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Quantum Entanglement to Extract a True Ground State from an Approximate Ground State

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We propose a procedure to extract a true ground state from an approximate ground state by using quantum entanglement. We use plural ancilla qubits with hierarchical structure, intending to gradually improve the precision of the approximation. We demonstrate the procedure for quantum systems with a few qubits that derive from the (1+1)-dimensional Schwinger model by classically emulated digital quantum simulation. Although we use for simplicity an approximate ground state prepared by adiabatic quantum computation, our procedure is applicable to any approximate ground state that is a superposition of a true ground state and excited states. Our procedure is applicable for an N-qubits Hamiltonian with a nondegenerate ground state.

Subject Index A61, A64

1. Introduction

Thanks to the recent development of quantum computers by superconductivity, some quantum systems have been analyzed by the universal type of quantum computers [1,2]. The (1+1)-dimensional Schwinger model [3] has been analyzed by analogue quantum simulation [4,5]. Recently, the (1+1)-dimensional Schwinger model has been studied with the IBM Eagle processor [6], which has more than 100 quantum bits, by the variational quantum eigensolver [7]. The classically emulated quantum simulator also has been used to analyze quantum systems. The classically emulated quantum simulator is free from noises and is suited for precise analysis of not-so-large quantum systems by classical computers. Recently, the (1+1)-dimensional Schwinger model has also been analyzed by the classically emulated quantum simulator [8–10]. By virtue of the analysis of the classically emulated quantum simulator, an unknown region of parameters for the (1+1)-dimensional massive Schwinger model with a topological θ-term has been explored. If theoretically established results can be reproduced, the analysis by the quantum simulator will be reliable. Some exact results are known for the (1+1)-dimensional Schwinger model in the massless case [11] with $\theta = 0$. For the (1+1)-dimensional Schwinger model, in the massless case the vacuum expectation value of $\bar{\psi}\psi$ has been computed as $\langle \mathrm{vac}|\bar{\psi}\psi|\mathrm{vac}\rangle = -\frac{e^{\gamma}}{2\pi}\frac{g}{\sqrt{\pi}} \approx -0.160g$ [12], where $\gamma \approx 0.5772$ is the Euler constant and g is the coupling constant between the fermionic fields and the electric field. Chakraborty et al. [8] have computed the value $\langle \mathrm{vac}|\bar{\psi}\psi|\mathrm{vac}\rangle$ by an extrapolation with respect to the number of qubits. Their result is consistent with the exact result within a kind of systematic error.

It also has been recognized that in the quantum simulation an approximate vacuum prepared by adiabatic quantum computation [13,14] slightly differs from a true vacuum [8,15]. It seems

that for the quantum simulation the conditions of the quantum adiabatic theorem [16,17] are not completely satisfied. Vacuum expectation values of some physical quantities computed with this approximate vacuum oscillate periodically in time, which will mean that the approximate vacuum is a superposition of a true vacuum and some excited states. Accordingly, vacuum expectation values of some physical quantities also slightly differ from the exact values and their variances are not so small. Therefore, it will be an important task to establish a procedure to separate a true vacuum from excited states in quantum simulation of this kind. Recently, the author of the present paper has proposed a procedure to improve the approximate vacuum [18]. The influence of excited states has been diminished and the variances of the vacuum expectation values have been suppressed. In the previous paper, we used a priori an exact value of ground state energy to improve the approximate vacuum. In this regard, the previous procedure is not self-contained.

In this paper, we propose a procedure to extract a true ground state from an approximate ground state, which is a superposition of a true ground state and excited states. Although we only simulate up to a three-qubits system, our procedure, in principle, is applicable for an N-qubits Hamiltonian with a nondegenerate ground state. We only use ancilla qubits and an approximate value of a ground state energy. We introduce concatenated ancilla qubits to exclude excited states iteratively from the approximate ground state. We do not intend to diagonalize a Hamiltonian by unitary transformations on the physical system. But, we separate a true ground state of a system from excited states by unitary transformations controlled by the ancilla qubits. By proper unitary transformations controlled by the ancilla qubits, we intend to transform the state $(\alpha|E_0\rangle + \Sigma_i\beta_i|E_i\rangle)|0\rangle|0\rangle\cdots|0\rangle$ into a quantum entangled state $\alpha|E_0\rangle|0\rangle|0\rangle\cdots|0\rangle + \Sigma_i\beta_i|E_i\rangle||1\rangle\rangle_i$, where $|E_0\rangle$ is a true ground state, $|E_i\rangle$ is the i-th excited state, $|0\rangle|0\rangle\cdots|0\rangle$ is the initial state of the ancilla qubits, and $||1\rangle\rangle_i$ is a state where at least one of the ancilla qubits is in the state $|1\rangle$. This time, we do not a priori use the exact value of the ground state energy. We use an approximate value of a ground state energy to improve the approximate ground state and update the approximate value of the ground state energy. We can iterate this improving procedure as many times as there are ancilla qubits in the system. We estimate ground state expectation values of physical quantities from the iteratively improved approximate ground state. As an initial approximate ground state we adopt the approximate ground state prepared by adiabatic quantum computation and we use its energy as the initial approximate value of the true ground state. For simplicity, we examine a one-qubit system, a two-qubits system, and a three-qubits system that derive from the (1+1)-dimensional Schwinger model. Our procedure can in principle be applicable irrespective of the number of qubits and is not restricted to the approximate ground state prepared by adiabatic quantum computation.

2. One-qubit system

2.1. *Hamiltonian and an approximate ground state prepared by adiabatic quantum computation*

We consider the following simple one-qubit Hamiltonian:

$$\hat{H}_T = X + JZ, \tag{1}$$

where X and Z stand for the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and J is a real parameter. We mainly concentrate on the typical case $J = 1$, which will be the easiest case

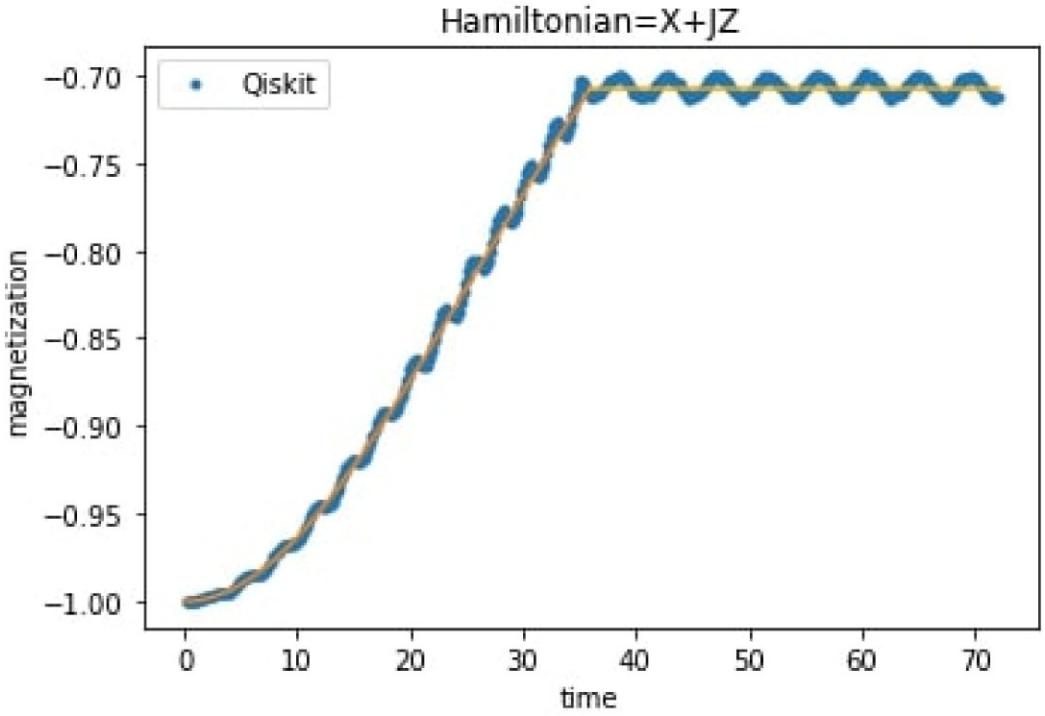

Fig. 1. Ground state expectation value of Z for $J = 1$ by IBM Qiskit qasm-simulator. The parameter $0 \leq s \leq 1$ is mapped to the time period $0 \leq t \leq 36$ and this time period is divided into 144 small time intervals for simulation. The number of trials is 10^6. The time period $36 \leq t \leq 72$ exhibits the time evolution by the constant Hamiltonian $\hat{H}_T$. We have used the second-order Suzuki–Trotter formula [19,20]. We have started from the initial state $|1\rangle$.

to theoretically analyze. To prepare an approximate ground state of $\hat{H}_T$ by adiabatic quantum computation we introduce the initial Hamiltonian $\hat{H}_0 = Z$. We have the following Hamiltonian $\hat{H}_A(s)$ to carry out the quantum adiabatic computation:

$$\hat{H}_A(s) = (1-s)\hat{H}_0 + s\hat{H}_T = (1-s)Z + s(X + JZ), \tag{2}$$

where s is a parameter that varies gradually from $s = 0$ to $s = 1$. We set $s = \frac{t}{T}$ for an adequate time period T. We start from the initial state $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. the ground state of $\hat{H}_0$. For $\hat{H}_0$, the state $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the excited state. We compute a ground state expectation value of Z at each s. We show a simulation result in Fig. 1. After the time $t = T$, the system time develops under the constant Hamiltonian H_T. If the true ground state is prepared the ground state expectation value of Z should be a constant. In the result in Fig. 1 the ground state expectation value of Z oscillates regularly. This means that the prepared state by the present adiabatic quantum computation is not a true ground state. In computer simulation we use a discrete time and the Suzuki–Trotter formula [19,20], which may be the cause of deviation from a true ground state. Recently, errors that derive from the Suzuki–Trotter formula in quantum simulation have been studied [21,22]. In the following we examine the approximate ground state prepared by adiabatic quantum computation for the case $J = 1$.

For $J = 1$, the ground state of H_T is

$$|E_0\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}(|0\rangle - (\sqrt{2}+1)|1\rangle), \tag{3}$$

and the first excited state of H_T is

$$|E_1\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}(|0\rangle + (\sqrt{2}-1)|1\rangle). \tag{4}$$

We see that $\langle E_0|H_T|E_0\rangle = -\langle E_1|H_T|E_1\rangle = -\sqrt{2}$, $\langle E_0|Z|E_0\rangle = -\langle E_1|Z|E_1\rangle = -\frac{\sqrt{2}}{2}$, and $\langle E_0|Z|E_1\rangle = \frac{\sqrt{2}}{2}$.

After the adiabatic state preparation process, the observable Z time develops as

$$e^{i\hat{H}_T t}Ze^{-i\hat{H}_T t} = e^{-iJHt}Ze^{iJHt} = \frac{1}{2}(X+Z) + \frac{1}{\sqrt{2}}Y\sin 2\sqrt{2}t + \frac{1}{2}(Z-X)\cos 2\sqrt{2}t. \tag{5}$$

At the time $t = T$ if we have a state $|\psi(t=T)\rangle = |\psi(0)\rangle = |\psi_0\rangle = \alpha|E_0\rangle + \beta|E_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, instead of the desired state $|E_0\rangle$, we have at a time $t(\geq T)$

$$\langle\psi(t)|Z|\psi(t)\rangle = -\frac{1}{\sqrt{2}}(1-2|\beta|^2) + \sqrt{2}|\alpha\beta|\cos(2\sqrt{2}(t-T)+\theta), \tag{6}$$

where we have again set $\alpha\beta^* = |\alpha\beta^*|e^{i\theta}$. A procedure to obtain an approximate ground state expectation value of Z is to take the time average of $\langle\psi(t)|Z|\psi(t)\rangle$ over the time region $t \geq T$. In this procedure, however, the approximate value is always accompanied by the $O(|\beta|^2)$ systematic error, although it may be small compared with $O(|\beta|)$. We propose another attempt to diminish not only the $O(|\beta|)$ error but also the $O(|\beta|^2)$ error.

To obtain ground state expectation values of physical quantities with high accuracy from the approximate vacuum $|\psi_0\rangle$, we introduce multiple ancilla qubits in the state $|0\rangle|0\rangle\cdots|0\rangle$. Our purpose is to transform the state $(\alpha|E_0\rangle + \beta|E_1\rangle)|0\rangle$ into the quantum entangled state $\alpha|E_0\rangle|0\rangle + \beta|E_1\rangle|1\rangle$. We compute an approximate ground state energy E_{00} from the approximate ground state $|\psi_0\rangle$ by $E_{00} = \langle\psi_0|\hat{H}_T|\psi_0\rangle$. We compute a time parameter θ_0 that satisfies $\theta_0 E_{00} = \frac{\pi}{2}$. We transform the first ancilla qubit by the Hadamard operator $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ to the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Taking the first ancilla qubit as the control qubit, we operate $U(\theta_0) = ie^{-i\theta_0\hat{H}_T}$ on the physical state $|\psi_0\rangle$. Then, we again Hadamard transform the first ancilla qubit. We call this series of operations a twirling operation. By this twirling operation, we can partially exclude the excited state. The initial state $(\alpha|E_0\rangle + \beta|E_1\rangle)|0\rangle$ is expected to approach the quantum entangled state $\alpha|E_0\rangle|0\rangle + \beta|E_1\rangle|1\rangle$. In practice, after the first twirling operation the physical state and the first ancilla qubit will be in a state $c_1|\psi_1\rangle|0\rangle + d_1|\varphi_1\rangle|1\rangle$, where $|\psi_1\rangle$ is a substitute of the true ground state $|E_0\rangle$ and $|\varphi_1\rangle$ is an obstructive state like the excited state $|E_1\rangle$. The first approximate values of physical quantities are computed as the expectation values by $|\psi_1\rangle$. We compute the second time parameter θ_1 by $\theta_1 E_{01} = \frac{\pi}{2}$, where $E_{01} = \langle\psi_1|\hat{H}_T|\psi_1\rangle$. Using θ_1 and the second ancilla qubit, we carry out the second twirling operation and obtain a state $c_2|\psi_2\rangle|0\rangle|0\rangle + d_2|\varphi_2\rangle||1\rangle\rangle$, where $||1\rangle\rangle$ means that at least one of the ancilla qubits is in the state $|1\rangle$. We call the states where ancilla qubits are in $||1\rangle\rangle$ excluded states. Whereas, we call the states where all of the ancilla qubits are in the state $|0\rangle$ active states. Using the state $|\psi_2\rangle$, we update the values E_{01} and θ_1 to E_{02} and θ_2, respectively. We can repeat this procedure as long as a fresh ancilla qubit is supplied.

Let us set $|\psi_j\rangle = \alpha_j|E_0\rangle + \beta_j|E_1\rangle$, $j = 0, 1, \cdots, m-1$, where $\alpha_0 = \alpha$, $\beta_0 = \beta$. By the $(j+1)$-th twirling operation (Fig. 2(a)), $|\psi_j\rangle$ and the $(j+1)$-th ancilla bit $|0\rangle$ transform as

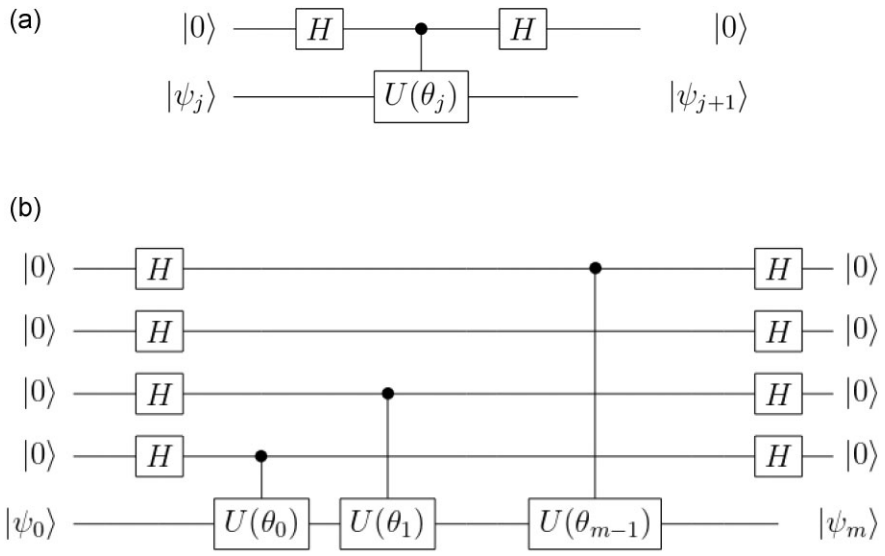

Fig. 2. (a) One-qubit j-th twirling operation. The operator U_j is defined by $U_j = ie^{-i\theta_j \hat{H}_T}$, where θ_j satisfies $\theta_j E_{0j} = \frac{\pi}{2}$ for E_{0j} that is computed by $E_{0j} = \langle\psi_j|\hat{H}_T|\psi_j\rangle$. By the Hadamard gate H the ancilla qubit is transformed as $|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The dot in the ancilla qubit means that the corresponding unitary transformation U is controlled by the ancilla qubit. By the controlled unitary transformation the physical quantum state and the ancilla qubit are transformed as $|\psi_j\rangle(|0\rangle + |1\rangle) \to |\psi_j\rangle|0\rangle + U|\psi_j\rangle|1\rangle$, and quantum entanglement is produced between the physical quantum state and the ancilla qubit. (b) m folds of one-qubit twirling operations.

$$
\begin{aligned}
|\psi_j\rangle|0\rangle &\xrightarrow{H} |\psi_j\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{C-U} |\psi_j\rangle\frac{1}{\sqrt{2}}|0\rangle + (\alpha_j ie^{-i\theta_j E_0} + \beta_j ie^{-i\theta_j E_1})\frac{1}{\sqrt{2}}|1\rangle \\
&\xrightarrow{H} |\psi_j\rangle\frac{1}{2}(|0\rangle + |1\rangle) + (\alpha_j ie^{-i\theta_j E_0} + \beta_j ie^{-i\theta_j E_1})\frac{1}{2}(|0\rangle - |1\rangle) \\
&= \left(\frac{1 + ie^{-i\theta_j E_0}}{2}\alpha_j|E_0\rangle + \frac{1 + ie^{-i\theta_j E_1}}{2}\beta_j|E_1\rangle\right)|0\rangle \\
&\quad \left(\frac{1 - ie^{-i\theta_j E_0}}{2}\alpha_j|E_0\rangle + \frac{1 - ie^{-i\theta_j E_1}}{2}\beta_j|E_1\rangle\right)|1\rangle \\
&\equiv c_{j+1}|\psi_{j+1}\rangle|0\rangle + d_{j+1}|\varphi_{j+1}\rangle|1\rangle, \qquad (7)
\end{aligned}
$$

where c_{j+1} and d_{j+1} are normalization constants. Since $\theta_j E_{0j} = \frac{\pi}{2}$ and E_{0j} is an approximate value of E_0, we have $\theta_j E_0 \simeq \frac{\pi}{2}$. Therefore, it is expected that $|\frac{\beta_{j+1}}{\alpha_{j+1}}| = |\frac{(1+ie^{-i\theta_j E_1})\beta_j}{(1+ie^{-i\theta_j E_0})\alpha_j}| < |\frac{\beta_j}{\alpha_j}|$ and the state accompanied by the ancilla qubits $|0\rangle|0\rangle\cdots|0\rangle$ approaches the true ground state $|E_0\rangle$. Our gate operation bears a resemblance to that of the quantum phase estimation [23,24].

The latter estimates the energy of a quantum state. In contrast with this our protocol separates a true ground state from excited states using an approximate ground state energy.

Let us consider the inequality $|\frac{\beta_{j+1}}{\alpha_{j+1}}| = |\frac{(1+ie^{-i\theta_j E_1})\beta_j}{(1+ie^{-i\theta_j E_0})\alpha_j}| < |\frac{\beta_j}{\alpha_j}|$ more closely. Since $\theta_j = \frac{\pi}{2E_{0j}}$, where E_{0j} is the j-th approximate value of the ground state energy E_0, we rewrite the ground state energy as $E_0 = E_{0j}(1 + \epsilon_{0j})$, where $\epsilon_{0j} = \frac{E_0 - E_{0j}}{E_{0j}}$. If the state $|\psi_j\rangle$ is a good approximate state of $|E_0\rangle$, ϵ_{0j} is expected to be small. In this case we have $1 + ie^{-i\theta_j E_0} = 1 + e^{-i\frac{\pi}{2}\epsilon_{0j}} \approx 2$. In contrast with this, rewriting $E_1 = E_{0j}(1 + \epsilon_{1j})$, where $\epsilon_{1j} = \frac{E_1 - E_{0j}}{E_{0j}}$, we have $1 + ie^{-i\theta_j E_1} = 1 + e^{-i\frac{\pi}{2}\epsilon_{1j}}$, which is equal to 2 only when $\epsilon_{1j} = 4n$ for some integer n. Therefore, except in the worst case

Table 1. Ground state expectation value of Z. The column of $j = 0$(adiabatic) indicates simulation results of the expectation value of Z generated by the IBM Qiskit qasm-simulator at the end of the naive adiabatic quantum computation ($s = 1$). The number j means that the j-th twirling operation has been done. The time interval θ_j has been divided into 100 small intervals. The number $10^7(10^8)$ means an average over 10 times for each set of $10^6(10^7)$ trials.

twirling operations	$j = 0$ (adiabatic)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
10^7	-0.71243	-0.70739	-0.70702	-0.70719	-0.70721	-0.70720
10^8	-0.71234	-0.70703	-0.70707	-0.70714	-0.70719	-0.70703

that rarely happens, we have $|1 + e^{-i\frac{\pi}{2}\epsilon_{1j}}| < 2$. Thus, in an ordinary case $|\frac{\beta_{j+1}}{\alpha_{j+1}}| < |\frac{\beta_j}{\alpha_j}|$ and $|\psi_{j+1}\rangle$ is closer to $|E_0\rangle$ than $|\psi_j\rangle$.

In summary, we repeat the following steps as long as we have unused fresh ancilla qubits (Fig. 2(b)):

- Evaluate $E_{0j} = \langle\psi_j|\hat{H}|\psi_j\rangle$.
- Perform Hadamard transformation H on the following fresh ancilla qubit.
- Perform unitary transformation $U(\theta_j)$ on the physical state controlled by the ancilla qubit, where $\theta_j = \frac{\pi}{2E_{0j}}$ (in this process the physical state and the ancilla qubit are entangled).
- Perform Hadamard transformation H on the ancilla qubit.
- We have a new total state in the form $c_{j+1}|\psi_{j+1}\rangle|0\rangle|0\rangle\cdots|0\rangle + \Sigma_i d_{j+1,i}|\varphi_{j+1,i}\rangle||1\rangle\rangle_{j+1,i}$.

2.2. *Simulation results*

In Table 1 we exhibit our simulation results for the case $J = 1$. The variance of measurement values of Z for the state $|E_0\rangle$ is $\sigma^2 = \frac{1}{2}$. The statistical error of the expectation value of Z for $n = 10^7$ trials is $\pm\frac{\sigma}{\sqrt{n}} = \pm 0.0002$. Therefore, roughly speaking, the expectation value of Z for an average over $n = 10^7$ trials will mainly be distributed over the range $-0.7073 \sim -0.7069$. On this point our simulation results agree with this theoretical value except for the no twirling operations case ($j = 0$). For $n = 10^8$ trials, $\pm\frac{\sigma}{\sqrt{n}} = \pm 0.00007$ and the main range of the distribution will be $-0.70719 \sim -0.70704$. Our simulation results again agree with this theoretical value except for the no twirling operations case. These results will mean that in $|\psi\rangle_j = \alpha_j|E_0\rangle + \beta_j|E_1\rangle$ the amplitudes β_j are sufficiently small for $j = 1, 2, \cdots, 5$, and our procedure works well.

We consider the number of the active states and the expectation value of the Hamiltonian from a slightly different point of view. For example, after the second twirling operation we will have the following quantum entangled state in which the ground state and the excited state are almost separated:

$$|\Psi\rangle = \alpha|E_0\rangle|0'_1\rangle|0'_2\rangle + \beta|E_1\rangle|\nearrow_1\rangle|\nearrow_2\rangle, \tag{8}$$

where

$$|0'_j\rangle = \frac{1}{2}(1 + e^{-i\frac{\pi}{2}\epsilon_{0j}})|0\rangle + \frac{1}{2}(1 - e^{-i\frac{\pi}{2}\epsilon_{0j}})|1\rangle, \tag{9}$$

$$|\nearrow_j\rangle = \frac{1}{2}(1 + e^{-i\frac{\pi}{2}\epsilon_{1j}})|0\rangle + \frac{1}{2}(1 - e^{-i\frac{\pi}{2}\epsilon_{1j}})|1\rangle. \tag{10}$$

Therefore, we have

$$|\Psi\rangle = \left(\alpha|E_0\rangle\frac{1}{4}(1+e^{-i\frac{\pi}{2}\epsilon_{00}})(1+e^{-i\frac{\pi}{2}\epsilon_{01}}) + \beta|E_1\rangle\frac{1}{4}(1+e^{-i\frac{\pi}{2}\epsilon_{10}})(1+e^{-i\frac{\pi}{2}\epsilon_{11}})\right)|0\rangle|0\rangle + |\varphi\rangle||1\rangle\rangle, \tag{11}$$

where $|\varphi\rangle$ stands for an excluded state. We only deal with the state in which the two ancilla qubits are in the state $|0\rangle|0\rangle$. Since the initial state $\alpha|E_0\rangle + \beta|E_1\rangle$ is an approximate vacuum, the energy ratio $\frac{E_0-E_{00}}{E_{00}} = \epsilon_{00}$ is expected to be small and $e^{-i\frac{\pi}{2}\epsilon_{00}} \approx 1$. Since E_{01} is obtained from an improved approximate ground state, $\frac{E_0-E_{01}}{E_{01}} = \epsilon_{01}$ is also expected to be small and $e^{-i\frac{\pi}{2}\epsilon_{01}} \approx 1$. In contrast with this, the possibilities that $e^{-i\frac{\pi}{2}\epsilon_{10}} \approx 1$ and $e^{-i\frac{\pi}{2}\epsilon_{11}} \approx 1$ will be very small. Therefore, except for the exceptional case that $e^{-i\frac{\pi}{2}\epsilon_{1j}} \approx 1$, the amplitude of the excited state $|E_1\rangle$ in the active state shrinks to zero by several iterations of the twirling operations. In this state we observe the two ancilla bits to be in the state $|0\rangle|0\rangle$ with the following probability:

$$||\langle 0|\langle 0|\Psi\rangle||^2 = |\alpha|^2\cos^2\frac{\pi}{4}\epsilon_{00}\cos^2\frac{\pi}{4}\epsilon_{01} + |\beta|^2\cos^2\frac{\pi}{4}\epsilon_{10}\cos^2\frac{\pi}{4}\epsilon_{11}. \tag{12}$$

The expectation value of the number of the active states is given by this probability times the trial number. For $J = 1$, the expectation value of the Hamiltonian $H_T \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ normalized by the probability $||\langle 0|\langle 0|\Psi\rangle||^2$ is

$$\frac{\langle\Psi|(H_T \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)|\Psi\rangle}{||\langle 0|\langle 0|\Psi\rangle||^2} = -\sqrt{2}\frac{|\alpha|^2\cos^2\frac{\pi}{4}\epsilon_{00}\cos^2\frac{\pi}{4}\epsilon_{01} - |\beta|^2\cos^2\frac{\pi}{4}\epsilon_{10}\cos^2\frac{\pi}{4}\epsilon_{11}}{|\alpha|^2\cos^2\frac{\pi}{4}\epsilon_{00}\cos^2\frac{\pi}{4}\epsilon_{01} + |\beta|^2\cos^2\frac{\pi}{4}\epsilon_{10}\cos^2\frac{\pi}{4}\epsilon_{11}}. \tag{13}$$

In the same way we have

$$\frac{\langle\Psi|(Z \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)|\Psi\rangle}{||\langle 0|\langle 0|\Psi\rangle||^2} = -\frac{\sqrt{2}}{2}\frac{|\alpha|^2\cos^2\frac{\pi}{4}\epsilon_{00}\cos^2\frac{\pi}{4}\epsilon_{01} - |\beta|^2\cos^2\frac{\pi}{4}\epsilon_{10}\cos^2\frac{\pi}{4}\epsilon_{11} - K}{|\alpha|^2\cos^2\frac{\pi}{4}\epsilon_{00}\cos^2\frac{\pi}{4}\epsilon_{01} + |\beta|^2\cos^2\frac{\pi}{4}\epsilon_{10}\cos^2\frac{\pi}{4}\epsilon_{11}}, \tag{14}$$

where K is the cross term:

$$\begin{aligned} K &= |\alpha\beta|\left(e^{i\theta}\frac{1+e^{i\frac{\pi}{2}\epsilon_{00}}}{2}\frac{1+e^{i\frac{\pi}{2}\epsilon_{01}}}{2}\frac{1+e^{-i\frac{\pi}{2}\epsilon_{10}}}{2}\frac{1+e^{-i\frac{\pi}{2}\epsilon_{11}}}{2} + \text{c.c.}\right) \\ &= 2|\alpha\beta|\cos\frac{\pi}{4}\epsilon_{00}\cos\frac{\pi}{4}\epsilon_{01}\cos\frac{\pi}{4}\epsilon_{10}\cos\frac{\pi}{4}\epsilon_{11}\cos\left(\theta + \frac{\pi}{4}\epsilon_{00} + \frac{\pi}{4}\epsilon_{01} - \frac{\pi}{4}\epsilon_{10} - \frac{\pi}{4}\epsilon_{11}\right). \end{aligned} \tag{15}$$

Since $|\cos\frac{\pi}{4}\epsilon_{00}\cos\frac{\pi}{4}\epsilon_{01}\cos\frac{\pi}{4}\epsilon_{10}\cos\frac{\pi}{4}\epsilon_{11}| < 1$ except for the singular values of $\epsilon_{01}, \epsilon_{11}$, the cross term K makes damping oscillation by the twirling operations.

After the third twirling operation we will have the state

$$|\Psi\rangle = \alpha|E_0\rangle|0'_1\rangle|0'_2\rangle|0'_3\rangle + \beta|E_1\rangle|\nearrow_1\rangle|\nearrow_2\rangle|\nearrow_3\rangle, \tag{16}$$

and so on. We will see that as the number of the twirling operation increases, the expectation value we find the ancilla qubits in the state $|0\rangle|0\rangle\cdots|0\rangle$ decreases, the expectation value of the Hamiltonian approaches $E_0 = -\sqrt{2}$, and the expectation value of Z approaches $-\frac{\sqrt{2}}{2}$.

We show in Table 2 our simulation results for the ground state expectation value of Z and the number of active states for a series of successive rounds of the twirling operation for $n = 10^8$ trials. For $j = 2, 3, 4$ the exact value $-\frac{\sqrt{2}}{2}$ of the ground state expectation value of Z lies in the 95% confidence interval. In this case $|\alpha| \gg |\beta|$ and it seems that we have almost reached a static state after the first twirling operation. The 95% confidence interval of the number of

Table 2. Ground state expectation value of Z and the number of active states for $J = 1$. The ground state expectation value of Z is obtained by averaging over $n = 10^8$ trials. We estimate the 95% confidence interval by $\pm 1.96 \times 2\sqrt{\frac{p(1-p)}{n}}$, where n is the number of active states and p is the probability $Z = 1$ is observed and $1 - p$ is the probability $Z = -1$ is observed when Z is measured in the basis $\{|0\rangle, |1\rangle\}$. For $j = 0, 1, \cdots, 5$, we find $1.96 \times 2\sqrt{\frac{p(1-p)}{n}} = 7 \times 10^{-5}$.

twirling operations	$j = 0$ (adiabatic)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\langle Z \rangle$	-0.71241 ± 7	-0.70704 ± 7	-0.70706 ± 7	-0.70707 ± 7	-0.70712 ± 7	-0.70699 ± 7
active states	100000000	99998535	99998549	99998547	99998442	99998525

active states is $\pm 1.96\sqrt{nq(1-q)} = \pm 75$, where $n = 10^8$ and $q = 0.99998535$ is the probability the active state is observed, which is calculated from the value for $j = 1$. The fluctuation of the number of active states in $j = 2, 3, 4, 5$ would represent the probabilistic fluctuation in the quantum measurement. These results will mean that in $|\psi\rangle_1 = \alpha_1|E_0\rangle + \beta_1|E_1\rangle$ the amplitude β_1 is sufficiently small. In the approximate ground state $|\psi\rangle = \alpha|E_0\rangle + \beta|E_1\rangle$ prepared by adiabatic quantum computation, the amplitude β is not so large. Therefore, the true ground state $|E_0\rangle$ has been almost extracted by the first twirling operation and this state has been maintained by the following twirling operations.

3. Twirling operations for an N-qubits Hamiltonian

In this section we introduce the twirling operations for an N-qubits Hamiltonian $\hat{H}$ that we use in this paper. Let us have an approximate ground state

$$|\psi\rangle = \alpha|E_0\rangle + \sum_{l=1}^{2^N-1} \beta_l|E_l\rangle \tag{17}$$

for the Hamiltonian $\hat{H}$. First, we measure the 0-th approximate value E_{00} of a ground state energy by $E_{00} = \langle\psi|\hat{H}|\psi\rangle$ and set $\theta_0 = \frac{\pi}{2E_{00}}$. Second, we prepare N ancilla qubits in the form $|0\rangle_1|0\rangle_2 \cdots |0\rangle_k \cdots |0\rangle_N$. Then, we Hadamard transform the k-th ancilla qubit $|0\rangle_k$. Using the k-th ancilla qubit as a control qubit, we apply the controlled operation $ie^{-i\theta_0\hat{H}}$ on the physical state 2^{k-1} times; the number 2^{k-1} is an example of a nonzero integer and does not have inevitability. We again Hadamard transform the k-th qubit. We perform this operation from the first ancilla qubit through the N-th ancilla qubit.

By this twirling operation, the total state $|\Psi\rangle$ transforms as

$$|\Psi\rangle = |\psi\rangle|0\rangle_1|0\rangle_2 \cdots |0\rangle_N$$

$$\to \alpha|E_0\rangle|0'\rangle_1|0'\rangle_2 \cdots |0'\rangle_N + \sum_{l=1}^{2^N-1} \beta_l|E_l\rangle|\nearrow_l\rangle_1|\nearrow_l\rangle_2 \cdots |\nearrow_l\rangle_N, \tag{18}$$

where

$$|0'\rangle_k = \frac{1}{2}\left(1 + e^{-i\frac{\pi}{2}2^{k-1}\epsilon_{00}}\right)|0\rangle + \frac{1}{2}\left(1 - e^{-i\frac{\pi}{2}2^{k-1}\epsilon_{00}}\right)|1\rangle, \tag{19}$$

$$|\nearrow_l\rangle_k = \frac{1}{2}\left(1 + e^{-i\frac{\pi}{2}2^{k-1}\epsilon_{l0}}\right)|0\rangle + \frac{1}{2}\left(1 - e^{-i\frac{\pi}{2}2^{k-1}\epsilon_{l0}}\right)|1\rangle, \tag{20}$$

with $\epsilon_{l0} = \frac{E_l - E_{00}}{E_{00}}$. After the first twirling operation, we have an unnormalized physical state in the active state $|0\rangle_1|0\rangle_2 \cdots |0\rangle_N$ as

$$|\psi_1\rangle = \alpha \prod_{k=1}^{N} e^{i\frac{\pi}{4}2^{k-1}\epsilon_{00}} \cos\left(\frac{\pi}{4}2^{k-1}\epsilon_{00}\right)|E_0\rangle + \sum_{l=1}^{2^N-1} \beta_l \prod_{k=1}^{N} e^{i\frac{\pi}{4}2^{k-1}\epsilon_{l0}} \cos\left(\frac{\pi}{4}2^{k-1}\epsilon_{l0}\right)|E_l\rangle. \quad (21)$$

Under the condition that $\epsilon_{00} \approx 0$, the relative amplitudes of the excited states diminish and the physical state approaches the ground state $|E_0\rangle$, except for the singular case $\epsilon_{l0} = 4n$ (n= an integer). Using $E_{01} = \langle\psi_1|\hat{H}|\psi_1\rangle$, we can perform the second twirling operation, and so on.

4. (1+1)-dimensional Schwinger model on a 1D spatial lattice

The (1+1)-dimensional Schwinger model on a finite-size lattice can be analyzed by quantum simulation [8], and the vacuum of the (1+1)-dimensional Schwinger model is nontrivial. Therefore, the (1+1)-dimensional Schwinger model is suitable for ascertaining whether our procedure works well. The (1+1)-dimensional Schwinger model with the θ-term is described in the natural unit system $\hbar = c = 1$ by the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi, \quad (22)$$

where $\gamma^0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the (1+1)-dimensional model the vector potential $\mathbf{A} = (A_0, A_1)$ only produces a 1D electric field. We use the metric $(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and we set the dielectric constant of the vacuum as $\epsilon_0 = 1$. We are interested in the simplest case $m = 0, \theta = 0$. In the temporal gauge $A_0 = 0$, the corresponding Hamiltonian is

$$\hat{H} = \int dx \left(-i\bar{\psi}\gamma^1(\partial_1 + igA_1)\psi + \frac{1}{2}\Pi^2\right), \quad (23)$$

where $\Pi = \dot{A}^1 = -E^1$. To investigate the vacuum of this Hamiltonian by quantum simulation, we formulate this Hamiltonian on a 1D spatial lattice with lattice spacing a. To avoid the species doubling problem, we introduce staggered fermions [25,26] with one-flavor. For $x = na$(n = even), we set $\psi(x) = (\psi_n\ 0)^T = \frac{1}{\sqrt{a}}(\chi_n\ 0)^T$. For even n, χ_n represents an annihilation operator of an antiparticle and $\chi_n^\dagger$ represents a creation operator of the antiparticle on the site $x = na$. For $x = na$(n = odd), we set $\psi(x) = (0\ \psi_n)^T = \frac{1}{\sqrt{a}}(0\ \chi_n)^T$. For odd n, $\chi_n^\dagger$ represents an annihilation operator of the particle and χ_n represents a creation operator of the particle on the site $x = na$. These fermionic operators satisfy the following anticommutation relations:

$$\{\chi_n^\dagger, \chi_m\} = \delta_{nm}, \qquad \{\chi_n, \chi_m\} = 0. \quad (24)$$

We introduce the link variable L_n by $L_n = -\frac{1}{g}\Pi(x = (n + \frac{1}{2})a)$ that lives on the link connecting the sites $x = na$ and $x = (n + 1)a$. Thus we have the lattice Hamiltonian, which leads to Eq. (19) in the limit $a \to 0$, as

$$\hat{H} = -iw\Sigma_{n=-\infty}^{\infty}(\chi_n^\dagger e^{iagA_1}\chi_{n+1} - \chi_{n+1}^\dagger e^{-iagA_1}\chi_n) + G\Sigma_{n=-\infty}^{\infty}L_n^2, \quad (25)$$

where $w = \frac{1}{2a}$ and $G = \frac{1}{2}g^2a$. Rewriting $e^{iagA_1}\chi_{n+1}$ as χ_{n+1}, which does not affect the anticommutation relations in Eq. (20), we have

$$\hat{H} = -iw\Sigma_{n=-\infty}^{\infty}(\chi_n^\dagger\chi_{n+1} - \chi_{n+1}^\dagger\chi_n) + G\Sigma_{n=-\infty}^{\infty}L_n^2. \quad (26)$$

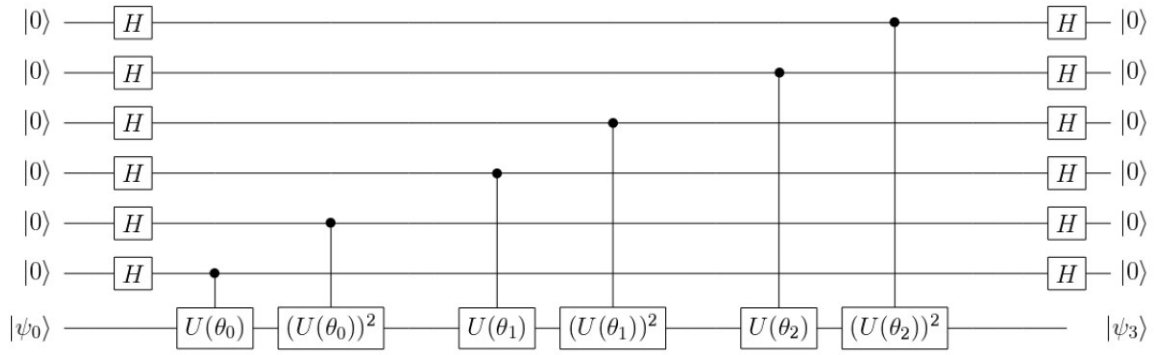

Fig. 3. Three folds of two-qubit twirling operations.

5. Two-qubits system derived from the (1+1)-dimensional Schwinger model

We consider the simplest case of Eq. (26); let us only have two sites $x = 0$ and $x = a$ on the spatial lattice, which we indicate by the numbers 0 and 1, respectively. We also adopt the fixed boundary condition; the electric field is zero out of the region $0 \leq x \leq a$ [8]. Gauss's law will hold on physical states. The discretized version of Gauss's law $\partial_1 E^1 = \rho = g : \psi^\dagger \psi :$ at $x = 0$ is

$$L_0 - L_{-1} = \chi_0^\dagger \chi_0 - \frac{1-(-1)^0}{2} = \chi_0^\dagger \chi_0. \tag{27}$$

From the fixed boundary condition $L_{-1} = 0$ and we have $L_0 = \chi_0^\dagger \chi_0$. Thus we have the following Hamiltonian on the two sites:

$$\hat{H} = -iw(\chi_0^\dagger \chi_1 - \chi_1^\dagger \chi_0) + G(\chi_0^\dagger \chi_0)^2. \tag{28}$$

Using the Jordan–Wigner transformation [27], we represent the fermionic variables χ_0 and χ_1 by spin variables:

$$\chi_0 = \frac{1}{2}(X_0 - iY_0), \qquad \chi_1 = \frac{1}{2}(-iZ_0)(X_1 - iY_1), \tag{29}$$

where $Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Thus we have the following simplest lattice Hamiltonian [8] up to a constant:

$$\hat{H} = \frac{1}{2}GZ_0 + \frac{1}{2}w(X_0X_1 + Y_0Y_1). \tag{30}$$

The first term in Eq. (30) is the electric field energy and the second term in Eq. (30) corresponds to the fermionic kinetic energy. For simplicity we study the following Hamiltonian:

$$\hat{H} = \frac{1}{2}(X_0X_1 + Y_0Y_1) + JZ_0, \tag{31}$$

where $J = \frac{G}{2w} = \frac{1}{2}g^2a^2$. The matrix representation of Eq. (27) is

$$\hat{H} = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & J & 1 & 0 \\ 0 & 1 & -J & 0 \\ 0 & 0 & 0 & -J \end{pmatrix}. \tag{32}$$

Table 3. Ground state expectation value of $\bar{Z}$ and the number of active states. The number of trials is 10^8 both for $J = 1$ and for $J = 2$. The numbers $j = 1, 2, \cdots, 6$ indicate the number of times that we have performed the twirling operation (Fig. 3). As for the adiabatic process, the initial Hamiltonian is $\hat{H}_0 = \frac{1}{2}(Z_0 - Z_1)$ and we have started from the initial ground state $|1\rangle_0|0\rangle_1$. The state $j = 0$ (adiabatic) is the initial approximate vacuum.

	$j = 0$(adiabatic)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
$10^8(J = 1)$	-0.71475	-0.70716	-0.70701	-0.70708	-0.70703	-0.70706	-0.70710
active state	10^8	99996822	99996734	99996858	99996733	99996846	99996864
$10^8(J = 2)$	-0.90014	-0.89438	-0.89435	-0.89440	-0.89442	-0.89441	-0.89439
active state	10^8	99995612	99995742	99995488	99995518	99995548	99995709

The smallest eigenvalue of Eq. (28) is $E_0 = -\sqrt{1 + J^2}$ and the corresponding eigenstate is

$$|E_0\rangle = \frac{1}{\sqrt{2(J^2 + J\sqrt{J^2 + 1} + 1)}}{}^t(0, 1, -\sqrt{J^2 + 1} - J, 0). \tag{33}$$

We are interested in the ground state expectation value of $\bar{Z} \equiv \frac{1}{2}(Z_0 - Z_1)$ [8]. We see that

$$\langle E_0|\bar{Z}|E_0\rangle = -\frac{J^2 + J\sqrt{J^2 + 1}}{J^2 + J\sqrt{J^2 + 1} + 1}, \tag{34}$$

which is $-\frac{1}{\sqrt{2}}$ for $J = 1$ and is $-\frac{2}{\sqrt{5}}$ for $J = 2$.

We show in Table 3 our simulation results for the vacuum expectation value of $\bar{Z}$ and the number of active states for a series of successive rounds of the twirling operation for $n = 10^8$ trials.

For $J = 1$, the theoretical value of the variance of $\bar{Z}$ is $\sigma^2 = \frac{2+\sqrt{2}}{4}$ when the ground state $|E_0\rangle$ is measured by the basis $\{|0\rangle, |1\rangle\}$. The standard deviation of the average of $\bar{Z}$ is $\frac{\sigma}{\sqrt{n}} = 0.000092\cdots$. Therefore, except for the no twirling operations case the average of $\bar{Z}$ is within one standard deviation from the theoretical value $-\frac{1}{\sqrt{2}}$. For $J = 2$, the theoretical value of the variance of $\bar{Z}$ is $\sigma^2 = \frac{1}{5}$ when the ground state $|E_0\rangle$ is measured by the basis $\{|0\rangle, |1\rangle\}$. The standard deviation of the average of $\bar{Z}$ is $\frac{\sigma}{\sqrt{n}} = 0.000045\cdots$. Therefore, except for the no twirling operations case the average of $\bar{Z}$ is within two standard deviations from the theoretical value $-\frac{2}{\sqrt{5}} = -0.8994427\cdots$. Thus, as for the expectation value of $\bar{Z}$, the shift of $O(|\beta|)$ seems to substantially be reduced by the first twirling operation. Therefore, it seems that the true ground state has substantially been extracted by the first twirling operation.

6. Three-qubits system derived from the (1+1)-dimensional Schwinger model

In this section we are not interested in the vacuum of the (1+1)-dimensional Schwinger model itself, but we are interested in the ground state of a three-qubits Hamiltonian that derives from the (1+1)-dimensional Schwinger model. We consider the next simplest case of Eq. (26); let us only have three sites $x = 0$, $x = a$, and $x = 2a$ on the spatial lattice, which we indicate by the numbers 0, 1, and 2, respectively. This time we also adopt the fixed boundary condition; the electric field is zero out of the region $0 \le x \le 2a$ [8]. This time we have the following three-qubits Hamiltonian:

$$\hat{H} = \frac{1}{2}G(Z_0 + Z_0Z_1) + \frac{1}{2}w(X_0X_1 + X_1X_2 + Y_0Y_1 + Y_1Y_2). \tag{35}$$

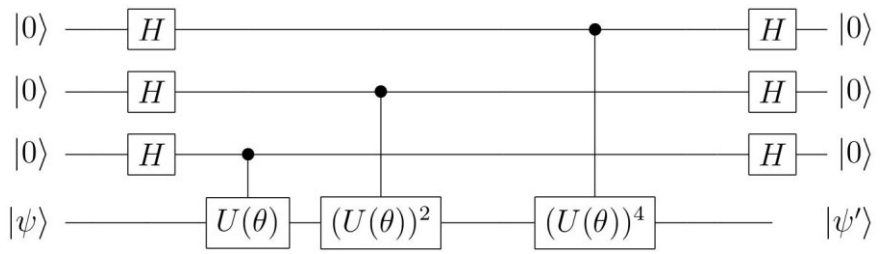

Fig. 4. One three-qubits twirling operation. We use three ancilla qubits for one twirling operation.

The corresponding dimensionless Hamiltonian is

$$\hat{H} = \frac{1}{2}(X_0X_1 + X_1X_2 + Y_0Y_1 + Y_1Y_2) + \frac{1}{2}J(Z_0 + Z_0Z_1). \tag{36}$$

Theoretically, we have the ground state

$$|E_0\rangle = \frac{1}{\sqrt{2J^2 + 2J\sqrt{2+J^2}+4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -J - \sqrt{2+J^2} \\ 1 \\ 0 \end{pmatrix}, \tag{37}$$

with eigenvalue $E_0 = -(J + \sqrt{2+J^2})$. For $J = 1$ we have $E_0 = -1 - \sqrt{3} = -2.73205\cdots$, and for $J = 2$ we have $E_0 = -1 - \sqrt{6} = -3.44948\cdots$. We also measure

$$\bar{Z} \equiv \frac{1}{3}(Z_0 - Z_1 + Z_2). \tag{38}$$

The theoretical vacuum expectation value of $\bar{Z}$ is

$$\langle \bar{Z} \rangle = \langle E_0|\bar{Z}|E_0\rangle = \frac{1}{2J^2 + 2J\sqrt{2+J^2}+4}\left(-\frac{4}{3} - 2J^2 - 2J\sqrt{2+J^2}\right). \tag{39}$$

For $J = 1$ we have $\langle \bar{Z} \rangle = -\frac{\frac{5}{3}+\sqrt{3}}{3+\sqrt{3}} = -0.71823\cdots$, and for $J = 2$ we have $\langle \bar{Z} \rangle = -(\frac{1}{3} + \frac{2}{9}\sqrt{6}) = -0.87766\cdots$.

We show simulation results in Table 4 for the cases $J = 1$ and $J = 2$. We see that the corrections of $\langle Z \rangle$ by the twirling operation are more significant than those of $\langle \hat{H} \rangle$, which agrees with the theoretical results for the one-qubit system when $|\beta|$ is small. As for $\langle \hat{H} \rangle$, the corrections are small and would be $O(|\beta_i|^2)$. It seems that the true ground state has substantially been extracted by the first and second twirling operations.

7. Summary and discussions

We have proposed a procedure to use quantum entanglement to extract a ground state from excited states for approximate ground states of the type $\alpha|E_0\rangle + \Sigma_i\beta_i|E_i\rangle$. We only use an approximate value of a ground state energy and concatenated ancilla qubits. Although we have used the approximate ground state prepared by adiabatic quantum computation for convenience, our procedure can be applied to any approximate ground state that is a superposition of a true ground state and excited states. We have developed the previous method presented by the present author [18] by concatenating the ancilla qubits and the twirling operations. By the

Table 4. Ground state expectation values of $\bar{Z}$ and $\hat{H}$. Before the twirling operations we have used adiabatic quantum computation and the state $j = 0$ (adiabatic) is the initial approximate ground state. The physical quantum state approaches $|E_0\rangle$ by the twirling operations (Fig. 4). We show averages over 10^8 trials. For the 4 twirling operations we need 12 fresh ancilla qubits.

$J = 1$	$j = 0$ (adiabatic)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	theoretical
$\langle\bar{Z}\rangle$	−0.712514	−0.71872	−0.71808	−0.71818	−0.71824	−0.71823
$\langle\hat{H}\rangle$	−2.73148	−2.73229	−2.73213	−2.73195	−2.73212	−2.73205
active states	10^8	99986679	99986373	99986107	99986210	–
$J = 2$	$j = 0$ (adiabatic)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	theoretical
$\langle\bar{Z}\rangle$	−0.88467	−0.87834	−0.87772	−0.87773	−0.87769	−0.87766
$\langle\hat{H}\rangle$	−4.44912	−4.44949	−4.44942	−4.44965	−4.44961	−4.44949
active states	10^8	99991476	999912973	99991366	99991307	–

concatenation it is expected that the approximate ground state approaches the true ground state more closely. We have introduced twirling operations for an N-qubits Hamiltonian. We have carried out quantum simulation by using IBM Qiskit for the one-qubit system, the two-qubits system, and the three-qubits system that derive from the (1+1)-dimensional Schwinger model. We have seen that the ground state expectation values we have obtained by the simulation agree with the theoretical values within the statistical errors.

We have simulated the one-qubit system, which has two energy eigenstates. We also have simulated the two-qubits system that derives from the (1+1)-dimensional Schwinger model. Although this system has four energy eigenstates, since we have started from the charge-neutral state and the Hamiltonian conserves the number of the charge, the approximate ground state is supposed to be a superposition of two charge-neutral states: the state in which no particles are excited and the state in which a pair of an electron and a positron is excited. Therefore, concerning the ground state, this system is substantially a two-state system. In our simulation results we have obtained satisfactory ground state expectation values by only one twirling operation. This may be peculiar to a two-state system. We also have examined the three-qubits system, which is not a two-state system. In this case we have seen that the combination of adiabatic quantum computation and the concatenated twirling operations seems effective to produce the true ground state $|E_0\rangle$. We can compute ground state expectation values of some physical quantities with high accuracy without algebraic calculation. We have simulated some simple systems that can be analyzed by algebraic calculation. Our procedure, however, in principle can be applied to more complicated systems that are difficult to analyze by algebraic calculation. In principle, our simulation procedure applies to any finite-dimensional Hamiltonian that is represented by Pauli matrices. Our procedure works well only for the case that the quantum system has a nondegenerate ground state. Our procedure would give a new quantum algorithm that uses quantum entanglement.

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