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# Hamiltonian analysis of Einstein–Chern–Simons gravity

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ABSTRACT

In this work we consider the construction of the Hamiltonian action for the transgressions field theory. The subspace separation method for Chern–Simons Hamiltonian is built and used to find the Hamiltonian for five-dimensional Einstein–Chern–Simons gravity. It is then shown that the Hamiltonian for Einstein gravity arises in the limit where the scale parameter $l$ approaches zero.



## 1. Introduction

In the context of the general relativity the spacetime is a dynamical object which has independent degrees of freedom and is governed by dynamical equations, namely the Einstein field equations. This means that in general relativity the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed space-time background. An action for gravity fulfilling these conditions, albeit only in odd-dimensional spacetime, $d = 2n + 1$, was proposed long ago by Chamseddine [1,2], which is given by a Chern–Simons form for the anti-de Sitter (AdS) algebra. Chern–Simons gravities have been extensively studied; see, for instance, Refs. [3–12].

If Chern–Simons theories are to provide the appropriate gauge-theory framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to general relativity.

Studies in this direction have been carried out in Refs. [13–16] (see also [17,18]). In these references it was found that standard, five-dimensional GR (without a cosmological constant) emerges as the $\ell \to 0$ limit of a CS theory for a certain Lie algebra $\mathfrak{B}_5$. Here $\ell$ is a length scale, a coupling constant that characterizes different regimes within the theory. The $\mathfrak{B}_5$ algebra, on the other hand, is constructed from the *AdS* algebra and a particular semigroup by means of the $S$-expansion procedure introduced in Ref. [19].

Black hole type solutions and the cosmological nature of the corresponding fields equations satisfy the same property, namely, that standard black-holes solutions and standard cosmological solutions emerge as the $\ell \to 0$ limit of the black-holes and cosmological solutions of the Einstein–Chern–Simons field equations [14–16].

The Einstein–Chern–Simons action was constructed using transgression forms and a method, known as subspace separation procedure [20]. This procedure is based on the iterative use of the Extended Cartan Homotopy Formula, and allows one to $(i)$ systematically split the Lagrangian in order to appropriately reflect the subspaces structure of the gauge algebra, and $(ii)$ separate the Lagrangian in bulk and boundary contributions.

However the Hamiltonian analysis of Einstein Chern–Simons gravity action as well as transgression forms is as far as we know an open problem.

In Ref. [21] was studied the Hamiltonian formulation of the Lanczos–Lovelock ($LL$) theory. The $LL$ theory is the most general theory of gravity in $d$ dimensions which leads to second-order field equations for the metric. The corresponding action, satisfying the criteria of general covariance and second-order field equations for $d > 4$ is a polynomial of degree $[d/2]$ in the curvature, has $[(d-1)/2]$ free parameters, which are not fixed from first principles.

In Ref. [6] was shown, using the first order formalism, that requiring the theory to have the maximum possible number of degrees of freedom, fixes these parameters in terms of the gravitational and the cosmological constants. In odd dimensions, the Lagrangian is a Chern–Simons forms for the *AdS* group. The vielbein and the spin connection can be viewed as different components of an $(A)dS$ or Poincare connection, so that its local symmetry is enlarged from Lorentz to (A)dS (or Poincare when $\Lambda = 0$).

The principal motivation of this work is, using the first order formalism, find the Hamiltonian formalism for a Chern–Simons theory leading to general relativity in a certain limit.

In the first-order approach, the independent dynamical variables are the vielbein ($e^a$) and the spin connection ($\omega^{ab}$), which obey first-order differential field equations. The standard second-order form can be obtained if the torsion equations are solved for the connection and eliminated in favor of the vielbein—this step,

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however, cannot be taken, in general, because the equations for $\omega^{ab}$ are not invertible for dimensions higher than four (for detail see Ref. [6]). This means it is not possible to find a Hamiltonian formulation of second order for a five-dimensional *AdS* as well as $\mathfrak{B}_5$ Chern–Simons theory.

The purpose of this work is to study (*i*) the Hamiltonian action for transgressions field theory, (*ii*) the subspace separation method in the context of Hamiltonian formalism, (*iii*) the Hamiltonian analysis of Einstein–Chern–Simons gravity, (*iv*) the relation between the Hamiltonian action of general relativity of Refs. [22, 23] and the Hamiltonian action for Einstein–Chern–Simons gravity.

This paper is organized as follows: In Sec. 2 the Hamiltonian analysis of the five-dimensional Chern–Simons theory is briefly reviewed. The Hamiltonian analysis of the transgressions form Lagrangians is considered in Sec. 3 where the Extended Cartan Homotopy Formula is reviewed and used to find the triangle equation in its Hamiltonian form. In Sec. 4 the subspace separation method for Chern–Simons Hamiltonian is built and used to find the Hamiltonian for five-dimensional Einstein–Chern–Simons gravity. It is then shown that the Hamiltonian for Einstein gravity of Refs. [22, 23] arises in the limit when the scale parameter $l$ approaches zero.

## 2. Hamiltonian analysis of the five-dimensional Chern–Simons theory

In this Section we briefly review of the Hamiltonian analysis of Chern–Simons theory studied in Refs. [24–26].

### 2.1. (4 + 1)-dimensional case

The Chern–Simons action in 4 + 1 dimensions is given by

$$\begin{aligned} S &= \int_M \mathcal{L}_{ChS4+1} \\ &= k\int_M \Big\langle A\wedge dA\wedge dA + \frac{3}{2}A\wedge A\wedge A\wedge dA \\ &\quad + \frac{3}{5}A\wedge A\wedge A\wedge A\wedge A\Big\rangle \\ &= k\int_M d^5x\varepsilon^{\mu\nu\rho\sigma\lambda}\Big\langle A_\mu\partial_\nu A_\rho\partial_\sigma A_\lambda + \frac{3}{2}A_\mu A_\nu A_\rho\partial_\sigma A_\lambda \\ &\quad + \frac{3}{5}A_\mu A_\nu A_\rho A_\sigma A_\lambda\Big\rangle, \end{aligned}$$

where $A = A^a_\mu T_a$ is the gauge connection field, $T_a$ are de generators of the corresponding gauge group, $\langle\cdots\rangle$ denote the symmetrized trace and $M$ is an orientable 5-dimensional manifold on which the connection $A$ is defined. If $M$ has the topology $M = \mathbb{R}\times\Sigma$ where $\mathbb{R}$ can be considered as the temporal line and $\Sigma$ as a spatial section, then we can split the gauge field in time and space components $A_\mu dx^\mu = Adx^0 + A_i dx^i$ $(i = 1, 2, 3, 4)$. The action then takes the form

$$S = k\int d^5x\varepsilon^{ijkl}g_{abc} \times\left(\frac{3}{4}A^a_0F^b_{ij}F^c_{kl} + \dot{A}^a_i\left(F^b_{jk}A^c_l - \frac{1}{4}f^c_{de}A^b_jA^d_kA^e_l\right)\right), \tag{1}$$

where $g_{abc} = \langle T_aT_bT_c\rangle$. Defining for convenience

$$K_a = -\frac{3}{4}k\varepsilon^{ijkl}g_{abc}F^b_{ij}F^c_{kl}, \tag{2}$$

$$l^i_a = k\varepsilon^{ijkl}g_{abc}\left(F^b_{jk}A^c_l - \frac{1}{4}f^c_{de}A^b_jA^d_kA^e_l\right),$$

we arrive to

$$S[A] = \int d^5x\left(\dot{A}^a_il^i_a - A^a_0K_a\right). \tag{3}$$

Since the canonical momenta are given by

$$\Pi^\mu_a = \frac{\partial\mathcal{L}_{cs2+1}}{\partial\dot{A}^a_\mu},$$

we find that the primary constraints are

$$\begin{aligned} \varphi_a &= \Pi^0_a \approx 0, \\ \phi^i_a &= \Pi^i_a - l^i_a \approx 0 \quad , \quad (i = 1, 2), \end{aligned}$$

and therefore the canonical Hamiltonian and the total Hamiltonian are then given by

$$H_c = \int d^4x\left(\dot{A}^a_\mu\Pi^\mu_a - \mathcal{L}\right) = \int d^4xA^a_0K_a.$$

## 3. Hamiltonian analysis of transgression field theory

In this section we consider the Hamiltonian analysis of transgression field theory introduced and studied in Refs. [27–30]. This results allow us to find the Hamiltonian triangular equation. This equation together with a method, known as subspace separation method, will be used to obtain the Chern–Simons Hamiltonian for the algebras $\mathfrak{so}(4,2)$ and $\mathfrak{B}_5$.

### 3.1. (2 + 1)-dimensional case

The action for a 3-dimensional transgression gauge field theory (TGFT) is given by

$$I^{(3)}_T[A,\bar{A}] = 2k\int_M\int_0^1 dt\langle\Theta F^t\rangle, \tag{4}$$

where $\Theta = A - \bar{A}$, with $A$ and $\bar{A}$ gauge connections, $A_t = \bar{A} + t\Theta$, $F_t = dA_t + A_tA_t$, $\langle\cdots\rangle$ denote the symmetrized trace and $M$ is an orientable 3-dimensional manifold on which the connection $A$ is defined. If $M$ has the topology $M = \mathbb{R}\times\Sigma$ where $\mathbb{R}$ can be considered as the temporal line and $\Sigma$ as a spatial section, then we can split the gauge field in time and space components

$$A_\mu dx^\mu = Adx^0 + A_idx^i\,(i = 1, 2), \tag{5}$$

introducing (5) into (4) we find

$$\begin{aligned} I^{(3)}_T[A_\mu,\bar{A}_\nu] &= \int_{I\times\Sigma} d^3x(k\int_0^1 dt\varepsilon^{\mu\nu\rho}\langle\theta_\mu F^t_{\nu\rho}\rangle) \\ &= \int_{I\times\Sigma} d^3x\mathcal{L}^{(3)}_T[A_\mu,\bar{A}_\nu], \end{aligned} \tag{6}$$

where the 3-dimensional TGFT Lagrangian is given by

$$\begin{aligned} \mathcal{L}^{(3)}_T[A_\mu,\bar{A}_\nu] &= k\int_0^1 dt\varepsilon^{\mu\nu\rho}\langle\theta_\mu F^t_{\nu\rho}\rangle \\ &= k\int_0^1 dt\varepsilon^{ij}g_{ab}[\theta^a_0(F^t_{ij})^b + 2(A^t_0)^aD^t_i\theta^b_j + 2(\dot{A}^t_i)^a\theta^b_j] \\ &\quad + B^{(3)}[A_\mu,\bar{A}_\nu], \end{aligned} \tag{7}$$

with $g_{ab} = \langle T_a T_b \rangle$, $B^{(3)}[A_\mu, \bar{A}_\mu] = \partial_i(-2k\int_0^1 dt\varepsilon^{ij} g_{ab}\theta_j^a (A_0^t)^b)$ is a boundary term and $D_i^t$ is the covariant derivatives for the $A_t$ connection.

From (7) we can see that when $\bar{A} = 0$ we obtain the 3-dimensional Chern–Simons Lagrangian. If $\bar{A} = 0$ we have $A_i^t = tA_i$, $\theta_i = A_i$ and therefore

$$\mathcal{L}_T^{(3)}[A_\mu, 0] = k\epsilon^{ij} g_{ab}(A^a F_{ij}^b + \dot{A}_i^a A_j^b) = \mathcal{L}_{CS}^{(3)}[A_\mu]. \tag{8}$$

Using the definition of Hamiltonian we find

$$\mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\nu] = -k\int_0^1 dt\varepsilon^{ij} g_{ab}[\theta_0^a (F_{ij}^t)^b + 2(A_0^t)^a D_i^t \theta_j^b]. \tag{9}$$

From (9) we can see that when $\bar{A} = 0$ we obtain the 3-dimensional Chern–Simons Hamiltonian of Refs. [25,26].

$$\mathcal{H}_T^{(3)}[A_\mu, 0] = \mathcal{H}_{ChS}^{(3)}[A_\mu]. \tag{10}$$

### 3.2. The 3-dimensional triangle equation

In the Lagrangian formalism the so called triangle equation is given by [27–30]

$$L_T^{(2n+1)}[A, \bar{A}] = L_T^{(2n+1)}[A, \tilde{A}] - L_T^{(2n+1)}[\bar{A}, \tilde{A}] - \kappa dQ^{(2n)}[A, \bar{A}, \tilde{A}], \tag{11}$$

where $L_T^{(2n+1)}[A, \bar{A}]$ is a transgression form and $\kappa$ is a constant. This equation can be read off as saying that a transgression form "interpolating" between $\bar{A}$ and $A$ may be written as the sum of two transgressions which introduce an intermediate, ancillary one-form $\tilde{A}$ plus a total derivative. It is important to note here that $\tilde{A}$ is completely arbitrary, and may be chosen according to convenience. From the triangle equation, and fixing the middle connection to zero ($\tilde{A} = 0$), we can see

$$L_T^{(2n+1)}[A, \bar{A}] = L_{ChS}^{(2n+1)}[A] - L_{ChS}^{(2n+1)}[\bar{A}] - \kappa dQ^{(2n)}[A, \bar{A}, 0]. \tag{12}$$

Using a method, known as subspace separation method, the equation (12) allows to construct the Chern–Simons Lagrangian for the $\mathfrak{B}$ algebra, from which emerges as $l \to 0$ the standard $(2n + 1)$-dimensional Lagrangian for general relativity.

In the case of the Hamiltonian formalism, triangular equation plays a similar role. The Hamiltonian triangular equation allows obtaining the Chern–Simons Hamiltonian for algebra $\mathfrak{B}$. This Hamiltonian empties into the Hamiltonian for five-dimensional relativity in the limit $l \to 0$. This is not achievable in the case of the Hamiltonian for AdS algebra.

Writing $L_T^{(2n+1)}[A, \bar{A}] = \mathcal{L}_T^{(2n+1)}[A_\mu, \bar{A}_\mu]\tilde{\mathbf{v}}^{(2n+1)}$ (where $\tilde{\mathbf{v}}^{(2n+1)}$ is the volume form) and integrating for $n = 1$ we have

$$\mathcal{L}_T^{(3)}[A_\mu, \bar{A}_\nu] = \mathcal{L}_{ChS}^{(3)}[A_\mu] - \mathcal{L}_{ChS}^{(3)}[\bar{A}_\mu] - \kappa dQ^{(2)}[A_\mu, \bar{A}_\nu]. \tag{13}$$

From (7) and (9) we can write

$$\mathcal{L}_T^{(3)}[A_\mu, \bar{A}_\mu] = P^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] - \mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\mu] + B^{(3)}[A_\mu, \bar{A}_\mu], \tag{14}$$

where

$$P^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = 2k\int_0^1 dt\varepsilon^{ij} g_{ab}(\dot{A}_i^t)^a \theta_j^b, \tag{15}$$

so that (13) takes the form

$$\mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_{ChS}^{(3)}[A_\mu] - \mathcal{H}_{ChS}^{(3)}[\bar{A}_\mu] + P^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] + P^{(3)}[\bar{A}_i, \dot{\bar{A}}_i] - P^{(3)}[A_i, \dot{A}_i] + \mathcal{B}^{(3)}[A_\mu, \bar{A}_\mu], \tag{16}$$

which can be written as

$$\mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_{ChS}^{(3)}[A_\mu] - \mathcal{H}_{ChS}^{(3)}[\bar{A}_\mu] + \tilde{P}^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] + \mathcal{B}^{(3)}[A_\mu, \bar{A}_\mu], \tag{17}$$

where

$$\tilde{P}^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = P^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] + P^{(3)}[\bar{A}_i, \dot{\bar{A}}_i] - P^{(3)}[A_i, \dot{A}_i]. \tag{18}$$

and

$$\mathcal{B}^{(3)}[A_\mu, \bar{A}_\mu] = B^{(3)}[A_\mu, \bar{A}_\mu] + B^{(3)}[\bar{A}_\mu] - B^{(3)}[A_\mu] + Q^{(3)}[A_\mu, \bar{A}_\mu], \tag{19}$$

with $Q^{(3)}[A_\mu, \bar{A}_\mu] = \kappa dQ^{(2)}[A_\mu, \bar{A}_\nu]$.

From here we can see that Eq. (17) is similar to Lagrangian triangular equation except for the term $\tilde{P}^{(3)}$ defined in equation (18). Since this term depends on the speed and the Hamiltonian can not dependent on it, it is convenient to analyze the aforementioned term. In fact, from

$$\tilde{P}^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = 2k\int_0^1 dt\varepsilon^{ij} g_{ab}[(\dot{A}_i^t)^a \theta_j^b + t\dot{\bar{A}}_i^a \bar{A}_j^b - t\dot{A}_i^a A_j^b] = k\varepsilon^{ij} g_{ab}[\dot{\bar{A}}_i^a A_j^b + \bar{A}_i^a \dot{A}_j^b], \tag{20}$$

we can see that $\tilde{P}^{(3)}$ is a boundary term

$$\tilde{P}^{(3)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = \partial_0(k\varepsilon^{ij} g_{ab}\bar{A}_i^a A_j^b), \tag{21}$$

so that the triangular equation (17) can be written as

$$\mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_{ChS}^{(3)}[A_\mu] - \mathcal{H}_{ChS}^{(3)}[\bar{A}_\mu] + \tilde{\mathcal{B}}^{(3)}[A_\mu, \bar{A}_\mu], \tag{22}$$

where $\tilde{\mathcal{B}}$ contains (21). This result can be extended to the case where the intermediate connection $\tilde{\mathbf{A}}$ is nonzero. In this case (22) takes the form

$$\mathcal{H}_T^{(3)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_T^{(3)}[A_\mu, \tilde{A}_\mu] - \mathcal{H}_T^{(3)}[\bar{A}_\mu, \tilde{A}_\mu] + \tilde{\mathcal{B}}^{(3)}[A_\mu, \bar{A}_\mu, \tilde{A}_\mu], \tag{23}$$

where the main difference with (22) resides in the fact that the term $\tilde{P}^{(3)}$, which appears in $\tilde{\mathcal{B}}^{(3)}[A_\mu, \bar{A}_\mu, \tilde{A}_\mu]$, contains two terms more than his counterpart (21).

The above results lead to show that the Hamiltonian transgression is the difference between two Chern–Simons Hamiltonians term plus a boundary term. The boundary term appears due to contributions of the Legendre transformation on the triangular Lagrangian equation. This result together to a method, known as subspace separation method, allows to construct the Chern–Simons Hamiltonian for an arbitrary Lie algebra.

### 3.3. (4 + 1)-dimensional case

In the (4 + 1)-dimensional case the action is given by

$$I_T^{(5)}[A, \bar{A}] = 3k\int_M \int_0^1 dt\langle \Theta F^t F^t \rangle. \tag{24}$$

Following the procedure of the above section we find

$$I_T^{(5)}[A,\bar{A}]=\frac{3}{4}k\int_{I\times\Sigma}d^5x\int_0^1 dt\varepsilon^{\mu\nu\rho\sigma\lambda}\langle\theta_\mu F_{\nu\rho}^tF_{\sigma\lambda}^t\rangle, \tag{25}$$

so that

$$\mathcal{L}_T^{(5)}[A_\mu,\bar{A}_\mu]=\frac{3}{4}k\int_0^1 dt\varepsilon^{\mu\nu\rho\sigma\lambda}\langle\theta_\mu F_{\nu\rho}^tF_{\sigma\lambda}^t\rangle. \tag{26}$$

Splitting the gauge field in time and space components $A_\mu dx^\mu=Adx^0+A_idx^i$ $(i=1,2,3,4)$, we find

$$\begin{aligned}\mathcal{L}_T^{(5)}&[A_\mu,\bar{A}_\mu]\\&=\frac{3}{4}k\int_0^1 dt\varepsilon^{0ijkl}\langle\theta_0F_{ij}^tF_{kl}^t-\theta_iF_{0j}^tF_{kl}^t+\theta_iF_{j0}^tF_{kl}^t-\theta_iF_{jk}^tF_{0l}^t\\&\quad+\theta_iF_{jk}^tF_{l0}^t\rangle\\&=k\int_0^1 dt\varepsilon^{0ijkl}g_{abc}(\frac{3}{4}\theta_0^a(F_{ij}^t)^b(F_{kl}^t)^c+3(A_0^t)^aD_i^t\theta_j^b(F_{kl}^t)^c\\&\quad+3(\dot{A}_i^t)^a\theta_j^b(F_{kl}^t)^c)+B^{(5)}[A_\mu,\bar{A}_\mu],\end{aligned} \tag{27}$$

where $g_{abc}=\langle T_aT_bT_c\rangle$ and $B^{(5)}[A_\mu,\bar{A}_\mu]=\partial_i(-3k\int_0^1 dt\varepsilon^{ijkl}g_{abc}\times(A_0^t)^a\theta_j^b(F_{kl}^t)^c)$ is a boundary term.

From (27) we can see that when $\bar{A}=0$ we obtain the 5-dimensional Chern–Simons Lagrangian.

$$\begin{aligned}\mathcal{L}_T^{(5)}[A_\mu,0]&=k\varepsilon^{0ijkl}g_{abc}(\frac{3}{4}A_0^a(F_{ij}^t)^b(F_{kl}^t)^c\\&\quad+\dot{A}_i{}^a\left(A_j^bF_{kl}^c-\frac{1}{4}f_{de}^cA_j^bA_k^dA_l^e\right)\\&=\mathcal{L}_{ChS}^{(5)}[A_\mu].\end{aligned} \tag{28}$$

Using the definition of Hamiltonian we find

$$\begin{aligned}\mathcal{H}_T^{(5)}[A_\mu,\bar{A}_\mu]&=-k\int_0^1 dt\varepsilon^{0ijkl}g_{abc}\\&\quad\times[\frac{3}{4}\theta_0^a(F_{ij}^t)^b(F_{kl}^t)^c+3(A_0^t)^aD_i^t\theta_j^b(F_{kl}^t)^c].\end{aligned} \tag{29}$$

From (29) we can see that when $\bar{A}=0$ we obtain the 5-dimensional Chern–Simons Hamiltonian

$$\mathcal{H}_T^{(5)}[A_\mu,0]=\mathcal{H}_{ChS}^{(5)}[A_\mu]. \tag{30}$$

### 3.4. Five-dimensional triangular equation

From (11) we have that the 5-dimensional triangular equation is given by

$$\mathcal{L}_T^{(5)}[A_\mu,\bar{A}_\mu]=\mathcal{L}_{ChS}^{(5)}[A_\mu]-\mathcal{L}_{ChS}^{(5)}[\bar{A}_\mu]-\kappa dQ^{(4)}[A_\mu,\bar{A}_\mu]. \tag{31}$$

From (27) and (29) we can see that

$$\begin{aligned}\mathcal{L}_T^{(5)}[A_\mu,\bar{A}_\mu]&=P^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]-\mathcal{H}_T^{(5)}[A_\mu,\bar{A}_\mu]\\&\quad+B^{(5)}[A_\mu,\bar{A}_\mu],\end{aligned} \tag{32}$$

where

$$P^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]=k\int_0^1 dt\varepsilon^{ijkl}g_{abc}3(\dot{A}_i^t)^a\theta_j^b(F_{kl}^t)^c, \tag{33}$$

so that (31) takes the form

$$\begin{aligned}\mathcal{H}_T^{(5)}[A_\mu,\bar{A}_\mu]&=\mathcal{H}_{ChS}^{(5)}[A_\mu]-\mathcal{H}_{ChS}^{(5)}[\bar{A}_\mu]+\tilde{P}^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]\\&\quad+\mathcal{B}^{(5)}[A_\mu,\bar{A}_\mu],\end{aligned} \tag{34}$$

where

$$\begin{aligned}\tilde{P}^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]&=P^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]+P^{(5)}[\bar{A}_i,\dot{\bar{A}}_i]\\&\quad-P^{(5)}[A_i,\dot{A}_i],\end{aligned} \tag{35}$$

and

$$\begin{aligned}\mathcal{B}^{(5)}[A_\mu,\bar{A}_\mu]&=B^{(5)}[A_\mu,\bar{A}_\mu]+B^{(5)}[\bar{A}_\mu]-B^{(5)}[A_\mu]\\&\quad+Q^{(5)}[A_\mu,\bar{A}_\mu],\end{aligned} \tag{36}$$

with $Q^{(5)}[A_\mu,\bar{A}_\mu]=\kappa dQ^{(4)}[A_\mu,\bar{A}_\mu]$.

From here we can see that the Eq. (34) is similar to Lagrangian triangular equation except for the term, $\tilde{P}^{(5)}$, defined in equation (35). Since this term depends on the velocity and the Hamiltonian can not dependent on it, it is convenient to analyze the aforementioned term. In fact, from

$$\begin{aligned}\tilde{P}^{(5)}[A_i,\bar{A}_i,\dot{A}_i,\dot{\bar{A}}_i]&=\partial_0[k\varepsilon^{ijkl}g_{abc}(\bar{A}_i^aA_j^b\partial_kA_l^c+\bar{A}_i^aA_j^b\partial_k\bar{A}_l^c\\&\quad+\frac{1}{4}f_{de}^c\bar{A}_i^aA_j^b\bar{A}_k^d\bar{A}_l^e+\frac{1}{4}f_{de}^c\bar{A}_i^aA_j^bA_k^dA_l^e)\\&\quad+\frac{1}{2}f_{de}^c\bar{A}_i^aA_j^bA_k^d\bar{A}_l^e+\frac{1}{8}f_{de}^c\bar{A}_i^a\bar{A}_j^bA_k^dA_l^e\\&\quad-\frac{1}{8}f_{de}^cA_i^aA_j^b\bar{A}_k^d\bar{A}_l^e)]\\&\quad+\partial_i[k\varepsilon^{ijkl}g_{abc}(\dot{\bar{A}}_j^aA_k^bA_l^c-\dot{A}_j^a\bar{A}_k^bA_l^c)],\end{aligned} \tag{37}$$

we have that the triangular equation is given by

$$\mathcal{H}_T^{(5)}[A_\mu,\bar{A}_\mu]=\mathcal{H}_{ChS}^{(5)}[A_\mu]-\mathcal{H}_{ChS}^{(5)}[\bar{A}_\mu]+\tilde{\mathcal{B}}^{(5)}[A_\mu,\bar{A}_\mu], \tag{38}$$

where $\tilde{\mathcal{B}}$ contains (37).

This result together with a procedure, known as subspace separation method, allows to construct the Chern–Simons Hamiltonian for the $\mathfrak{so}(4,2)$ and $\mathfrak{B}_5$ Lie algebra.

### 3.5. $(2n+1)$-dimensional case

The action for a $(2n+1)$-dimensional transgression gauge field theory is given by

$$I_T^{(2n+1)}[A_\mu,\bar{A}_\mu]=(n+1)k\int_M\int_0^1 dt\langle\Theta\mathbf{F}_t^n\rangle. \tag{39}$$

Following the same procedure of the above sections we find,

$$\begin{aligned}\mathcal{L}_T^{(2n+1)}[A_\mu,\bar{A}_\mu]&=\int_0^1 dt(\dot{A}_i^t)^a\mathcal{L}_a^i-\theta_0^a\mathcal{K}_a-(A_0^t)^a\mathcal{N}_a\\&\quad+B^{(2n+1)}[A_\mu,\bar{A}_\mu],\end{aligned} \tag{40}$$

where

$$\mathcal{L}_a^i = \frac{n(n+1)}{2^{n-1}} k\varepsilon^{ij_1\ldots j_{2n-1}} g_{ab_1\ldots b_n} \theta_{j_1}^{b_1} (F_{j_2 j_3}^t)^{b_1} \cdots (F_{j_{2n-2} j_{2n-1}}^t)^{b_n}, \tag{41}$$

$$\mathcal{K}_a = -\frac{(n+1)}{2^n} k\varepsilon^{ij_1\ldots j_{2n-1}} g_{ab_1\ldots b_n} (F_{ij_1}^t)^{b_1} \cdots (F_{j_{2n-2} j_{2n-1}}^t)^{b_n}, \tag{42}$$

$$\mathcal{N}_a = -\frac{n(n+1)}{2^{n-1}} k\varepsilon^{ij_1\ldots j_{2n-1}} g_{ab_1\ldots b_n} D_i^t \theta_{j_1}^{b_1} \times (F_{j_2 j_3}^t)^{b_2} \cdots (F_{j_{2n-2} j_{2n-1}}^t)^{b_n}, \tag{43}$$

$$B^{(2n+1)}[A_\mu, \bar{A}_\mu] = \partial_i (-3k \int_0^1 dt \varepsilon^{ij_1\ldots j_{2n-1}} g_{ab_1\ldots b_n} (\dot{A}_0^t)^a \theta_{j_1}^{b_1} \times (F_{j_2 j_3}^t)^{b_1} \cdots (F_{j_{2n-2} j_{2n-1}}^t)^{b_n}, \tag{44}$$

with $\varepsilon^{ij_1\ldots j_{2n-1}} = \varepsilon^{0ij_1\ldots j_{2n-1}}$ and $g_{ab_1\ldots b_n} = \langle T_a T_{b_1} \cdots T_{b_n} \rangle$. Using the definition of Hamiltonian we have

$$\mathcal{H}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] = \int_0^1 dt \left(\theta_0^a \mathcal{K}_a + (A_0^t)^a \mathcal{N}_a\right). \tag{45}$$

From (45) we can see that when $\bar{A} = 0$ we find the $(2n+1)$-dimensional Chern–Simons Hamiltonian

$$\mathcal{H}_{CS}^{(2n+1)}[A_\mu] = -\frac{k}{2^n}(n+1)\varepsilon^{ij_1\ldots j_{2n-1}} g_{ab_1\ldots b_n} A_0^a F_{ij_1}^{b_1} \cdots F_{j_{2n-2} j_{2n-1}}^{b_n} = \mathcal{H}_{ChS}^{(2n+1)}[A_\mu]. \tag{46}$$

I.e.,

$$\mathcal{H}_T^{(2n+1)}[A_\mu, 0] = \mathcal{H}_{ChS}^{(2n+1)}[A_\mu]. \tag{47}$$

### 3.6. The $(2n+1)$-dimensional triangle equation

Following the procedure used in the above sections we have

$$\mathcal{L}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] = \mathcal{L}_{ChS}^{(2n+1)}[A_\mu] - \mathcal{L}_{ChS}^{(2n+1)}[\bar{A}_\mu] - \kappa d Q^{(2n)}[A_\mu, \bar{A}_\mu]. \tag{48}$$

From (40) and (45) we have

$$\mathcal{L}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] = P^{(2n+1)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] - \mathcal{H}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] + B^{(2n+1)}[A_\mu, \bar{A}_\mu], \tag{49}$$

where

$$P^{(2n+1)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = \int_0^1 dt (\dot{A}_i^t)^a \mathcal{L}_a^i, \tag{50}$$

so that (48) takes the form

$$\mathcal{H}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_{ChS}^{(2n+1)}[A_\mu] - \mathcal{H}_{ChS}^{(2n+1)}[\bar{A}_\mu] + \tilde{P}^{(2n+1)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] + \mathcal{B}^{(2n+1)}[A_\mu, \bar{A}_\mu] \tag{51}$$

where,

$$\tilde{P}^{(2n+1)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] = P^{(2n+1)}[A_i, \bar{A}_i, \dot{A}_i, \dot{\bar{A}}_i] + P^{(2n+1)}[\bar{A}_i, \dot{\bar{A}}_i] - P^{(2n+1)}[A_i, \dot{A}_i]. \tag{52}$$

This term is a boundary term, so that the triangular equation is finally given by

$$\mathcal{H}_T^{(2n+1)}[A_\mu, \bar{A}_\mu] = \mathcal{H}_{ChS}^{(2n+1)}[A_\mu] - \mathcal{H}_{ChS}^{(2n+1)}[\bar{A}_\mu] + \tilde{\mathcal{B}}^{(2n+1)}[A_\mu, \bar{A}_\mu], \tag{53}$$

where

$$\tilde{\mathcal{B}}^{(2n+1}[A_\mu, \bar{A}_\mu] = B^{(2n+1)}[A_\mu, \bar{A}_\mu] + B^{(2n+1)}[\bar{A}_\mu] - B^{(2n+1)}[A_\mu] + Q^{(2n+1)}[A_\mu, \bar{A}_\mu] + \tilde{P}^{(2n+1)}[A_\mu, \bar{A}_\mu], \tag{54}$$

with

$$Q^{(2n+1)}[A_\mu, \bar{A}_\mu] = \kappa \partial_\lambda \left( n(n+1) \int_0^1 dt \int_0^t ds \varepsilon^{\lambda\mu\nu\rho_1\ldots\rho_{2n}} g_{abc_1\ldots c_{n-1}} \theta_\mu^a \bar{A}_\nu^b \times (F_{\rho_1\rho_2}^{st})^{c_1} \cdots (F_{\rho_{2n-1}\rho_{2n}}^{st})^{c_{n-1}} \right). \tag{55}$$

## 4. Hamiltonian analysis for Einstein gravity and AdS Chern–Simons gravity

### 4.1. Hamiltonian for five-dimensional general relativity

The action for five-dimensional Einstein gravity is given by

$$I_{EHC}^{(5)}[e, \omega] = \int_M \varepsilon_{abcde} R^{ab} e^c e^d e^e, \tag{56}$$

where $M$ is an orientable 5-dimensional manifold. If $M$ has the topology $I \times \Sigma$, then $M$ can be foliated in four-dimensional Cauchy surfaces along $I \in \mathbb{R}$, i.e., to carry out a separation $(4+1)$. Since $e^a = e^a{}_\mu dx^\mu$ and $R^{ab} = \frac{1}{2} R^{ab}{}_{\mu\nu} dx^\mu dx^\nu$ we have

$$I_{EHC}^{(5)}[e_\mu, \omega_\mu] = \int_I dt \int_\Sigma d^4x \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho\lambda} \varepsilon_{abcde} R^{ab}{}_{\mu\nu} e^c{}_\sigma e^d{}_\rho e^e{}_\lambda, \tag{57}$$

so that

$$\mathcal{L}_{EHC}^{(5)}[e_\mu, \omega_\mu] = \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho\lambda} \varepsilon_{abcde} R^{ab}{}_{\mu\nu} e^c{}_\sigma e^d{}_\rho e^e{}_\lambda. \tag{58}$$

This allow us split the fields in time and space components,

$$\begin{aligned}\mathcal{L}_{EHC}^{(5)}[e_\mu, \omega_\mu] &= \frac{1}{2} \varepsilon^{0ijkl} \varepsilon_{abcde} (R^{ab}{}_{0i} e^c{}_j e^d{}_k e^e{}_l - R^{ab}{}_{i0} e^c{}_j e^d{}_k e^e{}_l + 3R^{ab}{}_{ij} e^c{}_0 e^d{}_k e^e{}_l) \\ &= \varepsilon^{0ijkl} \varepsilon_{abcde} [\dot{e}_i^a \omega^{bc}{}_j e^d{}_k e^e{}_l + 3\omega^{ab}{}_0 \partial_i e^c{}_j e^d{}_k e^e{}_l + 2\omega^a{}_{f0} \omega^{fb}{}_i e^c{}_j e^d{}_k e^e{}_l + \frac{3}{2} R^{ab}{}_{ij} e^c{}_0 e^d{}_k e^e{}_l] \\ &\quad - \partial_i (\varepsilon^{0ijkl} \varepsilon_{abcde} \omega^{ab}{}_0 e^c{}_j e^d{}_k e^e{}_l).\end{aligned} \tag{59}$$

Considering that

$$\varepsilon^{0ijkl} T_{ij}^a = \varepsilon^{0ijkl} (2\partial_i e^a{}_j + 2\omega^a{}_{fi} e^f{}_j), \tag{60}$$

we find

$$\mathcal{L}_{EHC}^{(5)}[e_\mu, \omega_\mu] = \varepsilon^{0ijkl} \varepsilon_{abcde} [\dot{e}^a{}_i \omega^{bc}{}_j e^d{}_k e^e{}_l + \frac{3}{2} \omega^{ab}{}_0 T^c{}_{ij} e^d{}_k e^e{}_l + \frac{3}{2} R^{ab}{}_{ij} e^c{}_0 e^d{}_k e^e{}_l]. \tag{61}$$

A direct calculation allows us to obtain the Hamiltonian density for Einstein's gravity

$$\mathcal{H}^{(5)}_{EHC}[e_\mu,\omega_\mu] = -\frac{3}{2}\varepsilon^{0ijkl}\varepsilon_{abcde}[\omega^{ab}{}_0 T^c{}_{ij} e^d{}_k e^e{}_l + R^{ab}{}_{ij} e^c{}_0 e^d{}_k e^e{}_l]. \tag{62}$$

This means that (62) is the Hamiltonian for the five-dimensional general relativity in the first order formalism.

### 4.2. Hamiltonian for five-dimensional AdS Chern–Simons gravity

To construct the AdS Chern–Simons Hamiltonian we use the so called subspace separation method [20]. This procedure is based on the triangle equation (38), and embodies the following steps:

i) Identify the relevant subspaces present in the $\mathfrak{so}(4,2)$ algebra, i.e., write $\mathfrak{g} = V_1 \oplus V_2$, where

$$V_1 \to \{P_a\}, \quad V_2 \to \{J_{ab}\}. \tag{63}$$

ii) Write the connections as a sum of pieces valued on every subspace, i.e.,

$$A_\mu = \frac{1}{l} e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab}, \tag{64}$$

$$\bar{A}_\mu = \frac{1}{2}\omega^{ab}{}_\mu J_{ab}, \tag{65}$$

$$\tilde{A} = 0. \tag{66}$$

iii) The triangle equation (38) leads to

$$\mathcal{H}^{(5)}_{ChS}[A_\mu] = \mathcal{H}^{(5)}_{T}[A_\mu,\bar{A}_\mu] + \mathcal{H}^{(5)}_{ChS}[\bar{A}_\mu] - \tilde{\mathcal{B}}^{(5)}[A_\mu,\bar{A}_\mu]. \tag{67}$$

To calculate the terms present in (67) we use the following information:

- The non-vanishing components of the invariant tensor for five-dimensional *AdS* algebra are

$$\langle J_{ab} J_{cd} P_e \rangle = \frac{4}{3}\varepsilon_{abcde}.$$

- The connection $A^t_\mu$ and the tensors $\Theta_\mu$ and $F^t_{\mu\nu}$ are given by

$$\Theta_\mu = \frac{1}{l} e^a{}_\mu P_a, \tag{68}$$

$$A^t_\mu = \frac{t}{l} e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab}, \tag{69}$$

$$F^t_{\mu\nu} = \frac{t}{l} T^a{}_{[\mu\nu]} P_a + \left(\frac{1}{2} R^{ab}{}_{[\mu\nu]} + \frac{t^2}{l^2} e^a{}_{[\mu} e^a{}_{\nu]}\right) J_{ab}. \tag{70}$$

These results allow us to calculate the Hamiltonian present in the triangle equation and therefore

$$\begin{aligned}
&\mathcal{H}^{(AdS_5)}_{ChS}[e_\mu,\omega_\mu] \\
&= -k\varepsilon^{ijkl}\varepsilon_{abcde}\left(\frac{1}{4l} e^a{}_0 R^{bc}{}_{ij} R^{de}{}_{kl} + \frac{1}{l^3} e^a{}_0 R^{bc}{}_{ij} e^d{}_k e^e{}_l\right. \\
&\quad \left. + \frac{1}{l^5} e^a{}_0 e^b{}_i e^c{}_j e^d{}_k e^e{}_l + \frac{1}{2l}\omega^{ab}{}_0 R^{cd}{}_{ij} T^e{}_{kl} + \frac{1}{l^3}\omega^{ab}{}_0 T^c{}_{ij} e^d{}_k e^e{}_l\right).
\end{aligned}$$

From this Hamiltonian it is apparent that neither the $l \to \infty$ nor the $l \to 0$ limit yields the Hamiltonian for general relativity.

## 5. Hamiltonian for five-dimensional Einstein–Chern–Simons gravity

### 5.1. Lagrangian Einstein–Chern–Simons gravity

In Ref. [13] was shown that the standard, five-dimensional general relativity can be obtained from Chern–Simons gravity theory for a certain Lie algebra $\mathfrak{B}$ [13], whose generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$ satisfy the commutation relations given in Eq. (7) of Ref. [14]. This algebra was obtained from the anti de Sitter (AdS) algebra and a particular semigroup $S$ by means of the S-expansion procedure introduced in Ref. [19].

The Chern–Simons Lagrangian for the $\mathfrak{B}$ algebra, is built from the one-form gauge connection

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \tag{71}$$

and the two-form curvature

$$\begin{aligned}
F &= \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2}\left(\mathrm{D}_\omega k^{ab} + \frac{1}{l^2} e^a e^b\right) Z_{ab} \\
&\quad + \frac{1}{l}\left(\mathrm{D}_\omega h^a + k^a{}_b e^b\right) Z_a.
\end{aligned} \tag{72}$$

Following the dual procedure of $S$-expansion developed in Ref. [19] we find the five-dimensional Chern–Simons Lagrangian for the $\mathfrak{B}$ algebra is given by

$$\begin{aligned}
L^{(5)}_{\mathrm{EChS}} &= \alpha_1 l^2 \varepsilon_{abcde} e^a R^{bc} R^{de} \\
&\quad + \alpha_3 \varepsilon_{abcde}\left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e\right) \\
&\quad + \mathrm{d}B^{(4)}_{\mathrm{EChS}},
\end{aligned} \tag{73}$$

where the surface term $B^{(4)}_{\mathrm{EChS}}$ is given by

$$\begin{aligned}
&B^{(4)}_{\mathrm{EChS}} \\
&= \alpha_1 l^2 \varepsilon_{abcde} e^a \omega^{bc}\left(\frac{2}{3}\mathrm{d}\omega^{de} + \frac{1}{2}\omega^d{}_f \omega^{fe}\right) \\
&\quad + \alpha_3 \varepsilon_{abcde}\left[l^2\left(h^a \omega^{bc} + k^{ab} e^c\right)\left(\frac{2}{3}\mathrm{d}\omega^{de} + \frac{1}{2}\omega^d{}_f \omega^{fe}\right)\right. \\
&\qquad \left. + l^2 k^{ab}\omega^{cd}\left(\frac{2}{3}\mathrm{d}e^e + \frac{1}{2}\omega^d{}_f e^e\right) + \frac{1}{6} e^a e^b e^c \omega^{de}\right],
\end{aligned} \tag{74}$$

and where $\alpha_1$, $\alpha_3$ are parameters of the theory, $l$ is a coupling constant, $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c\omega^{cb}$ corresponds to the curvature 2-form in the first-order formalism related to the 1-form spin connection, and $e^a$, $h^a$ and $k^{ab}$ are others gauge fields presents in the theory.

The Lagrangian (73) shows that standard, five-dimensional General Relativity emerges as the $l \to 0$ limit of a Chern–Simons theory for the generalized Poincaré algebra $\mathfrak{B}$. Here $l$ is a length scale, a coupling constant that characterizes different regimes within the theory.

### 5.2. Hamiltonian for Einstein–Chern–Simons gravity

Now we will find the Hamiltonian for Einstein–Chern–Simons gravity, which is a Chern–Simons form for the so called $\mathfrak{B}$ algebra. Following the procedure of the above subsection, we have:

i) The relevant subspace of the $\mathfrak{B}$ algebra can be identify in the following form: $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, where

$$V_1 \to \{P_a\}, \; V_2 \to \{J_{ab}\}, \; V_3 \to \{Z_a\}, \; V_4 \to \{Z_{ab}\}. \tag{75}$$

ii) The connections $A_\mu$, $\bar{A}_\mu$, $\bar{A}^1_\mu$, $\bar{A}^2_\mu$ and $\tilde{A}_\mu$ are:

$$A_\mu = \frac{1}{l}e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab} + \frac{1}{l}h^a{}_\mu Z_a + \frac{1}{2}K^{ab}{}_\mu Z_{ab}, \quad (76)$$

$$\bar{A}^1_\mu = \frac{1}{l}e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab}, \quad (77)$$

$$\bar{A}^2_\mu = \frac{1}{2}\omega^{ab}{}_\mu J_{ab}, \quad (78)$$

$$.\tilde{A}_\mu = 0. \quad (79)$$

iii) So that the triangle equation leads to,

$$\mathcal{H}^{(5)}_{CS}[A_\mu] = \mathcal{H}^{(5)}_T[A_\mu, \bar{A}^1_\mu] + \mathcal{H}^{(5)}_T[\bar{A}^2_\mu, \tilde{A}_\mu] + \mathcal{H}^{(5)}_T[\bar{A}^1_\mu, \bar{A}^2_\mu] - \tilde{\mathcal{B}}^{(5)}[A_\mu, \bar{A}_\mu, \bar{A}^1_\mu, \bar{A}^2]. \quad (80)$$

To calculate the terms present in (80) we use the following information.

- The non-vanishing components of the invariant tensor for five-dimensional $\mathfrak{B}$ algebra are given by

$$\langle J_{ab} J_{cd} P_e \rangle = \alpha_1 \frac{4l^3}{3}\varepsilon_{abcde}, \quad (81)$$

$$\langle J_{ab} J_{cd} Z_e \rangle = \alpha_3 \frac{4l^3}{3}\varepsilon_{abcde}, \quad (82)$$

$$\langle J_{ab} Z_{cd} P_e \rangle = \alpha_3 \frac{4l^3}{3}\varepsilon_{abcde}. \quad (83)$$

- The connection $A^t_\mu$ and the tensors $\Theta_\mu$ and $F^t_{\mu\nu}$ are given by

$$\Theta_\mu(A_\mu, \bar{A}^1_\mu) = \frac{1}{l}h^a{}_\mu Z_a + \frac{1}{2}k^{ab}{}_\mu Z_{ab},$$

$$A^t_\mu(A_\mu, \bar{A}^1_\mu) = \frac{1}{l}e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab} + t\left(\frac{1}{l}h^a{}_\mu Z_a + \frac{1}{2}k^{ab}{}_\mu Z_{ab}\right),$$

$$F^t_{\mu\nu}(A_\mu, \bar{A}^1_\mu) = \frac{1}{l}T^a{}_{[\mu\nu]}P_a + \frac{1}{2}R^{ab}{}_{[\mu\nu]} + \frac{2t}{l}\left(D_{[\mu}h^a{}_{\nu]} + k^a{}_{b[\mu}e^b{}_{\nu]}\right)Z_a + \left(tD_{[\mu}k^{ab}{}_{\nu]} + \frac{1}{l^2}e^a{}_{[\mu}e^a{}_{\nu]}\right)Z_{ab},$$

$$\Theta_\mu(\bar{A}^2_\mu, 0) = \frac{1}{2}\omega^{ab}{}_\mu J_{ab},$$

$$A^t_\mu(\bar{A}^2_\mu, 0) = \frac{t}{2}\omega^{ab}{}_\mu J_{ab},$$

$$F^t_{\mu\nu}(\bar{A}^2_\mu, 0) = \frac{1}{2}\partial_{[\mu}\omega^{ab}{}_{\nu]}J_{ab} + \frac{t^2}{2}\omega^a{}_{c[\mu}\omega^{cb}{}_{\nu]}J_{ab},$$

$$\Theta_\mu(\bar{A}^1_\mu, \bar{A}^2_\mu) = \frac{1}{l}e^a{}_\mu P_a,$$

$$A^t_\mu(\bar{A}^1_\mu, \bar{A}^2_\mu) = \frac{t}{l}e^a{}_\mu P_a + \frac{1}{2}\omega^{ab}{}_\mu J_{ab},$$

$$F^t_{\mu\nu}(\bar{A}^1_\mu, \bar{A}^2_\mu) = \frac{t}{l}T^a{}_{[\mu\nu]}P_a + \frac{1}{2}R^{ab}{}_{[\mu\nu]}J_{ab} + \frac{t^2}{l^2}e^a{}_{[\mu}e^b{}_{\nu]}Z_{ab}.$$

These results allow us to calculate the Hamiltonian present in the triangle equation and therefore,

$$\mathcal{H}^{(\mathfrak{B}_5)}_{CS}[e_\mu, \omega_\mu, h_\mu, k_\mu] = -k\epsilon^{ijkl}\varepsilon_{abcde}\left[\frac{\alpha_1 l^2}{4}(e^a{}_0 R^{bc}{}_{ij}R^{de}{}_{kl} + 2\omega^{ab}{}_0 R^{cd}{}_{ij}T^e{}_{kl}) + \alpha_3(e^a{}_0 R^{bc}{}_{ij}e^d{}_k e^e{}_l + \omega^{ab}{}_0 T^c{}_{ij}e^d{}_k e^e{}_l + l^2 e^a{}_0 R^{bc}{}_{ij}D_k k^{de}{}_l + l^2\omega^{ab}{}_0 T^c{}_{ij}D_k k^{de}{}_l + l^2\omega^{ab}{}_0 R^{cd}{}_{ij}D_k h^e{}_l + l^2\omega^{ab}{}_0 R^{cd}{}_{ij}k^e{}_{fk}e^f{}_l + \frac{l^2}{2}k^{ab}{}_0 R^{cd}{}_{ij}T^e{}_{kl} + \frac{l^2}{2}h^a_0 R^{bc}{}_{ij}R^{de}{}_{kl})\right]. \quad (84)$$

From (84) we can see that in the limit $l \to 0$ we obtain the Hamiltonian for the five-dimensional general relativity in the first order formalism shown in (62),

$$\mathcal{H}^{(\mathfrak{B}_5)}_{CS}[e_\mu, \omega_\mu] = -k\alpha_3\varepsilon^{ijkl}\varepsilon_{abcde}[e^a{}_0 R^{bc}{}_{ij}e^d{}_k e^e{}_l + \omega^{ab}{}_0 T^c{}_{ij}e^d{}_k e^e{}_l]. \quad (85)$$

Studies of the first order Hamiltonian formalism, for 4-dimensional Einstein gravity can be found in Refs. [22,23], which coincide with (85) by reducing the dimension from five to four.

## 6. Concluding remarks

In this work the Hamiltonian analysis of the transgressions gauge fields theory was considered. The extended Cartan homotopy formula was reviewed and used to find the triangle equation in its Hamiltonian form.

The triangle equation leads to show that the Hamiltonian transgression is the difference between two Chern–Simons Hamiltonians term plus a boundary term. The boundary term appears due to contributions of the Legendre transformation on the triangular Lagrangian equation.

This result together to a method, known as subspace separation method, allows to construct the Chern–Simons Hamiltonian for the $\mathfrak{so}(4,2)$ and $\mathfrak{B}_5$ algebras.

Finally it was found that ($i$) from the Chern–Simons Hamiltonian for the $\mathfrak{so}(4,2)$ algebra it is apparent that neither the $l \to \infty$ nor the $l \to 0$ limit yields the Hamiltonian for general relativity. ($ii$) From the Chern–Simons Hamiltonian for the $\mathfrak{B}_5$ algebra we can see that the Hamiltonian for five-dimensional general relativity may indeed emerge when the scale parameter $l$ approaches zero.

Studies of the first order Hamiltonian formalism, for 4-dimensional Einstein gravity can be found in Refs. [22,23], which coincide with (85) by reducing the dimension from five to four.

It could be interesting to find a connection between the results obtained in this work (first-order formalism) with those of Teitelboim and Zanelli found in Ref. [21] (second-order formalism). The standard second-order form could be obtained if the torsion equations are solved for the connection and eliminated in favor of the vielbein. However, this is not possible in our case because the equations for $\omega^{ab}$ are not invertible for dimensions higher than four (for detail see Ref. [6]).

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