

Article

Supersymmetry and Exceptional Points

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Received: 26 April 2020; Accepted: 9 May 2020 ; Published: 1 June 2020



Abstract: The phenomenon of degeneracy of energy levels is often attributed either to an underlying (super)symmetry (SUSY), or to the presence of a Kato exceptional point (EP). In our paper a conceptual bridge between the two notions is proposed to be provided by the recent upgrade of the basic principles of quantum theory called, equivalently, \mathcal{PT} –symmetric or three-Hilbert-space (3HS) or quasi-Hermitian formulation in the current physical literature. Although the original purpose of the 3HS approach laid in the mere simplification of technicalities, it is shown here to serve also as a natural theoretical link between the apparently remote concepts of EPs and SUSY. An explicit illustration of their close mutual interplay is provided by the description of infinitely many supersymmetric, mutually non-equivalent and EP-separated regularized spiked harmonic oscillators.

Keywords: hiddenly Hermitian quantum Hamiltonians; Kato exceptional points; regularized singular potentials; hiddenly Hermitian supersymmetries; infinite family of regularized harmonic oscillators

1. Introduction

Ten years ago, people perceived the concepts of supersymmetry and of exceptional points as the two remote if not fully separate subjects of research. For example, during the international workshop “Supersymmetric Quantum Mechanics and Spectral Design” (18–30 July 2010, Benasque, Spain, organized by A. Andrianov et al [1], with proceedings [2]), just one talk was devoted to the role of exceptional points in supersymmetric quantum systems. In parallel, no speaker mentioned supersymmetry during the workshop “Exceptional points in physics” (2–5 November 2010, NITheP, Stellenbosch, South Africa, organized by W. D. Heiss [3]). Also, the same year, the participants of the 9th continuation of the series “Pseudo-Hermitian Hamiltonians in Quantum Physics” (21–24 June 2010, Zhejiang University, Hangzhou, China, organized by J.-D. Wu [4], with proceedings [5]) did not yet identify their field of interest as an emergent bridge between the two concepts, from either the mathematical and physical points of view.

At present, the situation is different. The use of the mathematical notion of exceptional points (EPs, [6]) makes an impression of being, in multiple branches of physics, ubiquitous [7,8]. The same growth of popularity became also characteristic for the concept of hidden Hermiticity *alias* pseudo-Hermiticity [9] or, better, quasi-Hermiticity [10,11], most widely known under the physics-emphasizing nicknames of “ \mathcal{PT} –symmetry” [12] or “gain-loss balance” [13]. In comparison, the concept of supersymmetry (SUSY, [14]) seems, undeservedly, overshadowed by these developments.

In our present paper we intend to demonstrate that the concept of SUSY witnesses a revival, in particular, due to its remarkable overlap with both of the EP- and \mathcal{PT} –related areas of research. The new conceptual bridges are emerging between these notions in a way which we intend to describe in what follows. On the SUSY side, in essence, the core of the mutual influence lies in the emergence of the new, unconventional realizations of the SUSY algebra in quantum mechanics. One witnesses

there a shift of attention from Hermitian to non-Hermitian Hamiltonians. On the EP side, in parallel, precisely such a shift makes the EP singularities accessible. Indeed, they are suddenly found as lying on boundaries of the relevant parametric domains admitting the standard probabilistic interpretation of the systems.

Our detailed scrutiny of this territory will proceed step by step. In Section 2 an introductory review of the formalism will be provided summarizing the basic features of the \mathcal{PT} —emphasizing quantum mechanics of closed systems. Subsequently, in Section 3 we will turn attention to a few most relevant features of the role of EPs in such a theory or theories. In Section 4 we will finally combine the preceding material in a climax outlining the bridges between SUSY and EPs. We will show, in essence, that the recent transfer of the applications of SUSY to quasi-Hermitian quantum systems extends the class of eligible models while still finding its natural limitations of applicability precisely at the same natural EP boundaries.

A few complementary aspects of our message will be also discussed in Section 5. We will emphasize there that there exists a close relationship between the EP-related properties of the SUSY and non-SUSY quantum systems. All these systems share the limitations of their unitarity and observability (i.e., of their probabilistic interpretation) strictly along the mathematical parameter-range boundaries formed by the EP singularities.

2. The Concept of Hiddenly Hermitian Hamiltonians

In the present paper our perception of quantum physics will be restricted to the theory in which a non-Hermitian but hiddenly Hermitian Hamiltonian H generates a *strictly unitary* evolution of quantum system in question (see the resolution of an apparent paradox as given in the early review of this approach to quantum mechanics in [11]).

2.1. Non-Hermitian Physics: Closed vs. Open Systems

In a concise introductory comment let us point out that the quantum systems under our present consideration are usually called “closed systems”. Carefully, one must separate the study of these models (with real energies) from the descriptions of other, manifestly non-unitary quantum systems (with complex energies) called “open”. Unfortunately, very similar language is often used in the literature on both the closed and open quantum dynamics. In a word of warning against possible misunderstandings let us recall, for illustration, the existence of potentially misleading overlaps in terminology. Often, people indiscriminately speak about “non-Hermitian quantum mechanics” having in mind *either* the resonances in unstable, open quantum systems [15,16] *or* the safely stable bound states in a closed, unitary setup [12].

Needless to add that the specific trademark logo of \mathcal{PT} —symmetry born in the context of quantum mechanics [17,18] and quantum field theory [19] became quickly popular also in the non-quantum world [13]. In optics, for example, the idea proved particularly successful [20]. Many research teams using Maxwell equations revealed that the propagation of light acquires remarkable properties when studied in a non-conservative medium characterized by \mathcal{PT} —symmetry redefined, ad hoc, as a balance between loss and gain. Thirdly, the same name “ \mathcal{PT} —symmetry” extended its implementations also to non-linear systems and equations, with the deepest relevance and applicability even in nanotechnologies [8].

Out of such a broad area of physics, our present attention will only be paid to the quantum closed systems.

2.2. Triplets of Hilbert Spaces: A Key to the Unitarity Paradox

The famous Stone’s theorem [21] implies that in a preselected Hilbert space (denoted, say, by a dedicated symbol \mathcal{K}), the Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad (1)$$

cannot describe a unitary (i.e., norm-preserving) evolution of a state $|\psi(t)\rangle$ whenever the Hamiltonian happens to be non-Hermitian, $H \neq H^\dagger$. Fortunately, in a way dating back to Dyson's studies of ferromagnetism [22] it became clear that in many realistic quantum models the non-unitarity of the evolution in \mathcal{K} may be declared, under certain conditions, irrelevant. Indeed, in many practical applications, it may make sense to treat H as a mere auxiliary isospectral avatar of a "true and correct" Hamiltonian

$$\mathfrak{h} = \Omega H \Omega^{-1} \quad (2)$$

which is "acceptable" (i.e., self-adjoint, $\mathfrak{h} = \mathfrak{h}^\dagger$) in *another*, true and correct, *physical* Hilbert space of states (to be denoted here by another dedicated symbol \mathcal{L}). In other words, the Dyson-attributed preconditioning (2) changes the picture while making the theory compatible with the Stone's theorem as well as with the conventional-textbook closed-system unitary-evolution requirements.

Dyson's numerous followers (cf., e.g., review [11]) were pragmatic practitioners who usually started their considerations from the principle of correspondence. Having specified a realistic (typically, a fermionic atomic-nucleus-model) Hamiltonian \mathfrak{h} acting in \mathcal{L} they needed to evaluate its low-lying spectrum. In such a (typically, variational) setting they revealed that the calculations may be perceptibly simplified, via preconditioning (2), after an educated *ad hoc* guess of a suitable invertible Hermiticity-breaking map $\Omega^{-1} : \mathcal{L} \rightarrow \mathcal{K}$. This map represented, typically, correlations between fermions living in \mathcal{L} in the language of certain effective bosons living in \mathcal{K} . In practice, such a strategy really accelerated the calculations yielding a perceptibly simplified picture of the underlying physical reality.

The Dyson-recommended preconditioning $\mathfrak{h} \rightarrow H$ is, almost without exceptions, motivated by the simplification of the solution of Schrödinger Equation (1). Naturally, the price to pay is not entirely negligible because the non-Hermiticity of H (which must still be assumed diagonalizable [9]) implies that its right and left eigenvectors are not just mutual Hermitian conjugates. For this reason (see also [23]) it makes sense to complement the bra-vector conjugates $\langle n |$ of the conventional right eigenkets $|n\rangle$ by the left-eigenbras of H (denoted as eigenbrabras $\langle\langle n |$) with, naturally, *different* conjugates $|n\rangle\langle n | \neq |n\rangle\langle n |$ (that is why we amended the notation).

For the same reason the full and explicit description of a pure time-dependent state of the systems requires not only the knowledge of the ket-state solutions $|\psi(t)\rangle$ of the evolution Equation (1) but also the complementary, non-Hermiticity-related knowledge of the brabra (or ketket) solutions of the following complementary evolution equation in \mathcal{K} ,

$$i \frac{d}{dt} |\psi(t)\rangle\langle\langle n | = H^\dagger |\psi(t)\rangle\langle\langle n |. \quad (3)$$

The two Schrödinger equations are interrelated by Equation (2) and by the fact that $\mathfrak{h} = \mathfrak{h}^\dagger$ in \mathcal{L} . Thus, in \mathcal{K} we can speak about quasi-Hermiticity of H [10] and write

$$H^\dagger \Theta = \Theta H, \quad \Theta = \Omega^\dagger \Omega. \quad (4)$$

In the stationary cases this observation makes the second Schrödinger equation redundant because we may set, simply, $|\psi(t)\rangle\langle\langle n | = \Theta |\psi(t)\rangle\langle n |$. Moreover, after we introduce such a biorthonormal basis $\{|n\rangle\langle n |, |n\rangle\langle\langle n |$ in \mathcal{K} which diagonalizes H , we have relations

$$H = \sum_n |n\rangle E_n \langle\langle n |, \quad I = \sum_n |n\rangle \langle n |. \quad (5)$$

We become able to define the third Hilbert space \mathcal{H} when changing the inner product (\bullet, \bullet) between any two elements ψ_a and ψ_b in \mathcal{K} . Thus, once we define

$$(\psi_a, \psi_b)_{\mathcal{K}} = \langle\psi_a | \psi_b \rangle, \quad (\psi_a, \psi_b)_{\mathcal{H}} = \langle\psi_a | \Theta | \psi_b \rangle, \quad \Theta = \sum_n |n\rangle \langle\langle n | \quad (6)$$

we may conclude that there are no reasons for calling H “non-Hermitian”. It is manifestly non-Hermitian in \mathcal{K} but this Hilbert space is unphysical [23]. A precise and detailed mathematical formulation of this conclusion was given in survey [24]:

Lemma 1. [24] *The observable predictions concerning a closed quantum system and using the space-Hamiltonian doublets (\mathcal{H}, H) or $(\mathcal{L}, \mathfrak{h})$ are equivalent.*

In the theories of closed quantum systems based on equivalences $(\mathcal{H}, H) \equiv (\mathcal{L}, \mathfrak{h})$ the terminology is still unsettled. The underlying innovative formulation of quantum mechanics is often called quasi-Hermitian [11] or, more or less equivalently, \mathcal{PT} –symmetric [12] alias three-Hilbert-space (3HS, [23]) alias crypto-Hermitian [25] alias pseudo-Hermitian [9]. Mathematicians, in contrast, seem to speak just about non-self-adjoint operators without adjectives [26]. In what follows, the acronym 3HS will be mostly employed.

2.3. Non-Uniqueness of Hilbert Space \mathcal{H}

In practice people usually make use of just one of three alternative consequences of the equivalence in Lemma 1. The first one is trivial: whenever the evaluation of predictions remains feasible *before* preconditioning (2) (i.e., in the initial representation $(\mathcal{L}, \mathfrak{h})$), there is no reason to leave the comfortable constructive single-space textbook recipes. The selection of one of the other two options, i.e., the description of reality in (\mathcal{H}, H) –representation is usually motivated either by the prohibitively complicated aspects of Hamiltonian \mathfrak{h} , or by a technical unfriendliness of Hilbert space \mathcal{L} .

The first version of the “non-Hermitian”, (\mathcal{H}, H) –based option was made popular, many years ago, by Dyson [22]. Its key technical ingredient may be seen in accessibility of an initial knowledge of mapping Ω . Still, even with this knowledge, the not too elementary transition from the traditional physical Hilbert space \mathcal{L} to its *ad hoc* auxiliary partner \mathcal{K} (in which the Hamiltonian becomes non-Hermitian) must really very strongly be motivated by the associated emergent technical simplifications [11].

In our present paper we intend to prefer the second “non-Hermitian” option which, in essence, means that one wants to work in a maximally user-friendly Hilbert space \mathcal{K} from the very beginning. The analysis then starts from a presumably physical diagonalizable Hamiltonian candidate H and from its diagonalization leading to Formula (5).

It is necessary to add that when we decided to set, in our preceding considerations, $|\psi(t)\rangle\rangle = \Theta |\psi(t)\rangle$, we simplified our analysis only at an expense of ignoring the fact that during the construction of the basis we did not take into considerations that either the biorthonormality and completeness relations are both invariant under a simultaneous rescaling of $|n\rangle \rightarrow |n\rangle / \varrho_n$ and $|n\rangle\rangle \rightarrow |n\rangle\rangle \times \varrho_n$. As we already emphasized in [27], this is a well known fact [11] which implies that the reconstruction of the physical Hilbert space \mathcal{H} (i.e., the formulation of the physical contents of the theory) is in fact ambiguous and, whenever one wishes to suppress this ambiguity, the recipe requires more information.

The “missing” information may find its origin in physics (one can introduce some other observables [28], cf. also reviews [11,12]). Some additional mathematical requirements may also help (in [29], for example, we proposed to minimize the anisotropy of the geometry in the physical Hilbert space). Still, in both contexts the essence of the problem is that the innocent-looking rescaling redefines the metric. Thus, just one of its special cases was used in Equation (6). The general Hamiltonian-compatible form of the physical metric is in fact non-unique and multiparametric,

$$\Theta = \Theta(\vec{\varrho}) = \sum_n |n\rangle\rangle \varrho_n^2 \langle\langle n| . \quad (7)$$

This implies that also the inner product (\bullet, \bullet) and space \mathcal{H} become variable, ambiguous and $\vec{\varrho}$ –dependent.

For practical purposes, there exist multiple methods of removal of such an unpleasant non-uniqueness of the theory. Their samples may be found, e.g., in [11,12] or in [29].

3. The Concept of Exceptional Points

One of the main distinguishing features of the parameter-dependent non-Hermitian Hamiltonians $H(\lambda)$ is that even in the absence of any symmetry (including, in a very prominent place, supersymmetry) one can still encounter a degeneracy of the energy levels in their spectra [30]. According to Kato [6], such a degeneracy proves to be of a fundamental importance in perturbation theory. For this reason he conjectured to call the underlying values of parameters exceptional points (EPs). In a way depending on the mathematical context these values $\lambda^{(EP)}$ were characterized either by their analytic-function background (i.e., by their connection with the branch point singularities, see the next paragraph), or by the role played in spectral theory. The latter area of mathematical applications also led to the Kato's best-known definition of the EPs as the points at which the algebraic multiplicity of an energy eigenvalue becomes different from its geometric multiplicity [6].

For the practical users of the idea it is sufficient to know that at the instant of the EP degeneracy of the energies one also observes a parallelization of some of the eigenvectors. Still, in the context of physics it took time before people imagined that the originally not too emphasized theoretical as well as experimental accessibility of these degeneracies could open a broad new area of many new EP-related phenomena [3,7,31].

3.1. The Birth of the Concept in Perturbation Theory

From the strictly historical point of view the current interest in exceptional points grew from several independent sources. All of them are, directly or indirectly, related to the abstract mathematical concept of an analytic function $\mathbb{F}(\lambda)$. One should remember that $\mathbb{F}(\lambda)$ is, in general, multivalued at its (complex) variable λ so that, strictly speaking, it is not a function but rather a collection of several single-valued functions which are represented, up to a suitable radius of convergence λ_{\max} , by the Taylor series. For a quick clarification of this concept it is sufficient to recall the square-root function $\mathbb{S}(\lambda)$ which is, in the complex plane of λ , two-valued. Its two branches $\mathbb{S}_{\pm}(\lambda) = \pm\sqrt{\lambda}$ are already single-valued functions which coincide at $\lambda = 0$. They may be defined, locally, by their respective Taylor series $T_{\pm}(\lambda)$. Thus, e.g., at a small complex ξ in $\lambda = \lambda(\xi) = 1/2 + \xi$ these two Taylor series are infinite,

$$\pm\sqrt{2}T_{\pm}(\lambda) = 1 + \xi - \frac{1}{2}\xi^2 + \frac{1}{2}\xi^3 - \frac{5}{8}\xi^4 + \dots,$$

and they both have the radius of convergence equal to $\xi_{\max} = 1/2$.

In Kato's book [6] the Taylor series represent the roots $z_n = z_n(\lambda)$ of secular equations $\mathbb{P}(z) = \det[H(\lambda) - zI] = 0$ where the operator $H(\lambda)$ (or, in simpler cases, an N by N matrix $H^{(N)}(\lambda)$) plays the role of quantum Hamiltonian. The explicit Taylor-series constructions of its bound-state-energy roots $z_n(\lambda) = E_n(\lambda) = E_n(0) + \lambda E'_n(0) + \dots$ are then deduced and shown to be of paramount importance in practice. What is essential is the knowledge of the radius of convergence λ_{\max} . A key to the answer is that the collection of roots $E_n(\lambda)$ often admits a unifying analytic-function reinterpretation $\mathbb{E}(\lambda)$. Its separate single-valued sheets represent energy levels $E_n(\lambda)$. The radius of convergence λ_{\max} acquires an elementary meaning of the distance of the origin from the nearest EP singularity.

3.2. EPs in Two by Two Effective Hamiltonians

In recent review paper [32] the authors emphasized that the study of certain complicated integrable models (IM) in $1 + 1$ dimensions may often profit from the so called ODE/IM correspondence. The acronym ODE abbreviates here the ordinary differential equations, and the benefits are not only bidirectional but also deeply relevant in the present context. This was demonstrated, earlier, in [33] where the same correspondence formed a background of the famous proof of reality of the bound-state

energy spectra of certain ordinary differential three-parametric quantum Hamiltonians H which were found non-Hermitian in $\mathcal{K} = L^2(\mathcal{S})$ (here the symbol \mathcal{S} stands for a suitable complex contour).

In a brief addendum [34] the authors of the proof paid more attention to the structure of the EP boundaries $\partial\mathcal{D}$ of the domain of parameters at which the spectrum remains real. They pointed out that in a small EP vicinity the passage of parameters through the boundary can be visualized via an effective two-by-two Hamiltonian

$$H^{(2)}(\delta) = \begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix}, \quad H^{(2)}(\delta) \begin{pmatrix} 1 \\ \eta \end{pmatrix} = \eta \begin{pmatrix} 1 \\ \eta \end{pmatrix}, \quad \eta = \eta_{\pm} = \pm\sqrt{\delta}. \quad (8)$$

This was finally the simplification which helped them to clarify some aspects of the reality proof in [33] including, in particular, the fact that in their model one can also detect a certain form of hidden supersymmetry.

All of these observations contributed to the motivation of our present study. First of all we imagined that a replacement of a complicated Hamiltonian by its “effective” two-by-two-matrix simulation offers a useful and universal mathematical tool of a qualitative description of the spectrum near its EP singularity. Secondly, the efficiency of the use of the effective N by N Hamiltonians seems to survive even a transition to the more complicated EP singularities, i.e., far beyond the class of their comparatively elementary ODE realizations [35]. Thirdly, even if one stays just with the most conventional ODE models, the effective-matrix simplifications may be expected to open a viable way towards a semi-explicit construction of the physical Hilbert space \mathcal{H} , i.e., of the last—and often omitted—step of the theory.

In the special case of ordinary differential Schrödinger equations, virtually all the other available methods of such a construction of \mathcal{H} and Θ seem to be prohibitively difficult [36]. In the approximation by the two by two matrix simulation (8), the latter last step remains elementary. Indeed, once we insert matrix $H^{(2)}(\delta)$ of Equation (8) together with a real and symmetric general matrix ansatz for the corresponding Hilbert-space metric $\Theta^{(2)}$ in the hidden Hermiticity condition (4) we obtain, with obvious proof, an exhaustive answer at any positive $\delta = \eta^2$ and positive η ,

$$\Theta = \Theta^{(2)}(\delta, b) = \begin{pmatrix} \eta & b \\ b & 1/\eta \end{pmatrix}. \quad (9)$$

Lemma 2. *The necessary positivity of metric (9) is guaranteed whenever its free real parameter b is such that $b^2 < 1$.*

3.3. EPs in Crypto-Hermitian Models with Local Potentials

In the physics-oriented literature, by far the most popular class of illustrative “non-Hermitian” Hamiltonians H with real spectra is being obtained via a judicious modification of some of the standard self-adjoint models of textbooks.

3.3.1. Regularized Harmonic Oscillator

In the above-cited papers [32–34] the authors studied a specific three-parametric family of non-Hermitian one-dimensional Schrödinger ODEs describing bound states in a local potential and with the standard eigenvalue-problem form $H(M, A, \ell)\psi = E\psi$ where

$$H(M, A, \ell) = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} - (ix)^{2M} - A(ix)^{M-1}$$

and where x moved along a suitable *ad hoc* complex contour. Using sophisticated mathematical methods these authors contributed to the clarification of the not quite expected fact that inside a suitable domain of parameter \mathcal{D} (with, as we mentioned above, EP boundary $\partial\mathcal{D}$) the bound-state spectrum $\{E_n\}$ proved real and discrete even though the potential itself was not real.

These observations were preceded by our letter [37] in which we showed that in the one-parametric special case of $H(1, 0, \ell)$ (which appeared to be exactly solvable), many results become obtainable via entirely elementary methods. Also, the exact diagonalizability of the special case with Hamiltonian $H(1, 0, \ell)$ or, equivalently, the exact solvability of the following Schrödinger ODE bound-state problem with $\alpha = \ell + 1/2 \geq 0$,

$$\left(-\frac{d^2}{dx^2} + x^2 - 2icx + \frac{\alpha^2 - 1/4}{(x - ic)^2} \right) \varphi(x) = (E + c^2) \varphi(x), \quad \varphi(x) \in \mathcal{K} = L_2(-\infty, \infty) \quad (10)$$

will prove useful, in what follows, for illustration of several nontrivial SUSY-EP correspondences.

In a preparatory step let us now briefly summarize some of the most relevant features of the bound-state solutions of Equation (10). First, in terms of the confluent hypergeometric special functions their general form is known,

$$\begin{aligned} \varphi(x) = C_+ (x - ic)^{-\alpha+1/2} e^{-(x-ic)^2/2} {}_1F_1 \left((2 - 2\alpha - E)/4, 1 - \alpha; (x - ic)^2 \right) + \\ + C_- (x - ic)^{\alpha+1/2} e^{-(x-ic)^2/2} {}_1F_1 \left((2 + 2\alpha - E)/4, 1 + \alpha; (x - ic)^2 \right). \end{aligned}$$

Secondly, these functions belong to $\mathcal{K} = L_2(-\infty, \infty)$ if and only if the infinite series terminates. This yields the compact formula for the wave functions

$$\varphi(x) = \text{const.} (x - ic)^{Q\alpha+1/2} e^{-(x-ic)^2/2} L_n^{(Q\alpha)} \left[(x - ic)^2 \right]. \quad (11)$$

They are numbered by the so called quasi-parity quantum number $Q = \pm 1$ and by another, common quantum number $n = 0, 1, \dots$. The power and exponential factors are accompanied here by the classical Laguerre polynomials,

$$\begin{aligned} L_0^\beta(z) &= 1, \\ L_1^\beta(z) &= \beta + 1 - z, \\ L_2^\beta(z) &= (\beta + 2 - z)^2 - (\beta + 2), \\ L_3^\beta(z) &= (\beta + 3 - z)^3 - 3(\beta + 3)(\beta + 3 - z) + 2(\beta + 3), \\ &\dots \end{aligned}$$

What is most important (cf. [38]) is that the operator Q with quasi-parity eigenvalues Q has later been reinterpreted as the operator of charge \mathcal{C} which defines, together with conventional parity \mathcal{P} , the most popular Hilbert-space special metric $\Theta = \mathcal{P}\mathcal{C}$ [9,12]. As a consequence, the values of energies are all real whenever $\alpha \geq 0$,

$$E = E_{Q,n} = 2 - 2Q\alpha + 4n, \quad Q = \pm 1, \quad n = 0, 1, 2, \dots \quad (12)$$

Nevertheless, a warning must follow: The validity of the strictly mathematical result (12) of Ref. [37] does not imply that all these real numbers can really be treated as bound-state energies.

The reasons are rather subtle, made explicit only long after the publication of [6]. Their essence lies in the emergence of EPs and in the ambiguity of metrics. The effect is most easily clarified via Figure 1. In this picture the leftmost, most obvious EP degeneracy is easily seen to occur at $\alpha = \alpha_0^{(EP)} = 0$. This is an instant of complexification, very often called spontaneous breakdown of \mathcal{PT} -symmetry [12]. This is a very special EP mathematical singularity. In [39] it has been given an alternative physical meaning and name of quantum phase transition of the first kind. Schematically, the underlying mechanism of the complexification is well explained by the two by two matrix model (8) of Ref. [34]. In the limit $\delta \rightarrow 0$ this matrix degenerates to the manifestly non-diagonalizable (i.e., unobservable and phenomenologically unacceptable) Jordan-block matrix $J^{(2)}(0)$.

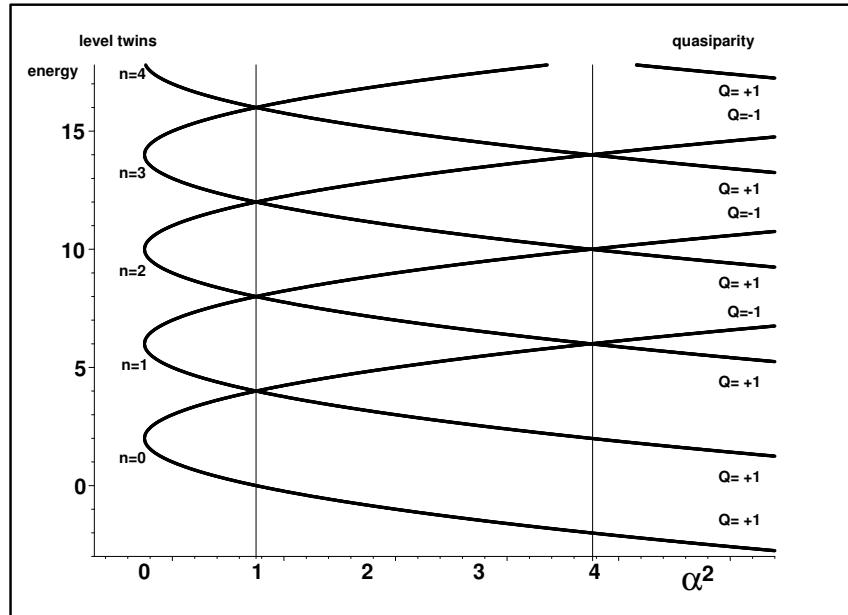


Figure 1. The α —dependence of the spectrum of the singular one-dimensional harmonic oscillator regularized by a complex shift of the axis of coordinates. The conventional limit with equidistant spectrum is obtained in the non-singular special case with $\alpha^2 = 1/4$. All integers α are exceptional points, $\{\alpha^{(EP)}\} = \mathbb{Z}$.

It is less easily seen that all the subsequent integers $\alpha = \alpha_k^{(EP)} = k = 1, 2, \dots$ are also exceptional points (cf. [37] for the proof). In [40] we explained their specific nature, and we proposed to call these unavoided level crossings without complexification the quantum phase transitions of the second kind. In the light of the Laguerre-polynomial solvability of the present model the quantum phase transition degeneracies of the second kind can even be given the form of exact identities

$$L_{n+1}^{(-1)} \left[(x - ic)^2 \right] = -(x - ic)^2 L_n^{(1)} \left[(x - ic)^2 \right]$$

etc.

3.3.2. Regularized Three-Body Calogero Model

In [41] we reinterpreted Schrödinger Equation (10) as a two-body special case of an integrable A —body Calogero model of the rational \mathcal{PT} —symmetric A_n series [42,43]. Although the discussion in [6] did not involve the question of EPs, we are now re-attracting attention to this family of models because their exact solvability again opens the way of our understanding the presence, localization and properties of the EPs in a less elementary setting.

For the sake of brevity let us only recall here the first nontrivial three-body version of the Hamiltonian

$$\tilde{H}^{(3)} = \sum_{i=1}^3 \left[-\frac{\partial^2}{\partial x_i^2} + \frac{3}{8} \omega^2 x_i^2 \right] + \frac{g}{(x_1 - x_2)^2} + \frac{g}{(x_2 - x_3)^2} + \frac{g}{(x_3 - x_1)^2}.$$

Interested readers may find all the technical details and formulae in [41]. Here we will only recall a few most relevant features of the model and add a few EP-related remarks.

First of all we will fix the units and set $\omega^2 = \frac{8}{3}$ yielding the compact formula

$$E_{n,k}^{(\pm)} = 4n + 6k \pm 6\alpha + 5, \quad \alpha = \frac{1}{2} \sqrt{1 + 2g} > 0, \quad n, k = 0, 1, \dots \quad (13)$$

which defines the spectrum. From our present point of view such a formula demonstrates, immediately, that the EP singularities may be expected to occur, in these integrable models, at any number of particles A . At the same time, an increase in the complexity of the spectrum at $A = 3$ (made explicit by the occurrence of an additional quantum number k) indicates that not all the unavoidable level crossings must necessarily be the EP singularities. Some of them just reflect the presence of a conventional (e.g., rotational [41]) symmetry in the corresponding partial differential Schrödinger equation. A conversion of Formula (13) into a picture (sampled by Figure 2 here) just reconfirms that in contrast to the model of the preceding subsection, some of the energy levels become degenerate due to the classical symmetries of interaction rather than due to the emergence of an EP singularity.

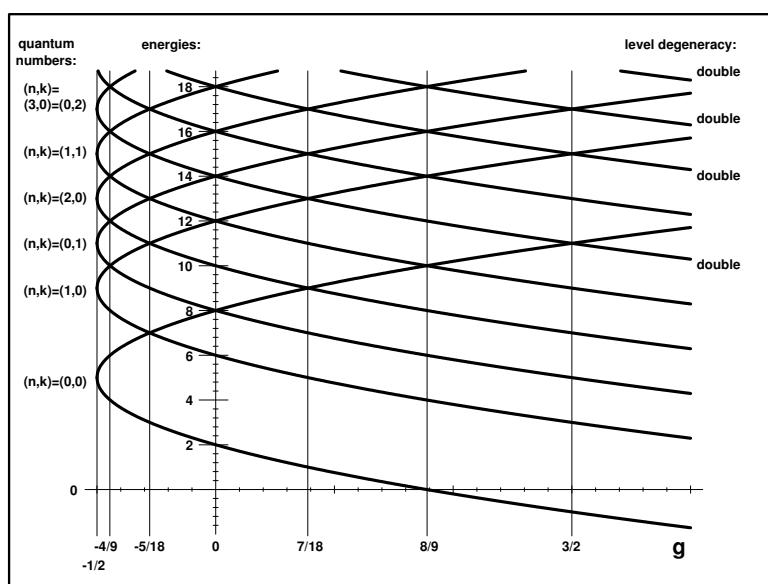


Figure 2. The degeneracies and coupling-dependence of the spectrum of the non-Hermitian three-particle Calogero model of Ref. [41]. Levels $E_{(n,k)} = 5 + 4n + 6k \pm 3\sqrt{1+2g}$ are numbered by a pair of integers $n, k = 0, 1, \dots$. All the visible exceptional points $g^{(EP)}$ are marked by vertical lines.

At the higher quantum numbers this phenomenon really does occur in a way mentioned, in Figure 2, in the right column of comments. Unfortunately, we do not have enough space here for a sufficiently exhaustive discussion of the differences or analogies between the qualitatively slightly different dynamical scenarios with $A = 2$ and $A = 3$ or $A \geq 3$. In fact, one of our main reasons is that, after one turns one's attention to the supersymmetric extensions of these models, the number of open questions starts growing rather quickly even during the study of the perceptibly simpler illustrative Calogerian model with $A = 2$.

4. The Concept of Hiddenly Hermitian Supersymmetry

Even though the SUSY-based arrangement of existing elementary particles into supermultiplets [44] did not work in particle physics [45], the idea itself proved extremely successful after its transfer to quantum mechanics. It helped to classify a family of exactly solvable models [46,47]. In the words of the latter reference, “in a supersymmetric theory commutators as well as anticommutators appear in the algebra of symmetry generators”. For our present purposes we shall use the explicit forms of these generators forming the specific Lie superalgebra $sl(1|1)$ (see Equation (18) below).

The same or similar ideas were also actively used during the birth of quasi-Hermitian theories [34,48–50]. Surprisingly enough, during or after all these promising developments, attention

remained restricted to the spectral problem, i.e., to the solution of the non-Hermitian Schrödinger equations in \mathcal{K} . Virtually no results were obtained in the direction of construction of the corresponding physical Hilbert space of states \mathcal{H} , i.e., of the amended operator $\Theta \neq I$ of the correct physical Hilbert-space metric.

The latter gap in the literature and, in particular, in the SUSY-related implementations of the 3HS theory is in fact not too surprising because the task is known to be truly difficult [9,36]. At the same time, the knowledge of the geometry of \mathcal{H} becomes crucial whenever the metric Θ deviates from the unit operator “too much” (see, e.g., the thorough analyses of this topic in [51,52]). For this reason, at least the use of qualitative considerations and approximate methods is needed when one decides to turn attention to the dynamical regime close to an EP extreme.

4.1. Harmonic-Oscillator Example

Let us now return to our quasi-Hermitian harmonic-oscillator model (10) and let us amend slightly the notation: In terms of the complex coordinate

$$r = r(x) = x - i c, \quad x \in (-\infty, \infty), \quad c > 0 \quad (14)$$

we shall consider the regularized quasi-Hermitian harmonic-oscillator Hamiltonians

$$H^{(\xi)} = -\frac{d^2}{dr^2} + \frac{\xi^2 - 1/4}{r^2} + r^2 \quad (15)$$

and their two shifted versions

$$H_{(L)} = H^{(\alpha)} - 2\gamma - 2, \quad H_{(R)} = H^{(\beta)} - 2\gamma \quad (16)$$

such that $\alpha = |\gamma|$ and $\beta = |\gamma + 1|$. In the light of the results of Section 3.3.1 we may conclude that all the integer values of γ are exceptional points, $\{\gamma^{(EP)}\} = \mathbb{Z}$. These values leave both of our sub-Hamiltonians non-diagonalizable so that they must be excluded from our consideration as manifestly unphysical.

Under this constraint, in terms of the so called superpotential

$$W = W^{(\gamma)}(r) = r - \frac{\gamma + 1/2}{r} \quad (17)$$

we may define operators $A = \partial_x + W$ and $B = -\partial_x + W$ and form the set of generators

$$\mathcal{G} = \begin{bmatrix} H_{(L)} & 0 \\ 0 & H_{(R)} \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad \tilde{\mathcal{Q}} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

of the well known Lie superalgebra $\text{sl}(1|1)$,

$$\{\mathcal{Q}, \tilde{\mathcal{Q}}\} = \mathcal{G}, \quad \{\mathcal{Q}, \mathcal{Q}\} = \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0, \quad [\mathcal{G}, \mathcal{Q}] = [\mathcal{G}, \tilde{\mathcal{Q}}] = 0. \quad (18)$$

Interested readers may find an exhaustive description of details in [53] (cf. also the present Equation (12) for energies).

In this setup our present intention is to pay attention to the occurrence and role of the EPs which were not properly treated in paper [53]. We will fill the gap in what follows. For such a purpose we will only need to recall, in explicit form, the existing results concerning the spectrum of the γ -dependent super-Hamiltonian $\mathcal{G} = \mathcal{G}(\gamma)$. Its parameter $\gamma \in \mathbb{R}$ taken from superpotential (17) is in fact the key quantity which determines the behavior of the (super)energies. In the light of this fact let us now split the real line of γ s into three subintervals.

4.1.1. Central Interval of $\gamma \in (-1, 0)$ with $\alpha = -\gamma$ and $\beta = \gamma + 1$.

As long as our super-Hamiltonian $\mathcal{G}(\gamma)$ is a direct sum of its “left” and “right” sub-Hamiltonians $H_{(L)}$ and $H_{(R)}$, we may denote the real N -th-level energy values (with $N = 0, 1, \dots$) by the respective pairs of symbols $[E_N^{(L)}, E_N^{(R)}]$ (at the positive quasi-parity $Q = +1$) and $[F_N^{(L)}, F_N^{(R)}]$ (at the negative quasi-parity $Q = -1$). In such an arrangement we may conclude:

Lemma 3. For $\gamma = \xi - 1/2 \in (-1, 0)$ the ground-state energy $E_0^{(L)} = 0$ is non-degenerate while the rest of the spectrum is doubly degenerate and such that

$$E_N^{(R)} = F_N^{(L)} = 4N + 2 - 4\xi < F_N^{(R)} = E_{N+1}^{(L)} = 4N, \quad N = 0, 1, \dots$$

Marginally, it is possible to add that the spectrum becomes equidistant at $\gamma + 1/2 = \xi = 0$.

4.1.2. The Right Half-Line of $\gamma = \alpha \in (0, \infty)$ with $\beta = \gamma + 1$.

As we indicated above, all the integer values of γ must be excluded from our considerations because at these values we have $\gamma = \gamma^{(EP)} \in \mathbb{Z}$ so that our quantum model loses any acceptable probabilistic interpretation [9].

Lemma 4. In every admissible open subinterval of $\gamma = \alpha = \beta - 1 \in (K, K + 1)$ with integer $K \geq 0$ the spectrum contains a single non-degenerate excited-state energy $F_0^{(L)} = 0$ which separates the energy spectrum into the negative ordered degenerate doublets

$$E_N^{(L)} = E_N^{(R)} = -4\gamma + 4N, \quad N = 0, 1, \dots, K$$

and the ordered pairs of the positive degenerate doublets

$$E_N^{(L)} = E_N^{(R)} = 4N - 4\gamma < F_{N-1}^{(R)} = F_N^{(L)} = 4N, \quad N = K + 1, K + 2, \dots$$

A note can be added that the ground state is now doubly degenerate. At the half-integer values of $\gamma = K + 1/2$ the positive doubly degenerate part of the spectrum becomes equidistant again.

4.1.3. The left half-line of $\gamma \in (-\infty, -1)$ with $\alpha = -\gamma$ and $\beta = -\gamma - 1$.

Lemma 5. In every admissible open subinterval of $\gamma = -\alpha = -\beta - 1 \in (-K - 1, -K)$ with integer $K \geq 1$ the spectrum contains a single non-degenerate ground-state energy $E_0^{(L)} = 0$ followed, at $K \geq 2$, by an anomalous $(K - 1)$ -plet of degenerate pairs of the lowest excited-state energies

$$E_{N+1}^{(L)} = E_N^{(R)} = 4N + 4, \quad N = 0, 1, \dots, K - 2.$$

The rest of the spectrum is then formed, at any $K \geq 1$, by the ordered pairs of degenerate doublets

$$E_{N+1}^{(L)} = E_N^{(R)} = 4N + 4 < F_N^{(L)} = F_N^{(R)} = 4N + 4 + 4\beta, \quad N = K - 1, K, \dots$$

At the half-integer values of $\gamma = -K - 1/2$ the equidistance now characterizes the non-anomalous part of the doubly degenerate spectrum, i.e., the spectrum without the lowermost $(2K - 1)$ -plet of levels (counting their SUSY-related degeneracy).

Corollary 1. In any preceding crypto-Hermitian harmonic-oscillator realization of SUSY algebra (18) the range of the admissible parameters γ always remains restricted to a finite interval of unit length, with the EP boundaries given by the two neighboring integers.

4.2. Regularized Hermitian Limit

In 1984 Jevicki and Rodrigues [54] pointed out that formalism of supersymmetric quantum mechanics breaks down for many phenomenologically interesting potentials with singularities (i.e., e.g., with the repulsive centrifugal-like barriers). In [53] we showed that what could help is a complex-shift regularization, and we asked what happens with the SUSY structures in the limit $c \rightarrow 0^+$. Our considerations were illustrated by the regularized harmonic oscillator of Equation (10). Thanks to the exact solvability of this model we were able to summarize our observations in Table Nr. 3 of Ref. [53] and in the appended spectrum-explaining comments.

From the present point of view the latter results deserve a comment. The point is that the centrifugal singularity remains regularized at any non-vanishing shift-parameter $c > 0$. In [6], we were interested also in the behavior of the system in the limit $c \rightarrow 0^+$. In this limit the regularization is removed so that the norm of at least some of the $c > 0$ solutions diverges. In [6], we reviewed the situation and we managed to show that some of the solutions remain acceptable by obeying the emergent standard boundary conditions in the origin. The resulting, drastically restricted family of normalizable $c = 0$ bound states is displayed in Figure 3.

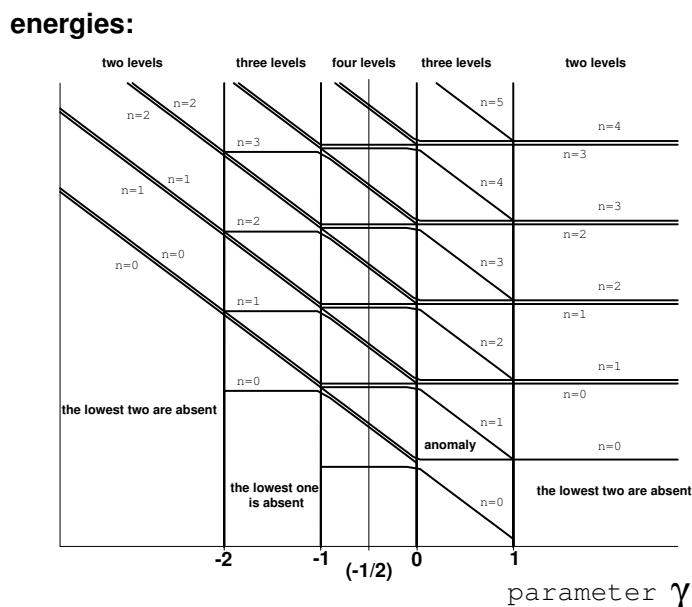


Figure 3. Degeneracies of the $c \rightarrow 0$ spectra of the quasi-Hermitian and supersymmetric harmonic oscillator (10). In the conventional regular $\gamma = -1/2$ special case the supersymmetric Hamiltonian becomes self-adjoint. Only four exceptional points $\gamma \in \{-2, -1, 0, 1\}$ remain phenomenologically relevant since the sets of bound states become incomplete out of the innermost admissible interval of $\gamma \in (-1, 0)$.

5. Discussion

5.1. SUSY and Local Potentials

Whenever the local potentials are being involved (which seems to be one of the characteristic aspects of many successful applications of various SUSY models), it will be absolutely necessary to return to the older [10] as well as newer [51,55] mathematical criticism of the very foundations of the 3HS formalism. Indeed, also in this more formal direction of research many important questions remain open at present [56].

In the field of mathematics, several interrelated challenges also survive concerning the constructions of the Hamiltonian-dependent Hilbert-space metrics $\Theta = \Theta(H)$ [29]. The abstract

freedom of our choice between alternative formulations of quantum mechanics (sampled by the pair (\mathcal{H}, H) and $(\mathcal{L}, \mathfrak{h})$ as mentioned in Lemma 1) is of a very restricted practical use. In applications, such a choice is usually more or less unique, determined by the criteria of feasibility of the constructions and calculations.

This is a pragmatic attitude which is being accepted in virtually all the applied hidden Hermiticity settings. A decisive factor leading to the choice of a specific representation (\mathcal{H}, H) is usually found in the user-friendly nature of the Schrödinger equations when solved in \mathcal{K} . In this sense the field of SUSY quantum mechanics seems particularly suitable and promising for future research, especially due to its deep algebraic as well as analytic background.

5.2. Analyticity and EPs

In the conventional quantum physics using self-adjoint Hamiltonians the EP singularities are often perceived as interesting and, perhaps, mathematically useful but not too phenomenologically important complex-plane points of coincidence of an $n \neq m$ pair of the analytically continued bound-state energies, $E_n(\lambda^{(EP)}) = E_m(\lambda^{(EP)})$ [35,57]. In a historical perspective this encouraged the study of operators of observables (i.e., most often, Hamiltonians) which were *analytic functions* of their parameters, $H(\lambda) = H(0) + \lambda H'(0) + \dots$. Such an assumption proved to be a natural source of overlaps between perturbation-theory mathematics and the PT-symmetry-related physics [58,59]. Typically, this helped to clarify the connections between the time-independent Schrödinger equations $H(\lambda)\psi_n(\lambda) = E_n(\lambda)\psi_n(\lambda)$ and their solutions based on the power-series Rayleigh-Schrödinger ansatzs

$$\psi_n(\lambda) = \psi_n(0) + \lambda \psi'_n(0) + \dots$$

and

$$E_n(\lambda) = E_n(0) + \lambda E_n^{(1)} + \dots$$

In a way discussed by Kato [6], at least a subset of all the (real or complex) energy eigenvalues $E_n(\lambda)$ may be then interpreted as evaluations of a *single* analytic, multisheeted function $\mathbb{E}(\lambda)$.

Kato also emphasized that a nice explicit illustration of the related mathematics may be then provided by certain *ad hoc* finite-dimensional N by N matrix toy-model Hamiltonians $H^{(N)}(\lambda)$. In the present context of the SUSY - EP correspondence this could inspire a further study of various quasi-Hermitian SUSY Hamiltonians and, in particular, their unphysical EP limits (plus their small perturbations – see, e.g., a few related remarks in [31]) in their canonical block-diagonal representations. For example, in a small vicinity of unperturbed $\alpha = \alpha_0^{(EP)} = 0$ in Equation (10) one could try to replace, approximately, the exact ODE Hamiltonian $H^{(\alpha)}$ of Equation (15) by its block-diagonal limit

$$H^{(\alpha)} = \left(\begin{array}{cc|cc|cc} 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 6 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 10 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{array} \right) + \mathcal{O}(\alpha)$$

where the blocks would be just the Jordan-block matrices $J^{(2)}(\eta)$ obtained from the matrix of Equation (8) in its non-diagonalizable extreme, $J^{(2)}(\eta) = \lim_{\delta \rightarrow 0} H^{(2)}(\delta) + \eta I^{(2)}$.

In the future applications involving some less elementary and, in particular, some supersymmetric models one may expect to obtain various less schematic and mathematically more interesting EP-related matrix structures. They could serve as approximations of realistic Hamiltonians along the lines studied, e.g., in [52].

5.3. Outlook

In our paper we skipped all outlines of physics behind the 3HS theory, and also of the multiple early contacts of this theory with SUSY (an extensive information on this topic may be found, e.g., in [60]). In any forthcoming phenomenology-oriented analysis more attention should certainly be paid, therefore, to the interlaced SUSY and EP aspects of the constructions of the physical Hilbert spaces of states \mathcal{H} as well as to the role played by some other, non-Hamiltonian operators of the hiddenly Hermitian observables.

During the future developments of the field of the SUSY-EP contacts one should certainly return, finally, also to the realistic-physics inspiration provided by the Dyson-inspired studies of non-SUSY interacting boson Hamiltonians H as well as by the recently revealed Dyson-unrelated hiddenly Hermitian but non-SUSY parallel developments involving, for example, the clusters-coupling Hamiltonians H [61].

Funding: This work is supported by the Faculty of Science of the University of Hradec Králové.

Acknowledgments: The author acknowledges the financial support from the Excellence project PřF UHK 2020 Nr. 2212.

Conflicts of Interest: The author declares no conflict of interest.

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