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Charge conservation, entropy current and gravitation

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We propose a new class of vector fields to construct a conserved charge in a general field theory whose energy–momentum tensor is covariantly conserved. We show that there always exists such a vector field in a given field theory even without global symmetry. We also argue that the conserved current constructed from the (asymptotically) timelike vector field can be identified with the entropy current of the system. As a piece of evidence we show that the conserved charge defined therefrom satisfies the first law of thermodynamics for an isotropic system with a suitable definition of temperature. We apply our formulation to several gravitational systems such as the expanding universe, Schwarzschild and Banãdos, Teitelboim and Zanelli (BTZ) black holes, and gravitational plane waves. We confirm the conservation of the proposed entropy density under any homogeneous and isotropic expansion of the universe, the precise reproduction of the Bekenstein–Hawking entropy incorporating the first law of thermodynamics, and the existence of gravitational plane wave carrying no charge, respectively. We also comment on the energy conservation during gravitational collapse in simple models.

Keywords: Conservation of entropy in general relativity; the first law of thermodynamics; expanding Universe; black hole entropy; exact gravitational plane wave; gravitational collapse.

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1. Introduction

A central mystery in theory of gravity is that while it is governed by fundamental physics laws, it contains black holes, which behave as thermodynamical objects.[1,2] In particular entropy of a well-known black hole has been shown to be given by the Bekenstein–Hawking formula[3–5] (except some cases such as an extremal one[6–8]): $S = A/4G_N$, where A is the area of the horizon and G_N is the Newton constant. This suggests that information of a black hole such as charges and dynamical degrees of freedom is localized at its surface rather than inside the horizon so as to behave as a membrane-like object.[9]

There have been various techniques invented to compute entropy of various types of black holes and to show their thermodynamic relations in and beyond the Einstein gravity.[10–13] (See also Refs. 14–18.) These approaches have been developed basically regarding charges including entropy as quasi-local ones evaluated by a surface integral, which enables one to evaluate a charge of black holes without knowing a charge distribution inside the horizon.

In the previous work to holographically realize a black hole with quantum correction,[19] the authors of this paper recognized that the quasi-local energy is not sufficient for precise evaluation of the total energy when matter distributes nontrivially in space–time. We reached a definition to evaluate a total charge of matter by a volume integration of its charge distribution, and proposed a precise definition available on a general curved space–time with Killing vector fields.[20] This definition was recognized in early time,[21,22] though the validity thereof has not been confirmed by explicit computation. We confirmed that this reproduces known results on mass and angular momentum for classic black holes, and that it gives an additional contribution to the known mass formula of any compact star obtained by quasi-local energy.[20] (See also Ref. 23.)

In this paper, we delve into the proposed definition of a matter charge and ask whether it can be extended to the case of geometry without any Killing vector field. We conclude that (i) a class of vector fields satisfying a particular partial differential equation, which includes the Killing vector fields if they exist, can make the charge conserved, (ii) there always exists such a vector field uniquely for a given initial condition in any field theory with energy–momentum tensor covariantly conserved and (iii) the conserved charge constructed from the particular vector field is nothing but the entropy of the system. We present our argument to reach this conclusion with application to several gravitational systems in what follows.

2. Charges on a General Space–Time

2.1. *Conservation condition*

Let us consider any field theory on a general curved space–time. We use Greek letters $\mu, \nu, \ldots$ to label an arbitrary fixed coordinate system x^μ, which run from 0 to $d-1$ with d the dimension of the space–time. In the previous paper,[20] we presented

a precise definition of a charge of matter associated with any generic vector field v^μ as

$$Q[v](t) := \int_{\Sigma_t} d^{d-1}x \sqrt{|g|}\, T^0{}_\nu v^\nu, \tag{1}$$

where $T^\mu{}_\nu$ is the matter energy momentum tensor given in the system, Σ_t is a hypersurface or a time slice at an arbitrarily fixed time $x^0 = t$. For instance, if one chooses a vector field as a generator of time translation and that of a space direction with suitable normalization, then the corresponding charges become energy and momentum in that direction, respectively. Superiority of this definition is its manifest general covariance.

It can be shown that the charge defined by (1) is conserved or time-independent when v^μ is a Killing vector field and the energy momentum tensor is covariantly conserved.[20–22] In this case the defined charge becomes a Noether charge corresponding to global symmetry of the system. We emphasize that our proposal can be applied not only to Einstein gravity but also to any other gravitational theories.

A question is whether a charge $Q[v]$ conserves if v is other than a Killing vector field. To answer this question we study the time evolution of the charge, which is computed as[20]

$$\frac{dQ[v]}{dt} = \int_{\Sigma_t} d^{d-1}\mathbf{x}\sqrt{|g|}T^\mu{}_\nu \nabla_\mu v^\nu. \tag{2}$$

Therefore, if a vector field satisfies the differential equation

$$T^\mu{}_\nu \nabla_\mu v^\nu = 0, \tag{3}$$

then it is sufficient for the charge $Q[v]$ to conserve. In this sense, we refer to Eq. (3) as the conservation condition for the vector v. In particular any Killing vector field trivially satisfies the conservation condition. This gives our conclusion (i).

2.2. *Intrinsic vector field*

Since the conservation condition (3) is a first-order linear partial differential equation, we can convert it to ordinary differential equations by the method of characteristics with a parameter τ. Therefore, for a given energy momentum tensor, there always exists a general solution at least locally in τ, which is determined uniquely once we fix an appropriate initial condition at $\tau = 0$ on a hypersurface Σ_{t_0} including a choice of a direction of the vector v.

To be more explicit, one can arbitrarily choose a reference vector field $\bar{v}^\mu(x)$ which is defined in the entire space–time. A simple example is $\bar{v}^\mu(x) := \frac{dx^\mu(\eta)}{d\eta}$ where η is a parameter to characterize the evolution of the hypersurface Σ_t. (Furthermore, if we choose η to be the global time x^0, the reference vector becomes $\bar{v}^\mu(x) = \delta^\mu_0$.) Then, we can look for the solution of the conservation condition in the form of $v^\mu(x) = c(x)\bar{v}^\mu(x)$ where $c(x)$ is a scalar function. Equation (3) reduces to

$$A^0(x)\partial_0 c(x) + \sum_{\mu\neq 0} A^\mu(x)\partial_\mu c(x) + B(x)c(x) = 0, \tag{4}$$

where $A^\mu(x) := T^\mu{}_\nu(x)\bar{v}^\nu(x)$ and $B(x) := T^\mu{}_\nu(x)\partial_\mu\bar{v}^\nu(x) + T^\mu{}_\nu(x)\Gamma^\mu_{\nu\lambda}\bar{v}^\lambda(x)$. This can be solved once $c(x^0, \mathbf{x})$ is given at some x^0 unless $A^0(x)$ identically vanishes in the entire space–time.[a] A choice of the initial condition depends on a physical setup of the system. Indeed $A^0(x) = T^0{}_\nu(x)\bar{v}^\nu(x) \neq 0$ is necessary to obtain a nontrivial charge as $Q[v] = \int_{\Sigma_t} d^{d-1}\mathbf{x}\sqrt{|g|}T^0{}_\nu(x^0, \mathbf{x})\bar{v}^\nu(x^0, \mathbf{x})c(x^0, \mathbf{x})$.

Since the time direction must be chosen so that a component of the stress tensor with time component does not vanish in any reasonable physical system, there always exists a vector field to satisfy (3) proportional to a timelike vector field (or an asymptotically timelike vector field in the presence of black holes). We refer to this (asymptotically) timelike vector field as an intrinsic vector field and denote it by ζ.[b] In this paper, we mainly consider a conserved charge associated with ζ, except Sec. 4.4 where both conserved energy and momentum are treated.

The above argument establishes our conclusion (ii) that there always exists a conserved quantity of the form (1) in any field theory on general curved space–time even without any global symmetry. Thus, this conserved charge is different from a Noether charge and must be very special and fundamental for the system. In the following section, we give physical interpretation for this charge.

3. Entropy and Entropy Current

What is the physical meaning of the conserved charge $Q[\zeta]$ associated with the intrinsic vector field? Our answer is *entropy* of the system,

$$S := Q[\zeta] = \int d^{d-1}x\, s^0, \tag{5}$$

where s^0 is the time component of an entropy current density defined by

$$s^\mu = \sqrt{|g|}T^\mu{}_\nu\zeta^\nu. \tag{6}$$

One can easily show that this current density satisfies the ordinary continuity equation $\partial_\mu s^\mu = 0$, thanks to the conservation condition (3) and the covariant conservation of the energy momentum tensor.

Let us give some remarks for this interpretation. The entropy is the most fundamental quantity and uniquely defined in a dynamical system, and therefore it is physically reasonable that the entropy is related to the conserved charge $Q[\zeta]$, which is also uniquely defined for a generic system without global symmetry. Since the fundamental physics law is expected to be reversible, it is natural that the entropy of the whole system including matter with gravitational interaction is conserved in a fundamental theory such as general relativity. To the contrary, if the entropy

[a]In the region where $A^0(x) = 0$, one cannot determine the time evolution of $c(x)$. However, since the charge density is proportional to $A^0(x)$ one does not need to know the value of $c(x)$.
[b]The existence of such a vector field for a spherically symmetric gravitational system was pointed out in Ref. 24. This vector field may be known as the Kodama vector. We have checked that the Kodama vector satisfies the conservation condition (3).

were not conserved, for example, in a general gravitational system, then it would mean that there exists extra matter producing entropy via unknown interaction ("fifth force"), which would be unreasonable. Further evidence will be presented in following sections.

3.1. *Analysis for geometry without horizon*

Let us analyze our proposal in the case where the coordinate system allows a globally well-defined unit time evolution vector field, which we denote by n^μ. In this case tensors can be decomposed into the longitudinal and transverse components with respect to this vector for each index. For example, the energy momentum tensor is decomposed as

$$T^\mu{}_\nu = \rho n^\mu n_\nu + P^\mu{}_\nu - n^\mu J_\nu - J^\mu n_\nu, \tag{7}$$

where $\rho := n_\mu T^\mu{}_\nu n^\nu, P^\mu{}_\nu := \bar{g}^\mu{}_\alpha T^\alpha{}_\beta \bar{g}^\beta{}_\nu, J_\nu := n_\alpha T^\alpha{}_\beta \bar{g}^\beta{}_\nu$, with $\bar{g}^\mu{}_\nu = \delta^\mu_\nu + n^\mu n_\nu$, and the metric tensor is decomposed into the Arnowitt, Deser and Misner (ADM) form as

$$ds^2 = -N^2 (dx^0)^2 + \bar{g}_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0), \tag{8}$$

where N, N^i are called a lapse function and a shift vector, respectively. Note that $J^0 = 0$ and $n_\mu = -N\delta^0_\mu$.

From the covariant conservation, we have $(\nabla_\mu T^\mu{}_\nu) n^\nu = 0$, which boils down to

$$\check{\partial}\rho + \rho K + P^\mu{}_\nu K^\nu{}_\mu + n^\nu \check{\nabla} J_\nu - \bar{\nabla}_\mu J^\mu = 0, \tag{9}$$

where $\check{\nabla} := n^\mu \nabla_\mu$, $\bar{\nabla}_\mu := \bar{g}^\rho{}_\mu \nabla_\rho$, $K^\nu{}_\mu := \bar{\nabla}_\mu n^\nu$ is the extrinsic curvature, and $K := K^\mu{}_\mu$. Since the intrinsic vector field is defined to be proportional to a time evolution vector field, we set $\zeta^\mu = -\beta n^\mu$, for which the conservation condition (3) reduces to

$$\rho\check{\partial}\beta - \beta P^\mu{}_\nu K^\nu{}_\mu + \beta J_\nu \check{\nabla} n^\nu - J^\mu \bar{\partial}_\mu \beta = 0. \tag{10}$$

Combining Eqs. (9) and (10) and taking a new coordinate η satisfying $n^\mu = \frac{\partial x^\mu}{\partial \eta}$, we obtain

$$\frac{d(\rho\beta)}{d\eta} + \rho\beta K - \bar{\nabla}_\mu (J^\mu \beta) = 0. \tag{11}$$

In the case with $J^\mu = 0$, this can be easily solved as

$$\beta = \beta_0 \frac{\rho_0}{\rho} \exp\left[-\int_{\eta_0}^{\eta} d\eta K \right], \tag{12}$$

where ρ_0, β_0 are initial values of ρ, β at $\eta = \eta_0$, respectively.

The entropy current density becomes

$$s^0 = \sqrt{\bar{g}}\rho\beta, \quad s^i = -\beta\sqrt{\bar{g}}(\rho N^i + N J^i), \tag{13}$$

where $\bar{g} := \det \bar{g}_{ij}$. Thus, the continuity equation becomes

$$\partial_0 s^0 = \partial_i (s^0 N^i + N\sqrt{|\bar{g}|} J^i \beta). \tag{14}$$

It would be amusing that the shift vector has a physical meaning of the flow velocity of the entropy density when $J^i = 0$. Note that the entropy density is locally conserved if $J^i = N^i = 0$.

3.2. *The first law of thermodynamics*

We now show that our entropy current density satisfies the first law of thermodynamics in isotropic systems, so that the shift vector vanishes and the matter energy momentum tensor is given by a perfect fluid, characterized by $P^\mu{}_\nu = P\bar{g}^\mu{}_\nu$ and $J_\mu = J^\mu = 0$ in (7), and thus $K = \frac{d\log\sqrt{|\bar{g}|}}{d\eta}$. In this situation, it follows from the continuity equation (14) that the entropy density itself becomes time-independent. Furthermore, since (10) in this case becomes

$$\rho\frac{d\beta}{d\eta} = \frac{\beta P}{v}\frac{dv}{d\eta}, \tag{15}$$

we can show that s^0 in (13) satisfies

$$\frac{ds^0}{d\eta} = \frac{du}{d\eta}\beta + u\frac{d\beta}{d\eta} = \left(\frac{du}{d\eta} + P\frac{dv}{d\eta}\right)\beta, \tag{16}$$

where $v := \sqrt{|\bar{g}|}$ is a volume density, and $u := \rho v$ is the (internal) energy density corresponding to the energy, a charge with the unit time evolution vector field as $E := Q[-n] = \int d^{d-1}x\, u$. This becomes exactly the first law of thermodynamics if β is identified with the inverse temperature, which proves our conclusion (iii). Note that the variation by η in Eq. (16) is realized by some dynamical process, which must satisfy the equation of motion for the matter, as well as the Einstein equation or its variant for gravity if the metric $g_{\mu\nu}$ is dynamical.

Our method determines both the entropy density s^0 and the inverse temperature β for the matter through the gravitational interaction with $g_{\mu\nu}$. An overall normalization for both is fixed as an initial condition for the intrinsic vector ζ in the system, but a ration s^0/β is free from such an ambiguity. Once the normalization is given, the dependence of the temperature on space–time is completely controlled by Eq. (3). This will be seen in Sec. 4.1.

3.3. *Case for black hole geometry*

Let us consider applying the above formulation to a black hole geometry, where the existence of globally well-defined unit timelike vector field is not guaranteed. Then the analysis done in Sec. 3.2 is not generally applicable, and we need to analyze the system by setting the intrinsic vector field $\zeta^\mu = -\zeta\delta^\mu_0$, where ζ is a function determined to satisfy the conservation condition (3).

It happens, however, that the matter energy momentum tensor in black hole geometry vanishes except singularity at the origin.[20] In such a case, the conservation condition is solved by any time-independent ζ, and there is no further way to determine it. In this paper, as a provisional prescription, we assume that the energy

conservation law holds to satisfy $\frac{1}{\beta}ds^0 = du + Pdv$ as in the previous case, and we determine ζ to satisfy the first law of thermodynamics with temperature given by the Hawking temperature identical to the surface gravity normalized by 2π.[4] We expect that this assumption is in principle verified by setting up a gravitational system such that a smooth geometry with matter gravitationally collapses into the black hole and keeping track of charges during the process. We leave this to future studies.

4. Applications

4.1. *Friedmann–Lemaître–Robertson–Walker metric*

Let us consider d-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) metric, which is given by $ds^2 = -(dx^0)^2 + a^2\tilde{g}_{ij}dx^i dx^j$, where a is the scale factor dependent only on time, and the Ricci tensor for $\tilde{g}$ is $\tilde{R}_{ij} = (d-2)k\tilde{g}_{ij}$ with $k = 1, 0, -1$ corresponding to a $(d-1)$-dimensional sphere, flat space, hyperbolic space, respectively. This is a model of homogeneous and isotropic expanding universe in Einstein gravity with cosmological constant Λ. In particular, the shift vector vanishes and the energy momentum tensor is given by a perfect fluid, whose density ρ and pressure P are determined by the Einstein or Friedmann equation as $\rho = \frac{1}{8\pi G_N}(\frac{(d-1)(d-2)}{2}\frac{k+\dot{a}^2}{a^2} - \Lambda), P = \frac{1}{8\pi G_N}((2-d)\{\frac{\ddot{a}}{a} + \frac{(d-3)}{2}\frac{k+\dot{a}^2}{a^2}\} + \Lambda)$, where $\dot{f} = \partial_0 f$.

Since there is a unit time evolution vector field as $n^\mu = -\delta^\mu_0$ and the system is isotropic, this is a case studied in Sec. 3.2, where not only the entropy but also the entropy density are conserved, and the first law of thermodynamics (16) holds. The entropy density is computed as $s^0 = a^{d-1}\sqrt{\tilde{g}}\beta\rho$, where β is calculated from (12) as $u\beta = u_0\beta_0 = \text{const}$ with $u = \rho\sqrt{\tilde{g}}a^{d-1}$, since $K = \partial_t \log a^{d-1}$. Thus, in the homogeneous and isotropic expanding universe, the energy density is proportional to the temperature, which decreases during the expansion of the system in any equation of state with nonzero pressure while temperature keeps constant in the case of the pressureless dust, as evident from $du = -Pdv$. The entropy density satisfies the thermodynamic relation (16). (See also Refs. 25, 26 and references therein.) Note that s^0 is constant even for the $k = 1$ (sphere) case, where the universe first expands, reaches its maximum size, and then contracts.

4.2. *Schwarzschild black hole*

We next evaluate the entropy of the Schwarzschild black hole in d-dimensional space–time with cosmological constant Λ. We employ the metric in the Eddington–Finkelstein coordinate as $ds^2 = -(1+u(r))(dx^0)^2 - 2udx^0dr + (1-u(r))dr^2 + r^2\tilde{g}_{ij}dx^i dx^j$, where $u(r) = -2\Lambda r^2/(d-2)(d-1) - r_0^{d-3}/r^{d-3}$ and r_0 is a constant corresponding to the radius of horizon when $\Lambda = 0$. Note that the metric is nonsingular even at $u = \pm 1$, and a constant t surface is always spacelike even inside the horizon

for nonnegative Λ. (For negative Λ, a constant t surface is spacelike inside the horizon but becomes timelike for large r satisfying $u(r) > 1$.) The energy defined by (1) with a Killing vector $v^\mu = -\delta^\mu_0$ for time translation reproduces the well-known black hole mass $M = V_{d-2}(d-2)r_0^{d-3}/(16\pi G_N)$, where $V_{d-2} := \int d^{d-2}x\sqrt{\det\, \tilde{g}_{ij}}$,[20] since

$$T^\mu{}_0 = -\rho\delta^\mu_0, \quad \rho := \frac{d-2}{16\pi G_N}\frac{r_0^{d-3}\delta(r)}{r^{d-2}}. \tag{17}$$

This is the case discussed in Sec. 3.3 that the conservation condition is solved by any time-independent intrinsic vector field because the energy momentum tensor vanishes except $r = 0$ as seen from (17). In this situation, the intrinsic vector field can be written as $\zeta^\mu = -\zeta(\mathbf{x})\delta^\mu_0$, where $\zeta(\mathbf{x})$ is an arbitrary smooth function of spacial coordinates $\mathbf{x}$. Then the entropy current defined by (5) is computed as $s^\mu = \zeta\sqrt{g}\rho\delta^\mu_0$, where we can take a constant $\zeta = \zeta(\mathbf{0})$ since ρ is proportional to $\delta(r)$. Note that the entropy density is localized at the singularity. Then the entropy given by (6) is evaluated as $S = M\zeta$. As discussed in Sec. 3.3, we determine ζ to satisfy the first law of thermodynamics $TdS = dM$ identifying T with the Hawking temperature $T = (d-3-\frac{2\Lambda r_H^2}{(d-2)})/4\pi r_H$, where r_H is the (outer) horizon radius determined by $u(r_H) = -1$.[27] Plugging $S = M\zeta$ into the thermodynamic relation reduces to a differential equation $\zeta + M\frac{d\zeta}{dM} = 1/T$, whose general solution is $\zeta = \frac{1}{M}\int\frac{dM}{T}$. Using $dM = \frac{(d-2)V_{d-2}r_H^{d-4}((d-3)L^2+(d-1)r_H^2)}{16\pi G_N L^2}dr_H$, the integration can be performed as $\zeta = \frac{V_{d-2}r_H^{d-2}}{4G_N M}$. Remark that we can fix the integration constant since $\zeta \to 0$ when $r_0 \to 0$ or $r_H \to 0$, which cannot be fixed only by the thermodynamic relation.[28] This reproduces the known result of the black hole entropy computed by Bekenstein–Hawking formula because the area of the horizon is given by $V_{d-2}r_H^{d-2}$.

4.3. *BTZ black hole*

Next application is to the BTZ black hole, whose metric is $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\phi - \omega(r)dt)^2$, where $f(r) = \frac{r^2}{L^2} - m\theta(r) + \frac{J^2}{4r^2}, \omega = \frac{J}{2r^2}$. It was shown in Ref. 20 that the energy and the angular momentum defined by using (1) choosing $v^\mu = -\delta^\mu_0$ and $v^\mu = \delta^\mu_\phi$ reproduce the well-known result $M = \frac{m}{8G_N}, P_\phi = \frac{J}{8G_N}$. The matter energy momentum tensor was computed as

$$T^0{}_\nu = \frac{\delta(r)}{16\pi G_N r}(-m\delta^0_\nu + J\delta^\phi_\nu). \tag{18}$$

As discussed in Sec. 3.3 and the case of the Schwarzschild black hole, the conservation condition is solved by $\zeta^\mu = -\zeta(\mathbf{x})\delta^\mu_0$, due to (18). This reduces the entropy density to $s^0 = \zeta m\delta(r)/16\pi G_N$, where $\zeta = \zeta(\mathbf{0})$. Then the entropy is evaluated as $S = M\zeta$. As stated in Sec. 3.3 we determine the value of ζ to satisfy the thermodynamic relation $\frac{\partial S}{\partial M} = 1/T$ with P_ϕ fixed. As in the case of Schwarzschild black hole, this can be solved as $\zeta = \frac{1}{M}\int\frac{dM}{T}$. In the BTZ black hole, the Hawking temperature is given by $T = \frac{r_+^2 - r_-^2}{2\pi L^2 r_+}$ with $r_\pm^2 = mL^2/2(1 \pm \sqrt{1-(J/mL)^2})$.[29] On the

other hand, $dM = \frac{r_+^2 - r_-^2}{r_+ 4L^2 G_N} dr_+$ with P_ϕ fixed. Employing these we can perform the integration as $\zeta = \frac{2\pi r_+}{4G_N M}$, where we fixed the integration constant by $\zeta \to 0$ when $r_+ \to 0$. This reproduces the known result of the black hole entropy computed by Bekenstein–Hawking formula since the horizon area is given by $2\pi r_+$. Then the chemical potential for the angular momentum is computed as $\mu_\phi = -T\frac{\partial S}{\partial P_\phi} = \frac{r_-}{r_+ L}$, since $\frac{\partial S}{\partial P_\phi} = -\frac{2\pi r_- L}{r_+^2 - r_-^2}$ with the mass M fixed, which also matches the known result.[29,30] The first law of thermodynamics is given by $TdS = dM - \mu_\phi dP_\phi$.

4.4. *Exact gravitational plane wave*

Our final example is the gravitational waves, called an exact plane wave solution discussed in old literature.[31] The metric is given by $ds^2 = e^{2\Omega(u)}(dx^2 - d\tau^2) + u^2(e^{2\beta(u)}dy^2 + e^{-2\beta(u)}dz^2)$ with $u = \tau - x$. From the Einstein equation, nonvanishing energy momentum tensors are computed as $T^x{}_x = T^\tau{}_x = -T^\tau{}_\tau = -T^x{}_\tau = \frac{e^{-2\Omega}}{4\pi u}(2\Omega' - u\beta'^2)$ with a prime being a u-derivative. Since the generator of time translation $v_0^\mu = -\delta_\tau^\mu$ and that of x-direction $v_x^\mu = \delta_x^\mu$ satisfy the conservation condition, energy and momentum in the x direction are conserved in this system. These are computed from (1) as $E = P_x = \frac{V_2}{4\pi}\int dx\, u(2\Omega' - u\beta'^2)$, where $V_2 = \int dy\, dz$. This implies that the energy and momentum vanish if $2\Omega' - u\beta'^2 = 0$, in which the matter energy momentum tensor and the Ricci tensor vanish while the Riemann curvature tensor does not when $(u\beta'' + 2\beta' - u^2\beta'^3)/u \neq 0$. Thus, we observe that there exists a gravitational plane wave which carries neither energy nor momentum, though the Riemann curvature tensor does not vanish everywhere.

If we choose functions Ω, β as $2\Omega' - u\beta'^2 = \frac{C}{u}\frac{\theta(u+\epsilon) - \theta(u-\epsilon)}{2\epsilon}$ with positive constants C and ϵ, we obtain $E = P_x = \frac{V_2 C}{4\pi}\int_{\tau-\epsilon}^{\tau+\epsilon} dx\, \frac{1}{2\epsilon} = \frac{V_2 C}{4\pi}$. The energy momentum tensor is nonzero only on $-\epsilon < \tau - x < \epsilon$, which propagates in the x space at the speed of light in the $\epsilon \to 0$ limit.

5. Discussion

In this paper, we have proposed a procedure to construct a conserved current in any field theory with covariantly conserved energy momentum tensor and interpreted it as the entropy current. Since any matter generates its energy momentum tensor, which couples to gravitational field, the entropy carried by matter never escapes from the censorship of *universal* gravitation. This may be a reason why we can find a conserved current in a generic gravitational theory. (See also Refs. 32 and 33.) In the applications to black holes, the entropy density is localized at the singularity, while the entropy is still compatible with the Bekenstein–Hawking formula. A deeper understanding of this "information map puzzle" is called for.

Although the presented formulation may look quite different from known conventional approaches, we confirm that it has successfully reproduced known results with a more unified standpoint. In Appendix A, we also analyze a few simple models for gravitational collapse in this formulation. An advantage may be in that

the energy distribution may become meaningful or at least visualizable in such a time-dependent system with event horizon emergent. We draw a picture of energy flow during the gravitational collapses while the total energy is conserved.

Gravitational systems analyzed in this paper are restricted to isotropic and homogeneous systems such as the FLRW model and a few classic black holes whose matter distribute only at the singularity. In these systems, in order to solve the conservation condition it is sufficient to choose the intrinsic vector field to be proportional to the time evolution vector field. However in a more general system, there needs to be more vector fields to characterize the system. In such a situation, the intrinsic vector field must be expanded by all such vector fields in general. As a result the simple thermodynamic relation for the entropy in Sec. 3.2 obtained in this paper will be modified significantly. It would be very interesting to find out the corresponding dynamical thermodynamic relation and the local inverse temperature in such a more general situation.

The application of the method developed in this paper is also open to any effective field theory on a fixed background metric. This application may shed new light to thermodynamics and hydrodynamics. (See also Ref. 34.)

While the entropy is always conserved in situations considered in this paper, there are several ways to violate the conservation of the entropy: by changing the boundary conditions of fields or by violating the covariant conservation of energy momentum tensor. In the latter case, while the effect may be compensated by finding a vector field v to satisfy $\nabla_\mu(T^\mu{}_\nu v^\nu) = 0$, the original conservation condition may be suitable to investigate the second law of thermodynamic.

We leave these interesting issues to future studies.

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Appendix A. Local Energy Density in Gravitational Collapse and Energy Conservation

In this appendix, simple gravitational collapses,[35] extended to an arbitrary dimensions with nonzero cosmological constant, are analyzed in terms of conserved charge proposed in this paper, which turns out to be the total energy of the system in these cases. We will see how energy distribution varies during the collapse while the total energy is conserved even after a black hole formation.

A.1. *Thick light shells*

We first consider a simple model of gravitational collapses for thick light shells,[35] whose metric in the Eddington–Finkelstein coordinate is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -(1+u)dt^2 - 2udtdr + (1-u)dr^2 + r^2\bar{g}_{ij}dx^i dx^j, \tag{A.1}$$

where $\bar{g}_{ij}$ is the $d-2$ metric for a sphere,

$$u(r,t) = \delta u(r,t) - \frac{\Lambda r^2}{(d-2)(d-1)}, \quad \delta u(r,t) := -\frac{m(r,t)}{r^{d-3}}, \tag{A.2}$$

$m(r,t) = M\theta(r)$, $M\theta(r)F(\frac{t+r}{\Delta})$, 0 at $t+r > \Delta$, $0 \le t+r \le \Delta$, $t+r < \Delta$, respectively, with a monotonically increasing function $F(x)$ satisfying $F(0)=0$ and $F(1)=1$. Thus, the energy momentum tensor is evaluated as

$$T^t{}_t = \frac{(d-2)}{16\pi G_N}\frac{(r^{d-3}\delta u)_r}{r^{d-2}}, \quad T^r{}_r = \frac{(d-2)}{16\pi G_N}\left[\frac{(r^{d-3}\delta u)_r}{r^{d-2}} - \frac{2(\delta u)_t}{r}\right], \tag{A.3}$$

$$T^t{}_r = \frac{(d-2)}{16\pi G_N}\frac{(\delta u)_t}{r} = -T^r{}_t, \quad T^i{}_j = \frac{\delta^i_j}{16\pi G_N}\left[\frac{(r^{d-3}\delta)_{rr}}{r^{d-3}} - \frac{2(r^{d-3}\delta)_{rt}}{r^{d-3}} + (\delta u)_{tt}\right], \tag{A.4}$$

and $\zeta^\mu = -\delta^\mu_t$ satisfies Eq. (3) for the intrinsic vector. The corresponding charge (energy) is conserved for all t as

$$E(t) = -\int d^{d-1}x\,\sqrt{|g|}\,T^t{}_t = \frac{(d-2)V_{d-2}}{16\pi G_N}M, \tag{A.5}$$

and Fig. A.1 (Left) show a distributions of local energy $-\sqrt{|g|}T^t{}_t$ with $F(x) = 3x^2 - 2x^3$.

A.2. *Zero pressure fluid* (*dust*)

Zero pressure fluid is handled by the Painlevé–Gullstrand coordinate as[35]

$$g_{\mu\nu}dx^\mu dx^\nu = (\psi^2-1)dt^2 - 2\psi dtdr + dr^2 + r^2\bar{g}_{ij}dx^i dx^j, \tag{A.6}$$

where $\psi(r,t) = -\sqrt{u(r,1)}$, $-\sqrt{u(r,\lambda)}$ at $t-\Delta+G(r,M) > 0$, $t-\Delta+G(r,M) \le 0$,

$$u(r,\lambda) = \frac{M\theta(r)\lambda}{r^{d-3}} + \frac{2\Lambda r^2}{(d-2)(d-1)}, \quad G(r,M) = \int_0^r \frac{ds}{\sqrt{u(s,1)}}, \tag{A.7}$$

and $0 \le \lambda \le \Delta$ is determined as a function of r,t by $t - h(\lambda) + G(r, M\lambda) = 0$ through a given monotonically increasing function of $h(\lambda)$ with $h(0) = 0$ and $h(1) = \Delta$.

As in the previous case, $\zeta^\mu = -\delta^\mu_t$ becomes the intrinsic vector, whose corresponding charge (energy) is conserved for all t. Figure A.1 (Right) shows a distribution of energy density, given by $\sqrt{|g|}\frac{d-2}{16\pi G_N}\frac{(r^{d-3}\delta\psi^2)_r}{r^{d-2}}$ with $\delta\psi^2 := \psi^2 - \frac{2\Lambda r^2}{(d-2)(d-1)}$ for $h(\lambda) = \Delta\lambda$ and $\Lambda = 0$.

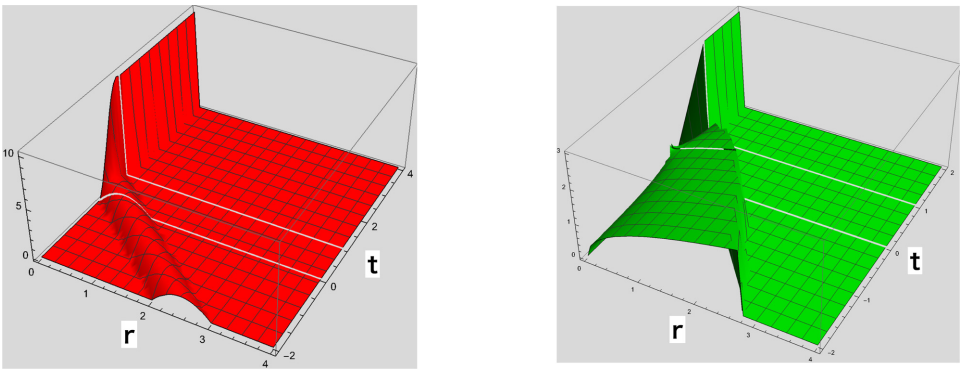

Fig. A.1. Energy density during the gravitational collapse as a function of r and t. The black hole formation at $r = 0$ starts at $t = 0$ and ends at $t = 1$. The δ function at the origin ($r = 0$) for the black hole is approximated by the step function for visibility. (Left) Thick light shells in the Eddington–Finkelstein coordinate. (Right) Zero pressure fluid in the Painlevé–Gullstrand coordinate.

A.3. *Matters with nonzero pressure*

As a final example, let us consider a gravitational collapse of matter with nonzero pressure, whose metric in the Vaidya coordinate is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r,v)dv^2 + 2dvdr + r^2\bar{g}_{ij}dx^i dx^j, \tag{A.8}$$

where $f(r,v) = 1 + \delta f - \frac{2\Lambda r^2}{(d-2)(d-1)}$ with $f(r,v) = 1 + \delta f - \frac{2\Lambda r^2}{(d-2)(d-1)}$, and $v = t + r$ with t in the Eddington–Finkelstein coordinate.[35] The energy momentum tensor becomes

$$\begin{aligned} T^v{}_v = T^r{}_r = \frac{d-2}{16\pi G_N}\frac{(r^{d-3}\delta f)_r}{r^{d-2}}, \quad T^r{}_v = -\frac{d-2}{16\pi G_N}\frac{(\delta f)_v}{r}, \\ T^i{}_j = \frac{\delta^i_j}{16\pi G_N}\frac{(r^{d-3}\delta f)_{rr}}{r^{d-3}}, \end{aligned} \tag{A.9}$$

and $T^v{}_r = 0$, which describes matters with nonzero pressure (nonperfect fluid) as well as the energy–momentum transfer due to $T^r{}_v$. Matters with nonzero pressure indeed describes infalling massive matters, since $v + r = t + 2r = \text{const}$ represents propagations slower than light. It is easy to see that $\zeta^\mu = -\delta^\mu_v$ is the intrinsic vector, so that the corresponding charge (energy) is conserved for all t.

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