

ON MATRIX *D*-BRANE DYNAMICS AND FUZZY SPHERES

by

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To Sophie, for her endless, unconditional love and support

Declaration

I hereby declare that the material presented in this thesis is a representation of my own personal work

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Abstract

This thesis deals with the interplay between the dynamics of certain D -brane systems, described by Matrix degrees of freedom, and a class of symmetric non-commutative spaces, namely fuzzy spheres. After a brief introduction to the main ideas and concepts, we describe two classes of configurations involving even-dimensional fuzzy spheres (S^{2k}), *i.e.* static $D1 \perp D(2k+1)$ BIonic brane intersections and collapsing spherical $D0$ - $D(2p)$ -brane bound states. Both scenarios admit macro- and microscopic realisations which overlap and agree in the large- N limit. We show that the above physics are commonly captured in terms of a Riemann surface description, the genus of which depends on the dimensionality of the sphere involved. The Riemann surfaces arise as complex orbits in complexified phase space and play an important role in recovering explicit analytic solutions. For the fuzzy- S^2 there is an $r \rightarrow 1/r$ duality, relating the time and space-dependent problems and described in terms of automorphisms of the Riemann surface. For the $D0$ - $D2$ system, we extend the large- N classical agreement, between higher and lower-dimensional pictures, to an agreement in quadratic fluctuations. In an appropriate scaling limit, the non-linearities of the DBI survive in both classical and quantum contexts, while massive open string modes and closed strings decouple. For the same case, we evaluate the full range of $1/N$ corrections, coming from implementing the symmetrised trace (STr) prescription for the microscopic non-abelian DBI action, and analyse how these affect the nature of the collapse. We extend our study to related systems involving odd-dimensional fuzzy spheres (S^{2k-1}) and recover a microscopic description similar to that for S^{2k} . The details are significantly more involved, due to STr effects entering the calculation even at large- N .

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Table of contents

DECLARATION	3
ABSTRACT	4
ACKNOWLEDGEMENTS	5
1 INTRODUCTION AND MOTIVATION	11
1.1 The abelian Dirac-Born-Infeld action	12
1.2 The non-abelian DBI action	14
1.3 Matrix Theory	17
1.4 Non-Commutative Geometry and Fuzzy Spheres	19
1.4.1 The Fuzzy- S^2	19
1.4.2 The Fuzzy- S^{2k}	21
1.4.3 The Fuzzy- S^{2k-1}	23
1.5 Non-Commutative gauge theory	26
1.6 The D -brane ‘dielectric’ effect	28
1.6.1 The static BIonic configurations	30
1.6.2 The time-dependent spherical configurations	31
2 FUZZY SPHERES AND LARGE-SMALL DUALITIES	33
2.1 Space and Time-Dependent Fuzzy- S^2	34
2.1.1 Non-abelian DBI description of non-static $D1 \perp D3$ funnels	34
2.1.2 The dual picture	36
2.1.3 Arrays of branes in space and Collapse/Expansion in time-dependence	37
2.1.4 Riemann Surface Technology	41
2.1.5 An $r \rightarrow 1/r$ duality between the Euclidean and Lorentzian DBI equations	46
2.2 Space and time-dependent fuzzy- S^4	47

2.2.1	Equivalence of the action for the $D1 \perp D5$ intersection	47
2.2.2	Solutions for space-dependent fuzzy spheres: Funnels	49
2.2.3	Solutions for time-dependent fuzzy spheres: Collapsing spheres	50
2.2.4	Geometry and Automorphisms of Hyper-elliptic curve for the fuzzy- S^4	52
2.2.5	Evaluation of integrals	56
2.3	Reduction of curve and inversion of integral for fuzzy- S^4	58
2.3.1	Series expansion of u_2 as a function of u_1	60
2.3.2	Evaluation of the time of collapse and the distance to blow up	61
2.3.3	Solution of the problem in terms of the u_2 variable and large-small duality	65
2.3.4	Lagrangeans for holomorphic differentials	68
2.4	Space and Time-dependent Fuzzy- S^6	69
3	SMALL-FLUCTUATION ANALYSIS	74
3.1	Yang-Mills type action for small fluctuations	75
3.1.1	Strong coupling radius	79
3.1.2	Overall transverse fluctuations and exactly solvable Schrödinger equation	81
3.2	Action for fluctuations from the $D0$ -brane non-abelian DBI	82
3.2.1	Geometry of fuzzy two-sphere: A brief review	83
3.2.2	The action for the gauge field and radial scalar	86
3.2.3	Scalar fluctuations for the reduced action	89
3.2.4	$1/N$ correction to the action	90
3.3	Scaling limits and Quantum Observables	91
3.3.1	The DBI-scaling	91
3.3.2	The $D0$ Yang-Mills (Matrix Theory) limit	94
3.3.3	Mixing with graviton scattering states	99
4	SYMMETRISED TRACE CORRECTIONS FOR NON-ABELIAN DBI	101
4.1	Lorentz invariance and the physical radius	102
4.2	The fuzzy- S^2 at finite- n	104
4.2.1	The $D1 \perp D3$ intersection at finite- n	106
4.2.2	Finite- N dynamics as a quotient of free multi-particle dynamics	107
4.3	Physical properties of the finite- N solutions	108
4.3.1	Special limits where finite- n and large- n formulae agree	108

4.3.2	Finite- N effects: Time of collapse, proper accelerations and violation of the dominant energy condition	109
4.3.3	Distance to blow up in $D1 \perp D3$	113
4.4	Towards a generalisation to higher fuzzy even-spheres	114
5	NON-ABELIAN DBI AND FUZZY ODD-DIMENSIONAL SPHERES	117
5.1	General fuzzy odd-sphere equations with $SO(D)$ symmetry	118
5.2	On the equations for fuzzy- S^3	122
5.2.1	Derivation of $[X_i, X_j]$ for fuzzy- S^3	124
5.3	On the equations for fuzzy- S^5	125
5.4	The Fuzzy- S^{2k-1} matrices and DBI with symmetrised trace	126
5.4.1	Fuzzy- S^3	126
5.4.2	Fuzzy- S^5	131
5.5	The large- N dynamics of fuzzy odd-spheres.	133
5.5.1	Constancy of the speed of light	135
5.5.2	Derivatives, no-bounce results, accelerations	136
5.6	The Matrix Theory (Yang-Mills) limit	137
5.7	Towards a dual 3-brane picture for fuzzy- S^3	138
5.7.1	Non-BPS systems in String Theory	138
5.7.2	The dual construction	140
6	CONCLUSIONS AND OUTLOOK	144
A	ASPECTS OF SPACE-TIME-DEPENDENCE OF THE S^2 FUNNEL	151
A.1	Lorentz Invariance of the BPS condition	151
A.2	Chern-Simons terms and D -brane charges for $D1 \perp D3$ system	154
B	THE COLLAPSING $D2$ RADIAL PROFILE FROM \wp -FUNCTIONS	156
C	GENERAL FORMULA FOR THE SYMMETRISED TRACE	159
D	CALCULATION OF STr FROM THE HIGHEST WEIGHT METHOD	166
D.1	Review of spin-half for $SO(3)$	167
D.2	Derivation of STr for the minimal $SO(2l+1)$ representation	168
D.3	Derivation of spin-one STr for $SO(3)$	169
D.4	Derivation of next-to-minimal representation for $SO(2l+1)$	170

E THE LARGE- n LIMIT OF THE FUZZY- S^3 PROJECTED ALGEBRA	172
F SOLUTIONS TO REDUCED ACTION AND SOLUTIONS TO DBI	175

List of Figures

2.1	Analytic plot of the Jacobi elliptic function solution for the static fuzzy- S^2 funnel array and the collapsing fuzzy- S^2 for $r_0 = 1$	40
2.2	Numerical plot of the Jacobi elliptic function solution for the static fuzzy- S^4 funnel array and the collapsing fuzzy- S^4 at large r_0	65

CHAPTER 1

INTRODUCTION AND MOTIVATION

String Theory has for a long time now been considered the most prominent candidate for describing Gravity, Quantum Mechanics and Matter in a unified way [1, 2]. It is realised by giving up the notion of point particles and assuming that the fundamental objects in the theory are open or closed strings, *i.e.* one dimensional, extended objects. The strings can oscillate giving rise to a number of different particle types, including one of spin two and zero mass, which is identified with the graviton. In the limit where the string length shrinks to zero, $\ell_s \rightarrow 0$, the strings look like localised particles. The theory makes startling predictions about the dimensionality of space-time. Superstring Theory in flat space only makes sense, at least at the perturbative level, in 10 dimensions, in which the strings also have fermionic excitations and give rise to a supersymmetric theory. In order to make contact with a four dimensional theory one needs to compactify on a six dimensional manifold. The geometry of the manifold then governs the low energy interactions of the theory. There exist five such supersymmetric, ten-dimensional String Theories, which have been shown to be vacuum states of a larger, eleven dimensional theory, called M-theory [3–5], and are related by a web of non-perturbative string dualities.

String Theories also turned out to contain solitonic, membrane-like objects of various dimensionalities, called Dirichlet branes (D -branes), the discovery of which was motivated by the above mentioned dualities [6–8]. A Dp -brane is a $p + 1$ dimensional hyper-plane in 10-dimensional space-time, upon which the open strings are allowed to end. This is possible even in theories where all strings are closed in the space-time bulk. The open string end-points satisfy von-Neumann boundary conditions in the longitudinal $p + 1$ directions and Dirichlet ones in the remaining $9 - p$ transverse co-ordinates. When a closed string touches the D -brane, it opens up and its end-points are free to move on the hyper-surface. Dp -branes are half-BPS objects and carry an elementary unit of charge with respect to the $p + 1$ form gauge potential, coming from the Ramond-Ramond (RR) sector of the type II superstring.

Dp -branes are dynamical objects, which naturally realise gauge theories on their world-volume. The massless spectrum of the open strings living on the brane can be described by a maximally supersymmetric $U(1)$ gauge theory in $p + 1$ dimensions. There is a vector field and the $9 - p$ massless real scalar fields that are present in the super-multiplet, can be identified with the Goldstone modes that are associated with the motion of the brane in the transverse directions. Therefore, when the branes are sitting on top of each other,

the vacuum expectation values (vevs) for the scalar fields vanish. One also has to include the supersymmetric fermionic partners. A collection of N parallel D -branes provide N^2 possibilities for the endpoints of open strings, since the latter can begin or end on any one of them. The spectrum is now described by $U(N)$ maximally supersymmetric gauge theory. When the number of branes N is large, the stack is a heavy macroscopic object, embedded into a theory of closed strings that contains gravity. This object will back-react on the geometry and curve space. It will be described by a classical metric and other background fields, which will include the $p + 1$ RR form potential.

The discovery of D -branes brought about a revolution in String Theory and the advent of a number of great theoretical successes. The relationship between the $U(N)$ supersymmetric world-volume theory and the type II supergravity description is at the heart of a gravity/gauge theory correspondence in the form of the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) duality. The original conjecture states that in the large- N limit, *à la* 't Hooft [9], $\mathcal{N} = 4$ superconformal $SU(N)$ Yang-Mills theory in four dimensions¹ is dual, in a strong/weak coupling sense, to String Theory on an $AdS_5 \times S^5$ background [10]. There have been many efforts to generalise this result. Through the latter, the AdS/CFT correspondence has provided us with new insights, motivations and results in the study of the geometry of Calabi-Yau manifolds. Furthermore, D -branes played an integral part in the statistical mechanical black hole entropy counting [11]. They have also given the inspiration for new cosmological scenarios, in the form of brane-worlds [12], provided a framework for a non-perturbative description of M-theory in terms of Matrix Theory [13] and suggested solutions to the Hierarchy problem, such as the Randall-Sundrum model [14, 15], to name but a few other examples. The rest of this chapter provides a more detailed review of some material that will lead naturally to the main content of this thesis.

1.1 The abelian Dirac-Born-Infeld action

The low-energy effective $U(1)$ gauge theory living on the world-volume M of a brane can be naturally derived by world-sheet CFT [16]. This can be carried out by considering a non-linear σ -model in conformal gauge, for which the co-ordinates satisfy mixed Dirichlet-von-Neumann boundary conditions, in accordance with the definition of a D -brane. It includes a boundary term with a $U(1)$ gauge field tangent to M and a set of fields perpendicular to it. The requirement for the vanishing of the β -functions for the closed string modes at lowest genus, provides the usual background bulk field equations of motion. A similar treatment for the open string contributions leads to the equations of motion for a Yang-Mills action. However, by working to lowest order in space-time curvature, field strength $H = dB$, extrinsic curvature of M and derivative of the field strength for the gauge field, one can recover the full stringy, $\alpha' = \ell_s^2$, corrections to the β -function equation. The

¹The extra $U(1)$ degree of freedom is associated with the centre of mass motion of the stack.

vanishing of this open string β -function exactly matches the equations of motion coming from the Dirac-Born-Infeld (DBI) action and provides a unique example of a case where α' corrections can be summed to all orders. The latter is a generalisation of the Born-Infeld (BI) action [17] including scalar fields. BI theory was initially introduced in the context of non-linear electrodynamics², in order to describe charged objects with finite total energy. There is another way to get the same action by evaluating a path integral in the presence of a D -brane [19–21]. This is the partition function of virtual open strings with mixed boundary conditions, propagating in a condensate of massless string modes plus some boundary background couplings. The path integral approach makes the T-duality properties of the resulting action more transparent and implies that all actions for $p < 9$ branes can be obtained by dimensionally reducing the ten dimensional BI action. The bosonic sector of the abelian $p + 1$ dimensional DBI action, which captures the open string dynamics on the Dp -brane world-volume, is given, in the so called ‘static’ gauge, by the expression

$$S_{DBI} = -T_p \int d^{p+1}\sigma \left(e^{-\phi} \sqrt{-\det(P[G+B]_{ab} + 2\pi\alpha' F_{ab})} \right) \quad (1.1.1)$$

The σ ’s are the D -brane world-volume co-ordinates, T_p is a constant, which corresponds to the brane tension, $P[G+B]$ is the pull-back of the bulk space-time metric and antisymmetric tensor respectively, ϕ is the dilaton and F the world-volume field strength. The ‘static’ gauge³ choice is related to the fixing of the gauge invariance associated with space-time diffeomorphisms, so as to align the world-volume of the brane with a surface where $\sigma^i = 0$, for $i = p + 1, \dots, 9$. Use of world-volume diffeomorphisms then allows us to match the world-volume co-ordinates with the remaining co-ordinates on that surface, $\sigma^a = x^a$, with $a = 0, \dots, p$. The transverse scalars Φ^i now capture the transverse displacements of the brane via the identification $\sigma^i = 2\pi\ell_s^2 \Phi^i$. In this gauge, the definition of the pull-back of a space-time tensor $E_{\mu\nu}$ onto the brane world-volume is

$$\begin{aligned} P[E]_{ab} &= E_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \\ &= E_{ab} + 2\pi\ell_s^2 E_{ai} \partial_b \Phi^i + 2\pi\ell_s^2 E_{ib} \partial_a \Phi^i + 4\pi^2 \ell_s^4 E_{ij} \partial_a \Phi^i \partial_b \Phi^j \end{aligned} \quad (1.1.2)$$

The part of the DBI action dealing with the fermionic super-partners of the bosonic fields is constrained by requiring invariance under a special fermionic gauge symmetry, called κ -symmetry. Upon gauge-fixing the latter, the global target space supersymmetry will combine with a field-dependent κ -transformation to give global world-volume super-

²For a review of non-linear electrodynamics and Born-Infeld theory we refer the reader to [18] and references therein.

³Also known in the literature as Monge gauge, perhaps more appropriately since it doesn’t only restrict to static configurations.

symmetry⁴. However for the rest of this thesis we will be concerned with the bosonic piece of the DBI action.

The square root structure of the BI action, in the context of non-linear electrodynamics, imposes a maximal field strength constraint on the allowed physical configurations. Since the DBI action is originating from the former via T-duality, it will also imply such restrictions. To be more precise, in the case of a $D0$ -brane, the action reduces to that of a relativistic particle, $S \sim \int dt \sqrt{1 - (\partial_t \Phi^s)^2}$, and the constraint is nothing but the relativistic bound on the particle's velocity [23]. This also alludes to the geometrical interpretation of the DBI action: It describes the world-volume swept out by the Dp -brane and encodes the low-energy dynamics.

In addition to the DBI part, the full D -brane action should contain couplings to the massless closed string RR fields. These not only include terms which are proportional to the appropriate $C^{(p+1)}$ RR potential, but also a number of others in the presence of non-trivial gauge or anti-symmetric background fields. These couplings to RR potentials of a lower form degree have a physical interpretation in terms of bound states of D -branes of different dimensions [24, 25] or of intersecting branes [26, 27]. All of these terms can be captured neatly in a second part of the action, the Chern-Simons (CS) term

$$S_{CS} = \mu_p \int_M P \left[\sum_p C^{(n)} e^B \right] \wedge e^{2\pi\alpha' F} \quad (1.1.3)$$

where μ_p is the associated p -brane charge and $P[\dots]$ implies once again the pull-back of the space-time fields onto the world-volume of the brane. Supersymmetry imposes that $\mu_p = \pm T_p$. The Chern-Simons part of the action is also invariant under T-duality.

1.2 The non-abelian DBI action

It is natural to try and extend the above treatment of the DBI effective action, to include configurations of N coincident D -branes [28, 29]. We have seen that the low-energy degrees of freedom will fit into a $U(N)$ representation. The vector A is now a matrix-valued non-abelian gauge field, which transforms in the adjoint representation, as are the transverse scalars

$$A_a = A_a^n T_n, \quad F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b], \quad D_a \Phi^i = \partial_a \Phi^i + i[A_a, \Phi^i] \quad (1.2.1)$$

with the T_n 's being hermitian $N \times N$ generators with $Tr(T_m T_n) = N \delta_{mn}$. Since we want the action to be a group scalar, we will have to trace over all group indices. However, this extension leads to a number of puzzles involving the validity of previous expressions which

⁴For more on the κ -symmetry transformation we refer to [22] and references therein.

now incorporate matrices, *e.g.* $\sigma^i = 2\pi\ell_s^2 \Phi^i$. These statements remain roughly correct, if we think of the case where the transverse scalars commute and can, therefore, be simultaneously diagonalised using the gauge symmetry. Then the diagonal eigenvalues of the matrices give the displacement of the branes in the respective direction. However, for a more general configuration, the spectrum of eigenvalues does not give an accurate description of the D -brane positions and it is clear that at short distances, classical commutative geometry ceases to be able to capture the physics of the system. We will return to this point soon. The gauge symmetry can also be used to interchange any pair of eigenvalues simultaneously, ensuring the fact that we cannot distinguish between the branes.

The derivation of the DBI for the non-abelian case is technically hard from the non-abelian σ -model point of view. Moreover, the discussion in the previous section was valid only in the limit where the derivatives of the field strength were negligible. The same will extend to higher derivatives of the scalar fields. However, there is an extra ambiguity in this construction, since $[D_a, D_b]F_{cd} = i[F_{ab}, F_{cd}]$. If we choose to only keep higher powers of F that are symmetric, it turns out that we not only manage to evade this problem but we can also calculate the non-abelian BI action from the path integral representation for the generating functional of the vector scattering amplitudes on the disc [30]. The above procedure is summarised by substituting the gauge trace with a *symmetrised trace* (STr) operator. This definition gives a natural and precise ordering for both parts of the action, as functionals of non-abelian fields. Once the non-abelian BI action is known, we can use the T-duality of string theory to get all the lower-dimensional D -brane actions, as we did for the abelian case [28, 31]. T-duality can affect a Dp -brane in two ways: When performed along a transverse direction x^{p+1} , it will increase its dimensionality to $D(p+1)$. When performed along a direction parallel to the brane, it will lower its dimensionality to $D(p-1)$. Under this action, the world-volume fields transform as

$$\Phi^{p+1} \rightarrow A_{p+1}, \quad A_p \rightarrow \Phi^p \quad (1.2.2)$$

respectively, while the rest of the components remain unchanged. By applying the T-duality rules carefully (since the background supergravity fields will also transform), one can reduce a ten dimensional BI action down to a $p+1$ dimensional DBI action. The result is summarised in the expression [28]

$$S_{NDBI} = -T_p \int d^{p+1}\sigma \, STr \left(e^{-\phi} \sqrt{\det(Q^i_j)} \right. \\ \left. \times \sqrt{-\det(P [E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + \lambda F_{ab})} \right) \quad (1.2.3)$$

where $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$, $Q^i_j \equiv \delta^i_j + i\lambda[\Phi^i, \Phi^k]E_{kj}$ and we have defined for brevity $\lambda = 2\pi\alpha'$. The first term under the square root, $\sqrt{\det(Q^i_j)}$, will account for the scalar potential coming from the above action, while the second one for the kinetic term contributions. The CS

piece of the action will also include new terms and will be given by

$$S_{CS} = \mu_p \int_M STr \left(P[e^{i\lambda \mathbf{i}_\Phi \mathbf{i}_\Phi} (\sum C^{(n)} e^B)] e^{\lambda F} \right) \quad (1.2.4)$$

where \mathbf{i}_Φ is the interior product by Φ^i and should be regarded as a vector in the transverse space. As an example, which will be useful later, consider the action of $\mathbf{i}_\Phi \mathbf{i}_\Phi$ on the n -form $C^{(n)} = \frac{1}{n!} C_{i_1 i_2 \dots i_n}^{(n)} dx^{i_1} dx^{i_2} \dots dx^{i_n}$. This will give

$$\mathbf{i}_\Phi \mathbf{i}_\Phi C^{(n)} = \frac{1}{2(n-2)!} [\Phi^{i_2}, \Phi^{i_1}] C_{i_1 i_2 i_3 \dots i_n}^{(n)} dx^{i_3} \dots dx^{i_n} \quad (1.2.5)$$

Of course, the above contribution will only be present for configurations where the scalar fields are matrix valued and cannot be simultaneously diagonalisable. In any other situation, these extra terms will vanish. Since we have identified, at least in some sense, the world-volume scalars with transverse spatial co-ordinates, the above requirement hints towards the possibility of introducing configurations, for which the space-time geometry becomes non-commutative. We will return to this point in the next section and see how the treatment of certain physical systems actually requires the introduction of such notions from geometry.

Another thing to note is the following: In the abelian case, the background supergravity fields are functions of all the space-time co-ordinates and therefore also functionals of the transverse scalars. In static gauge we can evaluate these background fields as a Taylor expansion around $x^i = 0$, without any problem. Take for example the metric tensor

$$\begin{aligned} G_{\mu\nu} &= e^{\lambda \Phi^i \partial_{x^i}} G_{\mu\nu}^0(\sigma^a, x^i)|_{x^i=0} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Phi^{i_1} \dots \Phi^{i_n} (\partial_{x^{i_1}} \dots \partial_{x^{i_n}}) G_{\mu\nu}^0(\sigma^a, x^i)|_{x^i=0} \end{aligned} \quad (1.2.6)$$

When we go to the non-abelian version of the action we should note that the above becomes a non-abelian Taylor expansion. Even though the expansion is by definition symmetric in the Φ 's, since the partial derivatives commute, there will be ordering issues in the full non-linear action, in which the implementation of the trace will also have to involve matrix-valued terms coming from the non-abelian F 's.

Let us concentrate more on the nature of the implementation of the STr operator. In the generic case, it will provide the symmetric average over all orderings of the F_{ab} , $D_a \Phi^i$ and $[\Phi^i, \Phi^j]$ fields. The latter are easily obtained from the higher dimensional field strength via T-duality. We also need to symmetrise over any individual Φ 's coming from a non-abelian Taylor expansion of the background fields. With this prescription, the low energy effective actions that we have presented can be verified by looking directly at string scattering amplitudes on the brane [32, 33]. However, even though this minimal extension of the abelian DBI is correct up to fourth order in F and is known to reproduce some highly non-trivial results, it does not agree with the full effective string action [34]. At order F^6

corrections involving commutators of the field strengths are needed in order to reproduce the full physics of the non-abelian fields in the infrared limit. There exist iterative techniques involving stable holomorphic bundles, which can evaluate all the higher derivative corrections and also the fermion contributions⁵.

1.3 Matrix Theory

As we have already mentioned, String Theories just probe perturbatively the physics around a vacuum, in a corner of the moduli space of compactifications of M-theory. However, most of that space lies beyond their reach and we still lack a proper description of M-theory. Amongst the little information that we possess is that, in the low-energy limit, the latter is described by eleven dimensional supergravity. That, in turn, suggests that matter in the theory takes the form of a membrane, charged under a three-form and its ‘electromagnetic’ dual, a five-brane. Type IIA String Theory and the collection of its perturbative and non-perturbative states, can be recovered by a Kaluza-Klein (KK) reduction along a space-like circle. The non-perturbative String Theory dualities, can then be used to relate IIA to IIB, type I and the heterotic theories. The non-perturbative links between the String Theories as well as the successes of D -branes, which are intrinsically non-perturbative objects, in the microscopic study of black hole physics hints that branes could be degrees of freedom more fundamental than even the strings themselves. The hope, therefore, is that some non-perturbative formulation of String Theory based on D -branes, would be able to uncover some, or even all, of the M-theory physics.

A big step towards realising such a formulation is Matrix Theory [13]. Here we give a brief review of the conjecture and some accompanying notions that will be useful in the following chapters but we will not attempt to give a comprehensive account of the vast literature that exists on the subject. We will, therefore, refer for details and further references to [39–41].

First, think of M-theory compactified on a space-like circle of radius R_s , in a sector of momentum $P = \frac{N}{R_s}$. In the limit where the space-like circle shrinks to zero, the lowest excitations with momentum P are N $D0$ -branes of type IIA. The Matrix Theory conjecture states that M-theory in the Infinite Momentum Frame (IMF) and in decompactified space, is described by the minimal Super-Yang-Mills (SYM) $D0$ -brane system with 16 supercharges, when the velocities are small, the string interactions weak and in the limit where $N, R_s, P \rightarrow \infty$. There is another formulation of the conjecture describing the Discrete Light Cone Quantisation (DLCQ) of M-theory compactified on a light-like circle of radius R and in a sector of light-cone momentum $P_- = \frac{N}{R}$ [42]. Then the decompactified theory is obtained by holding P_- fixed and taking $R, N \rightarrow \infty$. However, the correspondence between M and

⁵For the state-of-the-art in these calculations we refer to [35–37] and references therein. Also note that if we include fermions the results deviate from STr even at order α'^2 , see *e.g.* [38].

Matrix theory in this case should hold even at finite- N . The two frames are related by a large boost [43, 44].

One can deduce that in the above-mentioned limit of small velocities and weak interactions, the action of N $D0$ -branes can be obtained from the series expansion of (1.2.3). In order to keep up with the Matrix Theory literature we will switch to matrices X^i with explicit dimensions of length. In a flat background and if we gauge away the only component of the gauge potential A_0 , one ends up with the simple-looking action

$$S_M = \frac{1}{2g_s \ell_s} \int dt \operatorname{Tr} \left[\dot{X}^2 + \frac{1}{\lambda^2} [X^i, X^j] [X^i, X^j] + \text{fermions} \right] \quad (1.3.1)$$

Note that the STr in this case reduces to the simple trace operator, since in this approximation we only need to symmetrise over two objects at a time, which are already satisfying the cyclicity property. The X 's are $N \times N$ hermitian matrices, transforming in the adjoint representation of $U(N)$ and their indices run from $1, \dots, 9$. We conclude that the degrees of freedom of the theory are assembled into matrix degrees of freedom that include positions, for the diagonal elements, and stretched open strings between the branes, for the off-diagonal ones, plus fermionic super-partners. The classical static solutions of the above action are, of course, found by minimising the potential term $[X, X] = 0$ and the resulting configuration space is that of N identical particles moving in Euclidean space. It is clear that, if the conjecture is to be correct, this action should reproduce objects and scattering results from M-theory. Without going into the details, it is possible to obtain the super-graviton, the light-front super-membrane, the longitudinal five-brane and several properties of Schwarzschild black holes. Furthermore, the perturbative results of Matrix quantum mechanics can reproduce precisely all linearised classical supergravity interactions between arbitrary sources as well as the non-linear second order interactions in systems of two and three un-polarised gravitons, by one and two-loop calculations respectively. However, despite all of the above achievements, there are numerous problems that are still unanswered in the context of Matrix Theory, including the construction of the transverse five-brane, issues with compactifications on tori of $d > 5$, which lead to descriptions as complicated as the original M-theory, reproduction of general non-linear supergravity effects and the formulation of the theory on curved backgrounds (see [41] and references therein).

Quantisation of spherical membranes

A key concept that arises from the study of configurations in Matrix Theory, is that the geometry of the membrane world-volume is not conventional: It is described by non-commutative geometry, which is a generalisation of classical, or commutative, geometry, in much the same fashion as quantum mechanics generalises the phase space of classical mechanics. From the multitude of possible objects that can be reconstructed in the above context, we would like to focus on the compact super-membrane of spherical topology,

moving in non-compact space. These configurations have finite size and, therefore, finite energy but are unstable: They will eventually collapse into a black hole under the force of their surface tension, since there is nothing to prevent them from shrinking beyond their Schwarzschild radius. However, in the limit of very big (but not infinite) radius the time of collapse is very large and we can perform a semi-classical treatment. The first description of such a system in this context was by de Wit, Hoppe and Nicolai [45], who discretised (quantised) the spherical supermembrane in light-front co-ordinates, giving exactly the super matrix quantum mechanics that we have described. These authors found that in that gauge, the supermembrane has a residual invariance under area preserving diffeomorphisms on the world-volume. This group can be identified with $U(N)$ in the large- N limit. The essential point in that derivation was the construction of an exact correspondence between the functions on the spherical membrane and $U(N)$ matrices: Functions on the sphere, which are functions of the euclidean co-ordinates, can be described in Matrix Theory by the equivalent symmetrised polynomials in the generators of the N -dimensional representation of $SU(2)$. The appropriate matrix representation of a 2-sphere needed here and its higher dimensional generalisations will be the focus of the next section. In terms of the physics, after some finite time quantum effects will become important. The study of the behaviour of the membrane in the quantum domain and the resulting natural framework for studying black holes in Matrix Theory was investigated in [46].

1.4 Non-Commutative Geometry and Fuzzy Spheres

We have already discussed the necessity of introducing the concept of non-commutative geometry. The motivation was given by the Matrix Theory description of space-time by, generically, noncommuting matrix degrees of freedom. In this section, we will review the properties of the geometries that arise in the problem of the collapse of a compact, spherical membrane and its higher dimensional generalisations, namely fuzzy spheres of even and odd dimensionality.

1.4.1 The Fuzzy- S^2

Consider the algebra of complex-valued functions on the classical unit-sphere, $\mathcal{C}(S^2)$. This has a polynomial expansion given by

$$f(x^i) = f_0 + f_i x^i + \frac{1}{2} f_{ij} x^i x^j + \dots \quad (1.4.1)$$

where the indices run from $i = 1, 2, 3$ and the euclidean co-ordinates satisfy the constraint $x^i x^i = 1$. Note that the f tensors are traceless, due to the radial constraint, and symmetric, since the euclidean co-ordinates commute. We can perform a finite-dimensional truncation of this algebra, by constructing a sequence of non-commutative approximations to it. By

truncating all functions to the constant term, the algebra $\mathcal{C}(S^2)$ is reduced to the algebra of complex numbers $\hat{\mathcal{A}}_0(S^2) = \mathbb{C}$. The sphere is very poorly described and one can only distinguish a point. If we extend the truncation by also keeping terms linear in x^i , we end up with a four-dimensional vector space. In order to turn this vector space, $\hat{\mathcal{A}}_1(S^2)$, into an algebra we need to equip it with an appropriate product. That product is defined so that the algebra $\hat{\mathcal{A}}_1(S^2)$ is isomorphic to the algebra of 2×2 complex matrices $\text{Mat}_2(\mathbb{C})$, by setting $x^i = \sigma^i$, with σ^i the $SU(2)$ Pauli matrices. Once again, this description is not adequate: It only allows us to distinguish two points on the sphere, the north and the south pole, one for each of the eigenvalues of σ^3 , while the rest of the sphere is still indistinguishable. We denote these parts of the sphere as fuzzy. By incorporating the term which is quadratic in x^i we get a nine dimensional vector space, which can be equipped with a product, so that the set of functions $\hat{\mathcal{A}}_2(S^2)$ becomes equal to the algebra of 3×3 complex matrices, $\text{Mat}_3(\mathbb{C})$. This is achieved by identifying the x^i 's with the generators J^i of the 3-dimensional representation of the $SU(2)$ Lie algebra.

We can generalise the above procedure to suppressing the N th order in x 's. The number of independent components of completely symmetric, rank- r traceless tensors $f_{i_1 \dots i_r}$ is calculated by subtracting from the symmetric part the components that are obtained after taking a trace, *i.e.* the components of rank- $(r-2)$ symmetric tensors. Thus we will get that the number of independent components, for $r \geq 2$, is $\binom{r+2}{r} - \binom{r}{r-2} = 2r+1$. By summing over all ranks we get that the set of functions $\hat{\mathcal{A}}_n(S^2)$ is an $\sum_{r=0}^{N-1} (2r+1) = N^2$ dimensional vector space. We can therefore identify $\hat{\mathcal{A}}_n(S^2)$ with the algebra of $N \times N$ complex matrices $\text{Mat}_N(\mathbb{C})$, by replacing the initial x^i 's with the generators of the N -dimensional irreducible, spin- $\frac{n}{2}$ representation of the $SU(2)$ Lie algebra $[\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k$, where we have introduced $n = N - 1$, up to some normalisation factor. The radial constraint gives $\frac{1}{N}Tr(\alpha^i\alpha^i) = (N^2 - 1)$, where we have used the value for the quadratic Casimir of $SU(2)$, $\alpha^i\alpha^i = C \mathbb{1}_{N \times N} = (N^2 - 1) \mathbb{1}_{N \times N}$ and also $\frac{1}{N}Tr(\dots)$ in order to properly convert from matrices to euclidean co-ordinates and vice-versa. Agreement with the expected classical large- N limit fixes the normalisation to $X^i \equiv \frac{\alpha^i}{N}$. The latter gives the correct answer for the radial constraint $\frac{1}{N}Tr(X^iX^i) = 1 + \mathcal{O}(\frac{1}{N^2})$, up to $1/N$ corrections. This last result also urges us to take a closer look at the commutator

$$\begin{aligned} [X^i, X^j] &= \frac{1}{N^2} [\alpha^i, \alpha^j] \\ &= \frac{2i}{N^2} \epsilon^{ijk} \alpha^k \\ &= \frac{2i}{N} \epsilon^{ijk} X^k \end{aligned} \tag{1.4.2}$$

The commutator goes like $1/N$, *i.e.* the sphere becomes less fuzzy as N increases. We recover ordinary, commutative geometry as $N \rightarrow \infty$ [47]. It is clear that the ratio $1/N$ plays the same role as Planck's constant \hbar in quantum mechanics. However, we would like to note that the issue of convergence is very subtle and, even though for large- N we recover

the correct classical description in this example, that is not always the case as we will see very soon. What we have done here, can be viewed as a decomposition of the algebra $\hat{\mathcal{A}}_n(S^2)$ as a direct sum of irreducible representations of integer spin s , where $s = 0, \dots, n$, with unit multiplicity, *i.e.* $\hat{\mathcal{A}}_n(S^2) = \bigoplus_{s=0}^n V_s$.

1.4.2 The Fuzzy- S^{2k}

The correspondence between $\hat{\mathcal{A}}_n(S^2)$ and $\text{Mat}_N(\mathbb{C})$ points towards a structure that generalises to higher dimensions. The higher dimensional even-spheres are described by products of matrices that transform as vectors of the $SO(2k+1)$ group of S^{2k} and satisfy $X_\mu X_\mu = C \mathbb{1}_{N \times N}$, where $C = n(n+2k)$ is the quadratic Casimir of $SO(2k+1)$ [48, 49]. It is known from group theory that $SO(2k+1)$ representations can be put into 1–1 correspondence with Young diagrams, which are labelled by rows with respective lengths $\vec{r} = (r_1, r_2, \dots, r_k)$, obeying the relation $r_1 \geq r_2 \geq \dots \geq r_k$. Consider the vector space arising from the tensor product of B copies of the $(2k+1)$ -dimensional fundamental representation. The Young diagrams describe irreducible representations arising from a subspace of this vector space. If f_μ , with $\mu = 1, \dots, 2k+1$, are a set of basis vectors for the fundamental, we can write a basis vector for this tensor space as $f_{\mu_1} \otimes f_{\mu_2} \otimes \dots \otimes f_{\mu_B}$. A general vector can be described as sum over this basis $\sum_{\mu_1, \mu_2, \dots, \mu_B} A(\mu_1, \dots, \mu_B) f_{\mu_1} \otimes f_{\mu_2} \otimes \dots \otimes f_{\mu_B}$. The vectors of the irreducible representation are obtained by applying to these tensors A a tracelessness condition and a symmetrisation procedure corresponding to the Young diagram, which requires symmetrising along the rows and anti-symmetrising along the columns⁶. If we define the quantities

$$\begin{aligned} l_i &= r_i + k - i + \frac{1}{2} \\ m_i &= k - i + \frac{1}{2} \end{aligned} \tag{1.4.3}$$

where $i = 1, \dots, k$, the dimension of the irreducible representation can then be obtained via the neat formula

$$D(\vec{r}) = \prod_{i < j} \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \prod_i \frac{l_i}{m_i} \tag{1.4.4}$$

For the above representations constructed from tensor products of the $SO(2k+1)$ vector, the r_i 's are integers. Vectors with half-integer entries correspond to spinor representations, the dimensions of which are again described by the above expression (1.4.4).

The construction of the fuzzy- S^{2k} is then obtained by products of $N \times N$ matrices X_μ , with $\mu = 1, \dots, 2k+1$, which generate the full set of matrices. The X 's are defined by the

⁶The operator that encodes this symmetrisation procedure is called the Young symmetriser and is expressed as $\frac{1}{|S_R|} \frac{1}{|S_C|} \sum_{\sigma \in S_R} \sum_{\tau \in S_C} (-1)^\tau \sigma \tau$, where σ, τ are permutations in the symmetric groups S_R, S_C that act along the rows and the columns respectively and $|S_R|, |S_C|$ are their dimensions.

action of the respective Γ -matrix on the n -fold symmetric tensor product of the fundamental $2k$ -dimensional spinor V , $Sym(V^{\otimes n})$

$$X_\mu = \sum_{r=1}^n \rho_r(\Gamma_\mu) \quad (1.4.5)$$

where the operator $\rho_r(\Gamma_\mu)$ acts on the r -th copy of V in $Sym(V^{\otimes n})$. Different products of the X 's can be used to generate sets of matrices transforming under different representations of $SO(2k+1)$, with each tensor representation occurring once. Then the dimensions of these representations add up to N^2 , which is the size of the matrix algebra.

In the large- n limit one can see, after performing a counting, that the number of degrees of freedom described by the matrix algebra grows like $N \sim n^{k(k+1)}$. This is larger than the expected geometric degrees of freedom on the classical sphere, which go like n^{2k} . Indeed, in the large- n limit $\text{Mat}_N(\mathbb{C})$ actually approaches the algebra of functions on the coset $SO(2k+1)/U(k)$ [50]. If we wish to recover the algebra of functions on the classical manifold, the representations coming from the full matrix algebra need to be constrained to the ones corresponding to symmetric and traceless Young diagrams, *i.e.* Young diagrams for which only the first entry in the weight vector \vec{r} is nonzero, $r_i = 0$ for $i \neq 1$. In order to obtain this space, we need to project out all other representations, after which the new algebra will have a commutative but non-associative multiplication.

An example: The fuzzy- S^4

To make the above construction more concrete, we will describe the algebra for the case of the fuzzy four-sphere. The matrices X_μ act on vectors in the irreducible representation of $Spin(5)$ that we get by symmetrising the n th tensor power of the fundamental 4-dimensional spinor, $\vec{r} = (\frac{1}{2}, \frac{1}{2})$, corresponding to $\vec{r} = (\frac{n}{2}, \frac{n}{2})$. The dimension of the latter can be calculated, using (1.4.4), to be $N = \frac{1}{6}(n+1)(n+2)(n+3)$. The X_μ 's themselves transform as vectors of $SO(5)$, $\vec{r} = (1, 0)$. In order to better understand the correspondence between vectors in the irreducible representation and Young diagrams, consider the product of two matrices

$$\begin{aligned} X_{\mu_1} X_{\mu_2} &= \sum_{s_1=1}^n \rho_{s_1}(\Gamma_{\mu_1}) \sum_{s_2=1}^n \rho_{s_2}(\Gamma_{\mu_2}) \\ &= \sum_{s_1=s_2=s} \rho_s(\Gamma_{\mu_1} \Gamma_{\mu_2}) + \sum_{s_1 \neq s_2} \rho_{s_1}(\Gamma_{\mu_1}) \rho_{s_2}(\Gamma_{\mu_2}) \end{aligned} \quad (1.4.6)$$

For $\mu_1 \neq \mu_2$, and because of their symmetry properties⁷, each of the above sums can be put into 1 – 1 correspondence with the irreducible representations of $SO(5)$ labelled by the

⁷The first term is anti-symmetric under the exchange of $\mu_1 \leftrightarrow \mu_2$, while the second term is symmetric, since the ρ 's acting on $Sym(V^{\otimes n})$ commute.

Young diagrams $\vec{r} = (1, 1)$ and $\vec{r} = (2, 0)$ respectively. In such a manner, we can associate to each Young diagram an operator made of Γ -matrices acting in $Sym(V^{\otimes n})$. The most general irreducible representation V_{r_1, r_2} corresponding to a diagram with row lengths $\vec{r} = (r_1, r_2)$ can be written as

$$\sum_{\vec{s}} \rho_{s_1}(\Gamma_{\mu_1} \Gamma_{\mu_2}) \rho_{s_2}(\Gamma_{\mu_3} \Gamma_{\mu_4}) \dots \rho_{s_{r_2}}(\Gamma_{\mu_{2r_2-1}} \Gamma_{\mu_{2r_2}}) \rho_{r_2+1}(\Gamma_{\mu_{r_2+1}}) \dots \rho_{r_1}(\Gamma_{\mu_{r_1}}) \quad (1.4.7)$$

From the above it is also easy to check why representations of $SO(5)$ have Young diagrams with at most two rows: Any operator involving products of more than two Γ -matrices, can be re-written using the identity $\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$ in terms of fewer Γ 's. The complete set of irreducible representations of $SO(5)$ in $\text{Mat}_N(\mathbb{C})$ is given by summing over all $n \geq r_1 \geq r_2 \geq 0$. Upon doing this, one indeed recovers N^2 and the above operators give an $SO(5)$ -covariant basis for the matrix algebra. Therefore, we have found that

$$\text{Mat}_N(\mathbb{C}) = \bigoplus_{n \geq r_1 \geq r_2 \geq 0} V_{r_1, r_2} \quad (1.4.8)$$

In the large- n limit, the matrix algebra approaches the algebra of functions on the coset $SO(5)/U(2)$. The finite- n truncation of the algebra of functions on the classical four-sphere is obtained after projecting out all irreducible representations corresponding to diagrams with $r_2 \neq 0$. One can verify that the multiplication on this space is indeed non-associative at finite- n . There is also another description of the full matrix algebra at large- n in terms of a bundle over S^4 with fibre S^2 . This structure persists at finite- n , where the fuzzy- S^4 can be described as a fuzzy- S^2 fibre bundle over a ‘non-associative’ S^4 base space. This fibre bundle can also be understood as a $U(n+1)$ vector bundle over the ‘non-associative’ base [50]. Similar realisations of the fuzzy even-sphere algebra can be performed successfully for the fuzzy six-sphere and eight-sphere [48, 49].

1.4.3 The Fuzzy- S^{2k-1}

The extension of the construction to fuzzy odd-dimensional spheres is more complicated and involves reviewing some of the properties for $SO(2k)$, since this is now the appropriate isometry group. The tensor representations are once again labelled by integer valued weight vectors $\vec{r} = (r_1, \dots, r_k)$. However, r_k can now be either positive or negative. We will refer to representations with $r_k > 0$ as self-dual and those with $r_k < 0$ as anti-self-dual. The representations that are related by change of sign in r_k are conjugate. The dimension of these representations is calculated in a manner similar to $SO(2k+1)$, namely by defining

$$\begin{aligned} l_i &= r_i + k - i \\ m_i &= k - 1 \end{aligned} \quad (1.4.9)$$

and then by considering

$$D(\vec{r}) = \prod_{i < j} \frac{(l_i^2 - l_j^2)}{m_i^2 - m_j^2} \quad (1.4.10)$$

The matrix algebra will arise by considering products of matrices X_i . These satisfy $X_i X_i = C \mathbb{1}_{N \times N}$, where now $C = \frac{1}{2}(n+1)(n+2k-1)$ for $SO(2k)$, and are obtained by applying a projection to the matrices of a fuzzy even-sphere of one dimension higher. The matrices X_i are vectors of $SO(2k)$ and act on a reducible representation \mathcal{R}_n , obtained by taking a direct sum of two irreducible chiral representations of $SO(2k)$ \mathcal{R}_n^+ and \mathcal{R}_n^- . The construction with $r_k \neq 0$ once again involves applying the Young symmetrisers on traceless tensors. We will see how this works for the specific example of S^3 in a moment. In the large- n limit, the size of the full matrix algebra $End(\mathcal{R}_n)$ grows like $n^{(k-1)(k+2)}$ and remains non-commutative. However, $End(\mathcal{R}_n^\pm)$ do become commutative and approach the algebra of functions on the coset $\frac{SO(2k)}{U(k-1) \times U(1)}$. There is a description of this in terms of a $\frac{U(k)}{U(k-1) \times U(1)}$ fibre bundle over $SO(2k)/U(k)$. In this case, however, further interpretations of this structure will vary with dimensionality.

In order to attempt to recover the correct algebra of functions on the classical odd-sphere, one can define a projection to symmetric and traceless irreducible representations but also require invariance under a conjugation that exchanges positive and negative chiralities. We ought to note that the above general construction doesn't apply for the S^1 . This is a special case and, if defined via the above prescription, doesn't admit a classical limit, since the latter for the higher dimensional spheres depended on large irreducible representations for the SO groups [48, 49].

An example: The fuzzy- S^3

The fuzzy three-sphere algebra provides the best understood example of fuzzy odd-spheres [48, 49, 51]. It is obtained as a subspace of the fuzzy- S^4 by decomposing the symmetric tensor representations of $Spin(5)$ under $Spin(4)$ into a sum of reducible representations. The fuzzy- S^3 matrices are defined as a direct sum of irreducible representations of $Spin(4)$. Since $Spin(4) = SU(2) \times SU(2)$, the fundamental spinor V of $Spin(5)$ decomposes under $Spin(4)$ into irreducible representations labelled by spins (j_L, j_R) as $(1/2, 0) \oplus (0, 1/2) = P_+ V \oplus P_- V$. The P_\pm 's are positive and negative chirality projectors, defined for general k as $P_\pm = \frac{1}{2}(1 \pm \Gamma_{2k+1})$. We will call the direct sum of these irreducible representations $V = P_+ V \oplus P_- V = V_+ \oplus V_-$. The symmetric tensor representations of $Spin(4)$ that are needed, will be given, for every odd integer n , by restricting to a subspace of the fuzzy- S^4 , defined by the direct sum of the subspaces that we will call \mathcal{R}_n^\pm . The latter have $\frac{n+1}{2}$ factors of positive and $\frac{n-1}{2}$ factors of negative chirality and are irreducible representations of $SO(4)$. The total vector space is obtained by considering the direct sum $\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$. The projection operations in $End(Sym(V^{\otimes n}))$ that take us to \mathcal{R}_n^\pm are called $\mathcal{P}_{\mathcal{R}_n^\pm} =$

$((P_{\pm})^{\otimes \frac{n+1}{2}} (P_{\mp})^{\otimes \frac{n-1}{2}})_{sym}$ and ensure that we stay in that subspace. We can collect them into a single expression as $\mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^-}$. To summarise, the vector space we wish to consider is $\mathcal{R}_n = \mathcal{P}_{\mathcal{R}_n}(Sym(V^{\otimes n}))$.

The matrices are linear transformations in $End(\mathcal{R}_n)$

$$X_i = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \quad (1.4.11)$$

that map \mathcal{R}_n^+ to \mathcal{R}_n^- and vice-versa. This can be more precisely captured by saying that the X 's are sums of matrices in $Hom(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and $Hom(\mathcal{R}_n^-, \mathcal{R}_n^+)$, written as $X_i = \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-}$. Therefore, if the X 's are acting on the column \mathcal{R}_n with the \mathcal{R}_n^+ vectors arranged to be in the upper rows and \mathcal{R}_n^- on the lower ones, they should be matrices that are non-zero in the off-diagonal blocks. The same will hold for odd products of X 's. Even products will map \mathcal{R}_n^{\pm} back to \mathcal{R}_n^{\pm} and will be matrices in $End(\mathcal{R}_n^{\pm})$, with a block diagonal structure.

Based on taking products of matrices acting on \mathcal{R}_n we can construct a covariant $SO(4)$ basis, in terms of corresponding Young diagrams, in a fashion similar to that for the even-sphere cases. After a careful counting that we will not repeat here, the sum of the degrees of freedom from each of the four blocks for self-dual and anti-self-dual representations gives $N^2 = \frac{(n+1)^2(n+3)^2}{4}$, which is the correct answer for the dimensionality of the vector space, expected by using the formula (1.4.9) on \mathcal{R}_n [48]. However, in this case the co-ordinate matrices only generate a sub-algebra of $Mat_N(\mathbb{C})$. In the above mentioned counting, representations of the form $\mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r([\Gamma_i, \Gamma_j] P_-) \mathcal{P}_{\mathcal{R}_n^+}$ and $\mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r([\Gamma_i, \Gamma_j] P_+) \mathcal{P}_{\mathcal{R}_n^-}$ were included, which, while in $End(\mathcal{R}_n^{\pm})$, cannot be generated by products of X 's due to the nature of the projection. The sub-algebra generated by the co-ordinates, $\hat{\mathcal{A}}_N^{(g)}(S^3)$, should therefore be distinguished from the algebra $\hat{\mathcal{A}}_N(S^3) = Mat_N(\mathbb{C})$. The former contains symmetric representations with unit multiplicity, while the latter contains them with multiplicity two. The large- n behaviour of $End(\mathcal{R}_n^{\pm})$ approaches the algebra of functions on $\frac{SO(4)}{U(1) \times U(1)}$ and becomes commutative. This also has a simpler description in terms of functions on $S^2 \times S^2$. On the other hand, the $Hom(\mathcal{R}_n^{\pm}, \mathcal{R}_n^{\pm})$ approach a space of sections on $\frac{SO(4)}{U(1) \times U(1)}$, given by induced representations. We can attempt to recover the algebra of functions on the classical three-sphere, in the large- n limit, by restricting to representations with $r_2 = 0$ and also requiring that the matrices should be invariant under a conjugation that exchanges $\mathcal{R}_n^+ \leftrightarrow \mathcal{R}_n^-$ and $P_+ \leftrightarrow P_-$. However, even though the algebra is commutative and one recovers the correct number of degrees of freedom in this fashion, it becomes non-associative, with non-associativity persisting at large- n . This points to the need for a more sophisticated definition of the projection⁸. Overall similar steps can be followed in order to perform the construction for the fuzzy- S^5 [48, 49].

⁸More details on this issue and the description of an alternative product on the projected space of matrices are presented in App. E.

1.5 Non-Commutative gauge theory

We have already seen how Matrix Theory leads to introducing the notion of non-commutativity on the M-theory membrane and therefore on the D -brane world-volume via dimensional reduction. However, this is not the only scenario in which such effects can appear: Non-commutativity can also arise from open string theory. This should come as no surprise, since the effective world-volume theory is just re-capturing the virtual open string dynamics on the brane hyper-surface. Therefore, there should be an open string formulation of the same non-commutative phenomena. We will sketch how this works for flat D -branes. For a detailed treatment and review we refer to [52] and [53] respectively.

The bulk space-time field responsible for non-commutativity is the Kalb-Ramond field $B_{\mu\nu}$. In flat space a constant B -field is pure gauge and does not affect the closed string physics. In the presence of a flat D -brane, however, the gauge transformation for the B -field also acts on the world-volume gauge potential. We can, therefore, gauge it away and replace it by a background magnetic field. The bulk σ -model action for a world-sheet Σ of disc topology, having switched off the dilaton and for constant $B_{\mu\nu}$, is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{ij} \partial_a x^i \partial_b x^j - 2\pi i \alpha' B_{ij} \partial_a x^i \partial_b x^j) \quad (1.5.1)$$

with the boundary terms implicit up to a gauge transformation. Since the above action is quadratic, the world-sheet physics should be entirely determined by the propagator $\langle x^i(z) x^j(w) \rangle$. In order to recover the relevant quantity, we restrict to real z, w , which we will call t, s . After fixing the boundary conditions by varying the action, a short calculation yields the result for the boundary propagator

$$\langle x^i(t) x^j(s) \rangle = -\alpha' G^{ij} \log(t-s)^2 + \frac{i}{2} \theta^{ij} \epsilon(t-s) \quad (1.5.2)$$

where $\epsilon(t) = -1, 0, +1$ for $t < 0, t = 0, t > 0$ and

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} \right)_S^{ij} \quad (1.5.3)$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A^{ij} \quad (1.5.4)$$

The subscripts S, A denote the symmetric and anti-symmetric parts of the matrices respectively. We can now use the the standard relationship between world-sheet operators and space-time fields to interpret the object G^{ij} as the effective metric seen by the open strings on the D -brane world-volume. This is commonly referred to as the ‘open string metric’, to distinguish it from the space-time metric or ‘closed string metric’, g^{ij} . An equally simple interpretation can be given to the coefficient θ^{ij} : In CFT operator commutators can be calculated from the short distance behaviour of operator products by interpreting time

ordering as operator ordering. Then

$$[x^i(t), x^j(s)] = T(x^i(t-0)x^j(t) - x^i(t+0)x^j(t)) = i\theta^{ij} \quad (1.5.5)$$

with θ^{ij} a real constant antisymmetric matrix. Therefore, the end-points of the open string indeed live on a non-commutative space.

A similar process can be repeated for more general operators, carrying momenta p, q

$$V_p(t)V_q(s) = (t-s)^{2\alpha' G^{ij} p_i q_j} e^{-\frac{i}{2}\theta^{ij} p_i q_j} V_{p+q}(s) \quad (1.5.6)$$

Now by taking the limit $g_{ij} \rightarrow 0$, one essentially decouples closed strings and finds that $G^{ij} = 0$ and $\theta^{ij} = 2\pi\alpha'(B_{ij})^{-1}$. The dependence on the world-sheet coordinates s, t drops out and the OPE reduces to a conventional multiplication law, which by linearity extends to the product of two general functions and which we will denote with a \star -product

$$f(x) \star g(x) = e^{\frac{i}{2} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \zeta^j}} f(x + \xi) g(z + \zeta) |_{\xi=\zeta=0} \quad (1.5.7)$$

It turns out that, given the algebra (1.5.5), the above product provides the unique deformation of the algebra of functions on \mathbb{R}^n to a non-commutative, associative algebra \mathcal{A} . The expression (1.5.7) is known in the literature as the Moyal product and is used to define non-commutative gauge theory. The latter is defined by using the same formulae for the gauge transformations and the field strength \hat{F} as for ordinary gauge theory, but with ordinary matrix multiplication replaced by the \star -product. We have therefore recovered non-commutative gauge theory from open string theory. We should note that a non-commutative version of the DBI action can be obtained in a way similar to the one leading to the ordinary commutative case, by replacing products with \star -products, the pullback metric with G_{ij} and setting $B_{ij} = 0$. The only thing that we still need to determine is the pre-factor of the action and therefore the non-commutative coupling constant, which will involve the ‘open string coupling constant’ G_s . A subtlety, in this case, lies with the fact that the correct decoupling limit is no longer taken by just sending $\ell_s \rightarrow 0$: One needs to keep G^{ij} and θ^{ij} fixed.

A natural question that arises is how to reconcile this result and the fact that we have thus far been extensively describing how open strings give rise to conventional gauge theory on the world-volume. After all both commutative and non-commutative versions of the DBI were derived from the same starting point. It was realised in [52] that the two descriptions are capturing the same physics and we should somehow be able to map one to the other. The punch-line is that the confusion appears because of the definition of gauge invariance: In each case it arises from different choices of world-sheet regularisation. One could even make an ‘intermediate’ choice, which would lead to a non-commutative theory with a different value for the parameter θ . The fields in the resulting actions are then related by what is called a Seiberg-Witten map. This can be seen from the open string theory picture by

considering part of the B -field as a perturbation Φ . One then recovers the expressions

$$\left(\frac{1}{G + 2\pi\alpha'\Phi} \right)_S = \left(\frac{1}{g + 2\pi\alpha'B} \right)_S \quad (1.5.8)$$

$$\left(\frac{1}{G + 2\pi\alpha'\Phi} \right)_A = \left(\frac{1}{g + 2\pi\alpha'B} \right)_A - \frac{\theta}{2\pi\alpha'} \quad (1.5.9)$$

The Seiberg-Witten map also fixes the relationship between the gauge couplings

$$G_s = g_s \left(\frac{\det(G + 2\pi\alpha'\Phi)}{\det(g + 2\pi\alpha'B)} \right)^{1/2} \quad (1.5.10)$$

We, therefore, have a complete description of the effective action by replacing \hat{F} with $\hat{F} + \Phi$. The result is

$$S_{NCDBI} = \frac{1}{G_s(2\pi)^p \ell_s^{p+1}} \int Tr \sqrt{\det(G + 2\pi\alpha'(\hat{F} + \Phi))} \quad (1.5.11)$$

which lets us interpolate between the commutative ($\Phi = B$) and non-commutative ($\Phi = 0$) pictures, using the appropriate θ -dependent \star -product, appropriate G_s and effective metric G^{ij} .

1.6 The D -brane ‘dielectric’ effect

The appearance of non-commutative geometry on the world-volume of multiple D -branes allows for the appearance of a variety of very interesting physical effects. The most celebrated of this kind is Myers’ ‘dielectric’ effect [28, 29], which finds applications in a variety of contexts including giant gravitons and supersymmetric gauge theories via the AdS/CFT duality [54–57]. This is analogous to the electromagnetic dielectric effect, in which an external field induces the separation of charges in a neutral material. The separation causes the material to then polarise and carry a dipole, or possibly higher multi-pole moment. The brane realisation of this phenomenon is related to the inclusion of some background RR flux under which the Dp -branes are regarded as neutral, $F^{(n)}$ with $n > p + 2$. There are extra terms induced in the scalar potential and one finds new extrema, corresponding to noncommuting expectation values for the transverse scalar fields. Thus, the external field causes the brane to ‘polarise’ into a non-commutative, higher dimensional world-volume geometry.

We start with the scalar potential that one obtains from the world-volume action for N Dp -branes, by considering a flat space configuration with all other fields set to zero. The result for this is coming exclusively from the DBI term (1.2.3) and yields to lowest order in the parameter λ

$$V = T_p S Tr \sqrt{\det(Q^i_j)} = NT_p - \frac{T_p \lambda^2}{4} Tr[\Phi_i, \Phi_j][\Phi_i, \Phi_j] \quad (1.6.1)$$

It is obvious that there is a non-trivial set of static extrema to this potential, by considering constant, commuting matrices Φ . In that case, the transverse scalars can be simultaneously diagonalised and interpreted as describing the separations of N branes in the transverse space. The configuration is in equilibrium, since the system of parallel branes is half-BPS.

Now we will study a slightly different configuration and focus on N $D0$ branes for concreteness. We introduce to the action the non-trivial RR flux $F_{tijk}^{(4)} = -2f\epsilon_{ijk}$, where f is constant and $i, j, k \in \{1, 2, 3\}$. The flux will couple to commutators of Φ 's via the CS part of the action (1.2.4) and $C^{(3)}$. However, since the RR fields depend on all space-time co-ordinates, we need to use the non-abelian Taylor expansion to have an explicit dependence only in the world-volume co-ordinate t . The result, after performing an integration by parts and to lowest order in λ , is

$$\frac{i}{6}\mu_0 \operatorname{Tr}([\Phi_i, \Phi_j]\Phi_k)F_{tijk}^{(4)}(t) \quad (1.6.2)$$

The scalar potential coming from the DBI part of the action will remain the same. Combining the two terms, we get for

$$V(\Phi) = NT_0 - \frac{\lambda^2 T_0}{4} \operatorname{Tr}([\Phi_i, \Phi_j]^2) - \frac{i}{6}\lambda^2\mu_0 \operatorname{Tr}([\Phi_i, \Phi_j]\Phi_k)F_{tijk}^{(4)}(t) \quad (1.6.3)$$

Since in the static gauge $\mu_0 = T_0$, extremisation of this potential will yield the equation

$$[[\Phi_i, \Phi_j], \Phi_k] = -if\epsilon_{ijk}[\Phi_j, \Phi_k] \quad (1.6.4)$$

We will study two simple solutions of the above. The commuting Φ 's still solve the e.o.m. with a value for the potential $V_0 = NT_0$. However, if we instead use the Ansatz $\Phi_i = \frac{f}{2}\alpha_i$, where the α 's are the generators of an N -dimensional representation of the $SU(2)$ Lie algebra $[\alpha_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k$, we recover another solution. We have thus identified the transverse scalars with the co-ordinates on a fuzzy- S^2 . If we choose the irreducible representation, for which $\operatorname{Tr}(\alpha_i\alpha_i) = N(N^2 - 1)$, the potential will take the value

$$V_N = NT_0 \left(1 - \frac{\lambda^2 f^2}{6}(N^2 - 1) \right) \quad (1.6.5)$$

This is clearly lower than that for the commuting configuration. We could also have chosen reducible $N \times N$ representations of $SU(2)$ by considering a composition of lower dimensional representations. However, these will always yield smaller values of $\operatorname{Tr}(\alpha_i\alpha_i)$ and the irreducible fuzzy sphere configuration will be the ground state of the system. We recover a static fuzzy- S^2 configuration, stabilised at a physical radius given by the fuzzy sphere relation

$$R_{phys}^2 = \frac{\lambda^2}{N} \operatorname{Tr}(\Phi_i\Phi_i) = \lambda^2 N^2 \frac{f^2}{4} \left(1 - \frac{1}{N^2} \right) \quad (1.6.6)$$

The whole construction is very reminiscent of the Matrix Theory system of [46], in which there was a $D0$ - $D2$ bound state, although now studied in the DBI context. It is

interesting to observe that there also exists a higher dimensional description of the same non-commutative solution in terms of an abelian spherical $D2$ world-volume theory with N units of magnetic flux $F_{\theta\phi} = \frac{N}{2} \sin \theta$, corresponding to N $D0$ s dissolved inside the $D2$. One also needs to include the background RR four-form flux. We will not reproduce the whole calculation here, mainly because the main bulk of this thesis will be addressing very similar issues. The main point is that in the large- N limit, one finds perfect agreement between the two descriptions up to $\frac{1}{N^2}$ corrections. This common description of the physics in terms of two different world-volume theories will carry on for most of the examples that we will encounter. We will be referring to the higher-dimensional brane picture as *macroscopic* and the lower dimensional one as *microscopic* from now on. Of course, one should also investigate the regimes of validity of the two descriptions. Naively, the $D2$ calculations should be valid for $R_{phys} \gg \ell_s$, while the $D0$ ones for $R_{phys} \ll \ell_s$. However, upon requiring that the transverse scalar field commutators are small in the DBI scalar potential, so that the non-abelian Taylor expansion of the square root converges rapidly, one finds that the microscopic constraint is just $R_{phys} \ll \sqrt{N} \ell_s$. Therefore in the large- N limit there also is a significant overlap of the regimes of validity.

1.6.1 The static BIonic configurations

The above ideas can be extended to describe a class of static, orthogonally intersecting branes in Dirac-Born-Infeld theory, which also have both microscopic and macroscopic interpretations. From the higher dimensional point of view, these configurations have been known for some time in the form of spike solutions on the world-volume of flat branes, going under the name of BIons [18, 26, 27, 58]. For the simplest system of a $D3$ brane, these are described in terms of semi-infinite fundamental strings and/or D -strings extending out of the $D3$ world-volume in a transverse direction. There are both gauge fields and transverse scalars excited. The former correspond to electric/magnetic point charges on the brane, while the latter describe the deformation of the world-volume geometry. These solutions are BPS and seem to have a wide range of validity, even near the core of the spike where the fields are no longer slowly varying. The microscopic description of the magnetically charged BIon, is carried out by considering the non-abelian DBI theory of N D -strings, with a transverse scalar Ansatz that employs the fuzzy- S^2 $N \times N$ matrices in the irreducible representation, just as in the dielectric effect, but without any non-trivial background fields. An analysis of the world-volume supersymmetry gives a condition which can be recognised as the Nahm equation [59]. Solving this condition one recovers a simple solution for the radial profile of the form $R(\sigma) \sim \pm \frac{\lambda N}{\sigma - \sigma_\infty} (1 - 1/N^2)^{1/2}$, which describes a semi-infinite funnel with a fuzzy- S^2 cross-section that blows up into a larger flat brane/anti-brane of co-dimension two at $\sigma = \sigma_\infty$. These are BPS solutions of the full non-abelian DBI action. In the large- N limit one indeed recovers all the correct RR couplings, low energy dynamics and energy values. The regimes of validity of the two descriptions overlap in the same fashion

as was argued for the duality in the dielectric effect: The D -string description is valid for $R \ll \sqrt{N}\ell_s$, while the $D3$ for $R \gg \ell_s$. A very interesting fact is that this duality provides a physical realisation of the Nahm transform of the moduli space of BPS magnetic monopoles [29, 60]. Higher dimensional BIonic generalisations using higher dimensional even-spheres have also been considered in the literature [61, 62].

1.6.2 The time-dependent spherical configurations

Whilst we have a good description of static space-times and D -brane systems, only relatively recently has research made progress on the study of time-dependent processes. This is due to technical difficulties involved in formulating CFTs in time dependent backgrounds as well as in discovering more complicated time-dependent supergravity solutions. However, the study of time-dependence is imperative in the development of String Theory and a thorough understanding of this will allow for a complete description of cosmological evolution and black hole evaporation. It could also lead to new insights, in much the same way as the study of Quantum Field Theory on more general backgrounds resulted in bringing to light new effects like Hawking and Unruh radiation. Therefore, another natural and important path of investigation is that of the time-dependent collapse and overall fate of a compact spherical D -brane, via the study of the full DBI equations. The initial setup is similar to the one for the dielectric effect, with the difference lying in the absence of the background RR-flux. Therefore the brane will not be stabilised and will tend to collapse under its own tension. The investigation of the equations of motion yields a radial profile described by Jacobi elliptic functions [63]. There still exists a duality between the two pictures and an overlapping regime of validity. An extra complication arises from the necessity to take into account quantum effects in the regimes where the physical radius becomes very small.

In the rest of this thesis, we will be predominantly extending the study of the time-dependent dynamics of the above described spherical brane configurations to diverse dimensionalities, mainly from the microscopic perspective. In Ch. 2 we will see how the former are naturally related to the static BIonic intersections using the language of Riemann surfaces and the Jacobi Inversion problem. Ch. 3 deals with extending the classical agreement between the two pictures to the level of quadratic fluctuations for the fuzzy- S^2 . We will also describe the small R regime, in a certain scaling limit. In Ch. 4 we will elaborate on the $1/N^2$ corrections coming from the implementation of the STr prescription for the non-abelian DBI. Note that in all previous discussions these modifications were ignored, since we were considering the large- N limit. We will propose a full formula for obtaining these corrections for S^2 and study how they affect the dynamical evolution of our physical system. Ch. 5 deals with the properties of the microscopic D -brane action with a transverse co-ordinate matrix Ansatz corresponding to fuzzy odd-spheres, namely the fuzzy- S^3 and S^5 . The latter have a much more complicated structure than their even-dimensional counterparts. Finally, we close with conclusions and outlook. The material and results

presented throughout this work are based on [64–67].

CHAPTER 2

FUZZY SPHERES AND LARGE-SMALL DUALITIES

Fuzzy spheres of two, four, six dimensions [47, 48, 50, 68–73] arise in a variety of related contexts [46, 69, 74]. On the one hand they describe the cross-sections of fuzzy funnels appearing at the intersection of $D1$ -branes with $D3$, $D5$ or $D7$ -branes of Type IIB string theory [60–62, 75, 76]. In this context it is of interest to follow the spatial evolution of the size r of the fuzzy sphere as a function of the co-ordinate σ along the D -string. At the location of the higher dimensional brane, the cross-section of the funnel blows up. These equations for the funnel which arise either from the D -string or the $D(2p+1)$ -brane world-volume, can be generalised to allow for time-dependence as well as spatial dependence. The purely time-dependent solutions are relevant to the case of spherical bound states of $D0$ and $D2p$ -branes of Type IIA string theory.

In this chapter, we find that in the case of the fuzzy 2-sphere, there are purely spatial and purely time-dependent solutions described in terms of Jacobi elliptic functions. The spatial and time-profiles are closely related and the relation follows from an $r \rightarrow \frac{1}{r}$ duality. It is natural to introduce a complex variable $u_1 = \sigma - it$. For solutions described in terms of elliptic functions, the inversion symmetry is related to the property of complex multiplication $u_1 \rightarrow iu_1$. The periodic spatial solutions describe a configuration of alternating branes and anti-branes. At the location of the brane or anti-brane, the radius r of the funnel blows up. This is a well-understood blow-up, expected from the geometry of a 1-brane forming a 3-brane. The periodic solutions in time describe collapse followed by expansion of 2-branes. The collapse point is *a priori* a much more mysterious point, where the size of the fuzzy sphere is sub-stringy. Nevertheless the $r \rightarrow \frac{1}{r}$ duality, following from the equations of the Born-Infeld action, implies that the zeroes of the time evolution are directly related to the blow-up in the spatial profile.

In the case of the fuzzy 4-sphere, the functions defining the dynamics are naturally related to a genus 3 hyper-elliptic curve. Using the conservation laws of the spatial or time evolution, the time elapsed or distance along the 1-brane can be expressed in terms of an integral of a holomorphic differential on the genus 3 hyper-elliptic curve. The upper limit of the integral is the radius r . Inverting the integral to express r in terms of $u_1 = \sigma - it$ is a problem which can be related to the Jacobi Inversion problem, with a constraint. Because of the symmetries of the genus 3 curve, it can be mapped holomorphically to a genus 1

and a genus 2 curve. The genus 2 curve can be further mapped to a pair of genus 1 curves. The Jacobi Inversion problem expressed in terms of the genus 2 variables requires the introduction of a second complex variable u_2 and we find that there is a constraint which relates u_2 to u_1 . As a result, an implicit solution to the constrained Jacobi Inversion problem can be given in terms of ordinary (genus 1) Jacobi elliptic functions. The solution is implicit in the sense that the constraint involved is transcendental and is given in terms of elliptic functions. We give several checks of this solution, including a series expansion and calculations of the time of collapse or distance to blow-ups. The symmetries which allow the reduction of the problem to one involving lower genus Riemann surfaces also provide dualities of the type $r \rightarrow \frac{1}{r}$ which relate poles to zeroes.

We will also extend some of these discussions to the fuzzy 6-sphere. The space and time-dependence are related to integrals of a holomorphic differential on a genus 5 Riemann surface. A simple transformation relates the problem to genus 3. But we have not found a further reduction to genus one. The solution $r(t, \sigma)$ can still be related to a constrained Jacobi Inversion problem, which can be solved in terms of genus 3 Riemann theta functions. As far as large-small symmetries are concerned, the story is much the same as for S^4 in the limit of large ‘initial’ radius r_0 . In the time-dependent problem by ‘initial’ radius we mean the point where the radial velocity is zero. In the spatial problem, it is the place where the $\frac{dr}{d\sigma} = 0$. For general r_0 there are still inversion symmetries of the type $(1+r^4) \rightarrow (1+r^4)^{-1}$, but they involve fourth roots when expressed in terms of r , so are not as useful.

2.1 Space and Time-Dependent Fuzzy- S^2

The static system consisting of a set of N D -strings ending on an orthogonal $D3$ has been thoroughly studied [60, 61]. There exist two dual descriptions of the intersection at large- N , one from the $D1$ and one from the $D3$ world-volume point of view. In the $D1$ -picture it is described as a funnel of increasing radius as we approach the $D3$ brane, where the D -strings expand into a fuzzy- S^2 . In the $D3$ -picture the world-volume solution includes a BPS magnetic monopole and the Higgs field is interpreted as a transverse spike. Although the $D1$ picture is valid far from the $D3$ and the $D3$ picture close to it, there is a significant region of overlap which validates the duality. Here we will enlarge this discussion by lifting the static condition.

2.1.1 Non-abelian DBI description of non-static $D1 \perp D3$ funnels

We begin by considering the non-abelian DBI action of N D -strings in a flat background and with the gauge fields set to zero

$$S_{DBI}^{D1} = -T_1 \int d^2\sigma STr \sqrt{-\det \left(\eta_{ab} + \lambda^2 \partial_a \Phi_i Q_{ij}^{-1} \partial_b \Phi_j \right) \det(Q_{ij})} \quad (2.1.1)$$

where a, b world-volume indices, the Φ 's are world-volume scalars, $\lambda \equiv 2\pi\alpha' = 2\pi\ell_s^2$ and we recall that

$$Q_{ij} \equiv \delta_{ij} + i\lambda[\Phi_i, \Phi_j]$$

The expansion of this to leading order in λ yields the action

$$S_{DBI}^{D1} \simeq -T_1 \int d^2\sigma \left(N + \frac{\lambda^2}{2} STr \left(\partial^a \Phi_i \partial_a \Phi_i + \frac{1}{2} [\Phi_i, \Phi_j][\Phi_j, \Phi_i] \right) + \dots \right) \quad (2.1.2)$$

and the following equations of motion at lowest order, which are the Yang-Mills equations

$$\partial^a \partial_a \Phi_i = [\Phi_j, [\Phi_j, \Phi_i]] \quad (2.1.3)$$

We will consider the space-time-dependent Ansatz

$$\Phi_i = \hat{R}(\sigma, \tau) \alpha_i, \quad i = 1, 2, 3 \quad (2.1.4)$$

where the α_i 's are generators of the irreducible $N \times N$ matrix representation of the $SU(2)$ algebra

$$[\alpha_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k \quad (2.1.5)$$

with quadratic Casimir $\sum_{i=1}^3 (\alpha_i)^2 = C \mathbb{1}_{N \times N} = (N^2 - 1) \mathbb{1}_{N \times N}$. The resulting scalar field configuration describes a non-commutative fuzzy- S^2 with physical radius

$$R_{ph}^2(\sigma, \tau) = \frac{\lambda^2}{N} Tr[\Phi_i(\sigma, \tau)\Phi_i(\sigma, \tau)] = \frac{\lambda^2 C}{N} \hat{R}^2 \quad (2.1.6)$$

By replacing the Ansatz (2.1.4) into (2.1.1) we get the non-linear action

$$S = -T_1 \int d^2\sigma STr \sqrt{1 + \lambda^2 C \dot{\hat{R}}'^2 - \lambda^2 C \dot{\hat{R}}^2} \sqrt{1 + 4\lambda^2 C \hat{R}^4} \quad (2.1.7)$$

By varying this with respect to \hat{R} we recover the full equations of motion. Ignoring corrections that come from the application of the symmetrised-trace prescription, which are sub-leading at large- N [63], these are given by

$$2\lambda^2 C \dot{\hat{R}} \hat{R}' \dot{\hat{R}}' + \hat{R}''(1 - \lambda^2 C \dot{\hat{R}}^2) - \ddot{\hat{R}}(1 + \lambda^2 C \hat{R}'^2) = 8\hat{R}^3 \left(\frac{1 + \hat{R}'^2 \lambda^2 C - \dot{\hat{R}}^2 \lambda^2 C}{1 + 4\lambda^2 C \hat{R}^4} \right) \quad (2.1.8)$$

We can convert the above formula to dimensionless variables by considering the rescalings

$$r = \sqrt{2\lambda\sqrt{C}}\hat{R}, \quad \tilde{\tau} = \sqrt{\frac{2}{\lambda\sqrt{C}}}\tau, \quad \tilde{\sigma} = \sqrt{\frac{2}{\lambda\sqrt{C}}}\sigma \quad (2.1.9)$$

which will then imply

$$\begin{aligned}
 r^4 &= 4\lambda^2 C \hat{R}^4, \quad \left(\frac{\partial r}{\partial \tilde{\tau}} \right)^2 = \lambda^2 C \left(\frac{\partial \hat{R}}{\partial \tau} \right)^2, \quad \left(\frac{\partial r}{\partial \tilde{\sigma}} \right)^2 = \lambda^2 C \left(\frac{\partial \hat{R}}{\partial \sigma} \right)^2 \\
 \frac{\partial^2 r}{\partial \tilde{\tau}^2} &= \frac{1}{4} \left(\frac{\partial^2 \hat{R}}{\partial \tau^2} \right) (4\lambda^2 C)^{3/4}, \quad \frac{\partial^2 r}{\partial \tilde{\sigma}^2} = \frac{1}{4} \left(\frac{\partial^2 \hat{R}}{\partial \sigma^2} \right) (4\lambda^2 C)^{3/4} \\
 \frac{\partial^2 r}{\partial \tilde{\sigma} \partial \tilde{\tau}} &= \frac{1}{4} \left(\frac{\partial^2 \hat{R}}{\partial \sigma \partial \tau} \right) (4\lambda^2 C)^{3/4}
 \end{aligned} \tag{2.1.10}$$

The simplified DBI equations of motion can then be written in a Lorentz-invariant form

$$\partial_\mu \partial^\mu r + (\partial_\mu \partial^\mu r) (\partial_\nu r) (\partial^\nu r) - (\partial_\mu \partial^\nu r) (\partial_\nu r) (\partial^\mu r) = 2 r^3 \left(\frac{1 + (\partial_\mu r) (\partial^\mu r)}{1 + r^4} \right) \tag{2.1.11}$$

where μ and ν can take the values $\tilde{\sigma}, \tilde{\tau}$. Further aspects of Lorentz invariance and boosted solutions are discussed in App. A.1.

One can also write down the re-scaled action and energy density of the configuration, by making use of the dimensionless variables with the dots and primes implying differentiation by the re-scaled time and space respectively

$$\begin{aligned}
 \tilde{S} &= - \int d^2 \sigma \sqrt{(1 + r'^2 - \dot{r}^2)(1 + r^4)} \\
 E &= (1 + r'^2) \frac{\sqrt{1 + r^4}}{\sqrt{1 + r'^2 - \dot{r}^2}}
 \end{aligned} \tag{2.1.12}$$

2.1.2 The dual picture

We will now switch to the dual picture. The abelian DBI action for a $D3$ -brane with a general gauge field, a single transverse scalar in the x^9 direction, in a flat background is

$$S_{DBI}^{D3} = -T_3 \int d^4 \sigma \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \Phi \partial_b \Phi + \lambda F_{ab})} \tag{2.1.13}$$

The determinant, for a general gauge field, can be calculated and gives

$$\begin{aligned}
 S_{DBI}^{D3} &= -T_3 \int d^4 \sigma \sqrt{1 + \lambda^2 |\vec{B}|^2 + \lambda^2 (\vec{\nabla} \Phi)^2 + \lambda^4 (\vec{B} \cdot \vec{\nabla} \Phi)^2 - \lambda^2 |\vec{E}|^2 - \lambda^2 \dot{\Phi}^2} \\
 &\quad \sqrt{-\lambda^4 |\vec{E} \times \vec{\nabla} \Phi|^2 - \lambda^4 \dot{\Phi}^2 |\vec{B}|^2 - \lambda^4 (\vec{E} \cdot \vec{B})^2 + 2\lambda^4 \dot{\Phi} \vec{\nabla} \Phi \cdot (\vec{B} \times \vec{E})}
 \end{aligned} \tag{2.1.14}$$

from which one can derive the spherically symmetric equations of motion, in the absence of electric fields and with a single, radial component for the magnetic field on the $D3$, given

by the expression $\vec{B} = \mp \frac{N}{2R_{D3}} \hat{R}_{D3}$

$$\lambda\Phi''(1 - \lambda^2\dot{\Phi}^2) - \lambda\ddot{\Phi}(1 + \lambda^2\Phi'^2) + 2\lambda^3\Phi'\dot{\Phi}\dot{\Phi}' = -8R_{D3}^3\lambda\Phi' \frac{(1 + \lambda^2\Phi'^2 - \lambda^2\dot{\Phi}^2)}{(4R_{D3}^4 + \lambda^2N^2)} \quad (2.1.15)$$

Note that for this configuration, the field Φ only depends on the radial $D3$ co-ordinate R_{D3} and time t . Expression (2.1.15) then looks similar to (2.1.8) and we can show that it is indeed the same, when written in terms of $D1$ world-volume quantities. We consider total differentials of the fuzzy sphere physical radius $R_{ph}(\sigma, \tau)$

$$dR_{ph} = \frac{\partial R_{ph}}{\partial \sigma} \Big|_{\tau} d\sigma + \frac{\partial R_{ph}}{\partial \tau} \Big|_{\sigma} d\tau \quad (2.1.16)$$

and recover for constant R_{ph} and τ respectively

$$\frac{\partial \sigma}{\partial \tau} \Big|_{R_{ph}} = -\frac{\dot{R}_{ph}}{R'_{ph}} = \lambda\dot{\Phi}, \quad \frac{\partial \sigma}{\partial R_{ph}} \Big|_{\tau} = \frac{1}{R'_{ph}} = \lambda\Phi' \quad (2.1.17)$$

where we are making use of the identifications $R_{ph} = R_{D3}$, $\tau = t$ and $\sigma = \lambda\Phi$. Then the second order derivatives of σ are

$$\frac{\partial}{\partial R_{ph}} \Big|_{\tau} \frac{\partial \sigma}{\partial \tau} \Big|_{R_{ph}} = \lambda\dot{\Phi}', \quad \frac{\partial}{\partial \tau} \Big|_{R_{ph}} \frac{\partial \sigma}{\partial \tau} \Big|_{R_{ph}} = \lambda\ddot{\Phi} \quad \text{and} \quad \frac{\partial}{\partial R_{ph}} \Big|_{\tau} \frac{\partial \sigma}{\partial R_{ph}} \Big|_{\tau} = \lambda\Phi'' \quad (2.1.18)$$

The $D1$ -brane solution $R_{ph}(\sigma, t)$ can be inverted to give $\sigma(R_{ph}, t)$. By employing the following relations

$$\frac{\partial f(\sigma(R_{ph}, t), t)}{\partial t} \Big|_{R_{ph}} = \frac{\partial f}{\partial t} \Big|_{\sigma} + \frac{\partial f}{\partial \sigma} \Big|_t \frac{\partial \sigma}{\partial t} \Big|_{R_{ph}} \quad \text{and} \quad \frac{\partial f(\sigma(R_{ph}, t), t)}{\partial R_{ph}} \Big|_t = \frac{\partial f}{\partial \sigma} \Big|_t \frac{\partial \sigma}{\partial R_{ph}} \Big|_t \quad (2.1.19)$$

we have

$$\begin{aligned} \lambda\Phi'' &= -\frac{R''_{ph}}{R'^3_{ph}}, & \lambda\ddot{\Phi} &= -\frac{1}{R'^2_{ph}} \left(R'_{ph}\ddot{R}_{ph} - 2\dot{R}_{ph}\dot{R}'_{ph} + \dot{R}^2_{ph} \frac{R''_{ph}}{R'_{ph}} \right) \\ \text{and} \quad \lambda\dot{\Phi}' &= -\frac{1}{R'^2_{ph}} \left(\dot{R}'_{ph} - \dot{R}_{ph} \frac{R''_{ph}}{R'_{ph}} \right) \end{aligned} \quad (2.1.20)$$

By replacing these into (2.1.15), one recovers the exact non-linear equations of motion (2.1.8) in terms of the physical radius R_{ph} . This guarantees that any space-time-dependent solutions of (2.1.8) will have a corresponding dual solution on the $D3$ side.

2.1.3 Arrays of branes in space and Collapse/Expansion in time-dependence

We now restrict to purely time-dependent solutions of eq. (2.1.11). The resulting DBI equations of motion are identical to those coming from a Lagrangean which describes a

set of N $D0$ s, expanded into a fuzzy- S^2 . This configuration also has an equivalent dual DBI description in terms of a spherical $D2$ -brane with N -units of magnetic flux [63]. To simplify the notation, the re-scaled variables $\tilde{\tau}, \tilde{\sigma}$ of (2.1.9) will be called t, σ . Then the conserved energy density (2.1.12) (or energy in the $D0$ - $D2$ context) at large N is

$$E = \frac{\sqrt{1+r^4}}{\sqrt{1-\dot{r}^2}} \quad (2.1.21)$$

If r_0 is the initial radius of the collapsing configuration where $\dot{r} = 0$, $E = \sqrt{1+r_0^4}$ and we get

$$\dot{r}^2 = \frac{r_0^4 - r^4}{1 + r_0^4} \quad (2.1.22)$$

This allows us to write

$$\mp \int_0^t dt = \int_{r_0}^r \frac{\sqrt{1+r_0^4}}{\sqrt{r^4 - r_0^4}} dr \quad (2.1.23)$$

which can be inverted to give

$$r(t) = \pm r_0 C n \left(\tilde{t}, \frac{1}{\sqrt{2}} \right) \quad (2.1.24)$$

where $\tilde{t} = \frac{\sqrt{2}r_0 t}{\sqrt{r_0^4 + 1}}$. There is a choice of sign involved in going from (2.1.22) to (2.1.23). We will follow the convention that a collapsing sphere has negative velocity, while an expanding one positive. An analytic derivation of the above using the technology of elliptic functions is given App. B. Such solutions were first described in [77] and more recently in [46, 63, 78, 79].

The function $r(t)$ describes a $D2$ -brane of radius starting at $r = r_0$, at $t = 0$. It decreases to zero, then goes negative down to a minimum $-r_0$ and then increases back through zero to the initial position. The cycle is then repeated (see Fig. 1). The region of negative r is somewhat mysterious, but we believe the correct interpretation follows if, elaborating on (2.1.6), we define the physical radius as

$$R_{ph} = +\frac{\lambda}{\sqrt{N}} \sqrt{Tr(\Phi_i^2)} = \frac{\sqrt{\lambda} C^{\frac{1}{4}} r}{N \sqrt{2}} \quad (2.1.25)$$

in the region of positive r and as

$$R_{ph} = -\frac{\lambda}{\sqrt{N}} \sqrt{Tr(\Phi_i^2)} = -\frac{\sqrt{\lambda} C^{\frac{1}{4}} r}{N \sqrt{2}} \quad (2.1.26)$$

in the region of negative r . This guarantees that R_{ph} remains positive. The change in sign at 0 should not be viewed as a discontinuity that invalidates the use of the derivative expansion in the Dirac-Born-Infeld action, since the quantity that appears in the action is r (or \hat{R}) rather than R_{ph} . Continuity of the time derivative $\partial_t r$ at $r = 0$ also guarantees that the $D2$ -brane (or $D3$ -brane) charge is continuous. This interpretation is compatible with

the one in [60], where different signs of the \hat{R} were interpreted as corresponding to either a brane or an anti-brane emerging at the blow-up of the S^2 funnel.

Instead of dropping space dependence we can restrict ourselves to a static problem by making our Ansatz time independent. There is a conserved pressure $T^{\sigma\sigma}$

$$\frac{\partial T^{\sigma\sigma}}{\partial \sigma} = 0 \quad (2.1.27)$$

By plugging-in the correct expression we recover

$$\frac{\partial}{\partial \sigma} \sqrt{\frac{1+r^4}{1+r'^2}} = 0 \quad (2.1.28)$$

which can be combined with the initial condition $r' = 0$ at r_0 to give

$$r'^2 = \left(\frac{1+r^4}{1+r_0^4} \right) \quad (2.1.29)$$

The purely space-dependent and the purely time-dependent equations are related by Wick rotation $t \rightarrow i\sigma$. To apply this to the solution (2.1.24) we can use the identity

$$Cn\left(ix, \frac{1}{\sqrt{2}}\right) = \frac{1}{Cn(x, \frac{1}{\sqrt{2}})} \quad (2.1.30)$$

which is an example of a complex multiplication formula [80]. Therefore, the first order equation for the static configuration has solutions in terms of the Jacobi elliptic functions

$$r(\sigma) = \pm r_0 \frac{1}{Cn\left(\tilde{\sigma}, \frac{1}{\sqrt{2}}\right)} \quad (2.1.31)$$

where $\tilde{\sigma} = \frac{\sqrt{2}r_0\sigma}{\sqrt{r_0^4+1}}$ and it can be verified that these also satisfy the full DBI equations of motion. This solution is not BPS and does not satisfy the YM equations, as is further discussed in App. A.1.

The $r(\sigma)$ plot reveals that it represents an infinite, periodic, alternating brane-anti-brane array, with $D1$ -funnels extending between them. The values of σ where r blows up, *i.e.* the poles of the Cn -function, correspond to locations of $D3$ -branes and anti- $D3$ -branes. This follows because the derivative $\frac{\partial r}{\partial \sigma}$ changes sign between successive poles. This derivative appears in the computation of the $D3$ -charge from the Chern-Simons terms in the $D1$ -world-volume. On the left and right of a blow-up point the sign of the derivative is the same which is consistent with the fact that the charge of a brane measured from either the left or right should give the same answer. Alternatively, we can pick an oriented set of axes on one brane and transport it along the funnel to the neighbouring brane to find that the orientation has changed.

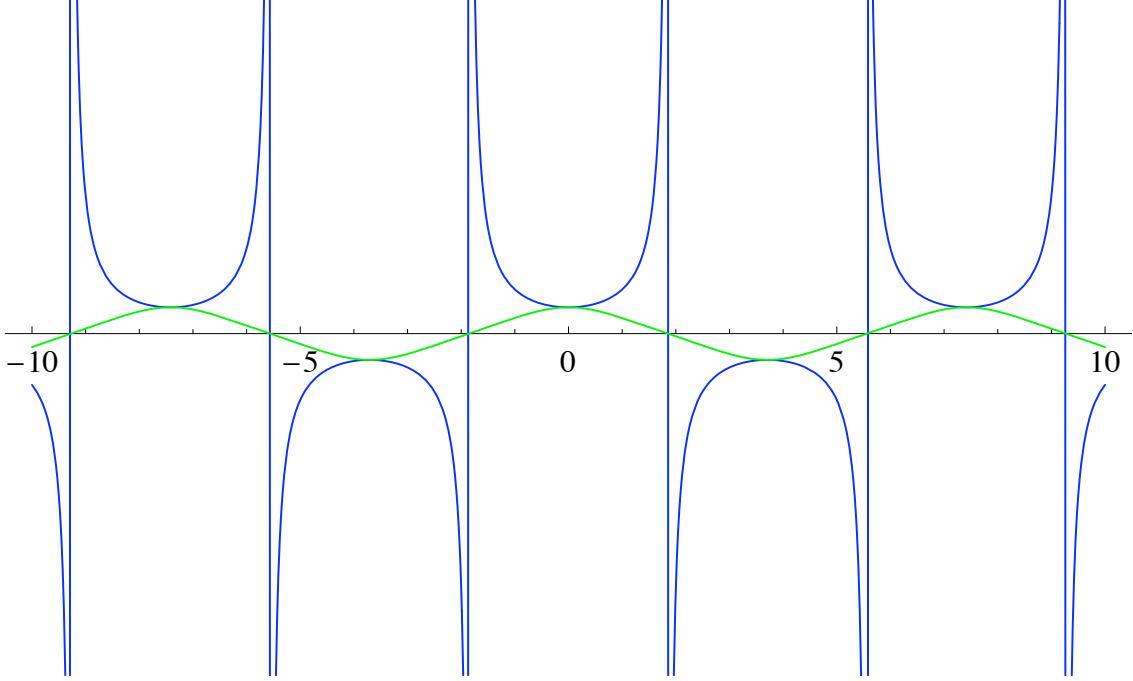


Figure 2.1: Analytic plot of the Jacobi elliptic function solution for the static fuzzy- S^2 funnel array and the collapsing fuzzy- S^2 for $r_0 = 1$.

This type of solution captures the known results of F and D -strings stretching between $D3$ and anti- $D3$ s [26, 27] by restricting to a half-period of the elliptic function in the space evolution. It is also possible to recover the BPS configurations of [61] which were obtained by considering the minimum energy condition of the static funnel, where $\dot{r} = 0$

$$\partial_\sigma \Phi_i = \pm \frac{i}{2} \epsilon_{ijk} [\Phi_j, \Phi_k] \quad (2.1.32)$$

This is equivalent to the Nahm equation [59] and is also the BPS condition. In dimensionless variables it translates to $r'^2 = r^4$ and has a solution in terms of $r = \pm 1/(\sigma - \sigma_\infty)$, with σ_∞ denoting the point in space where the funnel blows-up. We will restrict the general solution to a quarter-period and consider the expansion around the first blow-up which occurs at $Cn(K(\frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}})$, *i.e.* close to the $D3$ -brane. We get

$$\begin{aligned} r = \frac{r_0}{Cn\left(\frac{\sqrt{2}r_0\sigma}{\sqrt{1+r_0^4}}, \frac{1}{\sqrt{2}}\right)} &\simeq -\frac{r_0}{\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}r_0\sigma}{\sqrt{1+r_0^4}} - K(\frac{1}{\sqrt{2}})\right)} \\ &= -\frac{\sqrt{1+r_0^4}}{\sigma - \frac{\sqrt{1+r_0^4}K(\frac{1}{\sqrt{2}})}{\sqrt{2}r_0}} \end{aligned} \quad (2.1.33)$$

This is of the form $r = -\frac{\sqrt{1+r_0^4}}{\sigma - \sigma_\infty}$ which goes to $r = -\frac{1}{\sigma - \sigma_\infty}$ as $r_0 \rightarrow 0$.

2.1.4 Riemann Surface Technology

At this point we would like to introduce some of the Riemann surface technology that we will be using extensively for the rest of this chapter and the related theory of elliptic and hyper-elliptic functions. For more advanced material and reviews we refer the interested reader to [81–85].

Riemann surfaces and Algebraic curves

Riemann surfaces are complex manifolds of complex dimension 1. Here we will be mainly dealing with compact Riemann surfaces, *i.e.* Riemann surfaces that are compact as a topological space without boundary. The classification of compact topological surfaces is completely understood, the simplest example being the classical two-sphere. All other oriented compact topological surfaces can be obtained from S^2 by attaching a number of g handles, thus obtaining a surface of genus g . A surface of genus g has Euler characteristic $2 - 2g$. A set of marked points is an ordered set of distinct points (p_1, \dots, p_n) on the Riemann surface and two marked surfaces $C(p_1, \dots, p_n)$ and $C'(p'_1, \dots, p'_n)$ are said to be isomorphic if there exists a bi-holomorphic map (*i.e.* a bijective holomorphic map with a holomorphic inverse) $h : C \rightarrow C'$, such that $h(p_i) = p'_i \forall i$.

Now consider a Riemann surface M . One can always find a covering surface M^* , with fundamental group $\pi_1(M^*)$. Then $\pi_1(M^*)$ is isomorphic to a subgroup of $\pi_1(M)$. Actually, the covering surfaces of M are in bijective correspondence with conjugacy classes of subgroups of $\pi_1(M)$. When the latter subgroup of $\pi_1(M)$ is trivial, the covering space is the universal cover \tilde{M} . In the case that the subgroup is normal (N), there is a group $G \cong \pi_1(M)/N$ of fixed point free ($g \cdot x \neq x, \forall g \in G \setminus \{e\}$) holomorphic self-mappings of M^* such that $M^*/G \cong M$. A region Ω of M^* is said to be a fundamental domain of the G action, if the disjoint union of $g(\Omega)$ for $g \in G$ covers the entire surface, $M^* = \coprod_{g \in G} g(\Omega)$. The study of Riemann surfaces reduces to the study of fixed point free discontinuous groups of holomorphic self-mappings of the universal covers \tilde{M} (for a Riemann surface a finite group action is always properly discontinuous). It turns out that the only conformally distinct simply connected Riemann surfaces are the Riemann sphere \mathbb{CP}^1 , the upper half plane U and the complex plane \mathbb{C} . There is no non-trivial fixed point free holomorphic self-map for the sphere. In the case of the half plane, if the subgroup of the group of automorphisms $\text{Aut } \mathcal{H}$ is discontinuous, it is also discrete. The holomorphic self-mappings are $z \mapsto (az + b)/(cz + d)$, with $\{a, b, c, d\} \in \mathbb{R}$, $ad - cd = 1$ and the fixed point free condition $|a + d| \geq 2$. The Riemann surfaces with these covering spaces are exactly \mathcal{H}/G for discrete fixed point free subgroups G of $\text{Aut } \mathcal{H}$. Finally, the complex plane has the group of affine transformations as its automorphism group, namely $\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$ for $a \neq 0$. Now let's choose an element $\tau \in \mathbb{C}$ such that $\text{Im}(\tau) > 0$ and define a free abelian subgroup of \mathbb{C}

by

$$\Lambda_\tau = \mathbb{Z} \cdot \tau \oplus \mathbb{Z} \cdot 1 \subset \mathbb{C} \quad (2.1.34)$$

This is a lattice of rank 2. Then an elliptic curve of modulus τ is the quotient abelian group

$$E_\tau = \mathbb{C}/\Lambda_\tau \quad (2.1.35)$$

One can check that the natural action of Λ_τ through addition is properly discontinuous and fixed point free, therefore making E_τ a Riemann surface. Its fundamental domain is a parallelogram with vertices $0, 1, 1 + \tau$ and τ . Topologically E_τ is homeomorphic to a torus. Thus, an elliptic curve is a compact Riemann surface of genus 1. Conversely, if a Riemann surface is homeomorphic to a torus, then it is isomorphic to an elliptic curve. Note that τ and 1 form an \mathbb{R} -linear basis, such that the same lattice Λ_τ can be generated by some (ω_1, ω_2) as $\Lambda_\tau = \Lambda_{\omega_1, \omega_2} = \mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2$ with $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix}$ and $\{a, b, c, d\} \in SL(2, \mathbb{Z})$. We can therefore define the elliptic curve

$$E_\tau = \mathbb{C}/\Lambda_\tau = \mathbb{C}/(\mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2) \quad (2.1.36)$$

This is isomorphic to a curve obtained if we divide everything by ω_2 , since division by ω_2 is a holomorphic automorphism of \mathbb{C}

$$\tau \mapsto \mu = \frac{\omega_1}{\omega_2} = \frac{a\tau + b}{c\tau + d} \quad (2.1.37)$$

This is a linear fractional transformation and an example of a modular transformation. Note that we need $\text{Im}(\omega_1/\omega_2) > 0$. Actually, this linear fractional transformation is a holomorphic automorphism of the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$ and this allows us to define the moduli space of Riemann surfaces of genus 1, as $\mathcal{M}_{1,1} = \mathcal{H}/PSL(2, \mathbb{Z})$.

We can now define the Weierstrass elliptic function \wp , with periods ω_1, ω_2 as

$$\wp(z) = \wp(z | \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{m, n \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right) \quad (2.1.38)$$

This is a convergent holomorphic function on $\mathbb{C} \setminus \Lambda_{\omega_1, \omega_2}$, with double poles on the lattice points. Hence it is globally meromorphic on \mathbb{C} . Moreover, \wp is an even function, with double periodicity in ω_1, ω_2 and obeys the Weierstrass differential equation

$$(\wp(z)')^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3 \quad (2.1.39)$$

where g_2 and g_3 are constants that depend on ω_1, ω_2 , defined as

$$g_2(\omega_1, \omega_2) = 60 \sum_{m,n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m\omega_1 + n\omega_2)^4} \quad (2.1.40)$$

$$g_3(\omega_1, \omega_2) = 140 \sum_{m,n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m\omega_1 + n\omega_2)^6} \quad (2.1.41)$$

The Weierstrass differential equation implies that

$$z = \int dz = \int \frac{dz}{d\wp} d\wp = \int \frac{d\wp}{\wp'} = \int \frac{d\wp}{\sqrt{4(\wp)^3 - g_2\wp - g_3}} \quad (2.1.42)$$

However, this is the definition of an elliptic integral and the Weierstrass \wp -function is actually the inverse of the elliptic integral, an elliptic function. Note that all of the standard Jacobi elliptic functions can be given as functions of \wp of the same periodicity. Now, a meromorphic function is a holomorphic map into the Riemann sphere \mathbb{CP}^1 and the Weierstrass function actually defines a map from an elliptic curve onto \mathbb{CP}^1

$$\wp : E_{\omega_1, \omega_2} = \mathbb{C}/\Lambda_{\omega_1, \omega_2} \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \quad (2.1.43)$$

If we consider $X = \wp, Y = \wp'$ and denote by (e_1, e_2, e_3) the three roots of the polynomial equation $4X^3 - g_2X - g_3 = 0$, then except from the latter and ∞ of \mathbb{CP}^1 the map \wp is two-to-one and is a branched double covering of the Riemann sphere, ramified at the branch points e_1, e_2, e_3 and ∞ . Actually we can define a map from the elliptic curve into \mathbb{CP}^2 by

$$(\wp', \wp) : E_{\omega_1, \omega_2} = \mathbb{C}/\Lambda_{\omega_1, \omega_2} \longrightarrow \mathbb{CP}^2 \quad (2.1.44)$$

by mapping $(\wp'(z) : \wp(z) : 1) \in \mathbb{CP}^2$ for $E_{\omega_1, \omega_2} \ni z \neq 0$ and by mapping the origin to $(0 : 1 : 0) \in \mathbb{CP}^2$. In terms of the global co-ordinates $(X : Y : Z) \in \mathbb{CP}^2$ the image of the above map satisfies the homogeneous equation $Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$. The zero locus of this equation is a cubic curve, while its affine part is the locus of

$$Y^2 = 4X^3 - g_2X - g_3 \quad (2.1.45)$$

The above cubic curve is everywhere non-singular. The Riemann sphere has $PGL(2, \mathbb{C})$ as its automorphism group. We can use this in order to fix the branch points e_1, ∞, e_2 to $0, 1$ and ∞ respectively via the map $x \mapsto \frac{x-e_1}{x-e_2}$, up to an S_3 symmetry amongst e_1, e_2, e_3 . The fourth point will be mapped to some $e_3 \mapsto \lambda = \frac{e_3-e_1}{e_3-e_2}$, the value of which will depend on the above map and because of the S_3 symmetry could be any of the values $\lambda, \frac{1}{\lambda}, 1 - \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, 1 - \lambda, \frac{1}{1-\lambda}$. We can define a function

$$j(\lambda) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \quad (2.1.46)$$

which is called the j -invariant of an elliptic curve and is invariant under the effects of the S_3 symmetry. Curves with the same value of the j -invariant are isomorphic. It turns out that the moduli space of isomorphic curves will correspond to the moduli space of equivalent tori. Of course we could have mapped all three roots plus ∞ to some finite points, in which case we would recover a quartic curve. We have thus managed to encode all the properties of a torus, including its moduli space of complex structures, in a cubic or quartic algebraic curve Σ (hence the name elliptic curve). The corresponding quotient \mathbb{C}/Λ_τ is also called the Jacobian variety of the curve Σ . The notion of the Jacobian is important and generalises to higher genera.

Jacobi Inversion problem and ϑ -functions

The identification between Riemann surfaces and algebraic curves extends to genera higher than 1. Higher genus Riemann surfaces that satisfy the above condition are called hyper-elliptic. Every surface of genus $g < 2$ is hyper-elliptic¹. This does not hold for $g \geq 2$. We will use these results in order to introduce the Jacobi Inversion problem, which we will be implementing in our physical investigation [81, 83, 84, 86]. For any hyper-elliptic curve Σ of genus g , realised as a 2-sheeted cover over a Riemann sphere

$$y^2 = \sum_{i=0}^{2g+2} \lambda_i x^i = \prod_{i=1}^{g+1} (x - a_i) \prod_{i=1}^{g+1} (x - c_i) \quad (2.1.47)$$

with a 's and c 's being the branch points, between which we stretch the cuts, the system of integral equations

$$\begin{aligned} \int_{a_1}^{x_1} \omega_1 + \dots + \int_{a_g}^{x_g} \omega_1 &\equiv u_1 \\ &\vdots \quad \vdots \\ \int_{a_1}^{x_1} \omega_g + \dots + \int_{a_g}^{x_g} \omega_g &\equiv u_g \end{aligned}$$

describes the *invertible* Abel map, $\mathcal{U} : \Sigma^g / S_g \longrightarrow \text{Jac}(\Sigma)$, taking g symmetric points from the Riemann surface to the Jacobian of Σ . The Jacobi Inversion problem is the problem of finding the pre-image of the invertible Abel map.

The Jacobian of Σ is just $\mathbb{C}^g / \mathcal{L}$, where $\mathcal{L} = 2\zeta \oplus 2\zeta'$ is the lattice that is generated by the non-degenerate periods of the holomorphic differentials, or differentials of the first kind, defined on the surface

$$2\zeta = \oint_{A_k} \omega_l \quad \text{and} \quad 2\zeta' = \oint_{B_k} \omega_l \quad (2.1.48)$$

¹Most authors reserve the term hyper-elliptic for curves of $g \geq 2$, while using the term elliptic for $g = 1$. Every curve of genus zero is isomorphic to the Riemann sphere.

with

$$\omega_i = \frac{x^{i-1}dx}{y}, \quad i = 1, \dots, g \quad (2.1.49)$$

The A_k and B_k are the set of a- and b-cycles that provide a homology basis for the Riemann surface. Note that the Jacobian is a compact, commutative g -dimensional Lie group. The period matrix is the $g \times g$ matrix given by $\tau = \zeta^{-1}\zeta'$ and belongs to the Siegel upper half-space of degree g , \mathcal{H}_g , having positive imaginary part and being symmetric. It admits a transitive action of $PSp(2g, \mathbb{Z})$. There is also a set of associated canonical meromorphic differentials, or differentials of the second kind, which have a unique pole of second order

$$\xi_j = \sum_{k=j}^{2g+1-j} (k+1-j)\lambda_{k+1-j} \frac{x^k dx}{4y}, \quad j = 1, \dots, g \quad (2.1.50)$$

Their respective integrals over the A and B cycles are

$$2\eta = - \oint_{A_k} \xi_l \quad \text{and} \quad 2\eta' = - \oint_{B_k} \xi_l \quad (2.1.51)$$

Now consider $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^g$ two arbitrary vectors and define the periods

$$\Omega(\mathbf{m}, \mathbf{m}') = 2\zeta\mathbf{m} + 2\zeta'\mathbf{m}' \quad \text{and} \quad E(\mathbf{m}, \mathbf{m}') = 2\eta\mathbf{m} + 2\eta'\mathbf{m}' \quad (2.1.52)$$

We can define the fundamental Kleinian σ -function in terms of higher genus Riemann ϑ -functions

$$\sigma(\mathbf{u}) = Ce^{\mathbf{u}^T \kappa \mathbf{u}} \vartheta((2\omega^{-1})\mathbf{u} - \mathbf{K}_a|\tau) \quad (2.1.53)$$

where C is a constant, $\kappa = (2\zeta)^{-1}\eta$ and \mathbf{K}_a is the vector of Riemann constants with base point a , given by the Riemann vanishing theorem

$$\mathbf{K}_a = \sum_{k=1}^g \int_a^{a_i} d\mathbf{v} \quad (2.1.54)$$

with $d\mathbf{v} = (2\zeta^{-1})(\omega_1, \dots, \omega_g)^T$ being the set of *normalised* canonical holomorphic differentials. Furthermore, the genus- g Riemann ϑ -function with half-integer characteristics $[\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \varepsilon'_1 & \dots & \varepsilon'_g \\ \varepsilon_1 & \dots & \varepsilon_g \end{bmatrix} \in \mathbb{C}^{2g}$ is defined as

$$\vartheta[\varepsilon](\mathbf{v}|\tau) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp \pi i \{ (\mathbf{m} + \varepsilon')^T \tau (\mathbf{m} + \varepsilon') + 2(\mathbf{v} + \varepsilon)^T (\mathbf{m} + \varepsilon') \} \quad (2.1.55)$$

The general pre-image of the Abel map, and thus the solution to the Jacobi Inversion problem, can be given in a simple algebraic form as the roots of a polynomial in x

$$\mathbf{P}(x; \mathbf{u}) = x^g - x^{g-1} \wp_{g,g}(\mathbf{u}) - x^{g-2} \wp_{g,g-1}(\mathbf{u}) - \dots - \wp_{g,1}(\mathbf{u}) \quad (2.1.56)$$

Here $\mathbf{u} = (u_1, \dots, u_g)^T$ and the \wp 's are higher genus versions of the standard Weierstrass elliptic \wp -functions, which are defined as the logarithmic derivatives of the fundamental hyper-elliptic σ -functions

$$\wp_{ij}(\mathbf{u}) = -\frac{\partial^2 \ln \sigma(\mathbf{u})}{\partial u_i \partial u_j}, \quad \wp_{ijk}(\mathbf{u}) = -\frac{\partial^3 \ln \sigma(\mathbf{u})}{\partial u_i \partial u_j \partial u_k}, \dots, \quad i, j, k, \dots \in \{1, \dots, g\} \quad (2.1.57)$$

and have the nice periodicity properties

$$\wp_{ij}(\mathbf{u} + \Omega(\mathbf{m}, \mathbf{m}')) = \wp_{ij}(\mathbf{u}) \quad i, j \in \{1, \dots, g\} \quad (2.1.58)$$

For genera $g = 1, 2, 3$ the moduli space of Riemann surfaces are given by the expression $\mathcal{M}_{g,g} = \mathcal{H}_g / PSp(2g, \mathbb{Z})$. For $g = 4$ the solution was given by Schottky in terms of Riemann ϑ -functions. For $g \geq 5$ the period matrix can still be used in order to parametrise the moduli space but the process is significantly more involved and the answer can be given indirectly [87].

2.1.5 An $r \rightarrow 1/r$ duality between the Euclidean and Lorentzian DBI equations

After this digression we return to our physical problem. Defining $s = \frac{dr}{dt}$ the eq. (2.1.22) can be written as

$$s^2 = \frac{r_0^4 - r^4}{1 + r_0^4} \quad (2.1.59)$$

Viewing s and r as two complex variables constrained by one equation, we recognise a genus one Riemann surface. The quantity $\frac{dr}{s}$ which gives the infinitesimal time elapsed is an interesting geometrical quantity related to the Riemann surface, *i.e.* the holomorphic differential.

The curve (2.1.59) has a number of automorphisms of interest. One checks that $R = \frac{r_0^2}{r}$, $\tilde{s} = \frac{isr_0^2}{r^2}$ leaves the equation of the curve invariant. This automorphism of the curve leads directly to the complex multiplication identity (2.1.30), which relates the spatial and time-dependent solutions. Another, not unrelated, automorphism acts as $R \rightarrow \frac{1}{r}$, $R_0 \rightarrow \frac{1}{r_0}$, $\tilde{s} = \frac{is}{r^2}$. The relation between \tilde{s} and s is equivalent to a Wick rotation of the time variable. The transformation $r \rightarrow \frac{1}{r}$ can also be taken to act on the second order equation since it does not involve r_0 (which does not appear in the second order equation). The spatial BPS solution to the second order equation can be acted upon by this transformation. The outcome is a time-dependent solution describing a brane collapsing at the speed of light $r = \pm(t - t_\infty)$. This solution can be derived as an $r_0 \rightarrow \infty$ limit of the general time-dependent elliptic solution, in much the same way as the BPS solution was derived as an $r_0 \rightarrow 0$ limit of the spatial elliptic solutions.

The action of the $r \rightarrow \frac{1}{r}$ transformation on the second order equations can be seen explicitly. In the case of pure time-dependence, the equation of motion in dimensionless variables is

$$\ddot{r} = -2r^3 \left(\frac{1 - \dot{r}^2}{1 + r^4} \right) \quad (2.1.60)$$

while in the case of pure spatial dependence

$$r'' = 2r^3 \left(\frac{1 + r'^2}{1 + r^4} \right) \quad (2.1.61)$$

A substitution $r = 1/R$ can be used to transform (2.1.60) using

$$\begin{aligned} \dot{r} &= -\frac{1}{R^2} \dot{R} \\ \ddot{r} &= -\frac{1}{R^2} \ddot{R} + \frac{2}{R^3} \dot{R}^2 \end{aligned}$$

to get

$$-\frac{\ddot{R}}{R^2} + \frac{2}{R^3} (\dot{R})^2 = -\frac{2}{R^3} \frac{1 - \dot{R}^2}{1 + 1/R^4} \quad (2.1.62)$$

which can be simplified to

$$\ddot{R} = 2R^3 \left(\frac{1 + \dot{R}^2}{1 + R^4} \right) \quad (2.1.63)$$

So the effect of transforming $r \rightarrow \frac{1}{r}$ and renaming $t \rightarrow \sigma$ is the same as the substitution $t \rightarrow i\sigma$. This explains the relation between (2.1.24) and (2.1.31) which was previously obtained using the complex multiplication property (2.1.30) of Jacobi- C_n functions.

2.2 Space and time-dependent fuzzy- S^4

2.2.1 Equivalence of the action for the $D1 \perp D5$ intersection

We will extend the consideration of space and time dependent solutions to DBI for the case of the $D1 \perp D5$ intersection, which involves a fuzzy- S^4 , generalising the purely spatial discussion of [61]. The equations are also relevant to the time-dependence of fuzzy spherical $D0$ - $D4$ systems which have been studied in the Yang-Mills limit in [69]. On the $D1$ side, we will have five transverse scalar fields, satisfying the Ansatz

$$\Phi^i(\sigma, \tau) = \pm \hat{R}(\sigma, \tau) G^i, \quad i = 1, \dots, 5 \quad (2.2.1)$$

where the G^i 's are given by the action of $SO(5)$ gamma matrices on the totally symmetric n -fold tensor product of the basic spinor, the dimension of which is related to n by

$$N = \frac{(n+1)(n+2)(n+3)}{6} \quad (2.2.2)$$

The radial profile and the fuzzy- S^4 physical radius are again related by

$$R_{ph}(\sigma, \tau) = \sqrt{C} \lambda \hat{R}(\sigma, \tau) \quad (2.2.3)$$

where C is the Casimir $G^i G^i = C \mathbb{1}_{N \times N} = n(n+4) \mathbb{1}_{N \times N}$. By plugging the Ansatz (2.2.1) into the action and by considering the large- N behaviour of the configuration, one gets in dimensionless variables, defined just as for the S^2 case in (2.1.9),

$$S_1 = -N T_1 \int d^2\sigma \sqrt{1 + r'^2 - \dot{r}^2} (1 + r^4) \quad (2.2.4)$$

As in the $D1 \perp D3$ case, we can write the equations of motion for this configuration in a Lorentz-invariant way. The result is the same as before with the exception of the pre-factor on the right-hand-side

$$\partial_\mu \partial^\mu r + (\partial_\mu \partial^\mu r) (\partial_\nu r) (\partial^\nu r) - (\partial_\mu \partial^\nu r) (\partial_\nu r) (\partial^\mu r) = 4 r^3 \left(\frac{1 + (\partial_\mu r) (\partial^\mu r)}{1 + r^4} \right) \quad (2.2.5)$$

where again μ and ν can take the values σ, τ .

Let us now look at the $D5$ side: The world-volume action for n $D5$ -branes, with one transverse scalar excited, is

$$S_5 = -T_5 \int d^6\sigma STr \sqrt{-\det(G_{ab} + \lambda^2 \partial_a \Phi \partial_b \Phi + \lambda F_{ab})} \quad (2.2.6)$$

Introducing spherical co-ordinates with radius R_{D5} and angles α^i ($i = 1, \dots, 4$), we will have $ds^2 = -dt^2 + dR_{D5}^2 + R_{D5}^2 g_{ij} d\alpha^i d\alpha^j$, where g_{ij} is the metric of a unit four-sphere, the volume of which is given by $\int d^4\alpha \sqrt{g} = 8\pi^2/3$. We take the construction of [61] with homogeneous instantons and, keeping the same gauge field, we generalise $\Phi(R_{D5})$ to $\Phi(R_{D5}, t)$. We will not review this here but just state that by using the above, we can reduce the previous expression to

$$S_5 = -T_5 \int d^6\sigma \sqrt{g} \sqrt{1 + \lambda^2 \Phi'^2 - \lambda^2 \dot{\Phi}^2} \left(n R_{D5}^4 + \frac{3}{2} N \lambda^2 \right) \quad (2.2.7)$$

Then by implementing once again the relations (2.1.17) we recover

$$S_5 = -T_5 \int d^4\alpha dt dR_{ph} \sqrt{g} \sqrt{1 + \frac{1}{R_{ph}^2} - \frac{\dot{R}_{ph}^2}{R_{ph}^2}} \left(n R_{ph}^4 + \frac{3}{2} N \lambda^2 \right) \quad (2.2.8)$$

This can be easily manipulated to yield the following result

$$S_5 = -T_1 N \int d\tau d\sigma \sqrt{1 + R_{ph}'^2 - \dot{R}_{ph}^2} \left(1 + \frac{2n}{3N\lambda^2} R_{ph}^4 \right) \quad (2.2.9)$$

Using $\frac{2n}{3N} \sim \frac{4}{C}$, which holds for large- N and by once again employing dimensionless variables, this becomes

$$S_5 = -T_1 N \int d\tau d\sigma \sqrt{1 + r'^2 - \dot{r}^2} (1 + r^4) \quad (2.2.10)$$

which agrees with (2.2.4) and can be further simplified to the re-scaled action

$$\tilde{S}_5 = - \int d^2\sigma \sqrt{1 + r'^2 - \dot{r}^2} (1 + r^4) \quad (2.2.11)$$

Thus, every space-time-dependent solution described by the DBI equations of motion from the $D1$ world-volume will have an equivalent description on the $D5$ world-volume.

2.2.2 Solutions for space-dependent fuzzy spheres: Funnels

Assuming $\dot{r} = 0$, the action (2.2.11) becomes

$$\tilde{S}_5 = - \int d^2\sigma \sqrt{1 + r'^2} (1 + r^4) \quad (2.2.12)$$

There exists a conserved pressure $T^{\sigma\sigma} = P$ given by

$$P = \frac{1 + r^4}{\sqrt{1 + r'^2}} \quad (2.2.13)$$

which can be solved to give r'

$$\begin{aligned} \left(\frac{dr}{d\sigma} \right)^2 &= \frac{1}{P^2} - 1 + \frac{(1 + r^4)^2}{P^2} \\ &= \frac{(2r^4 + r^8) - (2r_0^4 + r_0^8)}{(1 + r_0^4)^2} \end{aligned}$$

Defining $\frac{dr}{d\sigma} = s$ this can be expressed as

$$s^2 = \frac{(r^4 - r_0^4)(r^4 - r_1^4)}{(1 + r_0^4)^2} \quad (2.2.14)$$

The roots in the equation above correspond to $(1 + r^4) = (1 + r_0^4)$ and $(1 + r^4) = -(1 + r_0^4)$. The second possibility gives $r_1^4 = -2 - r_0^4$. By differentiating the pressure, a second order differential equation can be derived

$$\begin{aligned} \frac{d^2r}{d\sigma^2} &= \frac{4r^3(1 + r'^2)}{(1 + r^4)} \\ &= \frac{4r^3(1 + r^4)}{P^2} \end{aligned} \quad (2.2.15)$$

An integral formula can be written for the distance along the $D1$ -brane using (2.2.14)

$$\int_0^\sigma d\sigma = \int_{r_0}^r dr \frac{(1+r_0^4)}{\sqrt{(2r^4+r^8)-(2r_0^4+r_0^8)}}, \quad (2.2.16)$$

where we have taken the zero of σ to be at the place $r = r_0$ where $r' = 0$. We deduce from the integral that there is a finite value of σ , denoted as Σ , where r has increased to infinity. This is similar to part of the spatial solution of the $D1 \perp D3$ system described by the C_n function (see Fig. 1). We can show that the full periodic structure analogous to that of the spatial $D1 \perp D3$ system follows in the $D1 \perp D5$ system, by using symmetries of the equations and the requirement that the derivative $\frac{dr}{d\sigma}$ is continuous. Note that (2.2.14) is symmetric under the operations

$$\begin{aligned} I_\sigma : r(\sigma) &\rightarrow r(-\sigma) \\ I_r : r(\sigma) &\rightarrow -r(\sigma) \\ T_\Sigma : r(\sigma) &\rightarrow r(\sigma - \Sigma) \end{aligned} \quad (2.2.17)$$

The branch with r increasing from r_0 to ∞ , as σ changes from 0 to Σ , can be acted on by $T_{2\Sigma} I_\sigma I_r$ to yield a branch where r increases from $-\infty$ to $-r_0$ over $\Sigma \leq \sigma \leq 2\Sigma$. Acting with $T_{2\Sigma} I_r$ gives a branch where r decreases from $-r_0$ to $-\infty$ over $2\Sigma \leq \sigma \leq 3\Sigma$. Finally a transformation of the original branch by $T_{4\Sigma} I_\sigma$ gives r decreasing from ∞ to r_0 over $3\Sigma \leq \sigma \leq 4\Sigma$. These four branches patch together to form a complete cycle.

implying

$$(1 + r_0^4)^2 = \frac{(1 + r^4)^2}{(1 - \dot{r}^2)} \quad (2.2.20)$$

This can be solved for the velocity

$$\begin{aligned} \left(\frac{dr}{dt} \right)^2 &= 1 - \frac{1}{E^2} - \frac{1}{E^2} (2r^4 + r^8) \\ &= \frac{(2r_0^4 + r_0^8) - (2r^4 + r^8)}{(1 + r_0^4)^2} \end{aligned}$$

Writing $s = \frac{dr}{dt}$

$$\begin{aligned} s^2 &= \frac{(1 + r_0^4)^2 - (1 + r^4)^2}{(1 + r_0^4)^2} \\ &= \frac{-(r^4 - r_0^4)(r^4 - r_1^4)}{(1 + r_0^4)^2} \end{aligned} \quad (2.2.21)$$

A trivial redefinition $s \rightarrow is$ relates the eq. (2.2.21) to (2.2.14). The time evolved can be written in terms of the radial distance

$$\int_0^t dt = \int_{r_0}^r dr \frac{(1 + r_0^4)}{\sqrt{(2r_0^4 + r_0^8) - (2r^4 + r^8)}} \quad (2.2.22)$$

Differentiating (2.2.21) gives a second order equation

$$\begin{aligned} \ddot{r} &= \frac{-4r^3(1 + r^4)}{E^2} \\ &= \frac{-4r^3(1 + r^4)}{(1 + r_0^4)^2} \end{aligned} \quad (2.2.23)$$

We can see from (2.2.22) that the time taken to start from $r = r_0$ and reach $r = 0$ is finite. We will call this finite time interval the time of collapse T . As in the spatial problem there are symmetries of the equation

$$\begin{aligned} I_t : r(t) &\rightarrow r(-t) \\ I_r : r(t) &\rightarrow -r(t) \\ T_T : r(t) &\rightarrow r(t - T) \end{aligned} \quad (2.2.24)$$

Following the same steps as in the spatial case, we can act successively with $I_{2\Sigma}I_tI_r$, $I_{2\Sigma}I_r$, $I_{3\Sigma}I_t$ and then with integer multiples of $4T$ to produce a periodic solution defined for positive and negative time. As discussed before and in analogy to the $D1 \perp D3$ scenario, this can be interpreted in terms of collapsing-expanding $D4$ -branes. The radius as a function of t has

the properties

$$\begin{aligned}
 r(t + 4mT) &= r(t) \\
 r(T) = r((2m+1)T) &= 0 \\
 r(t + 2mT) &= -r(t)
 \end{aligned} \tag{2.2.25}$$

where m is an arbitrary integer. We will see in the following that it will be useful to define a complex variable u_1 whose real part is related to σ and whose imaginary part is related to t as $u_1 = \sigma - it$. Unlike the case of the fuzzy two-sphere, we are not dealing simply with a variable u living on a torus. Rather it will become necessary to introduce a second complex variable u_2 such that the pair (u_1, u_2) lives on the Jacobian of a genus two-curve. It will also be natural to impose a constraint which amounts to looking at sub-varieties of the Jacobian. At the end of this it will, nevertheless, be possible to recover the sequence of zeroes and poles in (2.2.25) and (2.2.18).

2.2.4 Geometry and Automorphisms of Hyper-elliptic curve for the fuzzy- S^4

Eq. (2.2.21) defines a Riemann surface of genus 3. The integrals of interest (2.2.16) and (2.2.22), are integrals of a holomorphic differential along certain cycles of the Riemann surface. It is useful to recall the Riemann-Hurwitz formula

$$(2g - 2) = n(2G - 2) + B \tag{2.2.26}$$

which gives the genus g of the covering surface in terms of the genus G of the target and the number of branch points B . Since the RHS of (2.2.21) is a polynomial of degree 8 there are 8 points where $s = 0$, *i.e.* 8 branch points. Here $G = 0$ since the r co-ordinate can be viewed as living on the sphere, $n = 2$ and $B = 8$. So the integrals are defined on a genus $g = 3$ curve, which we will call Σ_3 .

After dividing out by r_0^8 on both sides of (2.2.21) and re-scaling $s = \frac{r_0^4 \tilde{s}}{1+r_0^4}$, $r^4 = r_0^4 \tilde{r}^4$, $r_1^4 = r_0^4 \tilde{r}_1^4$ we have

$$\tilde{s}^2 = (\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4) \tag{2.2.27}$$

There are three independent holomorphic differentials on this curve $\omega_1 = \frac{d\tilde{r}}{\tilde{s}}$, $\omega_2 = \frac{\tilde{r}d\tilde{r}}{\tilde{s}}$, $\omega_3 = \frac{\tilde{r}^2 d\tilde{r}}{\tilde{s}}$ (see for example [81, 88–90]). The infinitesimal time elapsed is $dt = \frac{dr}{s} = \frac{(1+r_0^4)}{r_0^3} \frac{d\tilde{r}}{\tilde{s}}$. It will be important in calculating the integrals (2.2.16) and (2.2.22), to understand the automorphisms of the Riemann surface Σ_3 . In the limit of large r_0 , \tilde{r}_1^4 approaches -1 and there is an automorphism

$$\begin{aligned}
 \tilde{r} &\rightarrow \frac{1}{\tilde{r}} \\
 \tilde{s} &\rightarrow \frac{i\tilde{s}}{\tilde{r}^4}
 \end{aligned}$$

which leaves the equation of the curve unchanged. It transforms the holomorphic differentials as follows

$$\begin{aligned}\omega_1 &\rightarrow i\omega_3 \\ \omega_2 &\rightarrow i\omega_2 \\ \omega_3 &\rightarrow i\omega_1\end{aligned}$$

For any finite r_0 there is a \mathbb{Z}_2 automorphism, $\tilde{r} \rightarrow -\tilde{r}$. Quotienting by this can be achieved by changing variables $\tilde{r}^2 = x$. The first and third holomorphic differentials ω_1 and ω_3 transform as follows

$$\begin{aligned}\frac{d\tilde{r}}{\sqrt{(\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4)}} &= \frac{dx}{\sqrt{4x(x^2 - 1)(x^2 - \tilde{r}_1^4)}} \\ \frac{\tilde{r}^2 d\tilde{r}}{\sqrt{(\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4)}} &= \frac{xdx}{\sqrt{4x(x^2 - 1)(x^2 - \tilde{r}_1^4)}}\end{aligned}\quad (2.2.28)$$

We can view this in terms of a map from a genus three curve (\tilde{r}, \tilde{s}) to a genus two curve (x, y)

$$\begin{aligned}x &= \tilde{r}^2 \\ y &= 2\tilde{r}\tilde{s}\end{aligned}\quad (2.2.29)$$

which implies

$$y^2 = 4\tilde{r}^2(\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4) = 4x(x^2 - 1)(x^2 - \tilde{r}_1^4)\quad (2.2.30)$$

This genus 2 curve (x, y) will be called Σ_2 . The holomorphic differentials are related by $\frac{d\tilde{r}}{\tilde{s}} = \frac{dx}{y}$ and $\frac{\tilde{r}^2 d\tilde{r}}{\tilde{s}} = \frac{xdx}{y}$. The map (2.2.29) has branching number $B = 0$, in agreement with (2.2.26). Even though $\frac{\partial x}{\partial \tilde{r}} = 0$ at $\tilde{r} = 0$, this is not a branch point, since x is not a good local co-ordinate for the curve (x, y) at $x = 0$. Rather, a good local co-ordinate is y which is linearly related to \tilde{r} and this means that there is no branch point.

The second holomorphic differential transforms as

$$\frac{\tilde{r} d\tilde{r}}{\sqrt{(\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4)}} = \frac{dx}{2\sqrt{(x^2 - 1)(x^2 - \tilde{r}_1^4)}}\quad (2.2.31)$$

This can be viewed in terms of a map

$$\begin{aligned}\tilde{x} &= \tilde{r}^2 \\ \tilde{y} &= \tilde{s}\end{aligned}\quad (2.2.32)$$

to a target torus (\tilde{x}, \tilde{y}) , obeying

$$\tilde{y}^2 = (\tilde{x}^2 - 1)(\tilde{x}^2 - \tilde{r}_1^4)\quad (2.2.33)$$

This map has $B = 4$, again in agreement with (2.2.26). There are two branch points at $(\tilde{r}, \tilde{s}) = (0, \pm \tilde{r}_1^2)$ corresponding to $(\tilde{x}, \tilde{y}) = (0, \pm \tilde{r}_1^2)$. Similarly we have two branch points at $\tilde{r} = \infty$. The region near $\tilde{r} = \infty$ is best studied by defining variables $\hat{r} = \tilde{r}^{-1}$ and $\hat{s} = \tilde{s}\tilde{r}^{-4}$ which re-express (2.2.27) as $\hat{s}^2 = (1 - \hat{r}^4)(1 - \hat{r}^4\tilde{r}_1^4)$. Similarly we define $\hat{x} = \tilde{x}^{-1}, \hat{y} = \tilde{y}\tilde{x}^{-2}$ which re-expresses (2.2.33) as $\hat{y}^2 = (1 - \hat{x}^2)(1 - \hat{x}^2\tilde{r}_1^4)$. Two branch points corresponding to $\tilde{r} = \infty$ are at $(\hat{x}, \hat{y}) = (0, \pm 1)$.

The genus two curve Σ_2 itself can be related to genus one curves. We first re-write the (x, y) eq. (2.2.30) as

$$y^2 = 4x(x-1)(x+1)(x-iR_1^2)(x+iR_1^2) \quad (2.2.34)$$

with $R_1^2 = -i\tilde{r}_1^2$ and real. This is of the special form [83, 84, 91]

$$y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\alpha\beta) \quad (2.2.35)$$

with $\alpha = -1$ and $\beta = iR_1^2$. Such curves have an automorphism T_1

$$\begin{aligned} T_1(x) \equiv X &= \frac{\alpha\beta}{x} = \frac{-iR_1^2}{x} \\ T_1(y) \equiv Y &= \frac{(\alpha\beta)^{\frac{3}{2}}y}{x^3} = \frac{e^{\frac{i\pi}{4}}R_1^3y}{x^3} \end{aligned} \quad (2.2.36)$$

which transforms the holomorphic differentials

$$\begin{aligned} T_1\left(\frac{dx}{y}\right) &= \frac{iXdX}{Y\sqrt{\beta}} \\ T_1\left(\frac{xdx}{y}\right) &= \frac{-i\sqrt{\beta}dX}{Y} \end{aligned} \quad (2.2.37)$$

The genus two curve (2.2.34) can be put in the form (2.2.35) in yet another way, by choosing $\alpha = -1, \beta = -iR_1^2$. This gives another automorphism T_2 which acts as

$$\begin{aligned} T_2(x) &= \frac{iR_1^2}{x} \\ T_2(y) &= \frac{e^{\frac{3i\pi}{4}}yR_1^3}{x^3} \end{aligned} \quad (2.2.38)$$

With either choice of (α, β) one is led to look for variables which are invariant under the automorphism and as a result one describes the genus 2 curve as a covering of *two* genus one curves (ξ_{\pm}, η_{\pm}) [83, 84, 91]

$$\eta_{\pm}^2 = \xi_{\pm}(1 - \xi_{\pm})(1 - k_{\pm}^2\xi_{\pm}) \quad \text{with} \quad k_{\pm}^2 = -\frac{(\sqrt{\alpha} \mp \sqrt{\beta})^2}{(1 - \alpha)(1 - \beta)} \quad (2.2.39)$$

The maps are given by

$$\begin{aligned}\xi_{\pm} &= \frac{(1-\alpha)(1-\beta)x}{(x-\alpha)(x-\beta)} \\ \eta_{\pm} &= -\sqrt{(1-\alpha)(1-\beta)} \frac{x \mp \sqrt{\alpha\beta}}{(x-\alpha)^2(x-\beta)^2} y\end{aligned}\quad (2.2.40)$$

This gives an isomorphism of the Jacobian of the genus 2 curve in terms of a product of the Jacobians of the genus 1 curves Σ_{\pm} . For the case $\alpha = -1$ we have $K(k_+) = K'(k_-)$. Since the complex structure is $\tau = i \frac{K'(k)}{K(k)}$, this means that the complex structures of Σ_+ and Σ_- are related by the $SL(2, \mathbb{Z})$ transformation $\tau \rightarrow -\frac{1}{\tau}$. Hence Σ_{\pm} have isomorphic complex structures.

As an aside we describe the full group of automorphisms of Σ_2 . It includes σ , the hyperelliptic involution, $\sigma(x) = x, \sigma(y) = -y$; T_3 , which acts as $T_3(x) = -x, T_3(y) = -iy$, and T_4 , which acts as $T_4(x) = -x, T_4(y) = iy$. Relations in this group of automorphisms are

$$\begin{aligned}T_1 T_2 &= T_3 \\ T_2 T_1 &= T_4 \\ T_3 T_4 = T_4 T_3 &= 1 \\ T_3^2 &= \sigma \\ T_3 \sigma &= T_4\end{aligned}\quad (2.2.41)$$

In the limit $r_0 \rightarrow \infty$, $R_1 = 1$ and the equation for the curve simplifies

$$y^2 = x(x^4 - 1) \quad (2.2.42)$$

As a result, the automorphism group is larger than at finite r_0 . The automorphism group is generated by U_1 and U_2 which act as follows

$$\begin{aligned}U_1(x) &= \frac{1}{x} \\ U_1(y) &= \frac{e^{\frac{i\pi}{2}} y}{x^3} \\ U_2(x) &= e^{\frac{i\pi}{2}} x \\ U_2(y) &= e^{\frac{i\pi}{4}} y\end{aligned}\quad (2.2.43)$$

If we write $U_1(x) = X, U_1(y) = Y$, we have the following action on the holomorphic differentials

$$\begin{aligned}\frac{dx}{y} &= -i \frac{X dX}{Y} \\ \frac{xdx}{y} &= -i \frac{dX}{Y}\end{aligned}\quad (2.2.44)$$

The action of U_2 on the holomorphic differentials is just

$$\begin{aligned} \frac{dx}{y} &\rightarrow \frac{e^{-i\pi/4} dx}{y} \\ \frac{xdx}{y} &\rightarrow \frac{e^{-3i\pi/4} xdx}{y} \end{aligned} \quad (2.2.45)$$

The automorphism group includes, as usual, the hyper-elliptic involution acting as $\sigma(x) = x, \sigma(y) = -y$. There is also an element U_3 acting as $U_3(x) = -x, U_3(y) = iy$. There are relations $U_1^2 = \sigma, U_2^2 = U_3$. In the large r_0 limit, $R_1 \rightarrow 1$, the formulae for T_1, T_2 from (2.2.36), (2.2.38) simplify and they can be written in terms of U_1, U_2 . Indeed we find that

$$\begin{aligned} T_1 &= \sigma U_2 U_1 \\ T_2 &= \sigma U_2^{-1} U_1 \end{aligned}$$

As an aside, we would like to note that the reduction techniques that we have used here fall under the general Weierstrass-Poincaré reduction theory [83, 84]. These simplifications are possible via second order non-trivial automorphisms and coverings of the higher genus curve. Equivalently, the reducibility condition emerges as a property of the period matrix of the appropriate Riemann surface. These reduction techniques are useful in finding the moduli spaces of Riemann surfaces of high genera, an otherwise complicated process. Knowledge of moduli spaces of Riemann surfaces is very important in string perturbation theory. For more details and recent results on these calculations we refer to [92].

2.2.5 Evaluation of integrals

The time integral (2.2.22) can be done in terms of Appell functions. The indefinite integral is

$$\frac{(1+r_0^4)r}{\sqrt{(2r_0^4+r_0^8)}} F_1 \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}; \frac{5}{4}; \frac{r^4}{r_0^4}, -\frac{r^4}{2+r_0^4} \right) \quad (2.2.46)$$

A quick way to get this is by using the Integrator [93] but we outline a derivation. Expanding the integrand of (2.2.22)

$$\begin{aligned} &\int dr (r_0^4 - r^4)^{-\frac{1}{2}} (r_0^4 + 2 + r^4)^{-\frac{1}{2}} \\ &= r_0^{-2} (2 + r_0^4)^{-\frac{1}{2}} \int dr \sum_{k,l=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-k)\Gamma(k+1)} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-l)\Gamma(l+1)} (-1)^k \left(\frac{r}{r_0}\right)^{4k} \left(\frac{r^4}{2+r_0^4}\right)^l \\ &= r_0^{-2} (2 + r_0^4)^{-\frac{1}{2}} r \sum_{k,l=0}^{\infty} \frac{1}{k!l!} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-k)} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-l)} \frac{1}{4k+4l+1} (-1)^k \left(\frac{r}{r_0}\right)^{4k} \left(\frac{r^4}{2+r_0^4}\right)^l \end{aligned}$$

Now use the following facts about Γ functions

$$\begin{aligned}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - k)} &= (-1)^k \frac{\Gamma(\frac{1}{2} + k)}{\Gamma(\frac{1}{2})} \\ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - l)} &= (-1)^l \frac{\Gamma(\frac{1}{2} + l)}{\Gamma(\frac{1}{2})} \\ \frac{1}{4(k + l + \frac{1}{4})} &= \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} \frac{\Gamma(\frac{1}{4} + k + l)}{\Gamma(\frac{5}{4} + k + l)}\end{aligned}$$

to recognise the series expansion of the Appell function [94]

$$F_1(a; b_1, b_2; c; z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l} (b_1)_k (b_2)_l z_1^k z_2^l}{(c)_{k+l} k! l!} \quad (2.2.47)$$

with arguments as given in (2.2.46) and where we have also made use of the Pochhammer symbols $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Using (2.2.46), the time taken to collapse from the initial radius r_0 to a smaller radius r is

$$t = \frac{(1 + r_0^4)r_0}{\sqrt{(2r_0^4 + r_0^8)}} F_1\left(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; \frac{5}{4}; 1, -\frac{r_0^4}{2 + r_0^4}\right) - \frac{(1 + r_0^4)r}{\sqrt{(2r_0^4 + r_0^8)}} F_1\left(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; \frac{5}{4}; \frac{r^4}{r_0^4}, -\frac{r^4}{2 + r_0^4}\right) \quad (2.2.48)$$

For the special values $r = r_0$, $z_1 = 1$, F_1 of (2.2.46) simplifies to

$$\frac{(1 + r_0^4)}{r_0 \sqrt{(2 + r_0^4)}} {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; 1\right) {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{3}{4}; -\frac{r_0^4}{2 + r_0^4}\right) \quad (2.2.49)$$

For $r = 0$, $F_1 = 1$ and the indefinite integral (2.2.46) evaluates to zero. Hence the time of collapse is given by (2.2.49) which can be simplified to

$$T = \frac{(1 + r_0^4)}{r_0 \sqrt{(2 + r_0^4)}} \frac{\Gamma(\frac{5}{4}) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{3}{4}; \frac{-r_0^4}{2 + r_0^4}\right) \quad (2.2.50)$$

For large r_0 the last argument of the hypergeometric function simplifies to -1 and we get

$$T = r_0 \frac{\Gamma(\frac{5}{4}) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} \frac{2^{-1/4} \Gamma(\frac{3}{4})}{\Gamma(\frac{5}{8}) \Gamma(\frac{5}{8})} = r_0 \sqrt{\pi} \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{8})} \sim 1.1636 \dots r_0 \quad (2.2.51)$$

It is also of interest to compute the interval in σ along the D -string from the minimum size of the funnel cross-section to the place where the funnel blows up. The indefinite integral (2.2.16) gives by direct evaluation, just as above

$$\frac{i(1 + r_0^4)}{r_0^2} \frac{r}{\sqrt{2 + r_0^4}} F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{5}{4}; \frac{r^4}{r_0^4}, -\frac{r^4}{2 + r_0^4}\right) \quad (2.2.52)$$

The distance to blow up is given by the difference of the last expression evaluated at infinity and at r_0 . We do not have an exact formula for the large r asymptotics of the Appell function at finite r_0 . We will thus be forced to take the large r_0 limit immediately. This will reduce (2.2.52) to

$$i r_0 r {}_2F_1\left(\frac{1}{8}, \frac{1}{2}, \frac{9}{8}; \frac{r^8}{r_0^8}\right) \quad (2.2.53)$$

and the distance to blow up will be

$$\frac{i r_0 \Gamma(\frac{3}{8}) \Gamma(\frac{9}{8})}{\sqrt{\pi}(-1)^{1/8}} - \frac{i r_0 \sqrt{\pi} \Gamma(\frac{9}{8})}{\Gamma(\frac{5}{8})} \quad (2.2.54)$$

The final result is

$$\Sigma = r_0(\sqrt{2} - 1)\sqrt{\pi} \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{8})} \sim 0.4819 \dots r_0 \quad (2.2.55)$$

Hence the full period 4Σ is $1.9276 \dots r_0$. The time of collapse is $1.1636 \dots r_0$. The space and time periods are no longer the same as was the case for fuzzy- S^2 , since there is a relative factor of $(\sqrt{2} - 1)$.

2.3 Reduction of the $g=3$ curve and inversion of the hyper-elliptic integral for the fuzzy- S^4

Whereas we have formulae for the time elapsed t in terms of r (2.2.48) or the $D1$ -co-ordinate σ as a function of r by using (2.2.52), it is desirable to have the inverse formulae expressing r as a function of σ and t . It will turn out that, as in the case of the fuzzy- S^2 , it will be useful to define a complex variable $u_1 = \sigma - it$. Whereas the u variable in the case of fuzzy- S^2 lives on a genus one curve, here the story will involve higher genus curves and will require the introduction of a second complex variable u_2 .

Let us revisit the σ -integral in the case of the fuzzy- S^4 . The integral we want to perform is

$$\sigma = \int_{r_0}^r \frac{(1 + r_0^4) dr}{\sqrt{(r^4 - r_0^4)(r^4 - r_1^4)}}$$

with $r_1^4 = -(2 + r_0^4)$. If we make the re-scaling $\tilde{r} = \frac{r}{r_0}$ and $\tilde{r}_1 = -\frac{2+r_0^4}{r_0^4}$ we get

$$\frac{\sigma r_0^3}{(1 + r_0^4)} = \int_1^{\tilde{r}} \frac{d\tilde{r}}{\sqrt{(\tilde{r}^4 - 1)(\tilde{r}^4 - \tilde{r}_1^4)}} \quad (2.3.1)$$

The RHS is the integral over the holomorphic differential on Σ_3 of the full curve. Using the reduction to Σ_2 by making a change of variables $\tilde{r}^2 = x$, we arrive at

$$\frac{2r_0^3 \sigma}{(1 + r_0^4)} = \int_1^x \frac{dx}{\sqrt{x(x^2 - 1)(x^2 - \tilde{r}_1^4)}} \quad (2.3.2)$$

Similar steps for the time-dependent fuzzy sphere give

$$-i \frac{2r_0^3 t}{(1+r_0^4)} = \int_1^x \frac{dx}{\sqrt{x(x^2-1)(x^2-\tilde{r}_1^4)}} \quad (2.3.3)$$

At this point it is useful to introduce a complex variable $u_1 = \sigma - it$. The inversion of the integrals (2.3.2), (2.3.3) is related to the Jacobi Inversion problem

$$\begin{aligned} \int_{x_0}^{x_1} \frac{dx}{y} + \int_{x_0}^{x_2} \frac{dx}{y} &= u_1 \\ \int_{x_0}^{x_1} \frac{xdx}{y} + \int_{x_0}^{x_2} \frac{xdx}{y} &= u_2 \end{aligned} \quad (2.3.4)$$

where x_0 is any fixed point on the Riemann surface Σ_2 . We will set $x_0 = 1$. By further fixing $x_2 = 1$ we recover the integral of interest in the first line

$$\begin{aligned} \int_1^{x_1} \frac{dx}{y} &= u_1 \\ \int_1^{x_1} \frac{xdx}{y} &= u_2 \end{aligned} \quad (2.3.5)$$

This is a constrained Jacobi Inversion problem which is related to a sub-variety² of the Jacobian of Σ_2 , denoted as $J(\Sigma_2)$. A naive attempt to consider the inversion of the first equation of (2.3.5) in isolation runs into difficulties with infinitesimal periods as explained on page 238 of [81].

By switching to the variables ξ_{\pm}, η_{\pm} defined in (2.2.40), the system (2.3.4) can be reduced to the sum of simple elliptic integrals

$$\begin{aligned} \int_{\xi_0}^{\xi_1} \frac{d\xi_+}{\eta_+} + \int_{\xi_0}^{\xi_2} \frac{d\xi_+}{\eta_+} &= u_+ \\ \int_{\xi_0}^{\xi_1} \frac{d\xi_-}{\eta_-} + \int_{\xi_0}^{\xi_2} \frac{d\xi_-}{\eta_-} &= u_- \end{aligned} \quad (2.3.6)$$

where $\xi_1 = \xi_{\pm}(x_1)$ and $\xi_2 = \xi_{\pm}(x_2)$. We have used

$$\begin{aligned} \frac{d\xi_{\pm}}{\eta_{\pm}} &= \sqrt{(1-\alpha)(1-\beta)}(x \pm \sqrt{\alpha\beta}) \frac{dx}{y} \\ u_{\pm} &= \sqrt{(1-\alpha)(1-\beta)}(u_2 \pm \sqrt{\alpha\beta} u_1) \end{aligned} \quad (2.3.7)$$

The first of the two integrals can be brought into the form

$$\int_1^{\sqrt{\xi_1}} \frac{2dz}{\sqrt{(1-z^2)(1-k_+^2 z^2)}} + \int_1^{\sqrt{\xi_2}} \frac{2dz}{\sqrt{(1-z^2)(1-k_+^2 z^2)}} = u_+ \quad (2.3.8)$$

²The geometry of such sub-varieties is discussed extensively in [95, 96]. One result is that (2.3.5) defines a complex analytic homeomorphism from Σ_2 to a complex analytic submanifold of $J(\Sigma_2)$.

by the substitution $\xi = z^2$ and then split to give

$$2 \int_0^{\sqrt{\xi_1}} \frac{dz}{\sqrt{(1-z^2)(1-k_+^2 z^2)}} + 2 \int_0^{\sqrt{\xi_2}} \frac{dz}{\sqrt{(1-z^2)(1-k_+^2 z^2)}} - 4 \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k_+^2 z^2)}} = u_+ \quad (2.3.9)$$

which are just

$$2Sn^{-1}(\sqrt{\xi_1}, k_+) + 2Sn^{-1}(\sqrt{\xi_2}, k_+) - 4K(k_+) = u_+ \quad (2.3.10)$$

Then by using the addition formulae for Sn^{-1} functions [97], we arrive at

$$Sn^{-1} \left(\frac{\sqrt{\xi_1} \sqrt{(1-\xi_2)(1-k_+^2 \xi_2)} + \sqrt{\xi_2} \sqrt{(1-\xi_1)(1-k_+^2 \xi_1)}}{1-k_+^2 \xi_1 \xi_2}, k_+ \right) = \frac{u_+}{2} + 2K(k_+) \quad (2.3.11)$$

Thus by setting $x_2 = 1$ we get $\xi_2 = 1$ and

$$\frac{1-\xi_1}{1-k_+^2 \xi_1} = Sn^2 \left(\frac{u_+}{2} + 2K(k_+), k_+ \right) \quad (2.3.12)$$

After using the fact that Sn is anti-periodic in $2K(k)$ and implementing half-argument formulae, we end up with

$$\frac{1+Cn(u_+, k_+)}{1+Dn(u_+, k_+)} = \xi_1 \quad (2.3.13)$$

Starting from the second integral of (2.3.6) the same steps lead to

$$\frac{1+Cn(u_-, k_-)}{1+Dn(u_-, k_-)} = \xi_1 \quad (2.3.14)$$

These expressions hold the answer to the Jacobi Inversion problem, although we still need to decouple u_1 and u_2 in the u_\pm 's.

2.3.1 Series expansion of u_2 as a function of u_1

The Eqs. (2.3.13) and (2.3.14) imply that u_+ and u_- are constrained by

$$\xi_1 = \frac{1+Cn(u_+, k_+)}{1+Dn(u_+, k_+)} = \frac{1+Cn(u_-, k_-)}{1+Dn(u_-, k_-)} \quad (2.3.15)$$

This can be used to solve for $u_2(u_1)$ (or $u_1(u_2)$). We do not have an explicit general solution to this transcendental constraint but we can solve it in a series around the initial radius $x = 1$. This corresponds to small times, *i.e.* corresponds to doing perturbation theory for

u_1 around zero and similarly for u_2 , as can be seen from the integrals (2.3.5). We find

$$\begin{aligned} u_2 &= u_1 + \frac{1}{6}(1+R_1^4)u_1^3 + \frac{1}{120}(1+R_1^4)(7+3R_1^4)u_1^5 \\ &\quad + \frac{1}{2520}(1+R_1^4)(65+66R_1^4+9R_1^8)u_1^7 + \mathcal{O}(u_1^9) \end{aligned} \quad (2.3.16)$$

As a consistency check, we can invert the two integrals independently and see whether the expansion (2.3.16) agrees with the results.

In more detail we have

$$\int_1^{x_1} \frac{dx}{\sqrt{x(x^2-1)(x^2+R_1^4)}} = u_1$$

which can be expanded and inverted to give

$$\begin{aligned} x_1 &= \frac{1}{2}(1+R_1^4)u_1^2 + \frac{1}{24}(1+R_1^4)(7+3R_1^4)u_1^4 + \frac{1}{360}(1+R_1^4)(65+66R_1^4+9R_1^8)u_1^6 \\ &\quad + \frac{1}{4320}(1+R_1^4)(4645+7479R_1^4+3087R_1^8+189R_1^{12})u_1^8 + \mathcal{O}(u_1^{10}) \end{aligned} \quad (2.3.17)$$

The same can be done for

$$\int_1^{x_1} \frac{xdx}{\sqrt{x(x^2-1)(x^2+R_1^4)}} = u_2$$

giving

$$\begin{aligned} x_1 &= \frac{1}{2}(1+R_1^4)u_2^2 - \frac{1}{24}(1+R_1^4)(R_1^4-3)u_2^4 + \frac{1}{360}(1+R_1^4)(9+6R_1^4+5R_1^8)u_2^6 \\ &\quad + \frac{1}{4320}(1+R_1^4)(189+63R_1^4-297R_1^8-235R_1^{12})u_2^8 + \mathcal{O}(u_2^{10}) \end{aligned} \quad (2.3.18)$$

The expressions (2.3.17) and (2.3.18) can be combined to give the desired expansion of $u_2(u_1)$. We find perfect agreement between this and (2.3.16).

2.3.2 Evaluation of the time of collapse and the distance to blow up

We can use the previous discussion to give a new calculation of the time of collapse and the distance to blow up, which can be shown to satisfy some non-trivial checks against the formulae in the preceding subsections. Consider (2.3.15) at the limits $x_1 = 1$ and $x_1 = 0$, where $\xi_1 = 1$ and $\xi_1 = 0$. These lead to the two equations

$$Cn(u_{\pm}, k_{\pm}) = 0 \quad \text{and} \quad Cn(u_{\pm}, k_{\pm}) = -1 \quad (2.3.19)$$

which give

$$u_{\pm}^{x_1=1} = 2(2m_{\pm} + 1)K(k_{\pm}) + 4in_{\pm}K'(k_{\pm}) \quad \text{and} \quad u_{\pm}^{x_1=0} = 0 \quad (2.3.20)$$

We should recall that for our case of $\alpha = -1$, it turns out that k_+ is the complementary modulus of k_- , *i.e.* $k_+^2 + k_-^2 = 1$, which in turn means that

$$K'(k_{\pm}) = K(k_{\mp}) \quad (2.3.21)$$

which is very useful in simplifying (2.3.20). From (2.3.7) we can write

$$u_1 = \frac{u_+ - u_-}{2\sqrt{(1-\alpha)(1-\beta)}\sqrt{\alpha\beta}} \quad (2.3.22)$$

Then by collecting like terms, the time of collapse or the distance to blow up, will be extracted from imaginary or real values of u_1

$$\begin{aligned} u_1 &= u_1^{x_1=1} - u_1^{x_1=0} \\ &= \frac{(2m_+ - 2in_- + 1)K(k_+) - (2m_- - 2in_+ + 1)K(k_-)}{\sqrt{-2(1-\beta)\beta}} \end{aligned} \quad (2.3.23)$$

In order to compare this with something that we already know, we will consider the large- r_0 limit, where $\beta = i$ and $k_{\pm}^2 = \frac{1 \mp \sqrt{2}}{2}$. For these values we find that

$$K(k_-) = (\sqrt{2} - i)K(k_+) \quad (2.3.24)$$

Note that we should be careful here and observe a subtlety: By taking $r_0 \rightarrow \infty$, we make k_- *exactly* real and larger than 1, while at very large values of r_0 , k_- has a small but non-zero positive imaginary part. It will be useful to recall the properties of the complete elliptic integral of the first kind $K(k_-)$, with k_- real. This function has a branch cut extending from 1 to ∞ . This means that the value above and below the cut will differ by a jump. This statement translates to [94]

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} K(k_- - i\epsilon) &= K(k_-) \\ \lim_{\epsilon \rightarrow 0^+} K(k_- + i\epsilon) &= K(k_-) + 2iK'(k_-) \end{aligned} \quad (2.3.25)$$

Note that we are using the conventions for $K(k_{\pm})$ of [97, 98], which differ from those of [94] by a $k \rightarrow k^2$. We can numerically verify that what we have corresponds to the second case and we will thus amend the above eq. (2.3.24) to

$$K(k_-) = (2i + \sqrt{2} - i)K(k_+) \quad (2.3.26)$$

We are now ready to proceed. After some minor algebra and with the use of $\tan \frac{3\pi}{8} = 1 + \sqrt{2}$,

we reach

$$u_1 = -2^{-3/4} \cos \frac{3\pi}{8} (A + iB) K(k_+) \quad (2.3.27)$$

where $A = A_1 + \sqrt{2}A_2$ and $B = B_1 + \sqrt{2}B_2$ with

$$\begin{aligned} A_1 &= 2m_+ + 2n_- + 2m_- - 6n_+ + 2, & A_2 &= 2n_- - 2n_+ \\ B_1 &= 2m_+ - 2n_- - 6m_- - 2n_+ - 2, & B_2 &= 2m_+ - 2m_- \end{aligned} \quad (2.3.28)$$

Now recall from (2.3.2) that since t is real and for the large- r_0 limit $t = -r_0 u_1 / 2i$, we need u_1 to be purely imaginary in order to reproduce the results that we already have. This can be done simply by setting $A = 0$, which is satisfied by

$$\begin{aligned} n_+ &= n_- \\ m_+ + m_- &= 2n_+ - 1 \end{aligned} \quad (2.3.29)$$

and has a simple solution if we choose $m_- = -1$ and $m_+ = n_+ = n_- = 0$.

Then (2.3.27) becomes

$$u_1 = -i 2^{-3/4} \cos \frac{3\pi}{8} (4 + 2\sqrt{2}) K(k_+) \quad (2.3.30)$$

and with the relation $u_1 = \sigma - it$ the time of collapse for large- r_0 follows

$$\begin{aligned} T &= \frac{r_0}{2} K \left(\sqrt{\frac{1 - \sqrt{2}}{2}} \right) \cos \frac{3\pi}{8} 2^{-3/4} (4 + 2\sqrt{2}) \\ &\simeq 2.3272 \dots \frac{r_0}{2} = 1.1636 \dots r_0 \end{aligned} \quad (2.3.31)$$

exactly what we got previously from the evaluation of the integral in terms of Appell functions. When the conditions (2.3.29) for the vanishing of A are satisfied, the expressions for B_1 and B_2 simplify to

$$B_1 = -4(2m_- + 1) \quad B_2 = -2(2m_- + 2n_+ + 1) \quad (2.3.32)$$

Setting $n_+ = 0$ we have B_1 and B_2 both proportional to $(2m_- + 1)$. This means that the first collapse to zero is repeated after every interval of twice the initial collapse time. This behaviour was anticipated using continuity and the symmetries (2.2.25).

Similarly, we can set $B = 0$ and recover a real expression which will correspond to the distance that it takes for the funnel which has a minimum cross-section of r_0 to grow to infinity. We first use the large r_0 limit. The reality conditions are

$$\begin{aligned} m_+ &= m_- \\ n_+ + n_- &= -(2m_- + 1) \end{aligned} \quad (2.3.33)$$

and a simple solution is the one with $m_- = m_+ = n_+ = 0$ and $n_- = -1$. Then

$$\begin{aligned}\Sigma &= r_0 K \left(\sqrt{\frac{1-\sqrt{2}}{2}} \right) \cos \frac{3\pi}{8} 2^{-3/4} \sqrt{2} \\ &\simeq 0.9639 \dots \frac{r_0}{2} = 0.4819 \dots r_0\end{aligned}\tag{2.3.34}$$

again in agreement with the previous results. Generally when the conditions for the vanishing of the imaginary part are satisfied, expressions for A_1 and A_2 simplify

$$A_1 = -8n_+ \quad A_2 = -2(2m_+ + 2n_+ + 1)$$

Setting $n_+ = 0$ gives a sequence of poles at distances proportional to $2m_+ + 1$. This was anticipated from symmetry and continuity using (2.2.17) and is shown in Fig. 2. Note that the pattern of zeroes along the time-axis is the same as for the fuzzy- S^2 . The pattern of poles along the space axis is also the same as for the fuzzy- S^2 . The difference is that the time from maximum radius to zero in the time evolution is not identical to the distance from minimum radius to infinite radius as in the case of fuzzy- S^2 . There is, nevertheless, a simple ratio of $(\sqrt{2} - 1)$ at the large r_0 limit.

It is also important to stress that we can use the above solutions to study the time of collapse for finite r_0 and then compare with what one gets from (2.2.50). We have checked numerically, for many values of r_0 between zero and ∞ , that the time given in (2.2.50) agrees to six decimal digits with the expression

$$T = \frac{(1+r_0^4)}{2r_0^3} \frac{(K(k_+) + K(k_-))}{\sqrt{-2\beta(1-\beta)}}\tag{2.3.35}$$

which reduces to (2.3.30) at large r_0 . Similarly for the distance to blow up we have

$$\Sigma = \frac{(1+r_0^4)}{2r_0^3} \frac{(2i+1)K(k_+) - K(k_-)}{\sqrt{-2\beta(1-\beta)}}\tag{2.3.36}$$

We believe the same numerical agreement to hold for finite r_0 . However, since the asymptotics of (2.2.52) for large r at finite r_0 were not well available with our mathematical software, we cannot confirm this.

So far we have considered the time of collapse where $r = 0$ (or the distance to blow up in the spatial case), but we can also consider t as a function of r for any finite r in terms of the Appell function (2.2.46). The inverse expression of r in terms of t , and more generally the complex variable $u_1 = \sigma - it$, is contained in (2.3.12) and the constraint (2.3.15).

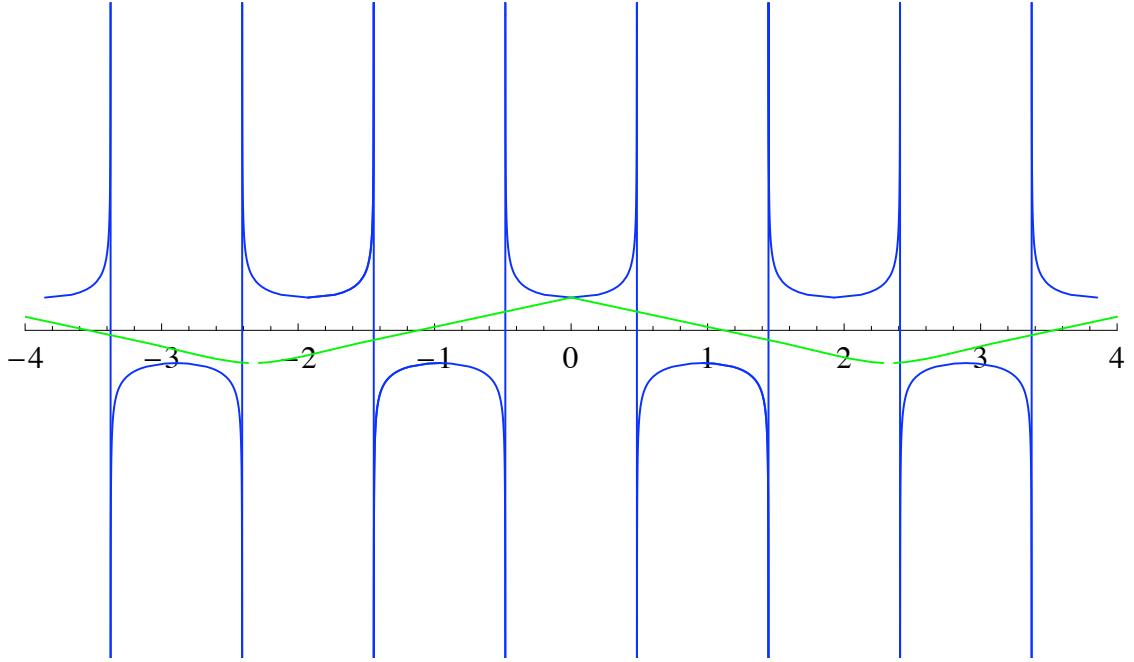


Figure 2.2: Numerical plot of the Jacobi elliptic function solution for the static fuzzy- S^4 funnel array and the collapsing fuzzy- S^4 at large r_0 .

2.3.3 Solution of the problem in terms of the u_2 variable and large-small duality

We have seen that inverting the integrals (2.2.22) and (2.2.16) requires the definition of a complex variable $u_1 = \sigma_1 - it_1$ whose real and imaginary parts are related to the space and time variables. In addition we have to introduce the second holomorphic differential, so that there are 2 complex variables u_1, u_2 . These variables are defined in terms of integrals and thus are subject to identifications by the period lattice \mathcal{L} of integrals around the a and b -cycles of the genus 2 Riemann surface, Σ_2 . They live on \mathbb{C}^2/\mathcal{L} , the Jacobian of Σ_2 . Introduction of a second point x_2 on the Riemann surface relates our problem to the standard Jacobi Inversion problem. The constraint $x_2 = 1$ restricts to a subvariety of the Jacobian. So far the constraint (2.3.15) has been viewed as determining u_2 in terms of u_1 . This allowed us to describe x_1 as a function of u_1 in the neighbourhood of $x_1 = 1$ and also to get new formulae for the time of collapse/distance to blow up which have been checked numerically against evaluation of the integrals using Appell functions. However, it is also of interest to consider using the constraint to solve u_1 in terms of u_2 and hence describe x_1 as a function of u_2 . The reason for this is that the automorphisms of the Riemann surface allow us to relate the large r (equivalently large x_1) behaviour of the spatial problem described in terms of the u_1 variable, to the small r (equivalently small x_1) behaviour of the time-dependent problem described in terms of the u_2 variable and vice versa. The relation is simpler in the large r_0 limit, hence we describe this first and then we return to the case of

finite r_0 .

The use of the u_2 variable is natural if we introduce another Lagrangean³. The original Lagrangean of interest and the new Lagrangean are coupled in the Jacobi Inversion problem (2.3.4) as well as the constrained version of the Jacobi Inversion problem (2.3.5) obtained by setting $x_2 = 1$. Just as u_1 is the complexified variable for the first Lagrangean, u_2 is the complexified space-time variable for the second Lagrangean. It will be convenient, for the following discussion, to define the relation between the second set of space-time variables t_2, σ_2 and the complex variable u_2 as $u_2 = -\sigma_2 + it_2$.

We have also seen that the automorphism U_1 of Sec. 2.2.4 maps du_1 to $-idu_2$ as in (2.2.44), hence takes the time dependent problem for the first kind of Lagrangean to the static one for the second mapping zeroes of a given periodicity to poles of the same periodicity. We will check this by investigating what happens to u_2 when we follow the collapse of x_1 down to zero along imaginary u_2 , or the blow-up of x_1 to infinity along real u_2 . It is clear from the integrals in (2.3.5) (and the series expansion (2.3.16)) that when x_1 decreases from 1 to 0, both u_1 and u_2 are imaginary and when x_1 increases from 1 to infinity, both are real.

Solving for u_2 , the second of the Eqs. (2.3.7) will give

$$u_2 = \frac{u_+ + u_-}{2\sqrt{(1-\alpha)(1-\beta)}} \quad (2.3.37)$$

and by following the steps leading to (2.3.27) and (2.3.28) we arrive at

$$u_2 = -2^{-3/4} \cos \frac{\pi}{8} (A' + iB') K(k_+) \quad (2.3.38)$$

Here we used $\tan \frac{\pi}{8} = \sqrt{2} - 1$. The A' 's and B' 's are given by

$$\begin{aligned} A'_1 &= 2m_+ + 2n_- + 2m_- - 6n_+ + 2, & A'_2 &= 2n_+ - 2n_- \\ B'_1 &= 2 + 6m_- + 2n_- - 2m_+ + 2n_+, & B'_2 &= 2m_+ - 2m_- \end{aligned} \quad (2.3.39)$$

Notice by comparing (2.3.28) and (2.3.39), that $A'_1 = A_1, B'_1 = -B_1$ and $A'_2 = -A_2, B'_2 = B_2$. By choosing $m_- = -1, m_+ = n_- = n_+ = 0$ we get

$$\begin{aligned} u_2 &= 2i(2 - \sqrt{2}) \cos \frac{\pi}{8} 2^{-3/4} K(k_+) \\ &\simeq i 0.9639 \dots \end{aligned} \quad (2.3.40)$$

which is *exactly* the same as in (2.3.34) in agreement with what we expect from the auto-

³We will look at these Lagrangeans in a following subsection. They are of the type (2.3.44), with $\alpha = 2$.

morphism. Similarly for the real case

$$\begin{aligned} u_2 &= -2\sqrt{2} \cos \frac{\pi}{8} 2^{-3/4} K(k_+) \\ &\simeq -2.3271\dots \end{aligned} \tag{2.3.41}$$

With the definition $u_2 = -\sigma_2 + it_2$ we see that we are getting the expected positive time of collapse in terms of the alternative time-variable and the expected positive distance to blow up with the alternative spatial variable. The matching of the A' and B' with A and B guarantees that x_1 , expressed as a function of u_2 , will have zeroes along the imaginary axis at odd integer multiples of a basic time of collapse and poles along the real axis at odd integer multiples of a basic distance to blow up. The time of collapse for u_2 is the same as the distance to blow up for u_1 and the distance to blow up for u_2 is the same as the time of collapse for u_1 . This gives a precise map between the behaviour at zero and the behaviour at infinity of the radius of the fuzzy sphere, generalising the relations that were found for the case of the fuzzy 2-sphere. This relation is expected from the automorphism U_1 for large r_0 described in Sec. 2.2.4, which maps u_1 to $-iu_2$ (2.2.44). This precise large-small relation for space and time-dependent fuzzy spheres is physically interesting. The physics of the large r limit is very well understood because it corresponds to the D -strings blowing up into a $D5$ -brane. The physics of the small r limit appears mysterious because it involves sub-stringy distances. We have shown that the two regions are closely related through the underlying Riemann surface which unifies the space and time aspects of the problem.

Large-small duality at finite r_0

We saw in the discussion above that the spatial problem of the fuzzy- S^4 evolving from a minimum size at r_0 to infinity can be related to the time-dependent problem of the fuzzy sphere collapsing from r_0 to zero. This large-small relation involves a map between the u_1 description and the u_2 description of the problem and uses a simplification which is valid at large r_0 . There continues to be a large-small duality at finite r_0 , but it is slightly more involved than the one at large r_0 . The difference is due to the nature of the automorphisms of the Riemann surface at finite r_0 and in the large r_0 limit. Indeed, the discussion above used the automorphism U_1 in a crucial way.

To describe the duality at finite r_0 , it is useful to think of a problem similar to the one we considered above, by choosing a different base-point in (2.3.5), *i.e.* $x_0 = -iR_1^2$

$$\begin{aligned} \int_{-iR_1^2}^{\tilde{x}_1} \frac{dx}{y} &= \tilde{u}_1 \\ \int_{-iR_1^2}^{\tilde{x}_1} \frac{xdx}{y} &= \tilde{u}_2 \end{aligned} \tag{2.3.42}$$

The upper limit is chosen, in the first instance, to vary along the negative imaginary axis

up to zero. A second problem is to consider \tilde{x}_1 extending along the negative imaginary axis down to infinity.

Applying the automorphism T_1 of (2.2.36) to the first line of (2.3.5) we have

$$\begin{aligned} u_1(x_1) &= \int_1^{x_1} \frac{dx}{y} \\ &= \frac{e^{\frac{i\pi}{4}}}{R_1} \int_{-iR_1^2}^{\tilde{x}_1 = -\frac{iR_1^2}{x}} \frac{X dX}{Y} \\ &= \frac{e^{\frac{i\pi}{4}}}{R_1} \tilde{u}_2 \left(\tilde{x}_1 = -\frac{iR_1^2}{x} \right) \end{aligned}$$

Similarly we have

$$u_2(x_1) = R_1 e^{\frac{i\pi}{4}} \tilde{u}_1 \left(\tilde{x}_1 = -\frac{iR_1^2}{x} \right) \quad (2.3.43)$$

As we saw earlier in this section, the solution to (2.3.5) can be described in terms of either the u_1 or the u_2 variable. Likewise the inversion of (2.3.42) can be expressed in terms of either \tilde{u}_1 or \tilde{u}_2 . The action of the automorphism T_1 described above implies that the solution of the spatial problem (2.3.5) where x_1 evolves from 1 to infinity along the real axis, when given in terms of the real part of the u_1 variable, maps to the evolution of \tilde{x}_1 from $-iR_1^2$ along the imaginary axis to zero, as described by the \tilde{u}_2 variable. Similarly the time-dependent problem of x_1 evolving from 1 to zero when described in terms of the imaginary part of the u_1 variable, maps to the evolution of \tilde{x}_1 along the imaginary axis from $-iR_1^2$ to infinity, as described by the \tilde{u}_2 variable. This shows that there continues to be a large-small duality at finite r_0 , but it relates the original problem with real x to a problem with imaginary x .

2.3.4 Lagrangeans for holomorphic differentials

We have seen that the space or time-dependence of the radius of the fuzzy- S^4 described by the Lagrangean (2.2.4) is given by integrating the holomorphic differential $\frac{dr}{s}$ on the curve (r, s) given by (2.2.21). There are more general holomorphic differentials, which enter on an equal footing in the geometry of the same Riemann surface. For the hyper-elliptic curves of the type we considered, they are of the form $\frac{r^\alpha dr}{s} \equiv dt_\alpha$, where α can take values from 1 to the genus of the curve. It is natural to ask if there are Lagrangeans such that the time elapsed or distance are given by the more general holomorphic differentials. This is indeed possible and the Lagrangean densities are

$$L_\alpha = -\sqrt{1 - r^{2\alpha}(\dot{r}^2 - r'^2)}(1 + r^4) \quad (2.3.44)$$

The automorphisms we described in the previous section, can be used to relate the time

evolution of interest, which appears as the imaginary part of u_1 , to the space evolution for $\alpha = 2$. The variables u_2 are related to the $\alpha = 2$ holomorphic differential of the genus three curve and hence to the $\alpha = 2$ Lagrangean above.

It is also interesting to explore how the second order equations following from L_α transform under inversion of r . Dropping the space dependence, the equation of motion following from (2.3.44) is

$$\ddot{r} = -\frac{4r^{3-2\alpha}}{1+r^4} + \frac{\dot{r}^2}{r(1+r^4)} \left((4-\alpha)r^4 - \alpha \right) \quad (2.3.45)$$

Under a transformation $R = \frac{1}{r}$, $\tau = t$, we get

$$\ddot{R} = \frac{4R^{3+2\alpha}}{1+R^4} + \frac{\dot{R}^2}{R(1+R^4)} \left((\alpha+2)R^4 + (\alpha-2) \right) \quad (2.3.46)$$

If we ignore, for the moment, the first term on the RHS, we see that the $\alpha = 0$ equation in (2.3.45) maps to the $\alpha = 2$ term. If we neglect the velocity dependent term on the RHS, the $\alpha = 0$ maps back to $\alpha = 0$.

Further consider the transformation $R = \frac{1}{r}$, $\frac{d\tau}{dt} = -\frac{1}{R^2}$, to get

$$\ddot{R} = \frac{4R^{7+2\alpha}}{1+R^4} + \frac{\dot{R}^2}{R(1+R^4)} \left((\alpha+4)R^4 + \alpha \right) \quad (2.3.47)$$

If we keep only the velocity dependent term on the RHS, we find that the case $\alpha = 0$ from (2.3.45) maps to the case $\alpha = 0$ of (2.3.47).

While these transformations include relations between the $\alpha = 0$ and $\alpha = 2$ Lagrangeans, they require neglecting some terms in the equation of motion so they would hold in special regimes where these are valid. The transformations involving r_0 , which we also described in the previous section, give a more direct relation between the functions solving equations from L_0 and L_2 . It would be interesting to find physical brane systems which directly lead to L_α for $\alpha \neq 0$.

2.4 Space and Time-dependent Fuzzy- S^6

Here we will briefly talk about the nature of the solution in the case of the $D1 \perp D7$ intersection, discussed by [62], which involves the fuzzy- S^6 . Starting from the D -string theory point of view, we will now have seven transverse scalar fields, given by the Ansatz

$$\Phi^i(\sigma, \tau) = \pm \hat{R}(\sigma, \tau) G^i, \quad i = 1, \dots, 7, \quad (2.4.1)$$

where the G^i 's are given by the action of the $SO(7)$ Γ^i 's on the symmetric and traceless n -fold tensor product of the basic spinor V , the dimension of which is related to n by [48]

$$N = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360} \quad (2.4.2)$$

Again, the radial profile and the fuzzy- S^6 physical radius are related by

$$R_{ph}(\sigma, \tau) = \sqrt{C} \lambda \hat{R}(\sigma, \tau) \quad (2.4.3)$$

with C the quadratic Casimir $G^i G^i = C \mathbb{1}_{N \times N} = n(n+6) \mathbb{1}_{N \times N}$. The time-dependent generalisation for the leading $1/N$ action of [62] can be written in dimensionless variables, which are once more defined as in (2.1.9),

$$S_1 = -N T_1 \int d^2\sigma \sqrt{1 + r'^2 - \dot{r}^2} (1 + r^4)^{3/2} \quad (2.4.4)$$

In a manner identical to the discussion for the fuzzy- S^2 and the fuzzy- S^4 , the equations of motion can be given in a Lorentz-invariant expression

$$\partial_\mu \partial^\mu r + (\partial_\mu \partial^\mu r) (\partial_\nu r) (\partial^\nu r) - (\partial_\mu \partial^\nu r) (\partial_\nu r) (\partial^\mu r) = 6 r^3 \left(\frac{1 + (\partial_\mu r) (\partial^\mu r)}{1 + r^4} \right) \quad (2.4.5)$$

At this point it is natural to propose that the form of the action and the equations of motion will generalise in a nice way for any fuzzy- S^{2k} sphere

$$S_1 = -T_1 \int d\sigma^2 S Tr \sqrt{(1 + r'^2 - \dot{r}^2) (1 + r^4)^k} \quad (2.4.6)$$

with the large- N equations of motion

$$\partial_\mu \partial^\mu r + (\partial_\mu \partial^\mu r) (\partial_\nu r) (\partial^\nu r) - (\partial_\mu \partial^\nu r) (\partial_\nu r) (\partial^\mu r) = 2k r^3 \left(\frac{1 + (\partial_\mu r) (\partial^\mu r)}{1 + r^4} \right) \quad (2.4.7)$$

As we saw in the previous cases, there will be a curve related to the blow-up of the funnel, derived by the conservation of pressure if we restrict to static configurations and also to the corresponding collapse of a $D6$ -brane by conservation of energy if we completely drop the space variable. We find that the curve determining the solutions is

$$s^2 = (r^4 - r_0^4)(r^4 - r_1^4)(r^4 - r_2^4) \quad (2.4.8)$$

which is of genus 5 and where a factor of $(1 + r_0^4)$ has been absorbed in the definition of s .

The roots are given by

$$\begin{aligned} (1 + r_0^4) &= u_0^4 \\ (1 + r_1^4) &= u_0^4 \eta \\ (1 + r_2^4) &= u_0^4 \eta^2 \end{aligned} \tag{2.4.9}$$

with $\eta = \exp \frac{2i\pi}{3}$.

• **Automorphism at large r_0 :**

At large r_0 , we have $r_1^4 = r_0^4 \eta$ and $r_2^4 = r_0^4 \eta^2$. Then there exists an automorphism

$$\begin{aligned} R &= \frac{r_0^2}{r} \\ S^2 &= -\frac{s^2 r_0^{12}}{r^{12}} \end{aligned}$$

It is convenient to define $\tilde{r} = \frac{r}{r_0}$, and $\tilde{s}^2 = s^2 r_0^{12}$. In these variables

$$\begin{aligned} \tilde{r} &\rightarrow \frac{1}{\tilde{r}} \\ \tilde{s} &\rightarrow \frac{i\tilde{s}}{\tilde{r}^6} \end{aligned}$$

and the action on the holomorphic differentials is

$$\begin{aligned} \omega_1 &\rightarrow i\omega_5 \\ \omega_2 &\rightarrow i\omega_4 \\ \omega_3 &\rightarrow i\omega_3 \\ \omega_4 &\rightarrow i\omega_2 \\ \omega_5 &\rightarrow i\omega_1 \end{aligned}$$

• **Automorphism at $r_0 = 0$:**

Now we have

$$s^2 = r^4(r^4 - r_1^4)(r^4 - r_2^4) \tag{2.4.10}$$

and there is an automorphism

$$\begin{aligned} R &= \frac{r_1 r_2}{r} \\ S^2 &= \frac{s^2 (r_1 r_2)^8}{r^{16}} \end{aligned}$$

In this limit $r_1^4 = (\eta - 1)$ and $r_2^4 = (\eta^2 - 1)$, so in the formulae above we can write $r_1^4 r_2^4 = 3$.

- **Symmetry at finite r_0 :**

It is useful to write the curve in terms of a variable u defined by $u^4 = (1 + r^4)$ and to write $u_0^4 = 1 + r_0^4$. Then we have

$$s^2 = (u^4 - u_0^4)(u^4 - u_0^4\eta)(u^4 - u_0^4\eta^2) \quad (2.4.11)$$

A symmetry is $v = \frac{u_0^2}{u}$, $\tilde{s}^2 = -\frac{s^2 u_0^{12}}{u^{12}}$. Expressing the symmetry in the r, s variables, we have R, S obeying the same eq. (2.4.8) with

$$\begin{aligned} (1 + R^4) &= \frac{(1 + r_0^4)^2}{(1 + r^4)} \\ S^2 &= -\frac{s^2(1 + r_0^4)^3}{(1 + r^4)^3} \end{aligned}$$

This reduces to the $R = \frac{r_0^2}{r}$ for $r \gg 1$, $R \gg 1$, $r_0 \gg 1$. Unfortunately R is not a rational function of r , but an algebraic function of r involving fourth roots, hence it is not a holomorphic or meromorphic function. Hence it is not possible to use this symmetry to map the holomorphic differentials of the genus 5 curve to those on genus 1 curves. We can still make the change of variables $x = r^2$ to get a reduction down to genus 3, but we have not been able to reduce this any further.

Since we cannot reduce the curve down to a product of genus one curves, we cannot relate the problem of inverting the hyper-elliptic integral to elliptic functions. We can nevertheless relate it to the Jacobi Inversion problem at genus 3. We consider variables u_1, u_2, u_3 defined as

$$\begin{aligned} \int_1^{x_1} \omega_1 + \int_1^{x_2} \omega_1 + \int_1^{x_3} \omega_1 &\equiv u_1 \\ \int_1^{x_1} \omega_2 + \int_1^{x_2} \omega_2 + \int_1^{x_3} \omega_2 &\equiv u_2 \\ \int_1^{x_1} \omega_3 + \int_1^{x_2} \omega_3 + \int_1^{x_3} \omega_3 &\equiv u_3 \end{aligned}$$

The variables u_1, u_2, u_3 live on the Jacobian of the genus three curve, which is a complex torus of the form \mathbb{C}^3/\mathcal{L} . The integrands appearing above live naturally on a Riemann surface which is a cover of the Riemann sphere, branched at 8 points. The lattice \mathcal{L} arises from doing the integrals around the a and b -cycles of the Riemann surface. The Eqs. (2.4.12) can be inverted to express

$$\begin{aligned} x_1 &= x_1(u_1, u_2, u_3) \\ x_2 &= x_2(u_1, u_2, u_3) \\ x_3 &= x_3(u_1, u_2, u_3) \end{aligned}$$

where $x_i(u_1, u_2, u_3)$ can be given in terms of genus three theta functions, or equivalently in terms of hyper-elliptic Kleinian functions [86]. The system (2.4.12) simplifies if we set $x_2 = x_3 = 1$. These simplified equations define a sub-variety of the Jacobian which is isomorphic to the genus 3 Riemann surface we started with [95]. The constraints $x_2 = x_3 = 1$ can be used to solve, at least locally near $x_1 = 1$, for u_2, u_3 in terms of u_1 . Then we can write $x_1(u_1, u_2, u_3)$ as $x_1(u_1)$. This program was carried out explicitly in Sec. 2.3, where the higher genus theta functions degenerated into expressions in terms of ordinary elliptic functions thanks to the reduction of the genus three curve we started with, to a product of genus one curves.

We can make use of the hyper-elliptic function technology of Sec.2.1.4 and give at least an implicit solution for the S^6 at the stage where we have been able to reduce the problem from one of genus-5, to one of genus-3. By equating the general polynomial with roots x_1, x_2, x_3 to the polynomial with \wp -coefficients, we get

$$\begin{aligned} x_1 + x_2 + x_3 &= \wp_{33}(u_1, u_2, u_3) \\ x_1 x_2 x_3 &= \wp_{23}(u_1, u_2, u_3) \\ x_1 x_2 + x_2 x_3 + x_3 x_1 &= -\wp_{13}(u_1, u_2, u_3) \end{aligned} \tag{2.4.12}$$

As we have already mentioned, we can fix two of the three points to be $x_2 = x_3 = 1$, to get

$$\begin{aligned} x_1 + 2 &= \wp_{33}(u_1, u_2, u_3) \\ x_1 &= \wp_{23}(u_1, u_2, u_3) \\ 2x_1 + 2 &= -\wp_{13}(u_1, u_2, u_3) \end{aligned} \tag{2.4.13}$$

This implies two transcendental constraints which can be used to get a solution for x_1 , in terms of the \wp 's as functions of u_1 .

CHAPTER 3

SMALL-FLUCTUATION ANALYSIS

So far we have studied the classical actions for a class of static configurations of orthogonal D -brane intersections and time dependent spherical brane collapse in diverse dimensions, related to the geometry of fuzzy even-spheres. It is natural to explore whether the equivalence at the level of classical solutions extends to an equivalence at the level of quadratic fluctuations. In this chapter we study the fluctuations of the time-dependent $D0$ - $D2$ brane system. We consider the action for fluctuations using the $D2$ -brane action. We find that the result is neatly expressed in terms of the open string variables of Seiberg and Witten [52]. The quadratic action is a ($U(1)$) Yang-Mills theory with a time-dependent coupling, effective metric and a Θ -parameter. The radial scalar couples to the Yang-Mills gauge field. We analyse the wave equation for the scalar fluctuations and identify a critical radius for the fuzzy sphere where strong coupling effects set in. This radius is different for different values of the angular momentum of the excitations. The fluctuation equation for scalars transverse to the \mathbb{R}^3 containing the embedded sphere turns out to belong to the class of solvable Lamé equations. It is very interesting that such an integrable structure appears in a non-supersymmetric context.

We will obtain the quadratic action for fluctuations on the sphere from the non-abelian symmetrised trace action of N $D0$ -branes. We find precise agreement with the action obtained from the $D2$ -side. The fact that the commutators $[\Phi_i, \Phi_j]$ contain terms which scale differently with N means that we need to keep $1/N$ terms from commutators of fields. The non-commutative geometry of the fuzzy sphere is reviewed and applied to this derivation [99]. We observe that the mass term for the radial scalar we obtain can also be calculated from the reduced action for the radial variable. This simple calculation is extended to higher dimensional fuzzy spheres and shows similar qualitative features.

We will also describe a DBI scaling limit, where $N \rightarrow \infty$, $g_s \rightarrow 0$ and $\ell_s \rightarrow 0$ keeping fixed the quantities $L = \ell_s \sqrt{\pi N}$, $\tilde{g}_s = g_s \sqrt{N}$ along with specified radius variables and gauge coupling constants. In this limit, the non-linearities of the gauge coupling, which have a square root structure coming from the DBI action, survive. We discuss the physical meaning of this scaling and its connection with the DKPS limit [100], which is important in the BFSS Matrix Model proposal for M-theory [13].

3.1 Yang-Mills type action for small fluctuations

When the spherical membrane is sufficiently large, we may use the Dirac-Born-Infeld action to obtain a small-fluctuation action about the time-dependent solution of [63]

$$-\frac{1}{4\pi^2 g_s^2 \ell_s^3} \int dt d\theta d\phi \sqrt{-\det(h_{\mu\nu} + \lambda F_{\mu\nu})} \quad (3.1.1)$$

where $\lambda = 2\pi\ell_s^2$; $h_{\mu\nu}$ is the induced metric on the brane and $F_{\mu\nu}$ describes the gauge field strength on the membrane. The gauge field configuration on the brane consists of a uniform background magnetic field, $B_{\theta\phi} = N \sin\theta/2$, and the fluctuations $f_{\mu\nu}$: $F_{\mu\nu} = (B + f)_{\mu\nu}$. The background magnetic field results from the original N $D0$ -branes, which dissolve into uniform magnetic flux inside the $D2$ -brane.

To quadratic order in the fluctuations, the action will involve a Maxwell field coupled together with a radial scalar field controlling the size and shape of the membrane. The parameters of this theory will be time-dependent because we are expanding about a time-dependent solution to the equations of motion. For the radial field we write $\tilde{R} = R + \lambda(1 - \dot{R}^2)^{1/2}\chi(t, \theta, \phi)$, where R satisfies the classical equations of motion and χ describes the fluctuations. The normalisation is chosen for later convenience. We also take into consideration scalar fluctuations in the directions transverse to the \mathbb{R}^3 containing the embedded S^2 of the brane world-volume, described by six scalar fields $\lambda \xi_m(t, \theta, \phi)$. Using the equations of motion, we have that the background field R satisfies the following conservation law equation [63]

$$1 - \dot{R}^2 = \frac{R^4 + N^2\lambda^2/4}{R_0^4 + N^2\lambda^2/4} = \frac{R^4 + L^4}{R_0^4 + L^4} \quad (3.1.2)$$

We have introduced the physical length L defined by

$$L^2 = \frac{N\lambda}{2} \quad (3.1.3)$$

which simplifies formulae and plays an important role in the scaling discussion of Sec. 3.3. Here R_0 can be thought of as the initial radius of the brane at which the collapsing rate \dot{R} is zero. The solution $R(t)$ to (3.1.2) decreases from R_0 to zero, goes negative and then oscillates back to its initial value. It was argued, using the $D0$ -brane picture in Ch. 2, that the physical radius R_{phys} should be interpreted as the modulus of R . Hence this is a periodic collapsing/expanding membrane, which reaches zero size and expands again. The finite time of collapse is given by

$$\bar{t} = c \frac{\sqrt{R_0^4 + L^4}}{R_0} \quad (3.1.4)$$

where the numerical constant c is given by $K(1/\sqrt{2})/\sqrt{2}$, with K a complete elliptic integral.

To leading (zero) order in the fluctuations, the induced metric $h_{\mu\nu}$ on the brane is given by

$$ds^2 = -(1 - \dot{R}^2)dt^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.1.5)$$

From the form of the induced metric we see that the proper time T measured by a clock co-moving with the brane is related to the closed string frame time t by a varying boost factor

$$dt = \frac{dT}{\sqrt{1 - \dot{R}^2}} \quad (3.1.6)$$

So an observer co-moving with the collapsing brane concludes that the collapse is actually occurring faster. In terms of proper time, the metric takes the form of a closed three-dimensional Robertson-Walker cosmology

$$ds^2 = -dT^2 + R^2(T)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.1.7)$$

with scale factor R . The analogue of the Friedman equation is the conservation law (3.1.2).

Expanding the DBI action to quadratic order in the fluctuations we obtain the following:

$$S_2 = - \int dt d\theta d\phi \frac{\sqrt{-G}}{2g_{YM}^2} \left[\frac{1}{2} G^{\mu\alpha} G^{\nu\beta} f_{\mu\nu} f_{\alpha\beta} + G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + m^2 \chi^2 + G^{\mu\nu} \partial_\mu \xi_m \partial_\nu \xi_m \right] \quad (3.1.8)$$

The effective metric $G_{\mu\nu}$ seen by the fluctuations is given by

$$ds_{open}^2 = -(1 - \dot{R}^2)dt^2 + \frac{R^4 + L^4}{R^2} d\Omega^2 \quad (3.1.9)$$

As we will see it is precisely the open string metric defined by Seiberg and Witten [52] in the presence of background B-fields. The coupling constant is given by

$$g_{YM}^2 = \frac{g_s}{\ell_s} \frac{\sqrt{R^4 + L^4}}{R^2} \quad (3.1.10)$$

and the mass of the scalar field is given by

$$m^2 = \frac{6R^2}{(1 - \dot{R}^2)(R^4 + L^4)^2} (L^4 - R^4) \quad (3.1.11)$$

As expected, linear terms in the fluctuations add to total derivatives once we use the equations of motion for the scale factor R .

The set-up here differs from the original set-up of Seiberg-Witten [52] in that we have a non-constant B-field, $B_{\theta\phi} = N \sin \theta/2$. However the basic observation that in the presence of a background magnetic field, the open strings on the brane see a different metric $G_{\mu\nu}$

from the closed string frame metric $h_{\mu\nu}$ ¹

$$\begin{aligned} h_{00} &= -(1 - \dot{R}^2) \\ h_{\theta\theta} &= R^2 \\ h_{\phi\phi} &= R^2 \sin^2 \theta \end{aligned} \quad (3.1.12)$$

continues to be true. The metric $G_{\mu\nu}$ is indeed related to $h_{\mu\nu}$ by

$$G_{00} = h_{00}, \quad G_{ab} = h_{ab} - \lambda^2 (Bh^{-1}B)_{ab} \quad (3.1.13)$$

or

$$G_{\mu\nu} = h_{\mu\nu} - \lambda^2 (Bh^{-1}B)_{\mu\nu} \quad (3.1.14)$$

The open string metric (3.1.9) is qualitatively different from the closed string metric. Despite the fact that the original induced metric $h_{\mu\nu}$ becomes singular when the brane collapses to zero size, the open string metric $G_{\mu\nu}$ is never singular. To see this, let us compute the area of the spherical brane in the open string frame. This is given by

$$A = 4\pi \left(R^2 + \frac{L^4}{R^2} \right) \quad (3.1.15)$$

As R varies, this function has a minimum at $R = L$, at which $A_{min} = 4\pi N\lambda$ and the density of $D0$ -branes is precisely at its maximum $1/4\pi\lambda$; that is, of order one in string units. Effectively, the open strings cannot resolve the constituent $D0$ -branes at distance scales shorter than the string length.

The coupling constant can be expressed as $g_{YM}^2 = G_s \ell_s^{-1}$, where

$$G_s = g_s \left(\frac{\det G_{\mu\nu}}{\det(h_{\mu\nu} + \lambda B_{\mu\nu})} \right)^{1/2} = g_s \frac{\sqrt{R^4 + L^4}}{R^2} \quad (3.1.16)$$

So as R decreases, the open strings on the brane eventually become strongly coupled.

There is also a time-dependent vacuum energy density

$$S_0 = -\frac{1}{4\pi^2 g_s \ell_s^3} \int dt d\theta d\phi \sqrt{-G} \frac{R^2}{\sqrt{R^4 + L^4}} \quad (3.1.17)$$

This vacuum energy density can be interpreted as the effective tension of the brane in the open string frame. In terms of the $D2$ -brane tension $T_0 = 1/4\pi^2 \ell_s^3$, this is given by

$$T_{eff} = T_0 \frac{R^2}{\sqrt{R^4 + L^4}} \quad (3.1.18)$$

¹More precisely, the metric $h_{\mu\nu}$ is induced on the brane due to its embedding and motion in the background flat closed string geometry. Distances on the brane defined by using $h_{\mu\nu}$ are also those measured by closed string probes. Thus we shall call $h_{\mu\nu}$ the ‘closed string metric’.

We see that the brane becomes effectively tensionless as $R \rightarrow 0$. This is another indication that the theory eventually becomes strongly coupled. The mass of the scalar field χ is a measure of the supersymmetry breaking scale of the theory. Supersymmetry is broken because the brane is compact: the mass tends to zero as $R \rightarrow \infty$.

There is a linear term

$$S_1 = \frac{1}{2\lambda^2} \int dt d\theta d\phi \frac{\sqrt{-G}}{g_{YM}^2} \Theta^{ab} f_{ab} \quad (3.1.19)$$

which is a total derivative, and can be dropped if we restrict to gauge fields of trivial first Chern class. It is noteworthy that the open string Θ -parameter, given by the standard formulae in terms of closed string frame parameters [52], is precisely what appears here,

$$\Theta^{ab} = \lambda \left(\frac{1}{h + \lambda B} \right)_A^{ab} \quad (3.1.20)$$

In terms of R this is given by

$$\Theta^{\theta\phi} = -\frac{2}{N} \frac{L^4}{(R^4 + L^4) \sin \theta} \quad (3.1.21)$$

The interpretation of Θ as a non-commutativity parameter will be made more clear later on in this chapter. Notice that this attains its maximum value as $R \rightarrow 0$, at which point $\Theta \sim 2/N \sin \theta$ being equal to the inverse background magnetic field.

In addition, there is a non-zero mixing term between the field strength $f_{\mu\nu}$ and the scalar field χ to quadratic order in the fluctuations. This is given by

$$\begin{aligned} S_{int} &= - \int dt d\theta d\phi \frac{\sqrt{-G}}{\lambda g_{YM}^2} \frac{2R^3}{\sqrt{1 - \dot{R}^2} \left(\frac{\lambda^2 N^2}{4} + R^4 \right)} \chi \Theta^{ab} f_{ab} \\ &= - \int dt d\theta d\phi \frac{\sqrt{-G}}{L^2 g_{YM}^2} \frac{R^3}{\sqrt{1 - \dot{R}^2} (L^4 + R^4)} \chi (N \Theta^{ab}) f_{ab} \end{aligned} \quad (3.1.22)$$

The second line makes it clear that this term is of order one if we consider the physical scaling limit² $\ell_s \rightarrow 0$, $N \rightarrow \infty$, $g_s \rightarrow 0$ while keeping R and L fixed. Therefore, it is comparable to the other terms appearing in the fluctuation analysis. In performing various integrations by parts we have made extensive use of the fact that the combination $(\sqrt{-G} \Theta^{\theta\phi})/g_{YM}^2$ is given by

$$\frac{\sqrt{-G}}{g_{YM}^2} \Theta^{\theta\phi} = -\frac{\lambda^2 N \ell_s}{2g_s} \sqrt{\frac{1 - \dot{R}^2}{R^4 + \frac{\lambda^2 N^2}{4}}} = -\frac{\lambda^2 N \ell_s}{2g_s} \frac{1}{\sqrt{R_0^4 + L^4}} \quad (3.1.23)$$

which is time-independent.

Thus, in the open string frame, the effective metric and non-commutativity parameter

²We will examine this limit more extensively soon.

are well behaved all the way through the evolution of the brane. The coupling constant diverges as $R \rightarrow 0$. From the point of view of open string matter probes on the brane, the sphere contracts to a finite size and then expands again as can be seen from eq. (3.1.15). But the expansion results eventually in a strongly coupled phase.

The ‘open string’ parameters $G_{\mu\nu}$, G_s and Θ appearing in the above action are the ones which more naturally would appear in the description of the brane degrees of freedom in terms of non-commutative field variables. We shall show in the next sections how such a description is realised if we replace the smooth membrane configuration (and the uniform background magnetic field) with a system of N $D0$ -branes, and re-derive the effective action for the fluctuations from the non-abelian DBI action of the $D0$ -brane system in the large- N limit. In the $D0$ -brane description the non-commutative variables are $N \times N$ matrices; alternatively, the non-commutative variables can be expressed in terms of functions on a fuzzy sphere whose co-ordinates are non-commutative [99].

One may turn off the scalar fluctuations χ and consider only fluctuations of the gauge field on spherical branes. In this set up one has a continuum fluid description of the $D0$ -branes on the collapsing brane. Indeed the gauge invariant field strength $F_{\mu\nu}$ describes the density and currents of the particles³. This continuum description eventually breaks down for two reasons: Firstly the non-commutativity parameter increases, indicating that the fuzziness in area spreads over larger distances. Secondly the gauge field fluctuations become strongly coupled.

3.1.1 Strong coupling radius

Let us now determine the size of the brane at which the strong coupling phenomenon appears. First notice that the coupling constant g_{YM}^2 is dimensionful, with units of energy. Thus the dimensionless effective coupling constant is given by g_{YM}^2/E_{proper} , where E_{proper} is a typical proper energy scale of the fluctuating modes. The dependence of the effective coupling constant on the energy reminds us that in $2+1$ dimensions the Yang Mills theory is weakly coupled in the ultraviolet and strongly coupled in the infrared. Because of the spherical symmetry of the background solution, angular momentum is conserved including interactions. Thus as the brane collapses, we may determine the relevant proper energy scale in terms of the angular momentum quantum numbers characterising the fluctuating modes.

To this end, let us examine the massless wave equation, as it arises for example for the transverse scalar fluctuations

$$\partial_\mu \left(\frac{\sqrt{-G}}{g_{YM}^2} G^{\mu\nu} \partial_\nu \xi \right) = 0 \quad (3.1.24)$$

³Such fluid descriptions are given in the brane constructions of [101, 102].

In terms of angular momentum quantum numbers, this becomes

$$\frac{1}{(1 - \dot{R}^2)} \partial_t^2 \tilde{\xi} + \frac{R^2 l(l+1)}{(R^4 + L^4)} \tilde{\xi} = 0 \quad (3.1.25)$$

where we have set $\xi = \tilde{\xi}(t)Y_{lm}$ with Y_{lm} being the appropriate spherical harmonic.

The proper energy is given approximately by

$$E_{proper} \sim \frac{R\sqrt{l(l+1)}}{\sqrt{(R^4 + L^4)}} \quad (3.1.26)$$

As the brane collapses the wavelength of massless modes is actually red-shifted! This is essentially because of the form of the effective open string metric.

Now we let the brane collapse to a size $R \ll \sqrt{N}\ell_s$. At smaller values of the radius the effective coupling constant becomes

$$g_{eff}^2 \sim \frac{g_s N^2 \ell_s^3}{R^3 \sqrt{l(l+1)}} \quad (3.1.27)$$

Clearly this becomes of order one when R approaches the strong coupling radius R_s

$$R_s = g_s^{1/3} \ell_s \left(\frac{N^2}{\sqrt{l(l+1)}} \right)^{1/3} = L \left(\frac{g_s \sqrt{N}}{\sqrt{l(l+1)}} \right)^{1/3} \quad (3.1.28)$$

Notice the appearance of $\ell_{11} = g_s^{1/3} \ell_s$, the characteristic scale of Matrix Theory. For l close to the cutoff N , $R_s \sim N^{1/3} \ell_{11}$, which is the estimated size of the quantum ground state of N $D0$ -branes [103, 104]. In general R_s involves an effective N given by $N_{eff} \sim N^2 / \sqrt{l(l+1)}$. We shall discuss these special values of the radius in more detail when we describe the membrane after taking various interesting limits for the parameters appearing in (3.1.28).

The coupling constant of the theory (3.1.8) is time-dependent. We can instead choose to work with a fixed coupling constant absorbing the time-dependence solely in the effective metric if we perform a suitable conformal transformation. By defining $\tilde{G}_{\mu\nu} = \Lambda G_{\mu\nu}$, the gauge field kinetic term gets multiplied by a factor of $\Lambda^{1/2}$. Then we can re-define the coupling constant: $\tilde{g}_{YM}^2 = g_{YM}^2 / \sqrt{\Lambda}$. The conformal transformation requires also suitable re-scalings of the fields χ and ξ_m as well as appropriate redefinitions of the various dimensionful parameters of the theory such as m^2 and the non-commutativity parameter Θ^{ab} .

Choosing $\Lambda = (L^4 + R^4)/R^4$, the transformed coupling becomes

$$\tilde{g}_{YM}^2 = g_s \ell_s^{-1} \quad (3.1.29)$$

and so it is time independent. The open string metric in this frame is still non singular.

However, the relevant dimensionless coupling is still the effective coupling g_{eff}^2 , eq. (3.1.27), which for small radii remains large. The effect of the conformal transformation gets rid of the time-dependence in the coupling constant but also red-shifts E_{proper} by a factor of $\Lambda^{-1/2}$. Therefore, we cannot escape the strong coupling regime in this fashion.

3.1.2 Overall transverse fluctuations and exactly solvable Schrödinger equation

Another interesting feature of (3.1.25) is that it is an integrable problem. Using $(1 - \dot{R}^2) = (R^4 + L^4)/(R_0^4 + L^4)$ the wave equation becomes

$$\partial_t^2 \tilde{\xi} + l(l+1) \frac{R^2}{R_0^4 + L^4} \tilde{\xi} = 0 \quad (3.1.30)$$

Substituting the solution for the scale factor R , which is known in terms of the Jacobi elliptic function as $R = R_0 Cn \left(\frac{t\sqrt{2}R_0}{\sqrt{R_0^4 + L^4}}, \frac{1}{\sqrt{2}} \right)$, we have

$$\partial_t^2 \tilde{\xi} + l(l+1) \frac{R_0^2}{R_0^4 + L^4} Cn^2 \left(\frac{t\sqrt{2}R_0}{\sqrt{R_0^4 + L^4}}, \frac{1}{\sqrt{2}} \right) \tilde{\xi} = 0 \quad (3.1.31)$$

We remind that, as shown in Ch. 2, the solution to the classical problem is related to an underlying elliptic curve. For this specific case we can explicitly express the Jacobi- Cn function in terms of Weierstrass- \wp functions of the underlying curve. The following relation is true for this case

$$Cn^2 \left(\sqrt{2}u, \frac{1}{\sqrt{2}} \right) = \frac{\wp(u; 4, 0) - 1}{\wp(u; 4, 0) + 1} \quad (3.1.32)$$

For these specific functions the following identity also holds

$$\wp(u + \Omega'; 4, 0) = -\frac{\wp(u; 4, 0) - 1}{\wp(u; 4, 0) + 1} \quad (3.1.33)$$

where Ω' is the purely imaginary half period of the relevant elliptic curve in its Weierstrass form, as given in App. B by

$$\Omega' = i \int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}} \quad (3.1.34)$$

After a re-scaling of time $t = u\sqrt{L^4 + R_0^4}/R_0$ we end up with

$$\partial_u^2 \tilde{\xi} + l(l+1) Cn^2 \left(u\sqrt{2}, \frac{1}{\sqrt{2}} \right) \tilde{\xi} = \partial_u^2 \tilde{\xi} - l(l+1) \wp(u + \Omega'; 4, 0) \tilde{\xi} = 0 \quad (3.1.35)$$

This is exactly the g -gap Lamé equation for the ground state of the corresponding one-dimensional quantum mechanical problem, which has solutions in terms of ratios of Weier-

strass σ -functions⁴.

A related solvable Schrödinger problem arises in the one-loop computation of the Euclidean path integral. This requires the computation of the determinant of the operator

$$-\partial_\tau^2 + \frac{R(i\tau)^2}{\sqrt{R_0^4 + L^4}} l(l+1) \quad (3.1.36)$$

where we have performed an analytic continuation $t \rightarrow i\tau$. The eigenvalues of the operator are determined by

$$-\partial_\tau^2 \tilde{\xi} + \frac{R(i\tau)^2}{\sqrt{R_0^4 + L^4}} l(l+1) \tilde{\xi} = \lambda \tilde{\xi} \quad (3.1.37)$$

We have already shown that $R(i\tau) = 1/R(\tau)$ and that $R^2(i\tau) = \wp(\tau - \Omega; 4, 0)$ where $\Omega = \int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}}$. Hence the eigenvalue equation becomes

$$-\partial_\tau^2 \tilde{\xi} + l(l+1) \wp(\tau - \Omega; 4, 0) \tilde{\xi} = \lambda \tilde{\xi} \quad (3.1.38)$$

where the eigenstates are also obtained in terms of σ -functions.

We postpone a detailed description and physical interpretation of the solutions of (3.1.32) and (3.1.38) for future work. It is intriguing that eq. (3.1.32) has appeared in the literature on re-heating at the end of inflation [106, 107]. The physical meaning of this similarity, between fluctuation equations for collapsing $D0$ - $D2$ systems and those of re-heating, remains to be found.

3.2 Action for fluctuations from the $D0$ -brane non-abelian DBI

The non-abelian DBI action for zero branes [28, 30] is given by

$$S = -\frac{1}{g_s \ell_s} \int dt STr \sqrt{-\det(M)} \quad (3.2.1)$$

where

$$M = \begin{pmatrix} -1 & \lambda \partial_t \Phi_j \\ -\lambda \partial_t \Phi_i & Q_{ij} \end{pmatrix} \quad (3.2.2)$$

and $Q_{ij} = \delta_{ij} + i\lambda \Phi_{ij}$ with the abbreviation $\Phi_{ij} = [\Phi_i, \Phi_j]$. The determinant of M , when the only non-zero scalars lie in the $i, j, k \in \{1, 2, 3\}$ directions, is given by

$$\begin{aligned} -\det M &= 1 + \frac{\lambda^2}{2} \Phi_{ij} \Phi_{ji} - \lambda^2 (\partial_t \Phi_i)(\partial_t \Phi_i) \\ &\quad - \frac{\lambda^4}{2} (\partial_t \Phi_k)(\partial_t \Phi_k) \Phi_{ij} \Phi_{ji} + \lambda^4 (\partial_t \Phi_i) \Phi_{ij} \Phi_{jk} (\partial_t \Phi_k) \end{aligned} \quad (3.2.3)$$

⁴For an application in supersymmetric gauge theories see for example [105].

These terms suffice for the calculation of the quadratic action for the fluctuations involving the gauge field and the radial scalar. However, when we include fluctuations for the scalars Φ_m for $m = 4, \dots, 9$ we need the full 10×10 determinant. Fortunately, since we will only be interested in contributions up to quadratic order, the relevant terms will only be those of order up to λ^4

$$\begin{aligned} & \frac{\lambda^2}{2} \Phi_{im} \Phi_{mi} \left(1 + \frac{\lambda^2}{4} \Phi_{jk} \Phi_{kj} - \lambda^2 \partial_t(\Phi_i) \partial_t(\Phi_i) \right) - \lambda^2 \partial_t(\Phi_m) \partial_t(\Phi_m) \left(1 + \frac{\lambda^2}{2} \Phi_{ij} \Phi_{ji} \right) \\ & - \frac{\lambda^4}{4} \Phi_{mi} \Phi_{ij} \Phi_{jk} \Phi_{km} + \lambda^4 \partial_t(\Phi_i) \Phi_{im} \Phi_{mj} \partial_t(\Phi_j) - \lambda^4 \partial^t(\Phi_m) \Phi_{mi} \Phi_{ij} \partial_t(\Phi_j) \end{aligned} \quad (3.2.4)$$

The expansion with terms of order up to λ^8 is given in [62].

The $D2$ -brane solution is described by setting $\Phi_i = \hat{R}(t) X_i$, where the matrices X_i generate the N -dimensional irreducible representation of $SU(2)$. By substituting this Ansatz into the $D0$ -action, we can derive equations of motion which coincide with those derived from the $D2$ DBI-action [63]. In the correspondence we use

$$R^2 = \lambda^2 C(\hat{R})^2 \quad (3.2.5)$$

where C is the Casimir of the representation, $C = N^2$ in the large- N limit. Note that the square root form in the $D0$ -action is necessary to recover the correct time of collapse. If we use the $D0$ -brane Yang-Mills limit, we get the same functional form of the solution in terms of Jacobi- Cn functions, but the time of collapse for initial conditions where R_0 is large is incorrect. The correct time of collapse increases as R_0 increases towards infinity, whereas the Yang-Mills limit gives a time which decreases in this limit. We expand around the solution as follows

$$\begin{aligned} \Phi_i &= \hat{R} X_i + A_i \\ A_i &= 2\hat{R} K_i^a A_a + x_i \phi \\ \Phi_m &= \xi_m \end{aligned} \quad (3.2.6)$$

The decomposition in the second line above will be explained shortly. Throughout this chapter, we will be working in the $A_0 = 0$ gauge.

3.2.1 Geometry of fuzzy two-sphere: A brief review

We quickly remind some facts about the fuzzy sphere and its application in Matrix theories⁵. As before, the X_i 's are generators of the $SU(2)$ algebra satisfying

$$[X_i, X_j] = 2i\epsilon_{ijk} X_k \quad (3.2.7)$$

⁵See for example [47, 99].

With this normalisation of the generators, the Casimir in the N -dimensional irreducible representation is given by $X_i X_i = (N^2 - 1)$. If we define $x_i = X_i/N$, we see that

$$\begin{aligned} x_i x_i &= 1 \\ [x_i, x_j] &= 0 \end{aligned} \tag{3.2.8}$$

in the large- N limit. Hence, in the large- N limit the x_i 's reduce to Cartesian co-ordinates describing the embedding of a unit 2-sphere in \mathbb{R}^3 . For traceless symmetric tensors $a_{j_1 \dots j_l}$ the functions $a_{j_1 \dots j_l} x_{j_1} \dots x_{j_l}$ describe spherical harmonics in Cartesian co-ordinates. Since general (traceless) Hermitian matrices can be expanded in terms of (traceless) symmetric polynomials of the X_i 's, hence in terms of the x_i 's⁶, all our fluctuations such as A_i or transverse scalars such as ξ_m become fields on the sphere in the large- N limit. The expansion of A_i is given by

$$A_i = a_i + a_{i;j_1} x_j + a_{i;j_1 j_2} x_{j_1} x_{j_2} + \dots \tag{3.2.9}$$

We can write this as $A_i(t, \theta, \phi)$, with the time-dependence appearing in the coefficients a_i , $a_{i;j_1 \dots j_l}$ and the dependence on the angles arising from the polynomial of the x_i 's. At finite- N , two important things happen: The x_i 's become non-commutative and the spectrum of spherical harmonics is truncated at $N - 1$. We will be concerned, in the first instance, with the large- N limit.

The action of X_i on the unit normalised co-ordinates follows from the algebra (3.2.7)

$$[X_i, x_j] = 2i\epsilon_{ijk}x_k \tag{3.2.10}$$

and can be re-written

$$-2i\epsilon_{ipq}x_p\partial_q(x_j) \tag{3.2.11}$$

So the adjoint action of X_i can be written as

$$\begin{aligned} [X_i, \quad] &= -2iK_i = -2i\epsilon_{ipq}x_p\partial_q \\ &= -2iK_i^a\partial_a \end{aligned} \tag{3.2.12}$$

We have used Killing vectors K_i defined by $K_i = \epsilon_{ipq}x_p\partial_q$, which obey $x_i K_i = 0$. They are tangential to the sphere and can be expanded as $K_i^a \partial_a$, where a runs over θ, ϕ . The components K_i^a have been used in (3.2.6) to pick out the tangential gauge field components, and the radial component ϕ defined in (3.2.6) obeys $\phi = x_i A_i$. It is useful to write down

⁶The latter give the correctly normalised spherical harmonics as we will explain later.

the explicit components of K_i . The Killing vectors K_i^a are given by

$$\begin{aligned} K_1^\theta &= -\sin\phi & K_1^\phi &= -\cot\theta \cos\phi \\ K_2^\theta &= \cos\phi & K_2^\phi &= -\cot\theta \sin\phi \\ K_3^\theta &= 0 & K_3^\phi &= 1 \end{aligned}$$

Some useful formulae are the following

$$\begin{aligned} K_i^a K_i^b &= \hat{h}^{ab} \\ x_i K_i^a &= 0 \\ K_i^a K_i^b \partial_a x_j \partial_b x_j &= 2 \\ \epsilon_{ijk} x_i K_j^a K_k^b &= \frac{\epsilon^{ab}}{\sin\theta} = \omega^{ab} \end{aligned} \tag{3.2.13}$$

where $\epsilon^{\theta\phi} = 1$. Here \hat{h}_{ab} is the round metric on the unit sphere and ω^{ab} is the inverse of the symplectic form. As a related remark, note that

$$\Theta^{ab} = -\frac{\lambda^2 N}{2(R^4 + \frac{\lambda^2 N^2}{4})} \epsilon_{ijk} x_i K_j^a K_k^b = -\frac{2}{N} \frac{L^4}{R^4 + L^4} \epsilon_{ijk} x_i K_j^a K_k^b \tag{3.2.14}$$

We will use these formulae to derive the action for the fluctuations A_a , ϕ geometrically as a field theory on the sphere in the large- N limit. We need one more ingredient. The $D0$ -brane action is expressed in terms of traces, which obey the $SU(2)$ invariance condition $Tr(\Phi) = Tr[X_i, \Phi]$. This can be used to show that if Φ is expressed as $\Phi = a + a_j x_j + a_{j_1 j_2} x_{j_1} x_{j_2} + \dots$, then the trace is just Na , *i.e.* it picks out the coefficient of the trivial $SU(2)$ representation. By using the similar $SU(2)$ invariance property of the standard sphere integral we have

$$\frac{Tr}{N} \rightarrow \frac{1}{4\pi} \int d\theta d\phi \sin\theta \tag{3.2.15}$$

This relation between traces and integrals makes it clear why we have chosen the Cartesian spherical harmonics to be symmetric traceless combinations of $x_{i_1} \dots x_{i_l} = (X_{i_1} \dots X_{i_l})/N^l$. Such spherical harmonics obey

$$\int d\Omega Y_{lm} Y_{l'm'} = \frac{Tr}{N} Y_{lm} Y_{l'm'} = \delta_{ll'} \delta_{mm'}$$

and are the appropriate functions to appear in (3.2.9).

We make some further general remarks on the calculation, before stating the result for the action obtained from the $D0$ -brane picture. Note that the last term in the expansion of the determinant (3.2.3) gives zero when we evaluate it on the Ansatz $\Phi_i = \hat{R}X_i$ used to obtain the solution, but it becomes non-trivial in calculating the action for the fluctuations $\Phi_i = \hat{R}X_i + A_i$. The zero appears because the symmetrised trace allows us to re-shuffle the

X_i with the $[X_i, X_j]$ for example. Using this property and the commutation relations gives the desired zero and hence leads to agreement between the effective actions for the radial variable, as derived from the $D2$ -brane picture.

3.2.2 The action for the gauge field and radial scalar

Using the Ansatz (3.2.6) we have

$$[\Phi_i, \Phi_j] = (\hat{R})^2 [X_i, X_j] + \hat{R} [X_i, A_j] + \hat{R} [A_i, X_j] + [A_i, A_j] \quad (3.2.16)$$

The first term scales like N , the second two terms are of order one in the large- N limit, while the last term is of order $1/N$. The last commutator term is sub-leading in $1/N$ since the x_i 's appearing in (3.2.9) commute in the strict large- N limit, as of (3.2.8). When computing terms such as the potential term $\sim [\Phi_i, \Phi_j]^2$, it is important to note that there are terms of order one coming from squaring $\hat{R} [X_i, A_j] + \hat{R} [A_i, X_j]$ as well as from the cross terms $(\hat{R})^2 [X_i, X_j] [A_i, A_j]$. For this reason, the underlying non-commutative geometry of the fuzzy 2-sphere is important in deriving even the leading terms in the dynamics of the fluctuations.

The first term in (3.2.16) is simplified by using the commutation relations to give

$$2i(\hat{R})^2 \epsilon_{ijk} X_k = 2iN(\hat{R})^2 \epsilon_{ijk} x_k \quad (3.2.17)$$

The second term can be written in the form

$$-4i(\hat{R})^2 K_i^a \partial_a (K_j^b A_b) - 2iK_i^a \partial_a (x_j \phi) \quad (3.2.18)$$

using (3.2.12). We can compute the leading $1/N$ correction arising from the commutator $[A_i, A_j]$ as follows. We can think of the unit normalised, non-commuting co-ordinates x_i as quantum angular momentum variables. Since their commutator is given by $[x_i, x_j] = (2i\epsilon_{ijk}x_k)/N$, the analogue of \hbar is given by $2/N$, which scales like the inverse of the spin of the $SU(2)$ representation. Thus the large- N limit is equivalent to the classical limit in this analogy, and in this case all matrix commutators $[A, B]$ can be approximated with ‘classical’ Poisson brackets as follows

$$[A, B] \rightarrow \frac{2i}{N} \{A, B\} \quad (3.2.19)$$

where $\{A, B\}$ is the Poisson bracket defined by

$$\{A, B\} = \omega^{ab} \partial_a A \partial_b B \quad (3.2.20)$$

using the inverse-symplectic form appearing in (3.2.13). As a check note that $\{x_i, x_j\} =$

$\epsilon_{ijk}x_k$. The commutator $[A_i, A_j]$ is then given by

$$[A_i, A_j] = \frac{2i}{N} \{A_i, A_j\} + O(1/N^2) = \frac{2i}{N} \omega^{ab} \partial_a A_i \partial_b A_j + O(1/N^2) \quad (3.2.21)$$

Substituting in (3.2.1) and expanding the square root, keeping up to quadratic terms in the field strength components $F_{ab} = \partial_a A_b - \partial_b A_a$ and $F_{0a} = \partial_t A_a$, we obtain

$$- \int dt d\theta d\phi \frac{\sqrt{-G}}{4g_{YM}^2} G^{\mu\alpha} G^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \quad (3.2.22)$$

where the effective metric and coupling constant are the ones appearing in Sec. 3.1. Hence we have recovered from the $D0$ -brane action (3.2.1) the first term of (3.1.8) obtained from the small-fluctuation expansion of the $D2$ -brane DBI action. We remark that in calculating the quadratic term in the spatial components of the field strength, the last term in (3.2.3) gives zero, but its contribution is important in getting the correct coefficient in front of F_{0a}^2 .

There is a term linear in F_{ab} given by

$$S_1 = \frac{1}{2\lambda^2} \int dt d\theta d\phi \frac{\sqrt{-G}}{g_{YM}^2} r^4 \Theta^{ab} F_{ab} \quad (3.2.23)$$

where r is the dimensionless radius variable, $r = R/L$. This differs from the linear term obtained from the $D2$ DBI action by the r^4 factor, but the whole term is a total derivative. As such it vanishes in the sector where the fluctuations do not change the net monopole charge of the background magnetic field. This is a reasonable restriction to put when analysing small fluctuations around a monopole configuration.

At first sight we could also have A_a^2 contributions, which would amount to a mass for the gauge field. Such terms coming from $\partial_t \Phi_i [\Phi_i, \Phi_j] [\Phi_j, \Phi_k] \partial_t \Phi_k$ cancel amongst themselves. The contributions from the other three terms of (3.2.3) cancel each other up to total derivatives, upon expanding the square root and also performing partial integrations in both the spatial and time directions. Here, we need to use the equation of motion for the scale factor R or \hat{R} .

A list of useful formulae is the following

$$\begin{aligned} [\Phi_i, \Phi_j] [\Phi_j, \Phi_i] &= 8N^2 \hat{R}^4 + 8\hat{R}^2 (\partial_a \phi) (\partial^a \phi) + 48\hat{R}^2 \phi^2 + 32\hat{R}^3 N \phi - 48\hat{R}^3 \frac{\epsilon^{ab}}{\sin \theta} F_{ab} \phi \\ &+ 32\hat{R}^3 \frac{\epsilon^{ab}}{\sin \theta} A_b (\partial_a \phi) - 16\hat{R}^4 N \frac{\epsilon^{ab}}{\sin \theta} F_{ab} + 16\hat{R}^4 F_{ab} F^{ab} + 64\hat{R}^4 A_a A^a \\ &- 64\hat{R}^4 \frac{1}{\sin^2 \theta} \left[\epsilon^{ab} (\partial_\theta A_a) (\partial_\phi A_b) + A_\phi (\partial_\phi A_\theta) \cot \theta \right] \end{aligned} \quad (3.2.24)$$

$$(\partial_t \Phi_i) (\partial_t \Phi_i) = N^2 \dot{\hat{R}}^2 - 4\hat{R}^2 F_{0a} F^{0a} + \dot{\phi}^2 + 2\dot{\hat{R}} N \dot{\phi} + 4(\dot{\hat{R}})^2 A_a A^a + 4\hat{R} \dot{\hat{R}} \partial_t (A_a A^a) \quad (3.2.25)$$

$$(\partial_t \Phi_i) [\Phi_i, \Phi_j] = 2i\hat{R}\dot{\hat{R}}NK_j^a(\partial_a \phi) + 4i\hat{R}^3N\epsilon_{ijp}x_pK_i^a(\partial_t A_a) \quad (3.2.26)$$

$$\begin{aligned} (\partial_t \Phi_i)[\Phi_i, \Phi_j][\Phi_j, \Phi_k](\partial_t \Phi_k) &= 4\hat{R}^2\dot{\hat{R}}^2N^2\hat{h}^{ab}(\partial_a \phi)(\partial_b \phi) + 16\hat{R}^6N^2\hat{h}^{ab}(\partial_t A_a)(\partial_t A_b) \\ &\quad + 8\dot{\hat{R}}\hat{R}^4N^2\omega^{ab}(\partial_t A_a)(\partial_b \phi) - 8\dot{\hat{R}}\hat{R}^4N^2\omega^{ab}(\partial_a \phi)(\partial_t A_b) \end{aligned} \quad (3.2.27)$$

To get the quadratic fluctuations we take a square root, expand, use the matrix correspondence between the trace and the integral over the sphere (3.2.15), and also employ the equations of motion. Note that, after taking the trace, the terms in the last line in (3.2.24) will combine with the linear term $-16\hat{R}^4N\frac{\epsilon^{ab}}{\sin \theta}F_{ab}$ to give

$$-16\hat{R}^4N\epsilon^{ab}\left(F_{ab} + i[A_a, A_b] + \frac{2}{N}(\partial_c \omega^{cd})(A_a \partial_d A_b)\right) \quad (3.2.28)$$

We see that F_{ab} gives a total derivative while the last two terms are not individually total derivatives but combine as such. The need for additional terms in the field strength, beyond the commutator $[A_a, A_b]$ was explained in [99]. The terms in (3.2.28) can be neglected when we are considering topologically trivial fluctuations.

It is important to note that this mass term only vanishes if we keep terms of the type $[X_i, X_j][A_i, A_j]$, which are order one terms obtained by multiplying the order N with the order $1/N$ commutators.

Next we turn to fluctuations involving the scalar field ϕ . The spatial part of the relativistic kinetic term is

$$-\frac{\ell_s}{2g_s} \int dt d\theta d\phi \sin \theta \frac{(2\lambda N\hat{R}^2)\hat{h}^{ab}\partial_a \phi \partial_b \phi}{\sqrt{(1 - \lambda^2 N^2\dot{\hat{R}}^2)(1 + 4\lambda^2 N^2\hat{R}^4)}} \quad (3.2.29)$$

This agrees with the $D2$ -calculation (3.1.8) if we make the natural identification

$$\phi = (1 - \lambda^2 N^2\dot{\hat{R}}^2)^{1/2}\chi \quad (3.2.30)$$

Following this, we can match the quadratic terms in $\partial_t \phi$ and we find again that we get the same answer as from the $D2$ -side. The overall kinetic term is given by

$$-\int dt d\theta d\phi \frac{\sqrt{-G}}{2g_{YM}^2} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \quad (3.2.31)$$

as in (3.1.8).

For the mass term of χ , we get

$$\begin{aligned} & -\frac{N}{4\pi\ell_s g_s} \int dt d\theta d\phi \sin\theta \frac{12\lambda^2 \hat{R}^2 (1 - 4N^2\lambda^2 \hat{R}^4)}{(1 + 4N^2\lambda^2 \hat{R}^4)^{3/2} \sqrt{1 - N^2\lambda^2 \dot{\hat{R}}^2}} \chi^2 \\ &= -\int dt d\theta d\phi \frac{\sqrt{-G}}{2g_{YM}^2} \frac{6R^2(L^4 - R^4)}{(L^4 + R^4)^2(1 - \dot{R}^2)} \chi^2 \end{aligned} \quad (3.2.32)$$

This agrees with the mass for χ in (3.1.8). Another thing to note here is that the determinant will also give contributions linear in ϕ and $\partial_t\phi$ and also terms quadratic in the scalar fluctuation of the form $\phi \partial_t\phi$. However, upon the expansion of the square root to quadratic order the overall linear factor of ϕ cancels. We recall that we are expecting the latter, since ϕ is a fluctuation around a background which solves the equations of motion. Upon conversion to the χ variable the kinetic term for ϕ will also contribute $\chi \partial_t\chi$ terms. Then by integrating by parts and dropping the respective total time derivative terms we end up with the appropriate mass for χ given above.

For the mixing terms between F_{ab} and ϕ , collecting all the relevant terms one gets

$$-\int dt d\theta d\phi \frac{\sqrt{-G}}{g_{YM}^2} \frac{8\hat{R}^3 N}{(1 + 4\lambda^2 N^2 \hat{R}^4) \sqrt{1 - \lambda^2 N^2 \dot{\hat{R}}^2}} \chi \Theta^{ab} F_{ab} \quad (3.2.33)$$

Once more, we get exact agreement with the *D2*-calculation (3.1.22). Finally the quadratic action for the scalars ξ_m obtained by expanding the terms in (3.2.4) is easily seen to be

$$-\int dt d\theta d\phi \frac{\sqrt{-G}}{g_{YM}^2} G^{ab} \partial_a \xi_m \partial_b \xi_m \quad (3.2.34)$$

which agrees with (3.1.8).

3.2.3 Scalar fluctuations for the reduced action

We expect to be able to reach the same results for the scalar fluctuations by just considering the large- N reduced action for the background fields as in [63]

$$S_2 = -\frac{2}{g_s \ell_s \lambda} \int dt \sqrt{1 - \dot{R}^2} \sqrt{R^4 + \frac{N^2 \lambda^2}{4}} \quad (3.2.35)$$

and consider adding fluctuations $R \rightarrow R + \lambda \sqrt{1 - \dot{R}^2} \chi$ as before. One gets

$$\begin{aligned} S_2^{\text{mass}} &= -\frac{2}{g_s \ell_s \lambda} \int dt \frac{\lambda^2 R^2 (3L^4 - 3R^4)}{2(L^4 + R^4)^{3/2} \sqrt{1 - \dot{R}^2}} \chi^2 \\ &= \frac{-4L\sqrt{\pi}}{\tilde{g}_s} \int dt \frac{R^2 (3L^4 - 3R^4)}{2(L^4 + R^4)^{3/2} \sqrt{1 - \dot{R}^2}} \chi^2 \end{aligned} \quad (3.2.36)$$

the same answer for the mass of the scalar fluctuation as by perturbing the full action (3.2.1), when written in terms of g_{YM}^2 and $\sqrt{-G}$.

We can make use of this result to check the behaviour of the scalar mass for higher even-spheres. The reduced action for the fuzzy S^4 that we wrote down in Ch. 2 is

$$S_4 = -\frac{4}{g_s \ell_s \lambda^2 N} \int dt \sqrt{1 - \dot{R}^2} \left(R^4 + \frac{\lambda^2 N^2}{4} \right) \quad (3.2.37)$$

Perturbing this will result to a mass

$$\begin{aligned} S_4^{\text{mass}} &= -\frac{4}{g_s \ell_s \lambda^2 N} \int dt \frac{2\lambda^2 R^2 (3L^4 - 5R^4)}{(L^4 + R^4) \sqrt{1 - \dot{R}^2}} \chi^2 \\ &= -\frac{8\sqrt{\pi}}{\tilde{g}_s L} \int dt \frac{R^2 (3L^4 - 5R^4)}{(L^4 + R^4) \sqrt{1 - \dot{R}^2}} \chi^2 \end{aligned} \quad (3.2.38)$$

where we have made use of the appropriate equations of motion.

We also have seen a similar behaviour for the S^6 . The reduced action is

$$S_6 = -\frac{8}{g_s \ell_s \lambda^3 N^2} \int dt \sqrt{1 - \dot{R}^2} \left(R^4 + \frac{\lambda^2 N^2}{4} \right)^{3/2} \quad (3.2.39)$$

and the result for the mass

$$\begin{aligned} S_6^{\text{mass}} &= -\frac{8}{g_s \ell_s \lambda^3 N^2} \int dt \frac{12\lambda^2 R^2 (3L^4 - 7R^4)}{\sqrt{L^4 + R^4} \sqrt{1 - \dot{R}^2}} \chi^2 \\ &= -\frac{48\sqrt{\pi}}{\tilde{g}_s L} \int dt \frac{R^2 (3L^4 - 7R^4)}{\sqrt{L^4 + R^4} \sqrt{1 - \dot{R}^2}} \chi^2 \end{aligned} \quad (3.2.40)$$

The physical behaviour remains the same for any k : for the pure $N = 0$ case the scalar mass squared is negative from the beginning of the collapse all the way down to zero. At finite (large) N there is a transition point which depends on the dimensionality k .

3.2.4 $1/N$ correction to the action

The derivation of the action from the $D0$ -brane side can easily be extended to include $1/N$ corrections. The net outcome will be a non-commutative gauge theory, where products are replaced by suitable star products. Two important features have to be noted. It is no longer consistent to assume $x_i K_i^a = K_i^a x_i$. This is because

$$\begin{aligned} [x_i, K_i^\theta] &= -\frac{2i}{N} \cot \theta \\ [x_i, K_i^\phi] &= 0 \end{aligned} \quad (3.2.41)$$

We can instead only assume $x_i K_i + K_i x_i = 0$. We also have a first correction to the Leibniz rule for the partial derivatives

$$\partial_a(FG) = (\partial_a F)G + F(\partial_b G) - \frac{i}{N}(\partial_a \omega^{bc})(\partial_b F)(\partial_c G) \quad (3.2.42)$$

This is consistent with

$$\partial_a[x_i, x_j] = [\partial_a x_i, x_j] + [x_i, \partial_a x_j] - \frac{i}{N}(\partial_a \omega^{bc})[\partial_b x_i, \partial_c x_j] \quad (3.2.43)$$

3.3 Scaling limits and Quantum Observables

Given the action we have derived from the $D0$ and $D2$ -sides, there are several limits to consider so as to describe the physics.

3.3.1 The DBI-scaling

Consider $g_s \rightarrow 0$, $\ell_s \rightarrow 0$, $N \rightarrow \infty$ keeping fixed

$$R, L = \ell_s \sqrt{\pi N}, g_s \sqrt{N} \equiv \tilde{g}_s \quad (3.3.1)$$

In this limit the following parameters appearing in the Lagrangean are fixed

$$\begin{aligned} g_{YM}^2 &= \frac{g_s}{\ell_s} \frac{\sqrt{R^4 + L^4}}{R^2} = \frac{\sqrt{\pi} \tilde{g}_s}{L} \frac{\sqrt{R^4 + L^4}}{R^2} \\ G_{00} &= \sqrt{1 - \dot{R}^2} \\ G_{ab} &= \frac{R^4 + L^4}{R^2} \hat{h}_{ab} \end{aligned} \quad (3.3.2)$$

We also keep fixed the energies and angular momenta of field quanta in the theory.

With this scaling all the quadratic terms of the field theory action on S^2 derived from the $D2$ -brane side in (3.1.8), (3.1.22), and reproduced in the previous section from the $D0$ -branes, remain fixed. Notice that all terms in (3.2.3) are also of order one and all of them contribute so as to obtain the small fluctuations action and the parameters of the theory given above. In addition, since in this limit $\ell_s \rightarrow 0$, massive open string modes on the branes decouple and we can neglect higher derivative corrections to the DBI action. Further, since $g_s \rightarrow 0$, we expect closed string emission to be negligible. This scaling should be compared to scalings studied in Matrix Theory in [43, 44, 100, 103, 108–110]. In the region $R \ll L$, we will consider the relation to the Matrix Theory limit below.

There are several interesting features of the limit (3.3.1). It allows us to neglect the finite size effects of the quantum $D0$ -brane bound state. The quantum field theory we have derived by expanding around the classical solution might be expected to be invalid in the

regime where the radius of the sphere reaches the size R_q of the quantum ground state of N $D0$ -branes. This has been estimated to be [103, 104]

$$\begin{aligned} R_q &= N^{1/3} g_s^{1/3} \ell_s \\ &= \frac{\tilde{g}_s L}{N^{1/3}} \end{aligned} \quad (3.3.3)$$

Clearly this is zero in the scaling limit, which gives us reason to believe that the DBI action is valid all the way to $R = 0$.

Another issue is gravitational back-reaction. This can be discussed by comparing the radius of the collapsing object to the gravitational radius of a black hole with the same net charge. This type of argument is used for example in [111] for studying collapsing domain walls in four dimensions. We find that in the scaling limit (3.3.1), gravitational back-reaction is negligible. To see this consider first the excess energy ΔE of the classical configuration above the ground state energy of N $D0$ -branes. For extremal black holes the horizon area is zero. For non-extremal ones, it is directly determined by the excess energy [112]

$$R_h^8 = g_s^{\frac{25}{14}} \ell_s^{\frac{121}{14}} \sqrt{N} (\Delta E)^{\frac{9}{14}} \quad (3.3.4)$$

Using

$$\begin{aligned} \Delta E &= \frac{N}{g_s \ell_s} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right) \\ &= \frac{N^2}{\tilde{g}_s L} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right) \end{aligned} \quad (3.3.5)$$

we find for the horizon radius

$$R_h^8 = N^{\frac{-24}{7}} \tilde{g}_s^{\frac{8}{7}} L^8 \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right)^{\frac{9}{14}} \quad (3.3.6)$$

which goes to zero in the large- N limit. This shows that gravitational back-reaction resulting in the formation of non-extremal black holes does not constrain the range of validity of the DBI action in our scaling limit.

Another black hole radius we may compare to is the Schwarzschild radius for an object having energy $N \sqrt{R_0^4 + L^4} / (g_s \ell_s L^2)$, as is the case for our membrane configuration. This comparison is more relevant in the limit $R \gg L$ where the $D0$ -brane density is small; in other words the charge density of the relevant black hole is small. In this case we expect the discussion of [111] to be most relevant. The Schwarzschild radius is given by

$R_{Sch} = (G_N E)^{1/7}$, or more explicitly

$$R_{Sch} = N^{-\frac{3}{7}} L \tilde{g}_s^{\frac{1}{7}} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} \right)^{\frac{1}{7}} \quad (3.3.7)$$

This is also zero in the scaling limit (3.3.1), and hence does not invalidate the DBI action.

Since R is time-dependent, the parameters of the theory are also time-dependent. We may consider correlators of gauge invariant operators

$$\langle \mathcal{O}(t_1, \sigma_1^a) \mathcal{O}(t_2, \sigma_2^a) \dots \rangle \quad (3.3.8)$$

where \mathcal{O} can be for example $Tr(F^2)$ or $Tr(\Phi^2)$, which use the field strength or transverse scalars. For times t_1, t_2, \dots corresponding via the classical solution to R near R_0 , the Yang-Mills coupling is small, and the approximation where non-linearities of the DBI have been neglected is a valid one. So we can compute such correlators perturbatively. When R approaches zero, the Yang-Mills coupling diverges, so we need to use the all-orders expansion of the DBI action. We have not computed the fluctuation action to all orders, but it is in principle contained in the full DBI action.

An interesting observable is $\langle 0 | \chi | 0 \rangle$ which gives quantum corrections to the classical path. In time-dependent backgrounds, one can typically define distinct early and late times vacua because positive and negative frequency modes at early and late times can be different. If we set up an early times vacuum in the ordinary manner, and write χ as a linear combination of early times creation and annihilation operators, the one point function of χ in the late times vacuum may be non-zero indicating particle production. The non-trivial relation between in and out-vacua is certainly to be expected for all the fields in the theory, since it is a generic feature of quantum fields in a time-dependent background [113]. Recent applications in the decay of unstable branes include [114–116].

We have argued that radiation into closed string states is negligible because their coupling constant $g_s \rightarrow 0$ in the scaling limit (3.3.1). In the context of open string tachyon condensation, describing brane decay, the zero coupling limit of closed string emission was shown not to approach zero as naively expected because of a divergence coming from a sum over stringy states [115]. Here we may hope to escape this difficulty because $\ell_s \rightarrow 0$ means that the infinite series of massive closed string states decouple and the Hagedorn temperature goes to infinity. Of course in the tachyonic context [115], the limit $\ell_s \rightarrow 0$ could not be taken since it would force the tachyon to be infinitely massive as well. To prove that there is no closed string production will require computation of the one-loop partition function in the theory expanded around the solution and showing that any non-vanishing imaginary part obtained in the limit (3.3.1) can be interpreted in terms of the DBI action (3.2.1). Such computations in a supersymmetric context are familiar in Matrix Theory. Recent work has also explored the non-supersymmetric context [117].

We have argued that open strings on the membrane eventually become strongly coupled when the physical radius is given by eq. (3.1.28). This special value for the radius remains fixed in our scaling limit: $R_s \sim (\tilde{g}_s)^{1/3}L$. It can be made arbitrarily small if we take \tilde{g}_s sufficiently small. But for any fixed value of this coupling, however small it is, strong coupling quantum effects are eventually needed to understand the subsequent membrane evolution. Quantum processes may cause the original brane with N units of $D0$ -brane charge to split into configurations of smaller charge. However such non-perturbative phenomena should be describable within the full non-abelian $D0$ -brane action (3.2.1).

We can also construct multi-membrane configurations. For example, we can construct m coincident spherical membranes if we start with the non-abelian DBI action of mN $D0$ -branes and replace the background values of the matrices Φ_i in (3.2.6) with the following block-diagonal forms [99]

$$\hat{R}X_i \rightarrow \hat{R}X_i \otimes \mathbb{1}_{m \times m} \quad (3.3.9)$$

The fluctuation matrices A_i are replaced by

$$A_i \rightarrow \sum_{\alpha=1}^{m^2} A_i^\alpha \otimes T^\alpha \quad (3.3.10)$$

where the $m \times m$ matrices T^α are generators of $U(m)$. Taking the large- N limit, while keeping m fixed, the action for the fluctuations should result in a non-abelian $U(m)$ gauge theory on a sphere describing a collection of m coincident spherical $D2$ -branes. The field strength of the $U(1)$ part of this gauge group attains a background value corresponding to mN units of flux on the sphere. We expect the effective metric and coupling constant of this theory to be given by the same formulae that we have derived before. Separate stacks of $D2$ -branes can be constructed by giving an appropriate vev to one of the transverse scalars; that is, by ‘Higgsing’ the $U(M)$ gauge group. The net background magnetic flux should now split appropriately amongst the separate stacks. Within this set-up, one can study non-perturbative instanton processes that result into transferring of $D0$ -branes from one membrane stack to another, as in [118]. The effective dimensionless coupling of such processes is given approximately by $g_{YM}^2/\langle\phi\rangle$, where $\langle\phi\rangle$ is the relevant Higgs vev. When the branes are large, that is $R > L$, this coupling can be kept small if we take \tilde{g}_s small, and such processes are exponentially suppressed. But when the radius becomes small, the theory becomes strongly coupled and such non-perturbative processes become relevant.

3.3.2 The $D0$ Yang-Mills (Matrix Theory) limit

In this limit, we take $R/L = r$ as well as r_0 to be small. We will show how the effective action for the fluctuations in this regime can be derived from the BFSS Matrix Model [13]. Earlier work on this model appears in [45, 119].

The effective parameters of the theory are $G_{\mu\nu}$, G_s and Θ^{ab} . In terms of the dimensionless radius variable r , these are given by

$$\begin{aligned} G_{00} &= -\sqrt{\frac{r^4+1}{r_0^4+1}}, & G_{ab} &= \frac{N\lambda}{2} \left(r^2 + \frac{1}{r^2} \right) \hat{h}_{ab} \\ G_s &= g_s \frac{\sqrt{r^4+1}}{r^2} \\ \Theta^{ab} &= -\frac{2\epsilon^{ab}}{N(1+r^4)\sin\theta} \end{aligned} \quad (3.3.11)$$

When $r, r_0 \ll 1$, these take the following ‘zero slope’ form [52, 120]

$$\begin{aligned} G_{00} \rightarrow \tilde{G}_{00} &= 1, & G_{ab} \rightarrow \tilde{G}_{ab} &= -\lambda^2 (Bh^{-1}B)_{ab} = \frac{N\lambda}{2r^2} \hat{h}_{ab} \\ G_s \rightarrow \tilde{G}_s &= g_s \sqrt{\det(\lambda B h^{-1})} = \frac{g_s}{r^2} \\ \Theta^{ab} \rightarrow \tilde{\Theta}^{ab} &= (B^{-1})^{ab} = -\frac{2\epsilon^{ab}}{N \sin\theta} \end{aligned} \quad (3.3.12)$$

Notice that the rate of collapse \dot{R} is given by

$$\dot{R}^2 = r_0^4 - r^4 \quad (3.3.13)$$

in this regime. In particular, this remains small throughout the collapse of the brane.

We can ‘derive’ these zero slope parameters from the effective action of the constituent $D0$ -branes in the small- r regime. The background fields scale as

$$\begin{aligned} \Phi_i &= \hat{R} X_i = \left(\frac{r}{2L} \right) X_i \\ \partial_t \Phi_i &= (\partial_t \hat{R}) X_i \sim \left(\frac{\sqrt{r_0^4 - r^4}}{2L^2} \right) X_i \end{aligned} \quad (3.3.14)$$

We assume a similar scaling behaviour for the fluctuations $A_i = 2\hat{R}K_i^a A_a + x_i\phi$ and their time derivatives $\partial_t A_i$ in the small- r regime. That is, we take the gauge field A_a to be of order one while the radial fluctuations $x_i\phi$ to be at most of the order r/L in magnitude. Similarly, the velocity fields $\partial_t A_a$ and $x_i\partial_t\phi$ are required to be of order r/L and r^2/L^2 respectively. This is a reasonable requirement for the behaviour of the fluctuations so as to keep them smaller or at least comparable to the background values of the fields. Then the full fields Φ_i and their time derivatives are sufficiently small in the small- r regime, and the $D0$ -brane effective action (3.2.1) takes the form of a $0+1$ dimensional Yang-Mills action

$$S = \frac{(2\pi)^2 \ell_s^3}{g_s} \int dt \left[Tr \left(\frac{1}{2} \partial_t \Phi_i \partial_t \Phi_i + \frac{1}{4} [\Phi_i, \Phi_j]^2 \right) - \frac{N}{\lambda^2} \right] \quad (3.3.15)$$

The second and third terms in (3.2.3) scale as r^4 in the limit, while the last two terms as

higher powers of r . In the small- r regime, we can neglect the last two terms of (3.2.3) and expand the square root of the DBI action dropping higher powers of r . We end up with an action that is quadratic in the time derivatives of the fields and quartic in the fields themselves⁷. Roughly speaking, in this regime each $D0$ -brane is moving slowly enough so that the non-relativistic, small velocity expansion of the DBI Lagrangean can be applied ending up with (3.3.15). This expansion is valid if we choose the initial radius parameter r_0 to be small enough, or the initial physical radius to satisfy $R_0 \ll L$. Essentially the Yang Mills regime is valid when the effective separation of neighbouring $D0$ -branes is smaller than the string scale throughout the collapse of the brane. Finally, in this regime the equation of motion for the scale factor is given by

$$\begin{aligned}\ddot{\hat{R}} + 8\hat{R}^3 &= 0 \\ \ddot{r} + \frac{2}{L^2}r^3 &= 0\end{aligned}\tag{3.3.16}$$

Setting $\Phi_i = \hat{R}X_i + A_i$, we can determine a matrix model for the fluctuating fields A_i . This matrix model is equivalent to a non-commutative $U(1)$ Yang Mills theory on a fuzzy sphere [99]. This correspondence maps hermitian matrices to functions on the sphere and replaces the matrix product with a suitable non-commutative star product. To see how the non-commutative gauge fields arise, we examine the transformation of the fluctuating matrices A_i under time independent infinitesimal $U(N)$ gauge transformations, which are symmetries of the action (3.3.15). Under such a gauge transformation, the matrices Φ_i and A_i transform as follows

$$\begin{aligned}\delta_\lambda \Phi_i &= i[\lambda, \Phi_i] \\ \delta_\lambda A_i &= -i\hat{R}[X_i, \lambda] + i[\lambda, A_i]\end{aligned}\tag{3.3.17}$$

with λ an $N \times N$ Hermitian matrix. Using eq. (3.2.12), the corresponding function on the sphere transforms as⁸

$$\delta_\lambda A_i = 2\hat{R}K_i^a \star \partial_a \lambda + i(\lambda \star A_i - A_i \star \lambda)\tag{3.3.18}$$

where λ is now taken to be a local function on the sphere. Thus we end up with a $U(1)$ non-commutative gauge transformation. The gauge covariant field strength is given by

$$F_{ij} = i\hat{R}[X_i, A_j] - i\hat{R}[X_j, A_i] + i[A_i, A_j] + 2\hat{R}\epsilon_{ijk}A_k = i[\Phi_i, \Phi_j] + 2\hat{R}\epsilon_{ijk}\Phi_k\tag{3.3.19}$$

The last equation makes gauge covariance manifest. The field strength F_{ij} is zero when the

⁷A similar expansion can be consistently carried out for the fields Φ_m that are transverse to the \mathbb{R}^3 where the membrane is embedded.

⁸We do not use different notation to distinguish the $N \times N$ hermitian matrices from their corresponding functions on the sphere. We hope the distinction is made clear from the context.

fluctuations are set to zero, while the commutator $[\Phi_i, \Phi_j]$ attains a background expectation value given by $\hat{R}^2[X_i, X_j]$.

In the commutative limit, the non-commutative gauge transformations (3.3.18) reduce to ordinary local $U(1)$ gauge transformations. As we already discussed, this is equivalent to a large- N limit. Decomposing $A_i = 2\hat{R}K_i^a A_a + x_i\phi$ we see that in the commutative limit, the tangential fields A_a transform as the components of a gauge field on the sphere, $\delta_\lambda A_a = \partial_a\lambda$, while the transverse field ϕ as a scalar. The full non-commutative gauge transformation (3.3.18) though mixes ϕ and the vector field A_a [99]; this is another manifestation of the fuzziness of the underlying space.

It is easy to see that in the commutative limit the field strength reduces to

$$F_{ij} \rightarrow 4\hat{R}^2 K_i^a K_j^b F_{ab} + 2\hat{R}(x_j K_i^a - x_i K_j^a) \partial_a \phi - 2\hat{R}\epsilon_{ijk} x_k \phi \quad (3.3.20)$$

The deformation arising from the underlying non-commutativity comes from the commutator piece $i[A_i, A_j]$ in (3.3.19). Up to the order of $1/N$, this deformation is given by (3.2.21), and can be re-written as

$$i[A_i, A_j] = \tilde{\Theta}^{ab} \partial_a A_i \partial_b A_j + \mathcal{O}(\tilde{\Theta}^2) \quad (3.3.21)$$

We conclude immediately that the underlying non-commutativity parameter is $\tilde{\Theta}$.

We can expand the $D0$ -brane Yang Mills action (3.3.15) to quadratic order in the fluctuations in the large- N limit. Having established the equivalence of the full $D0$ -DBI action with the $D2$ -brane action to this order in the fluctuations, all we need to do is to replace the effective metric, coupling constant and non-commutativity parameter with their ‘zero slope’ values (3.3.12). Of course, one can carry out the expansion directly using the action (3.3.15) and verify that the parameters of the theory in this regime are indeed given by $\tilde{G}_{\mu\nu}$, \tilde{g}_{YM}^2 and $\tilde{\Theta}$. The mass of the scalar field χ defined above eq. (3.2.31) is given by

$$m^2 = \frac{6r^2}{L^2} \quad (3.3.22)$$

in this regime and it is positive. Finally, the mixing term becomes

$$-\int dt d\theta d\phi \frac{\sqrt{-\tilde{G}}}{\tilde{g}_{YM}^2} \frac{r^3}{L^3} \chi (N\tilde{\Theta}^{ab}) F_{ab} \quad (3.3.23)$$

It is important to realise that non-linearities in the equations of motion, arising from interaction terms of higher than quadratic order in the non-relativistic Lagrangean (3.3.15), are all suppressed by factors of $1/N$. From the point of view of the $U(1)$ non-commutative field theory on the fuzzy sphere, all interaction terms arise from the non-commutative deformation of the field strength (3.3.19) and they end up being proportional to powers of

$\tilde{\Theta}$. It is easy then to see that non-linearities become important at angular momenta of order $l \sim N^{1/2}$ where $\tilde{\Theta}^{ab}\partial_a \otimes \partial_b$ is of order one. This fact was also emphasised in the analysis of [46]. From (3.1.28) then we see that such angular momentum modes become strongly coupled when

$$R \sim \ell_{11}N^{1/2} \quad (3.3.24)$$

or

$$r \sim g_s^{1/3} \quad (3.3.25)$$

Roughly, the strong coupling phenomenon occurs when in the closed string frame each $D0$ -brane occupies an area of order ℓ_{11}^2 , smaller than ℓ_s^2 .

In the scaling limit (3.3.1), the eleven dimensional Planck length tends to zero like $N^{-2/3}$ and the strong coupling radius (3.3.24) goes to zero. Thus in the limit (3.3.1) the evolution of such small branes, described by the $D0$ -brane Yang-Mills action (3.3.15), can be treated classically throughout the collapse of the brane. We can alternatively take a different scaling limit so as to probe the short eleven dimensional Planck scale, which sets the distance scale at which strong coupling quantum phenomena occur in our system in the non-relativistic regime.

We can take $g_s \rightarrow 0$ keeping R and ℓ_{11} fixed, and also N fixed and large. In this limit $L \rightarrow \infty$ like $g_s^{-1/3}$, so that r and also r_0 are small. The physical field variables $\lambda\Phi_i \sim RX_i/N$, and so they remain fixed in this limit. The same is true for their conjugate momenta. At the same time each individual $D0$ -brane is getting very heavy since $m_{D0} = 1/g_s\ell_s \sim g_s^{-2/3}/\ell_{11}$. Hence the $D0$ -branes are slowly moving in this limit. This limit is the famous DKPS limit [100, 103] in which the short distance scale probed by the $D0$ -branes is the eleven dimensional Planck scale. Closed strings decouple from the brane system. The same is true for excited massive open strings on the branes. This is because the energy of the fluctuating massless open string states is much smaller than the mass of excited open string oscillators in the limit [100, 103] and so massive open strings cannot get excited. Finally, in the BFSS limit [13] where the eleven circle radius is decompactified, the membrane we constructed is just a boosted spherical M-theory membrane.

The above strong coupling phenomenon occurs at a physical radius which is bigger than the size of the bound N $D0$ -brane quantum ground state by a factor of $N^{1/6}$. However angular momentum modes of order the cutoff N become strongly coupled when the physical radius R becomes comparable to the size of the ground state $\ell_{11}N^{1/3}$ as can be seen from (3.1.28). It is interesting that this scale which is expected to emerge from a complicated ground state solution of the $D0$ -brane Yang-Mills Hamiltonian also appears in the analysis of the linearised fluctuations of fuzzy spheres.

There is yet another simple way to see the ℓ_{11} length scale. It involves the application of the Heisenberg uncertainty principle to the reduced radial dynamics. The momentum

conjugate to R coming from the reduced action (3.2.35) is

$$\Pi_R = \frac{\sqrt{R_0^4 - R^4}}{g_s \ell_s^3 \pi} \quad (3.3.26)$$

With $(\Delta R) \Pi_R > \hbar$ and $\hbar \sim 1$, we get

$$(\Delta R) > \frac{g_s \ell_s^3 \pi}{\sqrt{R_0^4 - R^4}} \quad (3.3.27)$$

Evaluating the uncertainty at $R = 0$ and assuming the whole trajectory lies within the quantum regime, *i.e.* $R_0 \sim \Delta R$, we obtain a critical value for the initial radius $R_0 \sim R_c$ where $R_c \sim g_s^{1/3} \ell_s$, which is the eleven dimensional Planck scale. This simple analysis does not detect the $N^{1/3}$ factor that appears in the more complete analysis above.

The above discussion has focused on the region where R is much smaller than L . The region of $R \gg L$ or equivalently $L = \ell_s \sqrt{\pi N} \rightarrow 0$ is also of interest. In the strict $N = 0$ limit we have a $D2$ -brane without $D0$ -brane charge. The negative sign of the mass of the field χ that appears in (3.2.36) for $R > L$ also appears in the problem of fluctuations around the pure $D2$ -brane solution. This negative sign indicates that the zero mode of the field χ is tachyonic in this regime. When R_0 is larger than L , the tachyonic mass naively causes an exponential growth for the zero mode of the fluctuation χ . At this point, higher order corrections to the action involving the zero mode would become significant. However, the reduced action for the scalar dynamics has no exponentially growing solutions. This means that higher order terms stop this exponential growth. In fact, as R crosses L , the sign of the mass changes and we go into an oscillatory phase. This transition is reminiscent of a similar transition which occurs in the equation for fluctuations in inflationary scenarios, see for example [121]. In the case $R_0 \leq L$ the time evolution of the radial fluctuation does not encounter the tachyonic region.

3.3.3 Mixing with graviton scattering states

The key observable in the BFSS Matrix theory limit is the scattering matrix of $D0$ -brane bound ground states made of N_1, N_2, \dots, N_i $D0$ -branes, where N_i are all large. Since these interactions are governed by ℓ_{11} , which goes to zero in the scaling limit (3.3.1), such interactions amongst such states become irrelevant. However a simple estimate suggests that these states can mix with the fuzzy sphere states. Consider an $SU(2)$ representation of spin j with $N = 2j + 1$. Consider also matrices

$$U_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & b \pm iv \end{pmatrix}$$

The diagonal blocks are of size $N_1 \times N_1$ and $(N - N_1) \times (N - N_1)$. There are also the standard $N \times N$ $SU(2)$ matrices J_+, J_- , which act in this representation. In the fuzzy sphere configuration we set $X_\pm \equiv (X_1 \pm iX_2) = J_\pm$ while in the scattering configuration we set $X_\pm = U_\pm$. We calculate $Tr([J_+, U_-][J_-, U_+])$ and find this proportional to $\lambda^2(\hat{R})^2 N(N - N_1)$. If N_1 is a finite fraction of N then this goes like $\lambda^2(\hat{R})^2 N^2 \sim (\hat{R})^2 L^4$ in the large- N limit, which is of the same order as the terms in the quadratic action for the fluctuations we have computed. This indicates that the collapsing membrane can undergo transitions to these scattering states and conversely the scattering states can give rise to membranes.

CHAPTER 4

SYMMETRISED TRACE CORRECTIONS FOR NON-ABELIAN DBI

We saw in the introduction that the correct prescription for the non-abelian D -brane action requires the promotion of the trace over the gauge indices to a symmetrised trace (STr). In the context of the class of systems that we have thus far described, the implementation of such a procedure is complicated and is expected to give a series of $1/N$ corrections. Up to now, we have ignored such corrections since we have predominantly been working in the large- N limit. It is, however, imperative to explore how the symmetrised trace contributions are obtained systematically and how they affect the physics of the configurations under study. An initial attempt to calculate the effect of STr in the context of collapsing fuzzy- S^2 's was carried out in [63], where the first few terms in a $1/C$ expansion were extracted for general $SO(3)$ representations. For the spin-half representation in particular, the full set of corrections was recovered.

In this chapter we extend the calculation of symmetrised traces from the spin-half example of [63] to general representations of $SO(3)$. These results allow us to study in detail the finite- N physics of the time-dependent fuzzy two-sphere. We begin our finite- N analysis with a careful discussion on how to extract the physical radius from the matrices of the non-abelian Ansatz. The standard formula used in the Myers effect is $R^2 = \lambda^2 \operatorname{Tr}(\Phi_i \Phi_i)/N$. Requiring consistency with a constant speed of light, independent of N , leads us to propose an equation, which agrees with the standard formula in large- N commutative limits, but disagrees in general. We write down finite- N formulae for the energy and Lagrangean of the time-dependent fuzzy 2-sphere. We also give the conserved pressure which is relevant for the $D1 \perp D3$ system. We study the time of collapse as a function of N and find that in the region of large- N , for fixed initial radius R_0 , the time decreases as N decreases. However, at some point there is a turn-around in this trend and the time of collapse for spin-half is actually larger than at large- N . We also investigate the quantity $E^2 - p^2$, where E is the energy and p the momentum. This quantity is of interest when we view the time-dependent D -brane as a source for space-time fields. E is the T^{00} component of the stress tensor, and p is the T^{0r} component as we show by a generalisation of arguments previously used in the context of BFSS Matrix Theory. For the large- N formulae, $E^2 - p^2$ is always positive. At finite- N , this can be negative, although the speed of radial motion is less than the speed of light. Given the relation to the stress tensor, we can interpret this as a violation of the dominant energy condition. The other object of interest is the proper acceleration along the

trajectory of a collapsing $D2$ -brane. We find analytic and numerical evidence that there are regions of both large and small R , with small and relativistic velocities respectively, where the proper accelerations can be small. This is intriguing since the introduction of stringy and higher derivative effects in the small velocity region can be done with an adiabatic approximation, but it is interesting to consider approximation methods for the relativistic region.

We also discuss the higher fuzzy sphere case, by giving a formula for $STr(X_i X_i)^m$, in general irreducible representations of $SO(2k+1)$. This formula is motivated by some considerations surrounding D -brane charges and the ADHM construction, which are discussed in more detail in [122]. Some of the motivation is explained in App. C. This allows us a partial discussion of finite- N effects for higher fuzzy spheres and we are able to calculate their physical radius; however, in general one needs other symmetrised traces involving elements of the Lie algebra $so(2k+1)$.

The symmetrised trace prescription, which we study in detail in this chapter, is known to correctly match open string calculations up to the first two orders in an α' expansion, but the correct answer deviates from the $(\alpha')^3$ term onwards [34–37]. It is possible however that for certain special symmetric background configurations, it may give the correct physics to all orders. The D -brane charge computation discussed in App. C can be viewed as a possible indication in this direction. In any case, it is important to study the corrections coming from this prescription to all orders in order to be able to systematically modify it, if that becomes necessary when the correct non-abelian D -brane action is known. Conversely the physics of collapsing D -branes can be used to constrain the form of the non-abelian DBI.

4.1 Lorentz invariance and the physical radius

Once again our starting point will be the collapse of a cluster of N $D0$ -branes in the shape of a fuzzy- S^{2k} , in a flat background, which has a large- N dual description in terms of spherical $D(2k)$ branes with N units of flux. For quick reference, we give once more the form of the action for the microscopic non-abelian $D0$ description and some related quantities

$$S_0 = -\frac{1}{g_s \ell_s} \int dt STr \sqrt{-\det(M)} \quad (4.1.1)$$

where

$$M = \begin{pmatrix} -1 & \lambda \partial_t \Phi_j \\ -\lambda \partial_t \Phi_i & Q_{ij} \end{pmatrix}$$

and

$$Q_{ij} = \delta_{ij} + i\lambda[\Phi_i, \Phi_j]$$

We will consider the time-dependent Ansatz

$$\Phi_i = \hat{R}(t)X_i \quad (4.1.2)$$

The X_i are matrices obeying some algebra. The part of the action that depends purely on the time derivatives and survives when $\hat{R} = 0$ is

$$S_{D0} = \int dt STr \sqrt{1 - \lambda^2(\partial_t \Phi_i)^2} = \int dt STr \sqrt{1 - \lambda^2(\partial_t \hat{R})^2 X_i X_i} \quad (4.1.3)$$

For the fuzzy- S^2 , the $X_i = \alpha_i$, for $i = 1, 2, 3$, are generators of the N -dimensional irreducible representation of $su(2)$, where $N = n + 1$, and n is related to the spin J by $n = 2J$. In this case the algebra is

$$[\alpha_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k \quad (4.1.4)$$

and following [63], the action for N D0-branes can be reduced to

$$S_0 = -\frac{1}{g_s \ell_s} \int dt STr \sqrt{1 + 4\lambda^2 \hat{R}^4 \alpha_j \alpha_j} \sqrt{1 - \lambda^2(\partial_t \hat{R})^2 \alpha_i \alpha_i} \quad (4.1.5)$$

If we define the physical radius using

$$R_{phys}^2 = \lambda^2 \lim_{m \rightarrow \infty} \frac{STr(\Phi_i \Phi_i)^{m+1}}{STr(\Phi_j \Phi_j)^m} = \lambda^2 \hat{R}^2 \lim_{m \rightarrow \infty} \frac{STr(\alpha_i \alpha_i)^{m+1}}{STr(\alpha_j \alpha_j)^m} \quad (4.1.6)$$

we will find that the Lagrangean will be convergent for speeds between 0 and 1. The radius of convergence will be exactly one. This follows by applying the ratio test to the series expansion of

$$STr \sqrt{1 - \lambda^2 \dot{\hat{R}}^2 \alpha_i \alpha_i} \quad (4.1.7)$$

where a dot indicates differentiation with respect to time. This leads to

$$R_{phys}^2 = \lambda^2 \hat{R}^2 n^2 \quad (4.1.8)$$

Using explicit formulae for the symmetrised traces we will also see that, with this definition of the physical radius, the formulae for the Lagrangean and energy will have a first singularity at $\dot{\hat{R}}_{phys} = 1$. In the large- n limit, the definition of physical radius in (4.1.8) agrees with the one in [28] that we have used up to now, where R_{phys} is defined by $R_{phys}^2 = \frac{\lambda^2}{N} Tr(\Phi_i \Phi_i)$. Note that this definition of the physical radius will also be valid for the higher dimensional fuzzy spheres, and more generally in any matrix construction, where the terms in the non-abelian DBI action depending purely on the velocity are of the form $\sqrt{1 - \lambda^2 X_i X_i (\partial_t \hat{R})^2}$.

In what follows, the sums we get in expanding the square root are conveniently written

in terms of r, s , defined by $r^4 = 4\lambda^2 \dot{R}^4$ and $s^2 = \lambda^2 \dot{R}^2$. It is also useful to define

$$\begin{aligned} L^2 &= \frac{\lambda n}{2} \\ \hat{r}^2 &= \frac{R_{phys}^2}{L^2} = r^2 n \\ \hat{s}^2 &= s^2 n^2 \end{aligned} \tag{4.1.9}$$

The \hat{r} and \hat{s} variables approach the variables called r, s in the large- n discussion of [63] and of Ch. 2. Note, using (4.1.8), that

$$\dot{R}_{phys}^2 = s^2 n^2 = \hat{s}^2 \tag{4.1.10}$$

4.2 The fuzzy- S^2 at finite- n

For the fuzzy- S^2 , the relevant algebra is that of $su(2)$, eq. (4.1.4) above. We also have the quadratic Casimir $C = \alpha_i \alpha_i = (N^2 - 1) \mathbb{1}_{N \times N}$

We present here the result¹ of the full evaluation of the symmetrised trace for odd n

$$C(m, n) \equiv \frac{1}{n+1} STr(\alpha_i \alpha_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{(n+1)/2} (2i-1)^m \tag{4.2.1}$$

whilst for even n

$$C(m, n) \equiv \frac{1}{n+1} STr(\alpha_i \alpha_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{n/2} (2i)^m \tag{4.2.2}$$

For $m = 0$ the second expression doesn't have a correct analytic continuation and we will impose the value $STr(\alpha_i \alpha_i)^0 = 1$. The expression for $C(m, 1)$ was proved in [63]. A proof of (4.2.2) for $n = 2$ is given in App. D. The general formulae given above are conjectured on the basis of various examples, together with arguments related to D -brane charges. These are given in App. C. There is also a generalisation to the case of higher dimensional fuzzy spheres, described in Sec. 4.4 and App. C.

We will now use the results (4.2.1), (4.2.2), to obtain the symmetrised trace-corrected energy for a configuration of N time-dependent $D0$ -branes blown up to a fuzzy- S^2 . The

¹We hope at this point that no confusion will arise between the quadratic Casimir C and the combinatoric factor $C(m, n)$.

reduced action (4.1.5) can be expanded to give

$$\begin{aligned}\mathcal{L} &= -STr \sqrt{1 + 4\lambda^2 \hat{R}^4 \alpha_i \alpha_i} \sqrt{1 - \lambda^2 \dot{\hat{R}}^2 \alpha_i \alpha_i} \\ &= -STr \sqrt{1 + r^4 \alpha_i \alpha_i} \sqrt{1 - s^2 \alpha_i \alpha_i} \\ &= -STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (\alpha_i \alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} (-1)^m\end{aligned}\quad (4.2.3)$$

The expression for the energy then follows directly

$$\mathcal{E} = -STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (2m-1) (\alpha_i \alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} (-1)^m \quad (4.2.4)$$

and after applying the symmetrised trace results given above we get the finite- n corrected energy for any finite-dimensional irreducible representation of spin- $\frac{n}{2}$ for the fuzzy- S^2 .

For $n = 1, 2$ one finds

$$\frac{1}{2} \mathcal{E}_{n=1}(r, s) = \frac{1 + 2r^4 - r^4 s^2}{\sqrt{1 + r^4}(1 - s^2)^{3/2}} \quad (4.2.5)$$

$$\frac{1}{3} \mathcal{E}_{n=2}(r, s) = \frac{2}{3} \frac{(1 + 8r^4 - 16r^4 s^2)}{\sqrt{1 + 4r^4}(1 - 4s^2)^{3/2}} + \frac{1}{3} \quad (4.2.6)$$

We note that both of these expressions provide equations of motion which are solvable by solutions of the form $\hat{r} = t$.

For the case of general n , it can be checked that the energy can be written

$$\mathcal{E}_n(r, s) = \sum_{l=1}^{\frac{n+1}{2}} \frac{2 - 2(2l-1)^2 r^4 ((2l-1)^2 s^2 - 2)}{\sqrt{1 + (2l-1)^2 r^4}(1 - (2l-1)^2 s^2)^{3/2}} \quad (4.2.7)$$

for n -odd, while for n -even

$$\mathcal{E}_n(r, s) = 1 + \sum_{l=1}^{\frac{n}{2}} \frac{2 - 2(2l)^2 r^4 ((2l)^2 s^2 - 2)}{\sqrt{1 + (2l)^2 r^4}(1 - (2l)^2 s^2)^{3/2}} \quad (4.2.8)$$

Equivalently, the closed form expression for the Lagrangean for n -odd is

$$\mathcal{L}_n(r, s) = -2 \sum_{l=1}^{\frac{n+1}{2}} \frac{1 - 2(2l-1)^2 s^2 + (2l-1)^2 r^4 (2 - 3(2l-1)^2 s^2)}{\sqrt{1 + (2l-1)^2 r^4} \sqrt{1 - (2l-1)^2 s^2}} \quad (4.2.9)$$

whilst for n -even

$$\mathcal{L}_n(r, s) = -1 - 2 \sum_{l=1}^{\frac{n}{2}} \frac{1 - 2(2l)^2 s^2 + (2l)^2 r^4 (2 - 3(2l)^2 s^2)}{\sqrt{1 + (2l)^2 r^4} \sqrt{1 - (2l)^2 s^2}} \quad (4.2.10)$$

It is clear from these expressions that the equations of motion in the higher spin case will also admit the $\hat{r} = t$ solution. Note that, after performing the rescaling to physical variables (4.1.9) and (4.1.10), these energy functions and Lagrangeans have no singularity for fixed r , in the region $0 \leq \dot{R}_{phys} \leq 1$. As s increases from 0 the first singularity is at $s = \frac{1}{n}$, which corresponds to $\hat{s} = \dot{R}_{phys} = 1$. In this sense they are consistent with a fixed speed of light. However, they do not involve, for fixed r , the form $\sqrt{dt^2 - dr^2}$ and hence do not have an $so(1, 1)$ symmetry. It will be interesting to see if there are generalisations of $so(1, 1)$, possibly involving non-linear transformations of dt, dr , which can be viewed as symmetries.

4.2.1 The $D1 \perp D3$ intersection at finite- n

The relationship between the microscopic descriptions of the time dependent $D0$ - $D2$ system and the static $D1 \perp D3$ intersection was established in Ch. 2. There, the large- n behaviour of both systems was described by a genus one Riemann surface, which was a fixed orbit in complexified phase space. This was done by considering the conserved energy and pressure and complexifying the variables r and $\partial r = s$ respectively. Conservation of the energy-momentum tensor then yielded elliptic curves in r, s , involving a fixed parameter r_0 , which corresponded to the initial radius of the configuration. The actions for the two systems were related by a Wick rotation.

We can apply the symmetrised trace formula to also get exact results for the corrected pressure of the fuzzy- S^2 funnel configuration at finite- n . For our system we simply display the general result and the first two explicit cases

$$\mathcal{P} = STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (2m-1)(\alpha_i \alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} \quad (4.2.11)$$

$$\frac{1}{2} \mathcal{P}_{n=1}(r, s) = -\frac{1 + 2r^4 + r^4 s^2}{\sqrt{1 + r^4} (1 + s^2)^{3/2}} \quad (4.2.12)$$

$$\frac{1}{3} \mathcal{P}_{n=2}(r, s) = -\frac{2}{3} \frac{(1 + 8r^4 + 16r^4 s^2)}{\sqrt{1 + 4r^4} (1 + 4s^2)^{3/2}} - \frac{1}{3} \quad (4.2.13)$$

Note that the above formulae are related to the energies in (4.2.5) and (3.7) by a Wick rotation $s \rightarrow is$. Similar results to those for the time-dependent case hold for the exact expression of the pressure for the general spin- $\frac{n}{2}$ representation. Note again that these expressions will provide equations of motion which are solved by solutions of the form $\hat{r} = 1/\sigma$, where σ is the spatial $D1$ world-volume co-ordinate. An easy way to see this is to substitute $s^2 = r^4$ in (4.2.12), (4.2.13), to find that the pressures becomes independent of r and s . Since the higher spin results for the pressure are sums of the $n = 1$ or $n = 2$ cases, the argument extends.

4.2.2 Finite- N dynamics as a quotient of free multi-particle dynamics

Using the formulae above, we can see that the fuzzy- S^2 energy for general n is determined by the energy at $n = 1$. In the odd- n case

$$C(m, n) = \frac{2}{n+1} C(m, 1) \sum_{i_3=1}^{\frac{n+1}{2}} (2i_3 - 1)^{2m} = \frac{2}{n+1} (2m+1) \sum_{i_3=1}^{\frac{n+1}{2}} (2i_3 - 1)^{2m}$$

Using this form for $C(m, n)$ in the derivation of the energy, we get

$$\mathcal{E}_n(r, s) = \sum_{i_3=1}^{\frac{n+1}{2}} \mathcal{E}_{n=1} \left(r\sqrt{(2i_3 - 1)}, s(2i_3 - 1) \right) \quad (4.2.14)$$

Similarly, in the even- n case, we find

$$\mathcal{E}_n(r, s) = \sum_{i_3=1}^{\frac{n}{2}} \mathcal{E}_{n=2} \left(r\sqrt{i_3}, s(i_3) \right) \quad (4.2.15)$$

It is also possible to write $C(m, 2)$ in terms of $C(m, 1)$ as (for $m \neq 0$)

$$C(m, 2) = \frac{2^{2m+1}}{3} C(m, 1) = \frac{2^{2m+1}}{3} (2m+1) \quad (4.2.16)$$

Thus we can write $\mathcal{E}_n(r, s)$, for even n , in terms of the basic $\mathcal{E}_{n=1}(r, s)$ as

$$\mathcal{E}_n(r, s) = 1 + \sum_{i_3=1}^{\frac{n}{2}} \mathcal{E}_{n=1} \left(r\sqrt{(2i_3)}, s(2i_3) \right) \quad (4.2.17)$$

These expressions for the energy of spin- $\frac{n}{2}$ can be viewed as giving the energy in terms of a quotient of a multi-particle system, where the individual particles are associated with the spin-half system. For example, the energy function for $(n+1)/2$ free particles with dispersion relation determined by $\mathcal{E}_{n=1}$ is $\sum_i \mathcal{E}_{n=1}(r_i, s_i)$. By constraining the particles by $r_i = r\sqrt{2i+1}$, $s_i = s(2i+1)$ we recover precisely (4.2.14).

We can now use this result to resolve a question raised by [63] on the exotic bounces seen in the Lagrangeans obtained by keeping a finite number of terms in the $1/n$ expansion. With the first $1/n$ correction kept, the bounce appeared for a class of paths involving high velocities with $\gamma = \frac{1}{\sqrt{1-s^2}} \sim C^{1/4}$, near the limit of validity of the $1/n$ expansion. The bounce disappeared when two orders in the expansion were kept. It was clear that whether the bounces actually happened or not could only be determined by finite- n calculations. These exotic bounces would be apparent in constant energy contour plots for r, s as a zero in the first derivative $\partial r/\partial s$. In terms of the energies, this translates into the presence of a zero of $\partial E/\partial s$ for constant r . It is easy to show from the explicit forms of the energies that

these quantities are strictly positive for $n = 1$ and $n = 2$. Since the energy for every n can be written in terms of these, we conclude that there are no bounces for any finite- n . This resolves the question raised in [63] about the fate of these bounces at finite- n .

We note that the large- n limit of the formula for the energy provides us with a consistency check. In the large- n limit the sums above become integrals. For the odd- n case (even- n can be treated in a similar fashion), define $x = \frac{2i_3-1}{n} \sim \frac{2i_3}{n}$. Then the sum in (4.2.14) goes over to the integral

$$\frac{n}{2} \int_0^1 dx \frac{2 - 2x^2 n^2 r^4 (x^2 n^2 s^2 - 2)}{\sqrt{1 + x^2 n^2 r^4} (1 - x^2 s^2)^{3/2}} = \frac{n\sqrt{1 + r^4 n^2}}{\sqrt{1 - s^2 n^2}} \quad (4.2.18)$$

By switching to the \hat{r}, \hat{s} parameters the energy can be written as $\frac{n\sqrt{1+\hat{r}^4}}{\sqrt{1-\hat{s}^2}}$. This matches exactly the large- n limit used in [63].

4.3 Physical properties of the finite- N solutions

4.3.1 Special limits where finite- n and large- n formulae agree

In the above we compared the finite- n formula with the large- n limit. Here we consider the comparison between the fixed- n formula and the large- n one in some other limits. On physical grounds we expect some agreement. The $D0$ - $D2$ system at large \hat{r} and small velocity \hat{s} is expected to be correctly described by the $D2$ equations. These coincide with the large- n limit of the $D0$. In the $D1 \perp D3$ system, the large \hat{r} limit with large imaginary \hat{s} is also described by the $D3$.

Such an argument should extend to the finite- n case. Unfortunately, the common description of both types of systems in Ch. 2 by a genus one Riemann surface will not be as simple: The energy functions are more complicated and the resulting Riemann surfaces are of higher genus. We still expect the region of the finite- n curve, with large r and small, real s , to agree with the same limit of the large- n curve. We also expect the region of large r and large imaginary s to agree with large- n .

For concreteness consider odd- n . Indeed for large r , small s , (4.2.7) gives

$$\sum_{l=1}^{\frac{n+1}{2}} 4(2l-1)r^2 \sim n^2 r^2 = n\hat{r}^2 \quad (4.3.1)$$

This agrees with the result obtained from the $D2$ -brane Lagrangean [63] using (4.1.9) and (4.1.10). In this limit, both the genus one curve and the high genus finite- n curves degenerate to a pair of points. Now consider the energy functions in the limit of large \hat{r} and large imaginary \hat{s} . This is the right regime for comparison with the $D1 \perp D3$ system since this is described, at large- n and in the region of large \hat{r} , by $\hat{r} \sim \frac{1}{\sigma}$. This means that \hat{r} is large at

small σ , where $\frac{d\hat{r}}{d\sigma} \rightarrow i\hat{s}$ is large. Using the Wick rotation $s \rightarrow is$ (which takes us from the time-dependent system to the space-dependent system)

$$\mathcal{E} \rightarrow \mathcal{P} \sim \frac{nr^2}{s} = n\frac{\hat{r}^2}{\hat{s}} \quad (4.3.2)$$

which agrees with the same limit of the large- n curve. In this limit, both the large- n genus one curve and the finite- n curves of large genus degenerate to a genus zero curve.

The agreement in (4.3.1) between the $D0$ and $D2$ pictures is a stringy phenomenon. It follows from the fact that there is really one system, a bound state of $D0$ and $D2$ branes. A boundary conformal field theory would have boundary conditions that encode the presence of both the $D0$ and $D2$. In the large- N limit, the equations of motion coming from the $D0$ -effective action agree with the $D2$ -effective action description at all R_{phys} . This is because at large- N it is possible to specify the DBI-scaling of Ch. 3, where the regime of validity of both the $D0$ and $D2$ effective actions extends for all R_{phys} . This follows because the DBI scaling has $\ell_s \rightarrow 0$. Indeed it is easy to see that the effective open string metric (3.1.9) has the property that $\ell_s^2 G^{-1} = \frac{\ell_s^2 R_{phys}^2}{R_{phys}^4 + L^4}$ goes to zero when $N \rightarrow \infty$ with $L = \ell_s \sqrt{\pi N}$, R_{phys} fixed. This factor $\ell_s^2 G^{-1}$ controls higher derivative corrections for the open string degrees of freedom. At finite- N , we can keep $\ell_s^2 G^{-1}$ small, either when $R_{phys} \ll L$ or $R_{phys} \gg L$. Therefore, there are two regimes where the stringy description reduces to an effective field theory, where higher derivatives can be neglected. The agreement holds for specified regions of R_{phys} as well as \dot{R}_{phys} , because the requirement $\ell_s^2 G^{-1} \ll 1$ is not the only condition needed to ensure that higher derivatives can be neglected: We also require that the proper acceleration is small. At large R_{phys} , the magnetic flux density is small (as well as the higher derivatives being small) and the $D2$ -brane without non-commutativity is a good description. This is why the finite- N equations derived from the $D0$ -brane effective field theory agree with the abelian $D2$ -picture. For small R_{phys} , small \dot{R}_{phys} , we can also neglect higher derivatives. This is the region where the $D0$ Yang-Mills description is valid, or equivalently a strongly non-commutative $D2$ -picture.

4.3.2 Finite- N effects: Time of collapse, proper accelerations and violation of the dominant energy condition

We will consider the time of collapse as a function of n using the definition of the physical radius given at the beginning of this chapter. In order to facilitate comparison with the large- n system, we will be using \hat{r}, \hat{s} variables. To begin with, consider the dimensionless acceleration, which can be expressed as

$$-\gamma^3 \hat{s} \frac{\partial_{\hat{s}} \mathcal{E} |_{\hat{r}}}{\partial_{\hat{r}} \mathcal{E} |_{\hat{s}}} \quad (4.3.3)$$

with $\gamma = 1/\sqrt{1 - \hat{s}^2}$. As the sphere starts collapsing from $\hat{r} = \hat{r}_0$ down to $\hat{r} = 0$, the speed changes from $\hat{s} = 0$ to a value less than $\hat{s} = 1$. It is easy to see that the acceleration does not change sign in this region. Using the basic energy $\hat{\mathcal{E}} = \mathcal{E}/N$ from (4.2.5), we can write

$$\begin{aligned}\frac{\partial \hat{\mathcal{E}}_{n=1}(\hat{r}, \hat{s})}{\partial \hat{s}} &= \hat{s} \frac{(3(1 + \hat{r}^4) + \hat{r}^4(1 - \hat{s}^2))}{\sqrt{(1 + \hat{r}^4)(1 - \hat{s}^2)}^{\frac{5}{2}}}, \\ \frac{\partial \hat{\mathcal{E}}_{n=1}(\hat{r}, \hat{s})}{\partial \hat{r}} &= \frac{2\hat{r}^3}{(1 + \hat{r}^4)^{\frac{3}{2}}(1 - \hat{s}^2)^{\frac{3}{2}}} \left((1 + \hat{r}^4) + (1 - \hat{s}^2)(2 + \hat{r}^4) \right)\end{aligned}\quad (4.3.4)$$

Neither of the partial derivatives change sign in the range $\hat{s} = 0$ to 1 . Hence the speed \hat{s} increases monotonically. The same result is true for $n > 1$, since the energy functions for all these cases can be written as a sum of the energies at $n = 1$.

In the $n = 1$ case, $\hat{r} = r$, $\hat{s} = s$. For fixed r_0 the speed at $r = 0$ is given by

$$(1 - s^2|_{n=1}) = \frac{(1 + r_0^4)^{\frac{1}{3}}}{(1 + 2r_0^4)^{\frac{2}{3}}}\quad (4.3.5)$$

Comparing this with the large- n formula, which follows from conservation of energy

$$(1 - s^2|_{n=\infty}) = (1 + r_0^4)^{-1}\quad (4.3.6)$$

it is easy to see that

$$\left(\frac{(1 - s^2)|_{n=\infty}}{(1 - s^2)|_{n=1}} \right)^3 = \frac{(1 + 2r_0^4)^2}{(1 + r_0^4)^4} < 1\quad (4.3.7)$$

which establishes that the speed at $r = 0$ is larger for $n = \infty$.

We can strengthen this result to show that the speed of collapse at all $r < r_0$ is smaller for $n = 1$ than at $n = \infty$. For any $r < r_0$ we evaluate this energy function with the speed of collapse evaluated at $s^2 = \frac{r_0^4 - r^4}{r_0^4 + 1}$, which is the speed at the same r in the large- n problem. Let us define $F(r, r_0) = \hat{\mathcal{E}}_{n=1} \left(r, s = \sqrt{\frac{r_0^4 - r^4}{r_0^4 + 1}} \right)$. We compare this with $\hat{\mathcal{E}}_{n=1}(r, s)$ for s appropriate for the $n = 1$ problem, which is just $\frac{1+2r_0^4}{\sqrt{1+r_0^4}} \equiv G(r_0)$ by conservation of energy. We now use the fact, established above, that $\frac{\partial \hat{\mathcal{E}}_{n=1}}{\partial s}$ is positive for any real r . This means that we can show $s|_{n=1} < \sqrt{\frac{r_0^4 - r^4}{r_0^4 + 1}}$ by showing that $F(r, r_0) > G(r_0)$. A short calculation gives

$$F(r, r_0) - G(r_0) = \frac{r_0^4}{\sqrt{1 + r_0^4}(1 + r^4)}(r_0^4 - r^4)\quad (4.3.8)$$

It is clear that we have the desired inequality, showing that, at each r , the speed s in the $n = 1$ problem is smaller than the speed in the $n = \infty$ system. Hence the time of collapse is larger at $n = 1$. In the $n = 2$ case, we find that an exactly equivalent treatment proves

again that the collapse is slower than at large- n . However, this trend is not a general feature for all n . In the leading large- N limit, the time of collapse is given by the formula

$$\frac{T}{L} = \int dr \frac{\sqrt{1+r_0^4}}{\sqrt{r_0^4-r^4}} = \frac{K(\frac{1}{\sqrt{2}})}{\sqrt{2}} \frac{\sqrt{R^4+L^4}}{R} \quad (4.3.9)$$

For fixed ℓ_s , L decreases with decreasing N and as a result T decreases. When we include the first $1/N$ correction the time of collapse is [63]

$$\frac{T}{L} = \int dr \left[\frac{\sqrt{1+r_0^4}}{\sqrt{r_0^4-r^4}} + \frac{r_0^8}{6N^2(1+r_0^4)^{3/2}\sqrt{r_0^4-r^4}} - \frac{r_0^4(1+3(1+r_0^4))}{6N^2(1+r^4)\sqrt{1+r_0^4}\sqrt{r_0^4-r^4}} \right] \quad (4.3.10)$$

By performing numerical integration of the above for several values of the parameter r_0 and some large but finite values of N , we see that the time of collapse is smaller for the $1/N$ corrected case. This means that, in the region of large- N the time of collapse decreases as N decreases, with both the leading large- N formula and the $1/N$ correction being consistent with this trend. However, as we saw above the time of collapse at $n = 1$ and $n = 2$ are larger than at $n = \infty$. This means that there are one or more turning points in the time of collapse as a function of n .

The deceleration effect that arises in the comparison of $n = 1$ and $n = 2$ with large- n may have applications in cosmology. Deceleration mechanisms coming from DBI actions have been studied in the context of bulk causality in AdS/CFT [123, 124] and applied in the problem of satisfying slow roll conditions in stringy inflation [125]. Here we see that the finite- n effects result in a further deceleration in the region of small- n .

We turn to the proper acceleration which is important in checking the validity of our action. Since the DBI action is valid when higher derivatives are small, it is natural to demand that the proper acceleration, should be small (see for example [123]). The condition is $\gamma^3 \ell_s \partial_t^2 R_{phys} \ll 1$. In terms of the dimensionless variables it is $\gamma^3 (\partial_\tau^2 \hat{r}) \ll \sqrt{N}$. If we want a trajectory with initial radius \hat{r}_0 such that the proper acceleration always remains less than one through the collapse, then there is an upper bound on \hat{r}_0 (see for example Sec. 8 of [63]). The corresponding upper bound on the physical radius goes to infinity as $N \rightarrow \infty$, since $R_{phys} \sim \hat{r} \sqrt{\lambda N}$. For small \hat{r}_0 we are in the Matrix Theory limit and the effective action is valid. For large \hat{r}_0 and \hat{r} large, the acceleration is under control, $\alpha \sim 1/\hat{r}$ and the velocity will be close to zero. Interestingly, there will also be a class of trajectories parametrised by large \hat{r}_0 , which admit relativistic motion. Consider for example the $n = 1$ case (where $\hat{r} = r, \hat{s} = s$). The proper acceleration can be written as

$$\alpha = -\frac{2r^3}{1+r^4} \frac{-3+2s^2+r^4(s^2-2)}{\sqrt{1-s^2}(r^4(s^2-4)-3)} \quad (4.3.11)$$

For $s \sim 1$ and small r , this becomes

$$\alpha \simeq -\frac{2r^3}{3\sqrt{1-s^2}} \quad (4.3.12)$$

and $\sqrt{1-s^2}$ can be found from the energy at the same limits, in which (4.2.5) becomes

$$\sqrt{1-s^2} \simeq \frac{1}{(2r_0^2)^{1/3}} \quad (4.3.13)$$

Therefore, we can identify a region where the proper acceleration is small by restricting it to be of order $1/r_0$ for example

$$\alpha \simeq \frac{2r^3}{3}(2r_0^2)^{1/3} \sim \frac{1}{r_0} \quad (4.3.14)$$

This means that in regions where $r \sim r_0^{-5/9}$, we will have a relativistic limit described by the DBI, where stringy corrections can be neglected. This result also holds in the large- N limit. It will be interesting to develop a perturbative approximation which systematically includes stringy effects away from this region.

Another quantity of interest is the effective mass squared $E^2 - p^2$, where $p = \partial\mathcal{L}/\partial s$ is the radial conjugate momentum. It becomes negative for sufficiently large velocities. This includes the above regime of relativistic speeds and small radii. It is straightforward to see that if our collapsing configuration is considered as a source for space-time gravity, this implies a violation of the dominant energy condition. We quickly remind the reader of the various energy conditions for realistic matter [18, 126].

- **The Weak Energy Condition:**

Consider the D -dimensional energy-momentum tensor as a bilinear form $T_{\mu\nu}$ and the non-spacelike vectors V^μ . The Weak Energy Condition requires that $T_{\mu\nu}V^\mu V^\nu \geq 0$ and is equivalent to $T_{00} \geq 0$ in all Lorentz frames, *i.e.* the energy density should be non-negative. This is a fairly minimum requirement which can be violated in gauged supergravity theories, since the latter may contain scalars with negative potentials.

- **The Strong Energy Condition:**

We once again regard the energy-momentum tensor as a bilinear form and demand that $(T_{\mu\nu} - \frac{1}{D-2}G_{\mu\nu}T^\lambda_\lambda)V^\mu V^\nu \geq 0$. The physical interpretation of this is that locally gravity is attractive. As such, the condition implies that any cosmological term must be negative and is inconsistent with inflationary models of cosmology. Tensions or negative pressures can never be greater than $\frac{1}{D-1}$ times the energy density and therefore a medium that does not satisfy the Strong Energy Condition has to be supporting very large tensions or negative pressures, which render it highly unstable. It is incompatible with positive potentials for scalar fields but is satisfied by all supergravity models in all dimensions. The Strong Energy Condition is used to prove the

singularity theorems.

- **The Dominant Energy Condition:**

We now consider the energy-momentum tensor as an endomorphism T^μ_ν and require that if V^μ is inside or on the future light-cone, then so should $T^\mu_\nu V^\nu$. This translates in all orthonormal frames as $T_{00} \geq |T_{\mu\nu}|$. We may interpret this as requiring that matter may not move at super-luminal speeds. The Dominant Energy Condition is used to prove the Positive Energy Theorem, which says that the purely gravitational force between two isolated systems is attractive, *i.e.* prohibits the existence of long range gravitational repulsion. Note that the Strong Energy Condition rules out anti-gravity in a more local sense, by saying that the source of the Newtonian potential is always locally of the same sign. The Dominant Energy Condition implies causal propagation even in the case of time-dependent background metrics.

In the context of the BFSS Matrix model, it has been shown that for an action containing a background space-time $G_{IJ} = \eta_{IJ} + h_{IJ}$, in the linearised approximation, linear couplings in the fluctuation h_{0I} correspond to momentum in the X^I direction [31]. The same argument can be developed here for the non-Abelian DBI. We couple a small fluctuation h_{0r} , which in classical geometry we can write as $h_{0i} = h_{0r}x_i$ for the unit sphere. We replace x_i by α_i/n . The action for $D0$ -branes [28, 30] is generalised from (4.1.1) by replacing \dot{R} in $\lambda\partial_t\Phi_i = \lambda(\dot{R})\alpha_i = \frac{\dot{R}}{n}\alpha_i$ with $(\dot{R} + h_{0r})$. It is then clear that the variation with respect to \dot{R} , which gives p , is the same as the variation with respect to h_{0r} , which gives T^{0r} . Hence, the dominant energy condition will be violated, since $E < |p|$ is equivalent to $T^{00} < T^{0r}$. The violation of this condition by stringy D -brane matter can have profound consequences. For a discussion of possible applications in cosmology see [126].

4.3.3 Distance to blow up in $D1 \perp D3$

Comparisons between the finite and large- N results can be made in the spatial case using the conserved pressure. The arguments are similar to what we used for the time of collapse using the energy functions. Consider the case $n = 1$, and let $\hat{P} = P/N$. First calculate the derivative of the pressure

$$\frac{\partial \hat{P}}{\partial s} = \frac{s(4r^4 + r^4s^2 + 3)}{\sqrt{1 + r^4}(1 - s^2)^{5/2}} \quad (4.3.15)$$

This is clearly always positive. Now evaluate

$$\hat{P} \left(r, s = \frac{\sqrt{r^4 - r_0^4}}{\sqrt{1 + r_0^4}} \right) = -\frac{(1 + r_0^4)^{1/2}}{1 + r^4} (1 + r_0^4 + r^4) \quad (4.3.16)$$

This should be compared with $\hat{P}(r, s)$, evaluated for the value of s which solves the $n = 1$ equation of motion, which by conservation of pressure is $-\frac{(1+2r_0^4)}{\sqrt{1+r_0^4}}$. Take the difference to

find

$$\hat{P} \left(r, s = \frac{\sqrt{r^4 - r_0^4}}{\sqrt{1 + r_0^4}} \right) + \frac{(1 + 2r_0^4)}{\sqrt{1 + r_0^4}} = \frac{r_0^4(r^4 - r_0^4)}{\sqrt{1 + r_0^4}(1 + r^4)} \quad (4.3.17)$$

Thus at fixed r_0 and r , $\hat{P}_{n=1}$, when evaluated for the value of s which solves the large- n problem, is larger than when it is evaluated for the value of s which solves the $n = \infty$ problem. Since \hat{P} increases monotonically with s for fixed r , this shows that for fixed r_0 , and any r , s is always larger in the large- N problem. Since $\Sigma = \int dr/s$, this means the distance to blow up is smaller for $n = \infty$. Hence for fixed r_0 , the distance to blow up is larger at $n = 1$.

4.4 Towards a generalisation to higher fuzzy even-spheres

For generalisations to higher dimensional brane systems, and to higher dimensional fuzzy spheres [48, 50, 71, 74], it is of interest to derive an extension of the expressions for the symmetrised traces given above. In the general case, we define $N(k, n)$ to be the dimension of the irreducible representation of $SO(2k + 1)$ with Dynkin label $(\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2})$ which contains k entries. We then take $C(m, k, n)$ to be the action of the symmetrised trace on m pairs of matrices X_i , where $i = 1, \dots, 2k + 1$

$$C(m, k, n) = \frac{1}{N(k, n)} STr(X_i X_i)^m \quad (4.4.1)$$

Finding an expression for $C(m, k, n)$ is non-trivial. Investigations based upon intuition from the ADHM construction lead us to conjecture that for n odd

$$C(m, k, n) = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k - 1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \sum_{i_3=1}^{\frac{n+1}{2}} \left[\prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - \left(i_3 - \frac{1}{2} \right)^2 \right) (2i_3 - 1)^{2m} \right] \quad (4.4.2)$$

while for n even²

$$C(m, k, n) = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k - 1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \sum_{i_3=1}^{\frac{n}{2}} \left[\prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - i_3^2 \right) (2i_3)^{2m} \right] \quad (4.4.3)$$

We give the arguments leading to the expressions above in App. C.

For higher even-spheres there will be extra complications at finite- n . Consider the case of the fuzzy- S^4 for concreteness. The evaluation of the higher dimensional determinant in the corresponding non-abelian brane action will give expressions with higher products of

²For $m = 0$ the value $STr(X_i X_i)^0 = 1$ is once again imposed.

$\partial_t \Phi_i$ and $\Phi_{ij} \equiv [\Phi_i, \Phi_j]$

$$\begin{aligned} S = & -T_0 \int dt STr \left\{ 1 + \lambda^2 (\partial_t \Phi_i)^2 + 2\lambda^2 \Phi_{ij} \Phi_{ji} + 2\lambda^4 (\Phi_{ij} \Phi_{ji})^2 - 4\lambda^4 \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} + \right. \\ & \left. + 2\lambda^4 (\partial_t \Phi_i)^2 \Phi_{jk} \Phi_{kj} - 4\lambda^4 \partial_t \Phi_i \Phi_{ij} \Phi_{jk} \partial_t \Phi_k + \frac{\lambda^6}{4} (\epsilon_{ijklm} \partial_t \Phi_i \Phi_{jk} \Phi_{lm})^2 \right\}^{1/2} \end{aligned} \quad (4.4.4)$$

The Ansatz for the transverse scalars will still be

$$\Phi_i = \hat{R}(t) X_i$$

where now $i = 1, \dots, 5$ and the X^i 's are given by the action of $SO(5)$ gamma matrices on the totally symmetric n -fold tensor product of the basic spinor. After expanding the square root, the symmetrisation procedure should take place over all the X_i 's and $[X_i, X_j]$'s. However, the commutators of commutators $[[X, X], [X, X]]$ will give a non-trivial contribution, as opposed to what happens in the large- n limit where they are sub-leading and are taken to be zero. Therefore, in order to uncover the full answer for the finite- n fuzzy- S^4 it is not enough to just know the result of $STr(X_i X_i)^m$; we need to know the full $STr((X X)^{m_1} ([X, X][X, X]))^{m_2}$ with all possible contractions amongst the above. It would be clearly interesting to have the full answer for the fuzzy- S^4 . A similar story will apply for the higher even-dimensional fuzzy spheres.

Note, however, that for $\hat{R} = 0$ in (4.4.4) all the commutator terms Φ_{ij} will vanish, since they scale like \hat{R}^2 . This reduces the symmetrisation procedure to the one involving $X_i X_i$ and yields only one sum for the energy. The same will hold for any even-dimensional S^{2k} , resulting in the following general expression

$$\begin{aligned} \mathcal{E}_{n,k}(0, s) = & -STr \sum_{m=0}^{\infty} (-1)^m s^{2m} (2m-1) (X_i X_i)^m \binom{1/2}{m} \\ = & -N(k, n) \sum_{m=0}^{\infty} (-1)^m s^{2m} (2m-1) C(m, k, n) \binom{1/2}{m} \end{aligned} \quad (4.4.5)$$

Using (4.4.2), notice that in the odd- n case

$$C(m, k, n) = \prod_{i_2=1}^{2k-1} \frac{(1+i_2)}{(n+i_2)} C(m, k, 1) \sum_{i_3}^{\frac{n+1}{2}} \frac{f_{odd}(i_3, k, n)}{f_{odd}(1, k, 1)} (2i_3-1)^{2m} \quad (4.4.6)$$

The factor f_{odd} is

$$f_{odd}(i_3, k, n) = \prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - \left(i_3 - \frac{1}{2} \right)^2 \right) \quad (4.4.7)$$

Inserting this form for $C(m, k, n)$ in terms of $C(m, k, 1)$ we see that

$$\mathcal{E}_{n,k}(0, s) = N(n, k) \prod_{i_2=1}^{2k-1} \frac{(1+i_2)}{(n+i_2)} \sum_{i_3=1}^{\frac{n+1}{2}} \frac{f_{odd}(i_3, k, n)}{f_{odd}(1, k, 1)} \hat{\mathcal{E}}_{n=1,k}(0, s(2i_3-1)) \quad (4.4.8)$$

Similarly we derive, in the even- n case, that

$$\mathcal{E}_{n,k}(0, s) = N(n, k) \prod_{i_2=1}^{2k-1} \frac{(2+i_2)}{(n+i_2)} \sum_{i_3=1}^{\frac{n}{2}} \frac{f_{even}(i_3, k, n)}{f_{even}(1, k, 2)} \hat{\mathcal{E}}_{n=2,k}(0, s(i_3)) \quad (4.4.9)$$

where

$$f_{even}(i_3, k, n) = \prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - i_3^2 \right) \quad (4.4.10)$$

and $\hat{\mathcal{E}}$ is the energy density, *i.e.* the energy divided a factor of $N(n, k)$.

It is also possible to write $C(m, k, 2)$ in terms of $C(m, k, 1)$

$$\begin{aligned} C(m, k, 2) &= 2^{2m} C(m, k, 1) \prod_{i_4=1}^{k-1} \frac{i_4(i_4+2)}{i_4(i_4+1)} \prod_{i_2=1}^{2k-1} \frac{(i_2+1)}{(i_2+2)} \\ &= 2^{2m} C(m, k, 1) \frac{f_{even}(1, k, 2)}{f_{odd}(1, k, 1)} \prod_{i_2=1}^{2k-1} \frac{(i_2+1)}{(i_2+2)} \end{aligned} \quad (4.4.11)$$

which is valid for all values of $m \neq 0$.

It turns out to be possible to give explicit forms for the energy for the $n = 1$ and $n = 2$ case. Since the definition of the physical radius at finite- n , given at the beginning of this chapter, is also valid for higher dimensional fuzzy spheres, we can express the results in terms of the rescaled variables \hat{r} and \hat{s}

$$\begin{aligned} \hat{\mathcal{E}}_{n=1,k}(0, \hat{s}) &= \frac{1}{(1 - \hat{s}^2)^{\frac{2k+1}{2}}} \\ \hat{\mathcal{E}}_{n=2,k}(0, \hat{s}) &= \frac{1}{(1 - \hat{s}^2)^{\frac{2k+1}{2}}} \frac{(k+1)}{(2k+1)} \end{aligned} \quad (4.4.12)$$

When plugged into (4.4.8), (4.4.9) the above results provide a closed form for the energy at $\hat{r} = 0$, for any n and any k . A complete study of the time-dependent dynamics requires the evaluation of the energy functions for all \hat{r} , but the relative simplicity of (4.4.12) suggests that the computation of the required additional symmetrised traces might reveal a tractable extension.

CHAPTER 5

NON-ABELIAN DBI AND FUZZY ODD-DIMENSIONAL SPHERES

We have thus far focused on how fuzzy even-spheres enter certain configurations that arise from String Theory. It is natural to consider extensions of the ideas applied in those cases to systems involving fuzzy spheres of odd dimensionality. Fuzzy odd-spheres were constructed and studied earlier in [48, 49, 51]. In [127] the fuzzy 3-sphere algebra was expressed as a quantisation of the Nambu bracket. Subsequent work used the fuzzy 3-sphere in the context of $M2 \perp M5$ intersections [128–131].

This has provided a motivation to revisit fuzzy odd-spheres. In this chapter, we study the fuzzy odd-sphere equations in more detail and apply them to the time-dependent process of N $D0$ s blowing up into a fuzzy- S^3 and S^5 respectively. Compared to the study of fuzzy even-spheres, these phenomena turn out to be significantly more involved. Commutators of fuzzy odd-sphere matrices are not vanishing at large- N , hence calculating the symmetrised trace in that limit requires a non-trivial sum over orderings. Once these sums are performed, we find the surprising result that the time evolution of the fuzzy- S^3 is identical to that of the fuzzy- S^5 . After briefly reviewing the fuzzy- S^3 and higher-dimensional fuzzy odd-spheres we present a number of useful identities, which apply for odd-spheres of any dimensionality. We then focus on expressions for the particular cases of S^3 and S^5 and look at the dynamics of N coincident $D0$ -branes described by the Matrix DBI action. This is done by using an Ansatz involving the fuzzy odd-sphere Matrices and a time-dependent radius. This is inserted into the DBI action, to obtain a reduced action for the radius. We proceed to study the physical properties of these configurations, using the definition for the physical radius proposed in Ch. 4 and find that there will be no bounces of the trajectory for large- N . The characteristic length scale of the system is $L_{\text{odd}} = \sqrt{\pi} \ell_s$ and independent of N . We show that both the fuzzy- S^3 and S^5 solve the equations of motion in the Matrix Theory limit and yield solutions in terms of Jacobi elliptic functions. We also discuss a possible dual description of the fuzzy- S^3 , in terms of a non-BPS $D3$ -brane embedded in Euclidean space as a classical three-sphere.

5.1 General fuzzy odd-sphere equations with $SO(D)$ symmetry

We start with a quick summary of the construction of the fuzzy- S^3 and fuzzy- S^5 [48, 49] that was presented in Ch. 1. We are working with matrices constructed by taking the symmetric n -fold tensor product of $V = V_+ \oplus V_-$, where V_+ and V_- are the two-dimensional spinor representations of $SO(4)$, of respective positive and negative chirality. There are two projectors P_{\pm} , which project V onto V_{\pm} . In terms of the isomorphism $SO(4) = SU(2) \times SU(2)$ these have respective spin $(2j_L, 2j_R) = (1, 0)$ and $(2j_L, 2j_R) = (0, 1)$. The symmetrised tensor product space $Sym(V^{\otimes n})$, for every odd integer n , contains a subspace \mathcal{R}_n^+ with $(\frac{n+1}{2})$ factors of positive and $(\frac{n-1}{2})$ factors of negative chirality. This is an irreducible representation of $SO(4)$ labelled by $(2j_L, 2j_R) = (\frac{n+1}{2}, \frac{n-1}{2})$. The projector onto this subspace is in $End(Sym(V^{\otimes n}))$ and will be called $\mathcal{P}_{\mathcal{R}_n^+}$. Equivalently, there is a subspace \mathcal{R}_n^- with spins $(2j_L, 2j_R) = (\frac{n-1}{2}, \frac{n+1}{2})$ and projector $\mathcal{P}_{\mathcal{R}_n^-}$. The full space is then defined to be the direct sum $\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$. The projector for this space is $\mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} \oplus \mathcal{P}_{\mathcal{R}_n^-}$. The matrices X_i are in $End(\mathcal{R}_n)$

$$X_i = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n}$$

where $i = 1, \dots, 4$, mapping \mathcal{R}_{n+} to \mathcal{R}_{n-} and vice-versa. We can therefore re-express the above as a sum of matrices in $Hom(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and $Hom(\mathcal{R}_n^-, \mathcal{R}_n^+)$

$$X_i = \mathcal{P}_{\mathcal{R}_n^+} X_i \mathcal{P}_{\mathcal{R}_n^-} + \mathcal{P}_{\mathcal{R}_n^-} X_i \mathcal{P}_{\mathcal{R}_n^+}$$

The product $X_i^2 = C$ forms the quadratic Casimir¹ of $SO(4)$. There is a set of generators for the Matrix algebra

$$\begin{aligned} X_i^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \\ X_i^- &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} \\ X_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^+} \\ Y_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ij} P_-) \mathcal{P}_{\mathcal{R}_n^+} \\ X_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ij} P_-) \mathcal{P}_{\mathcal{R}_n^-} \\ Y_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^-} \end{aligned}$$

where

$$\Gamma_{ij} = \frac{1}{2}[\Gamma_i, \Gamma_j] \tag{5.1.1}$$

¹It is understood here that scalars of the isometry group are multiplying the $N \times N$ identity matrix.

The co-ordinates of the sphere can be written as $X_i = X_i^+ + X_i^-$ and one can also define the following combinations

$$\begin{aligned} X_{ij} &= X_{ij}^+ + X_{ij}^- \\ Y_{ij} &= Y_{ij}^+ + Y_{ij}^- \\ Y_i &= X_i^+ - X_i^- \\ \tilde{X}_{ij} &= X_{ij}^+ - X_{ij}^- \\ \tilde{Y}_{ij} &= Y_{ij}^+ - Y_{ij}^- \end{aligned} \tag{5.1.2}$$

The above generators in fact form an over-complete set. It was observed [132] that X_i, Y_i suffice as a set of generators. We have discussed in some detail in Ch. 1, how in the large- n limit, the full Matrix algebra turns out to contain more degrees of freedom than the algebra of functions on the classical three-sphere and how one can extend the projection that works for the fuzzy even-spheres to the odd-sphere case. This then gives rise to an algebra of functions with the right number of degrees of freedom at large- n . The projected Matrix algebra is commutative but becomes non-associative. However, non-associativity does not vanish² at large- n , unlike the fuzzy- S^{2k} case.

For general fuzzy odd-dimensional spheres, S^{2k-1} , the Matrix co-ordinates are matrices acting in a reducible representation $\mathcal{R}_n^+ \oplus \mathcal{R}_n^-$ of $SO(2k)$. The irreducible representations \mathcal{R}_\pm have respective weights $\vec{r} = (\frac{n}{2}, \dots, \frac{n}{2}, \frac{\pm 1}{2})$, with \vec{r} a k -dimensional vector. The matrices acting on the full space $\mathcal{R} = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$ can be decomposed into four blocks $End(\mathcal{R}_n^+)$, $End(\mathcal{R}_n^-)$, $Hom(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and $Hom(\mathcal{R}_n^-, \mathcal{R}_n^+)$.

We will use the above to construct a number of useful identities for any isometry group $SO(D)$, for $D = 2k$ even. There exist the following basic relationships [48, 49, 51]

$$\begin{aligned} (\Gamma_i \otimes \Gamma_i)(P_+ \otimes P_+) &= 0 \\ (\Gamma_i \otimes \Gamma_i)(P_- \otimes P_-) &= 0 \\ (\Gamma_i \otimes \Gamma_i)(P_+ \otimes P_-) &= 2(P_- \otimes P_+) \\ (\Gamma_i \otimes \Gamma_i)(P_- \otimes P_+) &= 2(P_+ \otimes P_-) \end{aligned} \tag{5.1.3}$$

For completeness, we give the explicit derivation. It is known from fuzzy even-spheres that $\sum_{\mu=1}^{2k+1} (\Gamma_\mu \otimes \Gamma_\mu)$ acting on the irreducible subspace (which requires subtracting traces for $k > 2$) of $Sym(V \otimes V)$ is equal to 1. For any vector v in this subspace, we have

$$(\Gamma_\mu \otimes \Gamma_\mu)v = v \tag{5.1.4}$$

Separating the sum over μ as $(\Gamma_i \otimes \Gamma_i) + (\Gamma_{2k+1} \otimes \Gamma_{2k+1})$, multiplying by $(P_+ \otimes P_-)$ from

²A discussion on the definition of this projection and the large- n behaviour of the associator can be found in App. E.

the left, and using the Clifford algebra relations proves the fourth equation above. The other equations are obtained similarly, by multiplying with an appropriate tensor product of projectors. From these we derive

$$\begin{aligned}
 X_i^2 &= \frac{(n+1)(n+D-1)}{2} \equiv C \\
 X_{ij}X_{ij} &= -\frac{D}{4}(n+1)(n+2D-3) \\
 Y_{ij}Y_{ij} &= -\frac{D}{4}(n-1)(n+2D-5) \\
 X_{ij}Y_{ij} &= \frac{(4-D)}{4}(n^2-1) \\
 X_iX_jX_iX_j &= (2-D)C \\
 [X_i, X_j] [X_j, X_i] &= 2C(C+D-2)
 \end{aligned} \tag{5.1.5}$$

and

$$\begin{aligned}
 [X_i, X_j] &= (n+D-1)X_{ij} - X_{ijk}X_k \\
 [X_j, [X_j, X_i]] &= 2(C+D-2)X_i \\
 X_jX_iX_j &= (2-D)X_i \\
 X_{ki}X_k &= X_kX_{ik} = -\frac{(n+2D-3)}{2}X_i \\
 Y_{ki}X_k &= X_kY_{ik} = \frac{(n-1)}{2}X_i \\
 X_jX_kX_iX_jX_k &= \frac{1}{4}(n^4 + 2n^3D + (D^2 + 6 - 2D)n^2 \\
 &\quad + (6D - 2D^2)n - 3D^2 + 18D - 23)X_i
 \end{aligned} \tag{5.1.6}$$

In the first equation of the second set we have used

$$X_{ijk} = \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_{ijk}P_+) \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_{ijk}P_-) \mathcal{P}_{\mathcal{R}_n^-} \tag{5.1.7}$$

where Γ_{ijk} is the normalised anti-symmetric product. It is useful to observe that

$$\mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2} \rho_{r_1}(\Gamma_k P_-) \rho_{r_2}(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} = X_k^- X_j^+ + X_{jk}^+ - \frac{(n+1)}{2} \delta_{jk} \tag{5.1.8}$$

It also follows that

$$\begin{aligned}
 \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2} \rho_{r_1}(\Gamma_k P_-) \rho_{r_2}(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} &+ (+ \leftrightarrow -) \\
 &= X_k X_j + X_{jk} - \frac{(n+1)}{2} \delta_{jk}
 \end{aligned} \tag{5.1.9}$$

where we have added the term obtained by switching the $+$ and $-$ from the first term.

These formulae can be used to calculate $X_i X_j X_k X_i$ ³

$$\begin{aligned}
 X_i X_j X_k X_i &= X_k X_j \left[\frac{(n-1)(n+D+1)}{2} + 2 \right] - 2 X_j X_k \\
 &\quad + \frac{(n+1)(n+D-1)}{2} (X_{jk} + Y_{jk} + \delta_{jk}) \\
 &= C(-n, -D) X_k X_j + 2[X_k, X_j] + C(n, D) (X_{jk} + Y_{jk} + \delta_{jk})
 \end{aligned} \tag{5.1.10}$$

In the last equality we have recognised that the coefficient of $(X_{jk} + Y_{jk} + \delta_{jk})$ has turned out to be $C = X_i^2$ (5.1.5). We also made explicit the dependence of C on n, D by writing $C = C(n, D)$, and observed that the other numerical coefficient on the RHS is $C(-n, -D)$.

In the large- n limit there are significant simplifications to the above matrix identities

$$\begin{aligned}
 X_m X_i X_m &= 0 \\
 X_m X_i X_j X_m &= C X_j X_i \\
 A_{ij} A_{jk} &= C(X_i X_k + X_k X_i) \\
 X_i X_{p_1} X_{p_2} \dots X_{p_{2k+1}} X_i &= 0 \\
 X_i X_{p_1} X_{p_2} \dots X_{p_{2k}} X_i &= C X_{p_2} \dots X_{p_{2k}} X_{p_1} \\
 [X_i X_j, X_k X_l] &= 0 \\
 A_{kl} X_m &= -X_m A_{kl} \\
 X_i X_j X_k &= X_k X_j X_i
 \end{aligned} \tag{5.1.11}$$

where, to avoid clutter, we have denoted

$$A_{ij} = [X_i, X_j] \tag{5.1.12}$$

and $C \sim \frac{n^2}{2}$. From these it follows that

$$\begin{aligned}
 X_m A_{ij} X_m &= -C A_{ij} \\
 [A_{ij}, A_{kl}] &= 0 \\
 A_{ij} A_{ji} &= 2C^2 \\
 A_{ij} A_{jk} A_{kl} A_{li} &= 2C^4
 \end{aligned} \tag{5.1.13}$$

As an example of how these simplifications occur, consider the last equality of (5.1.11). As explained at the beginning of this section, we can decompose a string of operators such as

³Products of these will appear in the computation of determinants in the following sections.

$X_i X_j X_k = X_i^+ X_j^- X_k^+ + X_i^- X_j^+ X_k^-$. Writing out $X_i^+ X_j^- X_k^+$

$$\begin{aligned}
 X_i^+ X_j^- X_k^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1, r_2, r_3} \rho_{r_1}(\Gamma_i P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_3}(\Gamma_k P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
 &= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2 \neq r_3} \rho_{r_1}(\Gamma_i P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_3}(\Gamma_k P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
 &= \mathcal{P}_{\mathcal{R}_n^-} \sum_{r_1 \neq r_2 \neq r_3} \rho_{r_3}(\Gamma_k P_+) \rho_{r_2}(\Gamma_j P_-) \rho_{r_1}(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \\
 &= X_k^+ X_j^- X_i^+
 \end{aligned} \tag{5.1.14}$$

In the second line we used the fact that the terms with coincident r 's, such as $r_1 = r_2$, are sub-leading in the large- n limit. There are $\mathcal{O}(n^3)$ terms of type $r_1 \neq r_2 \neq r_3$ while there are $\mathcal{O}(n^2)$ terms of type $r_1 = r_2 \neq r_3$ and $\mathcal{O}(n)$ terms of type $r_1 = r_2 = r_3$. In the third line, we used the fact that operators acting on non-coincident tensor factors commute. We find

$$\begin{aligned}
 X_i X_j X_k &= X_i^+ X_j^- X_k^+ + X_i^- X_j^+ X_k^- \\
 &= X_k^+ X_j^- X_i^+ + X_k^- X_j^+ X_i^- \\
 &= X_k X_j X_i
 \end{aligned} \tag{5.1.15}$$

Similar manipulations along with the basic relationships (5.1.3) lead to the rest of the formulae in (5.1.11).

5.2 On the equations for fuzzy- S^3

Specialising to the case of the fuzzy- S^3 we can deduce further Matrix identities. Squaring the generators

$$\begin{aligned}
 X_i^2 &= \frac{(n+1)(n+3)}{2} \\
 X_{ij}^2 &= -(n+1)(n+5) \\
 Y_{ij}^2 &= -(n-1)(n+3) \\
 X_{ij} Y_{ij} &= 0
 \end{aligned}$$

Note that $X_{ij} Y_{ij} = 0$ in the case of $SO(4)$. This product is not zero for general D . We also have

$$\begin{aligned}
 [X_i, X_j] &= \frac{(n+3)}{2} X_{ij} - \frac{(n+1)}{2} Y_{ij} \\
 X_j X_{ki} X_j &= \frac{(n+1)(n+5)}{2} Y_{ki} \\
 X_j Y_{ki} X_j &= \frac{(n-1)(n+3)}{2} X_{ki}
 \end{aligned}$$

$$\begin{aligned}
 X_{ki}X_k &= -\frac{n+5}{2}X_i \\
 Y_{ki}X_k &= \frac{n-1}{2}X_i \\
 X_kX_jX_k &= -2X_j \\
 X_jX_{ki}X_jX_k &= \frac{(n-1)(n+1)(n+5)}{4}X_i \\
 X_jY_{ki}X_jX_k &= -\frac{(n-1)(n+3)(n+5)}{4}X_i \\
 X_jX_kX_iX_jX_k &= \frac{1}{4}(n^2+4n-1)^2X_i
 \end{aligned} \tag{5.2.1}$$

In the second pair of equations of the above set, note that we might have expected $X_jX_{ki}X_j$ to be a linear combination of X_{ki} and Y_{ki} but only Y_{ki} appears. This follows directly from the transformation properties of these operators under $SO(4)$.

We can compute $X_jX_kX_iX_jX_k$ directly and get an answer which works for any D . Alternatively we can make use of the S^3 identities

$$\begin{aligned}
 X_jX_kX_iX_jX_k &= X_j([X_k, X_i] + X_iX_k)X_jX_k \\
 &= \frac{(n+3)}{2}X_jX_{ki}X_jX_k - \frac{(n+1)}{2}X_jY_{ki}X_jX_k + X_jX_iX_kX_jX_k
 \end{aligned} \tag{5.2.2}$$

Using the formulae (5.2.1) we see that the contributions from the first two terms are equal. The two computations of this object of course agree.

It is worth noting here that the decomposition of the commutator $[X_i, X_j]$ into a sum over X_{ij} and Y_{ij} should be expected. In [48] a complete $SO(4)$ covariant basis of matrices acting on \mathcal{R}_n was given in terms of operators corresponding to self-dual and anti-self dual Young diagrams. According to that analysis, the most general anti-symmetric tensor with two free indices should be a linear combination of the following structure

$$\sum_r \rho_r(\Gamma_{ij}) \tag{5.2.3}$$

with any allowed combination of P_{\pm} on \mathcal{R}_n^{\pm} , where the $SO(4)$ indices on the Γ 's have been suppressed for simplicity. Note that the coefficients multiplying the above basis elements include contractions with the appropriate δ and ϵ -tensors. For $SO(4)$ the antisymmetric two-index tensors are (anti)self-dual and $\epsilon_{ijkl}\Gamma_{kl}P_{\pm} = \pm 2\Gamma_{ij}P_{\pm}$, with $\Gamma_5P_{\pm} = \mp\Gamma_1\dots\Gamma_4P_{\pm}$. Contractions with δ are of course ruled out for symmetry reasons. As a consequence, everything can be expressed in terms of X_{ij} , Y_{ij} . The same procedure can be used to show that every composite object with one free $SO(4)$ index i can be reduced to be proportional

to X_i^\pm . The allowed linearly independent basis elements are

$$\begin{aligned} \sum_r \rho_r(\Gamma_i) \\ \sum_{r \neq s} \rho_r(\Gamma_{ij}) \rho_s(\Gamma_k) \delta_{jk} \sim \sum_s \rho_s(\Gamma_i) \\ \sum_{r \neq s} \rho_s(\Gamma_{jk}) \rho_s(\Gamma_l) \epsilon_{ijkl} \sim \sum_s \rho_s(\Gamma_i) \end{aligned} \quad (5.2.4)$$

It is easy to see explicitly that the last two quantities are proportional to X_i , when evaluated on \mathcal{R}_n^\pm . Since we should be able to express any object with one free index in terms of this basis, it will necessarily be proportional to X_i .

5.2.1 Derivation of $[X_i, X_j]$ for fuzzy- S^3

Here we include part of the calculation that led to the commutator decomposition for the fuzzy- S^3 , using two possible ways of evaluating $[X_i, X_j]^2$. One is a direct method and starts by writing out the commutators

$$\begin{aligned} [X_i, X_j][X_j, X_i] &= 2X_i X_j X_i X_j - 2X_i X_j X_j X_i \\ &= 2X_i^+ X_j^- X_i^+ X_j^- + 2X_i^- X_j^+ X_i^- X_j^+ - 2C^2 \end{aligned} \quad (5.2.5)$$

A straightforward calculation for any D gives

$$X_i^\mp X_j^\pm X_i^\mp X_j^\pm = -\frac{(D-2)(n+1)(n+D-1)}{2} \quad (5.2.6)$$

The two combine into

$$[X_i, X_j][X_j, X_i] = -\frac{(n+1)(n+D-1)}{2} [2D-4 + (n+1)(n+D-1)] \quad (5.2.7)$$

The other is based on the decomposition of any antisymmetric tensor of $SO(4)$ with two free indices in terms of the appropriate basis elements

$$[X_i, X_j] = \alpha X_{ij} + \beta Y_{ij} \quad (5.2.8)$$

where α and β are some yet undetermined coefficients. Then

$$[X_i, X_j][X_j, X_i] = \alpha^2 X_{ij} X_{ji} + \beta^2 Y_{ij} Y_{ji} \quad (5.2.9)$$

since for $D = 4$ we have seen that $X_{ij} Y_{ij} = 0$. Deriving the identities for $X_{ij} X_{ji}$ and $Y_{ij} Y_{ji}$ is a simple task. Using these results, it is easy to evaluate the above expression and compare

against what we get from the straightforward calculation. The outcome is

$$\alpha = \frac{(n+3)}{2} \quad \text{and} \quad \beta = -\frac{(n+1)}{2} \quad (5.2.10)$$

This result can also be checked against calculations of $X_{ij}[X_i, X_j]$ and $Y_{ij}[X_i, X_j]$.

5.3 On the equations for fuzzy- S^5

For the fuzzy- S^5 we only present a few specific identities that will appear in the following sections. The commutator decomposes into

$$[X_i, X_j] = (n+5)X_{ij} - X_{ijk}X_k \quad (5.3.1)$$

Alternatively we can express this as

$$X_i^\mp X_j^\pm - X_j^\mp X_i^\pm = (n+1)X_{ij}^\pm + \mathcal{P}_{\mathcal{R}_n} \frac{i}{6} \epsilon_{ijklmn} \left[\sum_{r \neq s} \rho_r(\Gamma_{lmn}) \rho_s(\Gamma_k) \right] X_7 \mathcal{P}_{\mathcal{R}_n} \quad (5.3.2)$$

where $\Gamma_7 P_\pm = \pm i \Gamma_1 \dots \Gamma_6 P_\pm$ and $X_7 \mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} - \mathcal{P}_{\mathcal{R}_n^-}$. There is no expression for $[X_i, X_j]$ as a linear combination of only X_{ij} and Y_{ij} , unlike the case of the fuzzy- S^3 . This is not surprising since the $SO(6)$ covariant basis for two-index antisymmetric tensors will now include terms of the form

$$\begin{aligned} & \sum_r \rho_r(\Gamma_{ij}) \\ & \sum_r \rho_r(\Gamma_{klmn}) \epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_{ij}) \\ & \sum_{r \neq s} \rho_r(\Gamma_{kl}) \rho_s(\Gamma_{mn}) \epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_{ij}) \\ & \sum_{r \neq s} \rho_r(\Gamma_{klm}) \rho_s(\Gamma_n) \epsilon_{ijklmn} \end{aligned} \quad (5.3.3)$$

Note that the last expression is not proportional to $\sum_r \rho_r(\Gamma \Gamma)$. We can once again show that any composite tensor with one free $SO(6)$ index i should be proportional to X_i^\pm , just as in the $SO(4)$ case. We have

$$\begin{aligned} & \sum_r \rho_r(\Gamma_i) \\ & \sum_{r \neq s} \rho_r(\Gamma_{ij}) \rho_s(\Gamma_k) \delta_{jk} \sim \sum_s \rho_s(\Gamma_i) \\ & \sum_{r \neq s} \rho_r(\Gamma_{jkl}) \rho_s(\Gamma_{lm}) \epsilon_{ijklmn} \sim \sum_s \rho_s(\Gamma_i) \end{aligned}$$

$$\sum_{r \neq s \neq t} \rho_r(\Gamma_{jk}) \rho_s(\Gamma_{kl}) \rho_t(\Gamma_m) \epsilon_{ijklmn} \sim \sum_t \rho_t(\Gamma_i) \quad (5.3.4)$$

as can be easily verified for any P_{\pm} combination on \mathcal{R}_n^{\pm} .

5.4 The Fuzzy- S^{2k-1} matrices and DBI with symmetrised trace

In this section we will substitute the Ansatz

$$\Phi_i = \hat{R}(\sigma, t) X_i \quad (5.4.1)$$

into the Matrix DBI action of $D1$ -branes to obtain an effective action for \hat{R} . We show in App. F that solutions to the reduced equations of motion also give solutions to the Matrix DBI equations of motion.

5.4.1 Fuzzy- S^3

The low energy effective action for N D -strings with no world-volume gauge field and in a flat background is given by the non-Abelian Dirac-Born-Infeld action

$$S = -T_1 \int d^2\sigma STr \sqrt{-\det \begin{bmatrix} \eta_{ab} & \lambda \partial_a \Phi_j \\ -\lambda \partial_b \Phi_i & Q_{ij} \end{bmatrix}} \equiv -T_1 \int d^2\sigma STr \sqrt{-\det(M)} \quad (5.4.2)$$

The determinant can be explicitly calculated keeping in mind the symmetrisation procedure. The result is

$$\begin{aligned} -\det(M) &= 1 + \frac{\lambda^2}{2} \Phi_{ij} \Phi_{ji} + \lambda^4 \left(\frac{1}{8} (\Phi_{ij} \Phi_{ji})^2 - \frac{1}{4} \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} \right) \\ &\quad + \lambda^2 \partial^a \Phi_i \partial_a \Phi_i + \lambda^4 \left(\frac{\partial^a \Phi_k \partial_a \Phi_k \Phi_{ij} \Phi_{ji}}{2} - \partial^a \Phi_i \Phi_{ij} \Phi_{jk} \partial_a \Phi_k \right) \end{aligned} \quad (5.4.3)$$

Considering the Ansatz (5.4.1) with $i = 1, \dots, 4$ describes the fuzzy- S^3 . The X_i 's are $N \times N$ matrices of $SO(4)$, as defined in Sec. 5.1. Their size is given by $N = \frac{1}{2}(n+1)(n+3)$. Substituting into (5.4.3) we get

$$\begin{aligned} -\det(M) &= 1 + \frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} + \frac{\lambda^4}{4} \hat{R}^8 \left(\frac{(A_{ij} A_{ji})^2}{2} - A_{ij} A_{jk} A_{kl} A_{li} \right) \\ &\quad + \lambda^2 (\partial^a \hat{R}) (\partial_a \hat{R}) X_i X_i + \lambda^4 \hat{R}^4 (\partial^a \hat{R}) (\partial_a \hat{R}) \left(\frac{X_k X_k A_{ij} A_{ji}}{2} - X_i A_{ij} A_{jk} X_k \right) \end{aligned} \quad (5.4.4)$$

At this point we need to implement the symmetrisation of the trace. In order to simplify

the problem, this procedure can be carried out in two steps. We first symmetrise the terms that lie under the square root. We then perform a binomial expansion and symmetrise again. The even-sphere cases that were initially considered in Chs. 2 and 3 didn't involve this complication, since the commutators $[X_i, X_j]$, $[A_{ij}, A_{kl}]$ and $[X_i, A_{jk}]$ turned out to be sub-leading in n . Thus, for large- N the square root argument was already symmetric and gave a simple result straight away. Here, however, the $[X_i, X_j]$ and $[X_i, A_{jk}]$ yield a leading- n contribution and the symmetrisation needs to be considered explicitly.

From now on, we will focus completely on the time-dependent problem of N type IIA $D0$ -branes and drop the σ -direction. Then the Ansatz (5.4.1) will be describing the dynamical effect of collapsing/expanding branes. Had we chosen to consider the static version of the above action, we would have a collection of coincident D -strings blowing up into a funnel of higher dimensional matter with an S^{2k-1} cross-section.

Vanishing symmetrised trace contributions

The terms involving only A 's are already symmetric, since the commutator of commutators, $[A_{ij}, A_{kl}]$, is sub-leading in the large- N limit. From (5.1.11) and (5.1.13) it follows immediately that the coefficient of \hat{R}^8 in (5.4.4) vanishes. The latter can be expressed as

$$\text{Sym}(A + B) = 0 \quad (5.4.5)$$

where $A = \frac{A_{ij}A_{ji}A_{kl}A_{lk}}{2} \equiv A_1A_2A_3A_4$, $B = A_{ij}A_{jk}A_{kl}A_{li} \equiv B_1B_2B_3B_4$. In this case, as we have already mentioned, the A_{ij} 's commute and $\text{Sym}(B) = B$, $\text{Sym}(A) = A$.

When we expand the square root we obtain terms of the form $\text{Sym}(C(A + B)^k)$ where C is a product of operators $C_1C_2\dots$, e.g. $C = (X_iX_i)^n$. It is easy to see that these also vanish. The symmetrised expression will contain terms of the form

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(C_1 \dots C_n (A_1 \dots A_4)^l (B_1 \dots B_4)^{k-l}) \quad (5.4.6)$$

The sum over σ will contain terms where the l copies of $A_1 \dots A_4$ and the $k - l$ copies of $B_1 \dots B_4$ are permuted amongst each other and also amongst the C factors. Due to the relations in (5.1.11) and (5.1.13), we can permute the $4l$ elements from A^l through the C 's and B 's to collect them back in the form of A^l . Likewise for B^{k-l} . Since A_{ij} elements commute with other A_{kl} and anti-commute with X_k , we will pick up, in this re-arrangement, a factor of (-1) raised to the number of times an A_{ij} type factor crosses an X_k factor. This is a factor that depends on the permutation σ and on k but not on l , since the number of A_{ij} 's coming from A and B do not depend on l . We call this factor $N(\sigma, k)$. The above

sum takes the form

$$\sum_{\sigma} N(\sigma, k) (C_1 \dots C_n) \sum_{l=0}^k \binom{k}{l} (A_1 \dots A_4)^l (B_1 \dots B_4)^{k-l} \quad (5.4.7)$$

This contains the expansion of $(A + B)^k$ with no permutations that could mix the A, B factors. Therefore it is zero.

Similarly, we can show that the coefficient of $\dot{R}^2 \dot{R}^2$ in the determinant (5.4.4) is zero. This requires a small calculation of summing over 24 permutations⁴. The relevant formulae are

$$\begin{aligned} X_i X_i A_{jk} A_{kj} &= 2C^3 \\ X_i A_{jk} X_i A_{kj} &= -2C^3 \\ X_i X_j A_{ik} A_{kj} &= C^3 \\ X_i A_{ik} X_j A_{kj} &= -C^3 \end{aligned} \quad (5.4.8)$$

which follow from (5.1.11). The outcome is again $\text{Sym}(A+B) = 0$, where $A = \frac{X_k X_k A_{ij} A_{ji}}{2} \equiv A_1 A_2 A_3 A_4$ and $B = X_k X_i A_{ij} A_{jk} \equiv B_1 B_2 B_3 B_4$. In this case, we do not have $\text{Sym}(A) = A, \text{Sym}(B) = B$, since the factors within A, B do not commute. We can repeat the above arguments to check $\text{Sym}(A+B)^k$. Start with a sum of the form

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(A^k B^{k-l}) \quad (5.4.9)$$

Because of the permutation in the sum, there will be terms where $X_i X_i A_{jk} A_{kj}$ has extra X 's interspersed in between. Such a term can be re-collected into the original form at the cost of introducing a sign factor, a factor $E(n_i) = (1 + (-1)^{n_i})/2$, where n_i is the number of X 's separating the two X_i , and also introducing a permutation of the remaining X 's. We will describe this process in more detail, but the important fact is that these factors will be the same when we are re-collecting $X_i X_k A_{ij} A_{jk}$, *i.e.* the index structure doesn't affect the combinatorics of the re-shuffling. The first step is to move the $A_{jk} A_{kj}$ all the way to the right, thus picking up a sign for the number of X 's one moves through during the process. We then have

$$\begin{aligned} &X_i (X_{p_1} \dots X_{p_m}) X_i (X_{q_1} \dots X_{q_n}) A_{jk} A_{kj} \\ &= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) X_i X_i (X_{q_1} \dots X_{q_n}) A_{jk} A_{kj} \\ &= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) X_i X_i A_{jk} A_{kj} \end{aligned} \quad (5.4.10)$$

In the last line $E(m) = (1 + (-1)^m)/2$ is 1 if m is even and zero otherwise. If instead we

⁴Or 6 if we fix one element using cyclicity.

are considering $X_i X_k A_{ij} A_{jk}$ we have

$$\begin{aligned}
 & X_i (X_{p_1} \dots X_{p_m}) X_k (X_{q_1} \dots X_{q_n}) A_{ij} A_{jk} \\
 &= X_i (X_{p_1} \dots X_{p_m}) X_k (A_{ij} A_{jk}) (X_{q_1} \dots X_{q_n}) \\
 &= X_i (X_{p_1} \dots X_{p_m}) X_k (C(X_i X_k + X_k X_i)) (X_{q_1} \dots X_{q_n}) \\
 &= X_i (X_{p_1} \dots X_{p_m}) C^2 X_i (X_{q_1} \dots X_{q_n}) \\
 &= E(m) C^3 (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) \\
 &= E(m) (X_{p_2} \dots X_{p_m} X_{p_1}) (X_{q_1} \dots X_{q_n}) X_i X_k A_{ij} A_{jk}
 \end{aligned} \tag{5.4.11}$$

The last lines of (5.4.10) and (5.4.11) show that the rules for re-collecting A and B from more complicated expressions, where their components have been separated by a permutation, are the same. Hence,

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(A^l B^{k-l}) \tag{5.4.12}$$

can be re-written, taking advantage of the fact that terms with different values of l only differ in substituting A with B . This does not affect the combinatorics of re-collecting. Finally, we get

$$\sum_{\sigma} F(\sigma, k) \sum_{l=0}^k \binom{k}{l} A^l B^{k-l} = 0 \tag{5.4.13}$$

The $F(\sigma, k)$ is obtained from collecting all the sign and E -factors that appeared in the discussion of the above re-shuffling.

When other operators, such as some generic C , are involved

$$\sum_{l=0}^k \binom{k}{l} \sum_{\sigma} \sigma(C A^l B^{k-l}) \tag{5.4.14}$$

the same argument shows that

$$\sum_{\sigma} \tilde{\sigma}(C) \sum_{l=0}^k \binom{k}{l} A^l B^{k-l} = 0 \tag{5.4.15}$$

Note that σ has been replaced by $\tilde{\sigma}$ because the process of re-collecting the powers of A, B , as in Eqs. (5.4.10) and (5.4.11), involve a re-shuffling of the remaining operators. The proof of $\text{Sym}(A + B) = 0$ can also be presented along the lines of the above argument. Then

$$\begin{aligned}
 X_i A_{jk} X_i A_{kj} &= -X_i X_i A_{jk} A_{kj} \\
 X_i A_{ij} X_k A_{jk} &= -X_i X_k A_{ij} A_{jk}
 \end{aligned} \tag{5.4.16}$$

which, combined with $X_i X_i A_{jk} A_{jk} = 2X_i X_k A_{ij} A_{jk}$, gives the vanishing result.

Non-vanishing symmetrised trace contributions

After the discussion in the last section, we are only left to consider

$$STr\sqrt{-\det(M)} = \sum_{m=0}^{\infty} STr \left(\frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} + \lambda^2 (\partial^a \hat{R})(\partial_a \hat{R}) X_i X_i \right)^m \binom{1/2}{m} \quad (5.4.17)$$

We derive the following formulae for symmetrised traces

$$\begin{aligned} \frac{STr(XX)^m}{N} &= C^m \frac{(m!)^2 2^m}{(2m)!} \\ \frac{STr((XX)^{m_1} (AA)^{m_2})}{N} &= 2^{m_1+m_2} C^{m_1+2m_2} \frac{(m_1+m_2)!}{m_2!} \frac{m_1!(2m_2)!}{(2m_1+2m_2)!} \\ \frac{STr(AA)^m}{N} &= 2^m C^{2m} \end{aligned} \quad (5.4.18)$$

To calculate the first line, note that we have to sum over all possible permutations of $X_{i_1} X_{i_1} X_{i_2} X_{i_2} \dots X_{i_m} X_{i_m}$. For all terms where the two X_{i_1} 's are separated by an even number of other X 's we can replace the pair by C . Whenever the two are separated by an odd number of X 's they give a sub-leading contribution and therefore can be set to zero in the large- N expansion. In doing the averaging we treat the two i_1 's as distinct objects and sum over $(2m)!$ permutations. For any ordering let us label the positions from 1 to $2m$. To get a non-zero answer, we need one set $i_1 \dots i_m$ to be distributed amongst m even places in $m!$ ways and another set of the same objects to be distributed amongst the odd positions in $m!$ ways. There is a factor 2^m for permutations of the two copies for each index. As a result we obtain the $C^m \frac{(m!)^2 2^m}{(2m)!}$.

Now consider the second line. By cyclicity we can always fix the first element to be an X . There are then $(2m_1 + 2m_2 - 1)!$ permutations of the $(2m_1 - 1)$ X -factors. As we have seen, in the large- N limit $AX = -XA$. Reading towards the right, starting from the first X , suppose we have p_1 A 's followed by an X , then p_2 A 's followed by an X , etc. This is weighted by $(-1)^{p_1+p_3+\dots+p_{2m_1-1}}$. Therefore, we sum over all partitions of $2m_2$, including a multiplicity for different orderings of the integers in the partition, and weighted by the above factor. This can be done by a mathematical package, such as Maple, in a variety of cases and gives

$$\frac{(m_1 + m_2 - 1)!}{m_2!(m_1 - 1)!} \frac{(2m_2)!(2m_1 - 1)!}{(2m_1 + 2m_2 - 1)!} \quad (5.4.19)$$

The denominator $(2m_1 + 2m_2 - 1)!$ comes from the number of permutations which keep one X fixed. The above can be re-written in a way symmetric under an $m_1 \leftrightarrow m_2$ exchange

$$\frac{(m_1 + m_2)!}{m_1!m_2!} \frac{(2m_1)!(2m_2)!}{(2m_1 + 2m_2)!} \quad (5.4.20)$$

The factor $\frac{(m_1!)^2 2^{m_1}}{(2m_1)!}$ comes from the sum over permutations of the X 's.

We describe another way to derive this result. This time we will not use cyclicity to fix the first element in the permutations of $(XX)^{m_1}(AA)^{m_2}$ to be X . Let there be p_1 A 's on the left, then one X , and p_2 A 's followed by another X and so on, until the last X is followed by p_{2m_1+1} X 's. We will evaluate this string by moving all the A 's to the left, picking up a sign factor $(-1)^{p_2+p_4+\dots+p_{2m_1}}$ in the process. This leads to a sum over $p_1 \dots p_{2m_1+1}$ which can be re-arranged by defining $P = p_2 + p_4 + \dots + p_{2m_1}$. P ranges from 0 to $2m_2$ and is the total number of A 's in the even slots. For each fixed P there is a combinatoric factor of $\tilde{C}(P, m_1) = \frac{(P+m_1-1)!}{P!m_1!}$ from arranging the P objects into m_1 slots. There is also a factor $\tilde{C}(2m_2 - P, m_1 + 1)$ from arranging the remaining $(2m_2 - P)$ A 's into the $m_1 + 1$ positions. These considerations lead to

$$\begin{aligned} & \sum_{p_{2m_1}=0}^{2m_2} \sum_{p_{2m_1-1}=0}^{2m_2-p_{2m_1}} \dots \sum_{p_1=0}^{2m_2-p_2-\dots-p_{2m_1}} (-1)^{p_2+p_4+\dots+p_{2m_1}} \\ &= \sum_{P=0}^{2m_2} (-1)^P \tilde{C}(P, m_1) \tilde{C}(2m_2 - P, m_1 + 1) \\ &= \frac{(m_1 + m_2)!}{m_1!m_2!} \end{aligned}$$

The factor obtained above is multiplied by $(2m_2)!$ since all permutations amongst the A 's give the same answer. Summing over permutations of X 's give the extra factor $2^{m_1}(m_1!)^2$. Finally, there is a normalising denominator of $(2m_1 + 2m_2)!$. Collecting these and the appropriate power of C gives

$$2^{m_2} C^{m_1+2m_2} \frac{(m_1 + m_2)!}{m_1!m_2!} \frac{2^{m_1}(m_1!)^2(2m_2)!}{(2m_1 + 2m_2)!} \quad (5.4.21)$$

which agrees with the second line of (5.4.18).

5.4.2 Fuzzy- S^5

We will now turn to the case of the fuzzy- S^5 . The starting action will be the same as in (5.4.2). However, the Ansatz incorporates six non-trivial transverse scalars $\Phi_i = \hat{R}(\sigma, t)X_i$, where $i = 1, \dots, 6$. The X_i 's are $N \times N$ matrices of $SO(6)$, as defined in Sec. 5.1, with their size given by $N = \frac{1}{192}(n+1)(n+3)^2(n+5)$. By truncating the problem to the purely time-dependent configuration, this system represents a dynamical process of N $D0$ -branes expanding into a fuzzy- S^5 and then collapsing towards a point. The static truncation provides an analogue of the static fuzzy- S^3 funnel, with a collection of N D -strings blowing up into a funnel with a fuzzy- S^5 cross-section.

The calculation of the determinant yields the following result

$$\begin{aligned}
 -\det(M) = & 1 + \frac{\lambda^2}{2} \Phi_{ij} \Phi_{ji} + \lambda^4 \left(\frac{1}{8} (\Phi_{ij} \Phi_{ji})^2 - \frac{1}{4} \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} \right) \\
 & + \lambda^6 \left(\frac{(\Phi_{ij} \Phi_{ji})^3}{48} - \frac{\Phi_{mn} \Phi_{nm} \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li}}{8} + \frac{\Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{lm} \Phi_{mn} \Phi_{ni}}{6} \right) \\
 & + \lambda^2 \partial^a \Phi_i \partial_a \Phi_i + \lambda^4 \left(\frac{\partial^a \Phi_k \partial_a \Phi_k \Phi_{ij} \Phi_{ji}}{2} - \partial^a \Phi_i \Phi_{ij} \Phi_{jk} \partial_a \Phi_k \right) \\
 & - \lambda^6 \left(\frac{\partial^a \Phi_m \partial_a \Phi_m \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li}}{4} - \frac{\partial^a \Phi_i \partial_a \Phi_i (\Phi_{jk} \Phi_{kj})^2}{8} \right. \\
 & \left. + \frac{\partial^a \Phi_i \Phi_{ij} \Phi_{jk} \partial_a \Phi_k \Phi_{ml} \Phi_{lm}}{2} - \partial^a \Phi_i \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{lm} \partial_a \Phi_m \right) \quad (5.4.22)
 \end{aligned}$$

Once again we will need to implement the symmetrisation procedure, just as we did for the fuzzy- S^3 . The structure of the terms within the square root for $D = 6$ is almost the same as for $D = 4$. The only difference is that we will now need to include expressions of the type $\text{Sym}(A + B + C)$ and $\text{Sym}(A + B + C + D)$, coming from the two new $\mathcal{O}(\lambda^6)$ terms. Consider the first of these for example. After expanding the square root, we will need the multinomial series expansion

$$\sum_{n_1, n_2, n_3 \geq 0} \frac{n!}{n_1! n_2! n_3!} A^{n_1} B^{n_2} C^{n_3} \quad \text{with} \quad n_1 + n_2 + n_3 = n \quad (5.4.23)$$

and analogously for the expression with four terms. The initial multinomial coefficient will separate out and we will then have to deal with the permutations, just as we did for the binomial terms. Note that the symmetrisation discussion in the former section did not make any use of the fact that $D = 4$. All the simplifications that occurred by taking the large- N limit and the combinatoric factors, which came from the re-shuffling of the operators, were derived for representations of $SO(2k)$ with k not specified. It is straightforward to see that the former arguments will carry through to this case. The effect of the permutations for any possible combination of terms will simply introduce a common pre-factor, multiplied by the multinomial coefficient of the original term. One can easily verify that, with the help of the large- N matrix identities from Sec. 5.1, all the terms multiplying powers of λ higher than 2 in (5.4.22) will give a zero contribution. Therefore, the substitution of the Ansatz will give

$$STr \sqrt{-\det(M)} = \sum_{m=0}^{\infty} STr \left(\frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} + \lambda^2 (\partial^a \hat{R}) (\partial_a \hat{R}) X_i X_i \right)^m \binom{1/2}{m}$$

This is, somewhat surprisingly, exactly what appeared in (5.4.17). It is intriguing that there is such a universal description for both the $D = 4$ and $D = 6$ problems.

5.5 The large- N dynamics of fuzzy odd-spheres.

The discussion of the previous section allows us to write the Lagrangean governing the collapse/expansion of the fuzzy 3-sphere as well as the fuzzy 5-sphere.

$$\begin{aligned}\mathcal{L} &= - \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{1/2}{m} \binom{m}{k} (-1)^{m-k} \left(\frac{\lambda^2 \hat{R}^4}{2} \right)^k (\lambda^2 \dot{\hat{R}}^2)^{m-k} STr \left[(A_{ij} A_{ji})^k (X_i X_i)^{m-k} \right] \\ &= - \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{1/2}{m} \binom{m}{k} (-1)^{m-k} s^{2(m-k)} r^{4k} \frac{m!}{(2m)!} \frac{(2k)!}{k!} 2^{m-k} (m-k)!\end{aligned}\quad (5.5.1)$$

where we work in dimensionless variables

$$\begin{aligned}r^4 &= \lambda^2 \hat{R}^4 C^2 \\ s^2 &= \lambda^2 \dot{\hat{R}}^2 C\end{aligned}\quad (5.5.2)$$

Alternatively we can express this as two infinite sums with $n+k = m$

$$\mathcal{L} = - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{1/2}{n+k} \binom{n+k}{k} (-1)^n s^{2n} r^{4k} \frac{(n+k)!}{(2(n+k))!} \frac{(2k)!}{k!} 2^n n! \quad (5.5.3)$$

From this we can calculate the energy of the configuration to get

$$E = - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (2n-1) \binom{1/2}{n+k} \binom{n+k}{k} (-1)^n s^{2n} r^{4k} \frac{(n+k)!}{(2(n+k))!} \frac{(2k)!}{k!} 2^n n! \quad (5.5.4)$$

The first sum can be done explicitly and gives

$$E = \sum_{n=0}^{\infty} \left(\frac{s^2}{2} \right)^n {}_2F_1 \left(\frac{1}{2}, n - \frac{1}{2}, n + \frac{1}{2}; -r^4 \right) \quad (5.5.5)$$

There is an identity for the ${}_2F_1$ function

$${}_2F_1(a, b, c; z) = (1-z)^{-a} {}_2F_1 \left(a, c-b, c; \frac{z}{z-1} \right), \quad \text{for } z \notin (1, \infty)$$

We can use this to re-express the energy sum as

$$E = \sum_{n=0}^{\infty} \left(\frac{s^2}{2} \right)^n \frac{1}{\sqrt{1+r^4}} {}_2F_1 \left(\frac{1}{2}, 1, n + \frac{1}{2}; \frac{r^4}{1+r^4} \right) \quad (5.5.6)$$

We need one more step to complete the evaluation. The integral representation for the hypergeometric function is

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \rho^{b-1} (1-\rho)^{-b+c-1} (1-\rho z)^{-a} d\rho$$

for $Re(c) > Re(b) > 0$ and $|Arg(1 - z)| < \pi$. These conditions are satisfied for $n \neq 0$ so we will split the sum into two parts. The $n = 0$ part simplifies to just $\sqrt{1 + r^4}$, while the rest is

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{s^2}{2}\right)^n \frac{1}{\sqrt{1+r^4}} {}_2F_1\left(\frac{1}{2}, 1, n + \frac{1}{2}; \frac{r^4}{1+r^4}\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{s^2}{2}\right)^n \frac{1}{\sqrt{1+r^4}} \left(n - \frac{1}{2}\right) \int_0^1 \frac{(1-\rho)^{n-\frac{3}{2}}}{\sqrt{(1-\rho\frac{r^4}{1+r^4})}} d\rho \end{aligned} \quad (5.5.7)$$

By first summing over n and then performing the integration over ρ we get a result, which when added to the $n = 0$ piece gives the full answer for the energy

$$\begin{aligned} E &= \frac{\sqrt{1+r^4} \sqrt{s^2/2+r^4} (r^4(s^2/2-1)-s^2/2) + r^4(s^2/2-1)(s^2/2) \tanh^{-1}\left(\sqrt{\frac{r^4+s^2/2}{r^4+1}}\right)}{(r^4+s^2/2)^{3/2}(s^2/2-1)} \\ &= \sqrt{1+r^4} \left(1 - \frac{s^4}{(s^2-2)(s^2+2r^4)}\right) + \frac{r^4 s^2}{2(r^4+s^2/2)^{3/2}} \tanh^{-1}\left(\sqrt{\frac{r^4+s^2/2}{r^4+1}}\right) \end{aligned} \quad (5.5.8)$$

We can use the same method to obtain the explicit form of the Lagrangean (5.5.3), restoring all the appropriate dimensionful parameters

$$S = -\frac{N}{g_s \ell_s} \int dt \left(\sqrt{1+r^4} - \frac{s^2}{2\sqrt{r^4+s^2/2}} \tanh^{-1} \frac{\sqrt{r^4+s^2/2}}{\sqrt{1+r^4}} \right) \quad (5.5.9)$$

We can perform expansions of the above expression for small values of the r and s variables

$$\begin{aligned} E(r \sim 0, s) &= \frac{1}{(1-s^2/2)} + r^4 \frac{1}{s^2-2} + \frac{\sqrt{2}r^4}{s} \tanh^{-1} \frac{s}{\sqrt{2}} + \dots \\ E(r, s \sim 0) &= \sqrt{1+r^4} + \frac{s^2}{2r^2} \tanh^{-1} \left(\sqrt{\frac{r^4}{r^4+1}} \right) + \dots \end{aligned} \quad (5.5.10)$$

Also for large r and small s

$$\begin{aligned} E(r, s) &= \left(r^2 + \frac{1}{2r^2} - \frac{1}{8r^6}\right) + \left(\frac{1}{8r^6} + \frac{\ln(2r^2)}{2r^2}\right) s^2 \\ &+ \left(\frac{3}{16r^6} - \frac{3\ln(2r^2)}{8r^6} + \frac{3}{8r^2}\right) s^4 + \left(\frac{5}{32r^2} - \frac{5}{32r^6}\right) s^6 + \dots \end{aligned} \quad (5.5.11)$$

One of the features of the fuzzy even-spheres, namely the fact that they admit $r = t$ type solutions is also true here. This statement translates into having an $s^2/2 = 1$ solution to the $\partial_t E = 0$ equations of motion. It is easy to check that this holds. It is unfortunate

that the energy formula includes an inverse hyperbolic tangent of (r, s) . This leads to a transcendental relationship between the two variables and prevents us from conducting an analysis similar to the one performed in Ch. 2, which would give the time of collapse, a possible solution to the radial profile and the configuration's periodicities.

5.5.1 Constancy of the speed of light

In order to explore the physical properties of our configurations, we will once again use the definition of the physical radius for any S^{2k-1} fuzzy sphere

$$\begin{aligned} R_{phys}^2 &= \lambda^2 \lim_{m \rightarrow \infty} \frac{STr(\Phi_i \Phi_i)^{m+1}}{STr(\Phi_i \Phi_i)^m} \\ &= \lambda^2 \hat{R}^2 \lim_{m \rightarrow \infty} \frac{STr(X_i X_i)^{m+1}}{STr(X_i X_i)^m} \\ &= \frac{\lambda^2 C \hat{R}^2}{2} \end{aligned}$$

where we have evaluated the matrix products in the large- N limit. This will guarantee that the series defining the Lagrangean will converge for $\dot{R}_{phys} = 1$. The definition was interpreted as an application of the principle of constancy of the speed of light in Ch. 4, where it also gave the correct results for the fuzzy even-sphere problems at both large and finite- N . We would like to highlight once again that the above is not the same thing as requiring local Lorentz invariance. This is because the form of the summed series is not the same as in special relativity. For example the action takes the form $\int dt (1 - |\dot{R}_{phys}| \tanh^{-1} |\dot{R}_{phys}|)$ at $R_{phys} = 0$, rather than the standard $\int dt \sqrt{1 - \dot{R}_{phys}^2}$, which appears in the large- N limit of the $D0$ - $D(2k)$ systems. We saw modifications of the standard relativistic form arising in the study of fuzzy even-spheres at finite- N in Ch. 4.

In terms of the (r, s) variables, which appear in the expressions for the energy in the previous section, we have

$$\begin{aligned} r &= \frac{\sqrt{2} R_{phys}}{\sqrt{\lambda}} = \frac{R_{phys}}{L_{\text{odd}}} \\ s &= \sqrt{2} \dot{R}_{phys} \end{aligned} \tag{5.5.12}$$

The physical singularity at $\dot{R}_{phys} = 1$ corresponds to $s^2 = 2$. For later convenience, we will further define the normalised dimensionless variables $\sqrt{2}(\hat{r}, \hat{s}) = (r, s)$. This implies that $\hat{s} = 1$ is the speed of light. The characteristic length scale of the system for fuzzy odd-spheres, appearing in (5.5.12), is $L_{\text{odd}} = \sqrt{\frac{\lambda}{2}}$. This should be contrasted with $L_{\text{even}} = \sqrt{\frac{N\lambda}{2}}$. In the fuzzy- S^2 problem we were able to keep L finite when $\ell_s \rightarrow 0$ while $\sqrt{N} \rightarrow \infty$. In the present case we cannot do so. If we take $\ell_s \rightarrow 0$ we lose the physics of the fuzzy odd-spheres. This is compatible with the idea that they should be related to some tachyonic configuration on an unstable higher dimensional dual brane. These tachyonic modes become infinitely

massive as $\ell_s \rightarrow 0$.

5.5.2 Derivatives, no-bounce results, accelerations

We can use the above results to get a picture of the nature of the collapse/expansion for the $D0$ s blown up into a fuzzy- S^{2k-1} . We can explore whether there will be a bounce in the trajectory, as was done in [63] and in Ch. 4 for the finite- N dynamics of fuzzy even-spheres, by looking for zeros of constant energy contour plots in (r, s) . This can be simply seen by a zero of the first derivative of the energy with respect to s , for constant r . For our case this is

$$\begin{aligned} \frac{\partial E}{\partial \hat{s}}|_{\hat{r}} = & \frac{2\sqrt{(1+4\hat{r}^4)\hat{s}^3}(\hat{s}^2 + \hat{r}^4(10-6\hat{s}^2))}{(1-\hat{s}^2)^2(4\hat{r}^4+\hat{s}^2)^2} \\ & + \frac{4\hat{r}^4 \hat{s} (8\hat{r}^4 - \hat{s}^2) \tanh^{-1}(\sqrt{\frac{4\hat{r}^4+\hat{s}^2}{1+4\hat{r}^4}})}{(4\hat{r}^4+\hat{s}^2)^{5/2}} \end{aligned} \quad (5.5.13)$$

We have checked numerically that the above expression is non-zero for $0 < \hat{s} < 1$ and $\hat{r} > 0$. Hence there will be no bounce and the configuration will classically collapse all the way to zero radius. In our treatment so far, we have assumed that higher derivative α' corrections can be neglected. This statement translates into requiring that higher commutators should be small. This condition gives $[\ell_s \Phi_i, [\ell_s \Phi_i,]] \ll 1$ and, with the use of $[X_i, [X_i,]] = n^2$ for large- n , it implies that $\hat{r} \ll \sqrt{\frac{\pi}{2}} \sim 1$. This corresponds to

$$R_{phys} < \ell_s \quad (5.5.14)$$

The other relevant quantity in investigating the validity of the action is the proper acceleration, which should be small. We remind that this is defined as

$$\alpha = \gamma^3 \frac{d^2 \hat{r}}{d\tau^2} = \gamma^3 \hat{s} \frac{d\hat{s}}{d\hat{r}} = -\gamma^3 \frac{\partial_{\hat{r}} E}{\partial_{\hat{s}} E}$$

with $\gamma = (1 - \hat{s}^2)^{-1/2}$. The derivative of the energy with respect to \hat{r} is

$$\begin{aligned} \frac{\partial E}{\partial \hat{r}}|_{\hat{s}} = & \frac{8\hat{r}^3(\hat{s}^4 + 16\hat{r}^8(\hat{s}^2 - 1) + 4\hat{r}^4\hat{s}^2(4\hat{s}^2 - 3))}{\sqrt{1+4\hat{r}^4}(\hat{s}^2 - 1)(4\hat{r}^4 + \hat{s}^2)^2} \\ & + \frac{16\hat{r}^3\hat{s}^2(\hat{s}^2 - 2\hat{r}^4) \tanh^{-1}(\sqrt{\frac{4\hat{r}^4+\hat{s}^2}{1+4\hat{r}^4}})}{(4\hat{r}^4 + \hat{s}^2)^{5/2}} \end{aligned} \quad (5.5.15)$$

From the above we get a complicated expression for the proper acceleration in \hat{r} and \hat{s} . Since the energy relation, combined with the boundary condition that the velocity at \hat{r}_0 is zero, gives a transcendental equation for \hat{r} and \hat{s} , we can't predict the behaviour of the velocity

for the duration of the collapse. However, for small \hat{r} the proper acceleration simplifies to

$$\alpha = \frac{4\hat{r}^3 (\hat{s} + 2(\hat{s}^2 - 1) \tanh^{-1}(\hat{s}))}{\hat{s}\sqrt{1 - \hat{s}^2}} + \mathcal{O}(\hat{r}^5) \quad (5.5.16)$$

In the small \hat{r} limit, we can see that the velocity \hat{s} will also be small throughout the trajectory⁵ and we will be within the Matrix Theory limit, which we describe in the next section. For these values the proper acceleration is small enough and the action is valid throughout the collapse. If we were to give the configuration some large initial velocity, we have numerical evidence that there will be a region where the proper acceleration is small enough for the action to be valid, but could change sign. We leave the further investigation of the relativistic regime for the future.

5.6 The Matrix Theory (Yang-Mills) limit

It is interesting to consider the Matrix Theory limit of the action for both the S^3 and the S^5 cases. Consider eq. (5.4.17)

$$S = -\frac{1}{g_s \ell_s} \int dt \text{ } STr \sqrt{1 + \frac{\lambda^2}{2} \hat{R}^4 A_{ij} A_{ji} - \lambda^2 \dot{\hat{R}}^2 X_i X_i}$$

For small \hat{R} and small $\dot{\hat{R}}$, the above yields

$$S = -\frac{N}{g_s \ell_s} \int dt \left(1 + \frac{\lambda^2}{4} \hat{R}^4 A_{ij} A_{ji} - \frac{\lambda^2}{2} \dot{\hat{R}}^2 X_i X_i \right) \quad (5.6.1)$$

The equations of motion for both S^3 and S^5 will be in dimensionless variables

$$\ddot{r} = -2r^3 \quad (5.6.2)$$

and will be solved exactly by radial profiles of the form

$$r(\tau) = r_0 \text{ } Cn \left(\sqrt{2} r_0 \tau, \frac{1}{\sqrt{2}} \right) \quad (5.6.3)$$

where r_0 is a parameter indicating the value of the initial radius at the beginning of the collapse. However, we need to prove that having a solution to the above reduced action is equivalent to solving the Matrix equations of motion. Starting from eq. (5.4.2) and doing a small λ expansion, we arrive at

$$S = -\frac{1}{g_s \ell_s} \int dt \text{ } Tr \left(1 + \frac{\lambda^2}{4} [\Phi_i, \Phi_j][\Phi_j, \Phi_i] - \frac{\lambda^2}{2} \partial_t \Phi_i \partial_t \Phi_i \right) \quad (5.6.4)$$

⁵If we take $r^4 \ll 1$ in (5.5.8) we find that $s \sim 0$.

The equations of motion are

$$\frac{\partial^2 \Phi_i}{\partial t^2} = -[\Phi_j, [\Phi_j, \Phi_i]] \quad (5.6.5)$$

and upon substituting the Ansatz, $\Phi_i = \hat{R}(t)X_i$, and using the matrix identities for any D we get

$$\ddot{\hat{R}} = -\hat{R}^3(C + D - 2) \quad (5.6.6)$$

It is easy to verify that, by directly substituting the Ansatz into (5.6.4) and then calculating the equations of motion, we will get the same result. Therefore, any solution of the reduced action will also be a solution of the full Matrix action for any N .

The time of collapse can be calculated by using conservation of energy, in the same way as for S^2 [63] and Ch. 2. Expressed in terms of the characteristic length-scale of the theory, the answer will be $T \sim L^2/R_0$. However, as we have already noted, the parameter L in the odd-sphere case does not contain a factor of \sqrt{N} . As a consequence, the collapse of fuzzy- S^3 and S^5 spheres at large- N and for $R_0 \ll L$, will occur much faster than for the fuzzy- S^2 .

5.7 Towards a dual 3-brane picture for fuzzy- S^3

In the spirit of the dualities that have been established between higher and lower dimensional branes for the case of even-spheres [60–63, 72], it is reasonable to expect that there should be a dual macroscopic description for the system of $D0$ -branes blowing up into a fuzzy- S^3 . The $D0$ s couple to the RR three-form potential, via a term proportional to $\int dt STr(C_{tij}^{(3)}[\Phi_j, \Phi_i] + C_{kij}^{(3)}\dot{\Phi}_k[\Phi_j, \Phi_i])$ in the Chern-Simons part of the action. They also couple to the five-form potential via $\int dt STr(C_{tijkl}^{(5)}[\Phi_i, \Phi_j][\Phi_k, \Phi_l])$, although by expanding the commutator terms and checking that the leading order term⁶ in $\epsilon_{ijkl}X_iX_jX_kX_l$ evaluates to zero, one sees that this is sub-leading in n . The overall trace will ensure that the net higher brane charge will be zero. Therefore, at large- n we will only see an effective coupling to the $D2$ -brane charge.

5.7.1 Non-BPS systems in String Theory

We will pause to briefly introduce the notions of unstable D -brane configurations⁷, which we will be using in the next section. Ordinary flat Dp -branes with p even/odd are stable objects that saturate the BPS bound, preserving half of the space-time supersymmetry, and couple to the RR potentials of IIA/IIB String Theories. They have a definite orientation and a well defined tension, given by $T_p = g_s^{-1}(2\pi)^{-p}\ell_s^{-(p+1)}$. All the open string modes that

⁶That is the one with no coincidences.

⁷For a complete review on open string tachyon dynamics see [133].

end on these branes have positive mass squared. Given a specific Dp -brane, we will call one which is oppositely oriented an anti- Dp -brane or $\bar{D}p$ -brane. If one considers a configuration consisting of a coincident Dp -brane and $\bar{D}p$ -brane, one discovers that the string that extends between the two has a tachyonic mode (there are actually two modes, since there are two possible orientations for strings stretching between the two branes), the mass squared of which is $m^2 = -1/2$. This is a tachyonic mode that escapes the GSO projection, since the latter prescription is opposite for strings that end on one kind of brane (either Dp or $\bar{D}p$) and strings that have one end on each, and the NS ground state is now part of the spectrum. The tachyon is interpreted as an instability of the system, which tends to flow to another more stable configuration.

Type II String Theories also contain single unstable (non-BPS) Dp -branes in their spectrum. One way of obtaining these is by orbifolding the Dp - $\bar{D}p$ system by $(-1)^{F_L}$, where F_L is the contribution to the space-time fermion number from the left-moving world-sheet sector. A linear combination of the two tachyonic modes from the brane-anti-brane system survives the $(-1)^{F_L}$ projection in the open string spectrum. The non-BPS branes are unoriented, have the ‘wrong’ dimensionality, *i.e.* odd/even p in IIA/IIB and also have a definite tension given by $\tilde{T}_p = g_s^{-1} \sqrt{2} (2\pi)^{-p} \ell_s^{-(p+1)}$. An important difference between these unstable branes and their conventional counterparts is that the former are neutral under the RR $(p+1)$ -form gauge fields of the type II theories.

The study of tachyon dynamics is non-trivial. The tachyon field couples to the infinite open string fields that appear in open string field theory and hence prevents us from examining it in isolation. Moreover, we cannot work in a low-energy approximation, since the mean value of the tachyon mass squared is of the same order as that of the other heavy modes of the string. Nevertheless, we can attempt to encode some of the classical behaviour in terms of an effective action, by formally integrating out all the positive mass squared fields that live on the world-volume. The massless fields that remain on the non-BPS Dp -brane world-volume include one gauge field and $(9-p)$ scalars, while for Dp - $\bar{D}p$ we have two $U(1)$ gauge fields and $2(9-p)$ transverse scalars. The existence of a single scalar tachyonic mode for the unstable brane shows that the effective theory must contain a real scalar field T with mass $m^2 = -1/2$. Equivalently, for the D - \bar{D} system there should be two real or one complex scalar of the same mass. There are interesting properties of the effective action that can be derived from a tree-level S-matrix analysis: The non-BPS brane tachyon effective action has a \mathbb{Z}_2 symmetry under the exchange $T \rightarrow -T$, while for D - \bar{D} it has a phase symmetry under $T \rightarrow e^{i\alpha}T$. One also needs to introduce the tachyon effective potential $V(T)$, which will be such that, for a space-time independent field configuration and with all massless fields set to zero, the effective action has the form $-\int d^{p+1}\sigma V(T)$. A straightforward consequence of the fact that the mass of T is negative is that there will be a maximum of the potential at $T = 0$. We can fix that maximum to be at $V(0) = 0$. The potential is therefore negative. The major question is now whether there exist minima of

this action. The answer is that there indeed exist a pair of global minima at some $T = \pm T_0$ for the unstable brane and a one parameter family of global minima $T = T_0 e^{i\alpha}$ for the D - \bar{D} system. At those minima, the tension of the original brane configurations are exactly cancelled by the negative potential contribution, so that the total energy density vanishes at the bottom of the tachyon potential

$$V(T_0) + \mathcal{E}_p = 0 \quad (5.7.1)$$

with $\mathcal{E}_p = \tilde{T}_p$ for the non-BPS brane and $\mathcal{E}_p = 2T_p$ for the D - \bar{D} system. Therefore, these minima correspond to the vacuum without any D -branes.

The effective action also admits static non-trivial classical solutions for the tachyon profile. These solutions represent lower dimensional branes. To be more precise, for the non-BPS D -brane there is a kink solution depending on a single spatial direction x , which interpolates between the two vacua $\pm T_0$ as $x \rightarrow \pm\infty$ and is centred around $x = 0$. The energy density is localised around a $(p-1)$ -dimensional sub-space and the kink solution actually describes a BPS $D(p-1)$ -brane in the same theory (or a $\bar{D}(p-1)$ -brane for the anti-kink solution). By setting the imaginary part of the complex scalar tachyon field for the Dp - $\bar{D}p$ system to zero, we recover a similar kink solution, describing a non-BPS $D(p-1)$ -brane in the same theory. On the other hand, by using both real and imaginary parts of the scalar one can construct a vortex solution, which depends on two spatial directions x and y . This is a co-dimension two solitonic configuration and describes a BPS $D(p-2)$ -brane in the same theory. Note that the validity of the tachyon condensation process (the rolling of the tachyon field to a lower energy configuration at the bottom of the potential) can be verified by evaluating the tension predicted by the effective action and comparing it with the one expected for the D -brane end-products. One could also consider larger numbers of coincident non-BPS branes or D - \bar{D} systems. Then a number of lower co-dimension D -branes will either arise by direct construction of higher dimensional static solitonic solutions or by a multi-step reduction. Consider as an example the case of 2 coincident non-BPS Dp -branes: One can either recover a BPS $D(p-3)$ -brane by constructing a 't Hooft-Polyakov monopole solution for the tachyon profile, or first get a kink and an anti-kink on each one and then another kink solution on the resulting D - \bar{D} system. This structure is not accidental and is related to a classification of D -brane charges via K-Theory [134, 135]. A consequence of this is that one can recover all BPS D -branes of type IIA by starting from a non-BPS $D9$ and all the ones from type IIB, starting from a $D9$ - $\bar{D}9$ system.

5.7.2 The dual construction

Returning to the problem of finding a macroscopic dual construction for the S^3 system, we propose that the simplest candidate is a collection of coincident $D3$ s embedded in Euclidean space as a classical 3-sphere, which in Type IIA are non-BPS. As a consequence, we expect

to have an effective field theory on the compact unstable $D3$ ($UD3$) world-volume, which will incorporate a real tachyon field and some $SU(M)$ non-trivial world-volume flux, where M is the number of $UD3$ s. The most general description will be in terms of the non-abelian, tachyonic Dp -brane action proposed in [136]. In the case at hand, there are no non-trivial transverse scalars and we are working in a flat background, in a spherical embedding. The DBI part of the $UD3$ -brane action will then reduce to

$$\begin{aligned} S &= -\frac{\mu_3}{g_s} \int d^4\sigma \text{STr} \left(V(T) \sqrt{-\det(P[G]_{ab} + \lambda F_{ab} + \lambda D_a T D_b T)} \right) \\ &= -\frac{\mu_3}{g_s} \int d^4\sigma \text{STr} \left(V(T) \sqrt{-\det(M)} \right) \end{aligned} \quad (5.7.2)$$

The symmetrisation procedure should be implemented amongst all non-abelian expressions of the form F_{ab} and $D_a T$ and on T of the potential, which is well approximated by $V(T) \sim e^{-T^2}$. The gauge field strength and covariant derivative of tachyons are

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a - i[A_a, A_b] \\ D_a T &= \partial_a T - i[A_a, T] \end{aligned} \quad (5.7.3)$$

It is straightforward to calculate the determinant. The result, in spherical co-ordinates $(\alpha_1, \alpha_2, \alpha_3)$, is

$$\begin{aligned} \sqrt{-\det(M)} &= \sqrt{g} \left[(1 - \dot{R}^2 - \lambda D_0 T D_0 T) \left(R^6 - \frac{\lambda^2 R^2}{2} F_{ij} F^{ji} \right) - R^4 \lambda^2 F_{0i} F_0^i \right. \\ &\quad + (1 - \dot{R}^2) \left(\lambda R^4 D_i T D^i T - \lambda^3 \left(\frac{D_i T D^i T F_{jk} F^{kj}}{2} - D^i T F_{ij} F^{jk} D_k T \right) \right) \\ &\quad \left. - 2\lambda^3 R^2 D_0 T D_j T F_{0i} F^{ij} + \frac{\lambda^4}{4} \left(\frac{F_{0i} F_0^i F_{jk} F^{kj}}{2} - F_{0i} F^{ij} F_{jk} F_0^k \right) \right]^{1/2} \end{aligned} \quad (5.7.4)$$

where $g = \sin^4 \alpha_1 \sin^2 \alpha_2$ is the determinant of the unit three-sphere metric.

The CS part of the non-BPS brane action which is of interest to us has been discussed in [137, 138] and is given in terms of the curvature of the super-connection

$$\begin{aligned} S_{CS} &\sim \mu_3 \int C \wedge \text{STr} e^{\frac{\mathcal{F}}{2\pi}} \\ &\sim \mu_3 \int C \wedge \text{STr} e^{\frac{1}{2\pi}(F - T^2 + DT)} \\ &\sim \frac{\mu_3}{8\pi^2} \int C^{(1)} \wedge \text{STr} ((F \wedge DT) W(T)) \end{aligned} \quad (5.7.5)$$

where only odd-forms are kept in the exponential expansion, the $DT \wedge DT \wedge DT$ term vanishes because of the overall symmetrisation and $W(T) = e^{-\frac{T^2}{2\pi}}$. There will also be a term proportional to $\int C^{(3)} \wedge \text{STr}(DT)$. If we want to match the two descriptions, we

should require that the overall $D0$ -charge is conserved. We therefore impose that

$$\frac{1}{8\pi^2} STr ((F \wedge DT)W(T)) = \frac{N}{\text{Vol}_{S^3}} \Omega_3 \quad (5.7.6)$$

where Ω_3 is the $SO(4)$ invariant volume form on the 3-sphere with angles $(\alpha_1, \alpha_2, \alpha_3)$, N is an integer and $\text{Vol}_{S^3} = 2\pi^2$. By restricting to the world-volume of a single brane, *i.e.* having a $U(1)$ gauge symmetry for the gauge and tachyon fields, this condition translates in components into the following expression

$$F_{[ij} \partial_{k]} T W(T) = 4N \epsilon_{ijk} \quad (5.7.7)$$

where $\epsilon_{\alpha_1 \alpha_2 \alpha_3} = \sqrt{g}$. Contraction with ϵ^{ijk} results in

$$\epsilon^{ijk} \partial_i T F_{kj} = -\frac{24N}{W(T)} \quad (5.7.8)$$

while contraction with $\partial^i T F^{kj}$ and then use of the above equation gives

$$\frac{\partial_i T \partial^i T F_{jk} F^{kj}}{2} - \partial^i T F_{ij} F^{jk} \partial_k T = -\frac{144N^2}{W(T)^2} \quad (5.7.9)$$

Using this last expression we can simplify the $UD3$ action just by $SO(4)$ symmetry of the charge conservation condition. We obtain

$$\begin{aligned} S = & -\frac{\mu_3}{g_s} \int d\sigma^4 \sqrt{g} V(T) \left[(1 - \dot{R}^2 - \lambda \dot{T}^2) \left(R^6 - \frac{\lambda^2 R^2}{2} F_{ij} F^{ji} \right) \right. \\ & - 2\lambda^3 R^2 \dot{T} \partial_j T F_{0i} F^{ij} - R^4 \lambda^2 F_{0i} F_0{}^i + (1 - \dot{R}^2) \left(\lambda R^4 \partial_i T \partial^i T + \frac{144\lambda^3 N^2}{W(T)^2} \right) \\ & \left. + \frac{\lambda^4}{4} \left(\frac{F_{0i} F_0{}^i F_{jk} F^{kj}}{2} - F_{0i} F^{ij} F_{jk} F_0{}^k \right) \right]^{1/2} \end{aligned} \quad (5.7.10)$$

The equations of motion for this configuration are involved. Nevertheless, note that all the individual terms in the above action should be scalars of $SO(4)$. This means that we should have both $\partial_i T \partial^i T$ and T^2 independent of the angles and a constant T cannot satisfy the spherical symmetry condition (5.7.6). Hence, there is no world-volume theory for a compact unstable brane system in three dimensions with $SO(4)$ symmetry and we should be looking at least at higher numbers of coincident non-BPS branes in order to find a dual description.

We may expect the matching between the $D0$ and the higher brane pictures to be more tricky than in the even-sphere case. In the even-sphere, the upper bound on the validity of the lower brane description increases with N , and allows an overlap with the higher brane description [60]. As can be seen from (5.5.14), the upper bound does not increase with N for the fuzzy odd-sphere case. The separation of the degrees of freedom of the fuzzy- S^3 Matrix algebra [49] into geometrical and internal also suggests that a simple relation to a

non-abelian theory is not possible. However, this does not preclude a relation via a non-trivial renormalisation group flow, analogous to that proposed in [128] in the context of the application of the fuzzy three-sphere to the $M2 \perp M5$ intersection.

CHAPTER 6

CONCLUSIONS AND OUTLOOK

We have studied the interplay between the non-commutative properties of fuzzy sphere geometry and the physics of non-abelian D -brane configurations. The space and time-dependence of fuzzy even-spheres S^2 , S^4 and S^6 are governed by equations which follow from the DBI action of D -branes. These fuzzy spheres can arise in bound states of $D0$ with $D2$, $D4$ or $D6$ on the one hand, or as cross-sections of fuzzy funnels formed by $D1$ s expanding into $D3$, $D5$ or $D7$ on the other. The purely time-dependent process has its simplest realisation as the collapsing-expanding $D0$ - $D(2p)$ bound state, although it also arises as a time-dependent process for the $D1 \perp D(2p+1)$ equations with no variation along the spatial $D1$ -direction.

In Ch. 2, we saw that the first order equations of motion are closely related to some Riemann surfaces and the infinitesimal time elapsed or the infinitesimal distance along the $D1$ s, are given by a holomorphic differential on these surfaces. We have given descriptions of the solutions in terms of elliptic functions or their higher genus generalisations. We showed that the space and time processes exhibit some very interesting large-small dualities closely related to the geometry of the Riemann surfaces.

In the case of S^2 , the Riemann surface has genus one and the duality is related to properties of Jacobi elliptic functions known as “complex multiplication formulae” [80]. The genus 1 Riemann surface that arises here has automorphisms, holomorphic maps to itself, which are responsible for these properties. We observed that the large-small duality is also directly related to a transformation property of the second order differential equations.

In the case of S^4 we derived formulae for the distance $\sigma(r)$ along the $D1$ -brane as a function of the radius of the funnel’s fuzzy sphere cross-section and the time elapsed $t(r)$ as a function of the radius. These formulae were expressed in terms of special functions such as Appell and hypergeometric functions. It was useful to combine space and time into a complex variable $u_1 = \sigma - it$ which was related to a genus three Riemann surface, having a number of automorphisms. These automorphisms were used to eventually relate the problem to a product of two genus one Riemann surfaces. After introducing a second complex variable u_2 , the problem of inverting the integrals to get a formula for the radius as a function of the complexified space-time variable u_1 was related to a classical problem in Riemann surfaces, the Jacobi Inversion problem. This can be solved, in general, using higher genus theta functions. The relation to a product of genus one surfaces in the case at hand, means that the Inversion problem can be expressed in terms of the standard elliptic

integrals. This allowed us to give a solution of the Inversion problem in terms of standard Jacobi elliptic functions. The solution involves a constraint in terms of elliptic functions, which can be used to solve u_2 in terms of u_1 , or vice-versa. This approach yields a new construction of $r(u_1)$ as a series around $r = r_0$, which agrees with a direct series inversion of the Appell function. It also gives new formulae for the time of collapse and distance to blow up in terms of sums of complete elliptic integrals, which were checked numerically to agree with the formulae in terms of hypergeometric functions.

The automorphism which allows a reduction of the problem to one involving genus one surfaces also relates the large- r behaviour of the spatial (time-dependent) solutions to the small- r behaviour of the time-dependent (spatial) problem. The introduction of the extra variable u_2 , required to make a connection to the Jacobi Inversion problem, also enters the description of this large-small duality.

Our discussion in the S^6 case was less complete, but a lot of the structure uncovered above continues to apply. The integrals giving $t(r)$ or $\sigma(r)$ are integrals of a holomorphic differential on a genus 5 Riemann surface. A simple $R = r^2$ transformation maps it to a holomorphic differential on a genus 3 curve. The Inversion problem of expressing r in terms of $u_1 = \sigma - it$ can be related to the Jacobi Inversion problem and a solution in terms of higher genus theta functions is outlined. We did not find any automorphisms of the genus 3 Riemann surface which would relate the problem to one involving holomorphic differentials on a genus 1 curve. There do continue to exist symmetry transformations of the curve but they are no longer holomorphic in terms of the (r, s) variables.

It will be interesting to consider the full CFT description of these funnels. Some steps towards the full CFT description of these systems have already been made [139]. There should be a boundary state describing the spatial configuration of $D1$ -branes forming a funnel which blows up into a $D3$ -brane. One could start with the CFT of the multiple $D1$ -branes and consider a boundary perturbation describing the funnel which opens into a $D3$. Alternatively we could start with a CFT for the $D3$ and introduce boundary perturbations corresponding to the magnetic field strength and the transverse scalar excited on the brane, which describe the $D1$ -spike. In either case the boundary perturbation will involve the elliptic functions which appeared in Sec. 2.1. Similarly the boundary perturbation for the fuzzy- S^4 case would involve elliptic functions associated with a pair of genus 1 Riemann surfaces, of the type described in Sec. 2.3. One such boundary state corresponds to the purely time-dependent solution and another to the purely space-dependent solution. If the large-small duality of the DBI action continues to hold in the CFT context, it would give a remarkable relation between the zero radius limit of time-dependent brane collapse and the well-understood blow-up of a $D1$ -brane funnel into a higher dimensional brane. Analytic continuations from Minkowski to Euclidean space in the context of boundary states have proved useful in studies of the rolling tachyon [133]. We expect similar applications of the analytic structures described here.

A complementary way to approach these dualities is to consider the supergravity description. The time-dependent system of a collapsing $D0$ - $D2$ bound state, for example, should correspond to a time-dependent supergravity background, when the back-reaction of space-time to the stress tensor of the $D0$ - $D2$ system is taken into account. A Wick rotation of this background can convert the time-like co-ordinate t to a space-like co-ordinate. Similarly the spatial funnel of a $D1 \perp D3$ system has a supergravity description with an S^2 having a size that depends on the distance along the direction of the D -string. The Wick rotated $D2$ -background should be related by a large-small duality to the $D1 \perp D3$ system. Similarly, the $D0$ - $D4$ system is related to the $D1 \perp D5$. The Jacobi Inversion problem which underlies our solutions has already arisen in the context of general relativity [140]. It will be interesting to see if this type of GR application of the Jacobi Inversion also arises in the space-time solutions corresponding to our brane configurations. Another perspective on the geometry of these configurations, including the boosted ones in App. A.1, and on the relations to S -branes, should be provided by considering the induced metric on the brane as for example in [141]. Generalisation of this discussion to funnels of \mathbb{CP}^2 and other cross-sections, as well as to funnels made of dyonic strings will be interesting.

The genus 3 curve for the S^4 was very special. It is related by the $R = r^2$ map to a product of genus 1 and a genus 2 curve. The resulting genus 2 curve itself has a Jacobian which is related to $K3$ Kummer surfaces [83] and factorises into a product of genus one curves. Special $K3$ s and complex multiplication also appear in the attractor mechanism [142, 143]. Whether the appearances of special $K3$ s in these two very different ways in String Theory are related is an intriguing question.

In Ch. 3 we focused on the analysis of fluctuations in the case of time-dependent $D0$ - $D2$ solutions and extended the agreement between the microscopic and macroscopic descriptions that exists at large- N . It will be interesting to see whether the comparison can be extended to higher orders in $1/N$, with an appropriate star product used on the higher brane and the Matrix product on the $D0$ -brane side interpreted as a star product on the sphere. We believe that the $1/N$ corrections that arise from the implementation of the STr on the fuzzy- S^2 should match the $1/N$ non-commutative deformations of the world-volume product on the higher dimensional brane. If such an agreement is to be recovered, one would be able to establish a correspondence between the $D0$ and $D2$ descriptions even at finite- N . The fact that the $D2$ open string metric does not diverge at radii smaller than the ones predicted by the naive limit $R \gg \ell_s$ provides an encouraging hint towards that realisation. However, the calculation of the small-fluctuation action, from both lower and higher world-volume pictures, is expected to be significantly more involved than the calculation presented here even to first next order in $1/N$. We leave such an investigation as an open question for future research. A similar analysis of small fluctuations can be performed for the spatial $D1 \perp D3$ configurations. Some aspects of this problem have already been studied in [75].

In the case of systems involving higher dimensional fuzzy spheres, such as $D0$ - $D4$

($D1 \perp D5$) systems or $D0$ - $D6$ ($D1 \perp D7$) systems, we expect on general grounds that there will be an abelian description based on a geometry of the form $SO(2k+1)/U(k)$ and a non-abelian description on the S^{2k} [48, 50, 144, 145]. A detailed fluctuation analysis of the kind studied here should allow for a more precise description of strong and weak coupling regimes. The flat space limit of our analysis of fluctuations about fuzzy sphere solutions should be related to the work in [146, 147].

The discussion of the scaling limit in Sec. 3.3 is very reminiscent of similar scaling limits in the context of BFSS Matrix Theory and the AdS/CFT duality. The difference is that here we are keeping the non-linearities coming from the non-abelian $D0$ -brane action (3.2.1). This action is of course less understood than the $0+1$ SYM of the BFSS Matrix Model or the $3+1$ SYM of the canonical AdS/CFT correspondence. For example a completely satisfactory supersymmetric version has yet to be written down, although some progress on this has been discussed in [30]. However it is significant that our scaling discussion of Subsec. 3.3.1 highlights the fact that the appropriate supersymmetrised non-abelian DBI action should provide a complete quantum mechanical description of the collapsing $D0$ - $D2$ system.

This may appear somewhat surprising, but we will argue is not unreasonable. When R is close to R_0 we have a Yang-Mills action at weak coupling, and quantum correlation functions can be computed in a weak coupling expansion. In the strict large- N limit, the Yang-Mills theory is commutative, while $1/N$ corrections amount to turning the background sphere into a non-commutative sphere. When the correlation functions are localised in regions where the radius is small, the Yang-Mills coupling is large and non-linearities in the fields become important. There must be a quantum mechanical framework which provides the continuation of the correlators to this region. Since ℓ_s has been taken to zero, massive string modes decouple and the only degrees of freedom left to quantise are those that appear in the non-abelian DBI for $D0$ -branes. String theory loops degenerate to loops of the fields in this action. The conjecture suggested by these arguments is that the fate of the collapsing $D0$ - $D2$ system in the zero radius region, and in the regime of parameters of Sec. 3.3, is contained in the quantum version of the supersymmetrised non-abelian $D0$ DBI. We have outlined a framework for calculating quantum corrections to the classical bouncing path, and discussed processes where one membrane splits into multiple membranes, or membranes mix with scattering states made of large bound states of zero-branes.

It will be interesting to look for a gravitational dual for the decoupled gauge theory of Sec. 3.3. One possibility is to start with a time-dependent multi- $D0$ brane solution of the type considered in [148, 149]. Then a $D2$ -brane could be introduced as a perturbation, as the $D5$ -brane was introduced in a background of $D3$ branes in [55]. The gravitational dual may shed light on the strong coupling regime of $R \rightarrow 0$. Another approach for a space-time gravitational description is to consider the spherical $D0$ - $D2$ system as a spherical shell, which causes a discontinuity in the extrinsic curvature due to its stress tensor and acts as a

monopole source for the two-form field strength due to the $D0$ -branes, and a dipole source for the four-form field strength due to the spherical $D2$ -brane. As long as the $D0$ -brane fluid description is valid, it should be possible to view the $D0$ branes as smeared on a sphere of time-dependent radius. Exploring the solutions and regimes of validity of these different gravitational descriptions will undoubtedly be intriguing, since the gravitational back-reaction of such time-dependent spherical brane bound states would provide possible avenues towards physically interesting time-dependent versions of gauge/gravity dualities. The paper [150] is an example where a gauge theory dual in a time-dependent set up is proposed.

Another possible avenue is to use the quadratic action that we obtained, to do one and higher loop computations of the partition function and correlators. As indicated by the connections to integrability in Sec. 3.1.2, these computations have interesting mathematical structure. It will be interesting to perturbatively incorporate the non-linearities in the fields, around the region of R close to R_0 . It will also be interesting to see if further results from integrable models play a role in the analysis of fluctuations, as Schrödinger equations with such potentials are well studied in the relevant literature [151].

In Ch. 4 we gave a detailed study of the finite- N effects for the time-dependent $D0$ - $D2$ fuzzy sphere system and the related $D1 \perp D3$ funnel. This involved calculating symmetrised traces of $SO(3)$ generators. The formulae have a surprising simplicity.

The $D0$ - $D2$ energy function $E(r, s)$ in the large- N limit looks like a relativistic particle with position dependent mass. This relativistic nature is modified at finite- N . Nevertheless our results are consistent with a fixed relativistic upper speed limit. This is guaranteed by an appropriate definition of the physical radius, which relies on the properties of symmetrised traces of large numbers of generators. We showed that the exotic bounces found in the large- N expansion in [63] do not occur. It was previously clear that these exotic bounces happened near the regime where the $1/N$ expansion was breaking down. The presence or absence of these could only be settled by a finite- N treatment, which we have provided in that chapter. We also compared the time of collapse of the finite- N system with that of the large- N system and found a finite- N deceleration effect for the first small values of N . The modified $E(r, s)$ relation allows us to define an effective squared mass which depends on both r, s . For certain regions in (r, s) space, it can be negative. When the $D0$ - $D2$ system is viewed as a source for gravity, a negative sign of this effective mass squared indicates that the brane acts as a gravitational source which violates the dominant energy condition.

We extended some of our discussion to the case of higher even fuzzy spheres with $SO(2k+1)$ symmetry. The results for symmetrised traces that we obtain can be used in a proposed calculation of charges in the $D1 \perp D(2k+1)$ system. They also provide further illustration of how the correct definition of the physical radius using symmetrised traces of large powers of Lie algebra generators gives consistency with a constant speed of light. A more complete discussion of the finite- N effects for the higher fuzzy spheres could start from these results.

In Ch. 5 we provided a set of formulae for general fuzzy odd-spheres and studied them as solutions to Matrix DBI $D0$ -brane actions and their Matrix Theory (Yang-Mills) limit. After implementing the symmetrised trace, which in the fuzzy odd-sphere case requires a non-trivial sum over orderings even at large- N , we found the same equations of motion for the fuzzy- S^3 and S^5 . We proved that solutions to the reduced DBI action also solve the full Matrix equations of motion. For the Matrix Theory limit, we gave exact expressions for these solutions in terms of Jacobi elliptic functions. The study of the physical properties of these systems showed that the classical collapse will proceed all the way to the origin.

Given that we have now established the fuzzy- S^3 (and S^5) as solutions to stringy Matrix Models, we can study the action for fluctuations. Using the remarks in [49] on the geometrical structure of the Matrix algebras, we expect that it should be possible to write the action in terms of fields on a higher dimensional geometry: $S^2 \times S^2$ for the S^3 case and $SO(6)/(U(2) \times U(1))$ in the case of S^5 . It will be intriguing to clarify the geometry and symmetries of this action.

A very interesting open problem is to identify a macroscopic large- N dual description of the fuzzy- S^3 and S^5 systems that we have described. The main difficulty lies in constructing $SO(2k)$ -invariant finite energy time-dependent solutions describing tachyons coupled to gauge fields on odd-dimensional spheres. The spherical symmetry restrictions that we discussed here should facilitate the task of investigating a non-abelian solution. A dual description, and agreement with the microscopic picture at the level of the action and/or equations of motion, would not only provide us with a new check of the current effective world-volume actions for non-BPS branes, but also give the possibility of constructing cosmological toy models of bouncing universes with three spatial dimensions. A non-trivial extension to the problem would be the addition of angular momentum. This could provide a stabilisation mechanism for fuzzy odd-spheres in the absence of the right RR fluxes, which provide a simple stabilisation mechanism for fuzzy even-spheres. The study of finite- N effects and the embedding of these systems in more general backgrounds [152, 153] provide other possible avenues for future research. It would also be interesting to get a better understanding of the relation between the fuzzy odd-sphere constructions considered here and those of [154, 155] involving fibrations over projective spaces.

To conclude, we hope that the collection of results presented in this thesis will shed new light into some of the non-trivial problems that still face String Theory. The study of time-dependent backgrounds in general is very important if we want to understand problems like the fate of the Big-Bang singularity in the early universe. Although the resolution of spacetime singularities is beyond the scope of traditional perturbative String Theory, there have been attempts to get holographic descriptions of the physics through the AdS/CFT correspondence. Time-dependent collapsing/expanding brane configurations provide toy models of Big-Bang type singularities, which could be examined in a similar context. The ‘quantum’ DBI prescription presented here, in the appropriate DBI scaling limit, suggests

a way of addressing the quantum nature of cosmological singularities. Alternatively, the $r \rightarrow \frac{1}{r}$ duality could provide yet another way of probing the singularity via the study of well understood dual static configurations. Moreover, this looks like a sort of generalisation of S^1 T-duality to S^2 . If it were found to extend to the full CFT, it could possibly be part of some larger duality group, which also involves a Wick rotation. This would be extremely interesting in its own right, since the discovery of such dualities (S,T,U) in the past has provided us with great insights on the nature of String and M-theory. Additionally, analytic continuations via a Wick rotation have often been used both as a means of ‘resolving’ space-time singularities as well as a solution generating mechanism from known static configurations. A motivation of such continuations from String Theory would be highly desirable. Finally, the STr results that emerged from the systems under study present a big step towards the realisation of a finite- N description of the elusive non-abelian DBI action, at least in the very specific case of D -brane configurations with $SO(3)$ isometry. This knowledge could greatly increase the possibilities of discovering the correct prescription in more general cases. It is indeed unfortunate that we still lack a full effective description of the world-volume dynamics on a stack of a finite number of coincident D -branes, given the important role that these systems play in all areas of modern research. Such knowledge would, undoubtedly, yield novel information in certain aspects of gauge/gravity dualities (*e.g.* finite- N effects in the AdS/CFT correspondence) but could also inspire new ways of constructing fundamental formulations of M-theory using multiple D -branes. We are greatly looking forward to investigating these exciting research avenues and believe that the answers to the broader questions posed here will come in due course.

APPENDIX A

ASPECTS OF SPACE-TIME-DEPENDENCE OF THE S^2 FUNNEL

A.1 Lorentz Invariance of the BPS condition

Here we will study the supersymmetry of the space and time-dependent system, in order to obtain a Lorentz invariant BPS condition. The vanishing of the variation of the gaugino on the world-volume of the $D1 \perp D3$ intersection will require that [60]

$$\delta\chi = \Gamma^{\mu\nu} F_{\mu\nu} \epsilon = 0 \quad (\text{A.1.1})$$

where μ, ν space-time indices, with

$$F_{ab} = 0 \quad F_{ai} = D_a \Phi_i \quad F_{ij} = i[\Phi_i, \Phi_j] \quad \text{and} \quad a = \sigma, \tau \quad , \quad i = 1, 2, 3 \quad (\text{A.1.2})$$

Here the spinor ϵ already satisfies the D -string projection $\Gamma^{\tau\sigma} \epsilon = \epsilon$ and multiply indexed Γ 's denote their normalised, fully antisymmetric product, *e.g.* $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$. Then

$$\left(i\Gamma^{jk}[\Phi^j, \Phi^k] + 2\Gamma^{ai}D_a \Phi^i \right) \epsilon = 0 \quad (\text{A.1.3})$$

We will consider the conjugate spinor

$$\overline{\delta\chi} = \delta\chi^\dagger \Gamma^0 \quad (\text{A.1.4})$$

and construct the invariant quantity

$$\delta\chi^\dagger \Gamma^0 \delta\chi = 0 \quad (\text{A.1.5})$$

where our conventions for the Γ -matrices are

$$(\Gamma^0)^2 = -1, \quad (\Gamma^i)^2 = 1 \quad \text{and} \quad (\Gamma^\mu)^\dagger = \Gamma^0 \Gamma^\mu \Gamma^0, \quad (\Gamma^{\mu\nu})^\dagger = \Gamma^0 \Gamma^{\mu\nu} \Gamma^0 \quad (\text{A.1.6})$$

Then (A.1.5) becomes

$$\epsilon^\dagger \left(-i(\Gamma^{\ell m})^\dagger [\Phi^\ell, \Phi^m] + 2(\Gamma^{bi})^\dagger D_b \Phi^i \right) \Gamma^0 \left(i\Gamma^{jk}[\Phi^j, \Phi^k] + 2\Gamma^{ai}D_a \Phi^i \right) \epsilon = 0 \quad (\text{A.1.7})$$

This will give three kinds of terms: Quadratic terms in derivatives, a quadratic term in commutators of Φ^i and cross-terms. The first of these will be

$$4(D_a\Phi^i)(D_b\Phi^j) \epsilon^\dagger (\Gamma^0\Gamma^{bj}\Gamma^0)(\Gamma^0\Gamma^{ai})\epsilon \quad (\text{A.1.8})$$

By making use of the Ansatz $\Phi^i = \hat{R}(\sigma, \tau)\alpha^i$ and by taking a trace over the gauge indices this becomes

$$4(\partial_a\hat{R})(\partial_b\hat{R}) \text{Tr} (\alpha^i\alpha^j) \epsilon^\dagger \left(\Gamma^0 \frac{1}{2} \{\Gamma^b, \Gamma^a\} \Gamma^j \Gamma^i + \Gamma^0 \frac{1}{2} [\Gamma^b, \Gamma^a] \Gamma^j \Gamma^i \right) \epsilon \quad (\text{A.1.9})$$

Since $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$, $\text{Tr} \alpha^i\alpha^j = \hat{c} \delta^{ij}$ for a constant \hat{c} and the expression is symmetric in a, b we get

$$12 \hat{c} (\partial_a\hat{R})(\partial^a\hat{R}) \left(\epsilon^\dagger \Gamma^0 \epsilon \right) \quad (\text{A.1.10})$$

The product of commutators will be

$$-[\Phi^\ell, \Phi^m][\Phi^j, \Phi^k] \epsilon^\dagger \Gamma^0 \Gamma^{\ell m} \Gamma^{jk} \epsilon \quad (\text{A.1.11})$$

which with the trace and the use of the Ansatz becomes

$$4\hat{c}\hat{R}^4 \varepsilon^{\ell mp} \varepsilon^{jkq} \delta^{pq} \epsilon^\dagger \Gamma^0 \Gamma^{\ell m} \Gamma^{jk} \epsilon \quad (\text{A.1.12})$$

Since we are looking for a supersymmetry condition for the intersection, we want the spinor ϵ to satisfy $\Gamma^{\tau ijk}\epsilon = \varepsilon^{ijk}\epsilon$ appropriate for the D3-brane. Using this fact we can re-write the above as

$$-4\hat{c}\hat{R}^4 (\delta^{\ell j}\delta^{mk} - \delta^{\ell k}\delta^{mj}) \varepsilon^{\ell mn} \varepsilon^{jkn} \epsilon^\dagger \Gamma^0 \epsilon \quad (\text{A.1.13})$$

and end up with

$$-12 \hat{c}(4\hat{R}^4) \left(\epsilon^\dagger \Gamma^0 \epsilon \right) \quad (\text{A.1.14})$$

Finally, the cross-terms

$$2i \left(\epsilon^\dagger \Gamma^0 \Gamma^{jk} \Gamma^{ai} [\Phi^j, \Phi^k] (D_a\Phi^i)\epsilon - \epsilon^\dagger \Gamma^0 \Gamma^{ai} \Gamma^{jk} (D_a\Phi^i)[\Phi^j, \Phi^k]\epsilon \right) \quad (\text{A.1.15})$$

and by following the same steps, will vanish, since $\text{Tr} \alpha^i\alpha^j\alpha^k \sim \varepsilon^{ijk}$ and the Γ -matrix commutator is $[\Gamma^i, \Gamma^{jk}] = (\delta^{ij}\Gamma^k - \delta^{ik}\Gamma^j)$. Collecting (A.1.10) and (A.1.14) we have

$$\epsilon^\dagger \Gamma^0 \epsilon \left((\partial^a\hat{R})(\partial_a\hat{R}) - 4 \hat{R}^4 \right) = 0$$

which implies

$$(\partial^a\hat{R})(\partial_a\hat{R}) - 4 \hat{R}^4 = 0 \quad (\text{A.1.16})$$

By employing dimensionless variables, the supersymmetry-preserving condition is simply

$$(\partial_\mu r)(\partial^\mu r) = r^4, \quad \mu = \sigma, \tau \quad (\text{A.1.17})$$

Thus, every configuration that satisfies (A.1.17) qualifies as a BPS state and therefore will be semi-classically stable. The special case $\hat{R}' = 2(\hat{R})^2$ was used in [60] and is related to Nahm's equations.

We will briefly look at some solutions of this first-order equation. It is not hard to see that the simplest ones can be put into the form

$$\hat{R}(\sigma, \tau) = \pm \frac{1}{2} \frac{1}{\gamma(\sigma - \sigma_\infty + \beta t)} \quad (\text{A.1.18})$$

where σ_∞ a constant, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and β another constant, which can be thought of as the ratio of the velocity of the system over the speed of light $c = 1$. Then the solution at hand is nothing but a boost of the static funnel

$$\hat{R} = \pm \frac{1}{2(\sigma - \sigma_\infty)} \quad (\text{A.1.19})$$

Moreover (A.1.18) will satisfy the full non-linear (2.1.8) and is therefore a proper solution of our space and time-dependent system. In this case we can see explicitly how this solution should also satisfy the YM equations of motion. By starting with the BPS-like relation (A.1.17), one gets by differentiation and algebraic manipulation

$$\partial_\mu \partial^\nu r (\partial_\nu r)(\partial^\mu r) = 2r^3 (\partial_\mu r)(\partial^\mu r), \quad (\text{A.1.20})$$

which if substituted in (2.1.11) and again with (A.1.17), will yield exactly $\partial_\mu \partial^\mu r = 2r^3$, which are the YM equations. Relations between the BPS condition, the DBI and YM equations were discussed in [156]. Therefore the class of space and time-dependent supersymmetric solutions to DBI *simultaneously* satisfy the BPS and the YM equations. Interestingly there exist solutions to one of YM or BPS which do not solve the DBI. For example, $r_{YM} = \frac{1}{\sqrt{2(\sigma^2 - \tau^2)}}$ solves the YM equations but not the BPS or DBI and $r_{BPS} = \frac{1}{\sqrt{\sigma^2 - \tau^2}}$ is just a solution to BPS.

The fact that we have recovered, in the context of space *and* time-dependent transverse scalars, a boosted static solution is something that we should have expected: The expression for the Born-Infeld action is, of course, Lorentz invariant. This guarantees that its extrema, and solutions to the equations of motion, although transforming accordingly under boosts, should still be valid solutions at the new points $x' = \Lambda^{-1}x$. The lowest energy configuration of the system is naturally the BPS one and a boost provides a generalisation that is still stable and time-dependent. A natural consequence of this is the boosted brane array, which

is indeed a solution of the space and time-dependent DBI equations of motion

$$r(\sigma, \tau) = \pm r_0 \frac{1}{Cn\left(\frac{\sqrt{2}r_0\gamma(\sigma-\beta\tau)}{\sqrt{r_0^4+1}}, \frac{1}{\sqrt{2}}\right)} \quad (\text{A.1.21})$$

and the boosted collapsing brane

$$r(\sigma, \tau) = \pm r_0 Cn\left(\frac{\sqrt{2}r_0\gamma(\tau-\beta\sigma)}{\sqrt{r_0^4+1}}, \frac{1}{\sqrt{2}}\right) \quad (\text{A.1.22})$$

A.2 Chern-Simons terms and D -brane charges for $D1 \perp D3$ system

We have seen that the equations of motion for a space and time-dependent fuzzy-funnel configuration have a solution both in the $D1$ and the $D3$ picture. The Chern-Simons part of the non-abelian DBI in a flat background, is not relevant in the search for the equations of motion. However, it would be a nice check to see whether the charge calculation also agrees when time-dependence of the scalars is introduced, as happens for the static case [60]. We will be assuming spherical symmetry of the solutions in what follows.

We expect to recover a coupling to a higher dimensional brane through the non-commutative, transverse scalars and the dielectric effect. This is indeed the case and one gets for the $D3$ -charge of the $D1$ configuration for *any* time-dependent solution, at large- N

$$S_{cs}^{D1} = \frac{2\mu_1}{\lambda} \int d\tau d\sigma R_{ph}^2 \left(C_{\sigma 123}^{(4)} \dot{R}_{ph} - C_{\tau 123}^{(4)} R'_{ph} \right) \quad (\text{A.2.1})$$

The Chern-Simons part of the low energy effective $D3$ -brane action reads

$$S_{CS} = \mu_3 \int STr P[C^{(4)}] + \mu_3 \int STr P[C^{(2)}] \wedge F + \mu_3 \int STr P[C^{(0)}] \wedge F \wedge F \quad (\text{A.2.2})$$

By considering a spherical co-ordinate embedding in static gauge, the calculation of the $D3$ charge will give

$$S_{CS}^{D3} = 4\pi\mu_3 \int dt dR_{D3} R_{D3}^2 \left[C_{t123}^{(4)} + C_{9123}^{(4)} \lambda \dot{\Phi} \right] \quad (\text{A.2.3})$$

where R_{D3}, t are world-volume indices and $1, 2, 3$ are space-time indices.

Following [61] we will make the identifications

- The physical radius of the fuzzy S^2 on the $D1$ side should correspond to the co-ordinate R_{D3} on the world-volume of the $D3$

$$R_{ph} \longleftrightarrow R_{D3} \quad (\text{A.2.4})$$

- The σ co-ordinate on the world-volume of the $D1$ should be analogous to the transverse

scalar Φ on the $D3$

$$\sigma \longleftrightarrow \lambda \Phi \quad (\text{A.2.5})$$

- We finally assume

$$\tau \longleftrightarrow t \quad (\text{A.2.6})$$

By thinking about the physical radius of the fuzzy sphere as a function of σ and t , using the relationships (2.1.17) and also the fact $(2\pi\sqrt{\alpha'})^{p'-p}\mu^{p'} = \mu^p$, we are able to write (A.2.3) in terms of $D1$ world-volume quantities. Everything works out nicely and one gets $S_{CS}^{D1} = S_{CS}^{D3}$.

APPENDIX B

THE COLLAPSING D_2 RADIAL PROFILE FROM \wp -FUNCTIONS

Here is a warm-up which makes use of the technology employed for the solution of the Jacobi Inversion problem, in terms of Weierstrass \wp -functions. We use this to recover the familiar result from the 2-sphere collapse.

The integral related to the time of collapse from an initial radius r_0 and in dimensionless variables is given in Ch. 2 by the expression

$$\int_{r_0}^{r'} \frac{\sqrt{1+r_0^4}}{\sqrt{r_0^4-r^4}} dr = -t \quad (\text{B.0.1})$$

After a re-scaling $\tilde{r} = \frac{r}{r_0}$ we have

$$\int_1^{\tilde{r}'} \frac{\sqrt{1+r_0^4}}{r_0 \sqrt{1-\tilde{r}^4}} d\tilde{r} = -t \quad (\text{B.0.2})$$

and

$$\int_1^{\tilde{r}'} \frac{d\tilde{r}}{\sqrt{\tilde{r}^4-1}} = -i \frac{r_0 t}{\sqrt{1+r_0^4}} = u_1 \quad (\text{B.0.3})$$

Next we will perform the substitution $\tilde{r}^2 = x$ ending up with

$$u_1 = \int_1^{x'} \frac{dx}{\sqrt{4x(x^2-1)}} \quad (\text{B.0.4})$$

This is a definite integral over the holomorphic differential of the elliptic curve

$$y^2 = 4x^3 - 4x \quad (\text{B.0.5})$$

with branch points at $x = 0, \pm 1$ and infinity.

It is easy to see that the former integral can be decomposed into the following two

$$u_1 = \int_1^{\infty} \frac{dx}{\sqrt{4x(x^2-1)}} - \int_{x'}^{\infty} \frac{dx}{\sqrt{4x(x^2-1)}} \quad (\text{B.0.6})$$

which are exactly the real half-period Ω of the surface and the inverse of the Weierstrass elliptic function $\wp^{-1}(x', g_2, g_3)$ with invariants $g_2 = 4, g_3 = 0$. Both the real and the

imaginary half-period (Ω, Ω') can be calculated by contour integration after we have defined a homology basis on the surface. In this case take the a -cycle to be the loop surrounding the points $(1, \infty)$ (or equivalently, by deformation, the points $(-1, 0)$) and the b -cycle the loop around $(0, 1)$ and across the two sheets. The results are

$$\Omega = \int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}} \quad \text{and} \quad \Omega' = i \int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}} \quad (\text{B.0.7})$$

with s a real, positive integration parameter. The modulus of the torus is then simply given by $\tau = \frac{\Omega'}{\Omega} = i$. There exists an alternative definition for this, namely $\tau = i \frac{K(k)}{K(k')}$, where $K(k)$ is the complete elliptic integral of the first kind and k, k' the elliptic modulus and the complementary modulus respectively, with $k^2 + k'^2 = 1$. This implies $K(k) = K(k')$ and $k = k' = 1/\sqrt{2}$ for the functions and integrals defined on this surface with this particular choice of cycles and cuts.

From the definition of $\wp(u)$ one can derive several relations between the latter and the Jacobi elliptic functions [97]. Take a general curve

$$\wp^{-1}(w) \equiv v = \int_w^\infty \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = \int_w^\infty \frac{dz}{\sqrt{4(z - e_1)(z - e_2)(z - e_3)}} \quad (\text{B.0.8})$$

and by setting

$$\gamma^2 = e_1 - e_3, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3} \quad \text{and} \quad z = e_3 + \frac{\gamma^2}{s^2}, \quad \text{for} \quad e_1 > e_2 > e_3 \quad (\text{B.0.9})$$

the integral becomes

$$v = \frac{1}{\gamma} \int_0^W \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \quad (\text{B.0.10})$$

This is simply the definition for the Jacobi elliptic integral of the first kind and is invertible with $Sn^{-1}(W) = \gamma v$. If we use this fact and (B.0.9) we get

$$\wp(v) = e_3 + \frac{\gamma^2}{Sn^2(\gamma v, k)} \quad (\text{B.0.11})$$

and for the example at hand, $e_1 = 1, e_2 = 0, e_3 = -1$ and the relationship becomes

$$\wp(v) = -1 + \frac{2}{Sn^2(\sqrt{2}v, \frac{1}{\sqrt{2}})} \quad (\text{B.0.12})$$

Returning to (B.0.6) we have

$$u_1 = \Omega - \wp^{-1}(x'; 4, 0) \quad (\text{B.0.13})$$

and

$$\begin{aligned} x' &= \wp(u_1 - \Omega; 4, 0) \\ &= 1 + \frac{2}{\wp(u_1; 4, 0) - 1} \end{aligned} \quad (\text{B.0.14})$$

where we have made use of the identity

$$\wp(v \pm \Omega) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(v) - e_1} \quad (\text{B.0.15})$$

Now from (B.0.12) and by using the following properties of elliptic functions

$$Cn(v, k) = \frac{1}{Cn(iv, k')} \quad \text{and} \quad Sn^2(v, k) + Cn^2(v, k) = 1 \quad (\text{B.0.16})$$

we obtain

$$x' = Cn^2 \left(\sqrt{2}iu_1, \frac{1}{\sqrt{2}} \right)$$

Converting back to the original quantities $x' = \tilde{r}'^2 = r'^2/r_0^2$ and by substituting $u_1 = -ir_0t/\sqrt{1+r_0^4}$, we recover the desired result

$$r' = r_0 Cn \left(\frac{r_0\sqrt{2}t}{\sqrt{1+r_0^4}}, \frac{1}{\sqrt{2}} \right) \quad (\text{B.0.17})$$

APPENDIX C

GENERAL FORMULA FOR THE SYMMETRISED TRACE

As in Ch. 4, we define $N(k, n)$ to be the dimension of the irreducible representation of $SO(2k+1)$ with Dynkin label $(\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2})$. These are the usual fuzzy sphere representations [48, 70] (for example, for $k=1$ the X_i are the elements of the Lie algebra of $SU(2)$ in the irreducible representation with spin $\frac{n}{2}$). Then

$$N(k, n) = \prod_{1 \leq i < j \leq k} \frac{n + 2k - (i + j) + 1}{2k - (i + j) + 1} \prod_{l=1}^k \frac{n + 2k - 2l + 1}{2k - 2l + 1} \quad (\text{C.0.1})$$

The symmetrised trace is defined to be the normalised sum over permutations of the matrices

$$STr(X_{i_1} \dots X_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} Tr(X_{\sigma(1)} \dots X_{\sigma(p)}) \quad (\text{C.0.2})$$

We have given earlier a conjecture for the symmetrised trace of m powers of the quadratic Casimir $X_i X_i = C \mathbb{1}_{N \times N}$, where $C = n(n + 2k)$. This is, for all m, k and n even

$$\frac{1}{N(k, n)} STr(X_i X_i)^m = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k-1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \sum_{i_3=1}^{\frac{n}{2}} \left[\prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - i_3^2 \right) (2i_3)^{2m} \right] \quad (\text{C.0.3})$$

where for $k = 1$ the product over $i_4 = 1, \dots, k-1$ is just defined to be equal to 1. Similarly for all m, k and for n odd we have proposed that

$$\frac{1}{N(k, n)} STr(X_i X_i)^m = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k-1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \sum_{i_3=1}^{\frac{n+1}{2}} \left[\prod_{i_4=1}^{k-1} \left(\left(\frac{n}{2} + i_4 \right)^2 - \left(i_3 - \frac{1}{2} \right)^2 \right) (2i_3 - 1)^{2m} \right] \quad (\text{C.0.4})$$

The argument leading to these conjectures follows. Firstly, it is possible to view the fuzzy sphere matrices X_i as the transverse co-ordinates of the world-volume theory of a stack of $D1$ -branes expanding into a stack of $D(2k+1)$ -branes [60–62]. There is also a dual realisation of this system in which the $D1$ -branes appear as a monopole in the world-volume theory of the $D(2k+1)$ -branes. The ADHM construction [157] can be used to construct the monopole dual to the fuzzy sphere transverse co-ordinates. If one takes the $N(k, n)$ -dimensional fuzzy sphere matrices representing a stack of $N(k, n)$ $D1$ -branes as ADHM data for a monopole,

then one naturally constructs a monopole defined on a stack of $N(k-1, n+1)$ $D(2k+1)$ -branes. The calculation showing that the charge for the monopole just constructed gives precisely $N(k, n)$ has also been done and provides a consistency check (this calculation extends some results in [72] and appears in [122]).

It is also possible to calculate the number of $D(2k+1)$ branes from the fuzzy sphere Ansatz for the transverse co-ordinates, by looking at the RR coupling on the D -string world-volume. In this case one does not get $N(k-1, n+1)$ as one would expect, but instead the quantity

$$N(k-1, n+1) \frac{\prod_{i=1}^{2k-1} (n+i)}{C^{k-\frac{1}{2}}} \quad (\text{C.0.5})$$

where $C = n(n+2k)$. This number of branes does agree with $N(k-1, n+1)$ for the first two orders in the large- n expansion

$$\text{Number of } D(2k+1)\text{-branes} = N(k-1, n+1) \left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \quad (\text{C.0.6})$$

Now consider this RR charge calculation more carefully. First take the $k=1$ case. Based on the ADHM construction we expect the number of $D3$ -branes to be $N(0, n+1) = 1$. However, eq. (C.0.5) suggests that the charge calculation gives for $k=1$ the answer

$$\frac{n+1}{C^{\frac{1}{2}}} \quad (\text{C.0.7})$$

Suppose that the numerator in the above is correct, but that the denominator is correct only at large- n and that it receives corrections at lower order to make the number of $D3$ -branes exactly one. Then these corrections need to satisfy

$$1 = (n+1)(C^{-\frac{1}{2}} + x_1 C^{-\frac{3}{2}} + x_2 C^{-\frac{5}{2}} + \dots) \quad (\text{C.0.8})$$

It is easy to show that we need $x_1 = -\frac{1}{2}$ and $x_2 = \frac{3}{8}$, by Taylor expanding and using that $C = n(n+2)$ for $k=1$. Therefore, we would like to have a group theoretic justification for the series

$$C^{-\frac{1}{2}} - \frac{1}{2}C^{-\frac{3}{2}} + \frac{3}{8}C^{-\frac{5}{2}} + \dots \quad (\text{C.0.9})$$

There exists a formula for the first three terms in the large- n expansion of the $k=1$ symmetrised trace operator [63], namely

$$\frac{1}{N(1, n)} STr(X_i X_i)^m = C^m - \frac{2}{3}m(m-1)C^{m-1} + \frac{2}{45}m(m-1)(m-2)(7m-1)C^{m-2} + \dots \quad (\text{C.0.10})$$

Now, if we make the choice $m = -\frac{1}{2}$ in (C.0.10) we get precisely (C.0.9). However, this suggests that if this choice is correct, then we should have an all orders prediction for the

action of the symmetrised trace operator. Thus, for $k=1$ we predict that

$$\frac{1}{N(1, n)} STr(X_i X_i)^m \Big|_{m=-\frac{1}{2}} \simeq \frac{1}{(n+1)} \quad (\text{C.0.11})$$

where for future reference we consider the left hand side to be equal to the symmetrised trace in a large- n series expansion, as appeared in [63].

Checking the conjecture (C.0.11) beyond the first three terms in a straightforward fashion, by techniques similar to those employed in [63], proves difficult. This involves either adding up a large number of chord diagrams, or complicated combinatorics if one uses the highest weight method.

An alternative approach involves first writing down the conjecture based on brane counting for general k , since the methods of [63] turn out to generalise from the $k=1$ to the general k case. The conjecture for general k , based on the brane counting, follows immediately from (C.0.5)

$$\frac{1}{N(k, n)} STr(X_i X_i)^m \Big|_{m=-k+\frac{1}{2}} \simeq \prod_{i=1}^{2k-1} \frac{1}{(n+i)} \quad (\text{C.0.12})$$

Note that the right hand side of this equation appears in the factor outside the sum in (C.0.3) and (C.0.4). Notice also that the above expression concerns the large- n expansion of the symmetrised trace considered at $m = -k + \frac{1}{2}$.

One can repeat the $k=1$ calculation of [63] for general k , to check the first three terms of this conjecture. A sketch of this calculation follows before displaying the full results. First we calculate $\frac{1}{N(k, n)} STr(X_i X_i)^m$ for $m = 2, 3, 4$. Then we find the first three terms in the symmetrised trace, large- n expansion using these results. Finally we can check that the conjecture (C.0.12) is true for the first three terms in the symmetrised trace large- n expansion, for general k as well as for $k=1$. We then proceed to calculate the fourth term in the expansion, for general k . To do this we need to calculate $\frac{1}{N(k, n)} STr(X_i X_i)^m$ for $m = 5, 6$. We then show that the fourth term in the large- n expansion of the symmetrised trace agrees precisely with (C.0.12).

In the following, we use the notation of [63], with each trace of a string of $2m$ X_i matrices arising here being represented by a chord diagram with m chords. This provides a convenient way to represent equivalent strings of matrices.

For the calculation of $STr(X_i X_i)^2$ there are three different strings of the four X_i matrices and two different chord diagrams. Two of the three strings correspond to the same chord diagram. In the following, the first column contains a fraction which is the multiplicity of the chord diagram in the list of strings divided by the total number of strings. The second column contains a picture of the chord diagram preceded by an example of a string in the equivalence class defined by this chord diagram. The evaluation of the chord diagram is the

final entry.

$$\begin{aligned} \frac{2}{3} \quad 1122 &= \text{Diagram with two vertical chords} = C^2 \\ \frac{1}{3} \quad 1212 &= (C - 4k) \text{Diagram with four chords} = C(C - 4k) \end{aligned}$$

Using this, one finds immediately that

$$\frac{1}{N(k, n)} STr(X_i X_i)^2 = C^2 - \frac{4}{3} Ck \quad (\text{C.0.13})$$

For $m = 3$ there are 15 different strings of X_i matrices and five different chord diagrams, which evaluate as follows

$$\begin{aligned} \frac{2}{15} \quad 112233 &= \text{Diagram with three chords} = C^3 \\ \frac{6}{15} \quad 112323 &= (C - 4k) \text{Diagram with four chords} = C^2(C - 4k) \\ \frac{3}{15} \quad 112332 &= \text{Diagram with three chords} = C^3 \\ \frac{3}{15} \quad 121323 &= (C - 4k) \text{Diagram with four chords} = C(C - 4k)^2 \\ \frac{1}{15} \quad 123123 &= \text{Diagram with six chords} = C^3 - 12kC^2 + 16k(k+1)C \end{aligned}$$

Thus we find that

$$\frac{1}{N(k, n)} STr(X_i X_i)^3 = C^3 - 4kC^2 + \frac{16}{15}k(4k+1)C \quad (\text{C.0.14})$$

For $m = 4$ there are 105 different strings and 18 different chord diagrams¹. We omit the details for simplicity. The final result is that

$$\frac{1}{N(k, n)} STr(X_i X_i)^4 = C^4 - 8kC^3 + \frac{16}{5}k(7k+2)C^2 - \frac{64}{105}k(34k^2 + 24k + 5)C \quad (\text{C.0.15})$$

For $m = 5$ there are 945 different strings of X_i matrices and 105 different chord diagrams.

¹We acknowledge the assistance of Simon Nickerson, for writing a computer programme used from this point onwards.

The result is

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^5 &= C^5 - \frac{40}{3}kC^4 + \frac{16}{3}(13k + 4)kC^3 - \frac{64}{63}(158k^2 + 126k + 31)kC^2 \\ &\quad + \frac{256}{945}(496k^3 + 672k^2 + 344k + 63)kC \end{aligned} \quad (\text{C.0.16})$$

For $m=6$ there are 10395 different strings of X_i matrices, and 902 different chord diagrams, and we find that

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^6 &= C^6 - 20kC^5 + \frac{16}{3}(31k + 10)kC^4 \\ &\quad - \frac{64}{63}(677k^2 + 582k + 157)kC^3 \\ &\quad + \frac{256}{315}(1726k^3 + 2616k^2 + 1541k + 336)kC^2 \\ &\quad - \frac{1024}{10395}(11056k^4 + 24256k^3 + 22046k^2 + 9476k + 1575)kC \end{aligned} \quad (\text{C.0.17})$$

Now we calculate the first four terms in the large- n expansion of $STr(X_i X_i)^m$. Suppose that the coefficient of C^{m-l} term in $STr(X_i X_i)^m$ is a polynomial in m of order $2l$. Then we have the following Ansatz: The known factors of these polynomials come from the fact that the series has to terminate so that there are never negative powers of C for $m=1, 2, 3, \dots$. Then

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^m &= C^m + y_1(k)m(m-1)C^{m-1} \\ &\quad + (y_2(k)m + y_3(k))m(m-1)(m-2)C^{m-2} \\ &\quad + (y_4(k)m^2 + y_5(k)m + y_6(k))m(m-1)(m-2)(m-3)C^{m-3} \\ &\quad + \mathcal{O}(C^{m-4}) \end{aligned}$$

The unknown functions $y_1(k), y_2(k), \dots, y_6(k)$ are found using the results of $STr(X_i X_i)^m$ for $m = 2, 3, 4, 5, 6$ calculated above. We get

$$\begin{aligned} y_1(k) &= -\frac{2}{3}k, \quad y_2(k) = \frac{2}{45}(5k+2)k \\ y_3(k) &= \frac{2}{45}(k-2)k, \quad y_4(k) = \frac{1}{2835}(-140k^2 - 168k - 64)k \\ y_5(k) &= \frac{1}{2835}(-84k^2 + 216k + 192)k, \quad y_6(k) = \frac{1}{2835}(128k^2 + 96k - 104)k \end{aligned} \quad (\text{C.0.18})$$

With this knowledge we are able to provide a check of the conjecture (C.0.12). First we

express the right-hand side of (C.0.12) as a function of C

$$\prod_{l=1}^{2k-1} \frac{1}{n+l} = \frac{1}{\sqrt{(k^2+C)}} \prod_{l=1}^{k-1} \frac{1}{C+2kl-l^2} = C^{-k+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{b_j}{C^j} \quad (\text{C.0.19})$$

where

$$\begin{aligned} b_0 &= 1 \\ b_1 &= -\frac{2}{3}k\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right) \\ b_2 &= \frac{1}{45}(10k^2-3k+2)k\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right) \\ b_3 &= \frac{1}{2835}(-24+34k-61k^2+56k^3-140k^4) \times \\ &\quad k\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)\left(k+\frac{5}{2}\right) \end{aligned} \quad (\text{C.0.20})$$

Now consider the left-hand side of (C.0.12) involving the large- n expansion of the expression $STr(X_i X_i)^m$, which we calculated above, but now we set $m = -k + \frac{1}{2}$. Expanding in inverse powers of C , we find that this becomes

$$\frac{1}{N(k, n)} STr(X_i X_i)^m \Big|_{m=-k+1/2} = C^{-k+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{b_j}{C^j}$$

with precisely the coefficients b_i given in (C.0.20). Given the extensive and non-trivial calculations required to obtain these results, we believe that there is strong evidence for the truth of (C.0.12).

For $k = 1$ the guess of the exact answer for n even is

$$\frac{1}{N(1, n)} STr(X_i X_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{\frac{n}{2}} (2i)^{2m} \quad (\text{C.0.21})$$

It is easy to show that (C.0.21) agrees with the first four orders in the large- n expansion (C.0.18) for $k = 1$. If we set $m = -\frac{1}{2}$ in this we get zero, because of the $(2m+1)$ factor. This might appear to contradict (C.0.11), but it is easy to show, using a large- n expansion, that if (C.0.21) is true then (C.0.11) holds to all orders. To calculate the large- n expansion of this sum we can use the Euler-Maclaurin formula. This approximates the sum by an integral, plus an infinite series of corrections involving the Bernoulli numbers B_{2p}

$$\begin{aligned} \sum_{i=1}^n f(i) &\simeq \int_0^{n+1} f(x) dx + \frac{1}{2}[f(n+1) - f(0)] \\ &\quad + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} [f^{(2p-1)}(n+1) - f^{(2p-1)}(0)] \end{aligned} \quad (\text{C.0.22})$$

We see from this calculation that for $k = 1$ the value $m = -\frac{1}{2}$ is very special. It is the only value of m for which the higher order terms in the Euler-MacLaurin large- n approximation of the sum in (C.0.21) are zero.

APPENDIX D

CALCULATION OF STr FROM THE HIGHEST WEIGHT METHOD

Results on finite- n symmetrised traces can be obtained by generalising the highest weight method of [63]. For the $SO(3)$ representations used in fuzzy 2-spheres we have

$$\frac{1}{2} STr_{J=1/2}(\alpha_i \alpha_i)^m = (2m + 1) \quad (\text{D.0.1})$$

where the $1/2$ comes from dividing with the dimension of the spin- $1/2$ representation. A similar factor will appear in all of the results below. The above result was derived in [63]. For the spin-one case, we will obtain

$$\frac{1}{3} STr_{J=1}(\alpha_i \alpha_i)^m = \frac{2^{2m+1}(2m + 1)}{3} \quad (\text{D.0.2})$$

These results can be generalised to representations of $SO(2l + 1)$ relevant for higher fuzzy spheres. The construction of higher dimensional fuzzy spheres uses irreducible representations of highest weight $(\frac{n}{2}, \dots, \frac{n}{2})$, as we have noted. For the minimal representation with $n = 1$ we have

$$\frac{1}{D_{n=1}} STr_{n=1}(X_i X_i) = \frac{(2l + 2m - 1)!!}{(2m - 1)!!(2l - 1)!!} \quad (\text{D.0.3})$$

Notice the interesting symmetry under the exchange of l and m . For the next-to-minimal irreducible representation with $n = 2$ we obtain:

$$\frac{1}{D_{n=2}} STr_{n=2}(X_i X_i) = 2^{2m}(l + 1) \frac{(2l + 2m - 1)!!}{(2m - 1)!!(2l + 1)!!} \quad (\text{D.0.4})$$

This is a generalisation of the spin-one case to higher orthogonal groups. It agrees with the formulae in Sec. C of the Appendix, with $l \rightarrow k$.

D.1 Review of spin-half for $SO(3)$

We will begin by recalling some facts about the derivation of the $n = 1$ case in [63]. The commutation relations can be expressed in terms of α_3, α_{\pm}

$$\begin{aligned}\alpha_{\pm} &= \frac{1}{\sqrt{2}}(\alpha_1 \pm i\alpha_2) \\ [\alpha_3, \alpha_{\pm}] &= \pm 2\alpha_{\pm} \\ [\alpha_+, \alpha_-] &= 2\alpha_3 \\ C &= \alpha_+\alpha_- + \alpha_-\alpha_+ + \alpha_3^2\end{aligned}\tag{D.1.1}$$

With these normalisations, the eigenvalues of α_3 in the spin-half representation are ± 1 and $\alpha_+\alpha_-$ is 1 on the highest weight state.

It is useful to define a quantity $\tilde{C}(p, q)$ which depends on two natural numbers p, q and counts the number of ways of separating p identical objects into q parts

$$\tilde{C}(p, q) = \frac{(p+q-1)!}{p!(q-1)!}\tag{D.1.2}$$

We begin by a review of the spin-half case, establishing a counting which will be used again in more complicated cases below. This relies on a sum

$$2^k \sum_{\hat{i}_{2k}}^{2n-2k} \cdots \sum_{\hat{i}_2=0}^{\hat{i}_3} \sum_{\hat{i}_1=0}^{\hat{i}_2} (-1)^{\hat{i}_1 + \hat{i}_2 + \cdots + \hat{i}_{2k}} = 2^k \frac{n!}{(n-k)!k!}$$

Recall that this sum was obtained by evaluating a sequence of generators of $SO(3)$ consisting of k pairs $\alpha_-\alpha_+$ and with powers of α_3 between these pairs

$$\alpha_3^{J_{2k+1}} \alpha_+ \alpha_3^{J_{2k}} \alpha_- \cdots \alpha_- \alpha_3^{J_3} \alpha_+ \alpha_3^{J_2} \alpha_- \alpha_3^{J_1}\tag{D.1.3}$$

We can move the powers of α_3 to the left to get factors $(\alpha_3 - 2)^{J_2 + J_4 + \cdots + J_{2k}}$. Moving the α_3 with powers $J_1, J_3 \dots$ gives $\alpha_3^{J_1 + J_3 + \cdots}$. The k powers of $\alpha_-\alpha_+$ gives 2^k . The above sum can be re-written

$$2^k \sum_{J_{2k+1}=0}^{2m-2k} \cdots \sum_{J_2=0}^{2m-2k-(J_3+\cdots+J_{2k+1})} \sum_{J_1=0}^{2m-2k-(J_2+\cdots+J_{2k+1})} (-1)^{J_2 + J_4 + \cdots + J_{2k}} = 2^k \frac{n!}{(n-k)!k!}\tag{D.1.4}$$

This includes a sum over $J_e = J_2 + J_4 + \dots + J_{2k}$. The summand does not depend on the individual J_2, J_4, \dots but only on the sum J_e which ranges from 0 to $2m - 2k$. The sum over J_2, J_4, \dots is the combinatoric factor, introduced above, which is the number of ways of splitting J_e identical objects into k parts, *i.e.* $\tilde{C}(J_e, k)$. The remaining $2m - 2k - J_e$ powers

of α_3 are distributed in $k + 1$ slots in $\tilde{C}(2m - 2k - J_e, k + 1)$ ways. Hence the sum (D.1.4) can be written more simply as

$$2^k \sum_{J_e=0}^{2m-2k} (-1)^{J_e} \tilde{C}(J_e, k) \tilde{C}(2m - 2k - J_e, k + 1) = 2^k \frac{m!}{(m - k)!k!} \quad (\text{D.1.5})$$

Then there is a sum over k from 0 to m , with weight

$$C(k, m) = \frac{2^k k! (2m - 2k)! m!}{(m - k)! (2m)!} \quad (\text{D.1.6})$$

which gives the final result $2m + 1$ [63]. Similar sums arise in the proofs below. In some cases, closed formulae for the sums are obtained experimentally.

D.2 Derivation of symmetrised trace for the minimal $SO(2l+1)$ representation

The Casimir of interest here is

$$X_\mu X_\mu = X_{2l+1}^2 + \sum_{i=1}^l \left(X_-^{(i)} X_+^{(i)} + X_+^{(i)} X_-^{(i)} \right) \quad (\text{D.2.1})$$

The patterns are similar to those above, with α_3 replaced by X_{2l+1} , and noting that here there are l “colours” of α_\pm which are $X_\pm^{(l)}$. All the states in the fundamental spinor are obtained by acting on a vacuum which is annihilated by l species of fermions. Generally we might expect patterns

$$\dots X_{2l+1}^{i_1} X_-^{(j_1)} X_{2l+1}^{i_2} X_+^{(j_2)} \dots \quad (\text{D.2.2})$$

In evaluating these, we can commute all the X_{2l+1} to the left. This results in shifts which do not depend on the value of j . It is easy to see that whenever $X_+^{(1)}$ is followed by $X_+^{(1)}$ we get zero because of the fermionic construction of the gamma matrices. $X_+^{(1)}$ cannot also be followed by $X_+^{(2)}$ because $X_+^{(1)} X_+^{(2)} + X_+^{(2)} X_+^{(1)} = 0$. So the pairs have to take the form $X_-^{(j)} X_+^{(j)}$ for fixed j . The sum we have to evaluate is

$$\begin{aligned} & \sum_{k=0}^m \sum_{J_e=0}^{2m-2k} (-1)^{J_e} \tilde{C}(J_e, k) \tilde{C}(2m - 2k - J_e, k) \tilde{C}(k, l) 2^k C(k, m) \\ &= \sum_{k=0}^m \binom{m}{k} 2^k C(k, m) \tilde{C}(k, l) \\ &= \frac{(2l + 2m - 1)!!}{(2m - 1)!!(2l - 1)!!} \end{aligned} \quad (\text{D.2.3})$$

The factors $\tilde{C}(J_e, k)$ and $\tilde{C}(2m - 2k - J_e, k)$ have the same origin as in the spin-half case. The factor (D.1.6) is now generalised to an l -colour version

$$C(k_1, k_2 \dots k_l; m) = 2^k \frac{(2m - 2k)!}{(2m)!} \frac{m!}{(m - k)!} k_1! k_2! \dots k_l! \quad (\text{D.2.4})$$

This has to be summed over k_1, \dots, k_l . For fixed $k = k_1 + \dots + k_l$ we have

$$\sum_{k_1 \dots k_l} C(k_1 \dots k_l, m) \frac{k!}{k_1! \dots k_l!} = \sum_{k_1 \dots k_l} C(k, m) = C(k, m) \tilde{C}(k, l) \quad (\text{D.2.5})$$

The combinatoric factor $\frac{k!}{k_1! \dots k_l!}$ in the second line above comes from the different ways of distributing the $k_1 \dots k_l$ pairs of $(-+)$ operators in the k positions along the line of operators. The subsequent sum amounts to calculating the number of ways of separating k objects into l parts which is given by $\tilde{C}(k, l)$. The $C(k, m)$ is familiar from (D.1.6). This sum can be done for various values of k, m and gives agreement with (D.0.3).

D.3 Derivation of spin-one symmetrised trace for $SO(3)$

For the spin-one case more patterns will arise. After an α_- acts on the highest weight, we get a state with $\alpha_3 = 0$ so that we have, for any positive r

$$\alpha_3^r \alpha_- |J = 1, \alpha_3 = 2\rangle = 0, \quad \forall r > 0 \quad (\text{D.3.1})$$

Hence any α_- can be followed immediately by α_+ . These neutral pairs of $(\alpha_+ \alpha_-)$ can be separated by powers of α_3 . Alternatively an α_- can be followed immediately by α_- . The effect of α_-^2 is to change the highest weight state to a lowest weight state. In describing the patterns we have written the “vacuum changing operator” on the second line, with the first line containing only neutral pairs separated by α_3 ’s. Let there be J_1 neutral pairs in this first line and L_1 powers of α_3 distributed between them. After the change of vacuum, we can have a sequence of $(\alpha_- \alpha_+)$ separated by powers of α_3 . Let there be a total of J_2 neutral pairs and L_2 α_3 ’s in the second line. At the beginning of the third line we have another vacuum changing operator α_+^2 which takes us back to the highest weight state. In the third line, we have J_3 neutral pairs and L_3 powers of α_3 . The equation below describes a general pattern with p pairs of vacuum changing operators. The total number of neutral pairs is $2p + J$ where $J = J_1 + J_2 + \dots + J_{2p+1}$. The general pattern of operators acting on

the vacuum is

$$\begin{aligned}
 & \# (\alpha_+ \alpha_-) \# (\alpha_+ \alpha_-) \# \cdots \# (\alpha_+ \alpha_-) \# \quad |J = 1, \alpha_3 = 2 > \\
 & \# (\alpha_- \alpha_+) \# (\alpha_- \alpha_+) \# \cdots \# (\alpha_- \alpha_+) \# \alpha_-^2 \\
 & \# (\alpha_+ \alpha_-) \# (\alpha_+ \alpha_-) \# \cdots \# (\alpha_+ \alpha_-) \# \alpha_+^2 \\
 & \quad \vdots \\
 & \# (\alpha_- \alpha_+) \# (\alpha_- \alpha_+) \# \cdots \# (\alpha_- \alpha_+) \# \alpha_-^2 \\
 & \# (\alpha_+ \alpha_-) \# (\alpha_+ \alpha_-) \# \cdots \# (\alpha_+ \alpha_-) \# \alpha_+^2
 \end{aligned} \tag{D.3.2}$$

where in the above the first line of operators acts on the state $|J = 1, \alpha_3 = 2 >$ first, then the second line acts, and so on. The symbols $\#$ represent powers of α_3 . We define $J_e = J_2 + J_4 + \cdots + J_{2p}$ which is the total number of $(-+)$ pairs on the even lines above. There is a combinatoric factor $\tilde{C}(J_e, p)$ for distributing J_e amongst the p entries, and a similar $\tilde{C}(J - J_e, p + 1)$ for the odd lines. The $L_e = L_2 + L_4 + \cdots + L_{2p}$ copies of α_3 can sit in $(J_2 + 1) + (J_4 + 1) + \cdots + (J_{2p} + 1)$ positions which gives a factor of $\tilde{C}(L_e, J_e + p)$. The $L_1 + L_3 + \cdots + L_{2p+1}$ can sit in $(J_1 + 1) + (J_3 + 1) + \cdots + (J_{2p+1} + 1) = J - J_e + p + 1$ positions, giving a factor $\tilde{C}(2m - 2J - 4p - L_e, J - J_e + p + 1)$. There is finally a factor $C(2p + J, m)$ defined in (D.1.6) which arises from the number of different ways the permutations of $2m$ indices can be specialised to yield a fixed pattern of $\alpha_+, \alpha_-, \alpha_3$

$$\begin{aligned}
 & \sum_{p=0}^{[m/2]} \sum_{J=0}^{m-2p} \sum_{J_e=0}^J \sum_{L_e=0}^{2m-4p-2J} \tilde{C}(J_e, p) \tilde{C}(J - J_e, p + 1) (-1)^{L_e} \tilde{C}(L_e, J_e + p) \times \\
 & \quad \tilde{C}(2m - 2J - 4p - L_e, J - J_e + p + 1) \times \\
 & \quad 2^{2m-2J-4p} Q(1, 1)^{J-J_e} Q(2, 1)^{J_e} Q(2, 2)^p C(2p + J, m)
 \end{aligned}$$

By doing the sums (using Maple for example) for various values of m we find $\frac{2^{2m+1}(2m+1)}{3}$. The factors $Q(i, j)$, denoted in [63] by $N(i, j)$, arise from evaluating the α_-, α_+ on the highest weight.

D.4 Derivation of next-to-minimal representation for $SO(2l + 1)$

The $n = 2$, general l patterns are again similar to the $n = 2, l = 1$ case except that the α_-, α_+ are replaced by coloured objects of l colours, *i.e.* the $X_{\pm}^{(j)}$. We also have the simple replacement of α_3 by X_{2l+1} .

We define linear combinations of the gamma matrices which are simply related to a set of l fermionic oscillators: $\Gamma_+^{(i)} = \frac{1}{\sqrt{2}}(\Gamma_{2i-1} + i\Gamma_{2i}) = \sqrt{2}a_i^\dagger$ and $\Gamma_-^{(i)} = \frac{1}{\sqrt{2}}(\Gamma_{2i-1} - i\Gamma_{2i}) = \sqrt{2}a_i$.

As usual X_i are expressed as operators acting on an n -fold tensor product, and

$$X_{\pm}^{(i)} = \sum_r \rho_r(\Gamma_{\pm}^{(i)}) \quad (\text{D.4.1})$$

Some useful facts are

$$\begin{aligned} X_{2l+1}^r X_+ |0\rangle &= 0, & X_{2l+1}^r X_+^2 |0\rangle &= (-2)^r X_+^2 |0\rangle \\ X_{2l+1}^r X_- X_+ |0\rangle &= X_- X_+ X_{2l+1}^r |0\rangle = (2)^r X_- X_+ |0\rangle \\ X_- X_+ X_{2l+1}^r X_+^2 |0\rangle &= 0, & Y_+ X_+^2 + X_+^2 Y_+ |0\rangle &= 0 \\ X_+ Y_+ X_+ |0\rangle &= 0, & X_- Y_+ X_+ |0\rangle &= 0 \\ X_+ X_- X_+^2 |0\rangle &= Q(2, 1) X_+^2 |0\rangle, & X_+ X_- X_+ Y_+ |0\rangle &= Q(2, 1) X_+ Y_+ |0\rangle \\ X_-^2 X_+^2 |0\rangle &= Q(2, 2) |0\rangle, & Y_- X_- X_+ Y_+ |0\rangle &= Q(2, 2) |0\rangle \end{aligned}$$

It is significant that the same $Q(2, 1), Q(2, 2)$ factors appear in the different places in the above equation. In the above X_+ stands for any of the l $X_+^{(i)}$'s. Any equation containing X_{\pm} and Y_{\pm} stands for any pair $X_{\pm}^{(i)}$ and $X_{\pm}^{(j)}$ for i, j distinct integers from 1 to l .

The general pattern is similar to (D.3.2) with the only difference that the $(\alpha_- \alpha_+)$ on the first line is replaced by any one $(X_-^{(i)} X_+^{(i)})$ for $i = 1, \dots, l$. The positive vacuum changing operators can be $(X_+^{(i)} X_+^{(j)})$, where i, j can be identical or different. For every such choice the allowed neutral pairs following them are $X_+^{(j)} X_-^{(i)}$ and the dual vacuum changing operator is $(X_-^{(j)} X_-^{(i)})$.

The summation we have to do is

$$\sum_{p=0}^{[m/2]} \sum_{J=0}^{m-2p} \sum_{L_e=0}^{2m-4p-2J} \sum_{J_e=0}^J \left(\begin{aligned} & C(2p+J, m) \tilde{C}(2p+J, l) \tilde{C}(J_e, p) \tilde{C}(J-J_e, p+1) \times \\ & (-1)^{L_e} \tilde{C}(L_e, J_e+p) \tilde{C}(2m-2J-4p-L_e, J-J_e+p+1) \times \\ & 2^{2m-2J-4p} Q(1, 1)^{J-J_e} Q(2, 1)^{J_e} Q(2, 2)^p \end{aligned} \right)$$

The Q -factors can be easily evaluated on the highest weight and then inserted into the above

$$Q(1, 1) = 4, \quad Q(2, 1) = 4 \quad Q(2, 2) = 16 \quad (\text{D.4.2})$$

By computing this for several values of m, l , we obtain (D.0.4). Note that both the $l = 1$ and the general l case will yield the correct value for $m = 0$, which is 1.

APPENDIX E

THE LARGE- n LIMIT OF THE FUZZY- S^3 PROJECTED ALGEBRA

In [49] it was proposed that there is a simple prescription for obtaining the space of functions on S^{2k-1} by performing a projection of the Matrix algebra onto symmetric and traceless representations of $SO(2k)$. The remaining representations should also be invariant under a \mathbb{Z}_2 action, which interchanges the positive and negative chiralities. This process projects out the Y_i 's, X_{ij}^\pm 's and Y_{ij}^\pm 's, while leaving the X_i 's and their symmetric products. The projected Matrix algebra is non-associative but commutative at finite- n , as is the case for even dimensional fuzzy spheres. Here we will show that in the large- n limit non-associativity persists, unlike the case of S^{2k} for which it vanishes.

We will begin by calculating the simplest associator, $X_i * X_j * X_k$, where $*$ stands for the standard non-associative product. This is

$$(X_i * X_j) * X_k - X_i * (X_j * X_k) \quad (\text{E.0.1})$$

with the implementation of the projection performed every time a product is calculated. The first matrix product gives

$$X_i \cdot X_j = \mathcal{P}_{\mathcal{R}_n^+} \left[\sum_{r=s} \rho_r(\Gamma_i \Gamma_j P_+) + \sum_{r \neq s} \rho_r(\Gamma_i P_-) \rho_s(\Gamma_j P_+) \right] \mathcal{P}_{\mathcal{R}_n^+} + (+ \leftrightarrow -) \quad (\text{E.0.2})$$

and after the projection

$$\begin{aligned} X_i * X_j &= \mathcal{P}_{\mathcal{R}_n^+} \sum_{r=s} \rho_r(\delta_{ij} P_+) \mathcal{P}_{\mathcal{R}_n^+} + \frac{1}{2} \mathcal{P}_{\mathcal{R}_n^+} \sum_{r \neq s} \rho_r(\Gamma_i P_-) \rho_s(\Gamma_j P_+) \mathcal{P}_{\mathcal{R}_n^+} \\ &\quad + \frac{1}{2} \mathcal{P}_{\mathcal{R}_n^+} \sum_{r \neq s} \rho_r(\Gamma_j P_-) \rho_s(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} + (+ \leftrightarrow -) \end{aligned} \quad (\text{E.0.3})$$

We then proceed to take the ordinary product with X_k .

Consider the 1-coincidence terms, where the Γ_k acts on the same tensor factor as Γ_i or Γ_j . The symmetric part of the product of Γ 's is clearly kept in the projected product, defined group theoretically above. The antisymmetric part has a traceless piece which transforms according to the Young diagram of row lengths (2, 1). The trace piece transforms in (1, 0) and has to be kept. The decomposition into traceless and trace parts for a 3-index tensor

antisymmetric in two indices is

$$\begin{aligned} A_{i[jk]} &= \left(A_{i[jk]} - \frac{1}{3} \delta_{ij} A_{l[lk]} - \frac{1}{3} \delta_{ik} A_{l[jl]} \right) + \left(\frac{1}{3} \delta_{ij} A_{l[lk]} + \frac{1}{3} \delta_{ik} A_{l[jl]} \right) \\ &= A'_{i[jk]} + \left(\frac{1}{3} \delta_{ij} A_{l[lk]} + \frac{1}{3} \delta_{ik} A_{l[jl]} \right) \end{aligned} \quad (\text{E.0.4})$$

with the normalisation fixed by taking extra contractions of the above. Using this, we obtain from the 1-coincidence terms

$$\frac{(2n+1)}{3} \delta_{ij} X_k + \frac{(n-1)}{6} (\delta_{jk} X_i + \delta_{ik} X_j) \quad (\text{E.0.5})$$

The terms with no coincidences, where the $\Gamma_i, \Gamma_j, \Gamma_k$ all act in different tensor factors, can be decomposed as

$$A_{(ij);k} = \frac{1}{3} (A_{(ij);k} + A_{(ik);j} + A_{(jk);i}) + \frac{1}{3} (2A_{(ij);k} - A_{(ik);j} - A_{(jk);i}) \quad (\text{E.0.6})$$

It can be verified, by applying the Young Symmetriser, that the first term corresponds to a symmetric Young diagram, while the second to a mixed symmetry one. The traceless part of the tensor $A_{(ij);k}$ in four dimensions can be evaluated to be

$$A_{(ij);k} - \frac{2\delta_{ij}}{9} A_{(ll);k} - \frac{\delta_{ik}}{9} A_{(lj);l} - \frac{\delta_{jk}}{9} A_{(il);l} \quad (\text{E.0.7})$$

Keeping the mixed symmetry trace part from the non-coincident terms, obtained when (E.0.3) multiplies X_k from the left, gives additional contributions. Adding these to (E.0.5) we get

$$(X_i * X_j) * X_k = \frac{(n^2 + 10n + 7)}{18} \delta_{ij} X_k - \frac{(n^2 - 8n + 7)}{36} (\delta_{jk} X_i + \delta_{ik} X_j) + S_{ijk} \quad (\text{E.0.8})$$

where S_{ijk} is the explicitly symmetrised product with no coincidences

$$S_{ijk} = \mathcal{P}_{\mathcal{R}_n^-} \sum_{r \neq s \neq t} \rho_r(\Gamma_{(i} P_+) \rho_s(\Gamma_j P_-) \rho_t(\Gamma_k) P_+) \mathcal{P}_{\mathcal{R}_n^+} \quad + \quad (+ \leftrightarrow -) \quad (\text{E.0.9})$$

Similarly we find that

$$X_i * (X_j * X_k) = \frac{(n^2 + 10n + 7)}{18} \delta_{jk} X_i - \frac{(n^2 - 8n + 7)}{36} (\delta_{ij} X_k + \delta_{ik} X_j) + S_{ijk} \quad (\text{E.0.10})$$

The difference is

$$(X_i * X_j) * X_k - X_i * (X_j * X_k) = \frac{n^2 + 4n + 7}{12} (\delta_{ij} X_k - \delta_{jk} X_i) \quad (\text{E.0.11})$$

The X 's should be renormalised in order to correspond to the classical sphere co-ordinates in the large- n limit. Since we have $X_i^2 \sim n^2/2$ for large- n and any D from (5.1.5), we define

the normalised matrices as

$$Z_i = \frac{\sqrt{2}}{n} X_i \quad (\text{E.0.12})$$

which gives $Z_i^2 = 1$. In this normalisation the associator becomes

$$(Z_i * Z_j) * Z_k - Z_i * (Z_j * Z_k) = \frac{1}{6} \left(1 + \frac{4}{n} + \frac{7}{n^2} \right) (\delta_{ij} Z_k - \delta_{jk} Z_i) \quad (\text{E.0.13})$$

and is obviously non-vanishing in the large- n limit.

More generally one can consider multiplying $(S_{i_1 \dots i_p} * S_{j_1 \dots j_q}) * S_{k_1 \dots k_r}$ and $S_{i_1 \dots i_p} * (S_{j_1 \dots j_q} * S_{k_1 \dots k_r})$. It is clear from the above discussion that the only terms of order one (after the normalisation) that can appear in the associator are the ones coming from terms with no coincidences. These products will, in the large- n limit, include terms which match the classical product on the space of functions on the sphere. But, as illustrated here, they will also include additional terms responsible for non-associativity even in the large- n limit. The $*$ -product (discussed above) on the projected space of Matrices transforming as symmetric representations is the most obvious one available: the matrix product followed by projection. We have shown that it does not become associative in the large- n limit. There is, however, another way to modify the matrix product which does become associative in the large- n limit. This involves keeping only the completely symmetric (in i, j, k etc.) part from the completely non-coincident terms and is a mild modification of the $*$ -product discussed above. One can imagine yet other modifications. An alternative method for defining a non-associative product for the fuzzy odd-sphere, which approaches the associative one in the large- n limit, would be to start with the even-sphere case for general even dimensions D (where the prescription of matrix product followed by multiplication does give vanishing non-associativity at large n) and then continue in D . Whether the latter product is related to the alternative product contemplated above is another question we will leave unanswered.

APPENDIX F

SOLUTIONS TO REDUCED ACTION AND SOLUTIONS TO DBI

In Sec. 5.6, we saw that solving the Matrix Theory equations of motion with the fuzzy odd-sphere Ansatz is equivalent to solving the equations for the reduced action. We show here that the same is true for the full DBI equations of motion. We will discuss the fuzzy- S^3 for concreteness, but the same proof applies to the case of the fuzzy- S^5 . Consider the action arising from the expansion of (5.4.3). The full Lagrangean will comprise of an infinite sum of ‘words’ W , consisting of products of $[\Phi, \Phi]$ ’s and $\partial_t \Phi$ ’s, $S = -T_1 \int dt S \text{Tr}(\sum W)$. Therefore, if

$$\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \hat{R}} = X_l \frac{\partial W}{\partial \Phi_l} \Big|_{\Phi_l = \hat{R}X_l} \quad (\text{F.0.1})$$

and

$$\partial_t \left(\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \dot{\hat{R}}} \right) = X_l \partial_t \left(\frac{\partial W}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} \quad (\text{F.0.2})$$

then

$$\left[\frac{\partial}{\partial \hat{R}} - \partial_t \left(\frac{\partial}{\partial \dot{\hat{R}}} \right) \right] \sum W(\Phi_i = \hat{R}X_i) = X_l \left[\left(\frac{\partial}{\partial \Phi_l} \right) \Big|_{\Phi_l = \hat{R}X_l} - \partial_t \left(\frac{\partial}{\partial (\partial_t \Phi_l)} \right) \Big|_{\Phi_l = \hat{R}X_l} \right] \sum W \quad (\text{F.0.3})$$

and a solution to the equations for the reduced action would also be a solution to the equations coming from the full Matrix action.

We will proceed by proving (F.0.1) and (F.0.2). Take a word consisting only of $[\Phi, \Phi]$ ’s and expand all the commutators. The result will be $W = (\Phi_{A_1} \dots \Phi_{A_m})$, where m is even and set equal to the contracted pairs of indices. Then define

$$\frac{\partial W}{\partial \Phi_l} \Big|_{\Phi = \hat{R}X} = \sum_{k=1}^m \hat{D}_{(1)}^l \left(\hat{C}_k W \right) \Big|_{\Phi = \hat{R}X} \quad (\text{F.0.4})$$

where the operator \hat{C}_k uses the cyclic property of the trace to rotate the k -th element, that is to be differentiated, to the first slot. $\hat{D}_{(1)}^l$ takes the derivative of the first term in the word with respect to Φ_l , then sets the index of its contracted partner equal to l . We have shown that any composite operator of $SO(4)$ and $SO(6)$ with one free index i will be proportional

to X_i . Therefore we will have that

$$\frac{\partial W}{\partial \Phi_l} |_{\Phi=\hat{R}X} = \hat{R}^{m-1} \sum_{k=1}^m \alpha_k(W) X_l \quad (\text{F.0.5})$$

where $\alpha_k(W)$ is some constant factor, which in general depends on the word W and k . One can see that $\alpha_k(W) = \alpha(W)$, is actually independent of k . If we multiply the contribution of the k -th term by X_l from the left

$$X_l \left[\hat{D}_{(1)}^l \left(\hat{C}_k W \right) \right] |_{\Phi=\hat{R}X} = \hat{R}^{m-1} \alpha_k(W) X_l X_l \quad (\text{F.0.6})$$

On the LHS we will now have again a Casimir of $SO(D)$

$$\hat{R}^{m-1} X_l \left[\hat{D}_{(1)}^l \left(\hat{C}_k W(\Phi \rightarrow X) \right) \right] = C \hat{R}^{m-1} \alpha_k(W) \quad (\text{F.0.7})$$

As such it will obey the cyclicity property and can be rotated back to form the original word with $\Phi \rightarrow X$

$$\begin{aligned} \alpha_k(W) &= \frac{1}{C} X_l \left[\hat{D}_{(1)}^l \left(\hat{C}_k W(\Phi \rightarrow X) \right) \right] \\ &= \frac{1}{C} W(\Phi \rightarrow X) \\ &= \frac{1}{C} (\alpha(W) C) \end{aligned} \quad (\text{F.0.8})$$

As a consequence, every contribution in the sum (F.0.4) is going to be the same and

$$X_l \frac{\partial W}{\partial \Phi_l} |_{\Phi=\hat{R}X} = m \hat{R}^{m-1} \alpha(W) C \quad (\text{F.0.9})$$

It is much easier to evaluate the LHS of (F.0.1) to get

$$\frac{\partial W(\Phi_i = \hat{R}X_i)}{\partial \hat{R}} = m C \alpha(W) \hat{R}^{m-1} \quad (\text{F.0.10})$$

Exactly the same procedure can be applied to words containing time derivatives, where m is now the number of Φ 's coming just from commutators and therefore (F.0.1) holds. Similar steps can be carried out for the words with $m \partial_t \Phi$ terms and $n \Phi$ terms coming from the expansion of commutators. We will have

$$\partial_t \left(\frac{\partial W}{\partial (\partial_t \Phi_l)} \right) |_{\Phi_l=\hat{R}X_l} = \partial_t \left(\sum_{k=1}^m \hat{S}_{(1)}^l \left(\hat{C}_k W \right) |_{\Phi=\hat{R}X} \right) \quad (\text{F.0.11})$$

where $\hat{S}_{(1)}^l$ takes the derivative of the first term in the word with respect to $\partial_t \Phi_l$, then sets

the index of its contracted partner equal to l . This will become

$$\partial_t \left(\sum_{k=1}^m \hat{S}_{(1)}^l \left(\hat{C}_k W \right) \Big|_{\Phi=\hat{R}X} \right) = \left(m \hat{R}^n \dot{\hat{R}}^{m-1} \alpha(W) X_l \right) . \quad (\text{F.0.12})$$

and, when multiplied by X_l , will result into what one would get from evaluation of the LHS of (F.0.2), namely

$$X_l \partial_t \left(\frac{\partial W}{\partial(\partial_t \Phi_l)} \right) \Big|_{\Phi_l=\hat{R}X_l} = C \alpha(W) m \left(\dot{\hat{R}}^{m-1} \hat{R}^n \right) . \quad (\text{F.0.13})$$

This completes the proof that any solutions to the reduced DBI equations of motion for S^3 and S^5 will also be solutions to the full Matrix equations of motion, for any N .

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