Bound geodesics in Kerr space time

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Abstract

We derive the analytical solutions of the bound timelike geodesic orbits in Kerr spacetime. The analytical solutions are expressed in terms of the elliptical integrals using Mino time λ. Mino time is useful to understand physical properties of Kerr geodesics since it decouples radial and polar motion of a particle. Then, we can estimate the fundamental frequencies of the orbits such as radial, polar and azimuthal motion, and the Fourier series of arbitrary functions of particle's orbits. In this paper, we derive the analytical expression of both the fundamental frequencies and the orbits in terms of the elliptical integrals using Mino time. We can use these analytical expressions to investigate physical properties of Kerr geodesics and immediately apply them to the estimation of gravitational waves from the extreme mass ration inspirals.

1 Introduction

One of the main subjects to understand the properties of Kerr black hole spacetime is the geodesic motion. A lot of studies of the geodesic motion in black hole spacetime are summarized in the classical text book of Chandrasekhar[1]. In the weak eld regime, the orbits of a particle is almost same as that of Newton gravity. In the strong eld regime, however, the orbits show quite complicated trajectories. For the case of bound geodesics, this can be explained by mismatches between fundamental frequencies of radial, $_r$, polar, and azimuthal-motion, $_\phi$. $_\phi$ shows the precession of orbital plane and $_\phi$ $_r$ shows the precession of orbital ellipse. These mismatches become larger as the particle goes to strong gravity region around black hole horizon or separatrix, which is the boundary between stable and unstable orbits. These relativistic e ects are studied for some cases and some sorts of extreme phenomena, such as horizon-skimming orbits[2, 3, 4] and zoom-whirl orbits[5], are found.

Therefore the fundamental frequencies play important roles to understand geodesic orbits. However, the coupling of both r and motions in geodesics equation has been preventing us from deriving the fundamental frequencies, $_r$, and $_\phi$, for bound geodesic orbits until recently. Using elegant Hamilton-Jacobi formalism, Schmidt[6] derived the fundamental frequencies without discussing the coupling of both r and motions. Mino[7] decoupled both r and motions introducing a new time parameter λ, which is called Mino time. Then Drasco and Hughes[8] rederived the fundamental frequencies using Mino time and showed how to compute the Fourier components of arbitrary functions of orbits. These facts enable us to compute gravitational waves from extreme mass ratio inspirals(EMRIs) in the case of bound geodesics.

In this paper, we derive the analytical expression of both the fundamental frequencies and the orbits in Kerr spacetime. This analytical expressions help us to discuss bound geodesics in Kerr spacetime. Moreover, it enables us to compute gravitational waves from EMRIs more accurately since we can compute the orbits with the arbitrary accuracy in principle. This paper is organized as follows. In Sec. 2, we review Kerr geodesics in Mino time. In Sec. 3, we brie y show how to derive the analytical expressions of the fundamental frequencies of bound geodesics in terms of the elliptic integrals. And we summarize this paper in Sec. 4. Throughout this paper, we use units with $G = c = 1$.

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2 Geodesic Orbits in Kerr Spacetime

The geodesic equations that describes particle's orbits in Kerr spacetime are given by

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r), \qquad \left(\frac{d\cos}{d\lambda}\right)^2 = \quad(\cos\ ),$$
$$\frac{dt}{d\lambda} = T_{\rm r}(r) + T\ (\cos\ ) + a\mathcal{L}_z, \qquad \frac{d}{d\lambda} = \ _{\rm r}(r) + \quad(\cos\ ) \quad a\mathcal{E}, \tag{1}$$

where the functions $R(r)$, $(\cos\ )$,$T_{\rm r}(r)$, $T\ (\cos\ )$, $_{\rm r}(r)$ and $(\cos\ )$ are de ned by

$$\begin{aligned} R(r) &= [P(r)]^2 \quad [r^2 + (a\mathcal{E} \quad \mathcal{L}_z)^2 + \mathcal{C}], \\ (\cos\ ) &= \mathcal{C} \quad (\mathcal{C} + a^2(1 \quad \mathcal{E}^2) + \mathcal{L}_z^2)\cos^2 \quad + a^2(1 \quad \mathcal{E}^2)\cos^4\ , \\ T_r(r) &= \frac{r^2+a^2}{}P(r), \quad T\ (\cos\ ) = \quad a^2\mathcal{E}(1 \quad \cos^2\ ), \\ _r(r) &= \frac{a}{}P(r), \qquad (\cos\ ) = \frac{\mathcal{L}_z}{1 \quad \cos^2}, \end{aligned}$$

with $P(r) = \mathcal{E}(r^2 + a^2) \quad a\mathcal{L}_z$, $= r^2 + a^2\cos^2$ and $= r^2 \quad 2Mr + a^2$. Here M and a are the mass and the angular momentum of the black hole, respectively. λ is Mino time de ned by $\lambda = \int d\ /\ $, where is proper time along geodesics. There are three constants of motion, $\mathcal{E}$, $\mathcal{L}_z$ and $\mathcal{C}$, which are the energy, the z-component of the angular momentum and the Carter constant per unit mass, respectively.

In Eq. (1), $dr/d\lambda$ depends only on r and $d\cos\ /d\lambda$ depends only on $\cos$. Thus, for the bound orbits, $r(\lambda)$ and $\cos\ (\lambda)$ become periodic functions which are independent of each other. The fundamental periods for the radial and polar motion, $_r$ and , with respect to λ are given by

$$_r = 2\int_{\rm r_{min}}^{\rm r_{max}} \frac{dr}{\sqrt{R(r)}}, \qquad = 4\int_0^{\cos\ _{\rm min}} \frac{d\cos}{\sqrt{\ (\cos\ )}}, \tag{2}$$

where

$$r_{\rm min} = \frac{pM}{1+e}, \quad r_{\rm max} = \frac{pM}{1\quad e}, \quad _{\rm inc} + ({\rm sgn}\,\mathcal{L}_z)\ _{\rm min} = \frac{}{2}. \tag{3}$$

Here $r_{\rm min}$ and $r_{\rm max}$ is periapsis and apoapsis for the radial motion respectively, and $_{\rm inc}$ is the inclination angle from the equatorial plane of black hole. Of course, $(\mathcal{E}, \mathcal{L}_z, \mathcal{C})$ are described by these orbital parameters $(p, e,\ _{\rm inc})$ and given in Ref.[6, 8].

Then the angular frequencies of the radial and the polar motion become

$$_r = \frac{2}{_r}, \qquad = \frac{2}{}. \tag{4}$$

Integral forms of $t(\lambda)$ and (λ) sre given by

$$t(\lambda) = \quad \lambda + t^{(r)}(\lambda) + t^{(\ )}(\lambda), \qquad (\lambda) = \ _\phi\lambda + \ ^{(r)}(\lambda) + \ ^{(\ )}(\lambda), \tag{5}$$

where and $_\phi$ are the frequencies of coordinate time t and with respect to λ respectively, which are given by

$$= \ _{t^{(r)}} + \ _{t^{(\theta)}} + a\mathcal{L}_z, \qquad _\phi = \ _{\phi^{(r)}} + \ _{\phi^{(\theta)}} \quad a\mathcal{E}, \tag{6}$$

where

$$\begin{aligned} _{t^{(r)}} &= \tfrac{2}{\Lambda_r}\int_{r_{\rm min}}^{r_{\rm max}} \frac{T_r(r)}{\sqrt{R(r)}}dr, \qquad _{t^{(\theta)}} = \tfrac{4}{\Lambda_\theta}\int_0^{\cos\ _{\rm min}} \frac{T_\theta(\cos\ )}{\sqrt{\Theta(\cos\ )}}d\cos\ , \\ _{\phi^{(r)}} &= \tfrac{2}{\Lambda_r}\int_{r_{\rm min}}^{r_{\rm max}} \frac{_r(r)}{\sqrt{R(r)}}dr, \qquad _{\phi^{(\theta)}} = \tfrac{4}{\Lambda_\theta}\int_0^{\cos\ _{\rm min}} \frac{_\theta(\cos\ )}{\sqrt{\Theta(\cos\ )}}d\cos\ , \end{aligned} \tag{7}$$

and $t^{(r)/(\ )}$ and $^{(r)/(\ )}$ are given by

$$t^{(r)}(\lambda) = \int_{r_{\rm min}}^{r[\ ]} \frac{T_{\rm r}(r) \quad _{t^{(r)}}}{\sqrt{R(r)}}dr, \quad t^{(\ )}(\lambda) = \int_0^{\cos\ [\ ]} \frac{T\ (\cos\ ) \quad _{t^{(\theta)}}}{\sqrt{\ (\cos\ )}}d\cos\ ,$$

$$ {}^{(r)}(\lambda) \;=\; \int_{r_{\min}}^{r[\;\;]} \frac{{}_{\mathrm{r}}(r) \qquad {}_{\phi(r)}}{\sqrt{R(r)}} dr, \quad {}^{(\;)}(\lambda) = \int_{0}^{\cos\;[\;\;]} \frac{(\cos\;\;) \qquad {}_{\phi(\theta)}}{\sqrt{\;\;(\cos\;\;)}} d\cos\;\;. \tag{8} $$

Eq. (5) shows that both $t(\lambda)$ and (λ) are decomposed into two parts, in which the rst term represents accumulation over λ-time and the last two terms represent oscillation from it with periods $2\;\;/\;\;_r$ and $2\;\;/\;\;$. We note that the frequencies with respect to λ are related to the frequencies with distant observer time as[8]

$$ {}_r = \frac{{}_r}{\;}, \qquad = \frac{\;}{\;}, \qquad {}_\phi = \frac{{}_\phi}{\;}. \tag{9} $$

3 Analytical solutions of bound geodesics

Since both $R(r)$ and $(\cos\;\;)$ are forth order polynomials, Eq. (2) can be expressed in terms of the elliptic integrals. It is useful if we know the four zero points of both $R(r)$ and $(\cos\;\;)$. We rewrite $R(r)$ and $(\cos\;\;)$ as[8]

$$ \begin{aligned} R(r) &= (1\quad\mathcal{E}^2)(r_1\quad r)(r\quad r_2)(r\quad r_3)(r\quad r_4), \\ (\cos\;\;) &= \mathcal{L}_z^2\;{}_0(z\qquad\cos^2\;\;)(z_+\quad\cos^2\;\;), \end{aligned} \tag{10} $$

where

$$ \begin{aligned} r_1 = \frac{pM}{1\quad e}, \quad r_2 &= \frac{pM}{1+e}, \quad r_3 = \frac{(A+B)+\sqrt{(A+B)^2\quad 4AB}}{2}, \quad r_4 = \frac{AB}{r_3}, \\ A+B &= \frac{2M}{1\quad\mathcal{E}^2}\quad(r_1+r_2), \quad AB = \frac{a^2\mathcal{C}}{(1\quad\mathcal{E}^2)\,r_1 r_2}, \end{aligned} \tag{11} $$

and where ${}_0 = a^2(1\quad\mathcal{E}^2)/\mathcal{L}_z^2$, $z\;\; = \cos^2\;\;_{\min}$ and $z_+ = \mathcal{C}/(\mathcal{L}_z^2\;{}_0 z\;\;)$. We note that two zero points, r_1 and r_2, of $R(r)$ are apoapsis and periapsis respectively and two zero point, z and z, of $(\cos\;\;)$ are ${}_{\min}$ and ${}_{\min}$ respectively. These zero points correspond to turning points, de ned in Eq. (3), of radial and polar motion. But the other two zero points of both $R(r)$ and $(\cos\;\;)$, r_3, r_4 and z_+, do not correspond to turning points of radial and polar motion, but represent zero points of them.

Using Eq. (10), we can express Eq. (2) in terms of the elliptic integrals as

$$ \begin{aligned} \int_{r_2}^{r} \frac{dr}{\sqrt{R(r)}} &= \frac{2}{\sqrt{(1\quad\mathcal{E}^2)(r_1\quad r_3)(r_2\quad r_4)}} F(\arcsin y_r, k_r), \\ \int_{0}^{\cos} \frac{d\cos}{\sqrt{\;\;(\cos\;\;)}} &= \frac{1}{\mathcal{L}_z\sqrt{\;{}_0 z_+}} F(\arcsin y\;\;, k\;\;), \end{aligned} \tag{12} $$

where

$$ \begin{aligned} y_r &= \sqrt{\frac{r_1\quad r_3}{r_1\quad r_2}\frac{r\quad r_2}{r\quad r_3}}, \quad k_r = \sqrt{\frac{r_1\quad r_2}{r_1\quad r_3}\frac{r_3\quad r_4}{r_2\quad r_4}}, \\ y &= \frac{\cos}{\sqrt{z}}, \quad k = \sqrt{\frac{z}{z_+}}, \end{aligned} \tag{13} $$

and $F(\varphi, k)$ is the incomplete elliptic integral of the rst kind.

Therefore orbital frequencies of radial and polar motion with respect to λ are given by

$$ {}_r = \frac{\sqrt{(1\quad\mathcal{E}^2)(r_1\quad r_3)(r_2\quad r_4)}}{2K(k_r)}, \qquad = \frac{\mathcal{L}_z\sqrt{\;{}_0 z_+}}{2K(k\;\;)}. \tag{14} $$

Here $K(k)$ is the complete elliptic integral of the rst kind de ned by $K(k) = F(\;\;/2, k)$. Using Eq. (13), it is straightforward to express and ${}_\phi$ in terms of the complete elliptic integrals. Explicit forms of and ${}_\phi$ will be given in elsewhere. Basically, geodesic orbits are derived by replacing the complete elliptic integrals of the fundamental frequencies with the elliptic integrals. Again, explicit forms of geodesic orbits will be given in elsewhere. We compare our analytical results of bound geodesics with that of numerical integration method as a consistency check in Fig. 1. This gure shows that the analytical solutions of geodesic equation in this paper exactly represent the solutions of bound geodesics in Kerr spacetime.

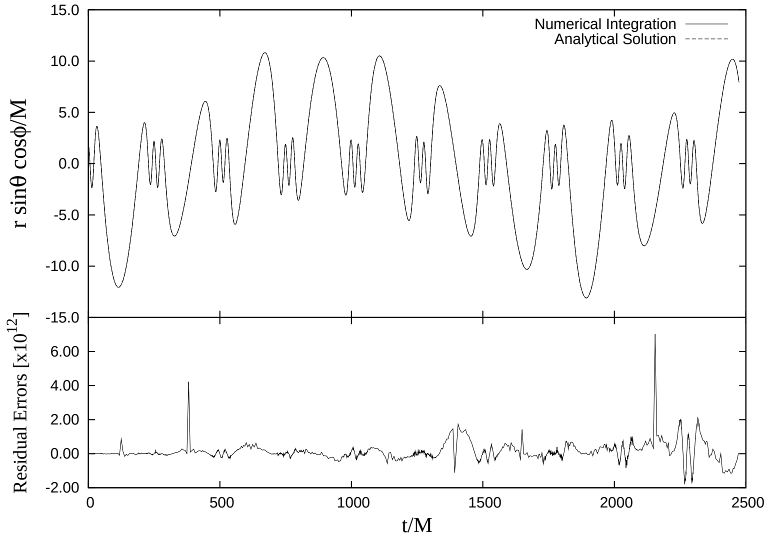

Figure 1: Plots of function $x(t) = r(t)\sin\theta(t)\cos\phi(t)$ using both analytical expressions and numerical integration method. In this figure, we set orbital elements as $a = 0.9M$, $p = 4M$, $e = 0.7$ and $\theta_{\rm inc} = 40^\circ$. Upper figure shows plots of both analytical solution, $x_{\rm A}(t)$, and numerical integration method, $x_{\rm N}(t)$. Lower figure shows the residual errors, $x_{\rm A}(t) - x_{\rm N}(t)$.

4 Summary

We derived the analytical solutions of the bound timelike geodesics in Kerr spacetime. We expressed these solutions in terms of the elliptic integrals using Mino time. Since Mino time decouples both r and θ-motion, it is straightforward to express the orbits in terms of the elliptic integrals if we suitably transform variables, r and θ.

We can apply these solutions to the computation of gravitational waves from extreme mass ratio inspirals, which are one of the main targets for space-based gravitational waves antenna LISA[9]. Using the analytical solution in this paper, we can compute the orbits arbitrary accuracy in principle. Thus it is very useful to compute gravitational waves from EMRI. We may also apply these solutions to investigate the properties of timelike and null geodesics in the case of both bound and unbound orbits. All of them will be discussed in future work.

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