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FIELD-ADAPTED COORDINATE TRANSFORMATIONS FOR ROTATING AND ACCELERATING BEAMS*

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Abstract

Many accelerators employ axisymmetric structures, such as RF cavities, induction cells, and solenoids, to accelerate and transport charged particle beams. To analyze the motion of the beam in solenoids, it is common to make a transformation to the rotating Larmor frame. In the presence of an electric field, this transformation can be modified to obtain further simplifications in the equation of motion. In this paper, we explore the use of a complex Larmor phase to simplify the equations of motion in the presence of simultaneous axial electric and magnetic fields, such as those found in the induction cells of a linear induction accelerator. We also analyze the corresponding envelope equation and find that the natural emittance in this frame can be expressed in terms of familiar quantities.

EQUATIONS OF MOTION

Our first task is to express the equations of motion in terms of the complex transverse coordinate $\zeta = x + iy$, with the fields in cylindrical coordinates. We begin with the Lorentz force equation $\dot{\vec{p}} = q\left(\vec{E} + c\vec{\beta}\times\vec{B}\right)$. Converting the time derivatives on the left-hand side to longitudinal derivatives (e.g. $\dot{\vec{x}} = c\beta_z\vec{x}'$), evaluating the cross-product $\vec{\beta}\times\vec{B}$ with $\vec{\beta} = \beta_z\left[x'\hat{x} + y'\hat{y} + \hat{z}\right]$, and expressing the space charge force as $-\nabla\phi_b/\gamma^2$ for a beam self-potential ϕ_b,[1] we obtain the equation

$$\begin{aligned}\vec{x}'' + \frac{(\gamma\beta_z)'}{\gamma\beta_z}\vec{x}' &= \frac{q}{p_z}\Bigg\{-\frac{1}{\gamma^2}\nabla\phi_b \\ &\quad + \left[\frac{x}{r}\frac{E_r}{\beta_z c} - \frac{y}{r}\frac{E_\theta}{\beta_z c} - \frac{y}{r}B_r - \frac{x}{r}B_\theta + y'B_z\right]\hat{x} \\ &\quad + \left[\frac{y}{r}\frac{E_r}{\beta_z c} + \frac{x}{r}\frac{E_\theta}{\beta_z c} + \frac{x}{r}B_r - \frac{y}{r}B_\theta - x'B_z\right]\hat{y} \\ &\quad + \left[\frac{E_z}{\beta_z c} + \frac{yx' - xy'}{r}B_r + \frac{xx' + yy'}{r}B_\theta\right]\hat{z}\Bigg\}.\end{aligned}$$

If we define $[B\rho] = p_z/q$, $E_\perp = E_r + iE_\theta$, $B_\perp = B_r + iB_\theta$, and $\partial_\perp = \partial_r + \frac{i}{r}\partial_\theta$, the transverse equations can be rewritten as

$$\begin{aligned}\zeta'' + \left[\frac{(\gamma\beta_z)'}{\gamma\beta_z} + i\frac{B_z}{[B\rho]}\right]\zeta' - \frac{E_\perp/\beta_z c + iB_\perp}{r[B\rho]}\zeta \\ = -\frac{q\partial_\perp\phi_b}{\beta_z^2\gamma^3 mc^2}\frac{\zeta}{r},\end{aligned} \tag{1}$$

and the longitudinal equation is

$$\begin{aligned}\frac{(\gamma\beta_z)'}{\gamma\beta_z} = \frac{1}{[B\rho]}\Bigg[\frac{E_{z0} - \partial_z\phi_b/\gamma^2}{\beta_z c} \\ + \frac{yx' - xy'}{r}B_r + \frac{xx' + yy'}{r}B_\theta\Bigg].\end{aligned} \tag{2}$$

LARMOR FRAME TRANSFORMATION WITH COMPLEX PHASE

At this point, the usual approach is to eliminate the $iB_z\zeta'$ term by transforming to the Larmor frame and defining $\tilde{\zeta} = \zeta e^{-i\psi(z)}$ with a real-valued phase $\psi(z)$. However, there is no reason this phase needs to be real-valued. We can modify the usual definition of the phase to eliminate the acceleration damping term from the equation as well, and this yields some convenient simplifications later on.

To transform the equation, we need the derivatives of ζ in terms of $\tilde{\zeta}$ and ψ:

$$\begin{aligned}\zeta' &= (\tilde{\zeta}' + i\psi'\tilde{\zeta})e^{i\psi} \\ \zeta'' &= \left[\tilde{\zeta}'' + 2i\psi'\tilde{\zeta}' + \left(i\psi'' - \psi'^2\right)\tilde{\zeta}\right]e^{i\psi}.\end{aligned}$$

Let

$$\begin{aligned}i\chi'(z) &= \frac{(\gamma\beta_z)'}{\gamma\beta_z} + i\frac{B_z}{[B\rho]} \\ \kappa(z) &= -\frac{E_\perp/\beta_z c + iB_\perp}{r[B\rho]}.\end{aligned}$$

Then the left-hand side of Eq. (1) becomes

$$\left[\tilde{\zeta}'' + i(2\psi' + \chi')\tilde{\zeta}' + \left(\kappa + i\psi'' - \psi'^2 - \chi'\psi'\right)\tilde{\zeta}\right]e^{i\psi}.$$

We can eliminate the $\tilde{\zeta}'$ term by choosing $\psi' = -\chi'/2$, or, in terms of physical variables,

$$\psi(z) - \psi(z_0) = -\frac{1}{2}\int_{z_0}^{z}\frac{B_{z0}}{[B\rho]}dz + \frac{i}{2}\ln\left(\frac{\gamma\beta_z}{\gamma_0\beta_{z0}}\right).$$

This differs from the usual real Larmor phase $\psi_L(z)$ by an adiabatic damping term. If we plug this into the transformation law $\zeta = \tilde{\zeta}e^{i\psi(z)}$, then the solution in the lab frame in terms of the solution in the modified Larmor frame $\mathscr{L}$ is

$$\zeta = \tilde{\zeta}e^{i[\psi_L(z) - \psi_L(z_0)]}\sqrt{\frac{\gamma_0\beta_{z0}}{\gamma\beta_z}}. \tag{3}$$

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[1] This form of the beam self-force is applicable to beams much longer than γ times the beam radius, i.e., the beam is long in its rest frame [1].

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The next step is to simplify the effective focusing term. Since $\psi' = -\chi'/2$,

$$\widetilde{\kappa} = \kappa + i\psi'' - \psi'^2 - \chi'\psi' = \kappa + i\psi'' + \psi'^2 .$$

If we neglect the effects of the longitudinal self-force (which should be small for a beam with a slowly-varying profile) and the longitudinal magnetic force (which is second order in x and x') on the transverse motion, then the longitudinal equation of motion allows us to express ψ' as

$$\psi' = i\frac{E_z/\beta_z c + iB_z}{2\,[B\rho]} . \tag{4}$$

Recognizing that the applied fields appear in the equations of motion as $F_\perp \equiv E_\perp/\beta_z c + iB_\perp$ and $F_z \equiv E_z/\beta_z c + iB_z$, we have

$$\kappa = -\frac{F_\perp}{r\,[B\rho]} \quad \text{and} \quad \psi' = i\frac{F_z}{2\,[B\rho]} .$$

Since $[B\rho]' / [B\rho] = (\gamma\beta_z)'/(\gamma\beta_z) \approx E_z/\beta_z c\,[B\rho]$,

$$-2i\psi'' = \frac{F_z'}{[B\rho]} - \frac{F_z E_z/\beta_z c}{[B\rho]^2} ,$$

so

$$\widetilde{\kappa} = -\frac{\frac{1}{r}F_\perp + \frac{1}{2}F_z'}{[B\rho]} + \frac{|F_z|^2}{4\,[B\rho]^2}.$$

Therefore,

$$\widetilde{\zeta}'' + \widetilde{\kappa}(z)\widetilde{\zeta} = -\frac{q\partial_\perp \phi_b}{\beta_z^2\gamma^3 mc^2}\frac{\widetilde{\zeta}}{r} , \tag{5}$$

where

$$\begin{aligned}
\widetilde{\kappa}(z) &= \frac{|F_z|^2}{4\,[B\rho]^2} - \frac{\frac{1}{r}F_\perp + \frac{1}{2}F_z'}{[B\rho]} ,\\
F_z &= \frac{E_z}{\beta_z c} + iB_z , \qquad F_\perp = \frac{E_\perp}{\beta_z c} + iB_\perp ,\\
E_\perp &= E_r + iE_\theta , \qquad B_\perp = B_r + iB_\theta ,\\
\partial_\perp &= \partial_r + \frac{i}{r}\partial_\theta .
\end{aligned}$$

Note that $F_z' = \left(\partial_z + x'\partial_x + y'\partial_y + \frac{1}{\beta_z c}\partial_t\right)F_z$. If the applied fields are linear and time-independent, this reduces to $F_z' = \partial_z F_z$.

LINEAR, TIME-INDEPENDENT, AXISYMMETRIC APPLIED FIELDS

If the applied fields are axisymmetric and time-independent, then only the axial and radial components are nonzero. In the linear approximation,

$$\vec{E} \approx E_{z0}\hat{z} - \frac{1}{2}rE_{z0}'\hat{r} \quad \text{and} \quad \vec{B} \approx B_{z0}\hat{z} - \frac{1}{2}rB_{z0}'\hat{r} ,$$

where $E_{z0}(z) \equiv E_z(z, r=0)$ and similarly for B_{z0}. Then

$$\frac{2}{r}F_\perp + F_z' = -\frac{\beta_z'}{\beta_z}\frac{E_{z0}}{\beta_z c} .$$

We can clean up the expression for $\widetilde{\kappa}$ by expressing β_z'/β_z in terms of E_{z0}. Assuming β_z is much larger than β_x and β_y,

$$\frac{\beta_z'}{\beta_z} \approx \frac{1}{\gamma^2}\frac{E_{z0}/\beta_z c}{[B\rho]} .$$

Therefore,

$$\widetilde{\kappa} = \frac{1}{4\,[B\rho]^2}\left[\left(1 + \frac{2}{\gamma^2}\right)\left(\frac{E_{z0}}{\beta_z c}\right)^2 + B_{z0}^2\right] . \tag{6}$$

Observe that $\widetilde{\kappa} \approx |\psi'|^2$ for $\gamma \gg 1$ in linear fields. Since the space charge term also becomes negligible in the ultrarelativistic limit,

$$\widetilde{\zeta}'' + \frac{E_{z0}^2 + c^2B_{z0}^2}{4\,[B\rho]^2\,c^2}\widetilde{\zeta} = 0 . \tag{7}$$

ENVELOPE EQUATION IN $\mathscr{L}$

In any real beam, we have a distribution of particles in six-dimensional phase space, often described with coordinates $(x, x', y, y', z, \delta)$, where $\delta = (p - p_0)/p_0$, and p_0 is the reference momentum. Let $\widetilde{\sigma} = \sqrt{\langle\widetilde{\zeta}^*\widetilde{\zeta}\rangle}$. Then

$$\begin{aligned}
\widetilde{\sigma}' &= \frac{\langle\widetilde{\zeta}^*\widetilde{\zeta}' + \widetilde{\zeta}'^*\widetilde{\zeta}\rangle}{2\widetilde{\sigma}} \qquad \text{and}\\
\widetilde{\sigma}'' &= \frac{1}{\widetilde{\sigma}}\left[\frac{1}{2}\langle\widetilde{\zeta}^*\widetilde{\zeta}'' + \widetilde{\zeta}''^*\widetilde{\zeta}\rangle + \langle\widetilde{\zeta}'\widetilde{\zeta}'^*\rangle - \frac{\langle\widetilde{\zeta}^*\widetilde{\zeta}' + \widetilde{\zeta}'^*\widetilde{\zeta}\rangle^2}{4\widetilde{\sigma}^2}\right] .
\end{aligned}$$

Observe that the first term in brackets in the expression for $\widetilde{\sigma}''$ is the real part of $\langle\widetilde{\zeta}^*\widetilde{\zeta}''\rangle$. Multiplying Eq. (5) by $\widetilde{\zeta}^*$ and taking the ensemble average, then taking the real part, and finally dividing by $\widetilde{\sigma}$, we find

$$\begin{aligned}
0 &= \widetilde{\sigma}'' - \frac{\langle\widetilde{\zeta}^*\widetilde{\zeta}\rangle\langle\widetilde{\zeta}'^*\widetilde{\zeta}'\rangle - \langle\Re\{\widetilde{\zeta}^*\widetilde{\zeta}'\}\rangle^2}{\widetilde{\sigma}^3} + \frac{\langle\Re\widetilde{\kappa}\widetilde{\zeta}^*\widetilde{\zeta}\rangle}{\widetilde{\sigma}}\\
&\quad + \frac{1}{\widetilde{\sigma}}\left\langle\frac{\frac{1}{r}\partial_\perp\phi_b}{\gamma^2\beta_z c\,[B\rho]}\widetilde{\zeta}^*\widetilde{\zeta}\right\rangle .
\end{aligned}$$

We can define the emittance in $\mathscr{L}$ as

$$\widetilde{\varepsilon} \equiv \sqrt{\langle\widetilde{\zeta}^*\widetilde{\zeta}\rangle\langle\widetilde{\zeta}'^*\widetilde{\zeta}'\rangle - \langle\Re\{\widetilde{\zeta}^*\widetilde{\zeta}'\}\rangle^2}$$

and rewrite the envelope equation as

$$\widetilde{\sigma}'' + \frac{\langle\Re\{\widetilde{\kappa}\}\widetilde{\zeta}^*\widetilde{\zeta}\rangle}{\widetilde{\sigma}} - \frac{\widetilde{\varepsilon}^2}{\widetilde{\sigma}^3} + \frac{1}{\widetilde{\sigma}}\left\langle\frac{\frac{1}{r}\partial_\perp\phi_b}{\gamma^2\beta_z c\,[B\rho]}\widetilde{\zeta}^*\widetilde{\zeta}\right\rangle = 0 .$$

If the applied fields are linear, then the focusing term is just $\widetilde{\kappa}(z)\widetilde{\sigma}$. Additionally, if the beam is axisymmetric (but not necessarily uniform), then the last term is $-P/\widetilde{\sigma}$, where $P = I_b/I_A\,(\gamma\beta_z)^2\,\gamma_0\beta_{z0}$, I_b is the beam current, and $I_A = 4\pi\varepsilon_0 mc^3/q$ is the Alfvèn current. In the case where both of these conditions are met, the envelope equation reduces to

$$\widetilde{\sigma}'' + \widetilde{\kappa}\widetilde{\sigma} - \frac{\widetilde{\varepsilon}}{\widetilde{\sigma}^3} - \frac{P}{\widetilde{\sigma}} = 0 .$$

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PROPERTIES OF $\tilde{\varepsilon}$

In terms of lab-frame quantities, it can be shown that

$$\begin{aligned}\tilde{\varepsilon} &= \frac{\gamma\beta_z}{\gamma_0\beta_{z0}}\sqrt{\langle|\zeta|^2\rangle\langle|\zeta'|^2\rangle - |\langle\zeta^*\zeta'\rangle|^2 + \langle r^2\left(\psi_L' - \theta'\right)\rangle^2} \\ &= \frac{\gamma\beta_z}{\gamma_0\beta_{z0}}\sqrt{\varepsilon_{LC}^2 + \langle p_\theta/p_z\rangle^2},\end{aligned}$$

where

$$\varepsilon_{LC} = \sqrt{\langle\vec{r}\cdot\vec{r}\rangle\langle\vec{r}'\cdot\vec{r}'\rangle - \langle\vec{r}\cdot\vec{r}'\rangle^2 - \langle(\vec{r}\times\vec{r}')\cdot\hat{z}\rangle^2}$$

is the eigenemittance [2, 3] of the rotating beam (sometimes referred to as the Lee-Cooper emittance [4]) and

$$p_\theta = \left[\vec{r}\times\left(\gamma\vec{\beta}mc + q\vec{A}\right)\right]\cdot\hat{z}$$

is the canonical angular momentum.

One of the most useful things to know about the emittance is when it is conserved. If we define $f_{sc} = \frac{1}{r}\partial_\perp\phi_b/\gamma^2\beta_z c\,[B\rho]$, then

$$\begin{aligned}\frac{1}{2}\frac{d\tilde{\varepsilon}^2}{ds} &= \left\langle\Re\tilde{\zeta}^*\tilde{\zeta}'\right\rangle\left\langle\Re\left(\tilde{\kappa}+f_{sc}\right)\tilde{\zeta}^*\tilde{\zeta}\right\rangle \\ &\quad - \left\langle\tilde{\zeta}^*\tilde{\zeta}\right\rangle\left\langle\Re\left(\tilde{\kappa}+f_{sc}\right)\tilde{\zeta}\tilde{\zeta}'^*\right\rangle.\end{aligned}$$

From this, we can see that if $\tilde{\kappa}+f_{sc}$ is uniform over the beam cross-section, $\tilde{\varepsilon}$ is conserved as a slice emittance, and if the entire beam sees the same $\tilde{\kappa}+f_{sc}$ as it passes through the beamline, $\tilde{\varepsilon}$ is conserved as a projected emittance. For example, this condition is satisfied for a monoenergetic, uniform beam subject to linear, time-independent applied fields (though the projected emittance is only conserved if the beam current is uniform along the beam's length). If the beam is non-uniform, $\tilde{\varepsilon}$ is still approximately conserved if space charge is negligible.

CONCLUSION

By analyzing the beam in a frame adapted to both the rotation and damping induced by the magnetic and electric fields, we can greatly simplify the analysis of charged particle beams in certain classes of accelerators. For example, in linear induction accelerators the beam is often simultaneously focused and accelerated, with solenoids integrated into the accelerating cells. The formalism presented here provides a straightforward way to obtain well-known results, including the form of the conserved emittance in solenoid-focused accelerators. Although the equation of motion becomes especially simple in axisymmetric fields, Eq. (5) does not assume such a constraint. Therefore, this formalism may also be used to account for the effects of multipole fields in systems that would otherwise be best analyzed in a rotating frame.

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REFERENCES

[1] J. Rosenzweig, C. Pellegrini, L. Serafini, C. Ternieden, and G. Travish, "Space-charge oscillations in a self-modulated electron beam in multi-undulator free-electron lasers," *Nucl. Instrum. Methods Phys. Res. A*, vol. 393, no. 1, pp. 376–379, 1997. doi:10.1016/S0168-9002(97)00516-0

[2] A. J. Dragt, F. Neri, and G. Rangarajan, "General moment invariants for linear hamiltonian systems," *Phys. Rev. A*, vol. 45, pp. 2572–2585, 1992. doi:10.1103/PhysRevA.45.2572

[3] K.-J. Kim, "Round-to-flat transformation of angular-momentum-dominated beams," *Phys. Rev. ST Accel. Beams*, vol. 6, p. 104 002, 2003. doi:10.1103/PhysRevSTAB.6.104002

[4] E. P. Lee and R. K. Cooper, "General Envelope Equation for Cylindrically Symmetric Charged Particle Beams," *Part. Accel.*, vol. 7, pp. 83–95, 1976, [Erratum: Part.Accel. 8, 67 (1977)].