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V.V.Nesterenko

THE SINGULAR LAGRANGIANS WITH HIGHER DERIVATIVES

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I. INTRODUCTION

The quantum field theories with Lagrangians containing the derivatives of the field functions higher than the first order have a bad reputation because of the ghost states with negative norm and as a consequence the possibility of unitarity violation /1/. But such theories have also attractive properties, in particular, the convergence of the corresponding Feynman diagrams is improved. Therefore, the gauge theories with higher derivatives /2-4/ and the gravity models with quadratic and higher-order curvature corrections to the Einstein-Hilbert action /5-11/ are considered. These theories are described by singular or degenerate Lagrangians with higher derivatives.

The quantization of the Yang-Mills fields has shown that the canonical quantization is the most suitable for the investigation of unitarity properties of the quantum gauge fields. This approach is based on the Hamiltonian description of the classical dynamics. The Hamiltonian formalism for the usual gauge fields is constructed with the aid of the Dirac theory of the generalized Hamiltonian systems with constraints of the first order /12-15/.

It is natural to explore the ghost-state problem and unitarity in theories with singular Lagrangians with higher derivatives in the framework of the canonical quantization as well. But for this purpose the Hamiltonian formalism for these theories must be constructed. In /17/ this problem has been solved for singular Lagrangians with higher derivatives of an arbitrary order. In the present paper another method of transition into the phase space is proposed and the connection of the Lagrangian and Hamiltonian descriptions is traced in more detail. For simplicity the degenerate Lagrangians of second order will only be considered.

The paper is organized as follows. In the second section the canonical variables are introduced and the definition of singular Lagrangians is given. In the third section the transition into the phase space is carried out and the secondary constraints by the Dirac method are searched. In the fourth section it is shown how one can get all the secondary constraints in the framework of the Lagrangian formalism and using the equations of motion in the Euler form. In the 5th section as an example, a generalization of the relativistic action of a point particle is considered: to the usual action proportional to the length of the world trajectory of a particle one adds the integral along this trajectory of its curvature [16]. The Hamiltonian description of the classical dynamics of this object is given and the transition to quantum theory is shortly discussed. In conclusion the unsolved problems in this approach are noted.

2. THE SINGULAR LAGRANGIANS OF SECOND ORDER

Let us consider a system with a finite number of degrees of freedom which equals n. Let $x = (x_1, x_2, \ldots, x_n)$ be generalized coordinates of this system and

$$L(x,\dot{x},\ddot{x}), \qquad \dot{x} \equiv \frac{dx(t)}{dt} \tag{2.1}$$

is its Lagrangian function. The Euler equations are

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}_i} = 0, \quad i = 1,2,\ldots,n. \tag{2.2}$$

The canonical variables for Lagrangian (2.1) are introduced in the following way

$$q_{1i} = x_i, \qquad q_{2i} = \dot{x}_i, \tag{2.3}$$

$$p_{1i} = \frac{\partial L}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} = \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial^2 L}{\partial \ddot{x}_i \partial x_j}\dot{x}_j - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \dot{x}_j}\ddot{x}_j - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \ddot{x}_j}\dddot{x}_j, \tag{2.4}$$

$$p_{2i} = \frac{\partial L}{\partial \ddot{x}_i}, \quad i,j = 1,2,\ldots,n. \tag{2.5}$$

As usual, the summation over repeated indices in the corresponding limits is supposed.

Lagrangian (2.1) is called nondegenerate if the canonical variables q_1, q_2, p_1, p_2 introduced according to (2.3)-(2.5) are independent, i.e. there are no equations of the form[1]

[1] It is supposed that equations (2.6) do not reduce to the form $g(q_1,q_2)=0$.

$$f(q_1,q_2,p_1,p_2) = 0 \tag{2.6}$$

which become identities with respect to x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$ after the substitution into them definitions (2.3)-(2.5). Otherwise, i.e. when the relations (2.6) are valid, Lagrangian (2.1) is called singular or degenerate.

The condition that the Lagrangian is nonsingular is obviously equivalent to the requirement that equations (2.4) and (2.5) can be solved uniquely with respect to the variables $\ddot{x}_i$ and $\dddot{x}_i$, $i = 1,\ldots,n$ in the form

$$\ddot{x}_i = \ddot{x}_i(q_1,q_2,p_2), \qquad \dddot{x}_i = \dddot{x}_i(q_1,q_2,p_1,p_2), \quad i = 1,\ldots,n. \tag{2.7}$$

For this solution it is necessary that in the whole range of variables x, $\dot{x}$, $\ddot{x}$ the following condition is fulfilled

$$\operatorname{rank}\|\Lambda_{ij}\| = n, \tag{2.8}$$

where

$$\Lambda_{ij}(x,\dot{x},\ddot{x}) = \frac{\partial^2 L}{\partial \ddot{x}_i \partial \ddot{x}_j}, \quad 1 \le i,j \le n. \tag{2.9}$$

If condition (2.8) is satisfied, then there are no relations (2.6). To prove this, let us suppose the opposite, i.e. let the constraint (2.6) take place, not all the derivatives $\partial f/\partial p_{1i}$, $1 \le i \le n$ vanishing simultaneously. Substituting the definitions (2.4) and (2.5) into (2.6) we get the identity with respect to x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$. Differentiation of this identity gives

$$\frac{\partial f}{\partial p_{1k}}\frac{\partial p_{1k}}{\partial \dddot{x}_j} = -\frac{\partial f}{\partial p_{1k}}\Lambda_{kj} = 0, \tag{2.10}$$

which obviously contradicts (2.8). If the function in (2.6) does not depend on p_1, then the derivatives $\partial f/\partial p_{2k}$, $k = 1,\ldots,n$ cannot vanish simultaneously. Differentiating (2.6) with respect to $\ddot{x}_j$ we obtain

$$\frac{\partial f}{\partial p_{2k}}\frac{\partial p_{2k}}{\partial \ddot{x}_j} = \frac{\partial f}{\partial p_{2k}}\Lambda_{kj} = 0, \tag{2.11}$$

which contradicts (2.8) again. Thus, the absence of relations (2.6) between the canonical variables is equivalent to the condition (2.8).

If Lagrangian (2.1) is nonsingular, then the Euler equations (2.2) due to condition (2.8) can be represented in the normal form

$$\overset{(IV)}{x}_i = \overset{(IV)}{x}(x,\dot{x},\ddot{x},\dddot{x}), \qquad 1 \le i \le n. \tag{2.12}$$

As early as the last century M.V.Ostrogradskii[17] has shown that for nondegenerate Lagrangians a system of n equations of the fourth order (2.2) or (2.12) is equivalent to a canonical system of 4n equations of the first order

$$\dot{q}_{1i} = \frac{\partial H}{\partial p_{1i}}, \qquad \dot{q}_{2i} = \frac{\partial H}{\partial p_{2i}},$$
$$\dot{p}_{1i} = -\frac{\partial H}{\partial q_{1i}}, \qquad \dot{p}_{2i} = -\frac{\partial H}{\partial q_{2i}}, \qquad 1 \le i \le n. \tag{2.13}$$

where the Hamiltonian H is defined by

$$H = p_1\dot{x} + p_2\ddot{x} - L(x,\dot{x},\ddot{x}). \tag{2.14}$$

It is important that H can be represented only as a function of the canonical variables q_1, q_2, p_1, p_2. Indeed, using (2.5) we get from (2.14)

$$dH = dp_1\dot{x} + p_1 d\dot{x} + dp_2\ddot{x} + p_2 d\ddot{x} -$$
$$-\frac{\partial L}{\partial x}dx - \frac{\partial L}{\partial \dot{x}}d\dot{x} - \frac{\partial L}{\partial \ddot{x}}d\ddot{x} = \tag{2.15}$$
$$= -\frac{\partial L}{\partial x}dq_1 + \left(p_1 - \frac{\partial L}{\partial \dot{x}}\right)dq_2 + \dot{q}_1 dp_1 + q_2 dp_2 .$$

Thus, dH depends only on the differentials of the canonical variables, this being right both for nondegenerate Lagrangians and for degenerate ones. In both the cases we have

$$H = H(q_1, q_2, p_1, p_2), \tag{2.16}$$

$$dH = \frac{\partial H}{\partial q_1}dq_1 + \frac{\partial H}{\partial q_2}dq_2 + \frac{\partial H}{\partial p_1}dp_1 + \frac{\partial H}{\partial p_2}dp_2 . \tag{2.17}$$

Substituting $p_1 - (\partial L/\partial \dot{x})$ into (2.15) according to (2.4) by $-\dot{p}_2$ and $\partial L/\partial x$ by virtue of the Euler equations (2.2) by $\dot{p}_1$ we equate the right-hand sides in (2.15) and (2.17)

$$-\dot{p}_1 dq_1 - \dot{p}_2 dq_2 + \dot{q}_1 dp_1 + \dot{q}_2 dp_2 =$$
$$= \frac{\partial H}{\partial q_1}dq_1 + \frac{\partial H}{\partial q_2}dq_2 + \frac{\partial H}{\partial p_1}dp_1 + \frac{\partial H}{\partial p_2}dp_2 . \tag{2.18}$$

Now we get

$$-\left(\dot{p}_1 + \frac{\partial H}{\partial q_1}\right)dq_1 - \left(\dot{p}_2 + \frac{\partial H}{\partial q_2}\right)dq_2 + \left(\dot{q}_1 - \frac{\partial H}{\partial p_1}\right)dp_1 +$$
$$+ \left(\dot{q}_2 - \frac{\partial H}{\partial p_2}\right)dp_2 = 0. \tag{2.19}$$

For nonsingular Lagrangians the canonical variables q_1, q_2, p_1, p_2 are independent and as a consequence are independent their differentials. This enables one to equate to zero the coefficients of each differential in (2.1) and to obtain the canonical equations (2.13). It was this way that was used by Ostrogradskii[17] for obtaining eqs.(2.13).

If the action corresponding to the Lagrangian (2.1) is invariant under transformation $t \to t + \varepsilon$, then according to the first Noether theorem[18] the quantity

$$E(x,\dot{x},\ddot{x},\dddot{x}) =$$
$$= H(q_1 = x,\ q_2 = \dot{x},\ p_1 = p_1(x,\dot{x},\ddot{x},\dddot{x}),\ p_2 = p_2(x,\dot{x},\ddot{x})) \tag{2.20}$$

is consered on solutions of the equations of motion (2.2). Therefore E can naturally be called the energy.

3. THE CONSTRAINTS IN THE PHASE SPACE AND THE GENERALIZED HAMILTONIAN EQUATIONS OF MOTION

Let the initial Lagrangian (2.1) be singular. We suppose that in the whole range of variables x, $\dot{x}$ and $\ddot{x}$ the condition

$$\mathrm{rank}\,\|\Lambda_{ij}\| = r = n - m_1 < n \tag{3.1}$$

is satisfied. In this case the Euler equations (2.2) represent a system of r equations of the fourth order and $m_1 = n - r$ equations containing no $\overset{(IV)}{x}$. These last m_1 equations will be called the Lagrangian constraints. They can be separated from system (2.2) in the following way. Let $\overset{a}{\xi}_i(x,\dot{x},\ddot{x})$, $a=1,\dots,m_1$, $i=1,\dots,n$ be eigenvectors of the matrix Λ defined by (2.9) with zero eigenvalues

$$\overset{a}{\xi}_i(x,\dot{x},\ddot{x})\Lambda_{ij}(x,\dot{x},\ddot{x}) = 0, \qquad (3.2)$$

$$1\le i,j\le n, \qquad 1\le a\le m_1.$$

The number of such vectors due to (3.1) equals m_1. Projecting the Euler equations (2.2) on these eigenvectors we get m_1 Lagrangian constraints

$$\underset{a}{B}(x,\dot{x},\ddot{x},\dddot{x}) = \overset{a}{\xi}_i\left(\frac{\partial L}{\partial x_i} - p_{1i}\right), \quad a=1,\dots,m_1. \qquad (3.3)$$

We suppose that the system of equations (2.2) is consistent. It will be satisfied, for example, in the case when the Lagrangian constraints containing no $\overset{(IV)}{x}$ define the invariant submanifold for equations of the fourth order in (2.2)[14].

Taking into account (3.1) one can immediately obtain m_1 constraints on q_1, q_2 and p_2. For this purpose relations (2.5) have to be solved for r variables $\ddot{x}$ in the form

$$\ddot{x}_\alpha = \ddot{x}_\alpha(q_1, q_2, p_{2\beta}, \ddot{x}_{r+1},\dots,\ddot{x}_n), \qquad 1\le\alpha,\beta\le r. \qquad (3.4)$$

Here we suppose that the first r rows and r columns of Λ are linearly independent. This can obviously be done always by a corresponding change of numeration of the variables x_i, $i = 1,\dots,n$. Substituting (3.2) into the rest m_1 relations (2.5) we get m_1 constraints in the form

$$p_{2r+a} = p_{2r+a}(q_1, q_2, p_{2\beta}), \qquad (3.5)$$

$$a = 1,\dots, m_1 = n - r, \qquad \beta = 1,\dots, r.$$

These constraints or the set of constraints equivalent to them will be written further in the following way

$$\varphi_a(q_1, q_2, p_2) = 0, \qquad a = 1,\dots, m_1. \qquad (3.6)$$

Constraints (3.5) or (3.6) by analogy with the Dirac generalized Hamiltonian dynamics for singular Lagrangians without higher derivatives[10-13] can naturally be called the primary constraints, as they are a consequence of the singularity condition (31) for Lagrangian (2.1) and the definition of canonical momenta (2.5) without using the equations of motion (2.2). After substitution the definition (2.3) and (2.5) into the constraints (3.6) the latter transform into m_1 identities for x, $\dot{x}$, $\ddot{x}$.

Replacing f in (2.11) by the primary constraints (3.6) one verifies that zero eigenvectors $\overset{a}{\xi}_i(x,\dot{x},\ddot{x})$, $1\le a\le m_1$, $1\le i\le n$ of the matrix Λ can always be chosen so that they transform by virtue of the definition (2.5) into the functions which depend only on the canonical variables q_1, q_2, p_2, i.e. the dependence on $\ddot{x}$ desappear. Without loss of generality one can put

$$\overset{a}{\xi}_i(q_1, q_2, p_2) = \frac{\partial\varphi_a(q_1,q_2,p_2)}{\partial p_{2i}}, \quad 1\le a\le m_1, \; 1\le i\le n. \qquad (3.7)$$

Let us try to transform the Euler equations (2.2) for singular Lagrangians into the phase space. For this purpose we replace the canonical momenta p_2 by their expressions in terms of q_1, q_2, $\dot{q}_2$ according to (2.5) in the left- and in the right-hand sides of the definition of the canonical Hamiltonian

$$H(q_1, q_2, p_1, p_2) = p_1\dot{q}_1 + p_2\dot{q}_2 - L(q_1, q_2, \dot{q}_2). \qquad (3.8)$$

As a result, we obtain an identity with respect to q_1, q_2, p_1 and $\dot{q}_2$. Differentiation of this identity with respect to $\dot{q}_2$ gives

$$\left(\frac{\partial\bar{H}}{\partial p_{2j}} - \dot{q}_{2j}\right)\frac{\partial\bar{p}_{2j}}{\partial\dot{q}_{2i}} = 0, \qquad 1\le i,j\le n. \qquad (3.9)$$

The bar means the replacement described above

$$\bar{f}(q_1,q_2,p_1,p_2) \equiv f\left(q_1,q_2,p_1,\frac{\partial L(q_1,q_2,\dot{q}_2)}{\partial\dot{q}_2}\right) \equiv F(q_1,q_2,p_1,\dot{q}_2). \qquad (3.10)$$

As $\partial\bar{p}_{2j}/\partial\dot{q}_{2i} = \Lambda_{ij}(q_1,q_2,\dot{q}_2)$, then it follows from (3.9) that the quantities

$$\dot{q}_{2j} - \frac{\partial\bar{H}}{\partial p_{2j}}, \qquad 1\le j\le n \qquad (3.11)$$

are the eigenvector of the matrix $\Lambda(q_1,q_2,\dot{q}_2)$ with zero eigenvalue. This vector can be decomposed over a complete set of zero eigenvectors of the matrix Λ

$$\dot q_{2j} - \overline{\frac{\partial H}{\partial p_{2j}}} = \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overset{a}{\xi}_j (q_1, q_2, \dot q_2) = \tag{3.12}$$

$$= \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overline{\frac{\partial \varphi_a(q_1, q_2, p_2)}{\partial p_{2j}}}.$$

Here we used eq.(3.7).

Let us substitute (2.5) into (3.8), differentiate the identity obtained with respect to q_2, and take into account the relation

$$p_1 + \dot p_2 = \partial L / \partial q_2 ,$$

which follows from (2.4) and (2.5). As a result, we get

$$\frac{\partial H}{\partial q_{2i}} + \dot p_{2i} = \left(\dot q_{2j} - \overline{\frac{\partial H}{\partial p_{2j}}}\right) \frac{\partial \bar p_{2j}}{\partial q_{2i}} = \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overline{\frac{\partial \varphi_a}{\partial p_{2j}}}\, \frac{\partial \bar p_{2j}}{\partial q_{2i}} . \tag{3.14}$$

Differentiation with respect to q_1 and q_2 of the identities, which appear upon transforming the primary constraints (3.6) by subtitution into them (2.5), gives

$$\overline{\frac{\partial \varphi_a}{\partial q_{si}}} = - \overline{\frac{\partial \varphi_a}{\partial p_{2j}}}\, \frac{\partial \bar p_{2j}}{\partial q_{si}} , \quad s = 1, 2, \quad 1 \le i, j \le n. \tag{3.15}$$

Now eq.(3.12) can be rewritten in the following form

$$\dot p_{2i} + \overline{\frac{\partial H}{\partial q_{2i}}} = - \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overline{\frac{\partial \varphi_a}{\partial q_{2i}}}, \quad 1 \le i \le n. \tag{3.16}$$

Taking into account that the Euler equations (2.2) can be cast in the form

$$\dot p_1 = \frac{\partial L}{\partial q_1}$$

we obtain

$$\dot p_{1i} + \overline{\frac{\partial H}{\partial q_{1i}}} = - \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overline{\frac{\partial \varphi_a}{\partial q_{1i}}} , \quad 1 \le i \le n. \tag{3.17}$$

Finally differentiation of (3.8) with respect to p_1 gives

$$\dot q_{1i} - \overline{\frac{\partial H}{\partial p_{1i}}} = 0, \qquad 1 \le i \le n . \tag{3.18}$$

We introduce now the Poisson brackets in the usual way

$$(f, g) = \frac{\partial f}{\partial q_{si}} \frac{\partial g}{\partial p_{si}} - \frac{\partial f}{\partial p_{si}} \frac{\partial g}{\partial q_{si}} , \quad s = 1, 2, \quad i = 1, \dots, n, \tag{3.19}$$

$$f = f(q_1, q_2, p_1, p_2), \qquad g = g(q_1, q_2, p_1, p_2).$$

Using them we can write eqs.(3.12), (3.16), (3.17) and (3.18) in the form

$$\dot z = \overline{(z, H)} + \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot q_2)\, \overline{(z, \varphi_a)} . \tag{3.20}$$

Here z means a complete set of the canonical variables q_1, q_2, p_1, p_2.

We remind that eqs. (3.20) are written in terms of the variables q_1, q_2, p_1, $\dot q_2$. The expressions (z, H) and (z, φ_a) can be transformed obviously into the phase space if we take into account (2.5). As a result, we get the functions of the canonical variables (z, H) and (z, φ_a) respectively[1]. The dependence on $\dot q_2$ in the functions $\lambda_a(q_1, q_2, p_1, \dot q_2)$ does not disappear by vertue of (2.5). In order to prove this, it is sufficient to act on the left-hand side and on the right-hand side of eq.(3.12) by the following linear differential operators[19]

$$\overset{a}{X} = \overset{a}{\xi}_j \frac{\partial}{\partial \dot q_{2j}} , \qquad a = 1, 2, \dots, m_1 . \tag{3.21}$$

This gives

$$\overset{a}{X} \lambda_b(q_1, q_2, p_1, \dot q_2) = \delta_{ab} \neq 0 . \tag{3.22}$$

If one takes the primary constraints in the resolved form (3.5), then the functions λ_a reduce in this case to $\dot q_{2\,l+a}$, $a = 1, \dots, m_1$.

Thus, the only way to transform eqs.(3.20) into the phase space is to try eliminate the functions $\lambda_a(q_1, q_2, p_1, \dot q_2)$ imposing the additional conditions on the solutions of these equations. From this point we are dealing actually with the Dirac system with primary constraints[12].

[1] If the Lagrangian L is nondegenerated, i.e. rank $\Lambda = n$, then if follows from (3.9) that (3.11) vanishes and in the right-hand sides of (3.16),(3.17) and (3.12) we have zeros. As a result, we get the canonical Ostragradskii equations (2.13).

But in the Dirac approach the equations of motion in the phase space were obtained by the Lagrangian method of indefinite multipliers. Therefore the functions λ_a were considered at first as unknown functions of time determined by additional conditions on the solutions of the equations of motion. One demands that the time derivatives of the primary constraints vanish on the solutions of these equations. As it is known, all the secondary constraints can be obtained in this way and some number of functions λ_a can be expressed in terms of the canonical variables. The remaining undetermined functions $\lambda_a(t)$ the number of which equals the number of the primary first-class constraints describe the functional freedom in the theory. But in the Dirac reasoning there are no convincing arguments why it is sufficient to take into account only the primary constraints in order to obtain the equations of motion in the phase space by the Lagrangian method of indefinite multipliers. In our opinion, the derivation of these equations by the defferentiation of the canonical Hamiltonian fills this gap. Another method of obtaining the equations of motion in the phase space for singular Lagrangians of arbitrary order which avoids this problem is developed in book[15].

So, we shall further follow the Dirac reasoning. Let us demand that the time derivatives of the primary constraints vanish on the solutions of eqs.(3.20)

$$\frac{d\overline{\varphi}_a}{dt} = \overline{(\varphi_a, H)}_{\varphi_c} + \sum_{b=1}^{m_1} \lambda_b(q_1, q_2, p_1, \dot{q}_2)\,\overline{(\varphi_a, \varphi_b)} \overset{\overline{\varphi}_c}{\approx} 0, \quad a, b, c = 1, \dots, m_1. \tag{3.23}$$

Here the sign $\approx$ means a weak equality when the conditions $\varphi_c = 0$ are satisfied. The expressions (φ_a, H) and (φ_a, φ_b) can be transformed into the phase space if we take into account (2.5). Hence one can express from (3.23) r_1 functions λ_a in terms of the canonical variables where

$$r_1 = \mathrm{rank}\, \| (\varphi_a, \varphi_b) \| _{\varphi_c = 0}. \tag{3.24}$$

The remaining $\mu_1 = m_1 - r_1$ equations in (3.21) give rise to μ_1 constraints on the canonical variables

$$\omega_{s_1}(q_1, q_2, p_1, p_2) = 0, \quad s_1 = 1, 2, \dots, \mu_1. \tag{3.25}$$

It is obvious in what way one has to change the consideration when some of eqs.(3.23) or all these equations are satisfied identically. Further it is necessary to demand that

$$\frac{\overline{d\omega_{s_1}}}{dt} \overset{\overline{\varphi}, \overline{\omega}}{\approx} 0, \quad s_1 = 1, \dots, \mu_1 \tag{3.26}$$

and so on. As a result, all the secondary constraints can be obtained in this way and m functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ remain undetermined in terms of the canonical variables, where m is the number of primary first-class constraints. The theory does not enable us to fix them, and they remain absolutely arbitrary functions of their arguments. Therefore one can consider them as arbitrary functions of time. As a result, eqs.(3.20) prove to be transformed into the phase space completely.

In order to get a right final result one, could have considered the functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ in (3.20) at the beginning as unknown functions of time. This enables us to go in the phase space immediately

$$\dot{z} = (z, H) + \sum_{a=1}^{m_1} \lambda_a(t)(z, \varphi_a). \tag{3.27}$$

The consideration of eqs.(3.20) at first in terms of the variables $q_1, q_2, p_1, \dot{q}_2$ given above justifies this procedure.

4. DERIVATION OF THE SECONDARY CONSTRAINTS IN THE FRAMEWORK OF THE LAGRANGIAN FORMALISM

In the preceding section the secondary constraints were obtained by a successive differentiation with respect to time of the primary constraints using the equations of motion in form (3.20) or (3.27). But for this purpose one can use the Euler equations in form (3.16a). As in the case of singular Lagrangians of the first order this way enables **us to obtain some additional information about the** secondary constraints[19] and trace the relation of the Lagrangian and Hamiltonian description[19-21].

Differentiation with respect to time of the left-hand sides in equations of primary constraints (3.6) gives

$$\frac{d}{dt}\varphi_a(q_1, q_2, p_2) = \frac{\partial \varphi_a}{\partial q_{1i}}\dot{q}_{1i} + \frac{\partial \varphi_a}{\partial q_{2i}}\dot{q}_{2i} + \frac{\partial \varphi_a}{\partial p_{2i}}\dot{p}_{2i}, \quad a = 1, \dots, m_1. \tag{4.1}$$

Now we replace the derivatives with respect to the coordinates q_1 and q_2 in (4.1) according to (3.19) and take into account (3.13). As a result, we get

$$\frac{d}{dt}\varphi_a(q_1,q_2,p_2) = -\frac{\partial\varphi_a}{\partial p_{2j}}\left(\frac{\partial^2 L}{\partial\ddot{x}_j\partial x_i}\dot{x}_i + \frac{\partial^2 L}{\partial\ddot{x}_j\partial\dot{x}_i}\ddot{x}_i - \frac{\partial L}{\partial\dot{x}_i} + p_{1i}\right), \quad (4.2)$$

$$a = 1,\dots,m_1 .$$

The expression in parentheses vanishes due to (2.4). Thus the derivative (d/dt) $\varphi_a(q_1, q_2, p_2)$ is equal to zero by virtue of the primary constraints (3.6) without using the equations of motion. In addition the equations

$$\frac{d}{dt}\varphi_a(q_1,q_2,p_2) = 0, \quad 1 \le a \le m_1 \quad (4.3)$$

are equivalent to the following relations

$$\xi^a_i(q_1,q_2,p_2)p_{1i} = \xi^a_i(q_1,q_2,p_2)\left(\frac{\partial L}{\partial\dot{x}_i} - \frac{\partial^2 L}{\partial\ddot{x}_i\partial x_j}\dot{x}_j - \frac{\partial^2 L}{\partial\ddot{x}_i\partial\dot{x}_j}\ddot{x}_j\right), \quad (4.4)$$

$$a = 1,\dots,m_1 .$$

Let us now investigate the question: what are the conditions under which eqs.(4.4) transform due to the definitions (2.3)-(2.5) into equations containing only the canonical variables q_1, q_2, p_1, p_2 and give, as a result, the secondary Hamiltonian constraints. For this purpose one has to act on the right-hand side of (4.5) by the operators (3.21). This gives[19]

$$\xi^a_i\,\xi^b_j\left(\frac{\partial^2 L}{\partial\dot{x}_i\partial\ddot{x}_j} - \frac{\partial^2 L}{\partial\ddot{x}_i\partial\dot{x}_j}\right)\varphi_c \approx (\varphi_b, \varphi_a), \quad (4.5)$$

$$a, b, c = 1,\dots,m_1 .$$

Hence, if there are the primary constraints which are in involution at least in a weak sense with the whole set of the primary constraints (3.6), then for the corresponding values of the index a in (3.14) the action of the operators (3.15) on the right-hand side of (3.14) gives zero. In this case the variables $\ddot{x}$ in the right-hand side of (3.14) can be eliminated by virtue of (2.5) and eqs.(4.5) give us the secondary constraints on the canonical variables. The number of these constraints is equal to the number of primary constraints which are in involution at least in a weak sense with the whole set of the primary constraints (3.6). Obviously, these constraints are the same secondary constraints (3.25) obtained in the preceding section by the Dirac method. From (4.4) it follows immediately that these constraints are linear in p_1 and they are obtained by projection of the definition (2.4) on the zero eigenvectors of the matrix Λ .

Further one must differentiate with respect to time the constraints (3.25)

$$\frac{d\omega_{s_1}}{dt} = \frac{\partial\omega_{s_1}}{\partial q_1}\dot{q}_1 + \frac{\partial\omega_{s_1}}{\partial q_2}\dot{q}_2 + \frac{\partial\omega_{s_1}}{\partial p_1}\dot{p}_1 + \frac{\partial\omega_{s_1}}{\partial p_2}\dot{p}_2 = 0, \quad (4.6)$$

$$s_1 = 1,\dots,\mu_1$$

and use (3.13) and equations of motion in form (3.16a). If using (2.5) we can eliminate $\ddot{x}$ from all the equations (4.6) or from some of them, then we get some more secondary constraints

$$\omega_{s_2}(q_1,q_2,p_1,p_2) = 0, \quad s_2 = \mu_1+1,\dots,\mu_2 . \quad (4.7)$$

This procedure of successive differentiation of the constraints must be continued until the appearance of the new constraints stops or the variables $\ddot{x}$ cannot be eliminated from all the equations

$$\frac{d}{dt}\omega_{s_{k+1}}(q_1,q_2,p_1,p_2) = 0, \quad s_{k+1} = \mu_k+1,\dots,\mu_{k+1} \quad (4.8)$$

using the definition (2.5). As a result, all the secondary constraints will be obtained

$$\omega_s(q_1,q_2,p_1,p_2) = 0, \quad s = 1,\dots,m_2, \quad (4.9)$$

$$m_2 = \mu_1+\mu_2+\dots+\mu_k .$$

Let us establish the relation between Hamiltonian and Lagrangian constraints. First of all we show that the differentiation with respect to time of eqs. (4.5), which leads to the first set of the secondary constraints (3.25), gives, by virtue of the equations of motion (2.2), the Lagrangian constraints (3.3). Equations (4.5) can be represented in the form

$$\xi^a_i\left(p_{1i} - \frac{\partial L}{\partial\dot{x}_i} + \frac{d}{dt}\frac{\partial L}{\partial\ddot{x}_i}\right) = 0, \quad a = 1,\dots,m_1 . \quad (4.10)$$

The differentiation with respect to time of the left-hand sides of these equalities gives

$$\xi^a_i\left(\dot{p}_{1i} - \frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial\ddot{x}_i}\right) + \left(\frac{d}{dt}\xi^a_i\right)\left(p_{1i} - \frac{\partial L}{\partial\dot{x}_i} + \frac{d}{dt}\frac{\partial L}{\partial\ddot{x}_i}\right) = 0. \quad (4.11)$$

In the first term in (4.11) we make the following substitution using equations of motion (2.2)

$$-\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial\ddot{x}_i} = -\frac{\partial L}{\partial x_i} . \quad (4.12)$$

The second term in (4.11) vanishes due to the definition (2.4). As a result, from (4.11) we get the Lagrangian constraints (3.3).

The procedure of differentiation with respect to time of the Lagrangian constraints is important for the Lagrangian formalism too. It is in fact the search of the invariant submanifold in the space with the coordinates $\{x, \dot{x}, \ddot{x}, \dddot{x}\}$. The Cauchy data for the Euler equations (2.2) must belong to this submanifold. Only for this constraint set of the initial data one can consistently formulate the Cauchy problem for eqs. (2.2).

It is clear by the construction that for the primary constraints (3.6) and for the first set of the secondary ones (3.25) there are no corresponding Lagrangian constraints, as the **substitution** of (2.4) and (2.5) into (3.6) and (3.25) gives the identities.

5. THE GENERALIZATION OF THE RELATIVISTIC POINT ACTION

As an example, we consider the following generalization of the point particle action[16]

$$S = -m\int ds + \alpha \int k\, ds, \tag{5.1}$$

where m is the mass of a point particle, ds is the differential of its world trajectory $ds^2 = dx_\mu dx^\mu$, k is the curvature of this trajectory $k^2 = (d^2x/ds^2)^2$, α is a dimensionless constant. With a given parametrization $x^\mu(\tau)$, $\mu = 0,1,2,\dots, D-1$ action (4.1) is rewritten in the form

$$S = -m\int \sqrt{\dot{x}^2}\, d\tau + \alpha \int \frac{\sqrt{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2}}{\dot{x}^2}\, d\tau, \tag{5.2}$$

$$\dot{x} = dx/d\tau.$$

The metric with the signature $\eta_{\mu\nu} = \mathrm{diag}\,(+, -, -, \dots)$ is used.

The matrix Λ defined in (2.9) in the case under consideration is given by

$$\Lambda_{\mu\nu} = \frac{\alpha}{\dot{x}^2\sqrt{g}}\left\{\dot{x}_\mu \dot{x}_\nu - \dot{x}^2\eta_{\mu\nu} - \frac{l_\mu l_\nu}{g}\right\}, \tag{5.3}$$

where $l_\mu = (\dot{x}\ddot{x})\dot{x}_\mu - \dot{x}^2\ddot{x}_\mu$, $g = (\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2$. As

$$\dot{x}^\mu l_\mu = 0, \quad l_\mu l^\mu = -g\dot{x}^2, \tag{5.4}$$

then it is easy to be convinced of that the matrix Λ has two eigenvectors with zero eigenvalues $\dot{x}^\mu$ and l^μ. Hence, four primary constraints must be in the theory.

Using the definition[1]

$$p_{2\mu} = -\frac{\partial L}{\partial \ddot{x}^\mu} = -\frac{\alpha}{\dot{x}^2}\frac{l_\mu}{\sqrt{g}} \tag{5.5}$$

and eqs. (4.4) we obtain the primary constraints corresponding to (3.4)

$$\varphi_1 = p_2 q_2 = 0, \tag{5.6}$$

$$\varphi_2 = p_2^2 q_2^2 + \alpha^2 = 0, \tag{5.7}$$

where $q_{2\mu} = \dot{x}_\mu$.

We get the secondary constraints in the model under consideration at first by a method described in section 4. **The Poisson brackets** will be defined as follows

$$(f,g) = \sum_{i=1}^{2}\left(\frac{\partial f}{\partial p_i^\mu}\frac{\partial g}{\partial q_{i\mu}} - \frac{\partial f}{\partial q_i^\mu}\frac{\partial g}{\partial p_{i\mu}}\right). \tag{5.8}$$

The primary constraints (5.6) and (5.7) are in involution between themselves in a strong sense (φ_1 , φ_2) = 0. Therefore two secondary constraints have to be which can be obtained by projection of the definition

$$p_{1\mu} = -\frac{\partial L}{\partial \dot{x}^\mu} + \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}^\mu} = -\frac{\partial L}{\partial \dot{x}^\mu} - \dot{p}_{2\mu} \tag{5.9}$$

on the zero eigenvectors of the matrix Λ: $\xi^1_\mu = \dot{x}_\mu = q_{2\mu}$, $\xi^2_\mu = l_\mu \sim p_\mu$. Projection of $q_{2\mu}$ on (5.9) gives

$$\omega_1 = p_1 q_2 - m\sqrt{q_2^2} = 0. \tag{5.10}$$

[1] The sign minus is introduced in order to get eq.(2.5) for the space-like components of p_2.

Finally, multiplying (5.9) by $p_{2\mu}$ we obtain

$$\omega_2 = p_1 p_2 = 0. \tag{5.11}$$

Differentiation with respect to time of (5.10) does not give new constraints. Differentiating (5.11) with respect to time and taking into account the equations of motion

$$\dot{p}_1 = 0 \tag{5.12}$$

and constraints (5.6)-(5.10) we obtain the expression

$$\frac{d\omega_2}{dt} = p_1 \dot{p}_2 = -p_1\left(p_1 + \frac{\partial L}{\partial \dot{x}}\right) = -p_1^2 + m^2 + \frac{\alpha}{\sqrt{q_2^2}}(p_1 q_2)\sqrt{g}. \tag{5.13}$$

One can not eliminate $\ddot{x}$ from (5.13) using (5.5). Indeed

$$\ddot{x}^{\mu}\frac{\partial g}{\partial \ddot{x}^{\mu}} = 2g \neq 0.$$

Thus the constraints (5.6), (5.7), (5.10) and (5.11) exhaust the whole set of constraints in the model under consideration. In contrast to the conclusion in[16,22] we have here four constraints.

It follows from definition (5.5) that

$$p_2 \ddot{x} = -\frac{\alpha}{\dot{x}^2}\sqrt{g}. \tag{5.14}$$

Therefore we get the following expression for the canonical Hamiltonian[1]

$$H = -p_1\dot{x} - p_2\ddot{x} - L = -p_1 q_2 + m\sqrt{q_2^2} = -\omega_1. \tag{5.15}$$

Let us evaluate the Poisson brackets between all the constraints and construct the matrix Δ

$$\Delta_{AB} = (\theta_A, \theta_B), \qquad 1 \le A, B \le 4, \tag{5.16}$$

$$\theta_1 = \varphi_1, \quad \theta_2 = \varphi_2, \quad \theta_3 = \omega_1, \quad \theta_4 = \omega_2.$$

On the submanifold M of the phase space defined by the constraints equations

$$\theta_A(q_1, q_2, p_1, p_2) = 0, \qquad A = 1,\dots,4 \tag{5.17}$$

[1] If we substitute in $H(q_1, q_2, p_1, p_2)$ the canonical momenta p_1 and p_2 by their expressions in terms of $x, \dot{x}, \ddot{x}, \dddot{x}$ according to (5.5) and (5.9) we get zero identically. It is the consequence of the invariance of the action (5.2) under the transformation $\tilde{\tau} = f(\tau)$ with the arbitrary function f.

the following elements of the matrix Δ are different from zero

$$\Delta_{24} = (\theta_2, \theta_4) = (\varphi_2, \omega_2) = -2p_2^2(p_1 q_2),$$
$$\Delta_{34} = (\theta_3, \theta_4) = (\omega_1, \omega_2) = -(p_1^2 - m^2). \tag{5.18}$$

Thus, we have on M rank Δ = 2. Hence, there are two first-class constrants and two second-class constrants in this theory. Let us pick out these constraints explicitly. For this purpose we go to the equivalent set of constraints[14]

$$\phi_s = \overset{s}{\xi}_A \theta_A, \qquad s = 1, 2,$$
$$\phi_3 = \theta_3 = \omega_1, \qquad \phi_4 = \theta_4 = \omega_2, \tag{5.19}$$

where $\overset{s}{\xi}_A$, s = 1,2, A = 1,...,4 are two zero eigenvectors of the matrix Δ. These vectors can be taken in the following form

$$\overset{1}{\xi}_1 = 1, \quad \overset{1}{\xi}_2 = \overset{1}{\xi}_3 = \overset{1}{\xi}_4 = 0,$$
$$\overset{2}{\xi}_1 = 0, \quad \overset{2}{\xi}_2 = -p_1^2 + m^2, \quad \overset{2}{\xi}_3 = 2p_2^2(p_1 q_2), \quad \overset{2}{\xi}_4 = 0. \tag{5.20}$$

As a result, we get the new set of constraints

$$\phi_1 = p_2 q_2 = 0,$$
$$\phi_2 = -(p_1^2 - m^2)(p_2^2 q_2^2 + \alpha^2) + 2p_2^2(p_1 q_2)(p_1 q_2 - m\sqrt{q_2^2}) = 0, \tag{5.21}$$
$$\phi_3 = p_1 q_2 - m\sqrt{q_2^2} = 0, \quad \phi_4 = p_1 p_2 = 0,$$

which are equivalent to the initial constraints $\theta_A = 0$, A = 1,...,4. It means that eqs.(5.21) define the same submanifold M in the phase space. But for constraints ϕ_A, A = 1,...,4 there is only one Poisson bracket different from zero on M

$$(\phi_3, \phi_4) = -(p_1^2 - m^2).$$

Thus, the constraints ϕ_1 and ϕ_2 are the first-class constraints. and ϕ_3, ϕ_4 are the second-class constrainst.

It is interesting to note that in the phase space there is the invariant submanifold defined by the constraints (5.21) and by the equation

$$\phi_5 = p_1^2 - m^2 = 0, \qquad (\phi_\alpha, \phi_\beta) \approx 0, \qquad \alpha, \beta = 1, \dots, 5.$$

Let us now obtain the secondary constraints in this model by the Dirac method. Taking into account (5.15) we get

$$(\varphi_1, H) + \sum_{a=1}^{2} \lambda_a (\varphi_1, \varphi_a) = (\varphi_1, H) = -\omega_1 = 0,$$

$$(\varphi_2, H) + \sum_{a=1}^{2} \lambda_a (\varphi_2, \varphi_a) = (\varphi_2, H) = -2(p_1 p_2) q_2^2 +$$

$$+ 2 q_2^2 \frac{(p_2 q_2)}{\sqrt{q_2^2}} \overset{\varphi_1}{\approx} -2(p_1 p_2) q_2^2 = -2 q_2^2 \omega_2 = 0.$$

The requirement of the stationarity of the secondary constraints ω_1 and ω_2 enables us to express λ_2 in terms of the canonical variables

$$\lambda_2 = \frac{p_1^2 - m^2}{2 p_2^2 (p_1 q_2)} .$$

The Hamiltonian which defines the dynamics in the phase space is

$$H_T = H + \lambda_1(t) \varphi_1 + \frac{p_1^2 - m^2}{2 p_2^2 (p_1 q_2)} \varphi_2 .$$

The quantization of this model should be made in the same way as in the case of the constrained Hamiltonian systems of the first order/12-15/.

6. CONCLUSION

The method proposed here enables one to construct the Hamiltonian formalism for systems described by singular Lagrangians of the second order. Obviously, the generalization of this procedure to singular Lagrangians containing the derivatives of higher order meets no principal difficulties.

It would be interesting to make clear the connection of the invariance properties of the initial degenerate action with the number of the Hamiltonian constraints in the theory and with the properties of their Poisson brackets.

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Нестеренко В.В. E2-87-9
Сингулярные лагранжианы с высшими производными

Построен гамильтонов формализм для систем, описываемых сингулярными лагранжианами второго порядка. Связи на канонические переменные могут быть определены двумя путями: 1) методом Дирака, 2) в рамках лагранжева формализма последовательным дифференцированием по времени первичных связей. Получены уравнения движения в фазовом пространстве. В качестве примера рассмотрено обобщенное действие релятивистской точечной частицы: к обычному действию, пропорциональному длине мировой траектории частицы, добавлен интеграл вдоль этой траектории от ее кривизны.

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Nesterenko V.V. E2-87-9
The Singular Lagrangians with Higher Derivatives

The Hamiltonian formalism for system with singular Lagrangians of the second order is constructed. The constraints on canonical variables can be found in two ways: first, by a Dirac method; second, in framework of the Lagrangian formalism by a successive differentiation with respect to time of the primary constraints. The equations of motion in phase space are obtained. As an example, a generalization of the relativistic point action is considered: to the usual action proportional to the length of the world trajectory of a point, one adds the integral along this trajectory of its curvature.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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