Realism-based nonlocality in neutrino oscillations

M Blasone[1,2]**, S De Siena**[3]**, C Matrella**[2]

[1] Dipartimento di Fisica, Università di Salerno, Via Giovanni Paolo II, 132 I-84084 Fisciano (SA), Italy.
[2] INFN, Sezione di Napoli, Gruppo collegato di Salerno, Italy.
[3] Università degli Studi di Salerno (Ret.), Fisciano, Italy.

E-mail: mblasone@unisa.it, silvio.desiena@gmail.com, cmatrella@unisa.it

Abstract.
We resort to the concepts of realism and indefiniteness introduced in Ref. [40], and based on the exploitation of ideal quantum tomography procedures. These concepts are connected to the existence of nonlocal correlations, and moreover allow to introduce a measure of nonlocality. In this paper we apply and test the approach of Ref. [40] in the physically relevant phenomenon of two-flavor neutrino oscillations, both in the plane-wave approximation and in the wave-packet approach, finding meaningful confirmations of the validity of the methodology, which allows to discriminate among distinct characters of different observables, and consequently on their physical relevance.

1. Introduction

The study of quantum correlations, developed in the field of quantum information and communication, has been recently translated and applied to other scientific areas such as elementary particles and structure of matter [1]–[36]. Particular attention has been devoted to define measures aimed to quantify the quantumness, and to identify a hierarchy among quantum correlations [37]–[39] with the nonlocal advantage of quantum coherence (NAQC) being the strongest. The quest is much simple if one considers pure bipartite quantum states, because in this case the concept of entanglement, quantified by the von Neumann entropy, encompasses all measures of quantum correlations. More challenging is the case of pure multi-partite states or of mixed states, or even more the case of mixed multi-partite states.

It is worth to be reported that for bipartite (pure or mixed) states one can identify the precise hierarchy among quantumness quantifiers that has shown in Fig.1, and many authors have studied these measures in different contexts.

In this paper we consider the approach of Ref.[40] in which the presence of nonlocal quantum correlations is associated to the dicotomy realism/irrealism (or indefiniteness), in terms of which a quantified content of nonlocality can be defined. The authors focus their attention not only on the quantum state, but also on observables and their measurements. In fact, starting from the premise than an observable is real after it is signal-measured, they envisage a thomography-based protocol that, given a quantum state, allows to propose a quantifier for the degree of indefiniteness of an observable. This allows to investigate quantum correlations and to signal nonlocality even for separable states, thus revealing nonlocal aspects that are not captured by Bell inequality violations [41]–[43]. Here we test the approach of Ref.[40] for a phenomenon of remarkable interest as the neutrino oscillations, which we describe both in the plane-wave approximation (pure state) and in the wave-packet approach (mixed state).

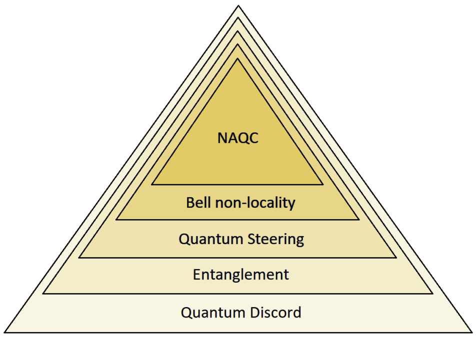

Figure 1: Hierarchy of quantum correlations (Figure adapted from Ref.[37]).

The paper is organized as follows. In Section 2 we briefly sum up the content of Ref.[40]. In Section 3 we test this approach for the phenomenon of neutrino oscillations, investigating nonlocal features. Conclusions follow. Finally, in Appendix some examples to clarify the topics are included.

2. Defining realism and indefiniteness

In Ref. [40] the following elements are considered: a state ρ of a generic quantum system associated to a separable Hilbert space $\mathcal{H}$, a subsystem of this global system associated to a Hilbert subspace $\mathcal{H}_1 \in \mathcal{H}$, and an observable (Hermitian operator) $\hat{O}_1$ on $\mathcal{H}_1$. The first step is to provide a definition of realism for the operator $\hat{O}_1$ and, using the complementary point of view, to quantify the amount of its possible indefiniteness. Then the final goal is to show that the presence of nonlocal quantum correlations implies a difference between the amount of indefiniteness of $\hat{O}_1$ with respect to the global state ρ, and with respect to the state of the subsystem.

Very roughly speaking, one can say that the observable $\hat{O}_1$ on $\mathcal{H}_1$ is "real" relative to the state ρ of the global system if "it allows the complete reconstruction of this state". To obtain a more precise definition of this criterion, in Ref. [40] two procedures are compared, both based on state tomography. Quantum state tomography is a tool to reconstruct the density matrix of an unknown ensemble of particles through a series of measurements [44, 45]. In order to eliminate statistical errors, a complete and exact reconstruction of a state would require an ideal tomography, realized by performing an infinite number of measurements on identically prepared systems. Although unrealistic, such a procedure can be considered in principle to obtain, as in Ref. [40], some physically sound concepts.

In the first procedure described in Ref. [40] one just repeats a state tomography as many times as necessary to obtain at last an ideal tomography that completely determines the density matrix ρ of the state. The second procedure is again based on state tomography, but it is modified in such a way that the observable $\hat{O}_1$ plays a "disturbing" role. In fact now, in every run of the procedure one places a secret measurement of this observable in between the preparation of the state and the tomography. In practice, the state is prepared, a secret measurement of the observable $\hat{O}_1$ is performed, and then the tomography is realized. Subsequently, this step is repeated again and again. Obviously, in general the disturbing role of the observable $\hat{O}_1$ can limit the precision with which the quantum state is reconstructed. To understand this point, we express the secretly measured observable $\hat{O}_1$ as

$$\hat{O}_1 = \sum_k \lambda_{1k}\, p_k^{(\hat{O}_1)} \tag{1}$$

where λ_{1k} and $p_k^{(\hat{O}_1)} = |v_k^{(\hat{O}_1)}\rangle\langle v_k^{(\hat{O}_1)}|$ are, respectively, the k-th eigenvalue and the projector (projectors) associated to to the eigenstate (eigenstates) $|v_k^{(\hat{O}_1)}\rangle$ relative to this eigenvalue. According to Quantum Theory, without any information on the secret measurements, the best description of the state of the system after the state tomography will be [46]

$$\Phi_{\hat{O}_1}(\rho) = \sum_k p_k^{(\hat{O}_1)} \rho\, p_k^{(\hat{O}_1)} = \sum_k P_k^{(\hat{O}_1)} p_k^{(\hat{O}_1)} \bigotimes \rho_{2|\lambda_{1k}}, \tag{2}$$

where

$$P_k^{(\hat{O}_1)} = Tr(p_k^{(\hat{O}_1)} \rho\, p_k^{(\hat{O}_1)}) \tag{3}$$

is the probability associated with this particular outcome of the measurement, and

$$\rho_{2|\lambda_{1k}} = \frac{Tr_1(p_k^{(\hat{O}_1)} \rho\, p_k^{(\hat{O}_1)})}{P_k^{(\hat{O}_1)}} \tag{4}$$

is the state of the rest of the system given the outcome $p_k^{(\hat{O}_1)}$.

On the basis of this approach, one thus can see that the best description of the state of the system obtained by following the second procedure provides a complete reconstruction of the quantum state if and only if

$$\Phi_{\hat{O}_1}(\rho) = \rho. \tag{5}$$

In this case the secret measurement of the observable $\hat{O}_1$ does not forbid the complete reconstruction of the state, and thus in Ref. [40] the observable $\hat{O}_1$ is defined as an "element of reality". When condition (5) is not satisfied, to the observable $\hat{O}_1$ can instead be associated a certain degree of indefiniteness, quantified in terms of entropic distance by the *indefiniteness measure*

$$\mathcal{J}(\hat{O}_1|\rho) = S(\Phi_{\hat{O}_1}(\rho)) - S(\rho). \tag{6}$$

The interesting point is that this definition can be connected to the presence of nonlocal quantum correlations, and can lead to a measure of nonlocality.

First of all, one can exploit definition (6) to show that the presence of nonlocal quantum correlations between the subsystem, labeled 1, and the rest of the global system, labeled 2, implies different values for "global" and "local" indefiniteness of an observable. In fact, one can evaluate indefiniteness of the observable $\hat{O}_1$ both with respect to the global state ρ (global indefiniteness) and with respect the reduced state $\rho^{(1)} = Tr_2\rho$ associated to the subspace $\mathcal{H}_1$. If one defines the quantity $\mathcal{D}_{[\hat{O}_1]}$ in terms of the mutual information $I_{1:2}$ as $\mathcal{D}_{[\hat{O}_1]}(\rho) = I_{1:2}(\rho) - I_{1:2}(\Phi_{\hat{O}_1}(\rho))$, it can be proved the relation [40]

$$\Delta\mathcal{J}(\hat{O}_1) \doteq \mathcal{J}(\hat{O}_1|\rho) - \mathcal{J}(\hat{O}_1|\rho^{(1)}) = \mathcal{D}_{[\hat{O}_1]}(\rho). \tag{7}$$

From this relation eventually follows the inequality

$$\Delta\mathcal{J}(\hat{O}_1) \geq \mathcal{D}_1(\rho), \tag{8}$$

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where $\mathcal{D}_1(\rho) = \min_{[\hat{O}_1]} \mathcal{D}_{[\hat{O}_1]}(\rho)$ is nothing but the quantum discord. Inequality (8) shows that the presence of nonlocal correlations (i. e. $\mathcal{D}_1(\rho) > 0$) has consequences on the indefiniteness property of the observable $\hat{O}_1$, and this prove the thesis of Ref. [40]. Incidentally, it is worth to be reported that in Ref. [47] it is shown that the concepts of local realism and local indefiniteness can lead in the case of pure states to complete complementarity relations. In fact, if one defines the amount of local realism as

$$\mathcal{R}(\hat{O}|\rho_1) = log_2 d_1 - \mathcal{J}(\hat{O}|\rho_1), \tag{9}$$

due to the fact that $\Phi_{\hat{O}}(\rho_1) = \rho_1^{diag}$, one has

$$\mathcal{R}(\hat{O}|\rho_1) = P_{vn}(\rho_A) + S_{vn}(\rho_A), \tag{10}$$

and

$$\mathcal{J}(\hat{O}|\rho_1) = C_{re}(\rho_A). \tag{11}$$

In Ref. [40] it is further shown that the property of indefiniteness implies a degree of nonlocality which can be quantified. In fact, one can consider two space-like separated subsystems 1 and 2, and investigate how a physical action on one subsystem influences the reality of the other. This leads to the concept of *minimal nonlocality* [40] defined by

$$\mathcal{N}_{min}(\rho) = \min_{\hat{O}_1, \hat{O}_2} \mathcal{N}(\hat{O}_1, \hat{O}_2|\rho), \tag{12}$$

where $\hat{O}_1, \hat{O}_2$ denotes operators on subsystems 1 and 2, and

$$\mathcal{N}(\hat{O}_1, \hat{O}_2|\rho) = \mathcal{J}(\hat{O}_1|\rho) - \mathcal{J}(\hat{O}_1|\Phi_{\hat{O}_2}(\rho)). \tag{13}$$

In later articles [41, 48] it is shown that this concept of nonlocality have interesting properties: it is is more resilient with repsect to local and bilocal weak measurements, and the set of states possessing this type of quantumness forms a strict superset of symmetrically discordant states and, therefore, of discordant, entangled, steerable, and Bell-nonlocal states.

This concludes the overview on Ref. [40], and in the following we will investigate realism/indefiniteness and nonlocality in the case of neutrino oscillations, where the relevant quantum correlation will be the quantum discord (see Fig.1).

3. Neutrino oscillations

Neutrino oscillations are a rare example of a macroscopically extended quantum phenomenon, due to the fact that a neutrino flavor state is a superposition of mass eigenstates with slightly different mass values. A neutrino which starts with a definite flavor (electronic ν_e, muonic ν_μ or tauonic ν_τ) subsequently evolves in time as a superposition of different flavors, among which it oscillates. The oscillations can be described in plane-wave approximation, where the neutrino state is a pure quantum state and only a time dependence is present. This description can be effective under some conditions, but in other cases is required a more realistic approach [23], where localization effects can be accounted for, namely the wave-packet approach [49, 50] where one starts with a dependence both on time and space, but where, due to the long time exposure of the detectors, it is convenient to consider an average in time of the density matrix operator. In this case a mixed state is obtained, and the ultimate dependence is on space.

Since the density matrices in the two approaches share the same general form, it is possible to follow a common procedure which we develop in the subsection 3.1. In subsections 3.2, 3.3 we will describe results in the case of plane-wave approximation and of wave-packet approach, respectively.

3.1. Common Procedure

In order to apply the concepts of the previous section, the two-flavor neutrino system can be described by a two-qubit system: $|\nu_e\rangle \equiv |10\rangle\,,\ |\nu_\mu\rangle \equiv |01\rangle$ [1]. Adopting this correspondence, the density matrix, associated to the bipartite neutrino state $|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)\,|\nu_\alpha\rangle + a_{\alpha\beta}(t)\,|\nu_\beta\rangle$, in the plane-wave approximation is

$$\rho^{(\alpha)}_{\alpha\beta}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{\alpha\alpha}(t)|^2 & a_{\alpha\alpha}(t)a^*_{\alpha\beta}(t) & 0 \\ 0 & a^*_{\alpha\alpha}(t)a_{\alpha\beta}(t) & |a_{\alpha\beta}(t)|^2 & 0 \\ 0 & 0 & 0 & 0, \end{pmatrix} \tag{14}$$

where $\alpha, \beta = e, \mu,\ \beta \neq \alpha$, and the time-dependent terms $a_{\eta,\eta'}(t)\ \eta, \eta' = \alpha, \beta$, denote the transition probability amplitudes. Then, obviously:

$$|a_{\alpha\alpha}(t)|^2 = P_{\alpha\alpha}(t),\ |a_{\alpha\beta}(t)|^2 = P_{\alpha\beta}(t) \tag{15}$$

and $P_{\alpha\alpha}(t),\ P_{\alpha\beta}(t)$ are the transition probabilities. Eq.(14) is referred to the situation in which a neutrino starts in the flavor α and then gives rise to a superposition of flavors $\alpha, \beta, \beta \neq \alpha$.

The density matrix[1] in the wave packet approach is given by [4]:

$$\rho_\alpha(x) = \sum_{k,j} U_{\alpha k} U^*_{\alpha j} f_{jk}(x)\,|\nu_j\rangle\,\langle\nu_k|\,, \tag{16}$$

where $f_{jk}(x) = \exp\left[-i\frac{\Delta m^2_{jk}x}{2E} - \left(\frac{\Delta m^2_{jk}x}{4\sqrt{2}E^2\sigma_x}\right)^2\right]$. We express $\rho_\alpha(x)$ in terms of flavor eigenstates by establishing the identification $|\nu_\alpha\rangle = |\delta_{\alpha e}\rangle_e\,|\delta_{\alpha\mu}\rangle_\mu\,|\delta_{\alpha\tau}\rangle_\tau$. By using the relation $|\nu_i\rangle = \sum_\alpha U_{\alpha i}\,|\nu_\alpha\rangle$, we can write:

$$\rho^{(\alpha)}_{e\mu\tau}(x) = \sum_{\beta\gamma} F^\alpha_{\beta\gamma}(x)\,|\delta_{\beta e}\delta_{\beta\mu}\delta_{\beta\tau}\rangle\,\langle\delta_{\gamma e}\delta_{\gamma\mu}\delta_{\gamma\tau}| \tag{17}$$

where $\alpha = e, \mu, \tau$ and

$$F^\alpha_{\beta\gamma}(x) = \sum_{kj} U^*_{\alpha j} U_{\alpha k} f_{jk}(x) U_{\beta j} U^*_{\gamma k}. \tag{18}$$

For example, the explicit form for the density matrix associated to an initial electronic bipartite neutrino state (two flavors), is given by:

$$\rho^{(e)}_{e\mu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^e_{ee}(x) & F^e_{e\mu}(x) & 0 \\ 0 & F^e_{\mu e}(x) & F^e_{\mu\mu}(x) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

We note that we can write with shorthand notation both density matrices (14), (19) in the common general form

$$\rho(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b^* & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

[1] Eqs.(16)-(18) are presented for the general case of three flavors. However, in the following, we will consider two-flavor reductions for the various experimental situations.

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Obviously, in the plane-wave case ($\rho = \rho_\alpha(t)$):

$$a = |a_{\alpha\alpha}(t)|^2, \; b = a_{\alpha\alpha}(t)a^*_{\alpha\beta}(t), \; c = |a_{\alpha\beta}(t)|^2, \tag{21}$$

while in the wave-packet approach ($\rho = \rho_{e\mu}(x)$):

$$a = F^e_{ee}(x), \; b = F^e_{e\mu}(x), \; c = F^e_{\mu\mu}(x). \tag{22}$$

The Pauli operators $\sigma_x, \sigma_y, \sigma_z$ and the identity matrix $\mathbf{1}_2$ provide a basis for the four-dimensional space with two-dimensional subspaces[2]. Then, we can consider the action of these operators on ρ and investigate the concepts introduced in the previous section. Here, we report the main results of this analysis. We remark that in all the plots the Pauli matrices are always referred to a subspace after reduction because we are investigating the influence of observables defined on a subsystem on the observables defined in the other subsystem.

- σ_x results to be completely not real when it acts on ρ, i.e. $\mathcal{J}(\sigma_x|\rho) = 1$, while its local indefiniteness is $\mathcal{J}(\sigma_x|\rho_1) = 1 + a\log_2 a + c\log_2 c$. The difference between global and local indefiniteness of σ_x is equal to $\Delta\mathcal{J}(\sigma_x) = \mathcal{D}_{[\sigma_x]}(\rho) = -a\log_2 a - c\log_2 c$, which indeed coincides with the quantum discord evaluated for ρ.
- σ_z results to be partially not real when it acts on ρ, i.e. $\mathcal{J}(\sigma_z|\rho) = -a\log_2 a - c\log_2 c + \lambda_+\log_2\lambda_+ + \lambda_-\log_2\lambda_-$, with $\lambda_\pm = \frac{1\pm\sqrt{1-4(ac-|b|^2)}}{2}$, while its local indefiniteness is $\mathcal{J}(\sigma_z|\rho_1) = 0$. The difference between global and local indefiniteness of σ_z is equal to $\Delta\mathcal{J}(\sigma_z) = \mathcal{D}_{[\sigma_z]}(\rho) = -a\log_2 a - c\log_2 c$, which coincides with the quantum discord evaluated for ρ.

Regarding the nonlocality indicator, we find:

- $\mathcal{N}(\sigma_x, \sigma_y|\rho) = \mathcal{N}(\sigma_y, \sigma_x|\rho) = -a\log_2\frac{a}{2} - c\log_2\frac{c}{2} + \frac{\lambda_-}{2}\log_2\frac{\lambda_-}{4} + \frac{\lambda_+}{2}\log_2\frac{\lambda_+}{4}$.
- $\mathcal{N}(\sigma_x, \sigma_z|\rho) = \mathcal{N}(\sigma_z, \sigma_x|\rho) = 1 + a\log_2\frac{a}{2} + c\log_2\frac{c}{2} - a\log_2 a - c\log_2 c$.
- $\mathcal{N}(\sigma_y, \sigma_z|\rho) = \mathcal{N}(\sigma_z, \sigma_y|\rho) = 1 + a\log_2\frac{a}{2} + c\log_2\frac{c}{2} - a\log_2 a - c\log_2 c$.

In the following sections we discuss these results for pure and mixed bipartite neutrino states in connection with parameters from some important neutrino oscillation experiments.

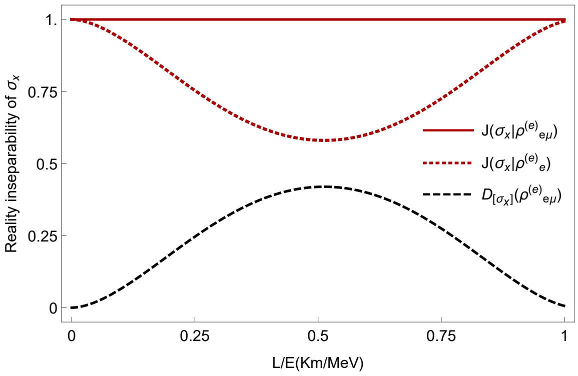
3.2. Plane-wave approximation

Now, by using the correspondence in Eq.(21) it is possible to discuss the previous results for the case of a two flavor pure neutrino state. In Fig.2 we plot the local and global indefiniteness of σ_x, σ_y and σ_z as functions of L/E for an initial electronic neutrino, by using parameters from Daya-Bay experiment (see Table).

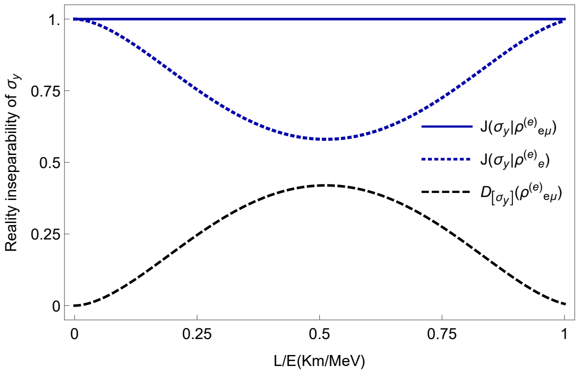
Daya-Bay	MINOS
$\Delta m^2_{ee} = 2.42^{+0.10}_{-0.11} \times 10^{-3} eV^2$	$\Delta m^2_{32} = 2.32^{+0.12}_{-0.08} \times 10^{-3} eV^2$
$\sin^2 2\theta_{13} = 0.084^{+0.005}_{-0.005}$	$\sin^2 2\theta_{23} = 0.95^{+0.035}_{-0.036}$
$L \in [364m, 1912m]$	$L = 735$ km
$E \in [1MeV, 8MeV]$	$E \in [0.5GeV, 50GeV]$

Let us analyze in particular the case of σ_z operator, which it is locally completely real. So we can predict with certainty the result of a measurement on the state. However, this realism is influenced by the other subsystem. In fact, an indefiniteness is introduced when correlations, quantified by a discord-like measure, between subsystems are present: in this case σ_z becomes

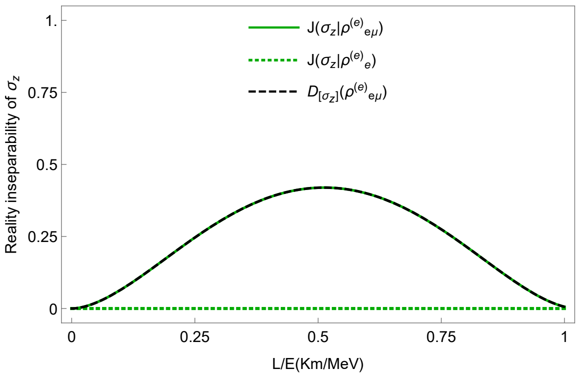
[2] We consider $\sigma_i = |\psi^e_{i+}\rangle\langle\psi^e_{i+}| - |\psi^e_{i-}\rangle\langle\psi^e_{i-}|$, $i = x, y, z$, when it acts on the reduced density matrix $\rho^{(e)}_e$, while $\sigma^{(e)}_i = (|\psi^e_{i+}\rangle\langle\psi^e_{i+}| - |\psi^e_{i-}\rangle\langle\psi^e_{i-}|) \otimes \mathbf{1}^{(\mu)}_\mathbf{2}$ if it acts on the global state $\rho^{(e)}_{e\mu}$.



(a) σ_x

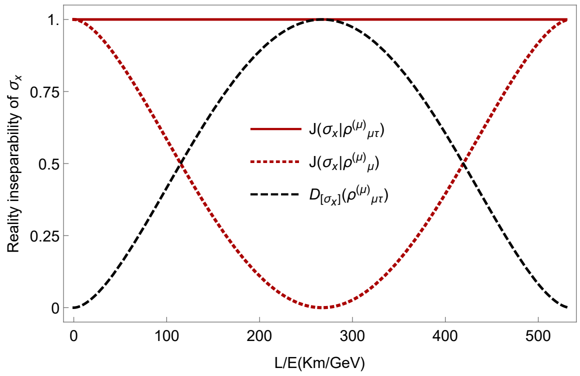


(b) σ_y

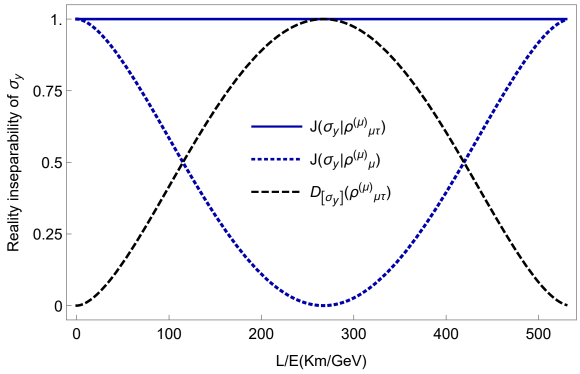


(c) σ_z

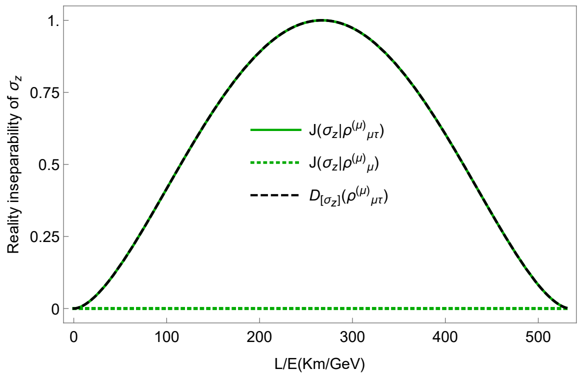
Figure 2: Reality inseparability of σ_x (a), σ_y (b), σ_z (c) for an initial electronic pure neutrino state (Daya-Bay).



(a) σ_x



(b) σ_y



(c) σ_z

Figure 3: Reality inseparability of σ_x (a), σ_y (b), σ_z (c) for an initial muonic pure neutrino state (MINOS).

partially not real on the global system. This demonstrates how the reality of an observable cannot be considered separately by the other subsystems. Similar considerations are valid for σ_x and σ_y, whose local indefiniteness is strongly dependent on the correlations between subsystems[3].

Furthermore, it is possible to observe that the terms $\mathcal{D}_{[\sigma_x]}(\rho^{(e)}_{e\mu})$, $\mathcal{D}_{[\sigma_y]}(\rho^{(e)}_{e\mu})$ and $\mathcal{D}_{[\sigma_z]}(\rho^{(e)}_{e\mu})$

[3] The physical meaning of the Pauli operators in this context can be understood in terms of number operators for neutrinos with different flavors in quantum mechanics, or in terms of flavor charge operators in a quantum field theory setting [3].

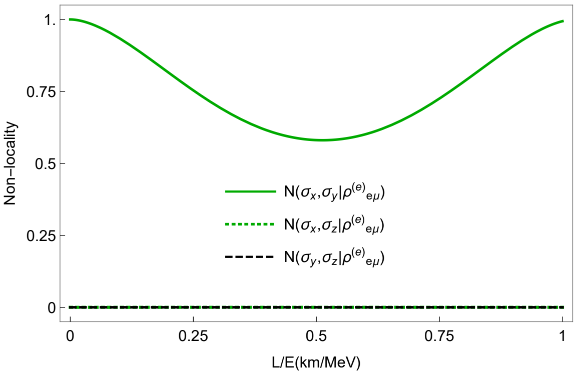


Figure 4: Nonlocality indicators for an initial electronic pure neutrino state (Daya-Bay).

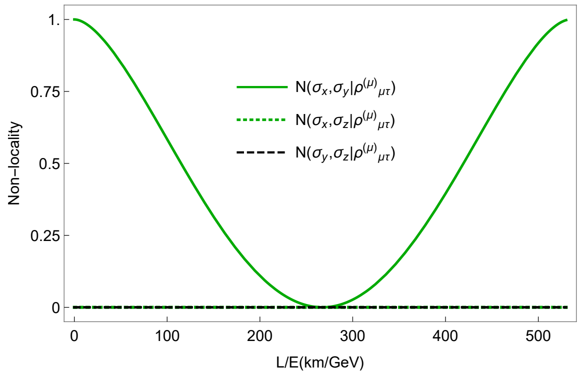


Figure 5: Nonlocality indicators for an initial muonic pure neutrino state (MINOS).

coincide among themselves and are equal to the quantum discord evaluated for the state under examination. Thus, all the operators minimize the discord-like measure, Eq.(7).

In Fig.3 it is possible to observe the local and global indefiniteness of σ_x, σ_y and σ_z as functions of L/E for an initial muonic neutrino, by using parameter from MINOS experiment (see Table). In this case, similar considerations as for the electronic case are valid. Here, we only point out that the different trends are due to the different values of the mixing angles associated to the experiments.

In Fig.4 we can observe the nonlocality indicators $\mathcal{N}(\sigma_x, \sigma_y|\rho^{(e)}_{e\mu})$, $\mathcal{N}(\sigma_x, \sigma_z|\rho^{(e)}_{e\mu})$ and $\mathcal{N}(\sigma_y, \sigma_z|\rho^{(e)}_{e\mu})$ for an initial electronic state. We can observe that the only non zero indicator is $\mathcal{N}(\sigma_x, \sigma_y|\rho^{(e)}_{e\mu})$. This means that the reality of the observable σ_x on one subsystem is influenced by unrevealed measurement of observable σ_y on the other subsystem and vice-versa. On the other hand, since $\mathcal{N}(\sigma_x, \sigma_z|\rho^{(e)}_{e\mu}) = \mathcal{N}(\sigma_y, \sigma_z|\rho^{(e)}_{e\mu}) = 0$, unrevealed measurements of observable σ_z on one subsystem cannot influence the reality of observables σ_x and σ_y on the other subsystem. Furthemore, from Figs. 2,4 we can observe that in the particular case of a neutrino bipartite state:

- $\mathcal{N}(\sigma_x, \sigma_y|\rho^{(e)}_{e\mu}) = J(\sigma_x|\rho_e) = J(\sigma_y|\rho_e)$,
- $\mathcal{N}(\sigma_x, \sigma_z|\rho^{(e)}_{e\mu}) = \mathcal{N}(\sigma_y, \sigma_z|\rho^{(e)}_{e\mu}) = J(\sigma_z|\rho_e)$.

In Fig.5 we can observe the nonlocality indicators $\mathcal{N}(\sigma_x, \sigma_y|\rho^{(\mu)}_{\mu\tau})$, $\mathcal{N}(\sigma_x, \sigma_z|\rho^{(\mu)}_{\mu\tau})$ and $\mathcal{N}(\sigma_y, \sigma_z|\rho^{(\mu)}_{\mu\tau})$ for an initial muonic state, by using parameter from MINOS experiment. For the $\nu_\mu \to \nu_\tau$ oscillation, analogous considerations as for the above Daya-Bay case are valid.

3.3. Wave-packet approach
By using the correspondence in Eq.(22) it is possible to discuss the previous results for the case of a two flavor mixed neutrino state, for which a wave packet approach for oscillations is considered. In Figs. 6, 7 we plot the local and global indefiniteness of σ_x, σ_y and σ_z as functions of space for an initial and muonic electronic neutrino, by using parameters from Daya-Bay and MINOS experiments, respectively. For the electronic case, the spatial extent of the neutrino wave packet we choose is $\sigma^x = 5 \cdot 10^{-6} m$, while for the muonic case we choose $\sigma^x = 7 \cdot 10^{-9} m$. These values are perfectly in agreement with the limits indicated in Ref.[51]. In Fig.5 we plot the nonlocality indicators $\mathcal{N}(\sigma_x, \sigma_y|\rho^{(\mu)}_{\mu\tau})$, $\mathcal{N}(\sigma_x, \sigma_z|\rho^{(\mu)}_{\mu\tau})$ and $\mathcal{N}(\sigma_y, \sigma_z|\rho^{(\mu)}_{\mu\tau})$ for an initial muonic mixed state.

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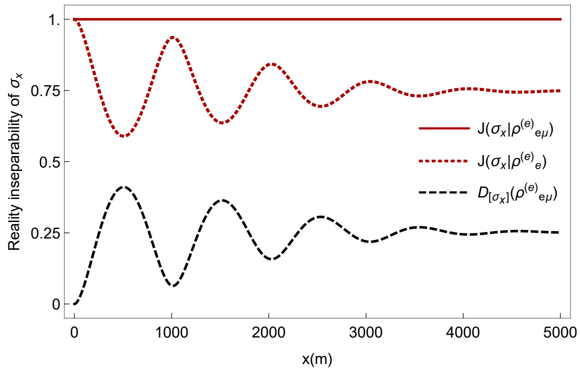
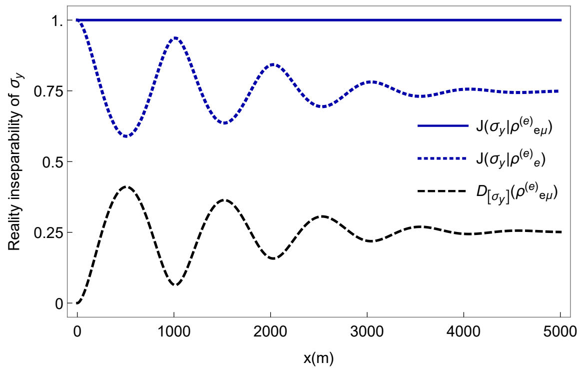
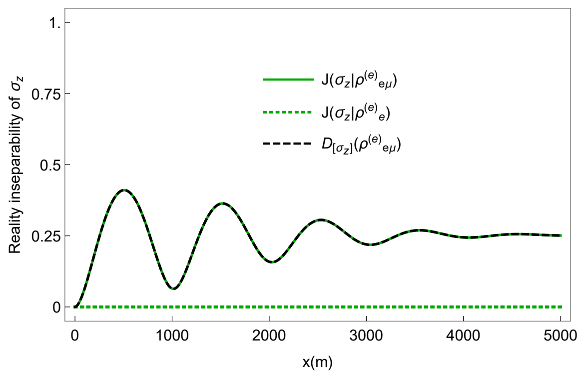


(a) σ_x

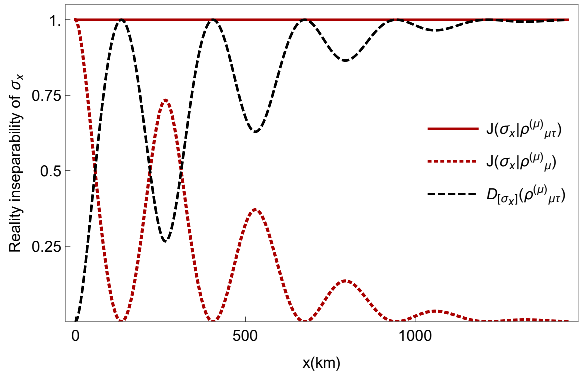


(b) σ_y

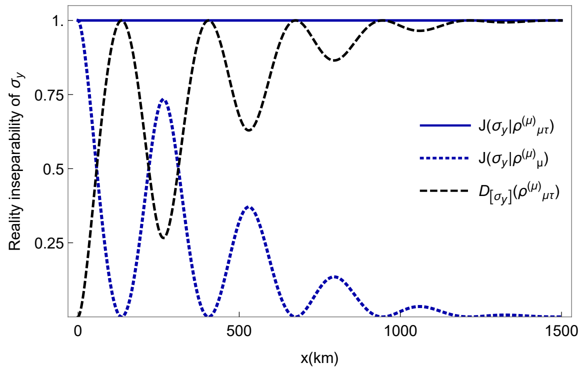


(c) σ_z

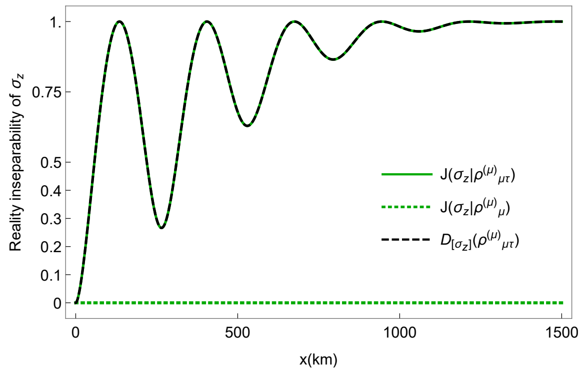
Figure 6: Reality inseparability of σ_x (a), σ_y (b), σ_z (c) for an initial electronic pure neutrino state in the wave packet approach (Daya-Bay).



(a) σ_x



(b) σ_y



(c) σ_z

Figure 7: Reality inseparability of σ_x (a), σ_y (b), σ_z (c) for an initial muonic pure neutrino state in the wave packet approach (MINOS).

The considerations made in the previous section continue to apply even in the case of a neutrino mixed state. We observe the different long distance behavior for the initial electronic and the muonic cases, due to the different values of the mixing angles associated to these oscillations.

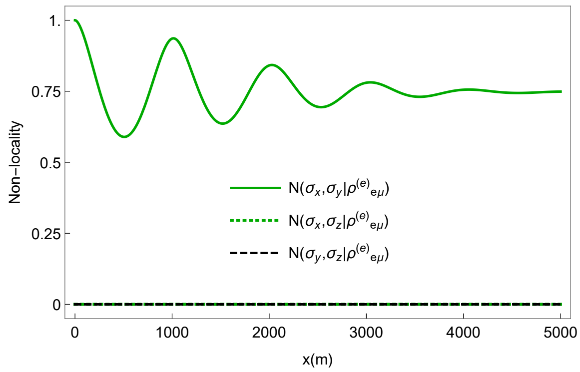


Figure 8: Nonlocality indicators for an initial electronic pure neutrino state in the wave packet approach (Daya Bay).

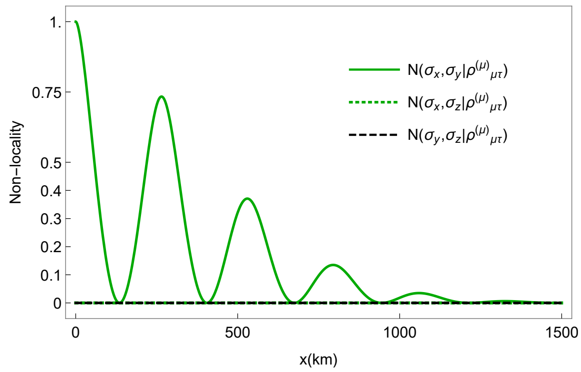


Figure 9: Nonlocality indicators for an initial muonic pure neutrino state in the wave packet approach (MINOS).

4. Conclusions

In this paper, we have analyzed the nonlocality associated to neutrino oscillations by exploiting the approach of Ref.[40], in which the concept of reality/indefiniteness of an operator is introduced. We have considered both pure and mixed neutrino bipartite states, for which we use a plane wave approximation and a wave packet approach, respectively. In both cases, it has been possible to conclude that the presence of correlations influences the realism of an operator for the neutrino system. In other words, the reality of an observable acting on one part of the neutrino bipartite system cannot be considered separately from the other part.

Again, we observe that the realism of an operator when it acts on one subsystem can be influenced by unrevealed measurements of another operator on the other subsystem. Whether this happens or not, it depends on the chosen operators. In particular, for the case at hand, a measurement of σ_z on one subsystem cannot influence the reality of operators σ_x, σ_y on the other subsystem and vice-versa, while a measurement of $\sigma_x(\sigma_y)$ influences the reality of $\sigma_y(\sigma_z)$.

These concepts and tools can be useful to identify physically relevant observables when one aims to go in to detail in investigating properties of quantum correlations in the physics of neutrino oscillations, with also an eye on possible future applications in quantum information protocols.

Appendix A.

First example

For pure states the discord coincides with entanglement, and it signals the absence of nonlocal correlations if it is zero. For the following separable pure state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |01\rangle].$$

For this state we obtain:

- $\mathcal{J}(\sigma_x|\rho) = 1$, $\mathcal{J}(\sigma_x|\rho_1) = 1$.
- $\mathcal{J}(\sigma_y|\rho) = 1$, $\mathcal{J}(\sigma_y|\rho_1) = 1$.
- $\mathcal{J}(\sigma_z|\rho) = 0$, $\mathcal{J}(\sigma_z|\rho_1) = 0$.

In the end, we can conclude that $\Delta\mathcal{J}(\sigma_x) = \Delta\mathcal{J}(\sigma_y) = \Delta\mathcal{J}(\sigma_z) = 0$. Again:

- $\mathcal{N}(\sigma_x, \sigma_y|\rho) = 1$.
- $\mathcal{N}(\sigma_x, \sigma_z|\rho) = 0$.

- $\mathcal{N}(\sigma_y, \sigma_z|\rho) = 0$.

To conclude, the minimum non-locality is given by $\mathcal{N}_{min} = 0$.

Second example
As second example we choose the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle].$$

For this state we obtain:

- $\mathcal{J}(\sigma_x|\rho) = 1$, $\mathcal{J}(\sigma_x|\rho_1) = 0$.
- $\mathcal{J}(\sigma_y|\rho) = 1$, $\mathcal{J}(\sigma_y|\rho_1) = 0$.
- $\mathcal{J}(\sigma_z|\rho) = 1$, $\mathcal{J}(\sigma_z|\rho_1) = 0$.

In the end, we can conclude that $\Delta\mathcal{J}(\sigma_x) = \Delta\mathcal{J}(\sigma_y) = \Delta\mathcal{J}(\sigma_z) = 1$. Again:

- $\mathcal{N}(\sigma_x, \sigma_y|\rho) = 1$.
- $\mathcal{N}(\sigma_x, \sigma_z|\rho) = 1$.
- $\mathcal{N}(\sigma_y, \sigma_z|\rho) = 1$.

To conclude, the minimum non-locality is given by $\mathcal{N}_{min} = 1$.

Third example
As third example we consider the maximal discord-separable mixed state:

$$\rho = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

For this state we obtain:

- $\mathcal{J}(\sigma_x|\rho) = \frac{1}{3}$, $\mathcal{J}(\sigma_x|\rho_1) = 0$.
- $\mathcal{J}(\sigma_y|\rho) = \frac{1}{3}$, $\mathcal{J}(\sigma_y|\rho_1) = 0$.
- $\mathcal{J}(\sigma_z|\rho) = \frac{1}{3}$, $\mathcal{J}(\sigma_z|\rho_1) = 0$.

In the end, we can conclude that $\Delta\mathcal{J}(\sigma_x) = \Delta\mathcal{J}(\sigma_y) = \Delta\mathcal{J}(\sigma_z) = \frac{1}{3}$ and coincide with the quantum discord [52]. Again:

- $\mathcal{N}(\sigma_x, \sigma_y|\rho) = 1.25$.
- $\mathcal{N}(\sigma_x, \sigma_z|\rho) = 1.25$.
- $\mathcal{N}(\sigma_y, \sigma_z|\rho) = 1.25$.

To conclude, the minimum non-locality is given by $\mathcal{N}_{min} = 1.25$.

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