Physics Letters B 771 (2017) 593–596

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Particle model with generalized Poincaré symmetry

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ARTICLE INFO

Article history:
Received 12 December 2016
Received in revised form 30 May 2017
Accepted 6 June 2017
Available online 10 June 2017
Editor: M. Cvetič

ABSTRACT

Using the techniques of nonlinear coset realization with a generalized Poincaré group, we construct a relativistic particle model, invariant under the generalized symmetries, providing a dynamical realization of the $\mathfrak{B}_5$ algebra.

1. Introduction

One of the typical approaches in theoretical physics consists on replacing the symmetry group of the spacetime, described by the Poincaré group, by another symmetry group consistent with possible fields which might be present on a given physical situation. This approach provides a geometrical description of the particle interaction with the mentioned additional fields (see e.g. [1]). Using the semigroup expansion method (S-expansion) introduced in [2], on the AdS algebra as the starting seed algebra, it is possible to construct a family of generalized Poincaré algebras by suitable elections of the semigroup [3]. These algebras are denoted by $\mathfrak{B}_m$ and have the structure of a semidirect sum between an ideal (constituted by the translations) and the Lorentz transformations. It is interesting to note that symmetries described by the Maxwell algebra (see e.g. [4,5,8]) correspond to the so-called $\mathfrak{B}_4$ algebra. The case of a dynamical realization of the Maxwell algebra was studied in [1,6,7]. It is natural to ask about the physical nature of the particle models that are obtained by considering similar constructions for the $\mathfrak{B}_m$ algebras with $m \geq 5$. The purpose of this paper is to shed some light on such problem for simplest case $m = 5$, by the construction of the particle action on the coset space $\mathcal{B}_5/SO(3,1)$.

The organization of this article is as follows: In Section 2 we will review the construction of a Maxwell ($\mathcal{B}_4$) invariant particle model, where the interaction term can be interpreted in a physical way as a constant EM background acting on the Minkowski spacetime. It is explicitly shown the $\mathcal{B}_4/SO(3,1)$ infinitesimal symmetries. Section 3 is devoted to reviewing the construction of $\mathfrak{B}_4$ and $\mathcal{B}_5$ algebras by the S-expansion of the AdS algebra. In Section 4 we construct the $\mathfrak{B}_5$ invariant particle model which constitutes the main result of this work. We present some possible extensions and further comments in Section 5.

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2. The Maxwell ($\mathcal{B}_4$) invariant particle model

2.1. The $\mathcal{B}_4/SO(3,1)$ space–time particle Lagrangian

A $\mathfrak{B}_4$ invariant particle model, defined in the Maxwell coset spacetime $\mathcal{B}_4/SO(3,1)$, can be constructed using the nonlinear realizations techniques [9,10]. Let us consider an element g in the coset spacetime parametrized as in [1,6] by

$$g = e^{-\phi^{ab} Z_{ab} - x^a P_a} = e^{-\phi^{ab} Z_{ab}} e^{-x^a P_a} , \tag{1}$$

where Z_{ab} and P_a are the coset generators. The Maurer–Cartan 1-form is given by

$$\Omega = -g dg = e^a P_a + \frac{1}{2} k^{ab} Z_{ab} , \tag{2}$$

where the components are

$$e^a = dx^a , \tag{3}$$

$$k^{ab} = d\phi^{ab} + \frac{1}{2}\left(x^a dx^b - x^b dx^a\right) . \tag{4}$$

We can therefore construct the following first-order Lagrangian

$$L = \pi_a x^a - \frac{e}{2}\left(\pi^2 + m^2\right) + \frac{1}{2} f_{ab}\left[\dot{\phi}^{ab} + \frac{1}{2}\left(x^a \dot{x}^b - x^b \dot{x}^a\right)\right] , \tag{5}$$

where the phase-space coordinates (x^a, π_a) are extended to $(x^a, \pi_a, \phi^{ab}, p_{ab}, f_{ab}, p^f_{ab})$ and e is the einbein. In order to keep the usual meaning of the x^a coordinates as Cartesian coordinates on Minkowski space, one is forced to implement the constraint $\frac{1}{2}\left(\pi^2 + m^2\right) = 0$. We can solve for π^a from its equation of motion (a procedure known as the inverse Higgs mechanism [11]) and replace it on (5) to construct the following $\mathcal{B}_4$ invariant Lagrangian

$$L = \frac{\dot{x}_a \dot{x}^a}{2e} - \frac{m^2}{2} e + \frac{1}{2} f_{ab}\left[\dot{\phi}^{ab} + \frac{1}{2}\left(x^a \dot{x}^b - x^b \dot{x}^a\right)\right] , \tag{6}$$

http://dx.doi.org/10.1016/j.physletb.2017.06.013

where the einbein e implements the diffeomorphism invariance. Since π^a is no more a dynamical coordinate, and by the constraints imposed by the canonical momentum definition, we can set

$$p_{ab} = f_{ab}, \quad p^f_{ab} = 0$$

and shrink the phase-space into $(x^a,\ \phi^{ab},\ f_{ab})$. In the proper time gauge, the equations of motions are

$$\delta x^a : m\ddot{x}_a = f_{ab}\dot{x}^b\,, \tag{7}$$

$$\delta f_{ab}:\ \dot{\phi}^{ab} = -\frac{1}{2}\left(x^a\dot{x}^b - x^b\dot{x}^a\right)\,, \tag{8}$$

$$\delta\phi^{ab}:\ \dot{f}_{ab} = 0\,. \tag{9}$$

Equation (9) implies that f_{ab} is a constant antisymmetric field density $f_{ab} = f^0_{ab}$, which breaks the Maxwell symmetry into a subalgebra known as the BCR algebra [4]. Substituting this information in equation (7) leads to an interpretation of the f_{ab} coordinates as a constant electromagnetic field tensor which produces an interaction over the relativistic free particle described by the Lorentz force. Equation (8) implies that $\dot{\phi}^{ab}$ is proportional to the angular or magnetic moment of the particle.

2.2. *The phase-space realizations of the $\mathfrak{B}_4$ algebra*

As in [6], using the nonlinear realization techniques [9,10] and using the approach described in [12], it is possible to construct the infinitesimal $\mathcal{B}_4/SO(3,1)$ symmetries

$$P_a:\ \delta x^a = \epsilon^a,\ \ \delta\phi^{ab} = -\frac{1}{2}\left(\epsilon^a x^b - \epsilon^b x^a\right),$$

$$Z_{ab}:\ \delta\phi^{ab} = \epsilon^{ab},$$

$$J_{ab}: \delta x^a = \lambda^a{}_b x^b,\ \ \delta\phi^{ab} = \lambda^{[a}{}_c\phi^{cb]},\ \ \delta f_{ab} = \lambda_{[a}{}^c f_{cb]}, \tag{10}$$

and where the Noether currents which realize the Maxwell algebra are

$$\mathcal{P}_a = p_a - \frac{1}{2}p_{ab}x^b,$$

$$\mathcal{Z}_{ab} = p_{ab},$$

$$\mathcal{J}_{ab} = p_{[a}x_{b]} + f_{[ac}\phi_{b]}{}^c + p^f_{[ac}f_{b]}{}^c, \tag{11}$$

where the coordinates p_{ab} and p^f_{ab} are fixed by $(p_{ab}, p^f_{ab}) = (f_{ab}, 0)$.

3. From the $\mathfrak{B}_4$ algebra to the $\mathfrak{B}_5$ algebra

The $\mathfrak{B}_4$ and $\mathfrak{B}_5$ algebra can be constructed using the S-expansion on the *AdS* algebra with a suitable choose of the semigroup [3]. If the semigroup is chosen as $S^{(2)}_E = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ with a null element $\lambda_3 = 0_S$ and the product rule defined by $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ where $\alpha+\beta < 3$, and $\lambda_\alpha\lambda_\beta = 0_S$ where $\alpha+\beta \geq 3$, after imposing the 0_S-reduction condition one obtains the $\mathfrak{B}_4$ algebra which coincides with the Maxwell algebra. On the other hand if the semigroup is chosen as $S^{(3)}_E = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and with an analogous product rule one obtains the $\mathfrak{B}_5$ algebra

$$[P_a, P_b] = Z_{ab},$$

$$[Z_{ab}, P_c] = \eta_{bc}Z_a - \eta_{ac}Z_b, \tag{12}$$

$$[J_{ab}, Z_{ab}] = \eta_{cb}Z_{ad} - \eta_{ca}Z_{bd} + \eta_{bd}Z_{ca} - \eta_{da}Z_{cb},$$

$$[J_{ab}, Z_c] = \eta_{bc}Z_a - \eta_{ac}Z_b,$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b,$$

$$[J_{ab}, J_{cd}] = \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{bd}J_{ca} - \eta_{da}J_{cb},$$

where the Z_{ab}, Z_a, P_a generators constitute a (translations) subgroup.

4. The $\mathcal{B}_5$ invariant particle model

4.1. *The $\mathcal{B}_5/SO(3,1)$ space–time particle Lagrangian*

In analogy with previous section, using the nonlinear realization techniques, one can construct a particle model defined on the space $\mathcal{B}_5/SO(3,1)$. Let us consider an element g of coset space $\mathcal{B}_5/SO(3,1)$ by

$$g = e^{-x^aP_a - \frac{1}{2}\phi^{ab}Z_{ab} - \theta^a Z_a} = e^{-x^aP_a}e^{-\frac{1}{2}\phi^{ab}Z_{ab}}e^{-\theta^a Z_a},$$

and computing the MC 1-form defined over the coset space–time

$$\Omega = -gdg = e^aP_a + \frac{1}{2}k^{ab}Z_{ab} + k^aZ_a,$$

where the components are

$$e^a = dx^a,$$

$$k^{ab} = d\phi^{ab} + \frac{1}{2}\left(x^a dx^b - x^b dx^a\right),$$

$$k^a = d\theta^a + \frac{1}{2}\left(\phi^{ab}dx_b - x_b d\phi^{ab}\right) + \frac{1}{6}x_c\left(x^c dx^a - x^a dx^c\right).$$

We can construct a first order Lagrangian. Interpreting the x^a coordinate in the same footing as a Minkowski space–time coordinate by the imposition of the $\frac{1}{2}\left(\pi^2 + m^2\right) = 0$ constraint, we obtain

$$L = \pi_a\dot{x}^a - \frac{e}{2}\left(\pi^2 + m^2\right) + \frac{1}{2}f_{ab}\left[\dot{\phi}^{ab} + \frac{1}{2}\left(x^a\dot{x}^b - x^b\dot{x}^a\right)\right]$$
$$- f_a\left[\dot{\theta}^a + \frac{1}{2}\left(\phi^{ab}\dot{x}_b - x_b\dot{\phi}^{ab}\right) + \frac{1}{6}x_c\left(x^c\dot{x}^a - x^a\dot{x}^c\right)\right]. \tag{13}$$

Using the inverse Higgs mechanism on the π^a coordinate

$$L = \frac{\dot{x}_a\dot{x}^a}{2e} - \frac{m^2}{2}e + \frac{1}{2}f_{ab}\left[\dot{\phi}^{ab} + \frac{1}{2}\left(x^a\dot{x}^b - x^b\dot{x}^a\right)\right]$$
$$- f_a\left[\dot{\theta}^a + \frac{1}{2}\left(\phi^{ab}\dot{x}_b - x_b\dot{\phi}^{ab}\right) + \frac{1}{6}x_c(x^c\dot{x}^a - \dot{x}^c x^a)\right]. \tag{14}$$

In the proper gauge, the equations of motion are

$$\delta x^a : m\ddot{x}_a = f_{ab}\dot{x}^b, \tag{15}$$

$$\delta f_{ab}:\ \dot{\phi}^{ab} = -\frac{1}{2}\left(x^a\dot{x}^b - x^b\dot{x}^a\right), \tag{16}$$

$$\delta f_a:\ \dot{\theta}^a = -\frac{1}{2}\phi^a{}_b\dot{x}^b + \frac{1}{6}\dot{\phi}^a{}_b x^b, \tag{17}$$

$$\delta\phi^{ab}:\ \dot{f}_{ab} = -2f_a\dot{x}_b, \tag{18}$$

$$\delta\theta^a:\ \dot{f}_a = 0. \tag{19}$$

The equation of motion (19) implies that f_a is a constant field density $f_a = f^0_a$ which plays an important role in the f_{ab} tensor field definition. Replacing into the Lagrangian (14), the $\mathfrak{B}_5$ symmetries breaks into a subgroup. As in the previous case, the equation of motion provides a description of a massive particle moving in a EM field, where the field is not constant.

4.2. The phase-space realizations of the $\mathfrak{B}_5$ algebra

Using the nonlinear realizations, the infinitesimal transformation can be constructed.

$$P_a: \delta x^a = \epsilon^a; \quad \delta\phi^{ab} = -\frac{1}{2}\left(\epsilon^a x^b - \epsilon^b x^a\right),$$
$$\delta\theta^a = \frac{1}{2}\phi^{ac}\epsilon_c + \frac{1}{2}\phi^a{}_c\phi^{cd}\epsilon_d + \frac{1}{12}x_c\left(\epsilon^a x^c - \epsilon^c x^a\right),$$
$$Z_{ab}: \delta\phi^{ab} = \epsilon^{ab}, \quad \delta\theta^a = -\frac{1}{2}\epsilon^{ac}x_c - \frac{1}{2}\phi^a{}_c\epsilon^{cd}x_d,$$
$$Z_a: \delta\theta^a = \rho^a,$$
$$J_{ab}: \delta x^a = \lambda^a{}_b x^b, \quad \delta\phi^{ab} = \lambda^{[a}{}_c\phi^{cb]}, \quad \delta\theta^a = \lambda^a{}_b\theta^b,$$
$$\delta f_{ab} = \lambda_{[a}{}^c f_{cb]}; \quad \delta f_a = \lambda_a{}^b f_b,$$

and the corresponding Noether currents are:

$$\mathcal{P}_a = p_a - \frac{1}{2}p_{ab}x^b + \frac{1}{2}\phi^{ab}f_b - \frac{1}{2}\phi_{ac}\phi^{cb}f_b - \frac{1}{12}x^b\left(f_a x_b - f_b x_a\right)$$
$$\mathcal{Z}_{ab} = p_{ab} + \frac{1}{2}\left(f_a x_b - f_b x_a\right) - \frac{1}{2}f_c\left(\phi^c{}_a x_b - \phi^c{}_b x_a\right)$$
$$\mathcal{Z}_a = -f_a$$
$$\mathcal{J}_{ab} = p_{[a}x_{b]} + f_{[ac}\phi_{b]}{}^c - f_{[a}\theta_{b]}$$

The algebra is dynamically realized, where the phase space coordinates $(p_{ab}, k_a, p^f_{ab}, k^f_a)$ are fixed by $(p_{ab}, k_a, p^f_{ab}, k^f_a) = (p_{ab}, -f_a, 0, 0)$. Note that the k_a coordinate is the canonical conjugate momenta for the θ^a variable. The constraints defined by this fixing, shrink the phase space from $(x^a, \pi_a, \phi^{ab}, p_{ab}, \theta^a, k_a, f_{ab}, p^f_{ab}, f_a, k^a_f)$ to $(x^a, \phi^{ab}, \theta^a, f_{ab}, f_a)$ where the last sets of coordinates are the dynamical ones.

5. Conclusions

In this paper we have constructed a relativistic particle model, invariant under the generalized Poincaré group ($\mathfrak{B}_5$) symmetries providing a dynamical realization of the $\mathfrak{B}_5$ algebra. This construction introduces a new dynamical field density called f_a which plays an important role in the generalization of the EM field Lorentz force.

The $\mathfrak{B}_5$ algebra realization can be achieved if the variables $(x^a, \phi^{ab}, f_{ab}, f_a)$ can be conceived as dynamical ones. The equations of motion for each variable (15)–(19) describe the state of a particle in the $\mathfrak{B}_5/SO(3,1)$ space where the equation for f_a breaks spontaneously the Lagrangian symmetry.

It is interesting to note that (i) there exists a physical and mathematical connection between the equations of motion (17) and (18) and the equations of motions (3.22) and (3.23) of the Ref. [13]. In fact, if we consider the transformation of coordinates

$$\theta^a = -\frac{1}{3}\xi_c{}^{ac} + \frac{1}{2}x_c\phi^{ac},$$

we have that the equation (17) is given by

$$\dot{\xi}_c{}^{ac} - 3\dot{x}_c\phi^{ac} - x_c\dot{\phi}^{ac} = 0, \tag{20}$$

and using the equation (16), we find that the equation (20) takes the form

$$\dot{\xi}_c{}^{bc} - 3\dot{x}_c\phi^{bc} + \frac{1}{2}x_c\left(x^b\dot{x}^c - x^b\dot{x}^a\right) = 0, \tag{21}$$

which corresponds to equation (3.23) of Ref. [13], with the contracted indices a and c.

Furthermore, if we consider the transformation

$$f_{abc} = \eta_{ac}f_b, \tag{22}$$

this means that

$$f_{bca} = \eta_{ba}f_c = \eta_{ab}f_c = f_{acb} = -f_{abc},$$

so that

$$f_{bca} + f_{abc} = 0. \tag{23}$$

From (23) and (22) we have

$$\dot{f}_{ab} = -2f_{cab}\dot{x}^c + \left(f_{bca} + f_{abc}\right)\dot{x}^c$$
$$= \left(-2f_{cab} + f_{abc} + f_{bca}\right)\dot{x}^c,$$

which corresponds to equation (3.22) of Ref. [13]. From (22) we can see that

$$f_{[abc]} = \frac{1}{3!}\left(f_{abc} - f_{acb} + f_{bca} - f_{bac} + f_{cab} - f_{cba}\right) = 0, \tag{24}$$

where this expression is related with the $Y_{[abc]} = 0$ condition of [13].

In addition, (ii) there is an explicit relation among the generalized Poincaré $\mathfrak{B}_5$ algebra and the second level extension of the Poincaré group studied in Ref. [13] and constructed in Ref. [14]. Let us consider now a relation between the generalized Poincaré $\mathfrak{B}_5$ algebra and the second level extension of the Poincaré group studied in Ref. [13]. From the second level extension of the Poincaré group [13] we can obtain the generalized Poincaré $\mathfrak{B}_5$ algebra. After we use the $Y_{abc} := \eta_{ac}Z_b$ basis transformation, where Y_{abc} is antisymmetric in b and c, and, symmetric in a and c, we have

$$[P_a, Z_{bc}] = i\left(2Y_{abc} - Y_{bca} - Y_{cab}\right),$$
$$= 2iY_{abc} - i\left(Y_{abc} + Y_{cab}\right). \tag{25}$$

From the Y_{abc} definition we can see,

$$Y_{bca} = -Y_{bac} = -Y_{cab} \Rightarrow Y_{bca} + Y_{cab} = 0. \tag{26}$$

Substituting (26) into (25)

$$[P_a, Z_{bc}] = 2iY_{abc}$$
$$= -i\left(\eta_{ab}Z_c - \eta_{ac}Z_b\right), \tag{27}$$

we obtain one of the commutation relations that defines the generalized Poincaré $\mathfrak{B}_5$. From the Eq. (3.15) of the Ref. [13] it is possible to obtain

$$[Y_{pab}, M_{cd}] = -i\big(\eta_{bc}Y_{pad} - \eta_{bd}Y_{pac} + \eta_{ad}Y_{pbc} - \eta_{ac}Y_{pbd} + \eta_{pc}Y_{dab} - \eta_{pd}Y_{cab}\big).$$

Multiplying it by η^{pd}

$$[\eta^{pb}Y_{pab}, M_{cd}] = -i\big(\eta_{bc}\eta^{pb}Y_{pad} - \eta_{bd}\eta^{pb}Y_{pac} + \eta_{ad}\eta^{pb}Y_{pbc} - \eta_{ac}\eta^{pb}Y_{pbd} + \eta_{pc}\eta^{pb}Y_{dab} - \eta_{pd}\eta^{pb}Y_{cab}\big)$$
$$= -i\left(\eta_{ad}\eta^{pb}Y_{pbc} - \eta_{ac}\eta^{pb}Y_{pbd}\right)$$

and using the same basis transformation, $Y_{pab} = \eta_{pb}Z_a$

$$\eta^{pb}Y_{pab} = \eta^{pb}\left(\eta_{pb}Z_a\right) = 4Z_a$$

we obtain

$$4[Z_a, M_{cd}] = -4i\left(\eta_{ac}Z_d - \eta_{ad}Z_c\right),$$

which can be written in the form

$$[Z_a, M_{bc}] = -i\left(\eta_{ab}Z_c - \eta_{ac}Z_b\right). \tag{28}$$

This result coincides with the Eq. (12) in the appropriate representation. Since Y_{abc} commutes with itself and with Z_{ab}, we find

$$[Z_a, Z_b] = [Z_a, Z_{bc}] = 0. \tag{29}$$

This means that from equations (27), (28) and (29), it is possible to obtain the generalized Poincaré $\mathfrak{B}_5$ algebra.

A natural extension of this work is the construction of a particle model for every $\mathfrak{B}_m$ and its supersymmetric extensions. Since the f_a is a constant background field density acting on the particle, our model supports, in addition with [1], the framework where the cosmological constant can be interpreted as a dynamical variable. This point of view in addition with the $\mathfrak{B}_m$ algebras may give an insight bridge between the problems related with the cosmological constant and the S-expansion. Another interesting possibility of this work is the construction of general identifications that unify the extended Poincaré algebras, constructed in Ref. [13] and the $\mathfrak{B}_m$ algebras.

Acknowledgements

This work was supported in part by FONDECYT grant No. 1130653 and in part by grant No. 22140440 from CONICYT (National Commission for Scientific and Technological Research, Spanish initials) and from Universidad de Concepción, Chile. The author is grateful to P. Salgado for introducing the topics covered in the present work.

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