Polynomial algebra associated to the Cartan subalgebra of G_2 in its enveloping algebra

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Abstract. The commutant of the Cartan subalgebra of G_2 in its enveloping algebra is determined, showing that it gives rise to a 66-dimensional polynomial algebra of sixth order. It is shown that the commutant of the regular subalgebra A_2 can be obtained by restriction. An illustration how to use these results for Hamiltonian systems with G_2 as spectrum generating algebra is given.

1. Introduction

During the last years, there has been a wide interest to find purely algebraic formulations of superintegrable systems, by associating them to polynomial algebras of various types, such as subalgebras of enveloping algebras, Racah algebras and generalizations [1, 2, 3]. In this context, large series of polynomial algebras related to the classical semisimple series have been studied [4, 5], trying to find characterizations that lead to new superintegrable systems. Most of these approaches are formulated for specific systems realized in terms of differential operators, which does not always allow us to determine easily all the structural properties. One advantage of the algebraic ansatz is the absence of a reference to a given realization, a fact that enables us to define a generic notion of algebraic Hamiltonians and their corresponding (algebraic) constants of the motion [6]. With this strategy, it was shown in [4] that the commutant of the Cartan subalgebra for the classical series A_n gives rise to a polynomial algebra deeply related to the Racah algebra $R(n)$, that further correspond to the symmetry algebras of generic superintegrable systems on spheres $\mathbb{S}^{n-1}$ (see [7]). The algebraic problem can easily be reformulated in analytical terms using the symmetric algebra [8]. This further shows the close relation to the so-called missing label problem [9], hence suggesting that spectrum generating algebras and dynamical symmetries, as special types of physically relevant systems [10], can also be studied via a purely algebraic formalism based on enveloping algebras. On the other hand, this approach connects with the embedding problem of Lie algebras in enveloping algebras, which is still not completely solved [11].

Besides the case of A_n, the commutant of the Cartan subalgebra for the remaining classical series has been systematically analyzed in [12], showing that the root systems play a relevant role in the structure of the resulting polynomial algebra. For the exceptional Lie algebras, this analysis is still missing, mainly due to computational obstructions. In this work, we inspect the case of the rank-two exceptional Lie algebra G_2, showing that it gives rise a a polynomial algebra of sixth order. We further show how to recover the polynomial algebras corresponding to (regular) subalgebras containing the Cartan subalgebra, and how to use these results for the construction of constants of the motion of Hamiltonians with spectrum generating algebra G_2.

2. The commutant of Lie subalgebras in enveloping algebras

Given a real or complex (semisimple) Lie algebra $\mathfrak{s}$, we denote its universal enveloping algebra by $\mathcal{U}(\mathfrak{s})$. For any positive integer p, we consider the subspace $\mathcal{U}_{(p)}(\mathfrak{g})$ generated by the monomials $X_1^{a_1}\dots X_n^{a_n}$ satisfying the numerical constraint $a_1+a_2+\cdots+a_n\leq p$, where $\{X_1,\dots,X_n\}$ is an arbitrary (ordered) basis of $\mathfrak{s}$. We say that an element $P\in\mathcal{U}(\mathfrak{s})$ has degree d if $d=\inf\left\{k\mid P\in\mathcal{U}_{(k)}(\mathfrak{s})\right\}$. As $\mathcal{U}(\mathfrak{s})$ is a naturally filtered algebra [13], for $p,q\geq 0$ the following inclusions hold

$$\mathcal{U}_{(0)}(\mathfrak{s})=\mathbb{C},\ \mathcal{U}_{(p)}(\mathfrak{s})\mathcal{U}_{(q)}(\mathfrak{s})\subset\mathcal{U}_{(p+q)}(\mathfrak{s}). \tag{1}$$

It can be shown that each $\mathcal{U}_{(p)}(\mathfrak{s})$ is a finite-dimensional representation of $\mathfrak{s}$, from which a decomposition of $\mathcal{U}(\mathfrak{s})$ as a sum of finite-dimensional representations of $\mathfrak{s}$ can be deduced (see [13]).

On the given basis, the adjoint action of $\mathfrak{s}$ on the enveloping algebra $\mathcal{U}(\mathfrak{s})$ and the associated symmetric algebra $S(\mathfrak{s})$ is given by (see e.g. [8])

$$\begin{aligned} P\in\mathcal{U}(\mathfrak{s})\mapsto\quad & P.X_i:=[X_i,P]=X_iP-PX_i\in\mathcal{U}(\mathfrak{s}),\\ P\left(x_1,\dots,x_n\right)\in S(\mathfrak{s})\mapsto\quad & \widehat{X}_i(P)=C_{ij}^k x_k\frac{\partial P}{\partial x_j}\in S(\mathfrak{s}), \end{aligned} \tag{2}$$

where C_{ij}^k denote the structure constants over the given basis. It can be easily verified that the differential operator $\widehat{X}_i=C_{ij}^k x_k\dfrac{\partial}{\partial x_j}$ is the infinitesimal generator of the 1-parameter subgroup associated to X_i through the coadjoint representation [14]. The symmetric algebra $S\left(\mathfrak{s}\right)$, that can be identified with $\mathbb{K}\left[x_1,\dots,x_n\right]$ ($\mathbb{K}=\mathbb{R},\mathbb{C}$), inherits naturally the structure of a Poisson algebra through the Berezin (or Lie–Poisson) bracket [15]

$$\{P,Q\}=C_{ij}^k x_k\frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial x_j},\ P,Q\in S\left(\mathfrak{s}\right). \tag{3}$$

By means of the standard symmetrization map

$$\Lambda\left(x_{j_1}\dots x_{j_p}\right)=\frac{1}{p!}\sum_{\sigma\in\boldsymbol{\Sigma}_p}X_{j_{\sigma(1)}}\dots X_{j_{\sigma(p)}}, \tag{4}$$

where Σ_p denotes the symmetric group of p letters, the canonical linear isomorphism Λ from $S(\mathfrak{s})$ onto $\mathcal{U}(\mathfrak{s})$ that commutes with the adjoint action is obtained [8]. It should however be observed that it is not in general an algebraic isomorphism [13]. If $S^{(p)}(\mathfrak{s})$ denotes the homogeneous polynomials of degree p, defining $\mathcal{U}^{(p)}(\mathfrak{s})=\Lambda\left(S^{(p)}(\mathfrak{s})\right)$ we get $\mathcal{U}_{(p)}(\mathfrak{s})=\sum_{k=0}^{p}\mathcal{U}^{(k)}(\mathfrak{s})$. From this it is straightforward to infer that for $P\in\mathcal{U}_{(p)}(\mathfrak{s}),Q\in\mathcal{U}_{(q)}(\mathfrak{s})$, the commutator satisfies

$$[P,Q]\in\mathcal{U}_{(p+q-1)}(\mathfrak{s}).$$

In this context, we define the commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ of an arbitrary subalgebra $\mathfrak{a}\subset\mathfrak{s}$ as the centralizer of the subalgebra in $\mathcal{U}(\mathfrak{g})$, i.e.

$$C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})=\left\{Q\in\mathcal{U}(\mathfrak{s})\mid[P,Q]=0,\quad\forall P\in\mathfrak{a}\right\}. \tag{5}$$

A special case is given for $\mathfrak{a} = \mathfrak{s}$, in which case we obtain the centre of $\mathcal{U}(\mathfrak{s})$

$$Z\left(\mathcal{U}(\mathfrak{s})\right) = \left\{P \in \mathcal{U}(\mathfrak{s}) \mid [\mathfrak{s}, P] = 0\right\}, \tag{6}$$

consisting of the invariant polynomials of $\mathfrak{s}$, i.e., their Casimir operators [14]. If a commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ is finitely generated (e.g., when it satisfies the Noetherian property), we can find a set $\{P_1, \ldots, P_s\}$ of linearly independent polynomials,[1] such that any element in $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ can be represented as

$$P_1^{a_1} P_2^{a_2} \ldots P_s^{a_s}, \quad a_i \in \mathbb{N} \cup 0, \tag{7}$$

where the scalars a_i satisfy certain algebraic relations (see [4] and references therein). In this situation, we say that the linear dimension of $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ is $\dim_L C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = s$, where the subindex L indicates linearity. In particular, $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ contains the Casimir operators of $\mathfrak{a}$, as well as those $C_1, \ldots, C_\ell$ of $\mathfrak{s}$. As these commute with each element in $\mathfrak{s}$, it follows that $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ possesses the structure of a free module over $\mathbb{C}\left[C_1, \ldots, C_\ell\right]$ (see [13] for details).

Computationally, it is more convenient to translate the problem of determining commutants to the analytic frame, using the canonical linear isomorphism Λ. More precisely, for $\mathfrak{a} \subset \mathfrak{s}$ we consider the centralizer

$$C_{S(\mathfrak{s})}\left(\mathfrak{a}\right) = \left\{Q \in S\left(\mathfrak{s}\right) \mid \{P, Q\} = 0,\ P \in \mathfrak{a}\right\}$$

with respect to the bracket (3). This enables us to compute these elements as the polynomials satisfying the linear first-order system of partial differential equations

$$\widehat{X_i}\left(Q\right) := \{x_i, Q\} = C_{ij}^k x_k \frac{\partial Q}{\partial x_j} = 0,\ 1 \leq i \leq m = \dim \mathfrak{a}, \tag{8}$$

where the $\{x_1, \ldots, x_m\}$ correspond to the coordinates in a dual basis of $\mathfrak{a}$ (see equation (2)). As is well known, the number of functionally independent solutions of (8) is given by $r_0 = \dim \mathfrak{s} - \mathrm{rank}(A)$, where A is the $m \times n$-matrix with entries $\left(C_{ij}^k x_k\right)$ corresponding to the equations of the subalgebra generators [14, 16]. The index m makes reference to the number of subalgebra generators, while n indicates the number of variables in which these are realized. Clearly, r_0 merely provides an upper bound for the number of independent polynomials, as the system may have rational or even transcendental solutions [8].

This approach can be used to construct formal superintegrable systems from algebraic structures (see e.g. [17]), leading to alternative notions of integrability and superintegrability [6]. In this frame, given $\mathfrak{a} \subset \mathfrak{s}$ and the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$, we define an algebraic Hamiltonian (with respect to the subalgebra) by means of the expression

$$\mathcal{H}_a = \sum_{i,j} \alpha_{ij} X_i X_j + \sum_k \beta_k X_k + \sum_\ell \gamma_\ell C_\ell, \quad X_i, X_j, X_k \in \mathfrak{a}; \quad \alpha_{ij}, \beta_k, \gamma_\ell \in \mathbb{K}, \tag{9}$$

where C_ℓ is a Casimir operator of $\mathfrak{s}$. Elements in the commutant correspond to constants of the motion of the Hamiltonian $\mathcal{H}_a$. The analytical counterpart has the form

$$\mathcal{H} = \sum_{i,j} \alpha_{ij} x_i x_j + \sum_k \beta_k x_k + \sum_\ell \gamma_\ell \mathcal{C}_\ell, \tag{10}$$

[1] The elements are generally functionally dependent, but it should be observed that these dependence relations are usually defined in the field of fractions, and not in $\mathcal{U}(\mathfrak{s})$ [8].

where $\mathcal{C}_\ell$ is the symmetric counterpart of C_ℓ . For a polynomial P_0 in the symmetric algebra and its image $P = \Lambda(P_0)$ in the enveloping algebra (see equation (4)), the Lie-Poisson bracket $\{\mathcal{H}_a, P_0\} = 0$ is satisfied.

Depending on the number of independent integrals of the motion obtained, two cases can arise:

(i) Polynomials in $C_{S(\mathfrak{s})}(\mathfrak{a})$ all have trivial Berezin bracket. Then $\mathcal{H}_a$ is integrable with an abelian symmetry algebra.

(ii) $C_{S(\mathfrak{s})}(\mathfrak{a})$ contains non-commuting elements. Then the algebraic Hamiltonian $\mathcal{H}_a$ has a non-commutative (super-)integrability property and the symmetry algebra is non-abelian and polynomial.

Now, considering a realization by differential operators of $\mathfrak{s}$, the previous operators eventually factorize or satisfy additional dependence relations, reducing the number of independent constants of the motion. Using this approach, several relevant superintegrable systems have been analyzed from an algebraic perspective [2, 4].

The argument can also be used for reduction chains of (reductive) Lie algebras. Suppose that we have a chain of subalgebras

$$\mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \cdots \subset \mathfrak{s}.$$

Then clearly $C_{S(\mathfrak{s})}(\mathfrak{s}_1)$ contains the commutant of any term $\mathfrak{s}_i$ in $S(\mathfrak{s})$, so that we obtain the dual chain

$$C_{S(\mathfrak{s})}(\mathfrak{s}_1) \supset C_{S(\mathfrak{s})}(\mathfrak{s}_2) \supset \cdots \supset C_{S(\mathfrak{s})}(\mathfrak{s}) . \tag{11}$$

As the last term corresponds to the Casimir invariants of $\mathfrak{s}$, it follows that any commutant is a (non-abelian) extension of $Z(S(\mathfrak{s}))$, consisting of the analytic analogues of the Casimir operators of $\mathfrak{s}$. In particular, the invariant operators of $\mathfrak{s}$ and the subalgebra $\mathfrak{a}$ can always be expressed as polynomials in the elements of the commutant.

3. Missing label operators

The computation of commutants in enveloping algebras can be seen as a special case, restricted to the class of polynomials, of the more general "internal labelling problem" (see e.g. [18, 19]). Restricting our attention to the case of semisimple algebras, which is the most relevant in this context [10], it is often convenient to describe the representations of a Lie algebra $\mathfrak{s}$ with respect to some distinguished subalgebra $\mathfrak{s}'$ of rank ℓ' that may correspond to some internal symmetry. Often, however, the labels provided by the chain are not sufficient to separate state degeneracies that may appear. The inner distinction of states of $\mathfrak{s}$-representations require $\frac{1}{2}(\dim\mathfrak{s}-\ell)$ labels,[2], from which the subalgebra $\mathfrak{s}'$ provides $\frac{1}{2}(\dim\mathfrak{s}'+\ell')$ labels. It may happen that $\mathfrak{s}'$ and $\mathfrak{s}$ have some Casimir operators (ℓ_0 in number) in common, i.e., that some invariants of $\mathfrak{s}'$ are actually $\mathfrak{s}$-invariant.[3] As the identity

$$\frac{1}{2}(\dim\mathfrak{s}-\ell) = n_0 + \frac{1}{2}(\dim\mathfrak{s}'+\ell') - \ell_0$$

must be satisfied, we conclude that the number n_0 of required inner labels is given by

$$n_0 = \frac{1}{2}\left(\dim\mathfrak{s} - \ell - \dim\mathfrak{s}' - \ell'\right) + \ell_0 \tag{12}$$

[2] The ℓ labels corresponding to eigenvalues of Casimir operators do not separate states, but only determine the representation as a whole.

[3] This situation can only appear if $\mathfrak{s}$ decomposes as a direct sum $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{m}$ of Lie algebras.

operators to separate the irreducible representations (IRs in short) of $\mathfrak{s}'$ that appear with multiplicity greater than one in the decomposition of Γ. Such operators commute with the generators of $\mathfrak{s}'$, and are commonly called missing label operators (MLO in short) or subgroup scalars. In the context of the symmetric algebra, missing label operators are obtained as solutions of the system (8). In particular, polynomial operators are clearly seen to belong to the commutant of $\mathfrak{s}'$ in the enveloping algebra of $\mathfrak{s}$, implying that $n_0 < r_0$. The converse also holds, namely, any element in the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$ is a subgroup scalar. In applications to representations, to prevent undesired interactions and to allow simultaneous diagonalization, these operators are additionally required to commute with each other [18]. In terms of commutants, this means that for labelling representations, we always use a maximal abelian subalgebra of the polynomial algebra $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$.

If $\mathfrak{s}' \subset \mathfrak{s}$ is an embedding of Lie algebras, it induces branching rules of representations [20]. In particular, the adjoint representation of $\mathfrak{s}$ decomposes as:

$$\mathrm{ad}(\mathfrak{s}) = \mathrm{ad}(\mathfrak{s}') \oplus R, \tag{13}$$

where R is a (generally reducible) representation of $\mathfrak{s}'$ called the characteristic representation. To compute the missing labels analytically, we can proceed as follows. Let $\{X_1, \ldots, X_m\}$ be a basis of $\mathfrak{s}'$ and extend it to a basis $\mathfrak{B} = \{X_1, \ldots, X_m, Y_1, \ldots Y_{n-m}\}$ of $\mathfrak{s}$, where $m = \dim(\mathfrak{s}')$. The brackets adopt the form:

$$[X_i, X_j] = C_{ij}^k X_k, \quad [X_i, Y_p] = D_{ip}^q Y_q, \quad [Y_p, Y_q] = E_{pq}^k X_k + F_{pq}^r Y_s,$$

where $i, j, k \in \{1, \ldots, m\}$ and $p, q, r \in \{1, \ldots, n-m\}$. Now we consider those differential operators that are associated to generators of $\mathfrak{s}'$, i.e., the system of PDEs

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j} - D_{ip}^q y_q \frac{\partial}{\partial y_p}, \quad 1 \leq i \leq m. \tag{14}$$

where $\{x_1, \ldots, x_m, y_1, \ldots, y_{n-m}\}$ are the coordinates in a dual basis of $\mathfrak{B}$. We observe that solutions F to the system (14) such that $\dfrac{\partial F}{\partial y_p} = 0$ for all $1 \leq p \leq n-m$ correspond to the Casimir invariants of the subalgebra, while a genuine missing label operator must explicitly depend on the variables $\{y_1, \ldots, y_{n-m}\}$. Now the system (14) has exactly $r_0 = n - r'$ independent (not necessarily polynomial) solutions, where r' denotes the rank of the $m \times n$ polynomial coefficient matrix. From these solutions, $\ell + \ell' - \ell_0$ correspond to the Casimir operators of either $\mathfrak{s}$ or $\mathfrak{s}'$, so that the number of available MLOs is given by $\chi = n - r' - \ell - \ell' + \ell_0$. It can be easily shown (see e.g. [21, 22]) that $m - r' = \ell_0$, which implies that $\chi = 2n_0$, showing that there are n_0 more labels available than required. It should however be noted that among these $2n_0$ solutions, at most n_0 can correspond to operators that commute with each other [9]. In particular, this implies that the maximal dimension of an abelian subalgebra of $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{s}')$ is upper-bounded by n_0.

4. The commutant of $\mathfrak{H}$ in G_2. Generators of G_2 in a A_2 basis

In this section, we extend the analysis of [12] for the classical series to the lowest-rank exceptional simple Lie algebra. We consider the (complex) Lie algebra G_2, whose set of positive roots is given by $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, as is well known [20]. For computational purposes, it is convenient to use an A_2-basis, corresponding to the regular embedding $A_2 \subset G_2$ [23]. Over such a basis, the adjoint representation $\Gamma[1,0]$ of G_2 decomposes as follows as sum of A_2-multiplets:

$$\Gamma[1,0] = \Lambda[1,1] + \Lambda[1,0] + \Lambda[0,1], \tag{15}$$

where $\Lambda[1,1]$ is the adjoint representation of A_2 and $\Lambda[1,0]$, $\Lambda[0,1]$ correspond to the quark and antiquark representation, respectively [24]. According to the decomposition, we label the generators as E_{ij}, a_k, b^l $(i,j,k,l = 1,2,3)$, where $E_{11}+E_{22}+E_{33} = 0$ holds. With this choice, A_2 is spanned by the operators E_{ij}, while $\{a_1, a_2, a_3\}$ correspond to the fundamental representation $\Lambda[1,0]$ and $\left\{b^1, b^2, b^3\right\}$ to its dual $\Lambda[0,1]$. The corresponding brackets are given by

$$\begin{aligned} &[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \quad [E_{ij}, a_k] = \delta_{jk}a_i, \quad \left[E_{ij}, b^k\right] = -\delta_{ik}b^j, \\ &[a_i, a_j] = -2\varepsilon_{ijk}b^k, \quad \left[b^i, b^j\right] = 2\varepsilon_{ijk}a_k, \quad \left[a_i, b^j\right] = 3E_{ij}. \end{aligned} \tag{16}$$

As generators of the Cartan subalgebra $\mathfrak{H}$ we choose the operators $H_1 = E_{11} - 2E_{22} + E_{33}$ and $H_2 = E_{22} - E_{33}$. The action is given in Table 1.

Table 1. Eigenvalues of $\mathfrak{H}$ over the basis (16)

X	E_{12}	E_{23}	E_{13}	E_{21}	E_{32}	E_{31}	a_1	a_2	a_3	b^1	b^2	b^3
$\lambda_1(X)$	3	3	0	−3	−3	0	1	−2	1	−1	2	−1
$\lambda_2(X)$	−1	−2	1	1	2	−1	0	1	−1	0	−1	1

Table 2. Commutators of G_2 in the A_2-basis

$[\circ,\circ]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_1	0	0	$3X_3$	$-3X_4$	$3X_5$	$-3X_6$	0	0	X_9	$-2X_{10}$	X_{11}	$-X_{12}$	$2X_{13}$	$-X_{14}$
X_2		0	$-X_3$	X_4	$-2X_5$	$2X_6$	X_7	$-X_8$	0	X_{10}	$-X_{11}$	0	$-X_{13}$	X_{14}
X_3			0	W_1	0	X_7	0	$-X_5$	0	X_9	0	$-X_{13}$	0	0
X_4				0	$-X_7$	0	X_6	0	X_{10}	0	0	0	$-X_{12}$	0
X_5					0	$-X_2$	$-X_3$	0	0	X_{11}	0	0	0	$-X_{13}$
X_6						0	0	X_4	0	0	X_{10}	0	$-X_{14}$	0
X_7							0	W_2	0	0	X_9	$-X_{14}$	0	0
X_8								0	X_{11}	0	0	0	0	$-X_{12}$
X_9									0	$-2X_{14}$	$2X_{13}$	W_3	$3X_3$	$3X_7$
X_{10}										0	$-2X_{12}$	$3X_4$	$-X_1$	$3X_6$
X_{11}											0	$3X_8$	$3X_5$	W_4
X_{12}												0	$2X_{11}$	$-2X_{10}$
X_{13}													0	$2X_9$
X_{14}														0
$W_1 = X_1 + X_2, W_2 = X_1 + 2X_2, W_3 = 2X_1 + 3X_2, W_4 = -X_1 - 3X_2$														

In order to determine the commutant $C_{\mathcal{U}(G_2)}(\mathfrak{H})$, we use the analytical counterpart, which corresponds to compute the centralizer $C_{S(G_2)}(\mathfrak{H})$ in the symmetric algebra of G_2 [4]. For computational purposes, it is more convenient to change the basis from $H_1, H_2, E_{12}, E_{21}, E_{23}, E_{32}, E_{13}, E_{31}, a_1, a_2, a_3, b^1, b^2, b^3$ to the indexed basis $X_1, \ldots, X_{14}$. Considering the corresponding coordinates $x_1, \ldots, x_{14}$ in the dual space G_2^*, polynomials in the centralizer $C_{S(G_2^*)}(\mathfrak{H})$ correspond to (polynomial) solutions of the linear first-order partial differential equations

$$\begin{aligned} \widehat{X}_1(F) =& 3x_3\partial_{x_3}F - 3x_4\partial_{x_4}F + 3x_5\partial_{x_5}F - 3x_6\partial_{x_6}F + x_9\partial_{x_9}F - 2x_{10}\partial_{x_{10}}F + x_{11}\partial_{x_{11}}F \\ &- x_{12}\partial_{x_{12}}F + 2x_{13}\partial_{x_{13}}F - x_{14}\partial_{x_{14}}F = 0, \\ \widehat{X}_2(F) =& - x_3\partial_{x_3}F + x_4\partial_{x_4}F - 2x_5\partial_{x_5}F + 2x_6\partial_{x_6}F + x_7\partial_{x_7}F - x_8\partial_{x_8}F + x_{10}\partial_{x_{10}}F \\ &- x_{11}\partial_{x_{11}}F - x_{13}\partial_{x_{13}}F + x_{14}\partial_{x_{14}}F = 0. \end{aligned} \tag{17}$$

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This system admits $r_0 = 14-2 = 12$ functionally independent solutions, as the coefficient matrix has rank two. As the Cartan subalgebra acts diagonally on the root vectors (see Table 1), we have that $\widehat{X}_s(x_k) = \lambda_s(x_k)x_k$ for any $1 \leq k \leq 14$ and $s = 1, 2$. It is hence immediate to verify that for any monomial $P = x_1^{\nu_1} \ldots x_{14}^{\nu_{14}}$ that satisfies (17), the following relations are fulfilled

$$\widehat{X}_s(P) = \sum_{k=1}^{14} \nu_k \lambda_s(x_k) P = 0, \quad s = 1, 2.$$

Therefore, we can restrict our attention to polynomials satisfying the eigenvalue constraints

$$\sum_{k=1}^{14} \nu_k \lambda_1(x_k) = 0, \quad \sum_{k=1}^{14} \nu_k \lambda_2(x_k) = 0. \tag{18}$$

With these numerical restrictions, we can proceed recursively, looking for the set $\mathcal{A}_d$ of monomials of degree $d \geq 1$ that satisfy the preceding condition. Proceeding along these lines, for each of the degrees $1 \leq d \leq 6$, and discarding the products of lower-order monomials, we obtain the following linearly independent monomials:

(i) $d = 1$

$$\mathcal{A}_1 = \{Q_1 = x_1, Q_2 = x_2\}$$

(ii) $d = 2$

$$\mathcal{A}_2 = \{Q_3 = x_3x_4, Q_4 = x_5x_6, Q_5 = x_7x_8, Q_6 = x_9x_{12}, Q_7 = x_{10}x_{13}, Q_8 = x_{11}x_{14}\}$$

We have that the cardinality of $\mathcal{A}_2$ is given by $|\mathcal{A}_2| = 6 = |\Phi^+|$. In particular, these elements correspond to the product of the root vectors X_α and $X_{-\alpha}$.

(iii) $d = 3$

$$\begin{aligned}\mathcal{A}_3 = \{&Q_9 = x_3x_6x_8, Q_{10} = x_3x_{10}x_{12}, Q_{11} = x_4x_5x_7, Q_{12} = x_4x_9x_{13}, Q_{13} = x_5x_{10}x_{14},\\ &Q_{14} = x_6x_{11}x_{13}, Q_{15} = x_7x_{11}x_{12}, Q_{16} = x_8x_9x_{14}, Q_{17} = x_9x_{10}x_{11}, Q_{18} = x_{12}x_{13}x_{14}\}\end{aligned}$$

$|\mathcal{A}_3| = 10$.

(iv) $d = 4$

$$\begin{aligned}\mathcal{A}_4 = \big\{&Q_{19} = x_3x_6x_{11}x_{12}, Q_{20} = x_3x_8x_{10}x_{14}, Q_{21} = x_3x_{10}^2x_{11}, Q_{22} = x_3x_{12}^2x_{14}, Q_{23} = x_4x_5x_9x_{14},\\ &Q_{24} = x_4x_7x_{11}x_{13}, Q_{25} = x_4x_9^2x_{11}, Q_{26} = x_4x_{13}^2x_{14}, Q_{27} = x_5x_7x_{10}x_{12}, Q_{28} = x_5x_9x_{10}^2,\\ &Q_{29} = x_5x_{12}x_{14}^2, Q_{30} = x_6x_8x_9x_{13}, Q_{31} = x_6x_9x_{11}^2, Q_{32} = x_6x_{12}x_{13}^2, Q_{33} = x_7x_{10}x_{11}^2,\\ &Q_{34} = x_7x_{12}x_{13}^2, Q_{35} = x_8x_{13}x_{14}^2, Q_{36} = x_8x_9^2x_{10}\big\}\end{aligned}$$

$|\mathcal{A}_4| = 18$.

(v) $d = 5$

$$\begin{aligned}\mathcal{A}_5 = \big\{&Q_{37} = x_3x_5x_{10}^3, Q_{38} = x_3x_6x_{10}x_{11}^2, Q_{39} = x_3x_6x_{12}^2x_{13}, Q_{40} = x_3x_7x_{13}^3, Q_{41} = x_3x_8x_9x_{10}^2,\\ &Q_{42} = x_3x_8x_{12}x_{14}^2, Q_{43} = x_4x_5x_9^2x_{10}, Q_{44} = x_4x_5x_{13}x_{14}^2, Q_{45} = x_4x_6x_{13}^3, Q_{46} = x_4x_7x_9x_{11}^2,\\ &Q_{47} = x_4x_7x_{12}x_{13}^2, Q_{48} = x_4x_8x_9^3, Q_{49} = x_5x_7x_{10}^2x_{11}, Q_{50} = x_6x_7x_{11}^3, Q_{51} = x_5x_7x_{12}^2x_{14},\\ &Q_{52} = x_5x_8x_{14}^3, Q_{53} = x_6x_8x_9^2x_{11}, Q_{54} = x_6x_8x_{13}^2x_{14}\big\}\end{aligned}$$

$|\mathcal{A}_5| = 18$.

(vi) $d = 6$

$$\begin{aligned}\mathcal{A}_6 = \{&Q_{55} = x_3^2x_6x_{12}^3, Q_{56} = x_3x_6^2x_{11}^3, Q_{57} = x_3x_8^2x_{14}^3, Q_{58} = x_3^2x_8x_{10}^3, Q_{59} = x_4x_5^2x_{14}^3,\\ &Q_{60} = x_4^2x_5x_9^3, Q_{61} = x_4^2x_7x_{13}^3, Q_{62} = x_4x_7^2x_{11}^3, Q_{63} = x_5^2x_7x_{10}^3, Q_{64} = x_5x_7^2x_{12}^3,\\ &Q_{65} = x_6^2x_8x_{13}^3, Q_{66} = x_6x_8^2x_9^3\}\end{aligned}$$

$|\mathcal{A}_6| = 12.$

It can be shown that for $d \geq 7$, any indecomposable monomial solution is a product of elements in the set $\mathcal{M} = \mathfrak{H} \cup \left(\bigcup_{k=1}^{6} \mathcal{A}_k\right)$. Indeed, due to the constraints (18), each monomial that satisfies the system has zero eigenvalue with respect to the generators of the Cartan subalgebra. Using the roots of G_2, this condition amounts to consider roots $\beta_1, \ldots, \beta_s \in \Phi$ such that $\beta_1 + \cdots + \beta_s = 0$ and $\beta_{i_1} + \cdots + \beta_{i_r} \neq 0$ for each set $\{i_1, \ldots, i_r\} \subset \{1, \ldots, s\}$, as otherwise the monomial is automatically decomposable. Now, as $3\alpha_1 + 2\alpha_2$ is the root of maximal height $\ell = 5$, the highest degree that can be obtained is six. Clearly, among the 66 monomials in $\mathcal{M}$, only 12 are functionally independent, which can be taken as $\{Q_1, \ldots, Q_9, Q_{12}, Q_{13}, Q_{14}\}$. In particular, the Casimir operators C_2', C_3' of A_2 and C_2, C_6 of G_2 are expressible as polynomials in the Q_i. More specifically

$$\begin{aligned}C_2' =& Q_1^2 + 3(Q_1Q_2 + Q_2^2) + 3(Q_4 + Q_7 + Q_8), \quad C_2 = C_2' + Q_3 + Q_5 + Q_6,\\ C_3' =& 2Q_1^3 + 9(Q_1^2Q_2 + Q_1Q_2^2) + 9Q_1(Q_7 - 2Q_4 + Q_8) - 27Q_2(Q_4 - Q_7) + 27(Q_{11} + Q_{12}).\end{aligned} \tag{19}$$

The expression for C_6 is skipped due to its length. As the number of linearly independent elements in the commutant exceeds that of functionally independent operators, elements in $\mathcal{M}$ must satisfy certain algebraic dependence relations. Such constraints can be found systematically fixing a degree d_0 and solving the equation

$$P = \sum_{s=1}^{r_0} \mu_s \prod_{k=1}^{66} Q_k^{a_{k,s}} = 0,$$

where Q_i denotes the i^{th} element in $\mathcal{M}$ and r_0 is the number of non-negative integer solutions of

$$d_0 = \sum_{k=1}^{66} a_{k,s} \deg(Q_k).$$

Up to degree five, a routine computation shows that there is no algebraic relation, while for $d = 6$, the following 29 algebraic dependence relations are found:

$$\begin{aligned}&Q_3Q_{20} - Q_{14}Q_{18} = 0, \quad Q_3Q_{22} - Q_{15}Q_{17} = 0, \quad Q_3Q_{24} - Q_{13}Q_{18} = 0,\\ &Q_3Q_{25} - Q_{13}Q_{17} = 0, \quad Q_3Q_{27} - Q_{15}Q_{16} = 0, \quad Q_3Q_{28} - Q_{14}Q_{16} = 0,\\ &Q_4Q_{32} - Q_{11}Q_{18} = 0, \quad Q_4Q_{36} - Q_{12}Q_{15} = 0, \quad Q_5Q_{19} - Q_{14}Q_{15} = 0,\\ &Q_5Q_{21} - Q_{17}Q_{18} = 0, \quad Q_5Q_{30} - Q_9Q_{15} = 0, \quad Q_5Q_{31} - Q_9Q_{17} = 0,\\ &Q_5Q_{34} - Q_{10}Q_{18} = 0, \quad Q_5Q_{35} - Q_{10}Q_{14} = 0, \quad Q_6Q_{23} - Q_{13}Q_{14} = 0,\\ &Q_6Q_{26} - Q_{16}Q_{17} = 0, \quad Q_6Q_{29} - Q_9Q_{14} = 0, \quad Q_6Q_{32} - Q_9Q_{16} = 0,\\ &Q_6Q_{33} - Q_{10}Q_{17} = 0, \quad Q_6Q_{36} - Q_{10}Q_{13} = 0, \quad Q_7Q_{30} - Q_{11}Q_{13} = 0,\\ &Q_7Q_{34} - Q_{12}Q_{16} = 0, \quad Q_8Q_{24} - Q_9Q_{12} = 0, \quad Q_8Q_{27} - Q_{10}Q_{11} = 0,\\ &Q_3Q_4Q_5 - Q_{15}Q_{18} = 0, \quad Q_3Q_5Q_6 - Q_{14}Q_{17} = 0, \quad Q_3Q_6Q_7 - Q_{13}Q_{16} = 0,\\ &Q_4Q_7Q_8 - Q_{11}Q_{12} = 0, \quad Q_5Q_6Q_8 - Q_9Q_{10} = 0.\end{aligned}$$

For higher degrees, additional algebraic relations appear, the explicit expression of which is skipped for brevity in the exposition. The monomials in $\mathcal{M}$, subjected to these algebraic relations, thus generate a polynomial algebra with respect to the Berezin bracket (3), such that a generic monomial in $\mathcal{M}$ can be expressed as

$$P = Q_1^{a_1} \dots Q_{66}^{a_{66}},$$

where the exponents a_i have to satisfy the algebraic relations. So, for example, the last of the relations above implies that $a_9 a_{10} = 0$, etc. It remains to determine the order of the polynomial algebra. As $\mathcal{H}$ commutes with any monomial, it generates the centre of the algebra, hence $\{\mathcal{A}_1, \mathcal{A}_j\} = 0$ for $2 \le j \le 6$. A routine computation shows that for any Berezin bracket $\{Q_i, Q_j\}$, the dependence on x_1, x_2 is at most linear, i.e.,

$$\frac{\partial^2 \{Q_i, Q_j\}}{\partial x_a \partial x_b} = 0, \quad a,b = 1,2; \quad 1 \le i < j \le 66.$$

For the remaining sets $\mathcal{A}_i$ of monomials, the following relations are obtained

$$\begin{aligned}
&\{\mathcal{A}_2, \mathcal{A}_2\} \subset \mathcal{A}_1\mathcal{A}_2 + \mathcal{A}_3,\\
&\{\mathcal{A}_2, \mathcal{A}_3\} \subset \mathcal{A}_1\mathcal{A}_3 + \mathcal{A}_2^2 + \mathcal{A}_4, \quad \{\mathcal{A}_2, \mathcal{A}_4\} \subset \mathcal{A}_1\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2^2 + \mathcal{A}_2\mathcal{A}_3 + \mathcal{A}_5,\\
&\{\mathcal{A}_2, \mathcal{A}_5\} \subset \mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 + \mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_2^3 + \mathcal{A}_3^2 + \mathcal{A}_6,\\
&\{\mathcal{A}_2, \mathcal{A}_6\} \subset \mathcal{A}_1\mathcal{A}_6 + \mathcal{A}_1\mathcal{A}_2^3 + \mathcal{A}_1\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_2^2\mathcal{A}_3 + \mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_5 + \mathcal{A}_6,\\
&\{\mathcal{A}_3, \mathcal{A}_3\} \subset \mathcal{A}_1\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2^2 + \mathcal{A}_2\mathcal{A}_3 + \mathcal{A}_5,\\
&\{\mathcal{A}_3, \mathcal{A}_4\} \subset \mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 + \mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_2^3 + \mathcal{A}_3^2 + \mathcal{A}_6,\\
&\{\mathcal{A}_3, \mathcal{A}_5\} \subset \mathcal{A}_1\mathcal{A}_6 + \mathcal{A}_1\mathcal{A}_2^3 + \mathcal{A}_1\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_2^2\mathcal{A}_3 + \mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_5 + \mathcal{A}_6,\\
&\{\mathcal{A}_3, \mathcal{A}_6\} \subset \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2^4 + \mathcal{A}_2\mathcal{A}_3^2 + \mathcal{A}_2^2\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_6 + \mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_4^2,\\
&\{\mathcal{A}_4, \mathcal{A}_4\} \subset \mathcal{A}_1\mathcal{A}_6 + \mathcal{A}_1\mathcal{A}_2^3 + \mathcal{A}_1\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_2^2\mathcal{A}_3 + \mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_5 + \mathcal{A}_6,\\
&\{\mathcal{A}_4, \mathcal{A}_5\} \subset \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2^4 + \mathcal{A}_2\mathcal{A}_3^2 + \mathcal{A}_2^2\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_6 + \mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_4^2, \qquad (20)\\
&\{\mathcal{A}_4, \mathcal{A}_6\} \subset \mathcal{A}_1\mathcal{A}_2^4 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_6 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_4^2 + \mathcal{A}_2^3\mathcal{A}_3 + \mathcal{A}_2^2\mathcal{A}_5\\
&\qquad + \mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_3^3 + \mathcal{A}_3\mathcal{A}_6\mathcal{A}_4\mathcal{A}_5,\\
&\{\mathcal{A}_5, \mathcal{A}_5\} \subset \mathcal{A}_1\mathcal{A}_2^4 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_6 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_4^2 + \mathcal{A}_2^3\mathcal{A}_3 + \mathcal{A}_2^2\mathcal{A}_5\\
&\qquad + \mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_3^3 + \mathcal{A}_3\mathcal{A}_6\mathcal{A}_4\mathcal{A}_5,\\
&\{\mathcal{A}_5, \mathcal{A}_6\} \subset \mathcal{A}_1\mathcal{A}_2^3\mathcal{A}_3 + \mathcal{A}_1\mathcal{A}_3^3 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_4\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_6 + \mathcal{A}_2^3\mathcal{A}_4\\
&\qquad + \mathcal{A}_2^5 + \mathcal{A}_2^2\mathcal{A}_3^2 + \mathcal{A}_2\mathcal{A}_4^2 + \mathcal{A}_2\mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_6^2 + \mathcal{A}_3^2\mathcal{A}_4 + \mathcal{A}_4\mathcal{A}_6 + \mathcal{A}_5^2,\\
&\{\mathcal{A}_6, \mathcal{A}_6\} \subset \mathcal{A}_1\mathcal{A}_2^5 + \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3^2 + \mathcal{A}_1\mathcal{A}_2^3\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_3^2\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_4^2 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_5 + \mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_6\\
&\qquad + \mathcal{A}_1\mathcal{A}_5^2 + \mathcal{A}_1\mathcal{A}_4\mathcal{A}_6 + \mathcal{A}_2^4\mathcal{A}_3 + \mathcal{A}_2^3\mathcal{A}_5 + \mathcal{A}_2^2\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_3\mathcal{A}_6 + \mathcal{A}_2\mathcal{A}_4\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_3^3\\
&\qquad + \mathcal{A}_3\mathcal{A}_4^2 + \mathcal{A}_3^3\mathcal{A}_5 + \mathcal{A}_5\mathcal{A}_6.
\end{aligned}$$

This shows that the polynomial algebra is of order six. The maximal order is obtained, for example, for the bracket

$$\{Q_{61}, Q_{65}\} = 4Q_1Q_4Q_6^3Q_8 - 18Q_1Q_4Q_6^2Q_8^2 + 8Q_2Q_4Q_6^3Q_8 - 17Q_2Q_4Q_6^2Q_8^2 - Q_2Q_6^3Q_8^2. \qquad (21)$$

We conclude that the commutant of the Cartan subalgebra in the enveloping algebra of G_2 gives rise to a 66-dimensional polynomial algebra of order six.

4.1. Recovery of the polynomial algebra associated to A_2

Considering the (regular) embedding chain $\mathfrak{H} \subset A_2 \subset G_2$, we know by (11) that $C_{S(G_2)}(\mathfrak{H})$ contains $C_{S(G_2)}(A_2)$, the elements of which correspond to the subgroup scalars of G_2 with respect to A_2. In this case, the subalgebra is generated by the elements $X_1, \ldots, X_8$, corresponding to $x_1, \ldots, x_8$ in the symmetric algebra. Thus, fixing these elements and setting the remaning coordinates equal to zero, the set of the $\mathcal{A}_i$ reduces to the following elements $Q_1, Q_2, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$, which are obviously still linearly independent. A straightforward computation shows that these elements satisfy the following Berezin brackets

$$
\begin{aligned}
\{Q_4, Q_7\} =& Q_{12} - Q_{11}, \quad \{Q_4, Q_8\} = Q_{11} - Q_{12}, \quad \{Q_4, Q_{11}\} = Q_4(Q_8 - Q_7) + Q_2 Q_{11}, \\
\{Q_4, Q_{12}\} =& Q_4(Q_8 - Q_8) - Q_2 Q_{11}, \quad \{Q_7, Q_{11}\} = (Q_1 + Q_2)Q_{11} + (Q_4 - Q_7)Q_{11}, \\
\{Q_7, Q_8\} =& Q_{12} - Q_{11}, \quad \{Q_7, Q_{12}\} = -(Q_1 + Q_2)Q_{12} + (Q_8 - Q_4)Q_7, \\
\{Q_8, Q_{11}\} =& (Q_4 + Q_7)Q_8 - (Q_1 + 2Q_2)Q_{11}, \quad \{Q_8, Q_{12}\} = (Q_1 + 2Q_2)Q_{12} + (Q_4 - Q_7)Q_8, \\
\{Q_{11}, Q_{12}\} =& Q_1 Q_4(Q_7 - Q_8) + Q_2 Q_4(2Q_7 - Q_8) + Q_2 Q_7 Q_8.
\end{aligned}
$$

As expected, this coincides with the polynomial algebra associated to the Lie algebra A_2 (see e.g. [2]).

4.2. Algebraic Hamiltonians and spectrum generating algebras

The construction of commutants is potentially useful for the construction of integrable systems whose Hamiltonian is given in terms of $\mathfrak{s}'$-generators. Indeed, if

$$
\mathcal{H} = \sum_{i,j=1}^{\dim \mathfrak{s}'} \alpha_{ij} X_i X_j + C_2 + C_6, \quad \alpha_{ij} \in \mathbb{K} \tag{22}
$$

is a fixed Hamiltonian, then the elements in the commutant $C_{S(G_2)}(\mathfrak{s}')$ of a regular (not necessarily maximal) subalgebra can be determined inspecting how the monomials of $C_{G_2}(\mathfrak{H})$ transform, and finding suitable linear combinations. In order to illustrate the procedure, let us consider the algebraic Hamiltonian

$$
\mathcal{H} = Q_1^2 + Q_1 Q_2 + Q_2^2 + Q_4 + Q_7 + Q_8, \tag{23}
$$

which is expressed in terms of A_2-generators. A long but routine computation shows that there are nine functionally independent polynomials commuting with $\mathcal{H}$, given by

$$
\begin{aligned}
&J_1 = Q_1, \quad J_2 = Q_2, \quad J_3 = Q_4, \quad J_4 = Q_7, \quad J_5 = Q_8, \quad J_6 = Q_{11}, \quad J_7 = Q_{12}, \\
&J_8 = -(Q_1 + Q_2)Q_3 - (Q_1 + 2Q_2)Q_5 + Q_9 + Q_{10} + Q_{13} + Q_{15} + Q_{16} - Q_{18}, \\
&J_9 = (Q_1 + 2Q_2)(Q_{13} + Q_{16}) + (Q_1 + Q_2)(Q_9 + Q_{10}) - Q_3 Q_8 - Q_5 Q_7 + Q_{14} + Q_{24} \\
&\qquad + Q_{27} + Q_{30} + Q_{32} + Q_{34} + Q_{36} + (Q_3 + Q_5)(Q_4 - 2Q_2^2 - 3Q_1 Q_2 - Q_1^2).
\end{aligned} \tag{24}
$$

With the first seven integrals the Casimir operators of A_2 are easily recovered, as it can be realized by looking at the expressions of C_2' and C_3' in equation (19), while the first integrals J_8 and J_9 involve variables not belonging to the subalgebra, generally leading to a non-abelian symmetry algebra. We observe that, for this Hamiltonian, we can see G_2 as the spectrum generating algebra [10], with $\mathcal{H}$ depending solely on the subalgebra A_2 (the Casimir operators of G_2 corresponding to an overall constant setting the zero of the energy), which also justifies the interest of this formal approach in the context of exactly solvable problems [1, 25, 26, 27].

5. Conclusions

Following the systematization initiated in [4] for A_ℓ and extended in [12] for the remaining classical series of semisimple Lie algebras, in this work we have analyzed the commutant of the Cartan subalgebra in the enveloping algebra of the rank-two exceptional Lie algebra G_2, giving rise to a polynomial algebra of order six and dimension 66. Due to the embedding chain (11), for any subalgebra $\mathfrak{H} \subset \mathfrak{s}' \subset G_2$, we have that the commutant of $\mathfrak{s}'$ can be determined from the Cartan commutant. In particular, this holds for the labelling operators required for the description of G_2-representations in a $\mathfrak{s}'$-basis. For the case of the regular subalgebra A_2, it has been shown how the polynomial algebra associated to the Cartan commutant in the enveloping algebra of A_2 can be recovered by restriction from $C_{S(G_2)}(\mathfrak{H})$ to $C_{S(G_2)}(A_2)$, and how this information can be used to derive constants of the motion for Hamiltonians expressed in terms of A_2-elements. This constitutes a special case of spectrum generating algebras, which could provide an alternative approach to the study of G_2 systems, by means of appropriate realizations by (higher) differential operators.

The construction of the commutant of the Cartan subalgebra in the enveloping algebra of the remaining simple exceptional algebras can be studied along the same lines. Results in this context would provide an alternative approach to the analysis of systems possessing an exceptional spectrum generating algebra, specially in connection with minimal realizations (see e.g. [28]). However, due to their particular structure, it is expected to be a computationally demanding problem, with very large dimensions and high order of the resulting polynomial algebra, an obstruction that suggests to search for alternative descriptions of these commutants. Work in this direction is currently in progress, in collaboration with I. Marquette, D. Latini, J. Zhang and Y-Z Zhang.

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