

Dark Range of ω In Brans-Dicke Gravity

Tekin Dereli^{1,2} and Yorgo Şenikoğlu²

¹ Department of Physics, Koç University, 34450 Sarıyer, İstanbul, Turkey

² Department of Basic Sciences, Faculty of Engineering and Natural Sciences, Maltepe University, 34857 Maltepe, İstanbul, Turkey

E-mail: tdereli@ku.edu.tr, yorgosenikoglu@maltepe.edu.tr

Abstract.

The variational field equations of Brans-Dicke scalar-tensor theory of gravitation that is coupled to a mass-varying sterile neutrino field are presented in Riemannian and non-Riemannian setting in the language of exterior differential forms over four-dimensional spacetime. In a gravitational plane wave geometry in Rosen coordinates, with the scalar field set equal to a constant, the field equations admit ghost neutrino solutions. These correspond to Majorana spinor configurations for which the total neutrino stress-energy-momentum tensor vanishes. We show here that in the absence of a sterile neutrino in the gravitational wave geometry, there are scalar field configurations for which the total scalar field stress-energy-momentum tensor vanishes for negative values of the Brans-Dicke parameter $-3/2 < \omega < 0$.

1. Introduction

General relativity is a geometrical theory of gravitation that is determined by the metric of spacetime. Brans-Dicke theory of gravity [1, 2, 3], a scalar-tensor theory, introduces an adjustment to the initial framework of Einstein in order to incorporate Mach's Principle by including a scalar field ϕ . The astrophysical observations supplied to scientist with the construct of low energy string theories with scalar fields are very efficient in depicting the large scale structure of spacetime and subatomic physics; so it is very reasonable to recognize them also at large scales.

Brans-Dicke theory covers the geodesic postulate as well; the motion of a test mass is determined by the geodesics with the gravitational effects of all particles embedded in the metric and the associated Levi-Civita connection. The assumption that the matter lagrangian is independent of the scalar field ϕ secures the equivalence principle. The string unification of gravity and quantum corrections to higher orders make researchers think on the property and reasonableness of the theory of gravitational interaction being depicted, by geometry alone. There are several hints that a non-Riemannian setting of spacetime may contribute a refined and coherent way to describe gravitational fields [4, 5]. A first order variation of the Brans-Dicke Lagrangian implies that the spacetime connection can admit a non-vanishing torsion. The field equations obtained are equivalent to up to a shift in the coupling constant [4].

Under the influence of gravity alone, non-spinning test masses follow geodesic equations of motion in a Riemannian spacetime and autoparallels of a connection with torsion [6] in a non-Riemannian geometry. The existence of other matter fields will reshape the geometry of spacetime, and in the presence spinorial matter, torsion will be picked up by differential equations [7, 8]. In this article, the gradient of the scalar field determines the space-time torsion, and a



particular choice of the scalar field applied to parallelly propagating gravitational plane waves will reduce the field equations to the vacuum Einstein equations. In particular, we are motivated on the interpretation in a Riemannian and non-Riemannian geometry. Explicitly, the Brans-Dicke parameter and the autoparallels of the connection with torsion which differ from the geodesic equations of motion will help us understand the coherence of the explanation.

The article is organized as follows: in Section 2, we present the Brans-Dicke theory of gravity with the gravitational field equations and the equations of motion of a non-spinning test mass in a non-Riemannian setting in the language of exterior differential forms. Section 3 details the plane wave solutions in Rosen coordinates and a particular choice of scalar field that reduce the field equations to source-free Einstein equations. In Section 4 we consider a mass-varying sterile neutrino field coupled to Brans-Dicke gravity. In particular two special sub-cases of solutions that describe ghost neutrinos on the one hand and ghost scalars on the other are discussed. Section 5 is devoted to concluding remarks.

2. Brans-Dicke scalar-tensor gravity

The Brans-Dicke field equations will be determined from the action $I = \int_M \mathcal{L}$, where M denotes the 4-dimensional spacetime manifold and the Brans-Dicke Lagrange density 4-form

$$\mathcal{L} = \frac{\alpha^2}{2} R_{ab} \wedge *(e^a \wedge e^b) - \frac{c}{2} d\alpha \wedge *d\alpha. \quad (1)$$

Here $\{e^a\}$'s are the co-frame 1-forms in terms of which the space-time metric is given by $g = \eta_{ab} e^a \otimes e^b$ with $\eta_{ab} = \text{diag}(-+++)$. $*$ stands for the Hodge map. The volume form is $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. We wrote the Brans-Dicke scalar field as $\phi = \alpha^2$ for convenience. Then $\{\omega^a_b\}$ denote the connection 1-forms that satisfy the Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = T^a \quad (2)$$

with the torsion 2-forms T^a . On the other hand the curvature 2-forms R^a_b of spacetime are given by

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b. \quad (3)$$

We vary the action independently with respect to the co-frame 1-forms e^a , connection 1-forms ω^a_b and the scalar field α . The corresponding coupled field equations in the non-Riemannian description turn out to be ($c \neq 0$):

$$\begin{aligned} -\frac{\alpha^2}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) &= c \tau_a[\alpha], \\ T^a &= e^a \wedge \frac{d\alpha}{\alpha}, \\ cd * d\alpha^2 &= 0; \end{aligned} \quad (4)$$

where

$$\tau_a[\alpha] = \frac{1}{2} (\iota_a d\alpha * d\alpha + d\alpha \wedge \iota_a * d\alpha) \equiv T_{ab} * e^b \quad (5)$$

are the energy-momentum 3-forms of the scalar field and ι_a denote the interior products that satisfy $\iota_a e^b = \delta^b_a$. We can now introduce the Levi-Civita connection 1-forms $\{\hat{\omega}^a_b\}$, fixed in a unique way by the metric tensor through the Cartan structure equations

$$de^a + \hat{\omega}^a_b \wedge e^b = 0. \quad (6)$$

Then in terms of them, the equivalent form of the Brans-Dicke field equations (4) in the Riemannian description turn out to be:

$$\begin{aligned} -\frac{\alpha^2}{2} \hat{R}^{bc} \wedge *(e_a \wedge e_b \wedge e_c) &= \frac{(c-6)}{2} (\iota_a d\alpha * d\alpha + d\alpha \wedge \iota_a * d\alpha) \\ &\quad + \hat{D}(\iota_a * d\alpha^2), \\ cd * d\alpha^2 &= 0, \end{aligned} \quad (7)$$

where \hat{D} denotes the covariant exterior derivative with respect to the Levi-Civita connection. Thus our coupling constant c can be related with the classical Brans-Dicke coupling constant ω as

$$c = 4\omega + 6. \quad (8)$$

We note that the right hand side of the Einstein field equations in a Riemannian setting is composed of two terms[9]. The first term involves the Brans-Dicke coupling constant as an over-all factor in front of the scalar field stress-energy-momentum tensor. This term comes from the kinetic part of the scalar field in the action. While the second term does not contain any parameter. In fact it comes from the geometry of the spacetime itself, however, it is only implicit in the non-Riemannian description. We wish to point out here that there exists such parameter values, that is, there is a *dark range of ω* , where these two terms on the right hand side cancel each other out and the field equations reduce to the ordinary source-free Einstein equations as we have demonstrated above. Thus in certain spacetime geometries the geodesic equations of motion are not affected by the scalar field at all in the conventional Riemannian approach. There is a non-trivial scalar field but the motion of non-spinning test masses are not influenced by it. We have a conceptual problem in the conventional Riemannian interpretation whereas in the non-Riemannian description there is none.

3. Gravitational plane waves

In order to demonstrate our point, we wish to study below a specific family of solutions of the Brans-Dicke field equations. Let us consider the plane fronted gravitational wave metric in Rosen coordinates (u, v, x, y) to solve the Brans-Dicke field equations[10, 11]:

$$g = 2du \otimes dv + \frac{dx \otimes dx}{f(u)^2} + \frac{dy \otimes dy}{h(u)^2}. \quad (9)$$

We further take a scalar field $\alpha = \alpha(u)$ and note that the scalar field equation to the system is identically satisfied. Working out the expressions for the curvature, torsion and the scalar field stress-energy-momentum tensors, the Einstein field equations to be solved reduce to the following second order differential equation:

$$\frac{f_{uu}}{f} - \frac{2f_u^2}{f^2} + \frac{h_{uu}}{h} - \frac{2h_u^2}{h} - \frac{2\alpha_{uu}}{\alpha} + \frac{4\alpha_u^2}{\alpha^2} = \frac{c\alpha_u^2}{\alpha^2}. \quad (10)$$

Therefore for a particular choice of α such that

$$\alpha(u) = \left((\omega + 1)(C_1 u + C_2) \right)^{\frac{1}{2(\omega+1)}}, \quad (11)$$

where C_1 and C_2 are arbitrary integration constants, all the α -dependent terms in (10) drop out and we recover the source-free Einstein field equation:

$$\left(\frac{f_{uu}}{f} - \frac{2f_u^2}{f^2} + \frac{h_{uu}}{h} - \frac{2h_u^2}{h} \right) = 0. \quad (12)$$

Let us further decorate this observation by explicitly giving the geodesic equations of motion as functions of proper time τ and constants of motion p_u, p_x, p_y neatly as [11]

$$\begin{aligned} u(\tau) &= u(0) + \frac{p_u \tau}{m}, \\ v(\tau) &= v(0) - \frac{m\tau}{2p_u} - \frac{p_x^2}{2p_u^2} \int_0^\tau f(u)^2 du - \frac{p_y^2}{2p_u^2} \int_0^\tau h(u)^2 du, \\ x(\tau) &= x(0) + \frac{p_x}{p_u} \int_0^\tau f(u)^2 du, \quad y(\tau) = y(0) + \frac{p_y}{p_u} \int_0^\tau h(u)^2 du. \end{aligned} \quad (13)$$

In a similar way the auto-parallel equations would be given as

$$\begin{aligned} \tau &= \frac{m}{p_u} \int_0^{\frac{p_u \tau}{m\alpha}} \alpha(u) du, \\ v(\tau) &= v(0) - \frac{m^2}{2p_u^2} \int_0^{\frac{p_u \tau}{m\alpha}} \alpha(u)^2 du - \frac{p_x^2}{2p_u^2} \int_0^{\frac{p_u \tau}{m\alpha}} f^2(u) du - \frac{p_y^2}{2p_u^2} \int_0^{\frac{p_u \tau}{m\alpha}} h^2(u) du, \\ x(\tau) &= x(0) + \frac{p_x}{p_u} \int_0^{\frac{p_u \tau}{m\alpha}} f^2(u) du, \quad y(\tau) = y(0) + \frac{p_y}{p_u} \int_0^{\frac{p_u \tau}{m\alpha}} h^2(u) du, \end{aligned} \quad (14)$$

In the former case a non-spinning test mass does not interact with the scalar field whereas in the latter case it clearly does.

Taking the Brans-Dicke field equations with torsion and looking at solutions, we have found a particular gravitational plane wave solution. The metric is the source-free (vacuum) Einstein metric and the terms coming from torsion and the stress-energy momentum tensor cancel out and the scalar field equation is satisfied. In the non-Riemannian description, we do not have any issues, on the right hand side, the energy momentum tensor is different than zero and positive definite. The left hand side of the field equations, the geometry is affected by torsion, given by the gradient of the scalar field. When we look at the autoparallel equations they are not the source-free Einstein geodesic equations. We find different equations of motion as expected. Had we insisted on the classical Brans-Dicke interpretation, this solution would be a solution again. But what kind of a solution? If one insists upon the conventional interpretation of the Brans-Dicke gravity, then the above class of gravitational wave solutions with $-3/2 < \omega < 0$ should be left aside as unphysical[6, 12, 13]. On the other we wish to argue below for a broader perspective in which such solutions may play a role, for instance, in a cosmological context.

4. A mass-varying sterile neutrino field coupled to scalar-tensor gravity

The starting point of our argument concerns the Einstein-massless scalar field theory coupled to a mass-varying sterile neutrino field. We discussed this model some years ago[14] in a cosmological context by a variational principle from the following action density 4-form in the Einstein picture: written in the Einstein picture:

$$\mathcal{L} = \frac{1}{2\kappa^2} \mathcal{R} * 1 - \frac{1}{2} d\Phi \wedge * d\Phi + V(\Phi) * 1 + Her \left\{ \frac{i}{2} \bar{\Psi} (\gamma \wedge * \nabla) \Psi \right\} - \frac{i}{2} M(\Phi) (\bar{\Psi} \Psi) * 1. \quad (15)$$

Here $\kappa^2 = 8\pi G$ is Newton's gravitational constant. Φ is a 0-form representing an acceloron scalar while Ψ is a Majorana 4-spinor that represents a sterile neutrino with a varying mass $M(\Phi)$. One may also introduce a non-trivial acceloron potential function $V(\Phi)$. Some classes of exact cosmological solutions of the corresponding variational field equation, for the cases $V(\Phi) = \pm V_0$, were constructed and shown to be consistent asymptotically as $t \rightarrow 0$ with the observed expansion of the flat 3-universe in Λ CDM models[14]. Let us consider the same model

as above instead in the Jordan-Brans-Dicke picture. In order to pass from the Einstein picture to the Jordan-Brans-Dicke picture we employ the following field re-definitions:

$$\tilde{g} = e^{2\sigma}g, \quad \tilde{\Psi} \equiv \psi = e^{-\frac{3}{2}\sigma}\Psi, \quad \tilde{\Phi} \equiv \phi = e^{-\sigma}\Phi \quad (16)$$

where we choose a scale function

$$\sigma = \omega \ln \Phi. \quad (17)$$

A real parameter ω is introduced which at the end turns out to be nothing but the Brans-Dicke coupling constant. Then the action density 4-form in the Jordan-Brans-Dicke picture, dropping the $\tilde{}$ s for notational ease, reads

$$\begin{aligned} \mathcal{L} &= \frac{\alpha^2}{2} \mathcal{R} * 1 - \omega d\alpha \wedge *d\alpha + \bar{V}(\alpha) * 1 \\ &+ Her \left\{ \frac{i}{2} \bar{\psi} (\gamma \wedge * \nabla) \psi \right\} - \frac{i}{2} \bar{M}(\alpha) (\bar{\psi} \psi) * 1 + 3\omega (\bar{\psi} * \gamma \psi) \wedge \frac{d\alpha}{\alpha}, \end{aligned} \quad (18)$$

where $\alpha^2 = \phi$. The scalar potential $\bar{V}(\alpha)$ and the varying-neutrino mass $\bar{M}(\alpha)$ are suitably re-defined. It is remarkable that the gradient of the Brans-Dicke scalar couples to the vector current associated with a sterile neutrino field with a particular coupling strength proportional to the Brans-Dicke coupling constant. An analogous vector coupling between the acceleron scalar and the sterile neutrino field was absent in our earlier work[14]. The above expression in the Jordan-Brans-Dicke picture suggests such a coupling of the Brans-Dicke scalar to the vector current of the sterile neutrino might have been introduced in the Einstein picture in the first place. Now, we are ready to discuss two sub-classes of special types of solutions in the Jordan-Brans-Dicke picture. We consider only the cases $\bar{V}(\alpha) = 0 = \bar{M}(\alpha)$. This is a simplifying assumption that may be relaxed.

(i) Suppose $\alpha = \frac{1}{\kappa}$ is a constant. Our coupled field equations above reduce to the system of Einstein-massless neutrino field equations. It had already been noted many years ago that the Einstein-massless neutrino field equations admit exact plane wave solutions for which the contribution of the neutrino stress-energy-momentum tensor on the right hand side of the Einstein field equations identically vanish[15, 16, 17, 18, 19]. Such field configurations are called *ghost neutrinos* because it seems as if the propagation of a non-vanishing neutrino wave has no effect on the spacetime metric. A consistent physical interpretation can be provided within the framework of Einstein-Cartan gravity with torsion coupled to a massless neutrino field[20, 21, 22, 23]. In what follows, it is our wish to point out a similar configuration that we call a *ghost acceleron* exists in the Einstein-Majorana fermion-scalar field theory expressed, however, in the Jordan-Brans-Dicke picture.

(ii) Suppose now the sterile neutrino field identically vanishes, $\psi = 0$. Then our coupled field equations reduce simply to the Brans-Dicke field equations. The left hand side ensures the Einstein vacuum equations while at the right hand side, we have already shown above that there are scalar field configurations for which the improvement term and the scalar stress-energy-momentum tensor cancel each other out. We might interpret these solutions as *ghost accelerons*. The fundamental problem arises when we consider the geodesic hypothesis. The presence of the scalar field is assumed not to affect the geodesics in this case. We have a ghost scalar in the same sense analogous to the ghost neutrino problem[18, 20]. The theory is not scale invariant, the metric is the vacuum Einstein metric and the test particle equations of motion are the geodesics that are implied by the source-free Einstein field equations. Thus test masses wouldn't be able to feel the scalar field. In the conventional interpretation of the Brans-Dicke gravity this is an issue. But in a non-Riemannian interpretation with torsion it is not. This observation we believe gives one more support for a non-Riemannian interpretation of Brans-Dicke gravity.

5. Concluding Remarks

In this article, we studied a gravitational plane wave solution of the Brans-Dicke field equations to point the consistency interpretation of the non-Riemannian setting. The spacetime geometry is relaxed to admit torsion, depending on the gradient of the scalar field. A particular choice of the scalar field α reduces the field equations to the source-free Einstein field equations. The clarification on the sign is of utmost importance. There are two terms that cancel out in the right hand side, one is the shifted energy-momentum tensor and another term that comes from the geometry. What we explicitly want to show is the existence of such a configuration for $0 < c < 6$ i.e. for a negative ω , specifically $-\frac{3}{2} < \omega < 0$. This may well be a dark energy candidate. On the other hand, we have detailed that if we try to explain the theory in a Riemannian setting, we come to a pathological case where there is a scalar field present that cancels, in a way, the right hand side of the Einstein equations and that in the same geodesic hypothesis the scalar test mass does not interact with this scalar field. If we adopt a more consistent hypothesis, that scalar test masses should follow auto-parallels of the connection with torsion, then the identification of the energy-momentum tensor and the equations of motion are coherent. Torsion explicitly affects as it should, the equations of motion of the scalar test masses. Brans-Dicke field equations and the equivalence principle should be given in a non-Riemannian setting to be viable, modest and coherent. That is why we have discussed this in a particular plane wave example with a negative Brans-Dicke parameter ω , dark ω , which can depict some form of dark energy.

6. Acknowledgement

Y.Ş. is grateful to Koç University for its hospitality and partial support.

7. References

- [1] C.H.Brans,R.H.Dicke,*Mach's principle and a relativistic theory of gravitation*,Phys.Rev.**124**,925(1961)
- [2] R.H.Dicke,*Mach's principle and invariance under transformation of units*,Phys.Rev.**125**,2163 (1962)
- [3] C.H. Brans,*Mach's principle and a relativistic theory of gravitation II*,Phys.Rev.**125**,2194 (1962)
- [4] T.Dereli,R.W.Tucker,*Weyl scalings and spinor matter interactions in scalar-tensor theories of gravitation*,Phys.Lett.**B110**(1982)206
- [5] P.Teyssandier,R.W.Tucker,C.Wang,*On an interpretation of non-Riemannian gravitation*,Acta Phys.Polon.**B29**(1998)987
- [6] T.Dereli,R.W.Tucker,*On the detection of scalar field induced space-time torsion*,Mod.Phys.Lett.**A17**(2002)421
- [7] D.Burton,T.Dereli,R.W.Tucker,*On the motion of matter in gravitational fields*, in **Symmetries in Gravity and Field Theory**, V.Aldaya, J.M.Cervero,Y Pilar Garcia (Editors) (Ediciones Universidad Salamanca, 2004) pp.237-249 [arXiv:gr-qc/0107017]
- [8] D.A.Burton,R.W.Tucker,C.H.Wang,*Spinning particles in scalar-tensor gravity*,Phys.Lett.**A372**(2008)3141
- [9] C.G.Callan Jr.,S.Coleman,R.Jackiw *A new improved energy momentum tensor*,Ann.Phys. **59**(1970)42-73
- [10] M.Adak, T.Dereli, Y.Şenikoğlu, *Non-Riemannian description of Robinson-Trautman spacetimes in Brans-Dicke gravity*, Int. J. Mod. Phys. **D 28**, 1950070 (2019)
- [11] T.Dereli,Y.Şenikoğlu, *Gravitational Plane Waves in a Non-Riemannian Description of Brans-Dicke Gravity*, Phys.Scr.**95**(2020)045223
- [12] V.Faraoni,*The limit $\omega \rightarrow \infty$ limit of Brans-Dicke theory*,Phys.Lett.**A245**(1998)26
- [13] V.Faraoni,J.Côté,*Two new approaches to the anomalous limit of Brans-Dicke to Einstein gravity*, Phys.Rev.**D99**(2019)064013
- [14] B.Balantekin,T.Dereli,*Exact cosmological solution of the coupled Einstein-Majorana fermion-scalar field equations*, Phys.Rev.**D75**(2007) 024039
- [15] J.Madore,*On the neutrino in general relativity*,Lett.Nuo.Cim.**5**(1972)48
- [16] J.B.Griffiths,*Gravitational radiation and neutrinos*,Comm.Math.Phys.**28**(1972)295
- [17] C.D.Collinson,P.B.Morris,*Spacetimes admitting neutrino fields with zero energy and momentum*,J.Phys.**A6**(1973)915
- [18] T.M.Davis,J.R.Ray *Ghost neutrinos in general relativity*, Phys.Rev.**D9**(1974)334
- [19] J.B.Griffiths,*On the Dirac fields in a curved spacetime*, J.Phys.**A12**(1979)2429

- [20] T.Dereli,R.W.Tucker *Exact neutrino solutions in the presence of torsion*,Phys.Lett.**A 82**,(1981)229
- [21] J.B.Griffiths,*Neutrino fields in Einstein-Cartan theory*, Gen.Rel.Grav.**13**(1981) 227
- [22] A.Dimakis,F.Müller-Hoissen,*Massive ghost Dirac equations in Einstein-Cartan theory*,Phys.Lett.**A92**(1982)431
- [23] A.Dimakis,F.Müller-Hoissen,*Solutions of the Einstein-Cartan-Dirac equations with vanishing energy-momentum tensor*,J.Math.Phys.**26**(1985)1040