Entangled neutrinos in a supernova

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Abstract. We consider the entanglement of neutrinos evolving adiabatically under the effect of vacuum oscillations and self interactions through decreasing neutrino density similar to a core collapse supernova. For an initial state which consists only of electron type neutrinos, we analytically calculate the asymptotic value of the entropy of entanglement between the lowest energy neutrino and the rest of the system as a function of the number of neutrinos. We find that, as we increase neutrino number the entanglement entropy grows at first. But after going through a maximum, it approaches to zero in the limit of infinite number of neutrinos. We find that the number of neutrinos for which the entropy maximizes depends on the mixing angle.

1. Introduction

Since neutrinos interact with each other via neutral current weak force, they can form a self interacting many-body system. Flavor evolution of neutrinos in such a system is different from the simple oscillations that each would display in vacuum. The emergent many-body effects due to the neutrino self interactions is usually referred to as the *collective flavor oscillations*. For the collective effects to be sizable, the mean free path of the neutrinos under self interactions should be comparable to or smaller than their vacuum oscillation wavelengths. It is believed that this condition is satisfied in some astrophysical environments such as the core collapse supernova [1, 2, 3, 4, 5], black hole accretion disks [6, 7, 8, 9, 10], neutron star mergers [11], and the Early Universe [12, 13, 14]. Collective flavor oscillations of neutrinos in such environments is extensively studied in the literature. For reviews see Refs. [15, 16].

Like most other many-body systems in physics, the system of self interacting neutrinos is usually studied by adopting the *mean field approximation*. This approximation replaces the picture of mutually interacting neutrinos with a collection of *independent* particles, each moving in an external *mean field*. This mean field is collectively formed by all neutrinos and averages the effect of the system on each neutrino. For self consistency, the mean field is recalculated at every time step as the neutrinos themselves evolve. For a detailed derivation and discussion in the context of neutrinos, see Ref. [17]. For a general introduction to mean field approximation, see Chapter II of Ref. [18].

Mean field approximation is a standard tool in various areas of physics where many-body interactions arise, such as condense matter systems, cold atoms, or atomic nuclei. The general

wisdom in those fields is that the mean field approximation becomes more and more accurate as the number of particles increase. But these systems differ from self interacting neutrinos in at least two curial aspects: First, they are usually found close to their ground states where the mean field changes slowly, allowing the scheme of averaging over the particles to work smoothly. Second, one usually has experimental access to these systems which makes it possible to compare mean field calculations with measurements. Neither of these is true in the case of self interacting neutrinos in astrophysical environments. The suitable set of initial states is far from equilibrium and fast oscillations in mean field is expected [19]. For recent studies on these fast oscillations see, for example, Refs. [20, 21].

These considerations led to a number of studies which addressed the adequacy of the mean field approximation in describing self interacting neutrinos [22, 23, 24, 17, 25, 26, 27, 28, 29]. The research on this subject revolves around the importance of entangled neutrino states in the dynamics. This is because the mean field approximation naturally eliminates all entangled states from the Hilbert space. After all, neutrinos cannot develop entanglement while they are evolving independently in an external field[1]. As a result, a large built up of entanglement in the original many-body system indicates that the mean field approximation is not applicable. In particular Refs. [27, 28, 29] calculate the von Neumann bi-partite entanglement entropy and conclude that this entropy increases with the number of neutrinos.

However, these studies are not conclusive: For n neutrinos, each with d flavor degrees of freedom, the Hilbert space dimension of the full many-body problem is d^n. As a result, solutions are feasible either for small numbers of neutrinos or with additional simplifying assumptions, For example, Ref. [27] employs relatively realistic conditions but includes only $n = 9$ neutrinos while Refs. [28, 29] can go up to $n = 128$ neutrinos in a simplified setup. But both cases have the potential to lead to artificial results. And in this paper, we present a specific example in which this is the case.

For this, we adopt a setup which is similar to the one considered in Ref. [27]: An effective two flavor mixing scheme, a decreasing neutrino density representing emission from an astrophysical source, and initial state which consists only of electron neutrinos in a box-like energy distribution. For this particular initial state, Ref. [27] observes that the entanglement entropy approaches to an asymptotic value as the neutrino density drops, and that this asymptotic value increases with the number of neutrinos up to $n = 9$. In this paper, by imposing the additional assumption that the evolution is adiabatic in this setup, we calculate the asymptotic value of the entanglement entropy analytically for any number of neutrinos. We find that, although the asymptotic entropy increases with n at first, it maximizes after a while and then starts to decrease. The particular number at which the asymptotic entropy maximizes depends on the adopted mixing angle. But we find that for all mixing angles, it eventually approaches to zero at large n.

It is important to note that Ref. [27] considers a variety of initial states and finds that the results depend strongly on the initial conditions. What allows us to analytically calculate the asymptotic entropy as a function of n is the fact that the initial state that we work with is symmetrical under the exchange of any two particles. This is a high degree of symmetry which makes it difficult to draw generic conclusions from our results.

But we think that this example still provides a useful point of view. In particular, we find that the number of neutrinos at which the entanglement entropy maximizes decreases with the mixing angle. For large mixing angles maximization happens at a few tens of neutrinos while for small mixing angles it happens after a few hundreds. This observation can be turned into a practical tool: Even if an analytical calculation is not possible for a different initial state, a numerical study can be carried out with a finite number of neutrinos in which one may try different mixing angles to see if this maximization occurs at a relatively small n. If it does, the

[1] However, the flavor evolutions of the neutrinos are still correlated because they are interacting with each other indirectly through their contributions to the mean field.

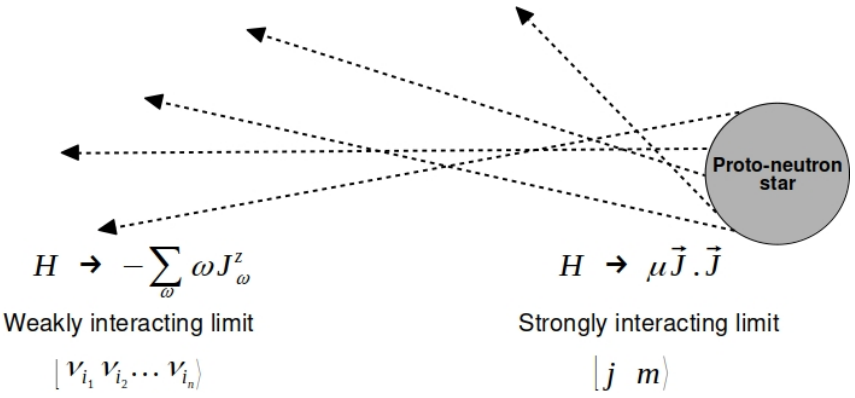

Figure 1: Schematic description of the neutrinos emitted from a proto-neutron star. Close to the center, the neutrino density is so large that neutrino-neutrino interactions dominate the dynamics. Away from the center, we can ignore the neutrino-neutrino interactions.

same effect that we observe here may be at play.

In what follows we introduce the model Hamiltonian describing the supernova neutrinos in Section II. In Section III, we discuss its spectrum and the adiabatic evolution of the particular initial state that we are interested in. In Section III we calculate the asymptotic value of the entanglement entropy as an analytical function of the number of neutrinos. In Section IV, we give our conclusions.

2. Hamiltonian

In this paper, we consider neutrinos emitted by a newly formed proto-neutron star at the center of a core collapse supernova as shown in Fig. 1. But this is only for the sake of providing a backdrop for our discussion. The actual model that we work with does not capture the rich physics of supernova neutrinos. Basically, we consider the proto-neutron star as a spherically symmetric source which emits only electron neutrinos, and assume that there are no other background particles. Although neutrino-neutrino interaction cross section depends on the angle between them [30], we take this angle dependence into account in an effective way using the *neutrino bulb model* which was developed in Ref. [31]. Also we assume that there are only two mixing neutrino flavors.

The mathematical formalism we use is the same as in Ref. [26]. It is built upon the neutrino isospin operator $\vec{J}_{\omega}$ which is a fictitious *spin operator* analogous to nuclear isospin. It is defined with respect to the doublet

$$\begin{pmatrix} |\nu_1(\omega)\rangle \\ |\nu_2(\omega)\rangle \end{pmatrix}. \tag{1}$$

Here, ν_i for $i = 1, 2$ denotes the neutrino eigenstate with the definite mass m_i. We use the vacuum oscillation frequency ω to label the neutrinos. It is related to the neutrino's energy E by

$$\omega = \frac{m_2^2 - m_1^2}{2E}. \tag{2}$$

For simplicity we assume that the oscillation modes are evenly spaced as

$$\omega = \omega_0, 2\omega_0, 3\omega_0, \ldots, n\omega_0, \tag{3}$$

and that each mode is occupied by only one neutrino. When we change the number of neutrinos, we do it in such a way that the highest frequency $n\omega_0$ is fixed.

By definition the components of the isospin operator obey the relations

$$J^z_\omega \, |\nu_1(\omega)\rangle = \frac{1}{2} \, |\nu_1(\omega)\rangle, \qquad J^z_\omega \, |\nu_2(\omega)\rangle = -\frac{1}{2} \, |\nu_2(\omega)\rangle, \qquad \vec{J}_\omega \cdot \vec{J}_\omega \, |\nu_i(\omega)\rangle = \frac{3}{4} \, |\nu_i(\omega)\rangle, \tag{4}$$

and satisfy $SU(2)$ commutators among themselves:

$$[J^+_\omega, J^-_{\omega'}] = 2\delta_{\omega\omega'} J^0_\omega, \qquad [J^0_\omega, J^\pm_{\omega'}] = \pm\delta_{\omega\omega'} J^\pm_\omega. \tag{5}$$

For a system of n neutrinos, we denote the total isospin by $\vec{J}$:

$$\vec{J} = \sum_\omega \vec{J}_\omega. \tag{6}$$

In terms of the isospin operators, the Hamiltonian describing their flavor evolution is given by

$$H(\mu) = \sum_\omega -\omega J^z_\omega + \mu(r)\vec{J} \cdot \vec{J}. \tag{7}$$

The first term of this Hamiltonian describes the vacuum oscillations and the second term describes the neutrino-neutrino interactions. Here r represents the distance from the center of the proto-neutron star. The effective interaction strength $\mu(r)$ is given by [30, 31]

$$\mu(r) = \frac{\sqrt{2}G_F}{V}(1 - \sqrt{1 - (R_\nu/r)^2})^2. \tag{8}$$

Here, G_F is the Fermi coupling constant, and V is the normalization volume used for box quantization of the neutrinos. By definition it is the volume in which the total probability of finding each neutrino is 1. Although the normalization volume appears explicitly in the Hamiltonian, what emerges from the calculations is always the neutrino density n/V. As the neutrinos expand from the source, V increases and the neutrino-neutrino interaction rate drops. The term in the parenthesis in Eq. (8) is the geometrical factor of the neutrino bulb model [31] with R_ν denoting the radius of the proto-neutron star. This factor serves two purposes: First, neutrino-neutrino interaction cross section depends on the angle between them, which, on the average, becomes smaller as the neutrinos expand from a source. Second, although the term $\vec{J} \cdot \vec{J}$ involves interactions between every pair of neutrinos, in reality not every neutrino can meet and interact with every other neutrino due to the geometry of the problem. Each neutrino can interact with only those emitted from a restricted part of the surface of the proto-neutron star. Also, while a given neutrino can meet and interact with many other neutrinos when it is near the surface, this is not the case when it is far away. The geometrical factor takes these into account by suitably scaling down the interaction rate with distance.

The Hamiltonian in Eq. (7) involves the neutrino mass difference but not the vacuum mixing angle because it is written in the mass basis. The mixing angle enters the problem as an initial condition when the neutrinos start their life in weak interaction eigenstates. These eigenstates are given by

$$\begin{aligned} |\nu_e(\omega)\rangle &= \cos\theta|\nu_1(\omega)\rangle + \sin\theta|\nu_2(\omega)\rangle, \\ |\nu_x(\omega)\rangle &= -\sin\theta|\nu_1(\omega)\rangle + \cos\theta|\nu_2(\omega)\rangle. \end{aligned} \tag{9}$$

Here ν_e denotes the electron flavor whereas ν_x denotes a general superposition of muon and tau flavors. We treat the mixing angle θ as a free parameter for illustration purposes. We also assume that the mass hierarchy is normal, i.e., $m_2 > m_1$.

3. Eigenstates and the adiabatic evolution

Generic evolution of the system requires knowledge of the evolution operator but it is difficult to calculate. However, adiabatic evolution requires only an understanding of the energy spectrum which is examined in detail in Refs. [32, 26]. In what follows we summarize the relevant points from these references.

The Hamiltonian in Eq. (7) conserves the total number of neutrinos, as well as the z component of the total isospin operator:

$$[H(\mu), J^z] = 0, \quad [H, N] = 0, \quad [N, J^z] = 0. \tag{10}$$

Here $\hat{N}$ is the total neutrino number operator. Its conservation is due to the fact that there are no terms creating or annihilating neutrinos in our formulation. We assume that neutrinos simply free stream outside the proto-neutron star. The commutation of J^z with the Hamiltonian is also evident: It commutes with the vacuum oscillation term because they are both isospin z components. It also commutes with the interaction term because the latter is the Casimir operator of the total isospin algebra. We denote the simultaneous eigenstates and corresponding eigenvalues of these operators as

$$\begin{aligned} H(\mu)\,|E(\mu), n, m\rangle &= E(\mu)\,|E(\mu), n, m\rangle, \\ \hat{N}\,|E(\mu), n, m\rangle &= n\,|E(\mu), n, m\rangle, \\ J^z\,|E(\mu), n, m\rangle &= m\,|E(\mu), n, m\rangle. \end{aligned} \tag{11}$$

Clearly the eigenvalue of the number operator, denoted by n, can be written as

$$n = n_1 + n_2, \tag{12}$$

where n_i denotes the total number of neutrinos of ν_i type for $i = 1, 2$. As can be seen from Eq. (4), the eigenvalue of J_z is then equal to

$$m = \frac{n_1 - n_2}{2}. \tag{13}$$

This tells us that each many-body eigenstate comes with a definite value for n_1 and n_2.

It is easy to find the many-body eigenstates and the corresponding eigenvalues of the Hamiltonian in the strongly and weakly interacting regimes. When the neutrinos are close to the surface of the proto-neutron star, the neutrino density so high that that we can take $\mu \to \infty$ limit and ignore the vacuum oscillation term in the Hamiltonian. (See Fig. 1.) In this limit, we have

$$\lim_{\mu\to\infty} H(\mu) = \mu \vec{J} \cdot \vec{J}. \tag{14}$$

This tells us that the total isospin states denoted by $|j\ m\rangle$ are the many-body eigenstates of the system, i.e.,

$$\lim_{\mu\to\infty} |E(\mu), n, m\rangle = |j\ m\rangle. \tag{15}$$

Here j denotes the total isospin quantum number which can take values starting from $j = 0$ or $1/2$ up to $j = n/2$, and m is given by Eq. (13). It may look like we lost a quantum number in Eq. (15), but this is due to the fact that total isospin quantum number j comes with many degeneracies when we add more than two isospins. The energy eigenvalues in this limit are given by

$$\lim_{\mu\to\infty} E(\mu) = \mu j(j+1). \tag{16}$$

Conversely, when neutrinos are far from the source, the neutrino density decreases so much that we can take $\mu \to 0$ limit in the Hamiltonian and ignore the interactions. In this limit, we have

$$\lim_{\mu\to 0} H(\mu) = -\sum_{\omega} \omega J^z_\omega \tag{17}$$

and the many-body eigenstates reduce to the products of individual mass eigenstates. In other words, they have the form

$$\lim_{\mu\to 0} |E(\mu), n, m\rangle = |\nu_{i_1}\ \nu_{i_2}\ \ldots\ \nu_{i_n}\rangle, \tag{18}$$

where $i_1, i_2, \ldots, i_n = 1, 2$. Note that when we write the single particle states is this form, we always arrange them in the order of increasing ω. Therefore, the first neutrino in Eq. (18) has the lowest oscillation frequency ω_0 whereas the last one has the highest oscillation frequency $n\omega_0$. The energy eigenvalues corresponding to the eigenstates given in Eq. (18) are

$$\lim_{\mu\to 0} E(\mu) = \frac{\omega_0}{2} \sum_k (-1)^{i_k} k, \tag{19}$$

where we used Eq. (3) and Eq. (4).

Under adiabatic conditions, the dynamical evolution of the eigenstates follow their own slow change with the external conditions. Here, the external condition is the interaction strength μ which decreases with neutrino density. Specifically, adiabaticity requires the flavor oscillation wavelength of the neutrinos to be much smaller than the distance scale at which the density changes. It is believed that this condition is satisfied in the core collapse supernovae. As a result, as we go from strongly to weakly interacting regime, the eigenstates in Eq. (15) should dynamically evolve into those given in Eq. (18). However, it is not immediately obvious which eigenstate evolves into which one because usually there are many crossings between energy eigenvalues. Therefore, one must also diagonalize the Hamiltonian in the intermediate regime in order to analytically connect the eigenstates of the strongly and weakly interacting regimes.

In the intermediate regime, the eigenstates and eigenvalues should be calculated using a numerical scheme. Fig. 2 shows the eigenvalues for $n = 10$ neutrinos as a function of the interaction strength μ. This figure is taken from Ref [26] and shows the eigenstates grouped according to the value of the conserved quantity J^z given in Eq. (13). Calculations are carried out using the Bethe ansatz technique which is based on the dynamical symmetries of the Hamiltonian in Eq. (7) examined in detail in Refs. Note that in Fig. 2 the eigenvalues are plotted in the direction of increasing μ. As the neutrinos go from the center of supernova to the outer regions as in Fig. 1, the eigenvalues should be traced from right to left [32, 26].

The insets in Fig. 2 show that these eigenvalues cross each other several times in the low μ region. However, for each m value, the highest energy eigenvalue never undergoes a crossing with the others. Our calculation of the entanglement entropy of the final state after an adiabatic evolution strongly depends on this observation. But before we move into the details, we must emphasize two points.

First, although our calculations depend on the assumption that it is true, there is no algebraic proof in the literature that the highest energy eigenstates of the Hamiltonian in Eq. (7) for each m are always non-degenerate. However, various proofs for special cases, and even *experimental evidence* exists in that direction. The neutrino Hamiltonian given in Eq. (7) is the same as the Hamiltonian describing pairing interactions between Cooper electrons in superconductors and cold atoms systems [32]. One just has to replace the neutrino isospin operators with pair quasispin operators, which also obey SU(2) commutation relations, and multiply the Hamiltonian with an overall minus sign. As a result, the highest energy eigenstates of the

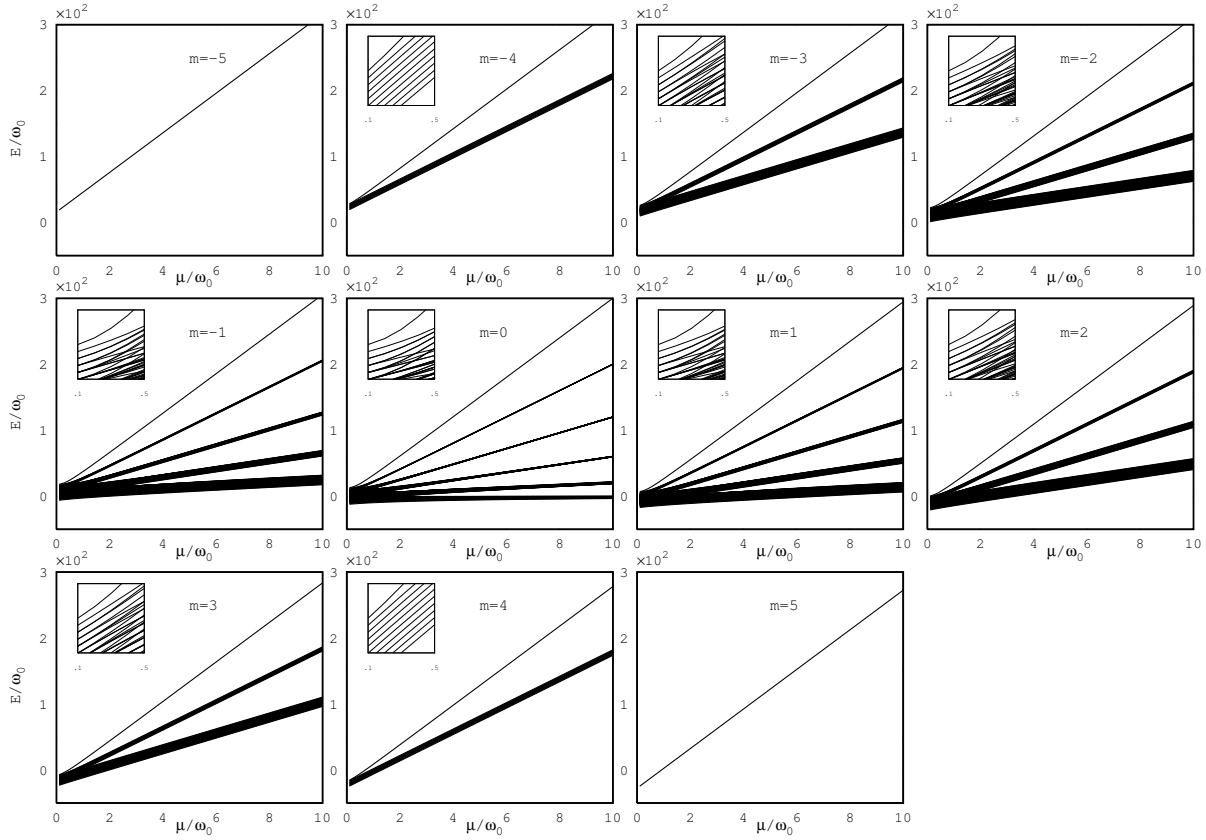

Figure 2: Eigenvalues for $n = 10$ neutrinos grouped according to the value of the conserved quantity J^z. Taken from Ref. [26].

neutrino Hamiltonian are the ground states of the pairing Hamiltonian. The non-degeneracy of these states is a well known phenomenon which is observed in numerical solutions with up to $n = 1600$ and $\kappa = 800$ in Ref. [33] for the case of equally spaced oscillation modes similar to the one introduced in Eq. (3). An analytical proof for large values of n and m is originally given by Gaudin [34], and later elaborated by Richardson [35].

One can also discuss this non-degeneracy from the point of view of experiments involving cold atomic systems. In such experiments one can control the strength of the pairing interaction and take the system adiabatically from weakly interacting regime to strongly interacting regime. This results in the famous Bardeen-Cooper-Schrieffer to Bose-Einstein condensation crossover [36]. The fact that it is a crossover, rather than a phase transition, indicates that the ground state of the system never undergoes a level crossing with one of the excited levels. A level crossing would cause abrupt changes in the measurable quantities which is not experimentally observed. For details see, for example, Refs. [37, 38, 39].

The second point to be emphasized is that the non-crossing is not required for our adiabaticity assumption. Although adiabatic evolution is generally identified with the presence of energy gaps, it is in fact possible for a system to evolve adiabatically even when energy eigenvalues cross each other which is called the gapless adiabaticity [40]. This is discussed in the context of self interacting neutrinos in Ref. [27]. In our case the non-crossing of the highest energy eigenstate is useful in identifying which eigenstate in the highly interacting limit is adiabatically connected to which eigenstate in the weakly adiabatic limit.

At the strongly interacting limit, Eq. (16) tells us that the highest energy eigenstate for a

given m is the one with the highest possible total isospin quantum number, which is $j = n/2$. At the weakly interacting limit, Eq. (19) tells us that the highest energy eigenstate is the one in which there are $n_1 = n/2 + m$ neutrinos of ν_1 type occupying the lowest oscillation frequencies, and $n_2 = n/2 - m$ neutrinos of ν_2 type occupying the highest oscillation frequencies. As a result, the states

$$|\frac{n}{2},\ m\rangle \longrightarrow |\underbrace{\nu_1,\ldots,\nu_1}_{\frac{n}{2}+m},\underbrace{\nu_2,\ldots,\nu_2}_{\frac{n}{2}-m}\rangle \tag{20}$$

are analytically connected through the decreasing interaction strength so that, under adiabatic conditions, the one on the left would evolve into the one on the right.

Theory of angular momentum addition tells us that the states with the highest total isospin quantum number are maximally symmetric. In other words, $|\frac{n}{2},\ m\rangle$ is invariant under the exchange of any two neutrinos. Those states with $j < n/2$ do not have this property. Considering that the state in which all neutrinos are ν_e is also symmetric under this exchange, one expects that this particular initial state can be decomposed entirely in terms of the $|\frac{n}{2},\ m\rangle$ states. That this is indeed the case can be shown by repeatedly applying Eq. (9), and using the relevant Clebsh-Gordon coefficients [26]:

$$|\psi\rangle_{\text{initial}} = |\nu_e,\nu_e,\ldots,\nu_e\rangle = \sum_{m=-n/2}^{n/2} (\cos\theta)^{(\frac{n}{2}+m)}(\sin\theta)^{(\frac{n}{2}-m)}\sqrt{\binom{n}{\frac{n}{2}+m}}|n/2,m\rangle. \tag{21}$$

Since this decomposition includes only the highest energy eigenstates in the $\mu \to \infty$ limit, its adiabatic evolution leads to

$$|\psi\rangle_{\text{final}} = \sum_{m=-n/2}^{n/2} (\cos\theta)^{(\frac{n}{2}+m)}(\sin\theta)^{(\frac{n}{2}-m)}\sqrt{\binom{n}{\frac{n}{2}+m}}e^{i\varphi_m}|\underbrace{\nu_1,\ldots,\nu_1}_{\frac{n}{2}+m},\underbrace{\nu_2,\ldots,\nu_2}_{\frac{n}{2}-m}\rangle \tag{22}$$

as per Eq. (20). Here φ_m are some phases (both dynamical and geometrical) whose exact forms are not needed here. Eq. (22) is an analytical expression for the asymptotic final state of the adiabatic evolution in $\mu \to 0$ limit. This is clearly an entangled many-body state whose entropy of entanglement is calculated in the next section.

4. Entanglement entropy

Entanglement entropy is a measure of entanglement between two parts of a system. We use von Neumann entanglement entropy which is defined as the regular thermodynamic entropy of either one of these parts. The idea is that, even when the full system is in a pure state, each part lacks a definite quantum state due to its entanglement to the other. Therefore individual parts would be in mixed states and have thermodynamic entropies.

Let us denote the two parts of the system by $\mathcal{A}$ and $\mathcal{B}$. The reduced density matrix $\rho_{\mathcal{A}}$ of part $\mathcal{A}$ is obtained by tracing the density matrix of the full system ρ over those degrees of freedom belonging to part $\mathcal{B}$, and vice versa:

$$\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}(\rho), \qquad \rho_{\mathcal{B}} = \text{Tr}_{\mathcal{A}}(\rho). \tag{23}$$

The entropy of the entanglement between parts $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$S = -\text{Tr}\left(\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}\right) = -\text{Tr}\left(\rho_{\mathcal{B}} \ln \rho_{\mathcal{B}}\right). \tag{24}$$

The entropy of entanglement is a conjoint quantity. Whether we trace over part $\mathcal{A}$ and calculate the entropy of part $\mathcal{B}$, or do the opposite, the result is the same as indicated by Eq. (24).

However, the entropy depends on how we partition the system. In particular, the zero value for entanglement entropy indicates that the quantum state $|\psi\rangle$ of the full system is a direct multiplication of pure states $|\psi\rangle_{\mathcal{A}}$ and $|\psi\rangle_{\mathcal{B}}$ of the parts i.e.,

$$|\psi\rangle = |\psi\rangle_{\mathcal{A}}|\psi\rangle_{\mathcal{B}}. \tag{25}$$

While this may be the case for one way of partitioning the system, it may not be so for another possible partition.

In our case, the full system is n neutrinos. We take part $\mathcal{A}$ to be the first $n-1$ neutrino and part B to be the last neutrino, which is the neutrino with the highest oscillation frequency. Initially entropy of entanglement between these parts is zero because the initial wave function of the full system given in Eq. (21) is factorizable. But this is clearly not the case in the final state given in Eq. (22). Tracing over all possible states of the first $n-1$ neutrino, one show that the reduced density matrix of the last neutrino is given by

$$\rho_{\mathcal{B}} = (\cos\theta)^{2n}|\nu_1\rangle\langle\nu_1| + (\cos\theta)^{2n-1}(\sin\theta)\sqrt{\binom{n}{n-1}}\left(e^{i(\varphi_{n-1}-\varphi_n)}|\nu_2\rangle\langle\nu_1| + e^{-i(\varphi_{n-1}-\varphi_n)}|\nu_1\rangle\langle\nu_2|\right)$$
$$+\sum_{k=0}^{n-1}(\cos\theta)^{2k}(\sin\theta)^{2n-2k}\binom{n}{k}|\nu_2\rangle\langle\nu_2|. \tag{26}$$

The trace of this density operator is equal to 1 as can be easily confirmed by noting that the sum of the coefficients of $|\nu_1\rangle\langle\nu_1|$ and $|\nu_2\rangle\langle\nu_2|$ is equal to $(\cos^2\theta + \sin^2\theta)^n$.

In general, the von Neumann entropy of a two-state density operator such as the one in Eq. (26), can be more easily calculated in terms of its polarization (or Bloch) vector [27]. This vector is defined by expanding the density operator in terms of the Pauli spin operators

$$\rho_{\mathcal{B}} = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right), \tag{27}$$

where

$$\sigma_1 = |\nu_1\rangle\langle\nu_2| + |\nu_2\rangle\langle\nu_1|, \quad \sigma_2 = -i|\nu_1\rangle\langle\nu_2| + i|\nu_2\rangle\langle\nu_1|, \quad \sigma_3 = |\nu_1\rangle\langle\nu_1| - |\nu_2\rangle\langle\nu_2|. \tag{28}$$

With this expansion, the eigenvalues of the density operator are easily identified as $(1 \pm P)/2$. This allows us to easily apply the formula in Eq. (24) in the basis where the density operator is diagonal. The result is

$$S = -\frac{1+P}{2}\ln\frac{(1+P)}{2} - \frac{1-P}{2}\ln\frac{(1-P)}{2}. \tag{29}$$

Note that since $0 \leq P \leq 1$, the entanglement entropy given in Eq. (29) is always positive. $P = 1$ limit corresponds to the case in which both $\mathcal{A}$ and $\mathcal{B}$ are in pure states because the eigenvalues of $\rho_{\mathcal{B}}$ are 1 and 0. In this limit Eq. (29) gives $S = 0$. In the opposite limit where $P = 0$, both eigenvalues of $\rho_{\mathcal{B}}$ are 1/2. This corresponds to the case that $\mathcal{A}$ and $\mathcal{B}$ are maximally entangled. In this limit Eq. (29) gives $S = \ln 2$ which is the maximum possible entropy for this particular partitioning of the system.

The value of the polarization vector can be directly calculated by substituting Eq. (26) into Eq. (27) and using the identity $x^n = e^{n\ln x}$. The result is

$$P = \sqrt{1 - 4e^{-2\beta n} + 4(1 + n\tan^2\theta)e^{-4\beta n}} \tag{30}$$

for $n \geq 2$, where $\beta = -\ln(\cos\theta)$. As can be seen from this formula, for finite n we always have $P < 1$ which indicates that the last neutrino is entangled to the rest of the system. This can be shown by noting that $0 \leq (1+n\tan^2\theta)e^{-2\beta n} \leq 1$. In fact, as we increase the number of neutrinos, P goes through a minimum at which point the entanglement entropy becomes maximum. This can be clearly seen in Fig. 3 where we plot the entanglement entropy S as a function of n obtained by substituting Eq. (30) in Eq. (29). The figure shows results for various mixing angles indicated next to the lines. For clarity, we plot large mixing angles on the left and small mixing angles on the right. We see that the points at which entropy becomes maximum depends on the adopted value of the mixing angle. For large mixing angles, maximization occurs at a few tens of neutrinos while for small mixing angles it occurs after a few hundreds of neutrinos.

However, Eq. (30) shows that in $n \to \infty$ limit we have $P \to 1$ and consequently $S \to 0$ for any value of the mixing angle, indicating that the last neutrino is not entangled to the rest of the system. For large mixing angles entropy approaches to zero quickly while for small mixing angles it decreases more slowly. This can be easily understood by looking at the final state in Eq. (22). The squared coefficients of the single particle states in that equation can be approximated by a Gaussian distribution as

$$(\cos\theta)^{(n+2m)}(\sin\theta)^{(n-2m)}\binom{n}{\frac{n}{2}+m} \approx \frac{1}{\sqrt{2\sigma^2\pi}}\exp\left(-\frac{(m-\bar{m})^2}{2\sigma^2}\right), \tag{31}$$

where

$$\bar{m} = n(\cos^2\theta - \frac{1}{2}), \qquad \sigma = \sqrt{n}\cos\theta\sin\theta. \tag{32}$$

As n increases, the fractional width of this Gaussian becomes smaller. This width represents the ratio of those terms which are considerably different from zero to the total number of terms. Since the former is proportional to $\sigma \sim \sqrt{n}$ whereas the latter is proportional to n, this ratio approaches to zero in $n \to \infty$ limit, indicating that

$$\lim_{n\to\infty}|\psi\rangle_{\text{final}} = e^{i\varphi_{\bar{m}}}|\underbrace{\nu_1,\ldots,\nu_1}_{\frac{n}{2}+\bar{m}},\underbrace{\nu_2,\ldots,\nu_2}_{\frac{n}{2}-\bar{m}}\rangle. \tag{33}$$

As a result, the final state is unentangled in the limit of a large number of particles.

5. Conclusions

In this study, we considered the entanglement of self interacting neutrinos in a setting similar to that of a core collapse supernova. Without taking matter effects into account, we focused on the adiabatic evolution of neutrinos under the vacuum oscillations and self interactions. For an initial condition in which all neutrinos are emitted as electron neutrinos from the surface of the proto-neutron star, we showed that the asymptotic value of the entanglement entropy far from the center can be calculated analytically. This calculation was possible due to the symmetry of the initial state under the exchange of any two neutrinos. This symmetry allows the initial state to project only on those states which undergo no level crossings during the evolution, and makes it possible to analytically trace its adiabatic evolution from the center of the supernova to the outer regions.

We find that the entropy of entanglement between the neutrino with the highest oscillation frequency and the rest of the system initially grows as we increase the number of particles in the system. However, after maximizing at a certain value, it approaches to zero in the $n \to \infty$ limit. We conclude that, for large numbers of neutrinos the final state is not entangled. We also find that the value at which the entanglement is maximum depends on the mixing angle. But the entanglement vanishes at large numbers for any value of the mixing angle.

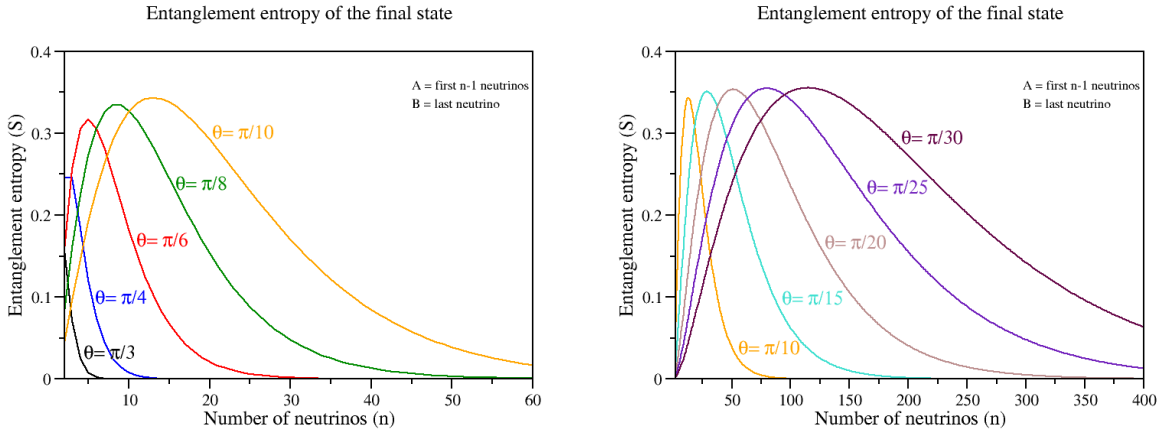

Figure 3: Asymptotic value of the entanglement entropy in $\mu \to 0$ limit as a function of the number of neutrinos n in the system obtained by substituting Eq. (30) in Eq. (29). For clarity, we plot the curves corresponding to larger mixing angles on the left and smaller mixing angles on the right. For large mixing angles, the entropy maximizes when we have a few tens of neutrinos. For small mixing angles it maximizes when we have a few hundreds of neutrinos.

It is not possible to draw generic conclusions from our study because our calculations depend significantly on the symmetry of the initial state. Ref. [27] finds that entanglement builds up in the system for a variety of initial states, including those which do not have the above-mentioned symmetry. In particular, those initial states which are not symmetric under the exchange of neutrinos would undergo level crossings. Whether the system evolves adiabatically or non-adiabatically through these crossings, the growth of the entanglement can be very different than the results obtained here.

Still, our results shine a new light on the problem. We believe that the dependence of the entanglement entropy on the mixing angle is our most important result. For those initial states for which analytical calculations are not possible, it may be best to try both large and small mixing angles. If one finds that the entropy increases in one case but decreases in the other, it is possible that a similar effect observed here is in effect. For example, Ref. [27] adopts the mixing angle $\theta \approx \pi/20$ and finds that the entanglement entropy increases until $n = 9$ neutrinos. Our results indicate that, for this mixing angle it would continue to increase until $n \approx 60$ before starting to decrease. However, for a mixing angle of $\theta = \pi/3$ it would start decreasing with n right away as can be seen from our Fig. 3. A similar comment may also be given on Refs. [28, 29] which adopt zero mixing angle and find that the entropy of entanglement still increases after 128 neutrinos. Although the setting and the initial state of Ref. [28, 29] are very different from our case (they adopt a constant neutrino density, only two energy modes, and a less symmetric initial state), it would be worthwhile to repeat this study for a finite mixing angle.

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