Quantum Algebras and Symmetries of Dynamical Systems

P. P. Kulish

Leningrad Branch Mathematical Institute AN USSR 191011 Leningrad, Fontanka 27, LOMI

The development of the quantum inverse problem method [1] and the study of solutions to the Yang–Baxter equation [2] gave rise to the notion of quantum groups and algebras (cf. [3,4] and references therein). These days quantum Lie groups and Lie algebras are popular topics in different branches of theoretical physics and modern mathematic physics as one can conclude from this Proceedings. The structure of this object with respect to many properties is quite similar or richer than the Lie group and Lie algebra one due to appearence of a deformation parameter q (representation theory, Clebsch–Gordon–Wiegner–Racah calculus, q–special functions, non–commutative differential geometry etc.). We will consider in this report the quantum algebras as symmetries of dynamical systems.

The simplest quantum algebra $V = sl_q(2)$ or quasitriangular Hopf algebra [5] depends on a parameter q and is generated by three elements $I, X_\pm$ satisfying relations

$$[I\,, X_\pm\,] = \pm X_\pm\,, \quad [X_+\,, X_-\,] = [2I]_q\,, \tag{1}$$

$$[2I]_q \equiv \frac{(q^{2I} - q^{-2I})}{(q - q^{-1})} \ . \tag{2}$$

The Hopf algebra structure on V is defined by maps of antipode (coinverse) $S : V \longrightarrow V$ (antihomomorphism), counit $\epsilon : V \longrightarrow \mathbb{C}$ and coproduct or comultiplication $\Delta : V \longrightarrow V \otimes V$ (algebra homomorphism) which satisfy a set of axioms [3-5] (we'll not discuss them here).

$$\begin{aligned}
S(1) &= 1\,, \quad S(I) = -I\,, \quad S(X_\pm) = -q^{\pm 1} X_\pm\,, \\
\epsilon(1) &= 1\,, \quad \epsilon(I) = \epsilon(X_+) = \epsilon(X_-) = 0\,, \\
\Delta(1) &= 1 \otimes 1\,, \quad \Delta(I) = I \otimes 1 + 1 \otimes I\,, \\
\Delta X_\pm &= X_\pm \otimes q^I + q^{-I} \otimes X_\pm \ .
\end{aligned} \tag{3}$$

For general parameter q (special values are roots of unity:$q^M = 1, M \in \mathbb{N}$) the finite dimensional representations of $sl_q(2)$ have the same structure as the Lie algebra $sl(2)$ ones (cf. [6]).

The contraction limit of spin j representation [7]

$$I = j - N\,, \quad \alpha^{\pm} = \lim_{j\to\infty} \frac{X_{\pm}}{\sqrt{[2j]_q}} \tag{4}$$

gives rise to a new algebra $A(q)$ with generators satisfying relations

$$[\,N\,,\,\alpha^{\pm}\,] = \pm\alpha^{\pm}\,, \quad [\,\alpha^{-}\,,\,\alpha^{+}\,] = q^{-2N}\ . \tag{5}$$

Introducing the operators

$$a = q^{\frac{N}{2}}\alpha^{-} = (a^{+})^{+}\,, \quad A = q^{N}\alpha^{-} = (A^{+})^{+}$$

one can rewrite the relations (5) as follows

$$aa^{+} - qaa^{+} = q^{-N}\,, \tag{6}$$

$$AA^{+} - qAA^{+} = 1\ . \tag{7}$$

As far a these transformations are invertible one can pick up any set of generators e.g. (a, a^{+}, N) to define an associative algebra $A(q)$. It is natural to give to these operators the name q–oscillator for in the limit $q \to 1$ (5) - (7) reproduce the usual harmonic oscillator [7-12]. In the limit $j \to \infty$ one obtains the irreducible representation of $A(q)$ which coinsides with the Fock space representation of the harmonic oscillator

$$a\varphi_0 = 0\,, \quad \varphi_n = \left(\sqrt{[n]_q\,!}\right)^{-1}\left(a^{+}\right)^{n}\varphi_0\,, \tag{8}$$

$$N\varphi_n = n\varphi_n\,, \quad a^{+}\varphi_{n-1} = \sqrt{[n]_q}\varphi_n\,, \quad [n]_q\,! = [1]_q[2]_q\ldots[n]_q$$

where we use the notation (2): $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. It is possible to express $a, a^{+}_{\frac{1}{2}}$ in $\mathcal{H}_F$ in terms of Bose–operators $b, b^{+} : a^{+} = \sqrt{[N]_q/N}b^{+}$. In particular $a^{+}a = [N]_q$ in $\mathcal{H}_F$. However the algebra $A(q)$ is not equivalent to the harmonic oscillator algebra for general q. It has nontrivial centre $z = q^{-N}([N]_q - a^{+}a)$ and others irreducible representations which are not equivalent to $\mathcal{H}_F$.

Taking as a Hamiltonian in $\mathcal{H}_F$ the operator $H = a^{+}a$ one has $U(1)$ as the symmetry group with the generator N and the quantum algebra $su_q(1,1)$ as the dynamical symmetry algebra with generators

$$K_0 = N + \frac{\alpha}{2}\,, \quad K_{-} = \sqrt{[N+\alpha]_q}a = (K_{+})^{+}\,, \tag{9}$$

$$[\,K_0\,,\,K_{\pm}\,] = \pm K_{\pm}\,, \quad [\,K_{+}\,,\,K_{-}\,] = -[2K_0]_q\ .$$

Let us concider a simple example of two q–oscillators as dynamical systems with the quantum algebra $su_q(2)$ as the symmetry of this system. The Hamiltonian is

$$H = a_1^{+}a_1q^{N_2} + q^{-N_1}a_2^{+}a_2\ . \tag{10}$$

It commutes with the generators of the $su_q(2)$

$$X_+ = a_1^+ a_2\,, \quad X_- = a_2^+ a_1\,, \quad I = \frac{1}{2}(N_1 - N_2)\ . \tag{11}$$

However this statement is correct only in the Fock space representation for the q–oscillators a_i, $i = 1, 2)$, where $H = [N_1 + N_2]_q$ due to the relations $a_i^+ a_i = [N_i]_q$. As a result the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is decomposed into direct sum of finite dimentional subspaces

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \sum_{k=0}^{\infty} V_k\,, \quad \dim V_k = k + 1\,, \tag{12}$$

where V_k are irreducible representations of the $su_q(2)$ with spin $j = k/2$. In the limit $q \to 1$ we reproduce the picture of two standart oscillators with the Lie algebra $su(2)$ as their symmetry. More of that the corresponding eigenstates $|k - m\rangle \otimes |m\rangle$, $m = 0, 1, \ldots, k$ of the Hamiltonian with the eigenvalue $[k]$ do not change being constructed from the eigenstates of N_i, $i = 1, 2$ which coincide with the number of operators of standart oscillators.

This construction can be easily generalized to the quantum algebra of higher rank e.g. $su_q(n)$ as the symmetry algebra of n dimensional q–oscillator. For $n = 3$ the Hamiltonian is

$$H = a_1^+ a_1 q^{N_2 + N_3} + a_2^+ a_2 q^{-N_1 + N_3} + a_3^+ a_3 q^{-N_1 - N_2}\ . \tag{13}$$

In the Fock space representation it is equal to $[N_1 + N_2 + N_3]$ and it commutes with the generators of the $su_q(3)$

$$X_1^+ = a_1^+ a_2 = (X_1^-)^+\,, \quad X_2^+ = a_2^+ a_3 = (X_2^-)^+\,, \tag{14}$$
$$I_1 = \frac{1}{2}(N_1 - N_2)\,, \quad I_2 = \frac{1}{2}(N_2 - N_3)\ .$$

The generator X_3^+ corresponding to the highest root is constructed using q–commutator or q–adjoint action

$$X_3^+ = (X_3^-)^+ = [\, X_1^+\,, X_2^+\,]_q = \mathrm{ad}_q(X_1^+)X_2^+ = X_1^+ X_2^+ - q X_2^+ X_1^+ \simeq a_1^+ a_3\ . \tag{15}$$

It is interesting to note that the Hamiltonians (10), (13) have interactions among different modes and their structure reminds the noncocommutative comultiplication of the quantum algebras.

There are more realistic integrable models wich posses quantum algebras as symmetry or dynamical symmetry algebras. They are non–affine Toda field theories [13], generalized Jaynes–Cummings model Hamiltonian of quantum optics [14] integrable spin models such as the XXZ model of spin $\frac{1}{2}$ [15] or its generalisation to higher spin [16], relativistic oscillator [17] and models of the conformal field theory such as Liouville equation and WZNW model [18].

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