psi_{\mathbf{k}}+\frac{k^2}{a^2}\psi_{\mathbf{k}}=0 \tag{165}
$$

We may rewrite it in conformal time

$$
\psi''_{\mathbf{k}}+2\left(\mathcal{H}-\frac{\phi''}{\phi'}\right)\psi'_{\mathbf{k}}+2\left(\mathcal{H}'-\mathcal{H}\frac{\phi''}{\phi'}\right)\psi_{\mathbf{k}}+k^2\,\psi_{\mathbf{k}}=0 \tag{166}
$$

and in terms of the slow-roll parameters ϵ and η

$$
\psi''_{\mathbf{k}}+2\,\mathcal{H}\left(\eta-\epsilon\right)\psi'_{\mathbf{k}}+2\,\mathcal{H}^2\left(\eta-2\epsilon\right)\psi_{\mathbf{k}}+k^2\,\psi_{\mathbf{k}}=0. \tag{167}
$$

Using the same logic leading to Eq. (92), from Eq. (165) we can infer that on superhorizon scales the gravitational potential ψ is nearly constant (up to a mild logarithmic time-dependence proportional to slow-roll parameters), that is $\dot{\psi}_{\mathbf{k}}\sim$ (slow-roll parameters)$\times\psi_{\mathbf{k}}$. This is hardly surprising, we know that fluctuations are frozen in on superhorizon scales.

Using Eq. (162), we can therefore relate the fluctuation of the gravitational potential ψ to the fluctuation of the inflaton field $\delta\phi$ on superhorizon scales

$$
\psi_{\mathbf{k}}\simeq\epsilon\,H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}\quad\text{(ON SUPERHORIZON SCALES)} \tag{168}
$$

This gives us the chance to compute the gauge-invariant comoving curvature perturbation $\mathcal{R}_{\mathbf{k}}$

$$
\mathcal{R}_{\mathbf{k}}=\psi_{\mathbf{k}}+H\,\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}=(1+\epsilon)\,\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}\simeq\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}. \tag{169}
$$

The power spectrum of the the comoving curvature perturbation $\mathcal{R}_{\mathbf{k}}$ then reads on superhorizon scales

$$
\mathcal{P}_{\mathcal{R}}=\frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}\left|\delta\phi_{\mathbf{k}}\right|^2=\frac{k^3}{4{m_{\rm Pl}}^2\epsilon\,\pi^2}\left|\delta\phi_{\mathbf{k}}\right|^2.
$$

What is left to evaluate is the time evolution of $\delta\phi_{\mathbf{k}}$. To do so, we consider the perturbed Klein-Gordon equation (155) in the longitudinal gauge (in cosmic time)

$$
\delta\ddot{\phi}_{\mathbf{k}}+3H\delta\dot{\phi}_{\mathbf{k}}+\frac{k^2}{a^2}\delta\phi_{\mathbf{k}}+V''\delta\phi_{\mathbf{k}}=-2\psi_{\mathbf{k}}V'+4\dot{\psi}_{\mathbf{k}}\dot{\phi}.
$$

Since on superhorizon scales $\left|4\dot{\psi}_{\mathbf{k}}\dot{\phi}\right| \ll |\psi_{\mathbf{k}}V'|$, using Eq. (168) and the relation $V' \simeq -3H\dot{\phi}$, we can rewrite the perturbed Klein-Gordon equation on superhorizon scales as

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(V'' + 6\epsilon H^2\right)\delta\phi_{\mathbf{k}} = 0.$$

We now introduce as usual the field $\delta\chi_{\mathbf{k}} = \delta\phi_{\mathbf{k}}/a$ and go to conformal time τ. The perturbed Klein-Gordon equation on superhorizon scales becomes, using Eq. (86),

$$\begin{aligned} \delta\chi''_{\mathbf{k}} &- \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\delta\chi_{\mathbf{k}} = 0, \\ \nu^2 &= \frac{9}{4} + 9\epsilon - 3\eta. \end{aligned} \tag{170}$$

Using what we have learned in the previous section, we conclude that

$$|\delta\phi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \quad \text{(ON SUPERHORIZON SCALES)}$$

which justifies our initial assumption that both the inflaton perturbation and the gravitational potential are nearly constant on superhorizon scale.

We may now compute the power spectrum of the comoving curvature perturbation on superhorizon scales

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2{m_{\rm Pl}}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \equiv A_{\mathcal{R}}^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}}$$

where we have defined the *spectral index* $n_{\mathcal{R}}$ of the comoving curvature perturbation as

$$\boxed{n_{\mathcal{R}} - 1 = \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k} = 3 - 2\nu = 2\eta - 6\epsilon.}$$

We conclude that inflation is responsible for the generation of adiabatic/curvature perturbations with an almost scale-independent spectrum.

From the curvature perturbation we can easily deduce the behaviour of the gravitational potential $\psi_{\mathbf{k}}$ from Eq. (162). The latter is solved by

$$\psi_{\mathbf{k}} = \frac{A(k)}{a} + \frac{4\pi G}{a}\int^t dt'\, a(t')\,\dot{\phi}(t')\,\delta\phi_{\mathbf{k}}(t') \simeq \frac{A(k)}{a} + \epsilon\,\mathcal{R}_{\mathbf{k}}.$$

We find that during inflation and on superhorizon scales the gravitational potential is the sum of a decreasing function plus a nearly constant in time piece proportional to the curvature perturbation. Notice in particular that in an exact de Sitter stage, that is $\epsilon = 0$, the gravitational potential is not sourced and any initial condition in the gravitational potential is washed out as a^{-1} during the inflationary stage.

Comment: We might have computed the spectral index of the spectrum $\mathcal{P}_{\mathcal{R}}(k)$ by first solving the equation for the perturbation $\delta\phi_{\mathbf{k}}$ in a di Sitter stage, with $H =$ constant ($\epsilon = \eta = 0$), whose solution is Eq. (75) and then taking into account the time-evolution of the Hubble rate and of ϕ introducing the subscript in H_k and $\dot{\phi}_k$. The time variation of the latter is determined by

$$\frac{d\ln\dot{\phi}_k}{d\ln k} = \left(\frac{d\ln\dot{\phi}_k}{dt}\right)\left(\frac{dt}{d\ln a}\right)\left(\frac{d\ln a}{d\ln k}\right) = \frac{\ddot{\phi}_k}{\dot{\phi}_k} \times \frac{1}{H} \times 1 = -\delta = \epsilon - \eta. \tag{171}$$

Correspondingly, $\dot{\phi}_k$ is the value of the time derivative of the inflaton field when a given wavelength $\sim k^{-1}$ crosses the horizon (from that point on the fluctuations remains frozen in). The curvature perturbation in such an approach would read

$$\mathcal{R}_{\mathbf{k}} \simeq \frac{H_k}{\dot{\phi}_k}\,\delta\phi_{\mathbf{k}} \simeq \frac{1}{2\pi}\left(\frac{H_k^2}{\dot{\phi}_k}\right).$$

Correspondigly

$$n_{\mathcal{R}} - 1 = \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k} = \frac{d\ln H_k^4}{d\ln k} - \frac{d\ln\dot{\phi}_k^2}{d\ln k} = -4\epsilon + (2\eta - 2\epsilon) = 2\eta - 6\epsilon$$

which reproduces our previous findings.

During inflation the curvature perturbation is generated on superhorizon scales with a spectrum which is nearly scale invariant, that is is nearly independent from the wavelength $\lambda = \pi/k$: the amplitude of the fluctuation on superhorizon scales does not (almost) depend upon the time at which the fluctuations crosses the horizon and becomes frozen in. The small tilt of the power spectrum arises from the fact that the inflaton field is massive, giving rise to a nonvanishing η and because during inflation the Hubble rate is not exactly constant, but nearly constant, where 'nearly' is quantified by the slow-roll parameters ϵ.

Comment: From what found so far, we may conclude that on superhorizon scales the comoving curvature perturbation $\mathcal{R}$ and the uniform-density gauge curvature ζ satisfy on superhorizon scales the relation

$$\dot{\mathcal{R}}_{\mathbf{k}} \simeq \dot{\zeta}_{\mathbf{k}} \simeq 0.$$

An independent argument of the fact that they are nearly constant on superhorizon scales is given in the Appendix A.

In this subsection we would like to show how the computation of the curvature perturbation can be performed in a gauge-invariant way. We first rewrite Einstein's equations in terms of Bardeen's potentials (160) and (161)

$$\delta G^0_0 = \frac{2}{a^2}\left(- 3\,\mathcal{H}\,(\mathcal{H}\,\Phi \,+\, \Psi') \,+\, \nabla^2\Psi \,+\, 3\,\mathcal{H}\,(-\mathcal{H}' \,+\, \mathcal{H}^2)\left(\frac{E'}{2} \,-\, B\right)\right), \tag{172}$$

$$\delta G^0_i = \frac{2}{a^2}\,\partial_i \left(\mathcal{H}\,\Phi \,+\, \Psi' \,+\, (\mathcal{H}' \,-\, \mathcal{H}^2)\left(\frac{E'}{2} \,-\, B\right)\right), \tag{173}$$

$$\begin{aligned}\delta G^i_j &= -\frac{2}{a^2}\left(\left((2\,\mathcal{H}' \,+\, 2\,\mathcal{H}^2)\,\Phi \,+\, \mathcal{H}\,\Phi' \,+\, \Psi'' \,+\, 2\,\mathcal{H}\,\Psi' \,+\, \frac{1}{2}\,\nabla^2 D\right)\delta^i_j\right.\\ &+ \left.(\mathcal{H}'' \,-\, \mathcal{H}\,\mathcal{H}' \,-\, \mathcal{H}^3)\left(\frac{E'}{2} \,-\, B\right)\delta^i_j \,-\, \frac{1}{2}\,\partial^i\partial_j D\right),\end{aligned} \tag{174}$$

with $D = \Phi - \Psi$. These quantities are not gauge-invariant, but using the gauge transformations described in subsection 7.6, we can easily generalize them to gauge-invariant quantities

$$\delta G^{(\mathrm{GI})0}_0 = \delta G^0_0 \,+\, (G^0_0)'\left(\frac{E'}{2} \,-\, B\right), \tag{175}$$

$$\delta G^{(\mathrm{GI})0}_i = \delta G^0_i \,+\, \left(G^0_i \,-\, \frac{1}{3}\,T^k_k\right)\partial_i\left(\frac{E'}{2} \,-\, B\right), \tag{176}$$

$$\delta G^{(\mathrm{GI})i}_j = \delta G^i_j \,+\, (G^i_j)'\left(\frac{E'}{2} \,-\, B\right) \tag{177}$$

and

$$\delta T^{(\mathrm{GI})0}_0 = \delta T^0_0 \,+\, (T^0_0)'\left(\frac{E'}{2} \,-\, B\right) = -\delta\rho^{(\mathrm{GI})}, \tag{178}$$

$$\delta T^{(\mathrm{GI})0}_i = \delta T^0_i \,+\, \left(T^0_i \,-\, \frac{1}{3}\,T^k_k\right)\partial_i\left(\frac{E'}{2} \,-\, B\right) = (\rho + p)\,a^{-1}\delta v^{(\mathrm{GI})}_i, \tag{179}$$

$$\delta T^{(\mathrm{GI})i}_j = \delta T^i_j \,+\, (T^i_j)'\left(\frac{E'}{2} \,-\, B\right) = \delta p^{(\mathrm{GI})} \tag{180}$$

where we have written the stress energy-momentum tensor as $T^{\mu\nu} = (\rho + p)\,u^\mu u^\nu + p\eta^{\mu\nu}$ with $u^\mu = (1, v^i)$.

Einstein's equations can now be written in a gauge-invariant way

$$\begin{aligned} &- \; 3\,\mathcal{H}\,(\mathcal{H}\,\Phi \,+\, \Psi') \,+\, \nabla^2\,\Psi \\ &= \; 4\,\pi\,G\left(-\Phi\,\phi'^2 \,+\, \delta\phi^{(\mathrm{GI})}\,\phi' \,+\right.\end{aligned}$$

$$\partial_i\,(\mathcal{H}\,\Phi \,+\, \Psi') \,=\, 4\,\pi\,G\left(\partial_i\,\delta\phi^{(\mathrm{GI})}\,\phi'\right),$$

$$\left((2\,\mathcal{H}' \,+\, \mathcal{H}^2)\,\Phi \,+\, \mathcal{H}\,\Phi' \,+\, \Psi'' \,+\, 2\,\mathcal{H}\,\Psi' \,+\, \frac{1}{2}\,\nabla^2 D\right)\delta^i_j \,-\, \frac{1}{2}\partial^i\partial_j D,$$

$$= \; -\,4\,\pi\,G\left(\Phi\,\phi'^2 \,-\, \delta\phi^{(\mathrm{GI})}\,\phi' \,+\right.$$

Taking $i \neq j$ from the third equation, we find $D = 0$, that is $\Psi = \Phi$ and from now on we can work with only the variable Φ. Using the background relation

$$2\left(\frac{a'}{a}\right)^2 - \frac{a''}{a} = 4\,\pi\,G\,\phi'^2 \tag{183}$$

we can rewrite the system of Eqs. (182) in the form

$$\begin{aligned}
\nabla^2\,\Phi - 3\,\mathcal{H}\,\Phi' - \left(\mathcal{H}' + 2\,\mathcal{H}^2\right) &= 4\,\pi\,G\left(\delta\phi^{(\mathrm{GI})}\,\phi' + \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\,;\\
\Phi' + \mathcal{H}\,\Phi &= 4\,\pi\,G\left(\delta\phi^{(\mathrm{GI})}\,\phi'\right)\,;\\
\Phi'' + 3\,\mathcal{H}\,\Phi' + \left(\mathcal{H}' + 2\,\mathcal{H}^2\right)\Phi &= 4\,\pi\,G\left(\delta\phi^{(\mathrm{GI})}\,\phi' - \delta\phi^{(\mathrm{GI})}\,\frac{\partial V}{\partial\phi}\,a^2\right)\,.
\end{aligned} \tag{184}$$

Substracting the first equation from the third, using the second equation to express $\delta\phi^{(\mathrm{GI})}$ as a function of Φ and Φ' and using the Klein-Gordon equation one finally finds the

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' - \nabla^2\,\Phi + 2\left(\mathcal{H}' - \mathcal{H}\,\frac{\phi''}{\phi'}\right)\Phi = 0\,, \tag{185}$$

for the gauge-invariant potential Φ.

We now introduce the gauge-invariant quantity

$$u \equiv a\,\delta\phi^{(\mathrm{GI})} + z\,\Psi\,, \tag{186}$$

$$z \equiv a\,\frac{\phi'}{\mathcal{H}} = a\frac{\dot{\phi}}{H}. \tag{187}$$

Notice that the variable u is equal to $-a\,Q$, the gauge-invariant inflaton perturbation on spatially flat gauges.

Eq. (185) becomes

$$u'' - \nabla^2\,u - \frac{z''}{z}\,u = 0\,, \tag{188}$$

while the two remaining equations of the system (184) can be written as

$$\nabla^2\,\Phi = 4\,\pi\,G\,\frac{\mathcal{H}}{a^2}\left(z\,u' - z'u\right)\,, \tag{189}$$

$$\left(\frac{a^2\,\Phi}{\mathcal{H}}\right)' = 4\,\pi\,G\,z\,u\,, \tag{190}$$

which allow to determine the variables Φ and $\delta\phi^{(\mathrm{GI})}$.

We have now to solve Eq. (188). First, we have to evaluate $\frac{z''}{z}$ in terms of the slow-roll parameters

$$\frac{z'}{\mathcal{H}z} = \frac{a'}{\mathcal{H}a} + \frac{\phi''}{\mathcal{H}\phi'} - \frac{\mathcal{H}'}{\mathcal{H}^2} = \epsilon + \frac{\phi''}{\mathcal{H}\phi'}.$$

We then deduce that

$$\delta \equiv 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}.$$

Keeping the slow-roll parameters constat in time (as we have mentioned, this corresponds to expand all quantities to first-order in the slow-roll parameters), we find

$$0 \simeq \delta' = \epsilon'(\simeq 0) - \frac{z''}{\mathcal{H}z} + \frac{z'\,\mathcal{H}'}{z\,\mathcal{H}^2} + \frac{(z')^2}{\mathcal{H}z^2},$$

from which we deduce

$$\frac{z''}{z} \simeq \frac{z'\,\mathcal{H}'}{z\,\mathcal{H}} + \frac{(z')^2}{z^2}.$$

Expanding in slow-roll parameters we find

$$\frac{z''}{z} \simeq (1+\epsilon-\delta)\,(1-\epsilon)\;\mathcal{H}^2 + (1+\epsilon-\delta)^2\;\mathcal{H}^2 \simeq \mathcal{H}^2\,(2+2\epsilon-3\delta)\,.$$

If we set

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right),$$

this corresponds to

$$\nu \simeq \frac{1}{2}\left[1 + 4\frac{(1+\epsilon-\delta)\,(2-\delta)}{(1-\epsilon)^2}\right]^{1/2} \simeq \frac{3}{2} + (2\epsilon-\delta) \simeq \frac{3}{2} + 3\epsilon - \eta.$$

On subhorizon scales ($k \gg aH$), the solution of equation (188) is obviously $u_{\mathbf{k}} \simeq e^{-ik\tau}/\sqrt{2k}$. Rewriting Eq. (190) as

$$\Phi_{\mathbf{k}} = -\frac{4\pi G a^2}{k^2}\frac{\dot{\phi}^2}{H}\left(\frac{H}{a\dot{\phi}}u_{\mathbf{k}}\right)^{\cdot},$$

we infer that on subhorizon scales

$$\Phi_{\mathbf{k}} \simeq i\,\frac{4\pi G\dot{\phi}}{\sqrt{2k^3}}\,e^{-i\frac{k}{a}}.$$

On superhorizon scales ($k \ll aH$), one obvious solution to Eq. (188) is $u_{\mathbf{k}} \propto z$. To find the other solution, we may set $u_{\mathbf{k}} = z\,\widetilde{u}_{\mathbf{k}}$, which satisfies the equation

$$\frac{\widetilde{u}''_{\mathbf{k}}}{\widetilde{u}'_{\mathbf{k}}} = -2\frac{z'}{z},$$

which gives

$$\widetilde{u}_{\mathbf{k}} = \int^{\tau}\frac{d\tau'}{z^2(\tau')}.$$

On superhorizon scales therefore we find

$$u_{\mathbf{k}} = c_1(k)\frac{a\dot{\phi}}{H} + c_2(k)\frac{a\dot{\phi}}{H}\int^t dt'\,\frac{H^2}{a^3\dot{\phi}^2}. \simeq c_1(k)\frac{a\dot{\phi}}{H} - c_2(k)\frac{1}{3a^2\dot{\phi}},$$

where the last passage has been performed supposing a de Sitter epoch, H = constant. The first piece is the constant mode $c_1(k)z$, while the second is the decreasing mode. To find the constant $c_1(k)$, we apply what we have learned in subsection 6.5. We know that on superhorizon scales the exact solution of equation (188) is

$$u_{\mathbf{k}} = \frac{\sqrt{\pi}}{2}\, e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}\,\sqrt{-\tau}\,H_\nu^{(1)}(-k\tau). \tag{191}$$

On superhorizon scales, since $H_\nu^{(1)}(x \ll 1) \sim \sqrt{2/\pi}\,e^{-i\frac{\pi}{2}}\,2^{\nu-\frac{3}{2}}\,(\Gamma(\nu_\chi)/\Gamma(3/2))\,x^{-\nu}$, the fluctuation (191) becomes

$$u_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}2^{\left(\nu-\frac{3}{2}\right)}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}\,(-k\tau)^{\frac{1}{2}-\nu}.$$

Therefore

$$c_1(k) = \lim_{k\to 0}\left|\frac{u_{\mathbf{k}}}{z}\right| = \frac{H}{a\dot{\phi}}\frac{1}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu} = \frac{H}{\dot{\phi}}\frac{1}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta-3\epsilon} \tag{192}$$

The last steps consist in relating the variable u to the comoving curvature $\mathcal{R}$ and to the gravitational potential Φ. The comoving curvature takes the form

$$\mathcal{R} \equiv -\Psi - \frac{H}{\phi'}\,\delta\phi^{(\mathrm{GI})} = -\frac{u}{z}. \tag{193}$$

Since $z = a\dot{\phi}/H = a\sqrt{2\epsilon}m_{\mathrm{Pl}}$, the power spectrum of the comoving curvature can be expressed on superhorizon scales as

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\,\pi^2}\left|\frac{u_{\mathbf{k}}}{z}\right|^2 = \frac{1}{2{m_{\mathrm{Pl}}}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \equiv A_{\mathcal{R}}^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \tag{194}$$

with

$$n_{\mathcal{R}} - 1 = 3 - 2\nu = 2\eta - 6\epsilon. \tag{195}$$

These results reproduce those found in the previous subsection.

The last step is to find the behaviour of the gauge-invariant potential Φ on superhorizon scales. If we recast equation (190) in the form

$$u_{\mathbf{k}} = \frac{1}{4\pi G}\frac{H}{\dot{\phi}}\left(\frac{a}{H}\Phi_{\mathbf{k}}\right)^{\cdot}, \tag{196}$$

we can infer that on superhorizon scales the nearly constant mode of the gravitational potential during inflation reads

$$\Phi_{\mathbf{k}} = c_1(k)\left[1 - \frac{H}{a}\int^t dt'\, a\,(t')\right] \simeq -c_1(k)\frac{\dot{H}}{H^2} = \epsilon\, c_1(k) \simeq \epsilon\frac{u_{\mathbf{k}}}{z} \simeq -\epsilon\,\mathcal{R}_{\mathbf{k}}. \tag{197}$$

Indeed, plugging this solution into Eq. (196), one reproduces $u_{\mathbf{k}} = c_1(k)\frac{a\dot{\phi}}{H}$.

4.6 Gravitational waves

Quantum fluctuations in the gravitational fields are generated in a similar fashion of that of the scalar perturbations discussed so far. A gravitational wave may be viewed as a ripple of spacetime in the FRW background metric (525) and in general the linear tensor perturbations may be written as

$$g_{\mu\nu} = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right],$$

where $|h_{ij}| \ll 1$. The tensor h_{ij} has six degrees of freedom, but, as we studied in subsection 7.1, the tensor perturbations are traceless, $\delta^{ij}h_{ij} = 0$, and transverse $\partial^i h_{ij} = 0$ $(i = 1, 2, 3)$. With these 4 constraints, there remain 2 physical degrees of freedom, or polarizations, which are usually indicated $\lambda = +, \times$. More precisely, we can write

$$h_{ij} = h_+\, e^+_{ij} + h_\times\, e^\times_{ij},$$

where e^+ and $e^\times$ are the polarization tensors which have the following properties

$$e_{ij} = e_{ji}, \quad k^i e_{ij} = 0, \quad , e_{ii} = 0,$$
$$e_{ij}(-\mathbf{k}, \lambda) = e^*_{ij}(\mathbf{k}, \lambda), \quad \sum_\lambda e^*_{ij}(\mathbf{k}, \lambda) e^{ij}(\mathbf{k}, \lambda) = 4.$$

Notice also that the tensors h_{ij} are gauge-invariant and therefore represent physical degrees of freedom.

If the stress-energy momentum tensor is diagonal, as the one provided by the inflaton potential $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$, the tensor modes do not have any source in their equation and their action can be written as

$$\frac{m_{\rm Pl}{}^2}{2}\int d^4x\, \sqrt{-g}\,\frac{1}{2}\partial_\sigma h_{ij}\, \partial^\sigma h_{ij},$$

that is the action of four independent massless scalar fields. The gauge-invariant tensor amplitude

$$v_{\mathbf{k}} = a m_{\rm Pl}\frac{1}{\sqrt{2}}\, h_{\mathbf{k}},$$

satisfies therefore the equation

$$v''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}} = 0,$$

which is the equation of motion of a massless scalar field in a quasi-de Sitter epoch. We can therefore make use of the results present in subsection 6.5 and Eq. (87) to conclude that on superhorizon scales the tensor modes scale like

$$|v_{\mathbf{k}}| = \left(\frac{H}{2\pi}\right)\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_T},$$

where

$$\nu_T \simeq \frac{3}{2} - \epsilon.$$

Since fluctuations are (nearly) frozen in on superhorizon scales, a way of characterizing the tensor pertubations is to compute the spectrum on scales larger than the horizon

$$\mathcal{P}_T(k) = \frac{k^3}{2\pi^2}\sum_{\lambda} |h_{\mathbf{k}}|^2 = 4 \times 2\frac{k^3}{2\pi^2}|v_{\mathbf{k}}|^2. \tag{198}$$

This gives the power spectrum on superhorizon scales

$$\boxed{\mathcal{P}_T(k) = \frac{8}{{m_{\rm Pl}}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_T} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}}$$

where we have defined the *spectral index* n_T of the tensor perturbations as

$$\boxed{n_T = \frac{d\ln \mathcal{P}_T}{d\ln k} = 3 - 2\nu_T = -2\epsilon.}$$

The tensor perturbation is almost scale-invariant. Notice that the amplitude of the tensor modes depends only on the value of the Hubble rate during inflation. This amounts to saying that it depends only on the energy scale $V^{1/4}$ associated to the inflaton potential. A detection of gravitational waves from inflation will be therefore a direct measurement of the energy scale associated to inflation.

The results obtained so far for the scalar and tensor perturbations allow to predict a *consistency relation* which holds for the models of inflation addressed, *i.e.* the models of inflation driven by one-single field ϕ. We define tensor-to-scalar amplitude ratio to be

$$r = \frac{\frac{1}{100}A_T^2}{\frac{4}{25}A_{\mathcal{R}}^2} = \frac{\frac{1}{100}8\left(\frac{H}{2\,\pi\,m_{\rm Pl}}\right)^2}{\frac{4}{25}(2\epsilon)^{-1}\left(\frac{H}{2\,\pi\,m_{\rm Pl}}\right)^2} = \epsilon.$$

This means that

$$r = -\frac{n_T}{2}$$

One-single models of inflation predict that during inflation driven by a single scalar field, the ratio between the amplitude of the tensor modes and that of the curvature perturbations is equal to minus one-half of the tilt of the spectrum of tensor modes. If this relation turns out to be falsified by the future measurements of the CMB anisotropies, this does not mean that inflation is wrong, but only that inflation has not been driven by only one field. Generalizations to two-field models of inflation can be found for instance in Refs. [19, 21].

4.7 The post-inflationary evolution of the cosmological perturbations

So far, we have computed the evolution of the cosmological perturbations within the horizon and outside the horizon during inflation. However, what we are really interested in is their evolution after inflation and to compute the amplitude of perturbations when they re-enter the horizon during radiation- or matter-domination.

To this purpouse, we use the following procedure. We use Eqs. (177) and (180) to write

$$\begin{aligned} \nabla^2 \Phi - 3\mathcal{H}\Phi' - \left(\mathcal{H}' + 2\mathcal{H}^2\right) &= 4\pi G a^2 \delta\rho^{(\mathrm{GI})}; & (199)\\ \Phi' + \mathcal{H}\Phi &= 4\pi G a^2 (\rho+p)\,\delta v^{(\mathrm{GI})}; & (200)\\ \Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi &= 4\pi G a^2 \delta p^{(\mathrm{GI})}. & (201)\end{aligned}$$

Combining these equations one finds

$$\Phi'' + 3\mathcal{H}\left(1+c_s^2\right)\Phi' - c_s^2\nabla^2\Phi + \left[2\mathcal{H}' + \left(1+c_s^2\right)\mathcal{H}^2\right]\Phi = 0, \tag{202}$$

where $c_s^2 = \dot{p}/\dot{\rho}$. This equation can be rewritten as

$$\dot{\mathcal{R}}_{\mathbf{k}} = 0$$

where we have set

$$\mathcal{R}_{\mathbf{k}} = -\Phi - \frac{2}{3}\frac{\mathcal{H}^{-1}\dot{\Phi} + \Phi}{(1+w)} - \Phi.$$

Here $w = p/\rho$. Notice that during inflation, when $p = \frac{1}{2}\dot{\phi}^2 - V$ and $\rho = \frac{1}{2}\dot{\phi}^2 + V$, $\mathcal{R}_{\mathbf{k}}$ takes the form

$$\mathcal{R}_{\mathbf{k}} = -\Phi_{\mathbf{k}} - \frac{1}{\epsilon\mathcal{H}}\left(\phi'_{\mathbf{k}} + \mathcal{H}\Phi_{\mathbf{k}}\right), \tag{203}$$

which using the equation

$$\Phi' + \mathcal{H}\,\Phi = 4\,\pi\,G\left(\delta\Phi^{(\rm GI)}\,\phi'\right)$$

reduces to the comoving curvature perturbation (193).

Eq. (202) is solved by

$$\Phi_{\mathbf{k}} = c_1(k)\left(1 - \frac{\mathcal{H}}{a^2}\int^\tau d\tau' a^2(\tau')\right) + c_2(k)\frac{\mathcal{H}}{a^2}\quad \text{(ON SUPERHORIZON SCALES)}$$

This nearly constant solution can be rewritten in cosmic time as

$$\Phi_{\mathbf{k}} = c_1(k)\left(1 - \frac{H}{a}\int^t dt' a(t')\right).$$

which is the same form of solution we found during inflation, see Eq. (197). This explains why we can choose the constant $c_1(k)$ to be the one given by Eq. (192), $c_1(k) = |u_{\mathbf{k}}/z|$ for superhorizon scales.

Since during radiation-domination $a \sim t^n$ with $n = 1/2$ and during matter-domination $a \sim t^n$ with $n = 2/3$, we can write

$$\Phi_{\mathbf{k}} = c_1(k)\left(1 - \frac{H}{a}\int^t dt' a(t')\right) = c_1(k)\left(1 - \frac{n}{n+1}\right) = \frac{c_1(k)}{n+1} = \mathcal{R}_{\mathbf{k}}\begin{cases}\frac{2}{3} & \text{RD}\\ \frac{3}{5} & \text{MD}\end{cases}$$

This relation tells us that, after a gravitational perturbation with a given wavelength is generated during inflation, it evolves on superhorizon scales after inflation simply slightly rescaling its amplitude. When the given wavelength re-enters the horizon, the amplitude of the gravitational potential depends upon the time of re-enter. If the perturbation re-enters the horizon when the universe is still dominated by radiation, then $\Phi_{\mathbf{k}} = \frac{2}{3}\mathcal{R}_{\mathbf{k}}$; if the perturbation re-enters the horizon when the universe is dominated by matter, then $\Phi_{\mathbf{k}} = \frac{3}{5}\mathcal{R}_{\mathbf{k}}$. For instance, the power spectrum of the gravitational perturbations during matter-domination reads

$$\mathcal{P}_\Phi = \left(\frac{3}{5}\right)^2 \mathcal{P}_\mathcal{R} = \left(\frac{3}{5}\right)^2 \frac{1}{2{m_{\rm Pl}}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_\mathcal{R}-1}$$

As the curvature perturbations enter the causal horizon during radiation- or matter-domination, they create density fluctuations $\delta\rho_{\mathbf{k}}$ via gravitational attractions of the potential wells. The density contrast $\delta_{\mathbf{k}} = \frac{\delta\rho_{\mathbf{k}}}{\overline{\rho}}$ can be deduced from Poisson equation

$$\frac{k^2}{a^2} = -4\pi G\,\delta\rho_{\mathbf{k}} = -4\pi G\,\frac{\delta\rho_{\mathbf{k}}}{\overline{\rho}}\,\overline{\rho} = \frac{3}{2}\,H^2\,\frac{\delta\rho_{\mathbf{k}}}{\overline{\rho}}$$

where $\overline{\rho}$ is the background average energy density. This means that

$$\delta_{\mathbf{k}} = \frac{2}{3}\left(\frac{k}{aH}\right)^2 \Phi_{\mathbf{k}}.$$

From this expression we can compute the power spectrum of matter density perturbations induced by inflation when they re-enter the horizon during matter-domination

$$\mathcal{P}_{\delta\rho} = \langle|\delta_{\mathbf{k}}|^2\rangle = A\left(\frac{k}{aH}\right)^n = \frac{2\pi^2}{k^3}\left(\frac{2}{5}\right)^2 A^2_{\mathcal{R}}\left(\frac{k}{aH}\right)^4\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}$$

from which we deduce that matter perturbations scale linearly with the wavenumber and have a scalar tilt

$$n = n_{\mathcal{R}} = 1 + 2\eta - 6\epsilon.$$

The primordial spectrum $\mathcal{P}_{\delta\rho}$ is of course reprocessed by gravitational instabilities after the universe becomes matter-dominated. Indeed, as we have seen in section 6, perturbations evolve after entering the horizon and the power spectrum will not remain constant. To see how the density contrast is reprocessed we have first to analyze how it evolves on superhorizon scales before horizon-crossing. We use the following trick. Consider a flat universe with average energy density $\overline{\rho}$. The corresponding Hubble rate is

$$H^2 = \frac{8\pi G}{3}\,\overline{\rho}.$$

A small positive fluctuation $\delta\rho$ will cause the universe to be closed

$$H^2 = \frac{8\pi G}{3}\left(\overline{\rho} + \delta\rho\right) - \frac{k}{a^2}.$$

Substracting the two equations we find

$$\frac{\delta\rho}{\rho} = \frac{3}{8\pi G}\frac{k}{a^2\rho} \sim \begin{cases} a^2 & \text{RD} \\ a & \text{MD} \end{cases}$$

Notice that $\Phi_{\mathbf{k}} \sim \delta\rho a^2/k^2 \sim (\delta\rho/\rho)\rho a^2/k^2 =$ constant for both RD and MD which confirms our previous findings.

When the matter densities enter the horizon, they do not increase appreciably before matter-domination because before matter-domination pressure is too large and does not allow the matter inhomogeneities to grow. On the other hand, the suppression of growth due to radiation is restricted to scales smaller than the horizon, while large-scale perturbations remain unaffected. This is the reason why the horizon size at equality sets an important scale for structure growth

$$k_{\rm EQ} = H^{-1}\left(a_{\rm EQ}\right) \simeq 0.08\,h\,{\rm Mpc}^{-1}.$$

Therefore, perturbations with $k \gg k_{\rm EQ}$ are perturbations which have entered the horizon before matter-domination and have remained nearly constant till equality. This means that they are suppressed with respect to those perturbations having $k \ll k_{\rm EQ}$ by a factor $(a_{\rm ENT}/a_{\rm EQ})^2 = (k_{\rm EQ}/k)^2$. If we define the transfer function $T(k)$ by the relation $\mathcal{R}_{\rm final} = T(k)\,\mathcal{R}_{\rm initial}$ we find therefore that roughly speaking

$$T(k) = \begin{cases} 1 & k \ll k_{\rm EQ}, \\ (k_{\rm EQ}/k)^2 & k \gg k_{\rm EQ}. \end{cases}$$

The corresponding power spectrum will be

$$\mathcal{P}_{\delta\rho}(k) \sim \begin{cases} \left(\frac{k}{aH}\right) & k \ll k_{\rm EQ}, \\ \left(\frac{k}{aH}\right)^{-3} & k \gg k_{\rm EQ}. \end{cases}$$

Of course, a more careful computation needs to include many other effects such as neutrino free-streeming, photon diffusion and the diffusion of baryons along with photons. It is encouraging however that this rough estimate turns out to be confirmed by present data on large scale structures [39].

Temperature fluctuations in the CMB arise due to five distinct physical effects: our peculiar velocity with respect to the cosmic rest frame; fluctuations in the gravitational potential on the last scattering surface; fluctuations intrinsic to the radiation field itself on the last-scattering surface; the peculiar velocity of the last-scattering surface and damping of anisotropies if the universe should be re-ionized after decoupling. The first effect gives rise to the dipole anisotropy. The second effect, known as the Sachs-Wolfe effect is the dominat contribution to the anisotropy on large-angular scales, $\theta \gg \theta_{\rm HOR} \sim 1^\circ$. The last three effects provide the dominant contributions to the anisotropy on small angular scales, $\theta \ll 1^\circ$.

We consider here the temperature fluctuations on large-angular scales that arise due to the Sachs-Wolfe effect. These anisotropies probe length scales that were superhorizon sized at photon decoupling and therefore insensitive to microphysical processes. On the contrary, they provide a probe of the original spectrum of primeveal fluctuations produced during inflation.

To proceed, we consider the CMB anisotropy measured at positions other than our own and at earlier times. This is called the brightness function $\Theta(t, \mathbf{x}, \mathbf{n}) \equiv \delta T(t, \mathbf{x}, \mathbf{n})/T(t)$. The photons with momentum $\mathbf{p}$ in a given range d^3p have intensity I proportional to $T^4(t, \mathbf{x}, \mathbf{n})$ and therefore $\delta I/I = 4\Theta$. The brightness function depends upon the direction $\mathbf{n}$ of the photon momentum or, equivalently, on the direction of observation $\mathbf{e} = -\mathbf{n}$. Because the CMB travels freely from the last-scattering, we can write

$$\frac{\delta T}{T} = \Theta\left(t_{\rm LS}, \mathbf{x}_{\rm LS}, \mathbf{n}\right) + \left(\frac{\delta T}{T}\right)_*,$$

where $\mathbf{x}_{\rm LS} = -x_{\rm LS}\mathbf{n}$ is the point of the origin of the photon coming from the direction $\mathbf{e}$. The comoving distance of the last-scattering distance is $x_{\rm LS} = 2/H_0$. The first term corresponds to the anisotropy already present at last scattering and the second term is the additional anisotropy acquired during the travel towards us, equal to minus the fractional pertubation in the redshift of the radiation. Notice that the separation between each term depends on the slicing, but the sum is not.

Consider the redshift perturbation on comoving slicing. We imagine the universe populated by comoving observers along the line of sight. The relative velocity of adjacent comoving observers is equal to their distance times the velocity gradient measured along $\mathbf{n}$ of the photon. In the unperturbed universe, we have $\mathbf{u} = H\mathbf{r}$, leading to the velocity gradient $u_{ij} = \partial u_i/\partial r_j = u_{ij} = H(t)\delta_{ij}$ with zero vorticity and shear. Including a peculiar velocity field as perturbation, $\mathbf{u} = H\mathbf{r} + \mathbf{v}$ and $u_{ij} = H(t)\delta_{ij} + \frac{1}{a}\frac{\partial v_i}{\partial v_j}$. The corresponding Doppler shift is

$$\frac{d\lambda}{\lambda} = \frac{da}{a} + n_i n_j \frac{\partial v_i}{\partial x_j} dx.$$

The perturbed FRW equation is

$$\delta H = \frac{1}{3} \nabla \cdot \mathbf{v},$$

while

$$(\delta\rho)^{\cdot} = -3\rho\delta H - 3H\delta\rho.$$

Instead of $\delta\rho$, let us work with the density contrast $\delta = \delta\rho/\rho$. Remembering that $\rho \sim a^{-3}$, we find that $\dot{\delta} = -3\delta H$, which give

$$\nabla \cdot \mathbf{v} = -\dot{\delta}_{\mathbf{k}}.$$

From Euler equation $\dot{\mathbf{u}} = -\rho^{-1}\nabla p - \nabla\Phi$, we deduce $\dot{\mathbf{v}} + H\mathbf{v} = -\nabla\Phi - \rho^{-1}\nabla p$. Therefore, for $a \sim t^{2/3}$ and negligible pressure gradient, since the gravitational potential is constant, we find

$$\mathbf{v} = -t\nabla\Phi$$

leading to

$$\left(\frac{\delta T}{T}\right)_* = \int_0^{x_{\rm LS}} \frac{t}{a} \frac{d^2\Phi}{dx^2}\, dx. \tag{204}$$

The photon trajectory is $ad\mathbf{x}/dt = \mathbf{n}$. Using $a \sim t^{2/3}$ gives

$$x(t) = \int_t^{t_0} \frac{dt'}{a} = 3\left(\frac{a_0}{t_0} - \frac{t}{a}\right).$$

Integrating by parts Eq. (204), we finally find

$$\left(\frac{\delta T}{T}\right)_* = \frac{1}{3}\left[\Phi(\mathbf{x}_{\rm LS}) - \Phi(0)\right] + \mathbf{e} \cdot \left[\mathbf{v}(0, t_0) - \mathbf{v}(\mathbf{x}_{\rm LS}, t_{\rm LS})\right].$$

Te potential at our position contributes only to the unobservable monopole and can be dropped. On scales outside the horizon, $\mathbf{v} = -t\nabla\Phi \sim 0$. The remaining term is the Sachs-Wolfe effect

$$\frac{\delta T(\mathbf{e})}{T} = \frac{1}{3}\Phi(\mathbf{x}_{\rm LS}) = \frac{1}{5}\mathcal{R}(\mathbf{x}_{\rm LS}).$$

Therefore, at large angular scales, the theory of cosmological perturbations predicts a remarkable simple formula relating the CMB anisotropy to the curvature perturbation generated during inflation.

In section 3, we have seen that the temperature anisotropy is commonly expanded in spherical harmonics $\frac{\Delta T}{T}(x_0, \tau_0, \mathbf{n}) = \sum_{\ell m} a_{\ell,m}(x_0) Y_{\ell m}(\mathbf{n})$, where x_0 and τ_0 are our position and the preset time, respectively, $\mathbf{n}$ is the direction of observation, ℓ's are the different multipoles and $\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell,\ell'} \delta_{m,m'} C_\ell$, where the deltas are due to the fact that the process that created the anisotropy is statistically isotropic. The C_ℓ are the so-called CMB power spectrum. For homogeneity and isotropy, the C_ℓ's are neither a function of x_0, nor of m. The two-point-correlation function is related to the C_l's according to Eq. (37).

For adiabatic perturbations we have seen that on large scales, larger than the horizon on the last-scattering surface (corresponding to angles larger than $\theta_{\rm HOR} \sim 1^\circ$) $\delta T/T = \frac{1}{3}\Phi(\mathbf{x}_{\rm LS})$ In Fourier transform

$$\frac{\delta T(\mathbf{k}, \tau_0, \mathbf{n})}{T} = \frac{1}{3}\Phi_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{n}(\tau_0 - \tau_{\rm LS})} \tag{205}$$

Using the decomposition

$$\exp(i\,\mathbf{k}\cdot\mathbf{n}(\tau_0 - \tau_{\rm LS})) = \sum_{\ell=0}^{\infty}(2\ell+1) i^\ell j_\ell(k(\tau_0 - \tau_{\rm LS})) P_\ell(\mathbf{k}\cdot\mathbf{n}) \tag{206}$$

where j_ℓ is the spherical Bessel function of order ℓ and substituting, we get

$$\Big\langle \frac{\delta T(x_0, \tau_0, \mathbf{n})}{T} \frac{\delta T(x_0, \tau_0, \mathbf{n}')}{T} \Big\rangle = \tag{207}$$

$$= \frac{1}{V}\int d^3x \Big\langle \frac{\delta T(x_0, \tau_0, \mathbf{n})}{T} \frac{\delta T(x_0, \tau_0, \mathbf{n}')}{T} \Big\rangle =$$

$$= \frac{1}{(2\pi)^3}\int d^3k \Big\langle \frac{\delta T(\mathbf{k}, \tau_0, \mathbf{n})}{T} \left(\frac{\delta T(\mathbf{k}, \tau_0, \mathbf{n}')}{T}\right)^* \Big\rangle =$$

$$= \frac{1}{(2\pi)^3}\int d^3k \Big(\Big\langle \frac{1}{3}|\Phi|^2 \Big\rangle \sum_{\ell,\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) j_\ell(k(\tau_0 - \tau_{\rm LS}))$$

$$j_{\ell'}(k(\tau_0 - \tau_{\rm LS})) P_\ell(\mathbf{k}\cdot\mathbf{n}) P_{\ell'}(\mathbf{k}'\cdot\mathbf{n}')\Big) \tag{208}$$

Inserting $P_\ell(\mathbf{k}\cdot\mathbf{n}) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{lm}(\mathbf{k}) Y_{\ell m}(\mathbf{n})$ and analogously for $P_\ell(\mathbf{k}'\cdot\mathbf{n}')$, integrating over the directions $d\Omega_k$ generates $\delta_{\ell\ell'}\delta_{mm'}\sum_m Y^*_{\ell m}(\mathbf{n}) Y_{\ell m}(\mathbf{n}')$. Using as well $\sum_m Y^*_{\ell m}(\mathbf{n}) Y_{\ell m}(\mathbf{n}') = \frac{2\ell+1}{4\pi} P_\ell(\mathbf{n}\cdot\mathbf{n}')$, we get

$$\Big\langle \frac{\delta T(x_0, \tau_0, \mathbf{n})}{T} \frac{\delta T(x_0, \tau_0, \mathbf{n}')}{T} \Big\rangle \tag{209}$$

$$= \Sigma_\ell \frac{2\ell+1}{4\pi} P_\ell(\mathbf{n}\cdot\mathbf{n}') \frac{2}{\pi}\int \frac{dk}{k}\Big\langle \frac{1}{9}|\Phi|^2 \Big\rangle k^3 j^2_\ell(k(\tau_0 - \tau_{\rm LS})).$$

Comparing this expression with that for the C_ℓ, we get the expression for the $C_\ell^{\rm AD}$, where the suffix "AD" stays for adiabatic

$$C_\ell^{\rm AD} = \frac{2}{\pi}\int \frac{dk}{k}\Big\langle \frac{1}{9}\,|\Phi|^2\,\Big\rangle k^3 j^2_\ell(k(\tau_0 - \tau_{\rm LS})) \tag{210}$$

which is valid for $2 \leq \ell \ll (\tau_0 - \tau_{\rm LS})/\tau_{\rm LS} \sim 100$.

If we generically indicate by $\langle |\Phi_{\mathbf{k}}|^2 \rangle k^3 = A^2\,(k\tau_0)^{n-1}$, we can perform the integration and get

$$\frac{\ell(\ell+1)C_\ell^{\rm AD}}{2\pi} = \left[\frac{\sqrt{\pi}}{2}\ell(\ell+1)\frac{\Gamma(\frac{3-n}{2})\Gamma(\ell+\frac{n-)}{2})}{\Gamma\left(\frac{4-n}{2}\right)\Gamma\left(\ell+\frac{5-n}{2}\right)}\right]\frac{A^2}{9}\left(\frac{H_0}{2}\right)^{n-1} \tag{211}$$

For $n \simeq 1$ and $\ell \gg 1$, we can approximate this expression to

$$\frac{\ell(\ell+1)C_l^{\rm AD}}{2\pi} = \frac{A^2}{9}. \tag{212}$$

This result shows that inflation predicts a very flat spectrum for low ℓ. This prediction has been confirmed by the COBE satellite [58]. Furthermore, since inflation predicts $\Phi_{\mathbf{k}} = \frac{3}{5}\mathcal{R}_{\mathbf{k}}$, we find that

$$\pi\,\ell(\ell+1)C_l^{\rm AD} = \frac{A_{\mathcal{R}}^2}{25} = \frac{1}{25}\frac{1}{2\,m_{\rm Pl}{}^2\,\epsilon}\left(\frac{H}{2\pi}\right)^2. \tag{213}$$

COBE data imply that $\frac{\ell(\ell+1)C_l^{\rm AD}}{2\pi} \simeq 10^{-10}$ or

$$\boxed{\left(\frac{V}{\epsilon}\right)^{1/4} \simeq 6.7\times 10^{16}\,{\rm GeV}}$$

Take for instance a model of chaotic inflation with quadratic potential $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$. Using Eq. (59) one easily computes that when there are ΔN e-foldings to go, the value of the inflaton field is $\phi^2_{\Delta N} = (\Delta N/2\pi G)$ and the corresponding value of ϵ is $1/(2\Delta N)$. Taking $\Delta N \simeq 60$ (corresponding to large-angle CMB anisotropies), one finds that COBE normalization imposes $m_\phi \simeq 10^{13}$ GeV.

We have learned that a stage of inflation during the early epochs of the evolution of the universe solves many drawbacks of the standard Big-Bang cosmology, such as the flatness or entropy problem and the horizon problem. Luckily, despite inflation occurs after a tiny bit after the bang, it leaves behind some observable predictions:

- *The universe should be flat*, that is the total density of all components of matter should sum to the critical energy density and $\Omega_0 = 1$. The current data on the CMB anisotropies offer a spectacular confirmation of such a prediction. The universe appears indeed to be spatially flat.

- *Primordial perturbations are adiabatic.* Inflation provides the seeds for the cosmological perturbations. In one-single field models of inflation, the perturbations are *adiabatic* or curvature pertrubations, *i.e.* they are fluctuations in the total energy density of the universe or, equivalently, scalar perturbations to the spacetime metric. Adiabaticity implies that the spatial distribution of each species in the universe is the same, that is the ratio of number densities of any two of these species is everywhere the same. Adiabatic perturbations predict a contribution to the CMB anisotropy which is related to the curvature perturbation $\mathcal{R}$ on large angles, $\delta T/T = \frac{1}{5}\,\mathcal{R}$, and are in excellent agreeement with the CMB data. Adiabatic perturbations can be contrasted to isocurvature perturbations which are fluctuations in the ratios between the various species in the universe. Isocurvature perturbations predict that on large angles $\delta T/T = -2\Phi$ and are presently ruled out, even though a certain amount of isocurvature perturbations, possibly correlated with the adiabatic fluctuations, cannot be excluded by present CMB data [13].

- *Primordial perturbations are almost scale-independent.* The primordial power spectrum predicted by inflation has a characteristic feature, it is almost scale-independent, that is the spectral index $n_{\mathcal{R}}$ is very close to unity. Possible deviations from exact scale-independence arise because during inflation the inflaton is not massless and the Hubble rate is not exactly constant. A recent analysis [23] shows that $n_{\mathcal{R}} = 0.97^{+0.08}_{-0.05}$, again in agreement with the theoretical prediction.

- *Primordial perturbations are nearly gaussian.* The fact that cosmological perturbations are tiny allow their analysis in terms of linear perturbation theory. Non-gaussian features are therefore suppressed since the non-linearities of the inflaton potential and of the metric perturbations are suppressed. Non-gaussian features are indeed present, but may appear only at the second-order in deviations from the homogeneous background solution and are therefore small [11]. This is rigously true only for one-single field models of inflation. Many-field models of inflation may give rise to some level of non-gaussianity [20]. If the next generation of satellites will detect a non-negligible amount of non-gaussianity in the CMB anisotropy, this will rule out one-single field models of inflation.

- *Production of gravitational waves.* A stochastic background of gravitational waves is produced during inflation in the very same way classical perturbations to the inflaton field are generated. The spectrum of such gravitational waves is again flat, *i.e.* scale-independent and the tensor-to-scalar amplitude ratio r is, at least in one-single models of inflation, related to the spectral index n_T by the consistency relation $r = -n_T/2$. A confirmation of such a relation would be a spectacular proof of one-single field models of inflation. The detection of the primordial stochastic background of gravitational waves from inflation is challenging, but would not only set the energy scale of inflation, but would also give the opportunity of discriminating among different models of inflation [42, 14].

5 High Energy Cosmological Models

String theory has long been viewed as an enterprise of little interest for experiments and observations. The energy scales usually considered to be relevant for strings are many orders of magnitude higher than what in the foreseeable future will be experimentally accessible. There are even some physicists who claim that the realm of string theory forever will be beyond the grasp of experimental science. Luckily, there are promising signs that the situation is about to change. Recent developments show that string theory can become accessible to observations much sooner than most people have ever hoped. The new player in the game is cosmology. For a long time an inexact patchwork of educated guesses and order of magnitude estimates, cosmology has developed into an exact science with a fruitful and rapid interaction between observations and theory. Much of the progress is based on the ever more precise observations of the CMBR, and measurements of how the expansion of the universe has changed with time. Thanks to these new observations it is now generally believed that the large scale structure of the universe can be traced back to microscopical physics near the Big Bang. In this way the universe works like a gigantic accelerator allowing us to study physics at the very highest energy scales, possibly even scales relevant for strings.

In the meantime, string theory has reached a maturity which allows for the formulation of realistic cosmological models. For a long time string theory focused on the physics of the very smallest scales. The problems, which were addressed, concerned the unification of forces, including gravity, and the compatibility of relativity and quantum mechanics. The idea was that once the fundamental microscopical laws were found the rest of physics would follow. In particular, cosmology was thought of as just another application of these fundamental laws. In later years the perspective has changed. Many now believe that the physics of the large and the small can not be separated, and that an understanding of unification not only is necessary for understanding the origin of the universe, but that an understanding of the origin of the universe is necessary in order to understand unification. To summarize, cosmology can be the key to the verification of string theory, and string theory can be what we need to solve several of the present puzzles in cosmology.

In this section we will give a review of recent attempts to connect string theory with cosmology. Any such attempt must, in one way or the other, be confronted with inflation, [77][78][79].[6] That is, the widely held view that the early universe underwent a period of exponential expansion. A complete theory of the early universe must either explain inflation or replace it with something else. This is also true for string theory, and I will therefore start out with a basic review of inflation focusing on those aspects useful for a string theorist wishing to enter the field. For a more complete introduction, and a complete list of references, I recommend [83]. Apart from standard material, I will briefly discuss the issue of transplanckian signatures. That is, the possibility of finding observational signatures of stringy or planckian physics in the CMBR.

We will then proceed with a discussion of the relation between string theory and inflation. Can strings give rise to inflation? W will review two sets of proposals: string cosmology and brane cosmology. The latter can be divided into two subproposals: models that generate inflation, and models that try to do with out inflation. We will also discuss some of the

[6]Other early ideas about inflation include [80][81][82].

difficulties encountered in constructing string theories in de Sitter space and briefly mention some important aspects of recent progress in this area. Finally, we will discuss the relevance of holography to cosmology.

The standard Big Bang model suffers from a number of annoying problems. One of them, *the flatness problem*, concerns the observation that the real density of the universe, ρ, long has been known to be very close to the critical density ρ_c. That is, $\Omega = \frac{\rho}{\rho_c}$ has been measured to be close to one. To understand the importance of this, we start with the Friedmann equation

$$H^2 = \frac{1}{3M_4^2}\rho - \frac{k}{a^2}, \tag{214}$$

where $M_4 = 1/\sqrt{8\pi G} \sim 2 \cdot 10^{18} GeV$ is the four dimensional (reduced) Planck mass. Furthermore, $H = \frac{\dot{a}}{a}$ is the Hubble constant and $a(t)$ the scale factor with the space time metric on the form

$$ds^2 = dt^2 - a^2 dS^2. \tag{215}$$

dS^2 is the comoving volume element of space with $k = 0$, $+1$ and -1 corresponding to flat, positively curved and negatively curved spaces respectively. We then rewrite the Friedmann equation as

$$\Omega - 1 = \frac{k}{a^2 H^2}, \tag{216}$$

and note that for any ordinary type of matter, $\frac{1}{a^2H^2}$ will *increase* with time. To see this, we use the continuum equation given by

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{217}$$

Assuming an equation of state of the form

$$p = w\rho, \tag{218}$$

where w is a constant, the continuum equation can be rewritten as

$$\frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0, \tag{219}$$

giving rise to

$$\rho \sim a^{-3(1+w)}. \tag{220}$$

If we start with $\Omega \sim 1$ $(k \sim 0)$ we have $H \sim 1/t^2$, and the Friedmann equation gives $a \sim t^{\frac{2}{3(1+w)}}$. As a consequence we finally find

$$\frac{1}{a^2H^2} \sim t^{2-\frac{4}{3(1+w)}}, \tag{221}$$

which clearly grows rapidly with time for any $w > -1/3$ – examples include pressureless dust with $w = 0$ and radiation with $w = 1/3$.

From the above one concludes that, unless the universe is *exactly* flat ($k = 0$) and, as a consequence, has *exactly* $\Omega = 1$, Ω will rapidly evolve away from $\Omega = 1$. If one starts with a value $\Omega < 1$, the value will decrease towards zero, while if $\Omega > 1$ the value of Ω will increase, and even diverge, if the expansion stops. In order to have a value close to 1 today, one would therefore expect to need a value of Ω even closer to 1 in the early universe. How close? Let us assume a radiation dominated universe up to the time $t_{rad} \sim 300000$ years, and thereafter matter domination. This is roughly the time when the universe became transparent and the time of origin of the CMBR. We can then, using (221) in two steps, estimate the amount of fine tuning at $t < t_{rad}$ to be

$$|\Omega(t) - 1| \sim \frac{t}{t_{rad}} \left(\frac{t_{rad}}{t_{now}} \right)^{2/3}. \tag{222}$$

With $t_{now} \sim 10^{10}$ years we find a fine tuning of one part in 10^{16} one second after the Big Bang, and one part in 10^{60} at planckian times $\sim 10^{-44}s$, if the deviation from $\Omega = 1$ is to remain small all the way up to present times. This is the flatness problem. That is, how can Ω be so close to one?

Another problem is the *horizon problem.* Regions of the universe, in particular sources of the CMBR at opposite points of the sky, look very similar even though, assuming normal radiation dominated expansion in the early universe, they can not have been in casual contact since the Big Bang. How is this possible? In the diagram it can be seen how points at the time when the CMBR was generated, all visible to us today, have not had time to communicate with each other. It is difficult to understand how the initial conditions at the Big Bang could be so extremely fine tuned.

A possible way out of the unnatural fine tuning implied by the flatness problem, would be some kind of mechanism at work in the early universe that dynamically drives Ω towards 1. This is where *inflation* comes in. Inflation corresponds to a period when $\frac{1}{a^2H^2}$ actually *decreases.* This is the case for an expanding universe if the scale factor a, that is, the distance between two test objects, increases faster than the horizon radius $1/H$. In a sense, one can say that the universe expands faster than the speed of light. In such a universe the redshift of any given object will increase with time as the object catches up with the cosmological horizon. Let us see how this works in more detail. A lightray in the metric

$$ds^2 = dt^2 - a^2 dx^2, \tag{223}$$

travels according to

$$x = \int_{t_0}^{t} \frac{dt}{a}, \tag{224}$$

between time of emission t_0, and time of observation t, where x is the comoving distance. If we follow a particular object we have $t_0 = t_0(t)$, while x is independent of t. Differentiating with respect to t, using $dx = 0$, we find

$$\frac{dt}{a(t)} - \frac{dt_0}{a(t_0)} = 0 \Longrightarrow \frac{dt_0}{dt} = \frac{a(t_0)}{a(t)}. \tag{225}$$

The redshift of a particular object, as a function of time, is defined by

$$z(t) = 1 + \frac{a(t)}{a(t_0(t))}. \tag{226}$$

Differentiation with respect to time t gives

$$\frac{dz}{dt} = \frac{\dot{a}(t)}{a(t_0(t))} - \frac{a(t)}{a(t_0(t))^2}\dot{a}(t_0(t))\frac{dt_0}{dt} = \frac{1}{a(t_0)}\left(a(t)H(t) - a(t_0(t))H(t_0)\right), \tag{227}$$

which is positive if $\frac{d}{dt}\frac{1}{a^2H^2} < 0$, as we set out to prove. Note that in a universe, which expands in the usual fashion, the redshift of a given object actually *decreases.*

Faster than light expansion also solves the horizon problem. The reason is, as explained above, that the expansion rate in a very definite sense is faster than the speed of light. Objects in causal contact can, through the expansion, be separated to distances larger than the Hubble radius. Eventually, when inflation stops, the Hubble radius will start growing faster than the expansion and the objects will return within their respective horizons. An observer not taking inflation into account will wrongly conclude that these objects have never before been in causal contact.

The simplest example of an inflating cosmology is a universe with $H = \text{const}$. Such a universe has $a(t) \sim e^{Ht}$ and is called a *de Sitter space time.*

We have now seen how inflation solves the problems of the Big Bang model, but how do we get inflation? The condition for inflation can be written

$$\frac{d}{dt}\frac{1}{a^2H^2} = \frac{d}{dt}\frac{1}{\dot{a}^2} = -\frac{2\ddot{a}}{\dot{a}^3} < 0, \tag{228}$$

or $\ddot{a} > 0$ (*if* $\dot{a} > 0$), that is, it corresponds to an *accelerating* expansion. Combining (214) (with $k = 0$) and (217), one can obtain another Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_4^2}(\rho + 3p), \tag{229}$$

from which it immediately follows that an accelerated universe requires matter with negative pressure. Luckily, this can be provided by a scalar field, the *inflaton*, which possesses a potential energy. Let us investigate this in more detail.

The Lagrangian for a scalar field is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right] = \int d^4x a^3\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}\nabla^2\phi - V(\phi)\right], \tag{230}$$

and the canonical energy momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}L. \tag{231}$$

In case of a homogenous inflaton field this reduces to an energy density given by

$$\rho = T_{00} = \dot{\phi}^2 - \frac{\dot{\phi}^2}{2} + V(\phi) = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{232}$$

and a pressure given by

$$p = \frac{1}{a^2} T_{xx} = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{233}$$

Note that x is the comoving coordinate – hence the rescaling of T_{xx} to obtain the physical pressure. We also have the equation of motion for the scalar field given by

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0. \tag{234}$$

At this point it is useful to introduce the *slow roll approximation.* That is, we assume

$$\dot{\phi}^2 << V(\phi), \tag{235}$$

or, in other words, $p \sim -\rho$. We also need to impose $\ddot{\phi} << V'(\phi)$, and as a consequence we therefore have

$$H^2 = \frac{1}{3M_4^2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \sim \frac{1}{3M_4^2} V(\phi) \tag{236}$$

$$3H\dot{\phi} \sim -V'(\phi). \tag{237}$$

The slow roll conditions are conveniently handled by introducing the *slow roll parameters*

$$\varepsilon = \frac{M_4^2}{2}\left(\frac{V'}{V}\right)^2 \tag{238}$$

$$\eta = M_4^2 \frac{V''}{V}. \tag{239}$$

It is a useful exercise to verify that the slow roll condition implies that the slow roll parameters are small. It is also true that inflation implies that the slow roll parameters are small.

How much inflation do we need to solve the problems of the Big Bang? According to (222) we need a fine tuning of 10^{60} at planckian times. If this is supposed to be achieved through exponential expansion, we must have

$$a^2 \sim e^{2Ht} = e^{2N} \sim 10^{60} \sim e^{140}. \tag{240}$$

That is, we find the required number of e-foldings, N, to be around 70. This gives a constraint on the potential as follows,

$$N = \ln \frac{a(t_e)}{a(t_i)} = \int_{t_i}^{t_e} H dt = \int_{\phi_i}^{\phi_e} H \frac{d\phi}{\dot{\phi}} = -\int_{\phi_i}^{\phi_e} \frac{3H^2}{V'} d\phi = -8\pi G \int_{\phi_i}^{\phi_e} \frac{V}{V'} d\phi \gtrsim 70. \tag{241}$$

Using the slow roll parameters we find

$$N = \frac{1}{\sqrt{2\varepsilon}} \frac{\phi_i - \phi_e}{M_4} \gtrsim 70, \tag{242}$$

and, as a consequence, one concludes that the inflationary potential needs to be rather flat.

Let us now consider a couple of explicit examples. The original works on inflation assumed potentials with local minima (old inflation), or very flat maxima (new inflation), in order to keep the inflaton away from the final, global, minima long enough to get the required number of e-foldings. Later it was realized that the potential can be of a very simple form. In fact, even a simple monomial like

$$V = \lambda M_4^{4-\alpha}\phi^\alpha, \tag{243}$$

can do the job. The reason is easy to understand from a quick look at (234). The second term in the equation, which is due to the expansion of the universe, works like a friction term that prevents the inflaton from rolling down too quickly preventing inflation from taking place. This is called *chaotic* inflation, [86].

For the particular potential above, we can calculate the slow roll parameters to be

$$\varepsilon = \frac{\alpha^2}{2}\frac{M_4^2}{\phi^2} \qquad \eta = \alpha\left(\alpha - 1\right)\frac{M_4^2}{\phi^2}. \tag{244}$$

Inflation starts at a large value of ϕ and the inflaton then rolls slowly towards the minimum with increasing ε and $|\eta|$. Inflation ends when the slow roll conditions no longer hold, i.e. when $\phi \sim \alpha M_4$. The number of e-foldings we obtain before this happens is given by

$$N = \frac{1}{M_4^2}\int_{\phi_e}^{\phi_i}\frac{\phi}{\alpha} \sim \frac{1}{2\alpha M_4^2}\phi_i^2 \Longrightarrow \phi_i \sim \sqrt{2\alpha N}M_4 \gg M_4. \tag{245}$$

At the start of inflation the slow roll parameters are given by

$$\varepsilon \sim \frac{\alpha}{4N} \qquad \eta \sim \frac{\alpha - 1}{2N}. \tag{246}$$

Another type of potential is

$$V = V_0 e^{-\sqrt{\frac{2}{p}}\frac{\phi}{M_4}}, \tag{247}$$

leading to *power inflation* with $a \sim t^p$. In this case the slow roll parameters are constant and given by

$$\varepsilon = \frac{1}{p}, \qquad \eta = \frac{2}{p}. \tag{248}$$

As a result, inflation continues forever with ϕ rolling to larger and larger values. In this case one needs an independent mechanism to end inflation.

How do we test inflation? The key is structure formation. An important reason to invoke inflation is to make the universe smooth and flat. In the real universe, however, there is a large amount of structure. This structure can be traced back to subtle variations in the matter distribution during the time when the CMBR was released. A naive application of inflation does, however, exclude such non-uniformity. So, from where does all the structure come? Actually, inflation itself supplies the answer provided we take quantum mechanics into account.

The main insight is that inflation magnifies microscopic quantum fluctuations into cosmic size, and thereby provides seeds for structure formation. The details of physics at the highest energy scales is therefore reflected in the distribution of galaxies and other structures on large scales. The fluctuations begin their life on the smallest scales and grow larger (in wavelength) as the universe expands. Eventually they become larger than the horizon and freeze. That is, different parts of a wave can no longer communicate with each other since light can not keep up with the expansion of the universe. This is a consequence of the fact that the scale factor grows faster than the horizon, which, as we have seen, is a defining property of an accelerating and inflating universe. At a later time, when inflation stops, the scale factor will start to grow slower than the horizon and the fluctuations will eventually come back within the causal horizon. The fluctuations will then start off acoustic waves in the plasma which will affect the CMBR. These imprints of the quantum fluctuations can be studied revealing important clues about physics at extremely high energies in the early universe.

Let us now investigate in more detail the predictions from inflation. We assume that the metric as well as the inflaton can be split into a classical background piece and a piece due to fluctuations according to

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}(\tau, \mathbf{x}) \tag{249}$$

$$\phi = \phi^{(0)} + \delta\phi(\tau, \mathbf{x}). \tag{250}$$

For convenience we have changed coordinates and introduced *conformal time*, τ, such that the metric is given by

$$ds^2 = a(\tau)^2 \left(dt^2 - d\mathbf{x}^2\right). \tag{251}$$

In these coordinates the scalar equation (234), ignoring the potential piece, becomes

$$\delta\phi_{\mathbf{k}}'' + 2\frac{a'}{a}\delta\phi_{\mathbf{k}}' + k^2\delta\phi_{\mathbf{k}} = 0, \tag{252}$$

where we have Fourier transformed in space and introduced the comoving momentum $\mathbf{k}$. The conventions are such that

$$\delta\phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} d^3k. \tag{253}$$

We have also introduced the notation $'$ for derivatives with respect to conformal time. If we then introduce the rescaled field $\mu = a\delta\phi$, the equation becomes

$$\mu_{\mathbf{k}}'' + \left(k^2 - \frac{a''}{a}\right)\mu_{\mathbf{k}} = 0. \tag{254}$$

Similarly, the metric fluctuations can be reduced to two polarizations obeying an equation identical to the one for the scalar fluctuations.

To proceed, treating the scalar and gravitational perturbations simultaneously, we assume that the scale factor depend on conformal time as

$$a \sim \tau^{1/2-\nu}, \tag{255}$$

where ν is a constant. An important example is $a \sim e^{Ht}$ with $H = \text{const.}$, where the change of coordinates gives

$$\frac{d\tau}{dt} = \frac{1}{a(t)} = e^{-Ht} \Longrightarrow a(\tau) = -\frac{1}{H\tau}, \tag{256}$$

and we find that $\nu = \frac{3}{2}$. Note that the physical range of τ is $-\infty < \tau < 0$. The equation for the fluctuations, with a of the form above, becomes

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right)\mu_{\mathbf{k}} = 0. \tag{257}$$

Luckily, this is a well known equation which is solved by Hankel functions. The general solution is given by

$$f_k(\tau) = \frac{\sqrt{-\tau\pi}}{2}\left(C_1(k)\,H_v^{(1)}(-k\tau) + C_2(k)\,H_v^{(2)}(-k\tau)\right), \tag{258}$$

where $C_1(k)$ and $C_2(k)$ are to be determined by initial conditions.

When quantizing this system (a nice treatment can be found in [103]) one needs to introduce oscillators $a_k(\tau)$ and $a^{\dagger}_{-k}(\tau)$ such that

$$\begin{aligned}\mu_{\mathbf{k}}(\tau) &= \frac{1}{\sqrt{2k}}\left(a_k(\tau) + a^{\dagger}_{-\mathbf{k}}(\tau)\right) \\ \pi_{\mathbf{k}}(\tau) &= \mu'_{\mathbf{k}}(\tau) + \frac{1}{\tau}\mu_{\mathbf{k}}(\tau) = -i\sqrt{\frac{k}{2}}\left(a_k(\tau) - a^{\dagger}_{-\mathbf{k}}(\tau)\right),\end{aligned} \tag{259}$$

obey standard commutation relations. The crux of the matter is that these oscillators are time dependent, and can be expressed in terms of oscillators at a specific moment in time using the Bogolubov transformations

$$\begin{aligned}a_{\mathbf{k}}(\tau) &= u_k(\tau)\,a_{\mathbf{k}}(\tau_0) + v_k(\tau)\,a^{\dagger}_{-\mathbf{k}}(\tau_0) \\ a^{\dagger}_{-\mathbf{k}}(\tau) &= u^*_k(\tau)\,a^{\dagger}_{-\mathbf{k}}(\tau_0) + v^*_k(\tau)\,a_{\mathbf{k}}(\tau_0),\end{aligned} \tag{260}$$

where

$$|u_k(\tau)|^2 - |v_k(\tau)|^2 = 1. \tag{261}$$

The latter equation makes sure that the canonical commutation relations are obeyed at all times if they are obeyed at τ_0. We can now write down the quantum field

$$\mu_{\mathbf{k}}(\tau) = f_k(\tau)\,a_{\mathbf{k}}(\tau_0) + f^*_k(\tau)\,a_{-\mathbf{k}}(\tau_0), \tag{262}$$

where

$$f_k(\tau) = \frac{1}{\sqrt{2k}}\left(u_k(\tau) + v^*_k(\tau)\right) \tag{263}$$

is given by (258).

But what are the initial conditions? The usual choice is to consider the infinite past and choose a state annihilated by the annihilation operator, i.e.

$$a_{\mathbf{k}}(\tau_0)\,|0,\tau_0\rangle = 0, \tag{264}$$

for $\tau_0 \to -\infty$. As we will see in the next section, there is much to say about this way to proceed, but let us, for the moment, continue according to common practise. From (259) we conclude that

$$\pi_{\mathbf{k}}(\tau_0) = -ik\mu_{\mathbf{k}}(\tau_0), \tag{265}$$

for $\tau_0 \to -\infty$. Since the Hankel functions asymptotically behave as

$$
\begin{aligned}
H_v^{(1)}(-k\tau) &\sim \sqrt{\frac{2}{-k\tau\pi}} e^{-ik\tau} \\
H_v^{(2)}(-k\tau) &\sim H_v^{(1)*}(-k\tau),
\end{aligned}
\tag{266}
$$

we find that the vacuum choice correspond to the choice $C_2(k) = 0$ (and $|C_1(k)| = 1$).

We have now fully determined the quantum fluctuations, and it is time to deduce what their effect will be on the CMBR. To do this, we compute the size of the fluctuations according to

$$
P(k) = \frac{4\pi k^3}{(2\pi)^3} \left\langle |\delta\phi_{\mathbf{k}}|^2 \right\rangle = \frac{k^3}{2\pi^2}\frac{1}{a^2} \left\langle |\mu_{\mathbf{k}}|^2 \right\rangle = \frac{k^3}{2\pi^2}\frac{1}{a^2} |f_k|^2 = \frac{k^3}{2\pi^2}\frac{1}{a^2}\frac{|-\tau\pi|}{4} \left| H_v^{(1)}(-k\tau) \right|^2 .
\tag{267}
$$

This we should evaluate at late times, that is, when $\tau \to 0$. In this limit the Hankel function behaves as

$$
H_v^{(1)}(-k\tau) \sim \sqrt{\frac{2}{\pi}} (-k\tau)^{-\nu},
\tag{268}
$$

and we find

$$
P \sim \frac{1}{4\pi^2}\frac{1}{a^2} (-\tau)^{1-2\nu} k^{3-2\nu} \sim \frac{1}{4\pi^2} H^2 k^{3-2\nu}.
\tag{269}
$$

Here we have used (255) to get rid off the τ dependence. Furthermore, if $\nu \sim 3/2$ and we have a slow roll, H is nearly constant and can be used to set the scale of the fluctuations. In particular, we find the well known scale invariant spectrum if $\nu = 3/2$,

$$
P = \frac{1}{4\pi^2} H^2.
\tag{270}
$$

This is more or less the whole story in case of the gravitational, or *tensor*, perturbations. As previously explained, the scalar fluctuations obey a similar equation, but the translation into the perturbation spectrum is a bit more involved. Basically, different values of ϕ lead to different times for the end of inflation according to $\delta t \sim \frac{\delta\phi}{\dot{\phi}} \sim \frac{H}{2\pi\dot{\phi}}$, see, e.g., [84]. If inflation ends later, the decay of vacuum energy, and hence the initiation of a more conventional cosmology with $H \sim 1/t$ and $\rho \sim 3M_4^2 H^2 \sim 1/t^2$, will be delayed. Therefore, we will find an enhanced density according to $\frac{\delta\rho}{\rho} \sim \frac{\delta t}{t} \sim \frac{H^2}{2\pi\dot{\phi}}$, and the relevant spectrum becomes, in this case,

$$
P_s \sim \left(\frac{H}{\dot{\phi}}\right)^2 \frac{1}{4\pi^2} H^2.
\tag{271}
$$

Comparing (270) and (271) we see that it is the scalar fluctuations that play the most important role. It should be stressed that the spectra, which we have obtained, are the *primordial* ones. To obtain the actual CMBR fluctuation spectra, including the acoustic peaks, which the primordial spectra give rise to, requires a lot more work which is outside the scope of this review.

To express deviations from scale invariance one introduces spectral indices according to

$$n_s - 1 = \frac{d \ln P_s}{d \ln k} = 3 - 2\nu_s \tag{272}$$

$$n_T = \frac{d \ln P_T}{d \ln k} = 3 - 2\nu_T, \tag{273}$$

where ν_s refers to the scalar perturbations and ν_T refers to the gravitational, or tensor, perturbations. While not clear from our simplified analysis, the ν's need not be the same in the two cases. Observations show that n_s is very close to 1, consistent with the basic ideas of inflation. Of extreme importance is to find any slight deviation from the scale invariant value which could give important information about the inflationary potential. Equally interesting would be to find a contribution from the gravitational background.

Inflation has turned out to be a wonderful opportunity to connect the physics of the large with physics of the small. Perhaps effects of physics beyond the Planck scale might be visible on cosmological scales in the spectrum of the CMBR fluctuations? This is the subject to which we now turn.

5.1 Transplanckian physics

As described in the previous section, quantum fluctuations play an important role in the theory of inflation. But how is the structure of these microscopic fluctuations determined? Is the standard argument that we have gone through really valid? In a time dependent background – where there are no global timelike Killing vectors – the definition of a vacuum is highly non trivial. In the ideal situation the time dependence is only transitionary, starting out with an initial, asymptotically Minkowsky like region, where it is possible to one define a unique initial in-vacuum. This vacuum will time evolve through the intermediate time dependent era, and then end up in a final Minkowsky like region. Typically, the initial vacuum will not evolve into the final vacuum but instead appear as an excited state with radiation. Technically, as I have explained, one says that the excited state is related to the vacuum through a *Bogolubov transformation*. A well known example is a star that collapses into a black hole and subsequently emits Hawking radiation.

Interestingly, a similar phenomena can be expected also during inflation. In this case, however, the situation is more tricky since the universe (in Robertson-Walker coordinates) is *always* expanding. How can we then choose an initial state in an unambiguous way? Luckily, the key feature of inflation, the accelerated expansion of the universe, can help out as we have already seen. If we follow a given fluctuation backwards in time far enough, its wavelength will become arbitrarily smaller than the horizon radius. This means that deviations from Minkowsky space will become less and less important, when it comes to defining the vacuum, and the vacuum becomes, in this way, essentially unique. This is the unique vacuum we used in the previous section, and it is sometimes called *the Bunch-Davies vacuum*. The fact that a unique vacuum is picked out is an important property of inflation and is one of several examples of how inflation does away with the need to choose initial conditions.

But, and this is the main point, the argument relies on an ability to follow a mode to infinitely small scales which, clearly, is not how it works in the real world. After all, it is generally believed that there exists a fundamental scale – Planckian or stringy – where physics could be completely different from what we are used to, and where we have very little control of what is happening. How does this affect the argument that the inflationary vacuum is unique? Could there be effects of new physics which will affect the predictions of inflation? In particular one could worry about changes in the predictions of the CMBR fluctuations. Several groups have investigated various ways of modifying high energy physics in order to look for such modifications, see, e.g., [12-27].

I will not discuss the specifics of the proposals of how to modify physics beyond the Planck scale. Instead I will take a different approach, following [91], and provide a typical and rather generic example of the kind of corrections one might expect due to changes in the low energy quantum state of the inflaton field due to the unknown high energy physics. To proceed along this direction, we need to find out *when* to impose the initial conditions for a mode with a given (constant) comoving momentum k. To do this, we use, as in the previous section, conformal time, given by $\tau = -\frac{1}{aH}$. We note that the physical momentum p and the comoving momentum k are related through

$$k = ap = -\frac{p}{\tau H}, \tag{274}$$

and impose the initial conditions when $p = \Lambda$. Λ is the energy scale, maybe the Planck scale or possibly the string scale, where fundamentally new physics becomes important. The basic idea is that we do not know what happens at higher energies, or shorter wavelengths, and therefore are forced to encode our ignorance in terms of initial conditions when the modes enter into the regime that we understand. The unknown high energy physics is usually referred to as *transplanckian*, and the hope is, obviously, that, e.g., string theory eventually will give us the means to derive these initial conditions. Proceeding with the calculation, we find the conformal time when the initial condition is imposed to be

$$\tau_0 = -\frac{\Lambda}{Hk}. \tag{275}$$

As we see, different modes will be created at different times, with a smaller linear size of the mode (larger k) implying a later time.

From the above it is clear that the choice of vacuum is a highly non trivial issue in a time dependent background. Without knowledge of the transplanckian physics we can only list various possibilities and investigate whether there is a typical size or signature of the new effects. A useful example is to choose the vacuum as determined by equation (264), but with an important difference. We do *not* take $\tau_0 \to -\infty$, but instead stop at the value of conformal time given by (275). This vacuum, which in general is different from the Bunch-Davies (note that for $\tau_0 \to -\infty$ the Bunch-Davies vacuum is recovered), should be viewed as a typical representative of natural initial conditions (in the sense explained above). It can be characterized as a vacuum corresponding to a minimum uncertainty in the product of the field and its conjugate momentum, [103], the vacuum with lowest energy (lower than the Bunch-Davies) [88], or as the instantaneous Minkowsky vacuum[7]. Therefore, it can be argued to be as special as the Bunch Davies vacuum, and there is no a priori reason for transplanckian physics to prefer one over the other.

[7] As observed in [104] the exact caracterization of the vacuum depends on the canonical variables used.

We have now a one parameter family of vacua with the single parameter given by the fundamental scale. What is the expected fluctuation power spectrum? Following [91] one finds

$$P(k) = \left(\frac{H}{\dot{\phi}}\right)^2 \left\langle |\phi_k(\tau)|^2 \right\rangle = \left(\frac{H}{\dot{\phi}}\right)^2 \frac{1}{a^2} \left\langle |\mu_k(\tau)|^2 \right\rangle \tag{276}$$

$$= \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \left(1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H}\right)\right), \tag{277}$$

with the standard case recovered when $\Lambda \to \infty$. The result should be viewed as a typical example of what to be expected from transplanckian physics if we allow for effects which at low energies reduce to modifications of the Bunch-Davies case. We note that the size of the correction is linear in H/Λ, and that a Hubble constant, which varies during inflation, gives rise to a modulation of the spectrum. As argued in [91], the modulation is expected to be a quite generic effect that is present regardless of the details of the transplanckian physics. (See also [93] for a discussion about this). After being created at the fundamental scale the modes oscillate a number of times before they freeze. The number of oscillations depend on the size of the inflationary horizon and therefore changes when H changes. A varying Hubble constant is crucial for a detectable signal, since a Hubble constant which does not vary during inflation would just imply a small change in the overall amplitude of the fluctuation spectrum and would not constitute a useful signal. Luckily, since the Hubble constant *is* expected to vary, the situation is much more interesting.

Let me now turn to a more detailed discussion of possible observable consequences. I will discuss what happens using the slow roll parameters. It is not difficult to show (using that H is to be evaluated when a given mode crosses the horizon, $k = aH$) that

$$\frac{dH}{dk} = -\frac{\varepsilon H}{k}, \tag{278}$$

which gives

$$H \sim k^{-\varepsilon}. \tag{279}$$

The k dependence of H will translate into a modulation of $P(k)$, with a periodicity given by

$$\frac{\Delta k}{k} \sim \frac{\pi H}{\varepsilon \Lambda}. \tag{280}$$

To be more specific, let me consider a realistic example. In the Hořava-Witten model [105], unification occurs at a scale roughly comparable with the string scale, the higher dimensional Planck scale, as well as the scale where the fifth dimension becomes visible. For a discussion and references see, e.g., [106] or [90]. As a rough estimate we therefore put $\Lambda = 2 \cdot 10^{16}$ GeV – a rather reasonable possibility within the framework of the heterotic string. Using that the Hubble constant during inflation can not be much larger than $H = 7 \cdot 10^{13}$ GeV, corresponding to $\varepsilon = 0.01$, we find

$$\frac{H}{\Lambda} \sim 0.004 \tag{281}$$

$$\frac{\Delta k}{k} = \Delta \ln k \sim 1. \tag{282}$$

This implies one oscillation per logarithmic interval in k, which fits comfortable within the parts of the spectrum covered by high-precision CMBR observation experiments.

As I have already emphasized, it is important to note that the transplanckian effects, regardless of their precise nature, have a rather generic signature in form of their modulation of the spectrum. If it had just been an overall shift or tilt of the amplitude, it would not have been possible to measure the effect even if it had been considerably larger than the percentage level. Instead, the only result would be a slight change in the inferred values of H and the slow roll parameters. With a definite signature, on the other hand, we can use several measurement points throughout the spectrum, as discussed in more detail in [96]. There it was argued that the upcoming Planck satellite might be able to detect transplanckian effects at the 10^{-3} level, which would put the Hořava-Witten model within range, or at least tantalizingly close. In this way one can also beat cosmic variance that otherwise would have limited the sensitivity to about 10^{-2} at best. Other discussions can be found in [97][98][99][100].

There has been extensive discussions of these results in the literature and their relevance for detectable transplanckian signatures. As pointed out in [92], the initial condition approach to the transplanckian problem allows for a discussion of many of the transplanckian effects in terms *the α-vacua*. These vacua have been known since a long time, [107], and corresponds to a family of vacua in de Sitter space which respects all the symmetries of the space time.

In [90][108] concerns were raised that there could be inconsistencies in field theories based on non trivial vacua of this sort. None of these problems are, however, necessarily relevant to the issue of transplanckian physics in cosmology for a very simple reason, as explained in [95]. The whole point with the transplanckian physics is to find out whether effects beyond quantum field theory can be relevant for the detailed structure of the fluctuation spectrum of the CMBR. In the real world we do expect quantum field theory to break down at high enough energy to be replaced by something else, presumably string theory. The modest proposal behind [91] is simply that we should allow for an uncertainty in our knowledge of physics near planckian scales. Several later works, e.g., [101][102], have confirmed this point of view and the CMBR remains a promising candidate for finding evidence of transplanckian physics.

5.2 String Theory Inflation

Much of contemporary cosmology has dealt with the construction of phenomenologically viable inflationary models with various potentials and number of inflaton fields. In the early days of inflationary theory there were hopes of incorporating inflation in more or less standard particle physics. Perhaps the inflaton was related to, say, the GUT-transition? Unfortunately this never worked out in a convincing way and, as a result, inflation lived its own life quite detached from the rest of theoretical particle physics.

Luckily, string theory is about to change all that. In string theory it is well known that parameters describing background geometries and compactifications, the moduli, are all promoted into scalar fields. There are, therefore, no lack of potential candidates for the inflaton, even though there are several difficult conditions to be met. For one thing, the potential of the inflaton must be extremely flat in order to allow for enough e-foldings. On the other hand, it can not be completely flat for the idea to work. In supersymmetric string theory there are many flat directions in the moduli space of solutions which could, it seems, serve as useful starting points. The hope would then be that these flat directions are lifted by non perturbative, supersymmetry breaking terms. Unfortunately, it is difficult to find these non perturbative corrections explicitly, and their expected form is anyway, in many cases, not of the right kind. In addition, there are also other problems to be solved. Apart from the flat, inflationary potential, one needs potentials that manage to fix dangerous moduli like those controlling the size of the extra dimensions. It is hard to see how realistic inflationary theories can be obtained without addressing this problem at the same time.

A little later I will explain some recent progress in the subject which suggests that realistic inflationary models can indeed be constructed using string moduli if one introduces branes. The idea is to use two stacks of branes separated by a certain distance, corresponding to the inflaton, in a higher dimensional space. As the branes move, the inflaton rolls, and when the branes collide inflation stops. This is a rapidly developing subject – for an early review see [76], and for more recent discussions, see [85], involving many aspects of string theory. But before discussing these promising ideas I will discuss a couple of other interesting approaches to cosmology.

First I will treat the attempts which go under the, somewhat unspecific, name of *string cosmology*, [109][110][111][112] (for a review see [113]). The idea is to make use of the *dilaton*, i.e. the field corresponding to the way the string coupling varies over space and time, and a variant of the string theoretical T-duality. The resulting theory fulfills the condition for inflation, albeit in an unorthodox way.

After this I will turn to models based on branes. Even if branes might very well be the key to realize inflation in string theory, they have, ironically, also been used to argue that string theory can provide an *alternative* to inflation. I will treat a couple of such proposals, the *ekpyrotic* and the *cyclic* universe where colliding branes again play an important role.

5.3 String Cosmology

String cosmology makes use of one of the most basic features of string theory, the dilaton. According to string theory the Hilbert action of general relativity is augmented by a new, dimensionless scalar field, the dilaton ϕ, and given by

$$S = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-\phi}\left(\mathcal{R}+\partial^\alpha\phi\partial_\alpha\phi\right), \tag{283}$$

where $\kappa_{10} = \frac{1}{2}(2\pi)^7\alpha'^4 \sim l_s^8$, and where the string coupling is related to the dilaton through $g_s^2 = e^\phi$. The action as given is written in the *string frame*. That is, the string length, l_s, is our fundamental unit and what we use as our measuring rod. This means that the Planck mass, the effective coefficient of the scalar curvature $\mathcal{R}$, varies with the dilaton. An alternative way to describe things is to use the *Einstein frame* which in many ways is physically more transparent than the string frame. In the Einstein frame it is the Planck length – which is more directly related to macroscopic physics through the strength of gravity – which is used as a fundamental unit. Let me explain how the frames are related to each other in a little more detail. To go from one frame to another, we note that the frames are, by definition, related through

$$\int d^D x\sqrt{-g}e^{-\phi}\mathcal{R} = \int d^D x\sqrt{-g_E}\left(\mathcal{R}_E + ...\right), \tag{284}$$

where

$$g_{\mu\nu} = e^{2\omega\phi}g_{E,\mu\nu}, \tag{285}$$

with the subscript E indicating Einstein frame, and furthermore

$$\sqrt{-g} = e^{D\omega\phi}\sqrt{-g_E}. \tag{286}$$

It follows from the definition of curvature that the scalar curvatures are related through

$$\mathcal{R} = e^{-2\omega\phi}\left(\mathcal{R}_E - 2\omega\left(D-1\right)\nabla^2\phi - \omega^2\left(D-2\right)\left(D-1\right)\partial^\alpha\phi\partial_\alpha\phi\right). \tag{287}$$

Hence we have that

$$\sqrt{-g}e^{-\phi}\mathcal{R} = e^{(D\omega-1-2\omega)\phi}\sqrt{-g_E}\left(\mathcal{R}_E - 2\omega\left(D-1\right)\nabla^2\phi - \omega^2\left(D-2\right)\left(D-1\right)\partial^\alpha\phi\partial_\alpha\phi\right), \tag{288}$$

and as a consequence we find

$$D\omega - 1 - 2\omega = 0 \Longrightarrow \omega = \frac{1}{D-2}. \tag{289}$$

The action in the Einstein frame finally becomes

$$S = -\frac{M_D^{D-2}}{2}\int d^D x\sqrt{-g_E}\left(\mathcal{R}_E - \frac{1}{D-2}\partial^\alpha\phi\partial_\alpha\phi\right), \tag{290}$$

where M_D is the D-dimensional Planck mass. We note that the sign of the kinetic term of the scalar field now is the familiar one.

If we consider a metric of FRW-form (223) generalized to D dimensions, we find

$$d^2s_E = e^{-2\omega\phi}d^2s = e^{-2\omega\phi}\left(dt^2 - a^2d\mathbf{x}^2\right) \equiv dt_E^2 - a_E^2 d\mathbf{x}^2, \tag{291}$$

where

$$a_E = e^{-\omega\phi}a \tag{292}$$

$$dt_E = e^{-\omega\phi}dt.$$

It is important to realize that the two frames are physically equivalent, even if things can, at a first glance, look rather different in the two frames. To fully appreciate string cosmology it is important to keep this in mind.

Let us now investigate the above action in more detail. I will perform the analysis in the string frame, and, for simplicity, assume a spatially homogenous RW-metric. One can readily check that the scalar curvature in these coordinates is given by

$$\mathcal{R} = -\left(D-1\right)\left(D-2\right)\frac{\dot{a}^2}{a^2} - 2\left(D-1\right)\frac{\ddot{a}}{a}. \tag{293}$$

The action looks rather innocent, but possesses a remarkable symmetry thanks to the presence of the stringy dilaton. The symmetry acts on the scale factor and the dilaton through the transformations

$$\begin{aligned} a(t) &\rightarrow 1/a(t) \\ \phi(t) &\rightarrow \phi(t) - 2(D-1)\ln a(t). \end{aligned} \tag{294}$$

It leaves the action invariant and assures that the solutions of the equations of motion have some very interesting properties that will be important for cosmology. To verify the symmetry, we note that

$$\begin{aligned} \sqrt{-g}e^{-\phi}\left(\mathcal{R}+\dot{\phi}^2\right) &= a^{D-1}e^{-\phi}\left(-(D-1)(D-2)\frac{\dot{a}^2}{a^2} - 2(D-1)\frac{\ddot{a}}{a} + \dot{\phi}^2\right) \\ &= a^{D-1}e^{-\phi}\left((D-1)(D-2)\frac{\dot{a}^2}{a^2} - 2(D-1)\frac{\dot{a}}{a}\dot{\phi} + \dot{\phi}^2\right) \\ &\quad + \text{total derivative} \end{aligned} \tag{295}$$

$$= a^{D-1}e^{-\phi}\left(-(D-1)\frac{\dot{a}^2}{a^2} + \left(\dot{\phi} - (D-1)\frac{\dot{a}}{a}\right)^2\right) \tag{296}$$

$$+ \text{total derivative} \tag{297}$$

Since we have

$$\begin{aligned} a^{D-1}e^{-\phi} &\rightarrow a^{-(D-1)}e^{-\phi+2(D-1)\ln a} = a^{D-1}e^{-\phi} \\ \frac{\dot{a}}{a} &\rightarrow a\frac{d}{dt}\left(\frac{1}{a}\right) = -\frac{\dot{a}}{a}, \end{aligned} \tag{298}$$

we find

$$\begin{aligned} & a^{D-1}e^{-\phi}\left(-(D-1)\frac{\dot{a}^2}{a^2} + \left(\dot{\phi} - (D-1)\frac{\dot{a}}{a}\right)^2\right) \\ & \rightarrow a^{D-1}e^{-\phi}\left(-(D-1)\frac{\dot{a}^2}{a^2} + \left(\dot{\phi} - 2(D-1)\frac{\dot{a}}{a} + (D-1)\frac{\dot{a}}{a}\right)^2\right), \end{aligned} \tag{299}$$

and hence an invariance of the action! In other words, if $a(t)$ and $\phi(t)$ solves the equations of motion, so does the transformed functions $1/a(t)$ and $\phi(t) - 2(D-1)\ln a(t)$.

To fully appreciate what is going on, and to understand the structure of the solutions, we need to note that there is yet another simple symmetry,

$$t \rightarrow -t, \tag{300}$$

i.e. time reversal invariance, which together with (294) tells an interesting story about possible cosmologies. Combining the two symmetries we can map out how various solutions are related to each other. If we first focus on the scale factor, we see how we from a given solution $a(t)$ can construct two new solutions according to

$$a(t) \to 1/a(t) \qquad H(t) \to -H(t) \tag{301}$$

$$a(t) \to a(-t) \qquad H(t) \to -H(-t). \tag{302}$$

The time $t = 0$ is referred to as the Big Bang and it is natural to allow for two eras, a pre and a post Big Bang. The basic idea of string cosmology is that physics can be traced back in time through the Big Bang into an earlier era, the pre Big Bang, where many of the initial conditions for the post Big Bang are determined in a natural and dynamical way.

It should be stressed that the whole set up is in line with the general picture of T-duality in string theory. According to T-duality, it is equivalent to compactify string theory on a small circle (compared with the string scale) and a large circle. In some sense large and small scales are, therefore, equivalent. Loosely applying this idea to the Big Bang, would suggest that if we trace the expansion far enough back in time, we are better off describing the universe as becoming bigger again, rather than smaller. As we will see, however, string cosmology suggests that we should take an *expanding* pre Big Bang theory and match it to an expanding post Big Bang. But, and this is an important but, this is the picture obtained in the *string frame*. The picture in the *Einstein frame*, as I will explain, is quite different with a contracting rather than an expanding pre Big Bang phase. This is precisely in line with the hand waving argument above.

Let us now work out a detailed example to get a better feeling for how the various cosmologies are related. In our example we add matter with a definite equation of state,

$$p = w\rho, \tag{303}$$

assuming an action of the form

$$S = -\frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(e^{-\phi}\left(\mathcal{R} + \partial^\alpha\phi\partial_\alpha\phi\right) + \text{matter}\right), \tag{304}$$

with, for simplicity, no explicit ϕ dependence in the matter piece. We will be using the Friedmann equations in the string frame, but, as an exercise, we start out in the more familiar Einstein frame where the Friedmann equations take the familiar form

$$H_E^2 = \frac{1}{3M_4^2}\left(\frac{M_4^2}{2}\frac{1}{2}\left(\frac{d\phi}{dt_E}\right)^2 + \rho_E\right), \tag{305}$$

where we have taken the prefactor of (290) into account (with $D = 4$), when we write down the energy density for the scalar field. It is now easy, using the relations (292), to translate this into the string frame. In particular we have

$$H_E = e^{\phi/2}\left(H - \frac{1}{2}\dot{\phi}\right) \tag{306}$$

$$\frac{d\phi}{dt_E} = e^{\phi/2}\dot{\phi} \tag{307}$$

$$\sqrt{-g_E}\rho_E = e^{-2\phi}\sqrt{-g}\rho_E = \sqrt{-g}\rho. \tag{308}$$

We finally obtain the Friedmann equation in the string frame as

$$H^2 = -\frac{1}{6}\dot{\phi}^2 + H\dot{\phi} + \frac{1}{3M_4^2}e^{\phi}\rho. \tag{309}$$

To proceed, we also need the continuum equation for matter which gives

$$\rho = \rho_0 a^{-3(1+w)}, \tag{310}$$

and the equation of motion for the dilaton obtained from the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0, \tag{311}$$

where

$$L = -a^3 e^{-\phi}\left(-6\frac{\dot{a}^2}{a^2} + 6\frac{\dot{a}}{a}\dot{\phi} - \dot{\phi}^2\right) = 6a\dot{a}^2 e^{-\phi} - 6\dot{a}a^2\dot{\phi}e^{-\phi} - a^3\dot{\phi}^2 e^{-\phi}. \tag{312}$$

Using an ansatz of the form

$$a \sim t^{\alpha} \tag{313}$$

$$\phi = \beta \ln t + \text{const.},$$

it is straightforward to derive, from (311), that

$$-12\alpha^2 - \beta^2 + 6\alpha\beta - 2\beta + 6\alpha = 0. \tag{314}$$

To fully determine α and β we need one more equation. Since both $H^2 \sim \dot{\phi}^2 \sim 1/t^2$ the same must be true for $e^{\phi}\rho$ according to the Friedmann equation. The continuum equation (219) then provides the missing relation

$$\beta - 3\left(1+w\right)\alpha = -2. \tag{315}$$

Finally, we can write down the solution to the latter two equations as

$$\alpha = \frac{2w}{1+3w^2} \tag{316}$$

$$\beta = \frac{6w-2}{1+3w^2}. \tag{317}$$

So far we have not said what kind of matter we are considering. But let me now, in order to be completely specific, assume that matter is in the form of radiation with $w = 1/3$. This gives

$$\alpha = 1/2 \qquad \beta = 0 \tag{318}$$

that is,

$$a \sim t^{1/2} \qquad \phi = \text{const.} \tag{319}$$

In other words, we have a standard radiation dominated, and non-inflationary, cosmology. In particular we have a decreasing Hubble constant given by

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} > 0, \tag{320}$$

with

$$\ddot{a} = -\frac{1}{4t^2} < 0 \qquad \frac{\ddot{a}}{\dot{a}} < 0. \tag{321}$$

Not much new, but at least we see that it is consistent to have a constant dilaton. It is now time to apply the symmetry transformations introduced above. We immediately find a new solution given by

$$a \sim (-t)^{-1/2} \qquad \phi = -6\ln(-t)^{1/2} + \text{const.}, \tag{322}$$

valid for $t < 0$.[8] We now have

$$H = \frac{\dot{a}}{a} = -\frac{1}{2t} > 0 \qquad \frac{\ddot{a}}{\dot{a}} = \frac{3}{2t^2} > 0 \tag{323}$$

with

$$\dot{H} = \frac{1}{2t^2} > 0, \tag{324}$$

i.e., a growing curvature. To summarize, we find an inflating universe with growing curvature and coupling as $t \to 0_-$, followed by a standard radiation dominated cosmology. In other words, a rather appealing cosmology. At least if we somehow can find a way of matching the two solutions at the Big Bang.

There is, however, another interesting twist to the story. As seen in the previous section, the description of the physics is quite different if we change to the Einstein frame. In our example, there is no real difference in the post Big Bang era between the two frames, since the dilaton is constant. In the pre-big bang phase, on the other hand, we find, using (292),

$$dt_E \sim (-t)^{3/2}\,dt \qquad \Longrightarrow \qquad t_E \sim -(-t)^{5/2} \tag{325}$$

$$a_E \sim (-t)^{3/2} \times (-t)^{-1/2} = -t \sim (-t_E)^{2/5}, \tag{326}$$

and as a consequence

$$a'_E \sim -(-t_E)^{-3/5} < 0 \tag{327}$$

$$a''_E \sim -(-t_E)^{-8/5} < 0. \tag{328}$$

The physical picture in the Einstein frame is therefore of a contracting rather than an expanding universe. Nevertheless we can rest assured that the physics will be equivalent.

In order to understand better what is going on, it is useful, following [113], to classify the various possibilities according to the following table:

[8] One should note that we have assumed that the matter piece also respects the symmetries. In the particular example that we study, this implies that the equation of state becomes $w = -1/3$ in the transformed theory.

Class I			
$\dot{a} > 0$	$\ddot{a} > 0$	$\dot{H} < 0$	standard inflation
Class II			
$\dot{a} > 0$	$\ddot{a} > 0$	$\dot{H} > 0$	superinflation
$\dot{a} < 0$	$\ddot{a} < 0$	$\dot{H} < 0$	collapse!

As in our example, superinflation and a collapsing universe can be different descriptions of the same physics in string and Einstein frames respectively.[9] It is interesting to see that the advantages of inflation can be obtained also in a contracting universe. The important thing is that the ratio of the radius of curvature and the scale factor becomes smaller with time.

As I have already hinted, a basic problem of string cosmology is how to match the pre and post Big Bang solutions. This is known as *the graceful exit problem.* As is clear from the examples above, the matching has to take place at strong coupling and little is known about how to achieve this. I will come back to the same problem in the next section, when I discuss some alternative models.

Another important issue is the CMBR-fluctuations. Let me continue to discuss the particular example introduced above. To apply the formulae of section 3.1.1. we need to go to conformal time. We find

$$\tau \sim -\left(-t_E\right)^{3/5} \tag{329}$$

and

$$a_E \sim (-\tau)^{2/3} \sim (-\tau)^{1/2-\nu_T} \qquad \Longrightarrow \qquad \nu_T = -1/6, \tag{330}$$

and from this

$$n_T = \frac{10}{3}. \tag{331}$$

That is, a blue spectrum for the gravitational perturbations not at all like the more or less scale invariant result of standard inflation. This is certainly an interesting prediction and could be a characteristic signal to look for if, and when, these perturbations become observationally accessible. Unfortunately, however, a similar spectrum can be derived also for the scalar fluctuations which dominate the CMBR. This is not at all in line with what observations show, and is one of the big problems with the simplest approaches to string cosmology. Some possible ways out of this dilemma is discussed in [113].

5.4 Brane cosmology

Basic setup In the middle 90's, it was realized that not only strings but also higher dimensional structures like membranes etc. play an important role in string theory. Moreover,

[9]One should note that superinflation driven by a standard scalar field is not possible in the Einstein frame. This will be discussed in a different context a bit later.

branes provide new possibilities to construct realistic cosmologies. Of particular interest is the idea to associate the Big Bang with a collision of brane worlds which I will discuss in some detail. This has been considered from two quite different points of view – either as an alternative to inflation or as a way of implementing inflation.

The first of the alternatives to inflation is the *ekpyrotic scenario*, [114][115][116]. It makes use of the Hořava-Witten interpretation of the heterotic $E_8 \times E_8$ string where there is an eleventh dimension separating two 9+1 dimensional brane worlds. The separation between the branes gives the string coupling in such a way that a small separation corresponds to weak coupling. We are assumed to be living on one of the branes, the visible brane, while the other brane is called the hidden brane. In the ekpyrotic scenario there is an additional brane in the bulk which is free to move. The configuration is assumed to be nearly supersymmetric, i.e. BPS, and therefore nearly stable – apart from a small potential which provides an attraction between the bulk brane and the visible brane.

The main idea behind the ekpyrotic scenario is to let the Big Bang correspond to a collision between the bulk brane and the visible brane. The homogeneity of the early universe, usually explained by inflation, is explained by the nearly BPS initial state. The bulk brane is almost parallel with the visible brane and the collision happens almost at the same time everywhere. From the point of view of physics on the visible brane, the era before the collision is a contracting universe, while the era after the collision (or Big Bang) is our expanding universe. Slight differences in collision time give rise to the crucial primordial spectrum of fluctuations. This represents a new mechanism, fundamentally different from the one of inflation.

An improved proposal is the *cyclic scenario*, [117], where one does away with the bulk brane and lets, instead, the visible and hidden branes collide. Actually, the branes are supposed to be able to pass through each other and, eventually, turn back for yet another collision. And so on, forever. The homogeneity is, in this model, explained not through initial conditions, but by a late time cosmological constant in each cycle. The cosmological constant provides an accelerated expansion that sweeps the universe clean of disturbances preparing it for a new cycle. The idea is that we presently are entering into such an era and, in this way, the model suggests an interesting role for the cosmological constant recently observed. In a way the cyclic universe make use of inflation of a kind, even though the energy scales involved are totally different. Note, however, that the quantum fluctuations during the inflationary stage in the cyclic universe will be irrelevant for the CMBR fluctuations due to the low energy scale.

Whatever description all of this has from the point of view of higher dimensions, there should also be an effective four dimensional picture. To study this, we start with the same action as in string cosmology, (283), but think of ϕ as a scalar field such that e^{ϕ} is proportional to the distance between the branes. The Big Crunch occurs when the distance between the branes vanish, that is when $\phi \to -\infty$, corresponding to a Big Crunch at weak coupling since, from the four dimensional point of view, e^{ϕ} is like a coupling. Note that this is just the opposite to what we have in string cosmology. We use the same ansatz as before, (313), in the Friedmann equation for an empty universe

$$H^2 = -\frac{1}{6}\dot{\phi}^2 + H\dot{\phi}, \tag{332}$$

to get

$$\alpha^2 = -\frac{1}{6}\beta^2 + \alpha\beta. \tag{333}$$

The two equations are solved by

$$\alpha = \pm\frac{1}{\sqrt{3}} \qquad \beta = \pm\sqrt{3} - 1, \tag{334}$$

that is,

$$a \sim t^{\varepsilon/\sqrt{3}} \qquad \phi = \left(\varepsilon\sqrt{3} - 1\right)\ln t, \tag{335}$$

where $\varepsilon = \pm 1$. If we had been doing string cosmology we would have applied the duality transformations of (294) (and (300)). This leads to

$$a \sim (-t)^{-\varepsilon/\sqrt{3}} \tag{336}$$

$$\phi = \left(\varepsilon\sqrt{3} - 1\right)\ln(-t) - 6\ln(-t)^{\varepsilon/\sqrt{3}} = \left(-\varepsilon\sqrt{3} - 1\right)\ln(-t)\,. \tag{337}$$

Clearly, this is essentially an exchange of the two solutions in (335). In string cosmology we would have made the choice

$$\begin{array}{ll} t < 0 & \varepsilon = -1 \\ t > 0 & \varepsilon = +1, \end{array} \tag{338}$$

with $t \to 0_- \Longrightarrow g_s \to +\infty$ and $t \to 0_+ \Longrightarrow g_s \to 0$. In the ekpyrotic universe, however, where the collision of branes corresponds to weak coupling, we have $\varepsilon = +1$ for *all* t!

To proceed, we note that for all t and ε, we have that

$$\frac{\dot{a}}{a} \sim \frac{\varepsilon}{\sqrt{3}t}. \tag{339}$$

Using this we find for string cosmology

$$\begin{array}{lll} t < 0 & \varepsilon = -1 & \frac{\dot{a}}{a} > 0 \\ t > 0 & \varepsilon = +1 & \frac{\dot{a}}{a} > 0, \end{array} \tag{340}$$

while the ekpyrotic universe has

$$\begin{array}{lll} t < 0 & \varepsilon = +1 & \frac{\dot{a}}{a} < 0 \\ t > 0 & \varepsilon = +1 & \frac{\dot{a}}{a} > 0. \end{array} \tag{341}$$

This was all in the string frame. In the Einstein frame we simply find

$$a_E \sim (-t_E)^{1/3} \qquad \text{for} \qquad t_E < 0 \tag{342}$$

$$a_E \sim t_E^{1/3} \qquad \text{for} \qquad t_E > 0 \tag{343}$$

if we follow the recipe provided earlier.[10] That is, regardless of whether we are considering string cosmology or the cyclic universe, we find a universe that first collapses and then expands. The difference is the behavior of the scalar field. One notes that the condition $\frac{\ddot{a}_E}{\dot{a}_E} > 0$ is fulfilled in the $t_E < 0$ era both for string cosmology and for the cyclic universe. In fact, the process with fluctuations crossing the horizon, and entering in a much later era, is common to standard inflation and the ekpyrotic/cyclic universe.

Can it work? The ekpyrotic/cyclic scenarios have been heavily criticized in the literature, see, e.g., [84][115]. I will briefly review some of this criticism. But let me begin by considering the generation of fluctuations in the ekpyrotic/cyclic universe. To do this, we make use of (234), which we expand to quadratic order to get

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \nabla^2\delta\phi + V''(0)\,\delta\phi = 0. \tag{344}$$

In conformal time, assuming spatial homogeneity, we find

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k + a^2 V''(0)\,\mu_k = 0. \tag{345}$$

Usually, the last term is ignored due to the flatness of the potential – a necessary condition for inflation. In the cyclic scenario, however, this is no longer the case. Instead, it is the term due to the expansion/contraction of the universe that should be ignored. The generation of fluctuations takes place when the universe is contracting very slowly, and the scale factor is more or less constant. A useful potential, with the correct properties, is

$$V(\phi) = -V_0 e^{-\phi/M_4}, \tag{346}$$

and we will look for a solutions with $a = \text{const}$. We then need to solve

$$\begin{aligned} \ddot{\phi} + V'(\phi) &= 0 \\ H^2 = \frac{1}{3M_4^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) &= 0 \end{aligned} \tag{347}$$

It is easy to verify that this works for

$$\dot{\phi} = \frac{2M_4}{t}, \tag{348}$$

and we find

$$\mu_{\mathbf{k}}'' + \left(k^2 - \frac{V_0}{M_4^2}e^{-\phi/M_4}\right)\mu_{\mathbf{k}} = 0, \tag{349}$$

or

$$\mu_{\mathbf{k}}'' + \left(k^2 - \frac{2}{t^2}\right)\mu_{\mathbf{k}} = 0, \tag{350}$$

[10] This corresponds to a universe filled by matter with equation of state given by $p = \rho$. This is precisely what one gets from a massless scalar field without potential.

which is precisely the same equation as derived in the context of inflation! However, there are some important differences. In the inflationary case, we need to rescale μ with the scale factor to get the original scalar field $\phi = \mu/a$. As we have already seen, the amplitude of the fluctuations in ϕ are always finite. In the ekpyrotic/cyclic case, however, the scale factor is essentially constant and we are stuck with the field μ whose amplitude *diverges* as $t \to 0$. We therefore need a cutoff near the moment of collision. This is a reflection of the tachyonic nature of the potential with $V''(0) < 0$. In inflation the classical perturbations are smeared thanks to the exponential expansion. In the ekpyrotic universe, however, thsi does not happen and the classical perturbations are amplified in just the same way as the quantum mechanical ones, [84][115]. Hence the need for fine tuning of initial conditions.

In case of the cyclic universe we must also investigate the claim that the cycles continue forever. A well known argument against an eternal, cyclic universe, comes from the second law of thermodynamics. With every cycle the entropy should increase and one would not expect an infinite number of cycles. In case of the brane based cyclic universe, it is argued that the exponential expansion due to the late time cosmological constant does the job through a rapid clean up which effectively provides an empty universe ready for the next cycle.

However, it is hard to see how this statement can be *exactly* true. From the point of view of a local observer it is true that any matter (carrying entropy) is heavily redshifted and pushed towards the cosmological horizon. But, as I will discuss in the chapter on holography, there is a limit on how much entropy can be stored by the horizon. When this limit is reached, there will be unavoidable consequences for the physics of the cyclic universe. As a result, the second law will eventually prevail after all. It is true, though, that the time scale for this to happen will be enormous.

Another crucial problem of the ekpyrotic/cyclic proposal is the bounce. Will the branes bounce off each other or will there be a devastating singularity? Unfortunately, it is well known that the necessary reversal from contraction to expansion is very difficult, if not impossible, to achieve. What is needed, is a Hubble constant which starts out negative and then becomes positive. In other words, we need a period with $\dot{H} > 0$. The problem is that we have the Friedmann equation

$$\dot{H} = -\frac{1}{2M_4^2}(\rho + p), \tag{351}$$

with a right hand side which for all reasonable types of matter is negative. An example is the scalar field in section 2.3., where the equations (232) and (233) yield

$$\dot{H} = -\frac{1}{2M_4^2}\dot{\phi}^2 < 0. \tag{352}$$

The same problem is also present in case of string cosmology, but in that case we at least can blame strong coupling and hope that, somehow, there is a way out. In the ekpyrotic scenario, everything happens at weak coupling suggesting that there is little chance of evading the contradiction.

There are also other arguments indicating that a singularity is the end and not a new beginning. The idea is that the creation of a new universe beyond the singularity inside of a black hole, would imply that black holes are information sinks as first suggested by Hawking.

However, it is now generally believed that string theory predicts that all information *is* getting back out from a black hole through the Hawking radiation. From this it is argued in [119] that no information can pass through a singularity into a new baby universe. The kind of bounce needed for the ekpyrotic or cyclic universe would therefore not take place. Very recently, [118], new arguments have been put forward where it is suggested that a cosmological singularity *can* be resolved in M-theory. It is fair to say, therefore, that there is no consensus in the field at the moment.

What, then, is the conclusion? The ekpyrotic universe represents a different paradigm without inflation where, instead, it is argued that high energy physics can naturally provide very special initial conditions all on its own. The cyclic universe does not do away with inflation completely but, in a very economical way, identifies inflation with the presence of a cosmological constant late in each cycle. Unfortunately, both scenarios face severe technical problems due to the difficulties in understanding the bounce. Whether or not string theory allows for a world beyond a time like singularity is of crucial importance, not only to cosmology.

Can inflation be realized using branes? As we have seen above the distance between two branes can be identified with a scalar field on the branes yielding interesting cosmologies. But instead of using this to construct an alternative to inflation, we will now try to identify the scalar field as the inflaton.

The first attempts to construct brane inflation used two sets of branes. If the configuration preserves supersymmetry there is no force between the branes and no potential for the scalar field. What happens is that there is a balance between the gravitational attraction between the branes and a repulsion due to the RR-charges of the branes. If supersymmetry is broken, however, the dilaton and the RR-fields obtain masses while the graviton remain massless. Hence the attraction wins and there is a force between the branes. In principle this could yield inflation if the resulting potential is of just the right form, [120]. Unfortunately, our understanding of string theory is not deep enough to enable us to perform trustworthy calculations with nonperturbative supersymmetry breaking. Actually, the situation is not unlike what we have in the ekpyrotic/cyclic scenario where the actual potential also is not very well known.

Another possibility, in the sense that we can perform reliable calculations to check the scenario, is to consider brane and anti-branes where supersymmetry is broken and there is a force already at tree level. In this case the branes have opposite charges and there is an attractive RR-force that adds to the gravitational force. Let us see how this works, [121][123][124].

We start out with the action of a Dp-brane in 10 dimensional space time. It is given by

$$S_D = -\int d^4x d^{p-3}y\sqrt{-h}\,[T_p + ...]\,, \tag{353}$$

where

$$T_p = \frac{1}{(2\pi)^p}\alpha'^{-\frac{p+1}{2}}e^{-\phi} \sim \frac{1}{g_s l_s^{p+1}}, \tag{354}$$

is the tension of the brane, $h_{\mu\nu}$ is the metric on the branes induced by the full metric $g_{\mu\nu}$, and x corresponds to our four space time coordinates while y are compact extra dimensions around which the D-branes are wrapped (if $p > 3$). The action for the $\bar{\text{D}}$p-brane is identical.

The position of the D-brane and the $\bar{\text{D}}$-brane in the transverse dimensions are denoted by z_1^m and z_2^m respectively, where $m = 1...d_\perp$ with $d_\perp = 10 - (p+1) = 9 - p$ as the number of transverse dimensions. We now define the relative position of the D and $\bar{\text{D}}$-branes as $z^m = z_1^m - z_2^m$, and their average position as $\bar{z}^m = \frac{z_1^m + z_2^m}{2}$. The two actions added together, can then be expanded as

$$S_D + S_{\bar{D}} = -\int d^4x d^{p-3}y \sqrt{-\bar{h}} T_p \left[2 + \frac{1}{4} g_{mn} \bar{h}^{\mu\nu} \partial_\mu z^m \partial_\nu z^n + ... \right], \tag{355}$$

where $\bar{h}_{\mu\nu}$ is evaluated on $\bar{z}^m$. We now have the kinetic terms for our inflaton field z, but what about the potential?

The potential energy is of the same form as the gravitational potential between two branes, that is, the energy per area is given by

$$\frac{E}{A_p} = -\beta \frac{1}{M_{10}^2} \frac{T_p^2}{z^{d_\perp - 2}}, \tag{356}$$

where $\beta = \frac{1}{8} \pi^{-d_\perp/2} \Gamma\left(\frac{d_\perp - 2}{2}\right)$, and $M_{10}^2 = e^{-2\phi} \kappa_{10}^{-2}$. Compactifying according to

$$\frac{M_{10}^2}{2} \int d^4x d^6y \sqrt{-g} \left[R + ... \right] \sim \frac{M_4^2}{2} \int d^4x \sqrt{-g} \left[R + ... \right], \tag{357}$$

we find

$$M_4^2 = M_{10}^2 V_\perp V_\parallel. \tag{358}$$

The six extra dimensions should be compact, and we assume that they have volumes given by

$$V_\perp \equiv r_\perp^{d_\perp}, \qquad V_\parallel \equiv r_\parallel^{p-3}. \tag{359}$$

The potential (including the mass density of the branes) can, after compactification, be written

$$V(z) = 2T_p V_\parallel - \frac{\beta}{M_{10}^2} \frac{T_p^2 V_\parallel}{z^{d_\perp - 2}} \equiv A - \frac{B}{z^{d_\perp - 2}}. \tag{360}$$

To complete the calculation, we also need to make sure that our inflaton has the correct normalization. Looking at (355) we see that we need to identify the canonically normalized scalar field as

$$\psi = \sqrt{\frac{T_p V_\parallel}{2}} z. \tag{361}$$

Can the resulting potential yield inflation? To answer this question, we need to evaluate the slow roll parameters. These are given by

$$\begin{aligned}
\varepsilon &= \frac{M_4^2}{2}\left(\frac{V'}{V}\right)^2 \sim \frac{M_4^2}{T_p V_\parallel}\left(\frac{B}{A}\left(d_\perp - 2\right)\frac{1}{z^{d_\perp - 1}}\right)^2 \\
&= \frac{\beta^2}{4}\left(d_\perp - 2\right)\frac{T_p}{M_{10}^2}\frac{1}{z^{d_\perp - 2}}\left(\frac{r_\perp}{z}\right)^{d_\perp} \sim g_s\left(\frac{l_s}{z}\right)^{d_\perp - 2}\left(\frac{r_\perp}{z}\right)^{d_\perp} \qquad (362)\\
\eta &= M_4^2\frac{V''}{V} \sim -\frac{2M_4^2}{T_p V_\parallel}\frac{B}{A}\left(d_\perp - 2\right)\left(d_\perp - 1\right)\frac{1}{z^{d_\perp}} \\
&= -\beta\left(d_\perp - 2\right)\left(d_\perp - 1\right)\left(\frac{r_\perp}{z}\right)^{d_\perp} \sim \left(\frac{r_\perp}{z}\right)^{d_\perp}, \qquad (363)
\end{aligned}$$

where we have made use of (358). The derivatives of V are taken with respect to ψ. From the requirement that η should be small, we immediately see that $z \gg r_\perp$ which, unfortunately, does not make sense. The branes can not be separated by a distance larger than the size of the compact dimension!

One possible way out, is to fine tune the positions of the D and $\bar{\mathrm{D}}$ to opposite sides of the compact dimension. From symmetry this must correspond to a meta-stable, forceless configuration. It can be shown that the potential close to the equilibrium position is such that slow roll and inflation is allowed. In the next section we will come back to other possibilities of obtaining realistic models.

It is also interesting to think about what will happen when the branes collide. From string theory we would expect the annihilation of the branes to be driven, from the perspective of the brane, by an open string tachyon. The field T corresponding to the tachyon becomes tachyonic when the distance between the branes is decreased to a string length, [125]. We therefore expect a potential of the form

$$V\left(z,T\right) = A\left(\frac{z^2}{l_s^2} - B\right)T^2 + CT^4 + V\left(z\right), \qquad (364)$$

where A, B and C are positive constants. Interestingly, this is just the kind of potential known from *hybrid inflation*, [126] . The original motivation for hybrid inflation was to generate enough e-foldings without the inflaton having to start out with values of the order of many Planck masses, recall (242). Contrary to a single field chaotic inflation, where all of the vacuum energy decays through the rolling inflaton, the decay in hybrid inflation takes place in two steps.

First, when z is large, the tachyon T is locked in a minimum at $T = 0$. The effective potential for the inflaton can, in the simplest case, be of the usual monomial type, but the minimum has a non-zero vacuum energy that can drive inflation. However, when z becomes small enough, $T = 0$ becomes unstable and rolls down to a new minimum. As a result, the vacuum energy decays away. In brane inflation, this corresponds to the annihilation of the branes.

Unfortunately we do not have a good understanding of what happens when the branes annihilate and how rehetaing takes place. That is, how all the matter now present in the universe is created out of the decaying vacuum energy. We must also make sure that all branes do not annihilate after the collision. There must be a net number of, say D-branes, remaining after all pairs of D and $\bar{\mathrm{D}}$ have annihilated.

5.5 Strings in de Sitter space

Recent observations give very strong indications that we are living in a universe with a positive cosmological constant, i.e. a de Sitter space. From the point of view of string theory this is quite surprising. In fact, it has been a long standing problem to formulate string theory in de Sitter space. Part of the difficulty has to do with supersymmetry. Contrary to the case of flat space time and a space time with a negative cosmological constant, i.e. anti de Sitter space, a positive cosmological constant goes together with super symmetry breaking. It has, therefore, not been possible to take advantage of the simplifications due to supersymmetry in constructing de Sitter space times. Another, more serious problem, is that string theory is naturally formulated using S-matrices. That is, we need an asymptotic lightlike infinity, like in Minkowsky space, to make sense of the scattering amplitudes produced by string theory. An exception is anti de Sitter space, where we have the option to describe physics holographically on the time like boundary. Unfortunately, neither of these possibilities are available in de Sitter space. Based on this, it was argued in [127] that an accelerated expansion, like the one due to a cosmological constant, necessarily is temporary.

A seemingly different problem in string theory is the stabilization of moduli. For instance, why is the size of the compact dimensions stable? Why do they not change in a substantial way during the evolution of the universe? As we will see in the following these two problems are not unrelated to each other.

Let us start out, following [128], with the action

$$S = \int d^D x \sqrt{-g} \left[-\frac{M_D^{D-2}}{2} \mathcal{R} + L(\psi) \right], \tag{365}$$

with $D = d + 4$, and metric

$$ds^2 = ds_4^2 + R^2(x)\, g_{dmn}(y)\, dy^m dy^n. \tag{366}$$

We dimensionally reduce to four dimensions and find

$$S = -\frac{M_D^{D-2}}{2} V_d \int d^4x \sqrt{-g_4} e^{d\phi(x)} \mathcal{R}_4 + V_d \int d^4x \sqrt{-g_4} e^{d\phi(x)} \int d^d y \sqrt{g_d} L(\psi), \tag{367}$$

where we have put

$$R(x) = R_0 e^{\phi(x)} \tag{368}$$

$$V_d = R_0^d. \tag{369}$$

Now let us rescale

$$g_{4,\mu\nu} \to e^{-d\phi(x)} g_{4,\mu\nu}. \tag{370}$$

This leads to

$$\sqrt{-g_4} \to e^{-2d\phi(x)} \sqrt{-g_4} \tag{371}$$

$$\mathcal{R}_4 \to e^{d\phi(x)} \mathcal{R}_4 + ... \tag{372}$$

and the action becomes

$$S=-\frac{M_4^2}{2}\int d^4x\sqrt{-g_4}\mathcal{R}_4+V_d\int d^4x\sqrt{-g_4}e^{-d\phi(x)}\int d^dy\sqrt{g_d}L\left(\psi\right). \tag{373}$$

It is important to note that the second term includes a factor $e^{-d\phi(x)}$ that decreases if the volume of the compact dimensions increases. This can only be compensated if $\int d^dy\sqrt{g_d}L\left(\psi\right)$ goes like the volume, i.e. like R^d. The metric g_d does not include any R-dependence which leave us with the density $L\left(\psi\right)$. There is no type of matter which has an energy density that grows with the volume of space. In fact, recalling (220) we see that such matter would have $w=-2$. The best we can do is to consider cases where there is effectively a cosmological constant. This can be obtained by wrapping a brane around the compact dimensions.

In general, we find that the energy approaches zero as the dimensions decompactify, and we end up, eventually in ten dimensional flat space time. There are several possibilities for how this can happen. In this case there is nothing that can prevent the compact dimensions from opening up, and the system rapidly rolls towards the decompactified case. Another possibility is that there is a minimum of the potential at negative energy, i.e., an anti de Sitter universe where the compact dimensions are stabilized. The much studied $AdS^5\times S^5$ is an example of this. Finally, there could be a (local) minimum with positive energy corresponding to a de Sitter universe. The size of the extra dimensions are now meta stable – eventually there will be a tunneling to the decompactified case.

Interestingly, the cases of figure 8 can be realized in string theory. In [129][130] type IIB with six dimensions compactified on Calabi-Yau spaces were studied. By turning on fluxes, the complex moduli of the internal spaces were stabilized, and in [131] it was noted that non perturbative string corrections can also fix the volume of the internal space. Hence we end up in a situation described by figure 8. Then, the authors of [131] added a number of $\bar{D}3$ branes, which increased the energy and corrected the potential to the one in figure 8.

Interestingly, the model also provides a way of realizing brane inflation. The trick is to make use of the fact that the $\bar{D}3$ branes are sitting at fixed positions on the internal manifold at the bottom of deep throats. If we add some $D3$ branes these will move down the throats attracted by the $\bar{D}3$ branes. Thanks to the redshift at the bottom of the throats, the problem of achieving slow roll in a compact dimension that I discussed previously is circumvented, [132].

It remains to construct realistic models within string theory that provide the right amount of inflation, the correct cosmological constant of today, as well as realistic particle physics. But the indications are certainly there that it should be possible. Are there any generic predictions? Most of the D-brane based string models discussed in the recent literature has an inflationary scale that is rather low. This means that ϵ is essentially zero and any deviation from scale invariance comes from η. From an observational point of view this is slightly disappointing for two reasons. First, a too small ε implies that contributions to the CMBR fluctuations from gravitational waves will be non-observable. Second, the magnitude of transplanckian effects in the CMBR, in the simplest and most generic scenarios, will be beyond detection. It is therefore of great interest to find out whether an almost vanishing ε is a robust prediction of string theory.

5.6 Holography

Holography is an intriguing possibility for finding connections between the smallest scales and cosmology. I will not give a review of all the various attempts to apply holography to cosmology. Some of the more original and interesting are discussed in [137]. Instead, I will describe a number of important and general features of holography that I find important to keep in mind. The subject is, unfortunately, full of contrary claims and confusions, and my aim is to put the subject on as solid ground as possible.

I will start out with a discussion of entropy bounds and the question of whether such bounds can provide useful restrictions on cosmology, not available by other means. My conclusion will be negative. Then I will proceed with a discussion of more intricate questions like complementarity. Here, the answer is not as clear cut, but my conclusion will, nevertheless, be that there is no known mechanism for how such effects could be made visible.

Holography has its origin in black hole physics and the discovery in the 70's by Bekenstein that black holes carry an entropy proportional to the area of the horizon, [133]. Bekenstein further argued that there are general bounds on the amount of entropy that can be contained in matter. The entropy bound, due to Bekenstein, that will serve as a starting point for my discussion states that in asymptotically flat space, [134], is

$$S \leq S_B = 2\pi E R, \tag{374}$$

where E is the energy contained in a volume with radius R. This is the *Bekenstein bound*. There are several arguments in support of the bound when gravity is weak [135], and it is widely believed to hold true for all reasonable physical systems. Furthermore, in the case of a black hole where $R = 2El_{pl}^2$, we have an entropy given by

$$S_{BH} = \frac{A}{4l_{pl}^2} = \frac{\pi R^2}{l_{pl}^2}, \tag{375}$$

which exactly saturates the Bekenstein bound. We will consequently put $\hbar = c = 1$, but explicitly write the Planck length, $l_{pl} = \sqrt{\frac{G\hbar}{c^3}}$, to keep track of effects due to gravity.

Beginning with [136], there have been many attempts to apply similar entropy bounds to cosmology and in particular to inflation, [137]. The idea has been to choose an appropriate volume and argue that the entropy contained within the volume must be limited by the area. An obvious problem in a cosmological setting is, however, that for a constant energy density a bound of this type will always be violated if the radius R of the volume is chosen to be big enough. In fact, this observation has been used to argue, choosing appropriate volumes, that holography puts meaningful limits on, e.g., inflation. However, as was explained in [138], it is not reasonable to discuss radii which are larger than the Hubble radius in the expanding universe. See also [139]. This, then, suggests that the maximum entropy in a volume of radius $R > r$, where r is the Hubble radius, is obtained by filling the volume with as many Hubble volumes as one can fit – all with a maximum entropy of $\frac{\pi r^2}{l_{pl}^2}$. This gives rise to the *Hubble bound*, which states that

$$S < S_H \sim \frac{R^3}{r^3}\frac{r^2}{l_{pl}^2} = \frac{R^3}{r l_{pl}^2}. \tag{376}$$

The introduction of the Hubble bound removes many of the initial confusions in the subject of holographic cosmology.

The Hubble bound is a bound on the entropy that can be contained in a volume much larger than the Hubble radius. It is, therefore, a bound that gives measurable consequences only if inflation stops allowing scales larger than the inflationary Hubble radius to become visible. Clearly, the notion of a cosmological horizon, and its corresponding area, does not play an important role from this point of view.

If we, on the other hand, want to discuss things from the point of view of what a local observer, who do not have time to wait for inflation to end, can measure, we must be more careful. In this case one has a cosmological horizon with an area that it is natural to give an entropic interpretation [140]. Since the area of the horizon grows when matter is passing out towards the horizon, from the point of view of the local observer, it is natural to expect the horizon to encode information about matter that, in its own reference frame, has passed to the *outside* of the cosmological horizon of the local observer. From the point of view of the observer, the matter will never be seen to leave but rather become more and more redshifted. The outside of the cosmological horizon should, therefore, be compared with the inside of a black hole. It follows that the horizon only indirectly provides bounds on entropy within the horizon as is nicely exemplified through the *D-bound* introduced in [141]. The cosmological horizon area in a de Sitter space with some extra matter is smaller than the horizon area in empty space. If the matter passes out through the horizon, the increase in area can be used to limit the entropy content in matter. This is the content of the D-bound which turns out to coincide with the Bekenstein bound. The D-bound, therefore has not, necessarily, that much to with de Sitter space or cosmology. It is more a way to use de Sitter space to derive a constraint on matter itself.

Let me now explain the nature and relations between the various entropy bounds a little bit better. In particular on what scales the entropy is stored. If we assume that all entropy is stored on short scales smaller than the horizon scale r, we can consider each of the horizon bubbles separately and use the Bekenstein bound (or D-bound) on each and everyone of these volumes. We conclude from this that the entropy, under the condition that it is present only on small scales, is limited by

$$S < S_{LB} = 2\pi E r, \tag{377}$$

which I will refer to as the *local Bekenstein bound*. It is interesting to compare this result with the entropy of a gas in thermal equilibrium. One then finds $S_g \lesssim Er$ for high temperatures where $T \gtrsim 1/r$, and $S_g \gtrsim Er$ for low temperatures where $T \lesssim 1/r$. This is quite natural and a consequence of the fact that most of the entropy in the gas is stored in wavelengths of the order of $1/T$. This means that the entropy for low temperatures is stored mostly in modes larger than the Hubble scale and can therefore violate the local Bekenstein bound S_{LB}.

The size of the horizon therefore limits the amount of information on scales larger than the Hubble scale, or, more precisely, the large scale information that once was accessible to the observer on small scales. If the horizon is smaller than its maximal value this is a sign that there is matter on small scales, and the difference limits the entropy (or information) stored in the matter. This is the role of the D-bound. We conclude, then, that a system with an entropy in excess of S_{LB} (but necessarily below S_H) must include entropy on scales larger than the horizon scale.

While the entropy bounds above are rather easy to understand, the way entropy can flow and change involve some more subtle issues. In the case of a diluting gas the expansion of the universe implies a flow of entropy out through the horizon, but as the gas eventually is completely diluted the flow of entropy taps off. Whether or not the horizon radius is changing, one will never be able to violate the Hubble bound or get an entropy flow through an apparent horizon violating the bound set by the area. A potentially more disturbing situation is obtained if we consider an empty universe (apart from a possibly changing cosmological constant), which can be traced arbitrarily far back in time, with entropy generated through the quantum fluctuations that are of importance for the CMBR. As discussed in several works, [103][142], there is an entropy production that can be associated with these fluctuations and one can worry that this will imply an entropy flow out through the horizon that eventually will exceed the bound set by the horizon. This is the essence of the argument put forward in [143].

To understand this better, one must have a more detailed understanding of the cause of the entropy. Entropy is always due to some kind of coarse graining where information is neglected. In the case of the inflationary quantum fluctuations we typically imagine, as I have explained, that the field starts out in a pure state – defined by some possibly transplackian physics – with a subsequent unitary evolution that keeps the state pure for all times. This is true whether we take the point of view of a local observer or use the global FRW-coordinates. To find an entropy we must introduce a notion of coarse graining. Various ways of coarse graining have been proposed, but they all imply an entropy that grows as the state gets more and more squeezed, [103][142]. It can be shown that most of this entropy is produced at large scales (when the modes are larger than the horizon), and well below the Hubble bound.

This is all in terms of the FRW-coordinates, but let us now take the point of view of the local observer. In this case the freedom to coarse grain is more limited. In order to generate entropy we must divide the system into two subsystems and trace out over one of the subsystems in order to generate entropy in the other. As an example consider a system with N degrees of freedom divided into two subsystems with N_1 and N_2 degrees of freedom, respectively, with $N = N_1 + N_2$ and $N_2 > N_1$. If the total system is in a pure state it is easy to show that the entropy in the larger subsystem is limited by the number of degrees of freedom in the smaller one, i.e. $S_2 < \ln N_1$.[11] Applied to our case, this means that the entropy flow towards the horizon must be balanced by other matter with a corresponding ability to carry entropy within the horizon. Since the amount of such matter is limited by the D-bound, the corresponding entropy flow is also limited. As a consequence, there can not be an accumulated flow of entropy out towards the horizon that is larger than the area of the horizon. For a similar conclusion see [144]. This does not mean that inflation can not go on for ever, nor that there can not be a steady production of entropy on large scales, but it does imply that the local observer will not be able to do an arbitrary amount of coarse graining.

To summarize: *from a local point of view the production of entropy in quantum fluctuations is limited by the ability to coarse grain; from a global point of view entropy is created on scales larger than the Hubble scale.*

[11] A simple proof can be found in [145] in the context of the black hole information paradox.

I have argued that holography, in the sense of putting limits on the entropy, does not constrain cosmology in any new way. It might still be a useful principle, but it does not contain anything beyond what is contained in the Bekenstein bound and the generalized second law which, in turn, seem to be automatically obeyed by the ordinary laws of physics. If we want to find truly new effects, we must go one step further and turn to the principle of *complementarity.* I will therefore investigate the possibilities of an *information paradox* and compare with the corresponding situation in the case of black holes.

In black hole physics, the emerging view is that a kind of complementarity principle is at work implying that two observers, one travelling into a black hole and the other remaining on the outside, have very different views of what is going on. According to the observer staying behind, the black hole explorer will experience temperatures approaching the Planck scale close to the horizon, and as a consequence, the black hole explorer will be completely evaporated and all information transferred into Hawking radiation. According to the explorer herself, however, nothing peculiar happens as she crosses the horizon. As explained in [146], the apparent paradox is resolved when one realizes that the two observers can never meet again to compare notes. Any attempts of the observers to communicate again, after the outside observer have extracted the information from the Hawking radiation, will necessarily make use of planckian energies and presumably fail.

An interesting question to pose is whether a similar mechanism could be at work also in de Sitter space. In order to investigate such a possibility, we will consider a scenario where at some moment in time the de Sitter phase is turned off and replaced by a non-accelerated $\Lambda = 0$ phase with ordinary matter. That is, an inflationary toy model. A possible information paradox, comes about if one assumes that an object receding towards the de Sitter horizon of an inertial de Sitter observer, will return its information content to the observer in the form of de Sitter radiation. If the cosmological constant turns off, the object itself will eventually return to the observers causal patch, and one has the threat of a duplication of information and therefore a paradox.

To come to terms with the paradox, let us focus on what an observer actually would see as an object recedes towards the horizon, [147]. Since the rate of the photons (emerging from the horizon) received by our observer is of order $1/R$, the time it would take for her to see the object burn will be extremely long. To find out how long, we will investigate what actually happens to the object (according to the observer). To do that we think of the horizon as an area consisting of R^2/l_{pl}^2 Planck cells, and remember that the photon has a wavelength of order Planck scale when emitted and can indeed resolve specific Planck cells.

Now, let us assume the object in question to be something really simple, with an information content much smaller than the R^2 number of degrees of freedom of the horizon. This would mean that only a few of the Planck cells are involved in encoding the object. In the extreme case of an object with entropy of $\mathcal{O}(1)$, one would need to wait until of the order R^2 photons have been emitted to be reasonably sure to see a photon coming from the burning of the object. In the other extreme, one can think of an object consisting of the order R^2 degrees of freedom. In this case it is clear that one has to wait until of the order R^2 photons have been emitted, in order for all parts of the object to have been burnt. Regardless of the size of the object, one has, therefore, to wait a time,

$$\tau \sim \frac{R^2}{l_{pl}^2} R = \frac{R^3}{l_{pl}^2} \sim \frac{1}{T^3 l_{pl}^2}, \tag{378}$$

in order to actually see the destruction.

If we now abruptly turn off the de Sitter phase, and let it be followed by a more standard cosmological evolution, we expect the object to eventually return to the observers causal patch at some time in the future [148]. If the time we wait before turning of inflation is shorter then the estimate above, this causes no problem. The object simply becomes visible again with a negligible amount of de Sitter radiation emitted. If we wait longer the situation is more confusing. The time we have estimated is the time it takes for an object to be irreversibly lost to the horizon, and it would be inconsistent for an unharmed, information loaded object to come back. After all, we have, with our own eyes, seen the object burn. This is, in fact, just the information paradox.

Luckily, the time scale we have estimated above is long enough for several interesting effects to take place which have the potential of removing the paradox. One argument goes as follows. Since the situation relation between the object and the observer is symmetric, it is clear that the object will be in as good, or bad, shape as the observer. Indeed, considering the symmetric situation we have between the observer and the object (being for example another observer) and the fact that they can meet again some time after the de Sitter phase has turned off, seems to imply that the estimated time should be the same for local objects as for those who approach the horizon, even from the perspective of one single observer.

So, let us now try to estimate the time it takes to break down a local object, bound to the observer. To do this, we reconsider the possibility that local interactions do give rise to a breakdown, but only if we take physics near the Planck scale into account. With an interaction rate given by $\Gamma = \sigma n v$, where the cross section is given by $\sigma \sim l_{pl}^2$, the number density of the radiation $n \sim T^3 \sim 1/R^3$ and the relative velocity $v = c = 1$, one finds the typical time τ it takes for this process to occur to be

$$1 \sim \sigma n v \tau \sim l_{pl}^2 \cdot 1/R^3 \cdot 1 \cdot \tau \;\Rightarrow\; \tau \sim \frac{R^3}{l_{pl}^2}. \tag{379}$$

This coincides, up to orders of one, with the previous result. Therefore, regardless of whether local objects or objects falling towards the horizon are concerned, the survival time will be the same. We argued above that this must be the case based on the symmetry between the observer and the object and by noting that, if the de Sitter phase is only temporary, they will eventually meet again. We find it encouraging that the above results are in agreement with this assessment.

The above analysis provides a possible escape route from the information paradox, since, as I have argued, it is very difficult for an observer to exist long enough to actually see any object being fully burnt by Hawking radiation. But this is not all, as observed in [148] there is a further obstacle to experiencing an information paradox. It can be shown that the return time for an object that has been falling towards the horizon a time $\tau \sim \frac{R^3}{l_{pl}^2}$, is of the order of the Poincare recurrence time, $\sim e^{R^2/l_{pl}^2}$ of the de Sitter space. That is, it exceeds the Poincare recurrence time of the detector.

What are the implications for inflation? In inflation the Hubble constant is constrained from observations to be no larger than $H \sim 10^{-4}\ M_4$. With this input the thermalization time for non-thermal excitations (α-vacua included) is found to be of order $\tau \sim R^3 = 1/H^3 \sim 10^{12}\ t_{pl}$. Comparing this with the time needed for the required number of e-foldings, which for 70 e-foldings is $t_{infl} \sim 70/H \sim 7 \cdot 10^5\ t_{pl}$, one concludes that the thermalization time allows for visible effects of non-thermal behavior in the CMBR, with room to spare. This is good news for the transplanckian signatures. On the other hand, with fluctuations leaving the horizon so close to the end of inflation, effects from holography and complementarity are expected to be subtle.

A fair conclusion is to say that so far holography has not yielded any useful restrictions on cosmology.

6 Trans-Planckian Effects on CMB

Inflation has nowadays become a standard ingredient for the description of the early Universe (see, e.g., Refs. [151]). In fact, it solves some of the problems of the standard big-bang scenario and also makes predictions about cosmic microwave background radiation (CMBR) anisotropies which are being measured with higher and higher precision. Further, it has been recently suggested that inflation might provide a window towards *trans*-Planckian physics [152] (for a partial list of subsequent works on this subject, see Refs. [158, 153, 154, 155, 156, 157]). The reason for this is that inflation magnifies all quantum fluctuations and, therefore, red-shifts originally trans-Planckian frequencies down to the range of low energy physics. This causes two main concerns: first of all, there is currently no universally accepted (if at all) theory of quantum gravity which allows us to describe the original quantum fluctuations in such an high energy regime; further, it is not clear whether the red-shifted trans-Planckian frequencies can indeed be observed with the precision of present and future experiments.

Regarding the first problem, one can take the pragmatic approach of modern renormalization theory and assume that quantum fluctuations are effectively described by quantum field theory after they have been red-shifted below the scale of quantum gravity, henceforth called Λ, and forget about their previous dynamics. Further, one can also take Λ as a constant throughout the evolution of the (homogeneous and isotropic) Universe, thus implicitly assuming the existence of some preferred reference frame (class of "cosmological" observers). The second problem is instead more of a phenomenological interest and needs actual investigation to find the size of corrections to the CMBR. It then seems that the answer depends on the details of the model that one considers and no general consensus has been reached so far. In fact, in Refs. [153, 154] it is claimed that such corrections can be at most of order $(H/\Lambda)^2$, where H is the Hubble parameter, hence too small to be detected. However, corrections are estimated of order H/Λ in Refs. [155, 156, 157]. Let us note that the first problem also plays an important role in this phenomenological respect, since it is the unknown trans-Planckian physics which fixes the "initial conditions" for the effective field theory description.

In Ref. [157], a principle of least uncertainty on the quantum fluctuations at the time of emergence from the Planckian domain (when the physical momentum $p \sim \Lambda$) was imposed. Without a good understanding of physics at the Planck scale, this can be regarded as an empirical way of accounting for new physics. Such a prescription fixes the initial vacuum (independently) for all frequency modes, and subsequent evolution is then obtained in the *sub*-Planckian domain by means of standard Bogolubov transformations (of course, neglecting the back-reaction) in de-Sitter space-time. In the present paper, we apply the same approach as in Ref. [157] to power-law inflation. This will allow us to check the final result against an inflationary model with time-dependent Hubble parameter.

6.1 The sub-Planckian effective theory

On the homogeneous and isotropic background

$$ds^2 = a^2(\eta)\,\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right] \tag{380}$$

the spatial Fourier components of the (rescaled) scalar field $\mu = a\,\phi$ (as well as tensor perturbations μ_T) satisfy

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0 \tag{381}$$

where primes denote derivative with respect to the conformal time $-\infty < \eta < 0$.

The index k is related to the physical momentum p by $k = a\,p$. Thus, a given mode with energy above the Planck scale in the far past would cross the fundamental scale Λ at the time η_k when

$$k = a(\eta_k)\,\Lambda \tag{382}$$

Strictly speaking, it is incorrect to regard such a mode as existing for $\eta < \eta_k$, since we do not have a theory for that case. What we will in fact consider is just the evolution for $\eta > \eta_k$.

6.2 Minimum uncertainty principle

Following Ref. [157], we shall impose that the mode k is put into being with minimum uncertainty at $\eta = \eta_k$, that is the vacuum satisfies in the Heisenberg picture (for the details see, e.g., Ref. [159])

$$\hat{\pi}_k(\eta_k)|\,0\,\rangle = i\,k\,\hat{\mu}_k(\eta_k)|\,0\,\rangle \tag{383}$$

where

$$\pi_k = \mu_k' - \frac{a'}{a}\,\mu_k \tag{384}$$

is the Fourier component of the momentum π conjugate to μ. We can write the scalar field and momentum at all times in terms of annihilation and creation operators for time dependent oscillators

$$\begin{aligned} \hat{\mu}_k(\eta) &= \tfrac{1}{\sqrt{2k}}\left[\hat{a}_k(\eta) + \hat{a}^\dagger_{-k}(\eta)\right] \\ \hat{\pi}_k(\eta) &= -i\sqrt{\tfrac{k}{2}}\left[\hat{a}_k(\eta) - \hat{a}^\dagger_{-k}(\eta)\right] \end{aligned} \tag{385}$$

The oscillators can be expressed in terms of their values at the time η_k through a Bogoliubov transformation

$$\begin{aligned} \hat{a}_k(\eta) &= u_k(\eta)\,\hat{a}_k(\eta_k) + v_k(\eta)\,\hat{a}^\dagger_{-k}(\eta_k) \\ \hat{a}^\dagger_{-k}(\eta) &= u_k^*(\eta)\,\hat{a}^\dagger_{-k}(\eta_k) + v_k^*(\eta)\,\hat{a}_k(\eta_k) \end{aligned} \tag{386}$$

Substituting this expression in (385) we obtain

$$\hat{\mu}_k(\eta) = f_k(\eta)\,\hat{a}_k(\eta_k) + f_k^*(\eta)\,\hat{a}^\dagger_{-k}(\eta_k)$$
$$i\,\hat{\pi}_k(\eta) = g_k(\eta)\,\hat{a}_k(\eta_k) - g_k^*(\eta)\,\hat{a}^\dagger_{-k}(\eta_k) \tag{387}$$

where

$$f_k(\eta) = \tfrac{1}{\sqrt{2k}}\,[u_k(\eta) + v_k^*(\eta)]$$
$$g_k(\eta) = \sqrt{\tfrac{k}{2}}\,[u_k(\eta) - v_k^*(\eta)] \tag{388}$$

and $f_k(\eta)$ is a solution of the mode equation (381). The condition (383) then reads

$$v_k(\eta_k) = \sqrt{\frac{k}{2}}\,f_k^*(\eta_k) - \frac{1}{\sqrt{2\,k}}\,g_k^*(\eta_k) = 0 \tag{389}$$

This requirement, together with the normalization condition

$$|u_k|^2 - |v_k|^2 = 1 \tag{390}$$

is sufficient to determine uniquely the initial state at $\eta = \eta_k$. The subsequent time evolution is then straightforward and one can estimate the power spectrum of fluctuations at a later time $\eta \gg \eta_k$ after the end of inflation,

$$P_\phi = \frac{P_\mu}{a^2} = \frac{k^3}{2\,\pi^2\,a^2}\,|f_k(\eta)|^2 \tag{391}$$

The above general formalism was applied to de-Sitter space-time in Ref. [157]. For that case, one has $a = -1/H\,\eta$ and the nice feature follows that

$$k\,\eta_k = -\frac{\Lambda}{H} \tag{392}$$

is a constant independent of k. This, in turn, allows to obtain an analytic expression for the initial state which satisfies Eq. (389) by suitably expanding for H/Λ small (i.e., $\eta_k \to -\infty$ for all k). We shall instead consider power-law inflation, where such a simplification does not occur.

6.3 Power-law inflation

In the proper time $dt = a\,d\eta$, power-law inflation is given by a scale factor $a \sim t^p$, in which $t_{\mathrm{p}} < t < t_{\mathrm{o}}$, with t_{p} of the order of the Planck time, $t_{\mathrm{o}} \gg t_{\mathrm{p}}$ is the time of the end of inflation, and $p \gg 1$ [155]. Upon changing to the conformal time, one obtains for the scale factor

$$a(\eta) = \left(\frac{\bar{\eta}}{\eta}\right)^q \tag{393}$$

where $q = p/(p-1)$, $\eta_{\rm p} < \eta \le \eta_{\rm o} < 0$ ($\eta_{\rm o}$ is the end of inflation) and the Hubble parameter is given by

$$H(\eta) = -q\,\frac{\eta^{q-1}}{\bar{\eta}^q} \tag{394}$$

The condition (382) now becomes

$$k\,\eta_k = \bar{\eta}\,\Lambda^{\frac{1}{q}}\,k^{1-\frac{1}{q}} \tag{395}$$

Since the right hand side depends on k (unless $q = 1$), it can be large or small depending on k, and an expansion for $-k\,\eta_k$ large is not generally valid.

For the scale factor (393) one has

$$\frac{a''}{a} = \frac{q\,(q+1)}{\eta^2} \tag{396}$$

and Eq. (381) can be solved exactly. One can write the general solution as

$$f_k = A_k\,\sqrt{-\eta}\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,\sqrt{-\eta}\,Y_{q+\frac{1}{2}}(-k\,\eta) \tag{397}$$

where J_ν and Y_ν are Bessel functions of the first and second kind [12], and A_k and B_k are complex constants. The Bogolubov coefficients are then given by

$$\begin{aligned} u_k &= \sqrt{-\frac{k\,\eta}{2}}\,\Big[A_k\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q+\frac{1}{2}}(-k\,\eta) \\ &\qquad -i\left(A_k\,J_{q-\frac{1}{2}}(-k\,\eta) + B_k\,J_{q-\frac{1}{2}}(-k\,\eta)\right)\Big] \\ v_k^* &= \sqrt{-\frac{k\,\eta}{2}}\,\Big[A_k\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q+\frac{1}{2}}(-k\,\eta) \\ &\qquad +i\left(A_k\,J_{q-\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q-\frac{1}{2}}(-k\,\eta)\right)\Big] \end{aligned}$$

The constants A_k and B_k can now be fixed by imposing the normalization condition (390) and Eq. (389). From Eq. (390) one obtains

$$A_k\,B_k^* - A_k^*\,B_k = -i\,\frac{\pi}{2} \tag{398}$$

and from Eq. (389),

$$A_k = -\frac{\bar{Y}_{q+\frac{1}{2}} + i\,\bar{Y}_{q-\frac{1}{2}}}{\bar{J}_{q+\frac{1}{2}} + i\,\bar{J}_{q-\frac{1}{2}}}\,B_k \tag{399}$$

where $\bar{J}_\nu \equiv J_\nu(-k\,\eta_k)$ and $\bar{Y}_\nu \equiv Y_\nu(-k\,\eta_k)$. From the combined equations one then obtains

$$\begin{aligned} |A_k|^2 &= -\frac{\pi^2}{8}\,k\,\eta_k\,\left[\bar{Y}^2_{q+\frac{1}{2}} + \bar{Y}^2_{q-\frac{1}{2}}\right] \\ |B_k|^2 &= -\frac{\pi^2}{8}\,k\,\eta_k\,\left[\bar{J}^2_{q+\frac{1}{2}} + \bar{J}^2_{q-\frac{1}{2}}\right] \\ \mathrm{Re}\,(A_k\,B_k^*) &= \frac{\pi^2}{8}\,k\,\eta_k\,\left(\bar{Y}_{q+\frac{1}{2}}\,\bar{J}_{q+\frac{1}{2}} + \bar{Y}_{q-\frac{1}{2}}\,\bar{J}_{q-\frac{1}{2}}\right) \end{aligned} \tag{400}$$

[12] We remark that such functions are real in the chosen domain of η.

We are finally in the position to compute the exact power spectrum at the time $\eta \le \eta_{\rm o}$, which is given by

$$
\begin{aligned}
P_\phi \quad &= \quad \frac{\eta_k\,\eta^{2\,q+1}\,k^4}{16\,\bar\eta^{2\,q}}\times \\
&\left\{\left[\bar Y_{q+\frac{1}{2}}\,J_{q+\frac{1}{2}}(-k\,\eta)-\bar J_{q+\frac{1}{2}}\,Y_{q+\frac{1}{2}}(-k\,\eta)\right]^2+\left[\bar Y_{q-\frac{1}{2}}\,J_{q+\frac{1}{2}}(-k\,\eta)-\bar J_{q-\frac{1}{2}}\,Y_{q+\frac{1}{2}}(-k\,\eta)\right]^2\right\}
\end{aligned}
\tag{401}
$$

The above expression can then be estimated for $\eta = \eta_{\rm o}$ (end of inflation) and $\eta_{\rm o} \to 0^-$. Since for $-k\,\eta_{\rm o} \ll 1$, the Bessel $Y_{q+\frac{1}{2}}$ dominates, one obtains, to leading order,

$$
P_\phi \simeq \frac{k^{3-2\,q}\,|\eta_k|}{2^{3-2\,q}\,|\bar\eta|^{2\,q}}\,\frac{\bar J^2_{q+\frac{1}{2}}+\bar J^2_{q-\frac{1}{2}}}{\sin^2\left(\pi\left(q+\frac{1}{2}\right)\right)\,\Gamma^2\left(\frac{1}{2}-q\right)}
\tag{402}
$$

If one further takes the limit $k\,\eta_k \to -\infty$ and expands to leading order for k small, the power spectrum becomes

$$
\begin{aligned}
P_\phi \quad &\simeq \frac{2^{2\,q-2}\,k^{2-2\,q}}{\pi\,|\bar\eta|^{2\,q}\,\cos^2(\pi\,q)\,\Gamma^2\left(\frac{1}{2}-q\right)}\left[1-\frac{H_k}{\Lambda}\,\sin(2\,\bar\eta\,\Lambda^{\frac{1}{q}}\,k^{1-\frac{1}{q}}+\pi\,q)\right] \\
&= P_{\rm PL}\left[1-\frac{H_k}{\Lambda}\sin\left(q\,\frac{2\,\Lambda}{H_k}+q\,\pi\right)\right]
\end{aligned}
\tag{403}
$$

where $H_k \equiv H(\eta_k)$ and we have factored out the expression $P_{\rm PL} \sim k^{2-2\,q}$ of the spectrum for power-law inflation [160] in the small $k\,\eta_{\rm o}$ regime (super-horizon scales) [161]. This result is thus in agreement with what was obtained for de-Sitter space-time in Ref. [157], as one can easily see by taking the limit $q \to 1$ ($p \to \infty$).

However, as we mentioned previously, $k\,\eta_k$ is not independent of k [see Eq. (395)]. The above expression therefore does not hold for all k, but just for those such that $-k\,\eta_k$ is large. Since it is very difficult to obtain general analytic estimates of the exact power spectrum for general values of k, in Fig. 10 we plot, for the exact expression of P_ϕ in Eq. (402), the ratio

$$
R_q = \frac{P_\phi - P_{\rm PL}}{P_{\rm PL}}
\tag{404}
$$

for $q = 2$, $3/2$ and $4/3$ (similar results are obtained for all values of $q \neq 1$). It is clear that for small k the oscillations in P_ϕ are relatively large around $P_{\rm PL}$, and this is precisely due to the dependence of $k\,\eta_k$ on k. The oscillations are then progressively damped for large k according to the approximate expression in Eq. (403) (and analogously to what is found in de-Sitter [157]). Note also that for increasing p (i.e. $q \to 1^+$), the wavelength of oscillations increases, as is shown in the approximation (403). Of course, one must keep in mind that only sub-horizon scales matter at the time η_k, for which $k \gg a\,H$, that is $|k\,\eta_k| \gg q$ (say of order λ). Hence, the relevant regions for different values of q are those with $k > \lambda^{q/(q-1)}$. In Fig. 10 we have set $\lambda = 10$ in order to obtain reasonably overlapping ranges, and the amplitude of the oscillations turns out to be of the order of a few percents inside the physical ranges (larger values of λ imply smaller oscillations). We considered a minimum uncertainty principle to fix, at an energy scale Λ, the vacuum of an effective (low energy) field theory. Such prescription involves the cut-off scale Λ for dealing with trans-Planckian energies, which therefore enters into the power spectrum of perturbations at later times. We have shown in some details that a Λ of the order of the Planck scale can affect appreciably the spectrum [see Eq. (403) and Fig. 10], in agreement with Refs. [155, 156, 157] by introducing a modulation of the spectrum, as may be clearly seen from the figure. This is a clear indication that trans-Planckian physics can lead to observable predictions in the cosmological models. We feel this is further evidence for the fact that trans-Planckian physics cannot be safely ignored in determining observable quantities such as features of the CMBR.

7 Baryogenesis

One of the most peculiar features of our Universe is the observed baryonic asymmetry. This can be conveniently characterized by the dimensionless number

$$\frac{n_B}{s} \equiv \eta \simeq 10^{-10} \tag{405}$$

where $n_B \equiv n_b - n_{\bar{b}}$ is the difference between the baryon and anti-baryon densities and s is the density of entropy. The consistency of primordial nucleosynthesis, which yields some of the most precise results in the standard model of cosmology, requires that η took the above value at the time when the light elements (*i.e.*, ^{3}He, ^{4}He, and ^{7}Li) were produced, and it is believed to have then remained the same up to the present epoch.

The necessary conditions for generating the baryonic asymmetry in quantum field theory were formulated by Sacharov in 1967 [162] (see also Ref. [163]) and can be summarized as follows:

1. Different interactions for particles and antiparticles, or, in other words, a violation of the C and CP symmetries;

2. Non-conservation of the baryonic charge;

3. Departure from thermal equilibrium.

The last condition results from an application of the CPT theorem [164, 165]. In fact, CPT invariance of quantum field theory in a static Minkowski space-time ensures that the energy spectra for baryons and anti-baryons are identical, leading consequently to identical distributions at thermal equilibrium. This explains why the baryon number asymmetry was required to be generated out of thermal equilibrium.

The so called mechanism of spontaneous baryogenesis [166, 167] uses the natural (strong) CPT non-invariance of the Universe during its early history to bypass this third condition. We know that an expanding Universe at finite temperature violates both Lorentz invariance and time reversal, and this can lead to effective CPT violating interactions [164, 165]. Thus the cosmological expansion of the early Universe leads us naturally to examine the possibility of generating the baryon asymmetry in thermal equilibrium. The main ingredient for implementing this mechanism is a scalar field ϕ with a derivative coupling to the baryonic current. If the current is not conserved and the time derivative of the scalar field has a non-vanishing expectation value, an effective chemical potential with opposite signs for baryons and anti-baryons is generated leading to an asymmetry even in thermal equilibrium.

The brane-world model with two branes proposed by Randall and Sundrum (RS) in Ref. [168] contains a metric degree of freedom called the *radion* which determines the distance between the two branes and appears as a scalar field ϕ on the branes. Cosmological solutions have also been examined rather extensively in this context. In particular, it has been shown that, when matter is added on one (or both) of the two branes, the standard Friedmann equation for the scale factor of the Universe is recovered (with possible corrections) provided the radion is suitably stabilized (see for example Refs. [169]-[172] and References therein). In this brane-world model, we therefore have both a scalar field (the radion) and the cosmological evolution as required by spontaneous baryogenesis, and we shall show that the radion field does in fact couple differently with baryons and anti-baryons. This scenario might therefore naturally reproduce the observed baryonic asymmetry [13].

In Section 7.1, we review in some details the mechanism of spontaneous baryogenesis. The cosmological solutions in the RS framework are discussed in Section 7.2 where the process of spontaneous baryogenesis driven by the radion is presented in general. Some more specific examples are also reported in Section 7.3. We then conclude and comment on our results.

We shall use units with $c = \hbar = k_{\rm B} = 1$, where $k_{\rm B}$ is the Boltzmann constant.

7.1 The Spontaneous Baryogenesis Mechanism

To illustrate the mechanism of spontaneous baryogenesis (see, *e.g.*, Refs. [166, 167] and [180]-[182]) let us consider a theory in which a neutral scalar field ϕ is coupled to the baryonic current J_B^μ according to the Lagrangian density

$$L_{\rm int} = \frac{\lambda'}{M_{\rm c}}\, J_B^\mu\, \partial_\mu \phi \tag{406}$$

where λ' is a coupling constant and M_c is a cut-off mass scale in the theory (presumably smaller than the Planck mass $M_{\rm Pl}$). Let us assume that ϕ is homogeneous, so that only the time derivative term contributes,

$$L_{\rm int} = \frac{\lambda'}{M_{\rm c}} \dot{\phi}\, n_B \equiv \mu(t)\, n_B \tag{407}$$

where $n_B = J_B^0$ is the baryon number density and $\mu(t)$ is to be regarded as an effective time-dependent chemical potential. This interpretation (see Ref. [183]) is valid if the current J_B^μ is not conserved (otherwise one could integrate the interaction term away) and if ϕ behaves as an external field which develops a slowly varying time derivative $\langle dot\phi \rangle \neq 0$ as the Universe expands. Since the chemical potential μ enters with opposite signs for baryons and anti-baryons, we have a net baryonic charge density in thermal equilibrium at the temperature T,

$$n_B(T;\mu) = \int \frac{d^3k}{(2\pi)^3}\, \left[f(k,\mu) - f(k,-\mu)\right] \tag{408}$$

where $\xi \equiv \mu/T$ is regarded as a parameter, and

$$f(k,\mu) = \frac{1}{\exp\left[\left(\sqrt{k^2+m^2} - \mu\right)/T\right] \pm 1} \tag{409}$$

is the phase-space thermal distribution [14] for particles with rest mass m and momentum k. For $|\xi| \ll 1$ we may expand Eq. (409) in powers of ξ to obtain

$$n_B(T;\mu) = \frac{g\,T^3}{6}\,\xi + O\left(\xi^2\right) \tag{410}$$

[13]Other mechanisms for baryogenesis in the context of brane-world models have been recently analyzed, e.g. in Refs. [173]-[179].

[14]Of course, the plus sign is for fermions and the minus sign for bosons.

where g is the number of degrees of freedom of the field corresponding to n_B. Upon substituting in for the expression of μ, one therefore finds

$$n_B(T;\mu) \simeq \frac{\lambda' g}{6\,M_c}\,T^2\,\langle\,\dot{\phi}\,\rangle \tag{411}$$

Regardless of the specific mechanisms which break baryon number conservation, we assume that there is a temperature T_F at which the baryon number violating processes become sufficiently rare so that n_B freezes out (T_F will in fact be called the *freezing temperature*). Once this temperature is reached as the universe cools down, one is left with a baryonic asymmetry whose value is given by Eq. (411) evaluated at $T = T_F$. The value of the parameter η remains unchanged in the subsequent evolution.

7.2 Radion Induced Spontaneous Baryogenesis

We have discussed how the mechanism of spontaneous baryogenesis may explain the observed baryonic asymmetry. We shall now argue that it might occur naturally in brane-world models. In particular, we shall consider the five-dimensional RS model of Ref. [168] perturbed by matter on one or both branes [170]-[172]. The reader is referred to Ref. [169] for more details on the framework and notation used hereafter.

In this model the metric can be written in the form

$$\begin{aligned} ds^2 = n^2(y,t)\,dt^2 - a^2(y,t)\,\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] - b^2(y,t)\,dy^2 \\ \equiv \tilde{g}_{AB}(x,y)\,dx^A\,dx^B \end{aligned} \tag{412}$$

where t is the time, x^i are the spatial coordinates along the branes and y is the extra-dimensional coordinate. In this formalism, the *Planck* brane is conventionally located at $y = 0$ and the *TeV* brane at $y = 1/2$. The Einstein equations are given by $G_{AB} = \kappa^2\,T_{AB}$, where $\kappa^2 = 1/(2\,M^3)$ and M is the five-dimensional Planck mass. The energy-momentum tensor T_{AB} contains a contribution from the bulk cosmological constant Λ of the form $T^{\rm bulk}_{AB} = \Lambda\,\tilde{g}_{AB}$ and a contribution from the matter on the two branes,

$$\begin{aligned} T_A^{\;B\ \rm branes} &= \frac{1}{b}\,\delta(y)\,{\rm diag}\left[V_* + \rho_*, V_* - p_*, V_* - p_*, V_* - p_*, 0\right] \\ &+\frac{1}{b}\,\delta(y-1/2)\,{\rm diag}\left[V + \rho, V - p, V - p, V - p, 0\right] \end{aligned} \tag{413}$$

where V_* is the (positive) tension of the Planck brane and V the (negative) tension on the TeV brane. We have correspondingly denoted by ρ_* and p_* the density and pressure of the matter localized on the positive tension (Planck) brane (assuming an equation of state of the form $p_* = w_*\,\rho_*$) and by ρ and p the density and pressure of the matter on the negative tension (TeV) brane. Once a stabilizing potential for the radion is included, the stress-energy tensor picks up an additional term and the solution of the Einstein equations may be written as a perturbation of the usual RS solution,

$$n(y) = a(y) = e^{-m_0\,b_0\,|y|} \tag{414}$$

with $V_* = 6\,m_0/\kappa^2 = -V$ and $\Lambda = -6\,m_0^2/\kappa$. We also recall that the constants b_0 and m_0 determine the effective four-dimensional Planck mass as $(8\,\pi\,G_N)^{-1} = M_{\rm Pl}^2 = (1-\Omega_0^2)/\kappa^2\,m_0$, where $\Omega_0 \equiv e^{-m_0\,b_0/2}$.

In order to obtain an effective action for the four-dimensional theory, one perturbs the metric about the RS solution in the form

$$
\begin{aligned}
a(t,y) &= a(t)\,e^{-m_0\,b(t)\,|y|}[1+\delta a(y,t)] \\
n(t,y) &= e^{-m_0\,b(t)\,|y|}[1+\delta n(y,t)] \\
b(t,y) &= b(t)[1+\delta b(y,t)]
\end{aligned}
\tag{415}
$$

and drops the metric perturbations which contribute only to second order in δa, δn and δb (see Refs. [170]-[172] for the effects of the latter). It is then useful to introduce the notation $\Omega(y,b(t)) = e^{-m_0\,b(t)\,|y|}$ and $\Omega_b \equiv \Omega(1/2,b(t))$ (Ω_b evaluated at $b=b_0$ is then given by Ω_0). By integrating over the fifth dimension, one obtains an effective action for the radion field. Further, upon examining the equations of motion for $b(t)$, one notes that, since Ω_b depends on b, the presence of matter on the two branes generates an effective potential for $b(t)$ given by

$$
V_{\rm eff}(b) = V_r(b) + \frac{f^4(b)}{4}\,\left[\rho_* - 3\,p_* + (\rho - 3\,p)\,\Omega_b^4\right]
\tag{416}
$$

with

$$
f(b) = \left(\frac{1-\Omega_0^2}{1-\Omega_b^2}\right)^{1/2}
\tag{417}
$$

The function $V_r = V_r(b(t))$ is the potential which would stabilize the radion at the value $b=b_0$ in the absence of matter. It can therefore be expanded near its minimum as [15]

$$
V_r(b) = \frac{1}{4}\,m_r^2\,\left(\frac{m_0\,b_0}{1-\Omega_0^2}\right)^2\,\Omega_0^2\,M_{\rm Pl}^2\,\left(\frac{b-b_0}{b_0}\right)^2
\tag{418}
$$

where

$$
m_r^2 = \frac{2\,(f^4\,V_r)''\,(b_0)}{3\,m_0^2\,M_{\rm Pl}^2\,\Omega_0^2}\,\left(1-\Omega_0^2\right)^2
\tag{419}
$$

and m_r is the effective radion mass. Thus the radion, in the presence of matter on the two branes, is stabilized to a shifted value $b_0 + \delta b$ determined by

$$
\frac{\delta b}{b_0} = \frac{1}{3}\left(\frac{1-\Omega_0^2}{m_0\,b_0}\right)\frac{\rho - 3p + \Omega_0^2\,(\rho_* - 3p_*)}{m_r^2\,\Omega_0^2\,M_{\rm Pl}^2}
\tag{420}
$$

[15]This expression follows from Eq. (4.12) of Ref. [169] by defining $\sqrt{3/2}\,\phi/\Lambda_W = m_0\,b$ (where $\Lambda_W = \Omega_0\,M_{\rm Pl} \simeq 1\,{\rm TeV}$).

where δb is the distance between the minima of $V_{\rm eff}$ with and without matter.

From the above expression, we see that, since the trace of the stress-energy tensor vanishes for radiation, even if the Universe on the TeV brane is in a radiation dominated era ($\rho \simeq 3\,p$), the radion evolution is determined by the behavior of massive matter on the TeV and Planck branes. This is the essential ingredient which allows for the possibility of inducing spontaneous baryogenesis. We note that the factor Ω_0^2 in front of ρ_* would make this term negligible for comparable energy densities on the TeV and Planck branes, but the fact that the natural energy scale on the Planck brane is of the order of $M_{\rm Pl}$ may nonetheless allow for a relevant contribution to baryogenesis from the Planck brane.

Let us now assume that the high energy Lagrangian for matter on the TeV brane contains an interaction term of the form given in Eq. (407),

$$L_{\rm int} = \lambda' \, m_0 \, \dot{b} \, n_B \tag{421}$$

where $\dot{b}$ now plays the role of $\langle\, \dot{\phi}\,\rangle$. Such a term is the same as that in Eq. (4.30) of Ref. [169]. On using Eq. (420) to estimate $\dot{b} \simeq \dot{\delta b}$, we finally obtain

$$\eta = \frac{m_0\,\dot{b}}{T} = \frac{1}{m_r^2\,\Omega_0^2\,M_{\rm Pl}^2} \left(\frac{1}{T}\right) \frac{d}{dt} \left[(\rho - 3\,p) + \Omega_0^2\,\left(\rho_* - 3\,p_*\right)\right] \tag{422}$$

Due to the expansion of the Universe, the time derivative on the right hand side of this equation will in general acquire a non vanishing (expectation) value and the baryonic symmetry is therefore dynamically broken in the model. We also note that the radion field is likely very massive [16], and one can then assume that the radion follows instantaneously any changes of the matter density.

7.3 Applications

In order to complete our analysis, we shall now estimate the baryonic asymmetry (422) at the freezing temperature $T_{\rm F}$ in three specific scenarios:

1. If the effect of matter on the Planck brane is negligible in Eq. (422), the condition to generate the observed baryonic asymmetry (405) can be estimated as

$$\left.\frac{1}{T}\,\frac{d}{dt}\,(\rho - 3\,p)\right|_{T_{\rm F}} > 10^{-10}\,{\rm TeV}^4 \tag{423}$$

   Let ρ_m be the energy density of any non-traceless component of the energy momentum tensor. By using the continuity equation and the Friedmann equation for a radiation dominated Universe up to an energy density of the order of $1\,{\rm TeV}^4$, which is roughly the limit of validity for the RS model, one obtains the requirement

$$\rho_m > 10^{-6}\,{\rm TeV}^4 \tag{424}$$

   for this mechanism to produce sufficient baryonic asymmetry.

[16] For example, if the radion is stabilized by the Goldberger-Wise mechanism [184], one has $m_r \simeq 1\,$TeV.

2. Since the Planck brane remains hidden, one can allow for a very large term proportional to $\dot{\rho}_*$ in Eq. (422) (we recall that matter energy on the Planck brane is allowed up to the Planck scale). The radion velocity would hence be larger than in the previous case, and the freezing temperature correspondingly lower. The bound in this case is

$$\frac{1}{T}\,\frac{d}{dt}\,(\rho_* - 3\,p_*)\bigg|_{T_{\rm F}} > 10^{-42}\,M_{\rm Pl}^4 \tag{425}$$

3. Another possibility is to consider the stage when the radion is still stabilizing towards the equilibrium value b_0 and the effect of matter on the branes is negligible. A typical radion velocity would be larger than in both previous cases, $m_0\,\dot{b} \simeq H(T)\,M_{\rm Pl}/\Lambda_W$, and

$$T > 10^2\,{\rm eV} \tag{426}$$

which allows the widest range of temperature among the three possibilities outlined.

Of course, the above list is not exhaustive and one could consider many other situations. For example, one could include more bulk fields or different couplings between the radion and brane fields. A complete analysis of all possible cases however goes beyond the scope of the present work and will not be given here. We have shown that the perturbations induced by the addition of matter on one (or both) of the two branes of a cosmological RS model with a stabilizing potential for the radion naturally lead to a non-vanishing expectation value for the velocity of the radion field. Since the latter couples with the baryonic current on the branes, this naturally induces the onset of spontaneous baryongenesis, as described by the general formula (422).

Having outlined the main ideas in the present paper, the next step would be to analyze all possible scenarios. For specific cases, it may in fact be possible to reproduce the observed baryonic asymmetry η in Eq. (405). Conversely, the required value of η can be viewed as a constraint that brane-world models must satisfy.

8 Brane Cosmology

Higher and higher precision data which are about to be collected in new experiments of particle physics and astrophysics in the next few years convey considerable attention to theories with extra dimensions. The main role of such theories, originally introduced in the 20's by Kaluza and Klein [186, 187], is to provide a connection between particle physics and gravity at some level. At a deeper level, string theory unifies all the interactions by means of some n-dimensional manifold (with $n > 4$) where the fundamental objects are supposedly living; at a more phenomenological level, models which assume the existence of extra dimensions, no matter their origin, are considered in order to solve some puzzles of particle physics, cosmology and astrophysics, giving rise to many possible observable consequences.

Originally proposed in order to solve the problem of the large hierarchy between Gravity and Standard Model scales, the Randall-Sundrum model of Ref. [188] (RS I) has acquired considerable relevance due to its stringy inspiration. It represents the prototype of the so-called brane-world and differs from previous models in that it constrains standard matter on a four-dimensional manifold (the brane) just letting gravity (and exotic matter) propagate everywhere. The RS I solution to the hierarchy problem needs one additional compactified (orbifolded) spatial dimension with two branes located at its fixed points, plus a negative cosmological constant filling the space between such branes (the bulk). The bulk cosmological constant Λ warps the extra dimension and generates the effective four-dimensional physical constants we measure. It was soon realized that the modifications to four-dimensional gravity induced by the fifth dimension may be reduced to such a short distance effect to be unobservable even in the presence of just one brane and infinite compactification radius (the RS II model of Ref. [189]).

The cosmological features of the RS models are nowadays being investigated even more than its particle physics consequences, due to the refined results lately obtained and to the major problems recent astrophysical data have revealed: the possible late time acceleration from supernovae, CMBR spectrum, dark matter and dark energy quests suggest either a full revision of the modern theoretical physics approach or the possibility of the existence of further, up to now ignored, ingredients such as the extra dimensions.

In particular the single brane RS II cosmological dynamics [190, 191] is known to generate $(\rho/V)^2$ corrections to standard Friedmann and acceleration equations, where ρ is the energy density of the fluid filling the brane and V is the constant brane tension. These corrections are negligible when $\rho \ll V$, the regime in which the RS II model is reliable and leads to standard cosmic evolution. The two brane RS I setup is much more involved: a stabilization mechanism for the distance between the branes, such as that of Ref. [192], is necessary to get the correct hierarchy in the absence of matter. Moreover, a bulk potential for the radion (the metric degree of freedom associated with the fifth dimension) is necessary to achieve solvable junction conditions when matter is present on the boundaries [193]. In this case, cosmological solutions to order ρ/V [194] are not sufficient to grasp the particular features of the background metric evolution originated by the extra dimension and one needs to investigate the effect of terms of order $(\rho/V)^2$ (as was done in Ref. [195]) or higher.

The aim of this article is to go beyond the first order approximation in brane cosmology for RS I models with two branes. Our approach will differ from Ref. [195] in that we do not consider a bulk scalar field to stabilize the radion but include an effective stabilizing potential directly into the equations (see also Ref. [196]). Consequently, our perturbative expansion is around the RS I solution. The calculations are then carried out in order to show how ρ^2 contributions to the four-dimensional Hubble parameter may affect the model (or may be unobservable). Such terms are expected as fingerprints of the fifth dimension in analogy with the single brane RS II framework. The latter case will also be studied as RS I in the limit when the distance between the branes diverges. Some hints about the possibility of an accelerated expansion driven by exotic fluids with pressure $p = w\,\rho$ and $w > 0$ will be presented, thus suggesting the necessity to go beyond the second order approximation.

The paper is organized as follows: in Section 9.1, we present the complete setup of the model under consideration; in Section 9.1 the second order ansatz is described and Einstein equations are perturbatively solved; in Section 9.2 cosmological consequences of the solutions are analyzed and compared to the known brane-world solutions; in Section 8.8 the analysis of the approximations is performed and, finally, in Section 8.9, some conclusions are drawn. For the five-dimensional metric g_{AB} we shall use the signature $(+,-,-,-,-)$, so that $g \equiv \det(g_{AB}) > 0$.

8.1 Einstein equations

Let us consider a RS I model perturbed by the presence of matter on the two branes. The bulk metric is given by

$$\begin{aligned} ds^2 &\equiv g_{AB}\,dx^A\,dx^B \\ &= n^2(y,t)\,dt^2 - a^2(y,t)\,dx^i\,dx^i - b^2(y,t)\,dy^2 \end{aligned} \tag{427}$$

The Einstein tensor for this metric is

$$G_{00} = 3\left\{\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a\,\dot b}{a\,b} - \frac{n^2}{b^2}\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 - \frac{a'\,b'}{a\,b}\right]\right\} \tag{428}$$

$$\begin{aligned} G_{ii} = &\frac{a^2}{b^2}\left[\left(\frac{a'}{a}\right)^2 + 2\frac{a'\,n'}{a\,n} - \frac{b'\,n'}{b\,n} - 2\frac{a'\,b'}{a\,b} + 2\frac{a''}{a} + \frac{n''}{n}\right] \\ &-\frac{a^2}{n^2}\left[\left(\frac{\dot a}{a}\right)^2 - 2\frac{\dot a\,\dot n}{a\,n} + 2\frac{\ddot a}{a} - \frac{\dot b}{b}\left(\frac{\dot n}{n} - 2\frac{\dot a}{a}\right) + \frac{\ddot b}{b}\right] \end{aligned} \tag{429}$$

$$G_{04} = 3\left[\frac{\dot a\,n'}{a\,n} + \frac{a'\,\dot b}{a\,b} - \frac{\dot a'}{a}\right] \tag{430}$$

$$G_{44} = 3\left\{\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{b^2}{n^2}\left[\frac{\dot a}{a}\left(\frac{\dot a}{a} - \frac{\dot n}{n}\right) + \frac{\ddot a}{a}\right]\right\} \tag{431}$$

where a prime denotes a derivative with respect to y and a dot a derivative with respect to the universal time t. The energy-momentum tensor in the bulk is that of an anti-de Sitter space with the addition of a term generated by a field which serves the purpose of stabilizing the distance between the two branes of the RS I model,

$$T^A_{\ B} = \Lambda\,g^A_{\ B} + \tilde T^A_{\ B} \tag{432}$$

where, as usual,

$$\tilde{T}_{AB} = -\frac{2}{\sqrt{g}} \frac{\delta \mathcal{L}_{\rm stab}}{\delta g^{AB}} \tag{433}$$

and $\mathcal{L}_{\rm stab}$ is the Lagrangian of the stabilizing field (for a scalar field, see e.g. Ref. [192, 194, 195]).

We shall consider the particular case in which the stabilizing mechanism can be effectively described by a harmonic potential for the radion, with an effective Lagrangian of the form [193]

$$\mathcal{L}_{\rm eff} = -\sqrt{g}\,\omega^2\,(b - b_0)^2 \equiv -\sqrt{g}\,U(b) \tag{434}$$

when the metric is written as in Eq. (525), and the potential U depends on the component g_{44}. The two 3-branes have opposite tensions and their contribution to the total energy-momentum tensor is given by

$$\begin{aligned} T^{A}_{i\,B} \;=\; & \frac{\delta(y - y_i)}{b} \\ & \times \mathrm{diag}\,(V_i + \rho_i, V_i - p_i, V_i - p_i, V_i - p_i, 0) \end{aligned} \tag{435}$$

where $i = p$, n, and $y_p = 0$ ($y_n = 1/2$) is the position of the positive (negative) tension brane. The Einstein equations in the bulk,

$$G_{AB} = k^2\,T_{AB} \tag{436}$$

form a system of four differential equations for the three independent functions $f_\alpha = (n, a, b)$. On using the Bianchi identity $\nabla_A G^{A0} = 0$, it is then straightforward to show that the three equations

$$\left\{ \begin{array}{l} G_{00} = k^2\,T_{00} \\ G_{04} = 0 \\ G_{44} = k^2\,T_{44} \end{array} \right. \tag{437}$$

are independent and form a complete set. This means that a solution to Eqs. (437) also solves the full set (436). Moreover, since G_{AB} is trivially conserved because of the Bianchi identities, the tensor T_{AB} is also automatically conserved regardless of the fact that the effective potential in Eq. (434) does not appear covariant.

For computational purposes, it is convenient to do some further manipulation. The first of Eqs. (437) can be replaced by [190, 191]

$$F'(y,t) + \frac{1}{6}\,k^2\,\left(\frac{\partial}{\partial y} a^4\right)\,T^0_{\;0} = 0 \tag{438}$$

with

$$F(y,t) = \frac{(a\,a')^2}{b^2} - \frac{(a\,\dot{a})^2}{n^2} \tag{439}$$

which, on intergating along the extra dimension, can be written as

$$\frac{\dot{a}^2}{a^2} - \frac{n^2\,a'^2}{b^2\,a^2} - \frac{k^2}{6} n^2\,T^0_{\ 0} + \frac{k^2\,n^2}{6\,a^4} \int a^4\,\left(T^0_{\ 0}\right)'\,\underset{\cdot}{\mathrm{y}} = \frac{n^2}{a^4}\,\tilde{c}(t) \tag{440}$$

where $\tilde{c}(t)$ is related to the boundary conditions at the branes. The conservation equation $\nabla_A T^{A4} = 0$ yields

$$b' = (b_0 - b)\left(\frac{n'}{n} + 3\frac{a'}{a} + 2\frac{b'}{b}\right) \tag{441}$$

which is identically satisfied by the solutions of the system (437). Instead of the three Eqs. (437), we shall therefore solve the equivalent system

$$\begin{cases} \frac{\dot{a}^2}{a^2} - \frac{n^2 a'^2}{b^2 a^2} - \frac{k^2}{6} n^2 T^0_{\ 0} + \frac{k^2 n^2}{6\,a^4} \int a^4\,(T^0_{\ 0})'\,\underset{\cdot}{\mathrm{y}} = \frac{n^2}{a^4}\tilde{c}(t) \\ \\ b' = (b_0 - b)\,\left(\frac{n'}{n} + 3\frac{a'}{a} + 2\frac{b'}{b}\right) \\ \\ G_{44} = k^2\,T_{44} \end{cases} \tag{442}$$

Moreover, bulk solutions must satisfy the boundary equations given by the junction conditions on the two branes,

$$\begin{cases} \lim_{y \to y_i^+} \frac{a'}{a} = -\frac{k^2}{6}\,(V_i + \rho_i)\;b\Big|_{y=y_i} \\ \\ \lim_{y \to y_i^+} \frac{n'}{n} = -\frac{k^2}{6}\,[V_i - (2 + 3\,w_i)\,\rho_i]\;b\Big|_{y=y_i} \end{cases} \tag{443}$$

where we assumed an equation of state for the vacuum perturbations of the form $p_i = w_i\,\rho_i$. When $\rho_i \to 0$ the RS I solution is fully recovered and one finds the usual warped static metric with $\tilde{c}(t) = 0$ and

$$\begin{aligned} n_{RS}(y,t) &= a_{RS}(y,t) = \exp\left(-m\,b_0\,|y|\right) \\ \\ b_{RS} &= b_0 \end{aligned} \tag{444}$$

which also require the well known fine-tuning

$$V_p = -V_n = \frac{6m}{k^2} \qquad \Lambda = -\frac{6m^2}{k^2} \tag{445}$$

8.2 The Low Density Expansion

A perturbative approach can be adopted in brane cosmology to investigate solutions to the Einstein equations by taking as a starting point the static RS I metric with $\rho_i = 0$ reviewed in the previous section. In fact, in the low density regime

$$\frac{\rho_i}{|V_i|} \ll 1 \tag{446}$$

one can express the corrections to the solution (444) to all orders in ρ_i/V_i by assuming that the metric functions $f_\alpha = (n, a, b)$ can be written as [17]

$$f_\alpha = f_{RS} + \delta f_\alpha \tag{447}$$

with $\delta f_\alpha \sim \sum_{n_i,n_j \geq 1} c_{n_i n_j}\, \rho_i^{n_i}\, \rho_j^{n_j}$.

In order to keep track of the various orders in the above expansion, it is useful to introduce an expansion parameter ϵ by replacing $\rho_i \to \epsilon\, \rho_i$ (and setting $\epsilon = 1$ at the end of the computation). We make the following *ansatz* for the metric,

$$n(y,t) = \exp\left(-m\, b_0\, |y|\right)\, \left[1 + \delta f_n(y,t)\right] \tag{448}$$

$$a(y,t) = a_h(t)\, \exp\left(-m\, b_0\, |y|\right)\, \left[1 + \delta f_a(y,t)\right] \tag{449}$$

$$b(y,t) = b_0 + \delta f_b(y,t) \tag{450}$$

so that the homogeneous scale factor $a_h(t)$ is factored out, and b_0 is the equilibrium point corresponding to the RS I model [see Eq. (444) above]. The solutions to the equations (529) can then be completely expressed in terms of the functions δf_α and $H_h \equiv \dot{a}_h/a_h$, which we expand to second order in ϵ as

$$\begin{aligned} \delta f_\alpha &\simeq \epsilon\, \left[f^{(1)}_{\alpha,p}(y)\, \rho_p + f^{(1)}_{\alpha,n}(y)\, \rho_n\right] \\ &\quad + \epsilon^2\, \left[f^{(2)}_{\alpha,p}(y)\, \rho_p^2 + f^{(2)}_{\alpha,n}(y)\, \rho_n^2 + f^{(2)}_{\alpha,m}(y)\, \rho_p\, \rho_n\right] \end{aligned} \tag{451}$$

$$\begin{aligned} H_h^2 &\simeq \epsilon\, \left(h^{(1)}_{h,p}\, \rho_p + h^{(1)}_{h,n}\, \rho_n\right) \\ &\quad + \epsilon^2\, \left(h^{(2)}_{h,p}\, \rho_p^2 + h^{(2)}_{h,n}\, \rho_n^2 + h^{(2)}_{h,m}\, \rho_p\, \rho_n\right) \end{aligned} \tag{452}$$

We also expand $c \equiv \tilde{c}/a_h^4$ as

$$\begin{aligned} c(t) &\simeq \epsilon\, \left(c^{(1)}_p\, \rho_p + c^{(1)}_n\, \rho_n\right) \\ &\quad + \epsilon^2\, \left(c^{(2)}_p\, \rho_p^2 + c^{(2)}_n\, \rho_n^2 + c^{(2)}_m\, \rho_p\, \rho_n\right) \end{aligned} \tag{453}$$

Note that the time dependence is just carried by the functions $\rho_i = \rho_i(t)$ and that, of all the coefficients appearing above, only the $f^{(n)}_{\alpha,i}$'s in Eq. (539) depend on y, whereas the others are constant.

[17]Note that the approximation which makes use of an effective Lagrangian is only compatible with second order calculations.

In order to proceed with the perturbative expansion, one also needs to expand $\dot{\rho}_i$. From the conservation equation [18]

$$\dot{\rho}_i = -3\, H(y_i,t)\ (1+w_i)\ \rho_i \tag{454}$$

it immediately follows that the time evolution of the matter densities is adiabatic, since $H \equiv \dot{a}/a \sim \rho^{1/2}$ and, therefore,

$$\frac{|\dot{\rho}_i|}{\rho_i^{5/4}} \sim \left(\frac{\rho_i}{|V_i|}\right)^{1/4} \ll 1 \tag{455}$$

If we now assume that, to second order in ϵ,

$$\begin{aligned} H^2(y,t) \ \simeq\ & \epsilon\left[h_p^{(1)}(y)\,\rho_p + h_n^{(1)}(y)\,\rho_n\right] \\ & +\epsilon^2\left[h_p^{(2)}(y)\,\rho_p^2 + h_n^{(2)}(y)\,\rho_n^2 + h_m^{(2)}(y)\,\rho_p\,\rho_n\right] \end{aligned} \tag{456}$$

the coefficients of the above expansion can be related to the corresponding ones in Eqs. (540) and (539) for $\alpha = a$ by equating the two expressions for H^2 at $y = y_i$ up to second order,

$$H^2(y_i,t) = \left(H_h(t) + \frac{\delta\dot{f}_a(y_i,t)}{1+\delta f_a(y_i,t)}\right)^2 \tag{457}$$

With the help of Eq. (454), one finally obtains

$$h_i^{(1)}(y) \ =\ h_{h,i}^{(1)} \tag{458}$$

$$h_i^{(2)}(y) \ =\ h_{h,i}^{(2)} - 6(1+w_i)\, h_{h,i}^{(1)}\, f_{a,i}^{(1)}(y) \tag{459}$$

$$\begin{aligned} h_m^{(2)}(y) \ =\ & h_{h,m}^{(2)} - 6(1+w_p)\, h_{h,n}^{(1)}\, f_{a,p}^{(1)}(y) \\ & -6(1+w_n)\, h_{h,p}^{(1)}\, f_{a,n}^{(1)}(y) \end{aligned} \tag{460}$$

8.3 First order results

In order to solve the bulk equations order by order, one has to substitute the previous expansion in the dynamical equations (529). This will allow us to determine explicitly the coefficients δf_α once the boundary conditions are imposed. Let us begin with first order equations.

At order ϵ, the constraint (441) reads

$$f_{b,p}^{(1)\prime}\,\rho_p + f_{b,n}^{(1)\prime}\,\rho_n - 4\,m\,b_0\left[f_{b,p}^{(1)}\,\rho_p + f_{b,n}^{(1)}\,\rho_n\right] = 0 \tag{461}$$

If we allow ρ_p and ρ_n to be arbitrary functions of the time, the above equation splits into two independent equations for the coefficients of ρ_i, and one finds

$$f_{b,i}^{(1)}(y) = b_i^{(1)}\ \exp\left(4\,m\,b_0\,y\right) \tag{462}$$

[18] Note that this conservation equation can be obtained by taking the limit $y \to y_i$ in the equation $G_{04} = 0$.

where $b_i^{(1)}$ are constant coefficients to be determined.

The functions $f_{a,i}^{(1)}(y)$'s can now be calculated by solving Eq. (440). Since the contribution of the stabilizing potential vanishes at order ϵ, the integral-differential equation (440) becomes the first order linear differential equation

$$\sum_i \mathrm{e}^{-2\,m\,b_0\,y}\,\rho_i\left(2\,m^2\,b_i^{(1)}\,\mathrm{e}^{4\,m\,b_0\,y} - b_0\,c_i^{(1)}\,\mathrm{e}^{4\,m\,b_0\,y} + b_0\,h_{h,i}^{(1)}\,\mathrm{e}^{2\,m\,b_0\,y} + 2\,m\,f_{a,i}^{(1)\prime}\right) = 0 \tag{463}$$

On solving for each coefficient of ρ_i independently, one obtains

$$f_{a,i}^{(1)} = \frac{1}{8}\,\mathrm{e}^{2\,m\,b_0\,y}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(\frac{c_i^{(1)}}{m^2} - 2\,\frac{b_i^{(1)}}{b_0}\right) - 2\,\frac{h_{h,i}^{(1)}}{m^2}\right] + c_{a,i}^{(1)} \tag{464}$$

Finally, one can expand the equation $G_{44} = k^2\,T_{44}$ to first order,

$$\begin{aligned}&\sum_i \rho_i\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6b_0h_{h,i}^{(1)} + 9b_0w_ih_{h,i}^{(1)} - 9b_0c_i^{(1)}\mathrm{e}^{2\,m\,b_0\,y} - 6m^2b_i^{(1)}\mathrm{e}^{2\,m\,b_0\,y} + 4k^2\omega^2b_0^2b_i^{(1)}\mathrm{e}^{2\,m\,b_0\,y}\right)\right]\\ &-6m\sum_i \rho_i f_{n,i}^{(1)} = 0\end{aligned} \tag{465}$$

and solve the two equations for $f_{n,i}^{(1)}(y)$. The result is

$$f_{n,i}^{(1)} = \frac{1}{2}\,\mathrm{e}^{2\,m\,b_0\,y}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(\frac{k^2\,\omega^2}{3\,m^2}\,b_0\,b_i^{(1)} - \frac{3\,c_i^{(1)}}{4\,m^2} - \frac{b_i^{(1)}}{2\,b_0} + \frac{(2+3\,w_i)}{2\,m^2}\,h_{h,i}^{(1)}\right)\right] + c_{n,i}^{(1)} \tag{466}$$

We are now left with six numerical coefficients

$$h_{h,i}^{(1)} \quad b_i^{(1)} \quad c_i^{(1)} \tag{467}$$

and four integration constants

$$c_{a,i}^{(1)} \quad c_{n,i}^{(1)} \tag{468}$$

In order to fix the above, one has to use the junction conditions. These four conditions, written in terms of the coefficients of ρ_i, form a system of eight equations: the discontinuity constraints for $a'(y,t)$ at $y = y_i$ imply

$$\begin{cases} \rho_p\left(3\,c_p^{(1)} - 3\,h_{h,p}^{(1)} + k^2\,m\right) + 3\,\rho_n\left(c_n^{(1)} - h_{h,n}^{(1)}\right) = 0 \\ \\ 3\,\rho_p\left(c_p^{(1)}\mathrm{e}^{m\,b_0} - h_{h,p}^{(1)}\right) \\ +\rho_n\left(3\,c_n^{(1)}\mathrm{e}^{2\,m\,b_0} - 3\,h_{h,n}^{(1)}\mathrm{e}^{m\,b_0} - k^2\,m\right) = 0 \end{cases} \tag{469}$$

whose solution is given by

$$h_{h,p}^{(1)} = \frac{k^2\,m\,\mathrm{e}^{m\,b_0}}{3\,(\mathrm{e}^{m\,b_0} - 1)} \qquad h_{h,n}^{(1)} = \frac{k^2\,m\,\mathrm{e}^{-m\,b_0}}{3\,(\mathrm{e}^{m\,b_0} - 1)} \tag{470}$$

$$c_p^{(1)} = \frac{k^2\,m}{3\,(\mathrm{e}^{m\,b_0} - 1)} \qquad c_n^{(1)} = \frac{k^2\,m\,\mathrm{e}^{-m\,b_0}}{3\,(\mathrm{e}^{m\,b_0} - 1)} \tag{471}$$

The two analogous constraints for $n'(y,t)$ are both equivalent to the equation

$$\begin{aligned} & e^{m\,b_0}\left[m\,(3\,w_p-1)+4\,\omega^2\,b_0\,b_p^{(1)}\left(e^{m\,b_0}-1\right)\right]\rho_p \\ & +\left[m\,(3\,w_n-1)+4\,\omega^2\,b_0\,b_n^{(1)}e^{m\,b_0}\left(e^{m\,b_0}-1\right)\right]\rho_n=0 \end{aligned} \tag{472}$$

which yields

$$\begin{aligned} b_p^{(1)} &= \frac{m\,(3\,w_p-1)}{4\,\omega^2\,b_0\,(e^{m\,b_0}-1)} \\ b_n^{(1)} &= \frac{m\,(3\,w_n-1)\,e^{-m\,b_0}}{4\,\omega^2\,b_0\,(e^{m\,b_0}-1)} \end{aligned} \tag{473}$$

We see that the junction conditions are not sufficient to determine the integration constants (468). Such freedom is in fact related to the gauge freedom in the choice of the initial value for the scale factor and time variable. Without loss of generality, and to simplify the second order calculations, we then set $c_{a,i}^{(1)}=0$. The values of the $c_{n,i}^{(1)}$'s are related to the choice of the time variable. Since one usually considers the negative tension brane in RS I as the four-dimensional "visible Universe", it is natural to use the proper time τ on this brane and choose the $c_{n,i}^{(1)}$'s so as to have $n(y_n,\tau)=1$. This can be achieved by setting

$$\begin{aligned} c_{n,p}^{(1)} &= -\frac{e^{2\,m\,b_0}\left[3\,m^2\,(3w_p-1)+2\,b_0^2\,k^2\,\omega^2\,(3w_p+2)\right]}{48\,m\,b_0^2\,\omega^2\,(e^{m\,b_0}-1)} \\ c_{n,n}^{(1)} &= \frac{e^{m\,b_0}\,\left[3\,m^2\,(3w_n-1)+2\,b_0^2\,k^2\,\omega^2\,\left(1-2e^{-m\,b_0}\right)\,(3w_n+2)\right]}{48\,m\,b_0^2\,\omega^2\,\left(e^{m\,b_0}-1\right)} \end{aligned} \tag{474}$$

and defining the time coordinate τ as

$$d\tau=\exp\left(-\frac{m\,b_0}{2}\right)\,\underset{.}{t} \tag{475}$$

With this choice, the cosmological Friedmann equations can be easily compared to standard ones.

Let us now comment on the first order results. As far as the radion perturbation is concerned, we found

$$\begin{aligned} \delta f_b \;=\; & \frac{m\,e^{4\,m\,b_0 y}}{4\,b_0\,\omega^2}\,\left[(1-3w_p)\,\rho_p+e^{-m\,b_0}(1-3w_n)\,\rho_n\right]\epsilon \\ & +\mathcal{O}(\epsilon^2) \end{aligned} \tag{476}$$

which was expected, as it is due to the known coupling of the radion with the trace of the energy-momentum tensor of brane matter. Traceless fluids, such as radiation, have no first order effect on the excitation of the radion. If one fills the branes with some pressureless fluid, the distance between the two branes grows. This effect, being counter-intuitive for the attractive nature of Newtonian gravity, is in fact a consequence of the form of the stabilizing potential. Its non trivial contribution to the bulk energy-momentum tensor at order ϵ is

$$T_{44} \sim -2\, b_0^3\, \omega^2\, \delta f_b \tag{477}$$

The first order expressions are identical to those for a static solution, as they can be obtained by neglecting $\dot\rho_i$. Every kind of matter on the branes thus acts so as to detune the brane tensions from the bulk cosmological constant and can be balanced by some constant pressure along the fifth dimension. Such pressure is given by the first order contribution of (477) which increases when δf_b decreases.

Note that (476) is proportional to the inverse of ω^2, which represents the effective spring constant coming from some stabilization mechanism. When such a constant diverges, the correction δf_b vanishes and the length of the fifth dimension is fixed as expected, even if there is a finite, ω-independent T_{44} pressure term. On the other hand, the correction to the scale factor,

$$\begin{aligned} \delta f_a \;=\; & \frac{\mathrm{e}^{2\,m\,b_0\,y}}{48\,m\,b_0^2\,\omega^2\,(\mathrm{e}^{m\,b_0}-1)} \left\{ \left[2\,b_0^2\,k^2\,\omega^2 \left(\mathrm{e}^{2\,m\,b_0\,y} - 2\,\mathrm{e}^{m\,b_0}\right) + 3\,m^2\,(3w_p-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right] \rho_p \right. \\ & \left. + \mathrm{e}^{-m\,b_0} \left[2\,b_0^2\,k^2\,\omega^2 \left(\mathrm{e}^{2\,m\,b_0\,y} - 2\right) + 3\,m^2 (3w_n-1) \mathrm{e}^{2\,m\,b_0\,y}\right] \rho_n \right\} \epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{478}$$

never vanishes when matter is present on the branes, even if that is trace-less. Furthermore a finite, non vanishing δf_a can be obtained in the limit of infinite spring constant, regardless of the matter equation of state.

The correction to the lapse function,

$$\begin{aligned} \delta f_n \;=\; & \frac{\mathrm{e}^{2\,m\,b_0\,y}}{48\,m\,b_0^2\,\omega^2\,(\mathrm{e}^{m\,b_0}-1)} \\ & \left\{ \left[2\,b_0^2\,k^2\,\omega^2\,(3w_p+2) \left(2\mathrm{e}^{m\,b_0} - \mathrm{e}^{2\,m\,b_0\,y}\right) + 3\,m^2\,(3w_p-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right] \rho_p \right. \\ & \left. + \mathrm{e}^{-m\,b_0} \left[2\,b_0^2\,k^2\,\omega^2\,(3w_n+2) \left(2 - \mathrm{e}^{2\,m\,b_0\,y}\right) + 3\,m^2 (3w_n-1)\,\mathrm{e}^{2\,m\,b_0\,y}\right] \rho_n \right\} \epsilon \\ & + \epsilon\, c_{n,p}^{(1)}\, \rho_p + \epsilon\, c_{n,n}^{(1)}\, \rho_n + \mathcal{O}(\epsilon^2) \end{aligned} \tag{479}$$

does not vanish when the branes are filled with trace-less matter. Note, however, that a vanishing correction can be obtained for some exotic fluid with $w_i = -2/3$ and negative pressure in the limit $\omega \to \infty$. Apart from these exceptions, one has non negligible corrections everywhere in the bulk.

In order to compare the first order results with the RS II case of a single brane, we must instead use the proper time on the positive tension brane. This is achieved by setting

$$\begin{aligned} c_{n,p}^{(1)} &= \frac{\left[3\,m^2(3w_p-1) + 2\,b_0^2\,k^2\,\omega^2 \left(1 - 2\mathrm{e}^{m\,b_0}\right)(3w_p+2)\right]}{48\,m\,b_0^2\,\omega^2\,(\mathrm{e}^{m\,b_0}-1)} \\ c_{n,n}^{(1)} &= \frac{\mathrm{e}^{-m\,b_0}\,\left[3\,m^2(3w_n-1) + 2\,b_0^2\,k^2\,\omega^2\,(3w_n+2)\right]}{48\,m\,b_0^2\,\omega^2\,(1-\mathrm{e}^{m\,b_0})} \end{aligned} \tag{480}$$

and letting $b_0 \to \infty$. The Friedmann equation is simply obtained by keeping $\mathcal{O}(\epsilon)$ terms in $H^2(y_i, t)$. Since the first order four-dimensional Hubble parameter is homogeneous, it reacts to all the sources along the y direction. On the positive tension brane one has, to first order in ϵ,

$$H_p^2 = \frac{m\,k^2\left(\mathrm{e}^{m\,b_0}\rho_p + \mathrm{e}^{-m\,b_0}\rho_n\right)}{3\left(\mathrm{e}^{m\,b_0}-1\right)}\,\epsilon \;\xrightarrow{b_0\to\infty}\; \frac{m\,k^2}{3}\,\rho_p\,\epsilon \tag{481}$$

regardless of the value of ω. When the negative tension brane is moved to infinity, its contribution goes to zero and one recovers the usual first order effect in brane cosmology.

On the visible brane, the Friedmann equation is slightly modified by the rescaled time parameter τ,

$$H_n^2 = \frac{m\,k^2\left(\mathrm{e}^{2\,m\,b_0}\rho_p + \rho_n\right)}{3\left(\mathrm{e}^{m\,b_0}-1\right)}\,\epsilon \;\xrightarrow{\rho_p\to 0}\; \frac{m\,k^2\,\rho_n\,\epsilon}{3\left(\mathrm{e}^{m\,b_0}-1\right)} \tag{482}$$

The acceleration equation is homogeneous as the Friedmann equation and becomes

$$\frac{\ddot a(y_i,t)}{a(y_i,t)} = \frac{m\,k^2\left[\mathrm{e}^{m\,b_0}(1+w_p)\rho_p + \mathrm{e}^{-m\,b_0}(1+w_n)\rho_n\right]}{6\left(1-\mathrm{e}^{m\,b_0}\right)}\epsilon \tag{483}$$

which has the weighted brane fluid energy densities as sources.

8.4 Second order results

We are now ready to evaluate $\mathcal{O}(\epsilon^2)$ corrections to the vacuum solution RS I. The procedure will be analogous to the one used for first order results in the previous section.

As a first step, one can impose the constraint (441) in order to find the dependence on y of the second order coefficients in δf_b. We are then left with three inhomogeneous equations obtained by setting to zero the coefficients of the independent matter densities in

$$\begin{aligned}
& f_{b,p}^{(2)\,\prime}\,\rho_p^2 + f_{b,n}^{(2)\,\prime}\,\rho_n^2 + f_{b,m}^{(2)\,\prime}\,\rho_p\,\rho_n - 4\,m\,b_0\left(f_{b,p}^{(2)}\,\rho_p^2 + f_{b,n}^{(2)}\,\rho_n^2 + f_{b,m}^{(2)}\,\rho_p\,\rho_n\right) \\
& + \frac{m\,(1-3\,w_p)^2\,\mathrm{e}^{6\,m\,b_0\,y}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - b_0^2\,k^2\,\omega^2\,\mathrm{e}^{m\,b_0}\right]\rho_p^2 \\
& + \frac{m\,(1-3\,w_n)^2\,\mathrm{e}^{2\,m\,b_0\,(3y-1)}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\left[\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - b_0^2\,k^2\,\omega^2\right]\rho_n^2 \\
& + \frac{m\,(1-3\,w_p)(1-3\,w_n)\,\mathrm{e}^{m\,b_0\,(6y-1)}}{24\,b_0^2\,\omega^4\,\left(\mathrm{e}^{m\,b_0}-1\right)}\left[2\,\mathrm{e}^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - b_0^2\,k^2\,\omega^2\left(1+\mathrm{e}^{m\,b_0}\right)\right]\rho_p\,\rho_n \\
& = 0
\end{aligned} \tag{484}$$

which contains the first order parameters previously determined. The solutions are

$$f_{b,p}^{(2)} = e^{4\,m\,b_0\,y} \left\{ b_p^{(2)} - \frac{(1-3w_p)^2\; e^{4\,m\,b_0\,y}}{96\,b_0^3\,\omega^4\,(e^{m\,b_0}-1)^2} \left[6\,m^2 + b_0^2\,k^2\,\omega^2 - 2\,b_0^2\,k^2\,\omega^2\,e^{m\,b_0\,(2y-3)}\right] \right\} \tag{485}$$

$$f_{b,n}^{(2)} = e^{4\,m\,b_0\,y} \left\{ b_n^{(2)} - \frac{(1-3w_n)^2\; e^{6\,m\,b_0\,(y-1)}}{96\,b_0^3\,\omega^4\,(e^{m\,b_0}-1)^2} \left[e^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right) - 2\,b_0^2\,k^2\,\omega^2\right] \right\} \tag{486}$$

$$f_{b,m}^{(2)} = e^{4\,m\,b_0\,y} \left\{ b_m^{(2)} - \frac{(1-3w_p)\,(1-3w_n)\; e^{m\,b_0\,(6y-5)}}{48\,b_0^3\,\omega^4\,(e^{m\,b_0}-1)^2} \left[e^{2\,m\,b_0\,y}\left(6\,m^2 + b_0^2\,k^2\,\omega^2\right)\right] \right\} \tag{487}$$

where

$$b_p^{(2)} \quad b_n^{(2)} \quad b_m^{(2)} \tag{488}$$

are integration constants to be determined from the junction conditions. Once we plug the $f_{b,i}^{(2)}$'s into the second order terms in Eq. (440), we get the equations for the $f_{a,i}^{(2)}$'s, which are not displayed for the sake of brevity. Finally, by solving $G_{44} = k^2\,T_{44}$ one obtains the corrections $f_{n,i}^{(2)}$'s. The results will contain six integration constants from the solutions of the first order differential equations for $f_{a,i}^{(2)}$ and $f_{n,i}^{(2)}$,

$$c_{a,i}^{(2)}\ , \quad c_{n,i}^{(2)}\ , \tag{489}$$

and nine parameters related to the radion, the four-dimensional Hubble parameter and $c(t)$ respectively,

$$h_{h,i}^{(2)}\ , \quad b_i^{(2)}\ , \quad c_i^{(2)}\ , \tag{490}$$

where i runs over p, n and m for second order quantities. Analogously to the first order case, one can fix the coefficients (490) by imposing the junction conditions, which form a system of nine independent equations. Nonetheless, one is again not able to fix the constants (489). We do not give their finale expressions due to their length. Here are the solutions obtained at the end of the calculations described above:

$$\begin{aligned} f_{a,p}^{(2)} &= \ldots\ldots\ldots\ldots \\ f_{a,n}^{(2)} &= \ldots\ldots\ldots\ldots \\ f_{a,m}^{(2)} &= \ldots\ldots\ldots\ldots \\ f_{n,p}^{(2)} &= \ldots\ldots\ldots\ldots \\ f_{n,n}^{(2)} &= \ldots\ldots\ldots\ldots \end{aligned} \tag{491}$$

8.5 Second order cosmology

Let us now look at the physical consequences of the above results as applied to two different cosmological scenarios. The Friedmann and acceleration equations will be showed and their phenomenology investigated in different regimes of ω and for different matter equations of state on the branes.

8.6 RS I

We begin from the case with two branes [19] for which we shall study how cosmology would be described by observers on the negative brane. In order to achieve that, one can use the corresponding proper time by fixing the $c^{(1)}_{n,i}$'s as in Eq. (474), choosing $c^{(2)}_{n,p}$, $c^{(2)}_{n,n}$ and $c^{(2)}_{n,m}$ and rescaling $n(y,t)$ to satisfy the condition $n(1/2,\tau)=1$. The Friedmann equation we are interested in is given by the second order expression of the Hubble parameter at $y=1/2$ as a function of τ,

$$\begin{aligned} H^2(1/2,\tau) &= \frac{k^2 m}{3\left(\mathrm{e}^{mb_0}-1\right)}\left(\rho_n+\mathrm{e}^{2mb_0}\rho_p\right)-(w_n+1)(3w_n-1)\frac{3m^2k^2\left(\mathrm{e}^{mb_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\rho_n^2 \qquad (492)\\ &-\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{mb_0}+1\right)}{48b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[18w_p^2-9w_p(w_n-1)-9w_n-1+\right]\right\}\mathrm{e}^{mb_0}\,\rho_p\,\rho_n\\ &+\left\{\frac{k^4}{36}+\frac{m^2k^2(3w_p-1)\left(\mathrm{e}^{mb_0}+1\right)}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[3w_p+7-4\mathrm{e}^{mb_0}(3w_p+4)\right]\right\}\mathrm{e}^{2mb_0}\,\rho_p^2 \end{aligned}$$

Note that this result does not depend on the integration constants $c^{(1)}_{a,i}$'s and $c^{(2)}_{a,i}$'s, which reflects the fact that the three-dimensional spatial curvature has been set to zero ab initio.

The Friedmann equation contains coefficients up to second order in the vacuum perturbations of both branes. First and second order contributions in Eq. (492) are consequences of the adiabatic regime of the five-dimensional dynamics which determines the value of the integration constant $\tilde{c}(t)$ in Eq. (440). The value of the latter is affected by the presence of matter on both branes through the junction conditions and, for instance, up to $\mathcal{O}(\epsilon^2)$ is given by

$$\begin{aligned} \lim_{\omega\to\infty} c^{(1)} &= \frac{m\,k^2\,\left(\rho_p+\mathrm{e}^{mb_0}\rho_n\right)}{3\,\left(\mathrm{e}^{m\,b_0}-1\right)} \qquad (493)\\ \lim_{\omega\to\infty} c^{(2)} &= \frac{k^4}{36}\left[\frac{\mathrm{e}^{2\,m\,b_0}+2\,\mathrm{e}^{m\,b_0}-1}{\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_p^2+\left(\frac{\mathrm{e}^{m\,b_0}+1}{\mathrm{e}^{m\,b_0}-1}\right)^2\mathrm{e}^{-m\,b_0}\,\rho_p\rho_n+\frac{\mathrm{e}^{-2\,m\,b_0}+2\,\mathrm{e}^{-m\,b_0}-1}{\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2\right] \end{aligned}$$

in the limit of infinite spring constant. Furthermore, as previously noted, this effect is also a consequence of the radion field potential acting as a source in Eq. (440). One can expect to cancel some terms in Eq. (492) by arbitrarily increasing the spring constant of the effective radion potential in order to decrease the radion shift from equilibrium. This mechanism partially works as the first order contributions to the bulk potential vanish along the time-time direction, whereas only second order terms, which depend on the matter equation of state, cancel when $\omega\to\infty$. One is then left with

$$\begin{aligned} \lim_{\omega\to\infty} H^2(1/2,\tau) &= \frac{m\,k^2\,\left(\rho_n+\mathrm{e}^{2\,m\,b_0}\rho_p\right)}{3\,\left(\mathrm{e}^{mb_0}-1\right)}\\ &-\frac{k^4}{36}\,\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n+\frac{k^4}{36}\,\mathrm{e}^{2\,m\,b_0}\,\rho_p^2 \qquad (494) \end{aligned}$$

[19]The unperturbed brane distance is taken to be finite and equal to $b_0/2$.

which is analogous to what has been obtained in Ref. [195]. The matter on the positive tension brane appears at second order with the role of some "dark" fluid and acts as a sort of Brans-Dicke field which adiabatically modifies the Newton constant perceived in the visible Universe. This behavior is somehow inherited by the dynamics of the radion which is known to modulate the strength of gravity on the visible brane.

Note that while first order coefficients are positive definite, irrespective of the brane tension, the sign of second order ones depends on the matter equation of state. Differently enough from unstabilized brane cosmology, this fact implies that leading order cosmological equations have the correct behavior on both branes. Letting the energy density $\rho_p \to 0$ in Eq. (492), one obtains

$$\begin{aligned} H^2(1/2,\tau) \;=\; & (w_n+1)\,(1-3w_n)\,\frac{3\,m^2\,k^2\left(\mathrm{e}^{m\,b_0}+1\right)}{32\,b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2 \\ & +\frac{m\,k^2\,\rho_n}{3\,\left(\mathrm{e}^{m\,b_0}-1\right)} \end{aligned} \tag{495}$$

which has the usual first order solution for both radiation and a cosmological constant on the visible brane. A matter dominated Universe would otherwise generate second order corrections.

We now come to the equation for the acceleration, which has the general form

$$\begin{aligned} &\frac{\ddot a(1/2,\tau)}{a(1/2,\tau)} = \\ &\frac{m\,k^2}{6\,(1-\mathrm{e}^{m\,b_0})}\left[(3w_n+1)\rho_n+\mathrm{e}^{2\,m\,b_0}(3w_p+1)\rho_p\right]+(w_n+1)\,(3w_n-1)\,\frac{3m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\rho_n^2 \\ &-\left\{\frac{k^4}{144}(3w_n+1)(3w_p+1)+\frac{m^2k^2}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\left[\mathrm{e}^{2\,m\,b_0}(3w_n-1)(18w_n^2+9w_pw_n)\right.\right. \\ &\quad +9\mathrm{e}^{m\,b_0}(6w_p^3+w_n^3+3w_p^2w_n+3w_pw_n^2+11w_p^2+11w_n^2+2w_pw_n-4) \\ &\quad \left.\left.+54w_p^3-27w_p^2w_n+99w_p^2-18w_pw_n+24w_p-13\right]\right\}\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n \\ &+\left\{\frac{k^4}{144}(3w_p+1)^2+\frac{m^2k^2(3w_p-1)\left(\mathrm{e}^{m\,b_0}+1\right)}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\left[\mathrm{e}^{m\,b_0}(27w_p^2+54w_p+23)-9w_p-5\right]\right\}\mathrm{e}^{2\,m\,b_0}\rho_p^2 \end{aligned} \tag{496}$$

and, for $\omega \to \infty$, reduces to

$$\begin{aligned} \frac{\ddot a(1/2,\tau)}{a(1/2,\tau)} \;=\; & \frac{m\,k^2}{6\,(1-\mathrm{e}^{m\,b_0})}\left[(3w_n+1)\rho_n+\mathrm{e}^{2\,m\,b_0}(3w_p+1)\,\rho_p\right] \\ & -\frac{k^4}{144}(3w_n+1)(3w_p+1)\mathrm{e}^{m\,b_0}\,\rho_p\,\rho_n+\frac{k^4}{144}(3w_p+1)^2\mathrm{e}^{2\,m\,b_0}\,\rho_p^2 \end{aligned} \tag{497}$$

which is again analogous to the result of Ref. [195]. The coefficient of ρ_p^2 in Eq. (497) is positive or zero in this limit and provides an accelerating contribution to the equation. The coefficient of the mixed term has a positive value when just one fluid has $w_i < -1/3$. For $\rho_p \to 0$, one is further left with

$$\frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = \frac{m\,k^2\,(3w_n+1)}{6\,(1-\mathrm{e}^{mb_0})}\,\rho_n + (w_n+1)\,(3w_n-1)\,(3w_n+2)\,\frac{3m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_n^2 \quad (498)$$

where the second order contribution is inversely proportional to ω and vanishes for radiation and a cosmological constant. This peculiarity leads to the standard cosmological evolution up to $\mathcal{O}(\rho_n^2)$ until the matter dominated era.

A singularity in the lapse function $n(y,t)$ for $b_0 \to \infty$ prevents us from analyzing the correct limit when the distance between the branes becomes infinite, we shall thus comment on this problem in the next subsection. On setting $\rho_p \to 0$, Eq. (494) admits the finite but trivial limit

$$\lim_{b_0\to\infty} H^2(1/2,\tau) = 0 \quad (499)$$

in which one also has

$$\lim_{b_0\to\infty} \frac{\ddot{a}(1/2,\tau)}{a(1/2,\tau)} = 0 \quad (500)$$

This result is due to the reduced strength of the gravitational interaction at infinity.

8.7 RS II

On can think of the RS II model as the limit of RS I in which the distance between the two branes becomes infinite, thus one expects that only ρ_p contributes in this limit. The cosmological proper time is now the one on the Planck brane and is recovered upon choosing

$$\begin{aligned} c_{n,p}^{(2)} &= \frac{1}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\left\{9m^4(3w_p-1)\left[18\mathrm{e}^{2mb_0}(3w_p^2+4w_p+1)+18\mathrm{e}^{mb_0}(3w_p^2+4w_p+1)\right.\right.\\ &\quad \left.-21w_p+7\right]+m^2b_0^2k^2\omega^2\left[6\mathrm{e}^{2mb_0}(162w_p^3+243w_p^2+60w_p-37)\right.\\ &\quad \left.-2\mathrm{e}^{mb_0}(162w_p^3+279w_p^2+192w_p-5)-81w_p^2+90w_p+75\right]\\ &\quad \left.+4b_0^4k^4\omega^4\left(1-2\mathrm{e}^{mb_0}\right)^2(6w_p+5)\right\} \end{aligned} \quad (501)$$

$$\begin{aligned} c_{n,n}^{(2)} &= \frac{\mathrm{e}^{-2mb_0}}{4608m^2b_0^4\omega^4\left(\mathrm{e}^{mb_0}-1\right)^2}\left\{9m^4(3w_n-1)\left[4\mathrm{e}^{2mb_0}(1-3w_n)+18\mathrm{e}^{mb_0}(3w_n^2+4w_n+1)\right.\right.\\ &\quad \left.+54w_n^2+63w_n+21\right]-m^2b_0^2k^2\omega^2\left[54\mathrm{e}^{2mb_0}(5w_n^2+2w_n-3)\right.\\ &\quad \left.-6\mathrm{e}^{mb_0}(162w_n^3+207w_n^2+12w_n-65)+324w_n^3+153w_n^2-102w_n-91\right]\\ &\quad \left.+4b_0^4k^4\omega^4\left[2\left(\mathrm{e}^{2mb_0}-2\mathrm{e}^{mb_0}\right)(9w_n^2+6w_n-1)+3(6w_n^2+6w_n+1)\right]\right\} \end{aligned} \quad (502)$$

$$
\begin{aligned}
c^{(2)}_{n,m} \;=\; & \frac{\mathrm{e}^{mb_0}}{2304 m^2 b_0^4 \omega^4 \left(\mathrm{e}^{mb_0}-1\right)^2} \Big\{ 9m^4 \Big[\mathrm{e}^{2mb_0}(3w_n-1)(18w_n^2+9w_pw_n+3w_p+27w_n+11) \\
& +9\mathrm{e}^{mb_0}(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+7w_p^2+7w_n^2+4w_pw_n-w_p-w_n-2) \\
& (3w_p-1)(18w_p^2+9w_pw_n+27w_p-6w_n+14)\Big] + m^2b_0^2k^2\omega^2 \Big[6\mathrm{e}^{2mb_0}(54w_n^2++27w_pw_n^2 \\
& +117w_n^2-36w_pw_n-33w_p+30w_n-19)+2\mathrm{e}^{mb_0}(162w_p^3-54w_n^3+81w_p^2w_n-27w_pw_n^2 \\
& +351w_p^2-117w_n^2+96w_p-30w_n-11)-108w_p^3-54w_p^2w_n-234w_p^2+9w_pw_n+39w_p-87w_n \\
& -4b_0^4k^4\omega^4 \Big[\mathrm{e}^{2mb_0}(9w_pw_n+3w_p+3w_n-1)+\left(1-2\mathrm{e}^{mb_0}\right)(3w_p+2)(3w_n+2)\Big]\Big\}
\end{aligned}
$$

together with Eq. (480). The Friedmann equation, for a finite b_0, reduces to

$$
\begin{aligned}
& H^2(0,t) = \qquad (504) \\
& \frac{mk^2}{3\left(e^{m\,b_0}-1\right)}\left(\mathrm{e}^{m\,b_0}\,\rho_p+\mathrm{e}^{-m\,b_0}\,\rho_n\right)+(w_p+1)\,(1-3w_p)\,\frac{3m^2k^2\mathrm{e}^{m\,b_0}\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\,\rho_p^2 \\
& -\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)\mathrm{e}^{m\,b_0}(18w_n^2-9w_pw_n-9w_p+9w_n-1)(9w_p^2+9w_p-4)}{24b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\right\}\mathrm{e}^{-m\,b_0}\,\rho_p\,\rho_n \\
& +\left\{\frac{k^4}{36}+\frac{m^2k^2\left(\mathrm{e}^{m\,b_0}+1\right)(3w_n-1)\left[\mathrm{e}^{m\,b_0}(3w_n+7)-4(3w_n+4)\right]}{96b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\right\}\mathrm{e}^{-2\,m\,b_0}\,\rho_n^2
\end{aligned}
$$

which is similar to Eq. (492) but has the finite, non trivial limit

$$
\lim_{b_0\to\infty} H^2(0,t) = \frac{m\,k^2}{3}\,\rho_p \qquad (505)
$$

This is precisely the standard Friedmann equation one has in four-dimensional cosmology. Furthermore, the equation for the acceleration to second order in terms of the time on the positive tension brane is given by

$$
\begin{aligned}
& \frac{\ddot a(0,t)}{a(0,t)}=\frac{m\,k^2\,\mathrm{e}^{m\,b_0}}{6\left(1-\mathrm{e}^{m\,b_0}\right)}\left[(3w_p+1)\rho_p+(3w_n-1)\mathrm{e}^{-2\,m\,b_0}\rho_n\right] \qquad (506) \\
& +(w_p+1)\,(3w_p-1)\,(3w_p+2)\,\frac{3m^2k^2\mathrm{e}^{m\,b_0}\left(\mathrm{e}^{m\,b_0}+1\right)}{32b_0^2\omega^2\left(\mathrm{e}^{m\,b_0}-1\right)^2}\rho_p^2 \\
& -\left\{\frac{k^4}{144}(3w_p+1)(3w_n+1)-\frac{m^2k^2}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[\mathrm{e}^{2mb_0}(54w_n^3+27w_pw_n^2+99w_n^2-18w_pw_n\right.\right. \\
& \quad -21w_p+24w_n-13)+9\mathrm{e}^{mb_0}(6w_p^3+6w_n^3+3w_p^2w_n+3w_pw_n^2+11w_p^2+11w_n^2+2w_pw_n \\
& \quad \left.\left.+w_p+w_n-4)+(3w_p-1)(18w_p^2+9w_pw_n+39w_p+15w_n+23)\right]\right\}\mathrm{e}^{-mb_0}\rho_p\rho_n \\
& +\left\{\frac{k^4}{144}(3w_n+1)^2-\frac{m^2k^2\left(\mathrm{e}^{mb_0}\right)(3w_n-1)}{96b_0^2\omega^2\left(\mathrm{e}^{mb_0}-1\right)^2}\left[\mathrm{e}^{mb_0}(9w_n+5)-27w_n^2+54w_n+23\right]\right\}\mathrm{e}^{-2mb_0}\rho_n^2
\end{aligned}
$$

and, compatibly with Eq. (505), yields

$$
\lim_{b_0\to\infty}\frac{\ddot a(0,t)}{a(0,t)} = -\frac{m\,k^2}{6}(3w_p+1)\,\rho_p \qquad (507)
$$

in the limit of infinite brane distance. Thus second order effects, typical of brane cosmology, become more and more negligible when the distance between the branes grows, which consequently leads to unobservable deviations from standard four-dimensional General Relativity.

8.8 Approximation analysis

The results obtained so far hold with the assumption that $\rho_i \ll M^4$ where M is, in general, the natural mass scale of the model. By taking all the mass parameters to such a natural scale, one expects the solutions to second order provide a good approximation for Eq. (529). In this regime, however, second order effects are certainly sub-leading and thus insufficient to significantly alter first order behavior. We shall hence present below a numerical analysis of the validity of our approximate solutions in the attempt to extend the range of the parameters in which ours results hold valid and widen the conclusions one can draw from second order expressions. In particular, we are interested in possible deviations from standard cosmological equations due to terms of $\mathcal{O}(\epsilon^2)$ in the first and third of Eqs. (437). Note that all numerical results will be obtained by setting the expansion parameter $\epsilon = 1$ as previously prescribed.

In order to test our approximations, since Eqs. (529) are not analytically solvable, we substitute the second order solutions into the exact Einstein equations to obtain an estimate of their non vanishing remainders which we then compare with the leading contributions (satisfying the corresponding approximate equations). Since the time dependence is contained in ρ_i in our expansions (539)-(453), it is also convenient to trade the time for ρ. We thus divide Eq. (440) into the following five terms

$$\eta_1(y,\rho) \equiv \frac{1}{n^2(y,t)}\frac{\dot{a}^2(y,t)}{a^2(y,t)} \tag{508}$$

$$\eta_2(y,\rho) \equiv -\frac{1}{b^2(y,t)}\left(\frac{a'(y,t)}{a(y,t)}\right)^2 \tag{509}$$

$$\eta_3(y,\rho) \equiv -\frac{k^2}{6}(\Lambda + U) \tag{510}$$

$$\eta_4(y,\rho) \equiv \frac{k^2}{6\,a^4(y,t)}\int^y a^4\ \left(T^0_{\ 0}\right)'\ \text{x} \tag{511}$$

$$\eta_5(y,\rho) \equiv -\frac{\tilde{c}(t)}{a^4(y,t)} \tag{512}$$

in which $t = t(\rho)$ in the right hand sides is understood as the time when $\rho = \rho_n$ (ρ_p will be chosen either equal to ρ_n or zero). The sum,

$$R_\eta(y,\rho) \equiv \sum_l \eta_l \tag{513}$$

evaluated on the second order solutions yields $R_\eta(y,\rho) = \mathcal{O}(\epsilon^3)$ as a measure of the corresponding error. Similarly, the third of Eqs. (529) may be written as the sum of the following six terms

$$\xi_1(y,\rho) \equiv \frac{1}{n^2(y,t)}\frac{\ddot a(y,t)}{a(y,t)} \tag{514}$$

$$\xi_2(y,\rho) \equiv -\frac{1}{b^2(y,t)}\left(\frac{a'(y,t)}{a(y,t)}\right)^2 \tag{515}$$

$$\xi_3(y,\rho) \equiv -\frac{1}{b^2(y,t)}\frac{a'(y,t)}{a(y,t)}\frac{n'(y,t)}{n(y,t)} \tag{516}$$

$$\xi_4(y,\rho) \equiv \frac{1}{n^2(y,t)}\left(\frac{\dot a^2(y,t)}{a^2(y,t)} - \frac{\dot a(y,t)\,\dot n(y,t)}{a(y,t)\,n(y,t)}\right) \tag{517}$$

$$\xi_5(y,\rho) \equiv -\frac{k^2}{3}\,(\Lambda+U) \tag{518}$$

$$\xi_6(y,\rho) \equiv -\frac{2}{3}\,k^2\,\omega^2\,b(y,t)\,\left[b(y,t)-b_0\right] \tag{519}$$

and

$$R_\xi(y,\rho) \equiv \sum_l \xi_l \tag{520}$$

with $R_\xi(y,\rho) = \mathcal{O}(\epsilon^3)$ for the same approximate solutions. One may now assume that approximate metric functions computed to $\mathcal{O}(\epsilon^2)$ are accurate approximations of exact solutions to Eqs. (529) if

$$|\eta_l| \gg |R_1| \quad \text{and} \quad |\xi_l| \gg |R_2| \tag{521}$$

for every term in Eqs. (508)-(519) evaluated to $\mathcal{O}(\epsilon^2)$. (It should actually be sufficient to satisfy the above conditions for the leading terms of each equation.) Throughout this section, where unspecified, a natural choice [20] of dimensionful parameters is considered.

Fig. 11 shows the functions η_l and ξ_l evaluated to second order in ϵ and the corresponding R_η and R_ξ for $\rho_i = 2\cdot 10^{-1}\,M^4$ and $w_i = 0.5$. The box in the first row on the left shows the absolute values of the leading terms η_2 and η_3, which are roughly one order of magnitude larger than the remaining terms η_l's displayed in the plot on the right along with R_η (the dotted line). In the second row of Fig. 11, the modula of the leading terms ξ_2, ξ_3 and ξ_5 are plotted on the left while the remaining coefficients ξ_l and $R_\xi(y)$ are presented on the right. Thanks to the relatively large amplitudes of the leading terms, one can rely on the second order approximation even when the ρ_i's are not too small, implying that a great improvement over first order results can be achieved in a regime where $\rho_i \sim 10^{-2}$. In this case the errors produced by truncating the expansion to $\mathcal{O}(\epsilon^2)$ are much smaller than one percent.

Let us further consider the particular case $\rho_p = 0$. This choice is made in order to study the cosmological consequences experienced on the negative tension brane (at $y = 1/2$) generated by second order terms proportional to ρ_n^2. We therefore use the proper time τ

[20] In terms of the fundamental scale M one has $m = M$, $k^2 = M^{-3}$, $b_0 = M^{-1}$, and $\omega^2 = M^7$.

on the negative tension brane [21]. In each graph in Fig. 12 we plot the squared Hubble parameter on the negative tension brane of Eq. (495) and the acceleration of Eq. (498) for a given energy density ρ_n as a function of its equation of state w_n. The plot on the left in the first row is for the small density $\rho_n = 10^{-2}\, M^4 \ll M^4$ and shows a behavior which is typical of standard four-dimensional cosmology. In fact, only $w_n < -1/3$ leads to an accelerating phase. This trend is modified by higher densities, as it emerges in the remaining graphs of Fig. 12. In particular when $\rho_n = 2 \cdot 10^{-1}\, M^4$ (plot on the left in the second row) the second order corrections seem to provide an accelerated regime for $0 < w_n < 1$. The acceleration appears amplified when $\rho_n = 6 \cdot 10^{-1}\, M^4 < M^4$ (plot on the right in the second row) or higher. Note however that the Hubble parameter, a positive definite quantity, constrains the region swept by w_n which is not allowed to reach unity. Finally the intermediate case is showed in the plot on the right the first row: due to the second order effect, H^2 exhibits a dependence on w_n otherwise not present.

The four regimes described above have to be tested with particular accuracy because the acceleration and Hubble parameter plotted in Fig. 12 are not the leading terms in Eqs. (529) and could thus be comparable with the remainders. In this case, it is somehow possible that the remainders significantly modify the behavior. We first note that $H^2 \sim \eta_1$ and $\ddot{a}/a \sim \xi_1$, as defined above in Eqs. (508) and (514), hence we can use η_1 and ξ_1 in place of H^2 and $\ddot{a}/a$ respectively. We then plot in Fig. 13 the ratio between the remainder R_η (evaluated to second order) and the squared Hubble parameter η_1 evaluated to first and second orders. The four plots show this ratio for different choices of ρ_n and w_n (the same as in Fig. 12 and in the same order) as a function of y. In particular, we choose $w_n = 0.95$ in order to explore regions where the acceleration has an unconventional behavior for $\rho_n = 10^{-2}\, M^4$, $10^{-1}\, M^4$, $2 \cdot 10^{-1}\, M^4$ and $w_n = 0.65$ for $\rho_n = 6 \cdot 10^{-1}\, M^4$. Apart from the last case, the corrections given by the neglected terms cannot significantly modify H^2 on the negative tension brane. In the first line of Fig. 14, the ratios between R_ξ (to second order) and the acceleration ξ_1 to first and second order are analogously plotted. For low enough energy densities, the second order expressions appear to be good approximations. On the other hand, in the second row it is shown that, in the unconventional regime of the acceleration (that is $\rho_n = 2 \cdot 10^{-1}\, M^4$ and $w_n = 0.95$, and $\rho_n = 6 \cdot 10^{-1}\, M^4$ and $w_n = 0.65$) the two terms are of the same order of magnitude. This shows that one should go beyond $\mathcal{O}(\epsilon^2)$ in order to determine the true behavior of $\ddot{a}/a$ for such equations of state.

8.9 Summary

We have computed approximate cosmological solutions of five-dimensional Einstein equations for Randall-Sundrum models in the presence of a radion effective potential. The calculations were performed up to the second order in the energy densities of the matter on the branes and assuming an adiabatic evolution of the system. Our approach differs from Ref. [195] in that we do not include a specific bulk field to achieve stabilization, and is therefore more general. Interestingly, their results are recovered in the limit of very large warp factor and radion mass. For the RS I model with matter localized only on the negative tension

[21] Let us recall that this is achieved by adopting the particular gauge choice for $c^{(1)}_{n,i}$ and $c^{(2)}_{n,i}$ which sets $f^{(1)}_{n,i}(t) = f^{(2)}_{n,i}(t) = 0$ and $c^{(1)}_{a,i} = c^{(2)}_{a,i} = 0$.

brane, we found negligible corrections for the Hubble parameter in the case of radiation or cosmological constant, thus supporting one of the main results of Ref. [195]. For RS II, we found negligible corrections for the equations of state just described and in the limit when the distance between the branes is taken to infinity.

On inspecting our results, we finally found some evidence of an accelerating phase for a wider range of values of the equation of state $p_n = w_n\,\rho_n$ on the negative tension brane if the distance between the branes is finite. However, one should then carry the computation to higher orders, since such an effect appears near the limit of validity of our perturbative expansion.

9 Gauss-Bonnet Brane Inflation

Following the approach used in [197] we analyze the cosmological implications of the GB action in a RS setup ([188, 189, 192, 198, 199])with radion stabilization through the addition of an effective term for the radion field to the total action. The Einstein equations are solved, assuming the adiabatic evolution of the fluids on the branes, up to the second order in ρ/V where ρ is the energy density of the fluids and V the tension of the branes. Finally the corrections to the Friedmann and acceleration equations are analyzed and compared to the case in which the GB term is switched off.

9.1 Einstein equations and Static Solutions

Let us start with the following action for the five-dimensional bulk dynamics

$$S_{bulk} = \frac{1}{k^2}\int d^5x\sqrt{-g}\left(\mathcal{L}_{EH} + \frac{\alpha}{2}\mathcal{L}_{GB}\right) \tag{522}$$

where

$$\mathcal{L}_{EH} = R + 2k^2(\Lambda - U) \tag{523}$$

$$\mathcal{L}_{GB} = R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD} \tag{524}$$

with U the effective contribution which stabilizes the size of the extra-dimension ([193, 194]). The boundary conditions for the above action are fixed by the brane contributions to the stress-energy tensor; in order to recover RS I when the GB contribution is switched off and no matter is present on the boundaries we consider two branes with different tensions located at the fifth dimension orbifold fixed points. If we choose the following ansatz for the metric

$$\begin{aligned} ds^2 &\equiv g_{AB}dx^A dx^b \\ &= -n^2(y,t)dt^2 + a^2(y,t)dx^i dx^i + b^2(y,t)dy^2 \end{aligned} \tag{525}$$

we can express the stabilizing contribution as

$$U = M(b(y,t) - r)^2 \tag{526}$$

where r is the expectation value of the radion field and the boundary terms are given by

$$T^{A}_{i\,B} = \frac{\delta(y-y_i)}{b}\mathrm{diag}(V_i+\rho_i, V_i-p_i, V_i-p_i, V_i-p_i) \tag{527}$$

where $i = p, n$, $y_i = 0, 1/2$ are the positions of the branes and V_i's are the brane tensions. By varying the total action one obtains the Einstein equations

$$G_{AB} + \alpha H_{AB} = k^2\left(\Lambda g_{AB} + \tilde{T}_{AB} + \sum_i (T_i)_{AB}\right) \tag{528}$$

where H_{AB} is the second order Lovelock tensor and $\tilde{T}^A_B = -\text{diag}(U, U, U, U, U + bdU/db)$ is the radion potential contribution. The equations (528), given the ansatz (525), form a system of four differential equations. Due to the fact G_{AB} and H_{AB} satisfy the Bianchi and Bach-Lanczos identities, respectively only three equations are independent. The bulk dynamics is thus determined by the system

$$\begin{cases} G_{tt} + \alpha H_{tt} = k^2 \left(\Lambda g_{tt} + \tilde{T}_{tt} \right) \\ \\ b' = (r - b) \left(\frac{n'}{n} + 3\frac{a'}{a} + 2\frac{b'}{b} \right) \\ \\ G_{yy} + \alpha H_{yy} = k^2 \left(\Lambda g_{yy} + \tilde{T}_{yy} \right) \end{cases} \tag{529}$$

(see [190, 191])where a prime denotes a derivative with respect to y. The boundary equations give the following junction conditions on the two branes:

$$\begin{cases} \lim_{y \to y_i^+} \left[\frac{4\alpha}{b^4} \left(\frac{a'}{a} \right)^3 - \frac{6}{b^2} \left(1 + 2\alpha \frac{\dot{a}^2}{a^2 n^2} \right) \frac{a'}{a} \right] = \frac{k^2}{b} \left(V_i + \rho_i \right) \Big|_{y = y_i} \\ \\ \lim_{y \to y_i^+} \left[\frac{4a'}{b^2 a} \left(1 + \frac{8\alpha \dot{b}^2}{b^2 n^2} - \frac{2\alpha \dot{a} \dot{n}}{a n^3} + \frac{4\alpha \dot{b} \dot{n}}{b n^3} + \frac{2\alpha \ddot{a}}{a n^2} - \frac{4\alpha \ddot{b}}{b n^2} \right) + \frac{2n'}{b^2 n} \left(1 + \frac{2\alpha \dot{a}^2}{a^2 n^2} + \frac{8\alpha \dot{a} \dot{b}}{a b n^2} + \frac{16\alpha \dot{a} \dot{n}}{a n^3} - \frac{8\alpha \ddot{a}}{a n^2} \right) \right. \\ \\ \left. \quad - \frac{4\alpha}{b^4} \left(\frac{a'}{a} \right)^2 \frac{n'}{n} - \frac{16\alpha}{b^2 n^2} \left(2\frac{\dot{b}}{b} + \frac{\dot{n}}{n} \right) \frac{\dot{a}'}{a} - \frac{16\alpha}{b^2 n^2} \left(\frac{\dot{a}}{a} \frac{\dot{n}'}{n} - \frac{\ddot{a}'}{a} \right) \right] = -\frac{k^2}{b} \left(V_i - p_i \right) \Big|_{y = y_i} \end{cases}$$

where a dot denotes a derivative with respect to the universal time t. Note that the eqs. (529-530) reduce to the standard case (without GB contribution) in the limit $\alpha \to 0$. Before investigating cosmology, in the setup just described, for $\rho_i \neq 0$, we should consider the case $\rho_i = 0$ in order to find the static solutions to perturb about. It is easy to verify that a warped metric still satisfies eqs.(529-530). In fact, if one makes the ansatz

$$a(y, t) = n(y, t) = \exp\left[-mry \right]; \quad b(y, t) = r \tag{531}$$

the Einstein equations and junction conditions are verified provided

$$m = \pm \sqrt{ \frac{1}{2\alpha} \left(1 \pm \sqrt{1 - \frac{2}{3} k^2 \alpha \Lambda} \right) } \tag{532}$$

with $\frac{2}{3} k^2 \alpha \Lambda \leq 1$ and

$$V_p = -V_n = \frac{6m}{k^2} \left(1 - \frac{2}{3} m^2 \alpha \right). \tag{533}$$

Apart from the two expected solutions which reduce to the static RS when $\alpha \to 0$, namely

$$m_{old} = \pm\sqrt{\frac{1}{2\alpha}\left(1-\sqrt{1-\frac{2}{3}k^2\alpha\Lambda}\right)} \overset{\alpha\to 0}{\longrightarrow} \pm k\sqrt{\frac{\Lambda}{6}}, \tag{534}$$

two additional solutions are obtained (see [200]) which are interestingly less sensitive to the bulk cosmological constant, since in the limit $\alpha \to 0$ the warp factor becomes independent of the bulk content

$$m_{new} = \pm\sqrt{\frac{1}{2\alpha}\left(1+\sqrt{1-\frac{2}{3}k^2\alpha\Lambda}\right)} \overset{\alpha\to 0}{\longrightarrow} \pm\frac{1}{\sqrt{\alpha}} \tag{535}$$

Furthermore, due to the GB terms, warped solutions are still present when the bulk is filled with a positive cosmological constant as (532) is still real in that case. Note that a tuning of brane tension is required only if one looks for static solutions: different tunings could be, as usual, treated as a perturbation $\rho_i = \Delta V_i$ and $p_i = -\Delta V_i$ and generate an expanding (contracting) phase which can be studied by means of the formalism we introduce in the next section.

9.2 The perturbed solutions

We now investigate the cosmological evolution in the brane-world with GB contribution. The calculations are based on the perturbative approach ($\rho_i/V_i \ll 1$) discussed in details in [197] (starting from a slightly different set of equations). The starting point is a static solution of the form (531) where [22] $m = |m_{old}|$ or $m = |m_{new}|$. When some kind of matter is added to the branes the system is detuned and evolves in time. The solution becomes

$$n(y,t) = \exp\left(-m\,r\,|y|\right)\,\left[1+\delta f_n(y,t)\right] \tag{536}$$

$$a(y,t) = a_h(t)\,\exp\left(-m\,r\,|y|\right)\,\left[1+\delta f_a(y,t)\right] \tag{537}$$

$$b(y,t) = r + \delta f_b(y,t) \tag{538}$$

and the differential equations (529) can be written in terms of $H_h \equiv \dot{a}_h/a_h$, δf_β (with $\beta = a, n, b$) and their derivatives. Moreover we note that when such a detuning is small, one can rely on the perturbative ansatz

$$\begin{aligned}
\delta f_\beta &\simeq \epsilon\,\left[f^{(1)}_{\beta,p}(y)\,\rho_p + f^{(1)}_{\beta,n}(y)\,\rho_n\right] \\
&\quad +\epsilon^2\,\left[f^{(2)}_{\beta,p}(y)\,\rho_p^2 + f^{(2)}_{\beta,n}(y)\,\rho_n^2 + f^{(2)}_{\beta,m}(y)\,\rho_p\,\rho_n\right] \qquad (539)\\
H_h^2 &\simeq \epsilon\,\left(h^{(1)}_{h,p}\,\rho_p + h^{(1)}_{h,n}\,\rho_n\right) \\
&\quad +\epsilon^2\,\left(h^{(2)}_{h,p}\,\rho_p^2 + h^{(2)}_{h,n}\,\rho_n^2 + h^{(2)}_{h,m}\,\rho_p\,\rho_n\right) \qquad (540)
\end{aligned}$$

[22] without loss of generality we choose the overall sign to be positive and consequently tune the brane at $y = 0$ with a positive tension and the brane at $y = 1/2$ with an opposite tension;

and expand these equations up to second order in ϵ. Note that the time dependence in the approximate solution above is encoded in $\rho_i(t)$ which evolves, as usual, satisfying the constraint of the continuity equation for a 4-dimensional fluid with equation of state $p_i = w_i\rho_i$ that is $\dot\rho_i = -3[\dot a(y_i,t)/a(y_i,t)](1+w_i)\rho_i$. One can finally, order by order in ϵ, solve eqs. (529), which now contain just derivatives with respect to y, and determine the integration constants and the unknowns parameters in (540) by making use of the boundary conditions (530). Iterating the procedure described above up to second order is almost straightforward (see [197] for details). Due to the fact that the ρ_i's evolve independently, one needs to solve 15 differential equations for the coefficients $f^{(1)}_{\beta,i}$, $f^{(2)}_{\beta,i}$. Once such bulk equations are solved, one is left with 25 coefficient to be fixed (10 of which derive from first order calculations) and the junctions conditions still to be imposed. It is possible to divide these coefficients into two categories: the ones which are related to the gauge freedom of the metric (they have to do with the definition of the time and of the three-dimensional scale factor $a(t)$), and the ones that are related to the boundary dynamics. The junction conditions form a system of 15 independent equations which determine just the dynamical coefficients. Five gauge coefficients can be fixed by arbitrarily defining a time evolution parameter and five (those related to the scale factor) may remain arbitrary since they are not present in the expressions of physical observables.
The Friedmann and the acceleration equations on the negative tension brane (the negative tension brane, in RS I setup with $m > 0$, is supposed to be the 4-dimensional space-time manifold in which we live) can thus be obtained by fixing $n(1/2,t) = 1$ and expressing $H^2 \equiv (\dot a(1/2,t)/a(1/2,t))^2$ and $\ddot a/a \equiv \ddot a(1/2,t)/a(1/2,t)$ as functions of the time measured on the same brane. This leads to (we let $\epsilon \to 1$ at the end of the calculations)

$$
\begin{aligned}
\frac{\ddot a}{a} = & -\frac{k^2 m\left[(3w_p+1)\mathrm{e}^{2mr}\rho_p + (3w_n+1)\rho_n\right]}{6\left(\mathrm{e}^{mr}-1\right)\left(2m^2\alpha+1\right)} + \frac{27k^2m^2\left(\mathrm{e}^{mr}+1\right)\left(w_n+1\right)\left(w_n+\frac{2}{3}\right)\left(w_n-\frac{1}{3}\right)}{32Mr^2\left(\mathrm{e}^{mr}-1\right)^2\left(2m^2\alpha+1\right)}\rho_n^2 \\
& + \left\{\frac{k^4}{144}(3w_n+1)(3w_p+1) + \frac{k^4m^2\alpha\left(\mathrm{e}^{mr}+3\right)\left(3w_p+1\right)\left(3w_n+1\right)}{72\left(\mathrm{e}^{mr}-1\right)} - \frac{m^2k^2\left(1-4m^4\alpha^2\right)}{96r^2M\left(\mathrm{e}^{mr}-1\right)^2}\right. \\
& \times\left[\mathrm{e}^{2mr}(3w_n-1)(18w_n^2+9w_pw_n+15w_p+39w_n+23) + 9\mathrm{e}^{mr}\left(6w_p^3+6w_n^3+3w_p^2w_n\right.\right. \\
& \left.+3w_pw_n^2+11w_p^2+11w_n^2+2w_pw_n+w_p+w_n-4\right) + 54w_p^3+27w_p^2w_n+99w_p^2-18w_pw_n \\
& \left.\left.+24w_p-21w_n-13\right]\right\}\frac{\mathrm{e}^{mr}\rho_p\rho_n}{\left(2m^2\alpha-1\right)\left(2m^2\alpha+1\right)^2} - \left\{\frac{m^2k^2(3w_p-1)\left(1-4m^4\alpha^2\right)\left(\mathrm{e}^{mr}+1\right)}{96r^2M\left(\mathrm{e}^{mr}-1\right)^2}\right. \\
& \left.\times\left[\mathrm{e}^{mr}(27w_p^2+54w_p+23)-9w_p-5\right] + \frac{k^4}{144}(3w_p+1)^2 - \frac{k^4m^2\alpha\left(1+3\mathrm{e}^{mr}\right)\left(3w_p+1\right)^2}{72\left(\mathrm{e}^{mr}-1\right)}\right\} \\
& \times\frac{\mathrm{e}^{2mr}\rho_p^2}{\left(2m^2\alpha-1\right)\left(2m^2\alpha+1\right)^2}
\end{aligned} \tag{541}
$$

$$
\begin{aligned}
&H^2 = \\
&\frac{k^2 m}{3\left(\mathrm{e}^{mr}-1\right)\left(2m^2\alpha+1\right)}\left(\mathrm{e}^{2mr}\rho_p+\rho_n\right)-\frac{3k^2m^2\left(1+\mathrm{e}^{mr}\right)\left(w_n+1\right)\left(w_n-\frac{1}{3}\right)}{32r^2M\left(\mathrm{e}^{mr}-1\right)^2\left(2m^2\alpha+1\right)}\rho_n^2+\left\{\frac{k^4m^2\alpha\left(\mathrm{e}^{mr}+3\right)}{18\left(\mathrm{e}^{mr}-1\right)}\right.\\
&\left.+\frac{k^4}{36}+\frac{m^2k^2\left(1-4m^4\alpha^2\right)\left(\mathrm{e}^{mr}+1\right)}{48r^2M\left(\mathrm{e}^{mr}-1\right)^2}\left[18w_p^2-9w_p(w_n-1)-9w_n-1+2\mathrm{e}^{mr}(9w_n^2+9w_n-4)\right]\right\}\\
&\times\frac{\mathrm{e}^{mr}\,\rho_p\,\rho_n}{\left(2m^2\alpha-1\right)\left(2m^2\alpha+1\right)^2}-\left\{\frac{m^2k^2(3w_p-1)\left(1-4m^4\alpha^2\right)\left(\mathrm{e}^{mr}+1\right)}{96r^2M\left(\mathrm{e}^{mr}-1\right)^2}\left[3w_p+7-4\mathrm{e}^{mr}(3w_p+4)\right]\right.\\
&\left.+\frac{k^4}{36}-\frac{k^4m^2\alpha\left(3\mathrm{e}^{mr}+1\right)}{18\left(\mathrm{e}^{mr}-1\right)}\right\}\frac{\mathrm{e}^{2mr}\,\rho_p^2}{\left(2m^2\alpha-1\right)\left(2m^2\alpha+1\right)^2}
\end{aligned} \tag{542}
$$

The above expressions are quite involved as they contain contributes from the fluids on both branes and reduce to the ones already found in [195, 197] for $\alpha \to 0$ and $m = m_{old}$. Note that, due to the choice of the time parameter, the coefficient of ρ_n^2 vanishes when ρ_n behaves as vacuum energy or radiation. As a consequence, when ρ_p is negligible, the second order terms, which would be otherwise responsible for a deviation from the standard cosmological evolution, vanish for $w_n = -1$ or $w_n = 1/3$. This important feature of brane-world scenarios with radion stabilization was already present in the case $\alpha = 0$ and is furthermore conserved when GB contribution is present. After some algebra one can partially absorb the GB coupling in (542)-(541) by redefining $\tilde{k}^2 \equiv k^2/(2m^2\alpha+1)$ which means that GB corrections can be observed only through the indirect contribution of positive tension brane matter. Furthermore note that α is always multiplied by m^2 and consequently the GB contributions vanish in the limit $\alpha \to 0$ only if we consider the case $m = m_{old}$. In the other case $m = m_{new}$, one has

$$\lim_{\alpha\to 0} \alpha \cdot m_{new}^2 = 1 \tag{543}$$

and the evolution obeys a dynamics which is modified with respect to the standard RS case. In this case, keeping just the leading contributions and letting $w_p \to -1$ (small brane detuning), one is left with

$$H^2 = \frac{k^2}{9}\left[m\tilde{\rho}_p - (3w_n+4)\left(3w_n-1\right)\quad\frac{m^2\tilde{\rho}_p\rho_n}{8r^2M}+\frac{m^2\tilde{\rho}_p^2}{2r^2M}\right] \tag{544}$$

$$\frac{\ddot{a}}{a} = \frac{k^2}{18}\left[2m\tilde{\rho}_p + (3w_n-1)\left(3w_n+4\right)\quad\left(3w_n+1\right)\frac{m^2\tilde{\rho}_p\rho_n}{8r^2M}+\frac{m^2\tilde{\rho}_p^2}{r^2M}\right] \tag{545}$$

where $\tilde{\rho}_p \equiv \mathrm{e}^{mr}\rho_p$ and one should keep $m\cdot\tilde{\rho}_p$ and $m\cdot\rho_n$ small. Apart from the contribution of $\tilde{\rho}_p$ to the effective cosmological constant, the term proportional to ρ_n^2 is negligible and one observes the usual contribution to the expansion rate at $w_n = -1$ or $w_n = 0$.
Finally we note that, when $M \to \infty$, some of the second order terms vanish, as they did in the standard case $\alpha = 0$. In fact these terms derive from the radion dynamics: such dynamics is sensible to the state equation of the fluids on the branes and generates a complicated w_i dependence in (542-541). In the limit $M \to \infty$ when the radion is fixed to the minimum of the stabilizing potential r it becomes trivial and these terms vanish.

9.3 Summary

We have examined the four-dimensional cosmological equations deriving from the Einstein equations for a brane-world with a stabilizing potential in the presence of a Gauss-Bonnet term in the action. We found solutions performing an expansion up to the second order in ρ/V in order to examine the cosmological behavior on the negative tension brane. The formalism can be easily extended to describe the positive tension brane as well.
Due to the Gauss-Bonnet extra terms, the system admits two different static solutions: one behaves as the usual RS when the Gauss-Bonnet coupling α goes to zero, while the second one has a warping factor independent of the bulk content in the limit $\alpha \to 0$.
At first order the deviations from standard (without GB term) equations can be reabsorbed with a redefinition of the 4-dimensional Newton constant. The same holds when one considers second order equations with a vanishing ρ_p. On the other hand, if one considers the contributions due to matter perturbations on the positive tension brane some deviations appear. An interesting feature emerging is that in the limit $\alpha \to 0$ such deviations are not swept away when perturbing the solution that does not reduce to the usual RS.

We would like to thank R. Casadio for the helpful comments.

10 Conclusions

We have described some features of High Energy Cosmology within the context of inflationary and extra-dimensional models of our early universe. We have applied the basic ingredients of these models to obtain predictions on corrections to the CMB spectrum and to build models potentially explaining the observed baryonic asymmetry and the current cosmological acceleration.

A Evolution of the curvature perturbation on superhorizon scales

In this appendix, we repeat the general arguments following from energy-momentum conservation given in Ref. [75] to show that the curvature perturbation on constant-time hypersurfaces ψ is constant on superhorizon scales if perturbations are adiabatic.

The constant-time hypersurfaces are orthogonal to the unit time-like vector field $n^\mu = (1-A, -\partial^i B/2)$. Local conservation of the energy-momentum tensor tells us that $T^\mu_{\nu;\mu} = 0$. The energy conservation equation $n^\nu T^\mu_{\nu;\mu} = 0$ for first-order density perturbations and on superhorizon scales give

$$\delta\dot{\rho} = -3H\left(\delta\rho + \delta p\right) - 3\dot{\psi}\left(\rho + p\right).$$

We write $\delta p = \delta p_{\rm nad} + c_s^2 \delta\rho$, where $\delta p_{\rm nad}$ is the non-adiabatic component of the pressure perturbation and $c_s^2 = \delta p_{\rm ad}/\delta\rho$ is the adiabatic one. In the uniform-density gauge $\psi = \zeta$ and $\delta\rho = 0$ and therefore $\delta p_{\rm ad} = 0$. The energy conservation equation implies

$$\dot{\zeta} = -\frac{H}{p+\rho}\,\delta p_{\rm nad}.$$

If perturbations are adiabatic, the curvature on uniform-density gauge is constant on superhorizon scales. The same holds for the comoving curvature $\mathcal{R}$ as the latter and ζ are equal on superhorizon scales, see section 7.

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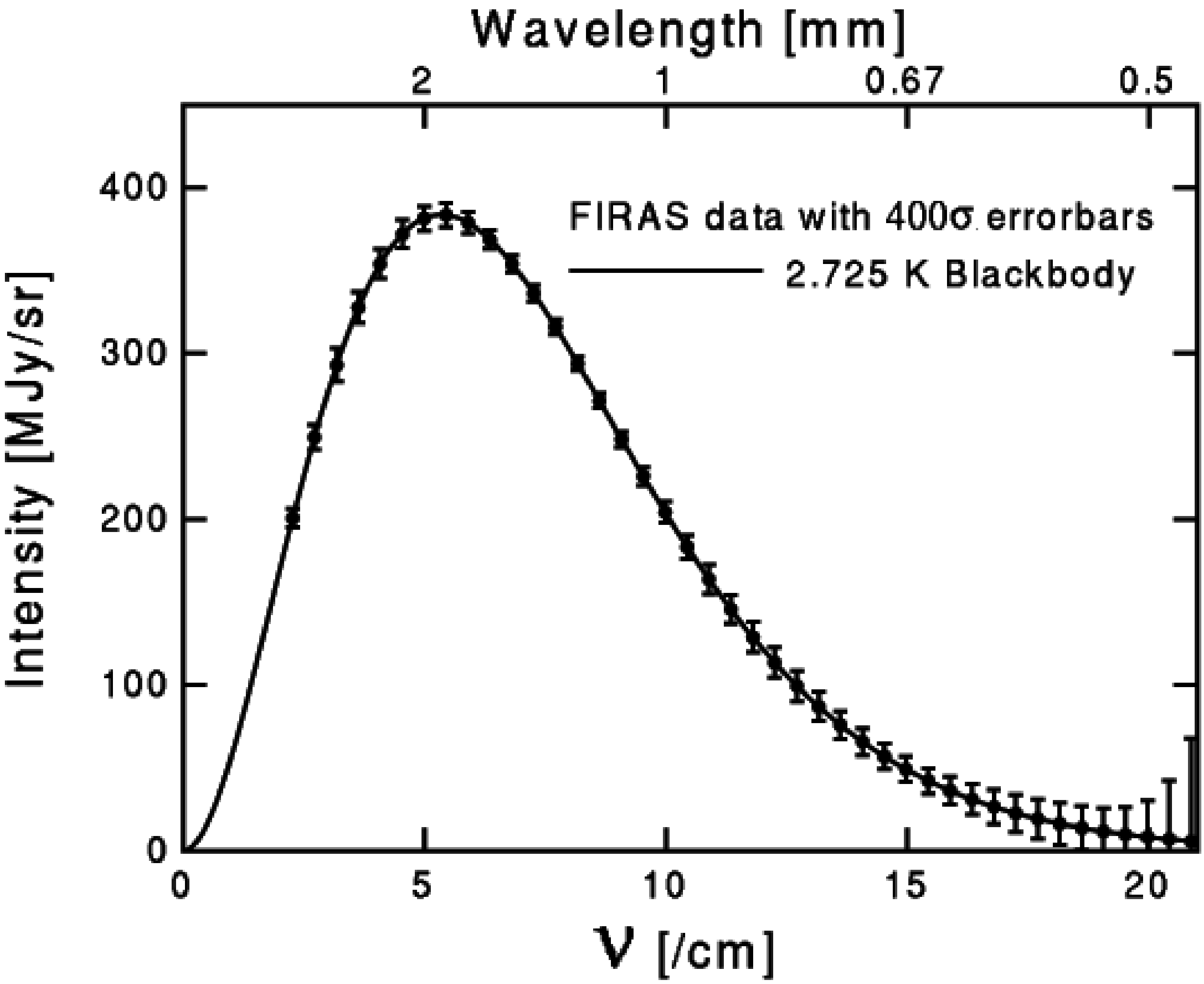

Figure 1: The black body spectrum of the cosmic background radiation

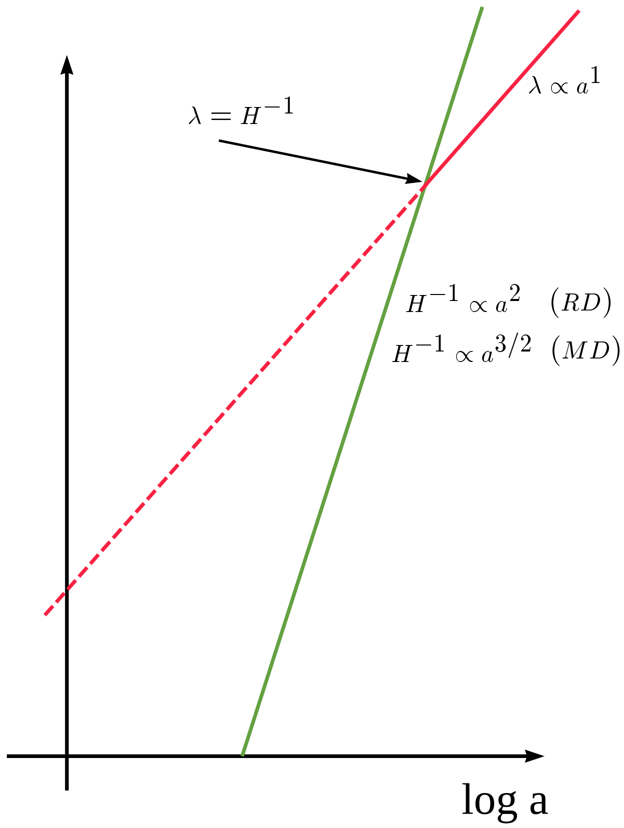

Figure 2: The horizon scale (green line) and a physical scale λ (red line) as function of the scale factor a. From Ref. [46].

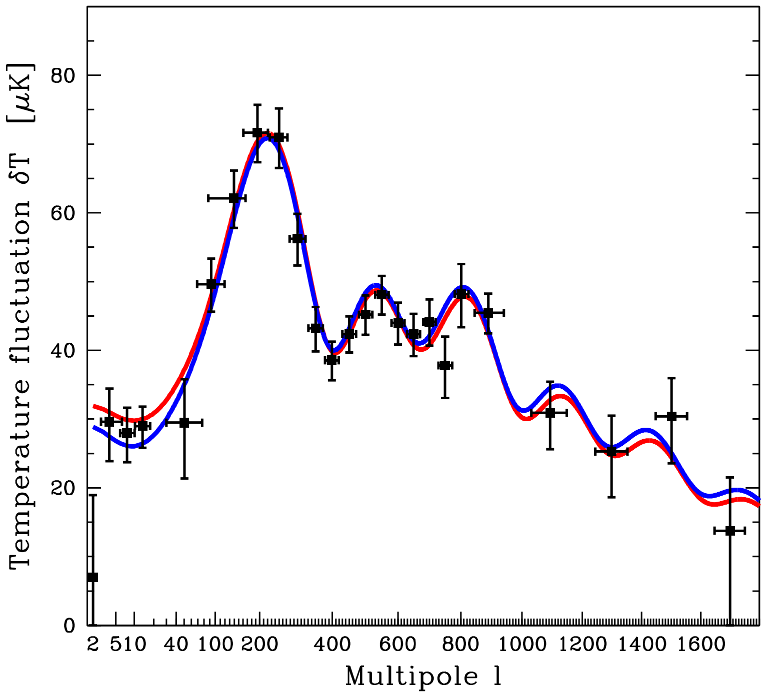

Figure 3: The CMBR anisotropy as function of ℓ. From Ref. [74].

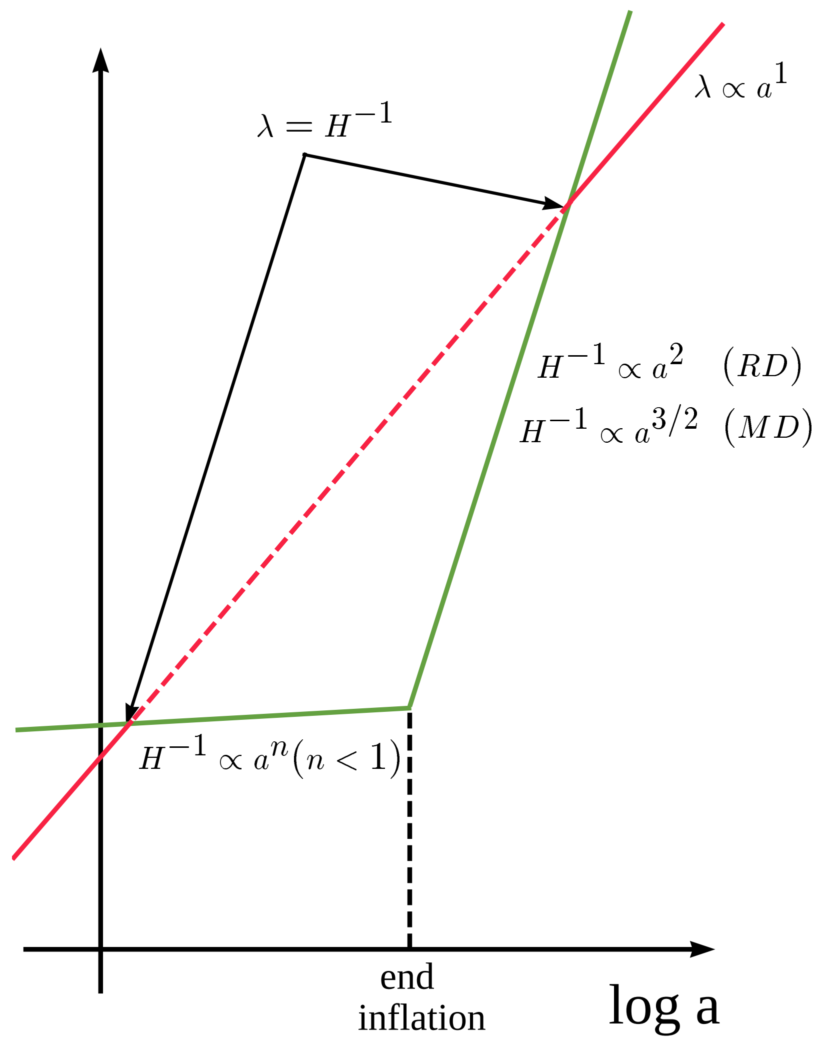

Figure 4: The behaviour of a generic scale λ and the horizon scale H^{-1} in the standard inflationary model. From Ref. [46].

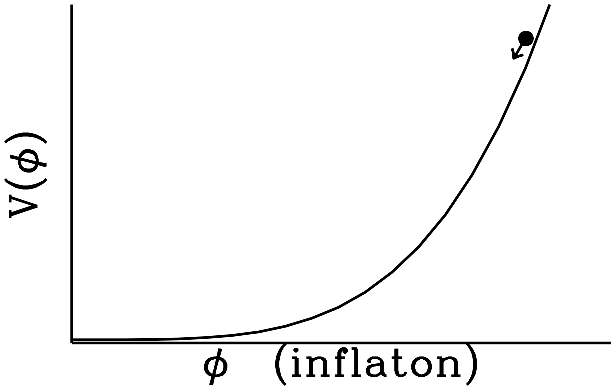

Figure 5: Large field models of inflation. From Ref. [46].

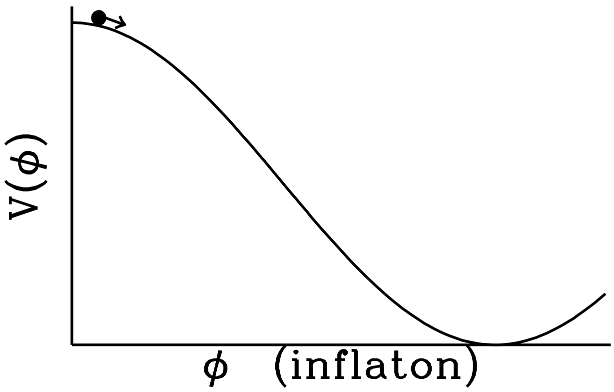

Figure 6: Small field models of inflation. From Ref. [46].

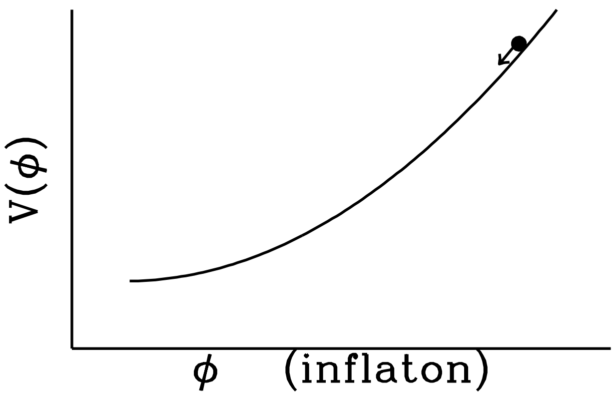

Figure 7: Hybrid field models of inflation. From Ref. [46].

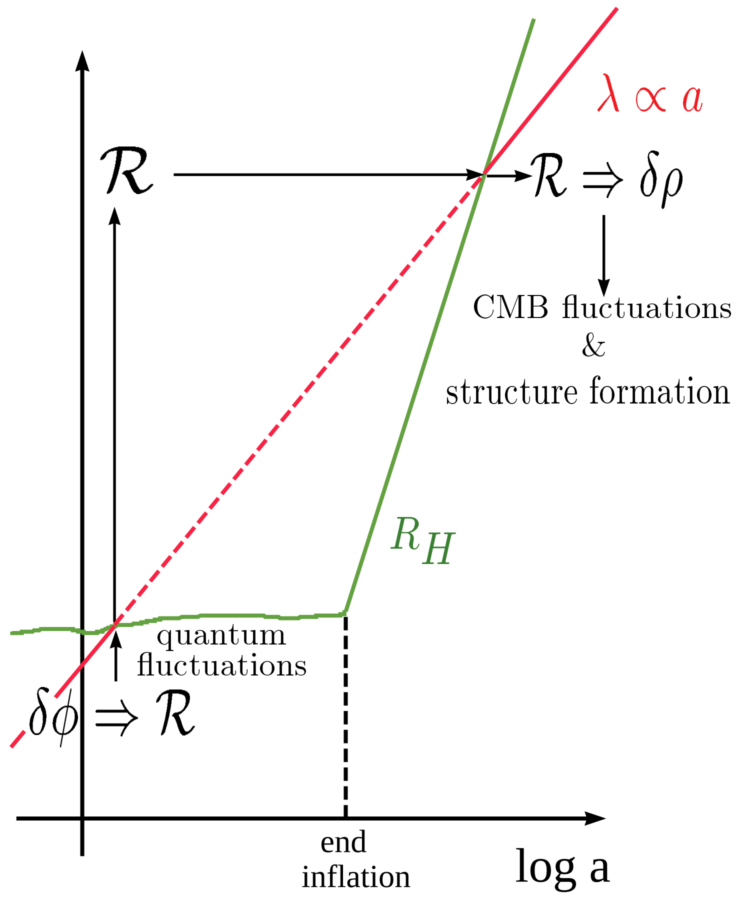

Figure 8: The horizon scale (green line) and a physical scale λ (red line) as function of the scale factor a. From Ref. [46].

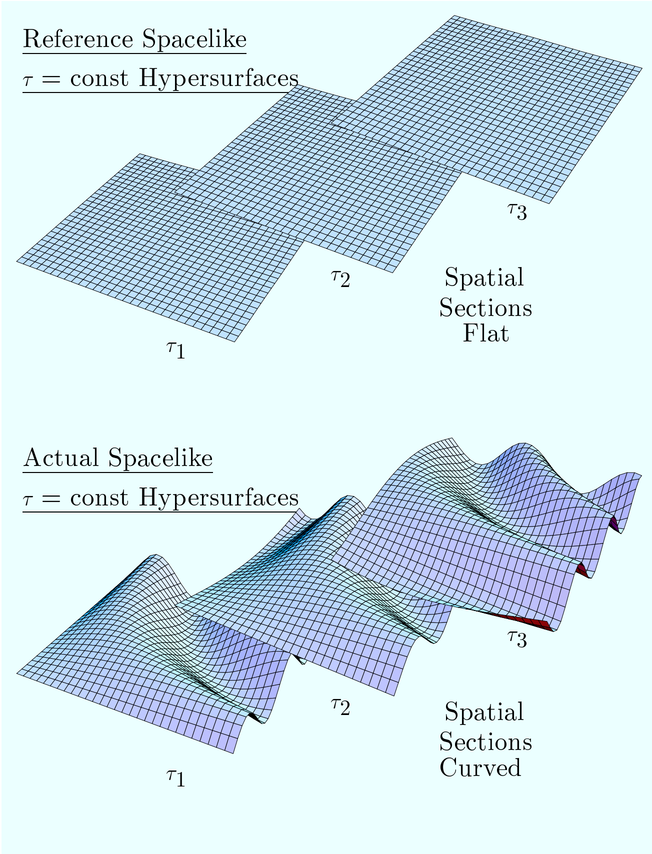

Figure 9: In the reference unperturbed universe, constant-time surfaces have constant spatial curvature (zero for a flat FRW model). In the actual perturbed universe, constant-time surfaces have spatially varying spatial curvature. From Ref. [46].

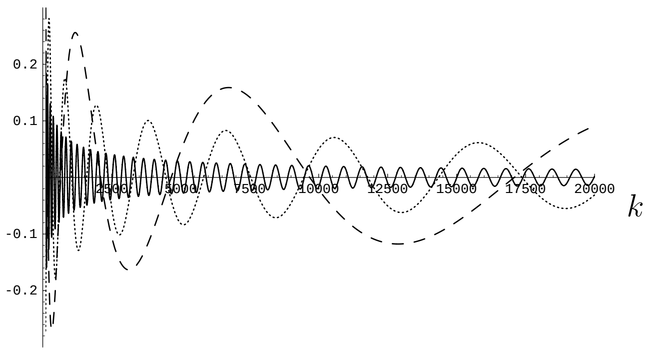

Figure 10: The ratio R_q for P_ϕ in Eq. (402) and $q = 2$ (solid line), $q = 3/2$ (dotted line) and $q = 4/3$ (dashed line). The momentum index k is in units with $\Lambda = \bar{\eta} = 1$ and the regions of physical interest are those for $k > 10^2$ ($q = 2$), $k > 10^3$ ($q = 3/2$) and $k > 10^4$ ($q = 4/3$).

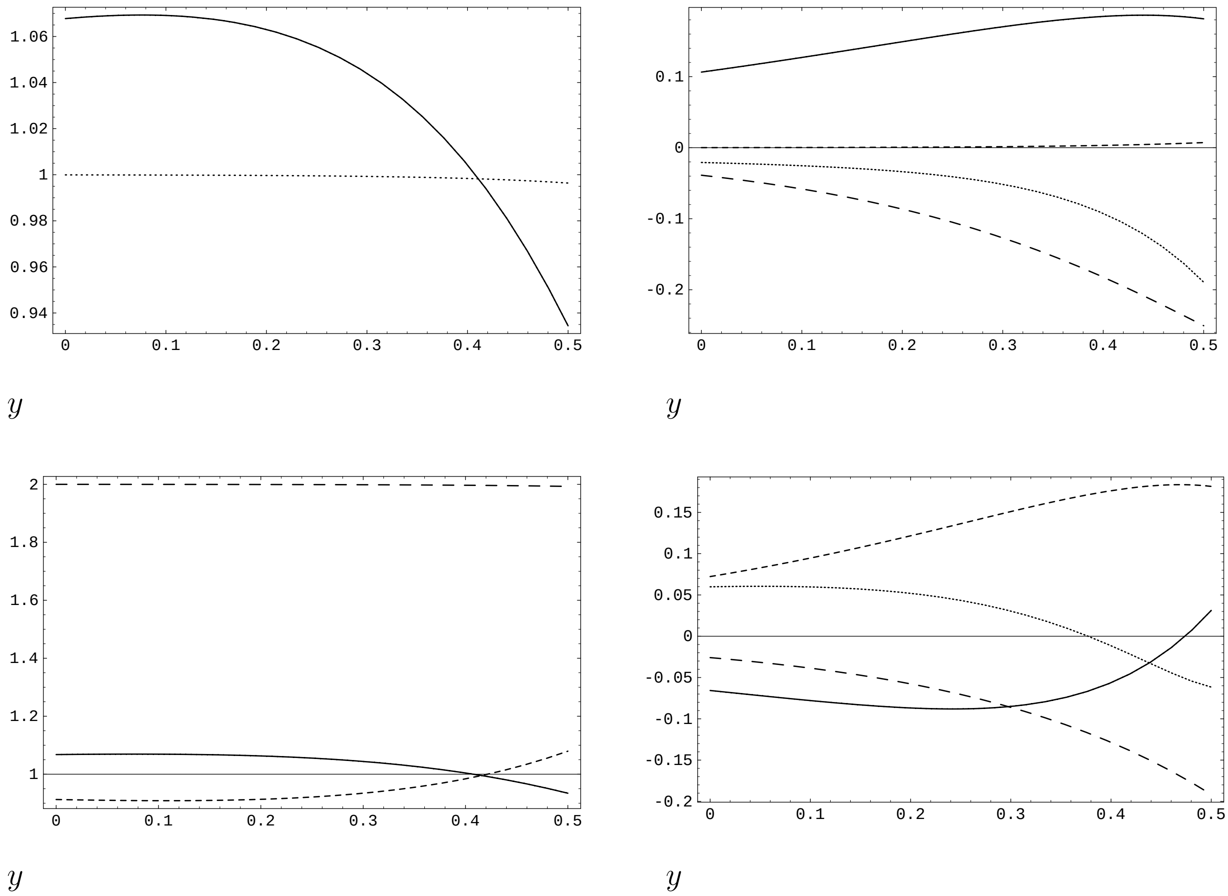

Figure 11: The graphs on the left contain the plot of the absolute values of the leading terms among η_l (above) and ξ_l (below) to $\mathcal{O}(\epsilon^2)$. The graphs on the right show the subleading terms among η_l (above) and ξ_l (below) and the corresponding remainders R_η and R_ξ to $\mathcal{O}(\epsilon^2)$ (dotted lines). All plots are for $\rho_i = 2 \cdot 10^{-1}\, M^4$ and $w_i = 0.5$ and cover all the bulk between the two branes.

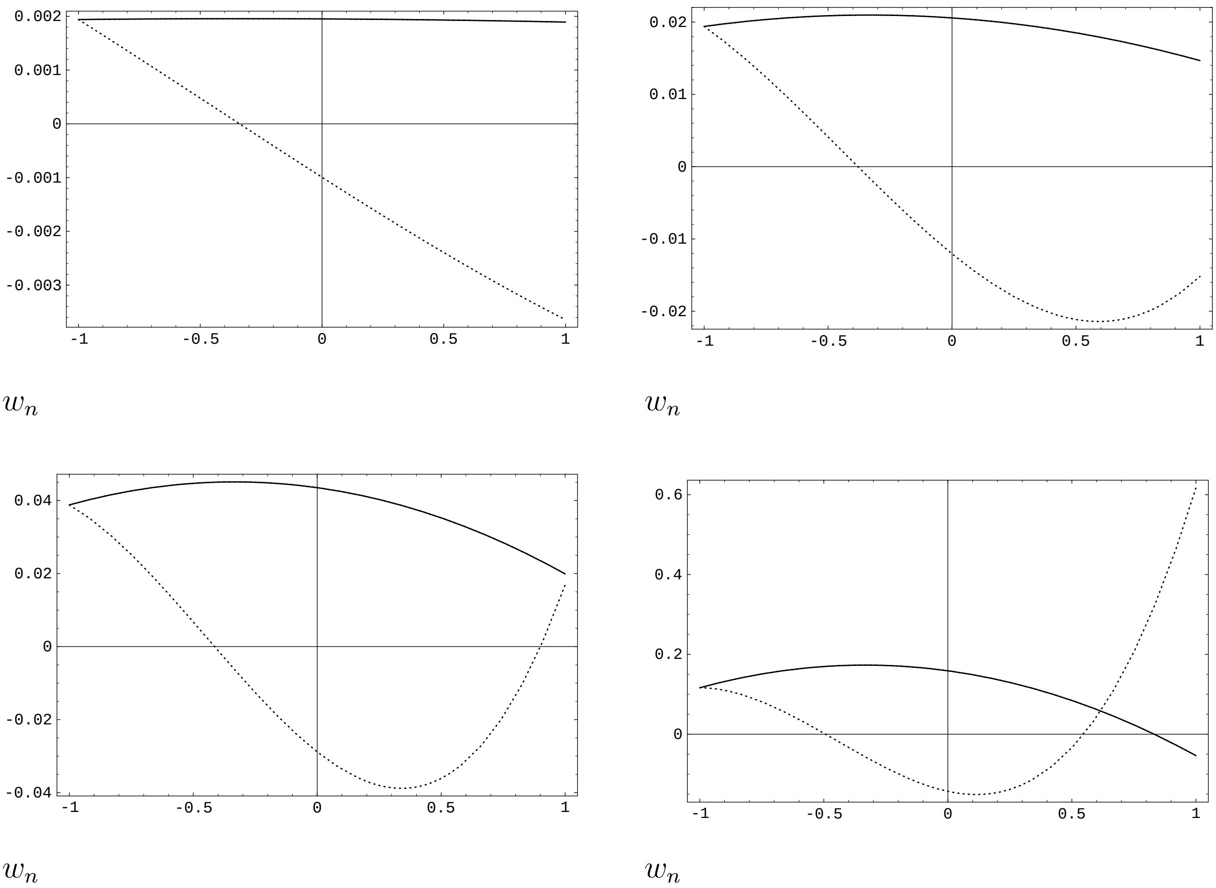

Figure 12: Plots of $H^2(1/2,\tau)$ (solid line) and $\ddot{a}(1/2,\tau)/a(1/2,\tau)$ (dotted line) to $\mathcal{O}(\epsilon^2)$ at a given time, as functions of w_n, for $\rho_n = 10^{-2}\, M^4$, $10^{-1}\, M^4$, $2 \cdot 10^{-1}\, M^4$ and $6 \cdot 10^{-1}\, M^4$ (from top left to bottom right).

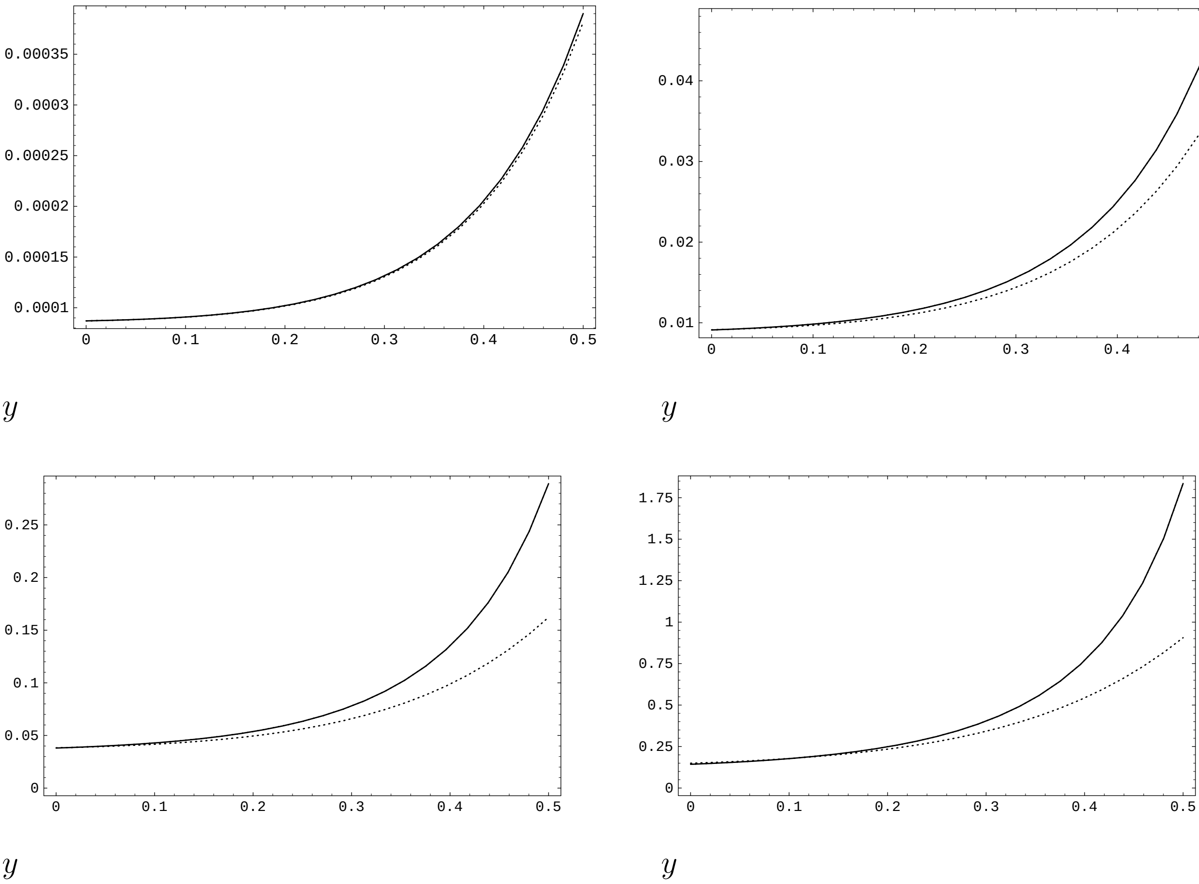

Figure 13: The two graphs in the first row show the ratio between R_η to $\mathcal{O}(\epsilon^2)$ and $\eta_1 \sim H^2$ to $\mathcal{O}(\epsilon)$ (dotted line) and to $\mathcal{O}(\epsilon^2)$ (solid line) at a given time, for $\rho_n = 10^{-2}\,M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 10^{-1}\,M^4$ and $w_n = 0.95$ (right). In the two graphs in the second row, the same ratios are are given for $\rho_n = 2 \cdot 10^{-1}\,M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 6 \cdot 10^{-1}\,M^4$ and $w_n = 0.65$ (right). The plots cover all the bulk between the two branes.

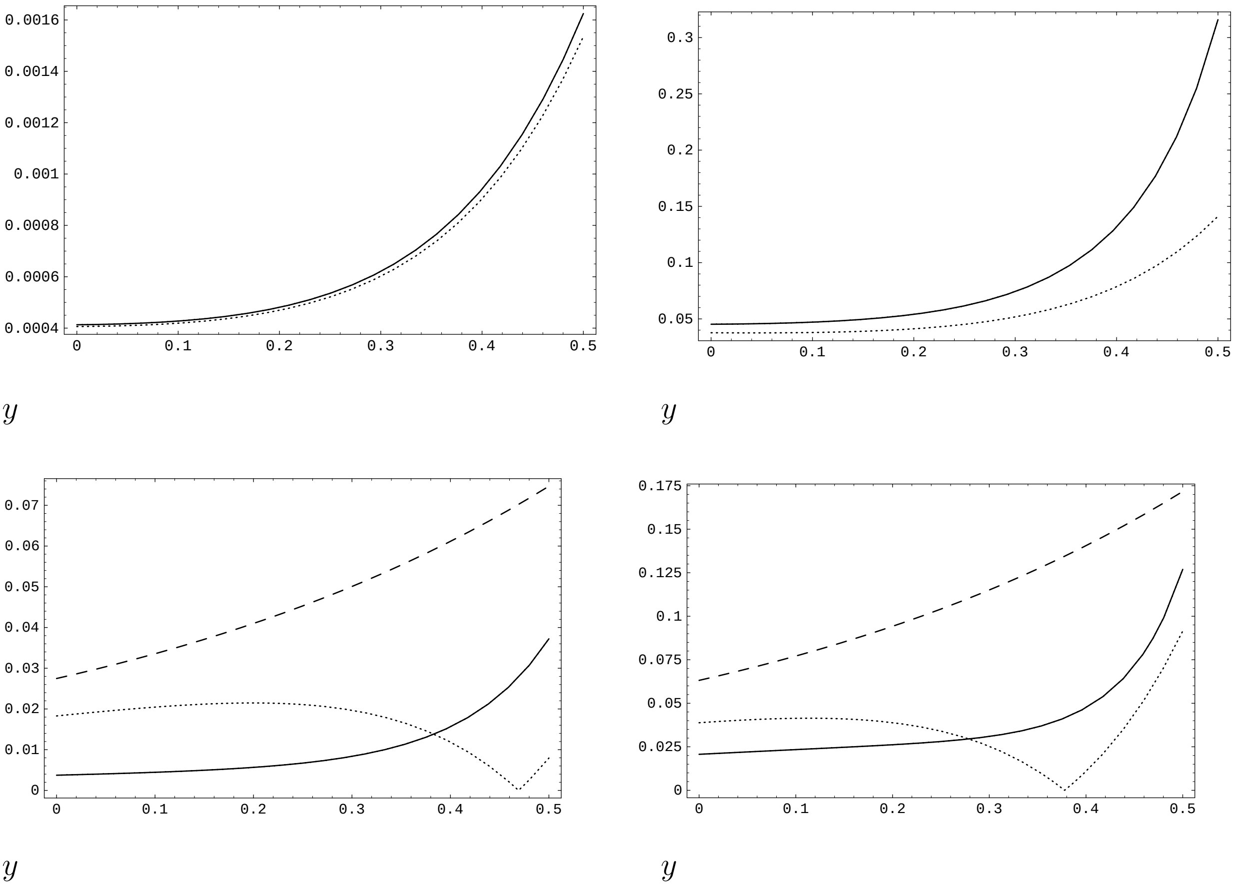

Figure 14: The two graphs in the first row show the ratio between R_ξ to $\mathcal{O}(\epsilon^2)$ and $\xi_1 \sim \ddot{a}/a$ to $\mathcal{O}(\epsilon)$ (dotted line) and to $\mathcal{O}(\epsilon^2)$ (solid line) at a given time, for $\rho_n = 10^{-2}\, M^4$ and $w_n = 0.95$ (left) and for $\rho_n = 10^{-1}\, M^4$ and $w_n = 0.95$ (right). In the two graphs in the second row, R_ξ to $\mathcal{O}(\epsilon^2)$ (solid line) is compared to the modula of ξ_1 to $\mathcal{O}(\epsilon)$ (dashed line) and to $\mathcal{O}(\epsilon^2)$ (dotted line) for $\rho_n = 2{\cdot}10^{-1}\, M^4$ and $w_n = 0.95$ (left), and for $\rho_n = 6{\cdot}10^{-1}\, M^4$ and $w_n = 0.65$ (right). The plots cover all the bulk between the two branes.