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Heavy quark expansion for heavy hadron lifetimes: completing the $1/m_b^3$ corrections

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Abstract: We complete the calculation of the contributions from the dimension six operators in the heavy quark expansion for the total lifetime of heavy hadrons. We give the leading order expressions for the Wilson coefficients of the Darwin term ρ_D and the spin-orbit term ρ_{LS}.

Keywords: Effective Field Theories, Heavy Quark Physics, Perturbative QCD

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Contents

1 Introduction

Inclusive weak decays of hadrons with a single heavy quark Q have been intensively studied over the last decades [1–3]. The most inclusive quantity is the lifetime of the ground state heavy hadrons which is determined by their weak decay [4–6]. The theoretical method is the heavy quark expansion (HQE) [7–9], which is a combined expansion in the strong coupling $\alpha_s(m_Q)$ [10] and inverse powers of the heavy quark mass [11]. The leading term of this expansion is free of any hadronic parameter and is given by the decay rate of the "free" heavy quark. The corrections to this statement appear only at order $1/m_Q^2$ and are given in terms of the residual kinetic energy μ_π^2 and the chromomagnetic moment μ_G^2 which are both of order $\Lambda_{\rm QCD}^2$.

Consequently these corrections should be at the level of a few percent, since the leading order result implies that all lifetimes of hadrons with a single heavy quark Q should be identical up to corrections of order $\Lambda_{\rm QCD}^2/m_Q^2$. In the early days of the HQE this was taken as an embarrassment, since the lifetimes of the bottom hadrons had not been measured yet, and the lifetimes between charmed hadrons differ by factors of two to five, which is related to the fact that the c-quark is too light for the HQE to be a good approximation for these observables [12].

Since then the methods have been refined and the HQE makes quite precise predictions for the lifetime pattern of bottom hadrons and qualitatively describes the pattern of charmed hadrons. In fact, assuming $\mathrm{SU}(2)_{\mathrm{flavour}}$ symmetry for the light quarks the lifetime differences between the three ground state mesons are driven by the terms of $1/m_Q^3$ and higher, in particular by the four-quark operators appearing at tree level, which involve light quarks of a particular flavour.

The progress in the HQE for lifetimes rests on two pillars. On the one hand, there are refinements in the HQE by including higher order terms in the $1/m_Q$ expansion [13], on the other hand there are perturbative calculations improving the Wilson coefficients appearing in the HQE. The higher orders in the HQE contain hadronic matrix elements, for which precise lattice predictions became available recently. Based on this, we are entering the precision era for these observables, in particular for the lifetime differences.

However, a few ingredients have not yet been worked out in detail, since they were believed to be irrelevant. For this reason, the full calculation of all terms appearing at $1/m_Q^3$ has not yet been done, not even at tree level, since it was assumed that such terms will be small and mainly independent of the light-quark flavour. In the present paper we complete the tree/level calculation of the $1/m_Q^3$ terms for the lifetime of a heavy hadron with a single heavy quark Q. While these contributions are known since some time for the inclusive semi-leptonic case, the full calculation of the terms at order $1/m_Q^3$ for the non-leptonic width was still missing.

In section 2 we describe the current status of bottom-hadron lifetimes. In section 3 we give a short description of the method of the calculation of the non-leptonic width. In section 4 we present our results, and discuss their implications in section 5.

2 Synopsis on the status of bottom-hadron lifetimes

The measurements of lifetimes and lifetime rations for bottom hadrons have become very precise of the last decades. The current (2019) experimental averages obtained by the Heavy Flavor Averaging Group (HFLAV) of the b-hadron lifetime ratios are [14]

$$\left.\frac{\tau(B_s)}{\tau(B_d)}\right|^{\mathrm{exp}} = 0.994 \pm 0.004\,, \quad \left.\frac{\tau(B^+)}{\tau(B_d)}\right|^{\mathrm{exp}} = 1.076 \pm 0.004\,, \quad \left.\frac{\tau(\Lambda_b)}{\tau(B_d)}\right|^{\mathrm{exp}} = 0.969 \pm 0.006\,, \tag{2.1}$$

which show that the experimental precision is indeed extremely high. Even higher precision seems to be achievable from the most recent results from LHCb [15] and ATLAS [16].

The theory precision should of course live up to these experimental advancements. The current status of the theoretical predictions is [17–19]:

$$\left.\frac{\tau(B_s)}{\tau(B_d)}\right|^{\mathrm{th}} = 1.0006 \pm 0.0025\,, \quad \left.\frac{\tau(B^+)}{\tau(B_d)}\right|^{\mathrm{th}} = 1.082^{+0.022}_{-0.026}\,, \quad \left.\frac{\tau(\Lambda_b)}{\tau(B_d)}\right|^{\mathrm{th}} = 0.935 \pm 0.054\,, \tag{2.2}$$

which shows that the HQE technique can be successfully applied to bottom hadron decays, allowing us to make precision predictions. Therefore, B-physics is entering in its precision era. To arrive at such precise theoretical values, several advancements have been made:

The leading term in the total decay rate, i.e. with the absence of power corrections, which describes the free b-quark decay and does not contain non-perturbative corrections, is currently known at NLO-QCD [3, 4, 20–25] and at NNLO-QCD in the massless case [26] for non-leptonic decays. For semi-leptonic decays the current precision is NNLO-QCD [27–36].

The contribution from the first power correction due to the dimension five kinetic and chromomagnetic operators is already known at LO-QCD for both semi-leptonic and non-leptonic decays [37–40]. For semi-leptonic decays NLO-QCD corrections are know as well [41–43].

The contribution from the second power correction due to the dimension six Darwin and spin-orbit operators is known at LO-QCD [44] and NLO-QCD [45] only for the semi-leptonic case. However, the ρ_D contribution for inclusive non-leptonic decays, is still missing. That is precisely the task we address in this work. In fact, previous studies focus on the four-quark operators appearing at this order which induce lifetime differences at tree level and which are parametrically enhanced by a phase space factor $16\pi^2$. However, our explicit calculation shows that the coefficient in front of ρ_D turns out to be enhanced and thus needs to be taken into account. The contribution from dimension six four-quark operators is known at NLO-QCD [6, 46, 47].

3 Outline of the calculation

In this section we give a brief outline of the calculation which in fact contains a few subtleties. A detailed description will be deferred to a more technical publication. We start from the effective Lagrangian for flavor changing transitions due to charged hadronic currents [48]

$$\mathcal{L}_{\rm eff} = -\frac{4G_F}{\sqrt{2}} V'_{\rm CKM} V^*_{\rm CKM} (C_1 \mathcal{O}_1 + C_2 \mathcal{O}_2) + {\rm h.c}\,, \tag{3.1}$$

where G_F is the Fermi constant, $\mathcal{O}_{1,2}$ are four-quark operators, and $V_{\rm CKM}, V'_{\rm CKM}$ are the corresponding Cabibbo-Kobayashi-Maskawa matrix elements describing weak mixing of quark generations. We consider only the tree-level four-quark operators of current-current type since the Wilson coefficients of these operators are the largest. The numerical values of the Wilson coefficients $C_{1,2}(\mu)$ at the scale $\mu = m_b$, where m_b is the value of the b-quark mass, are known in the SM with high precision mainly thanks to using the high order renormalization group improved QCD perturbation theory at the scales between m_b and M_W.

We are interested in weak decays of beauty hadrons mediated by the CKM leading transitions with the flavour structure $b \to c\bar{u}d$ and $b \to c\bar{c}s$. The latter decay is additionally slightly suppressed by the phase space available for the decay products due to the mass of the c-quarks. The canonical choice of the operator basis for the decays $b \to c\bar{q}_1 q_2$ reads [48]

$$\mathcal{O}_1 = (\bar{c}^i_L \gamma_\mu b^i_L)(\bar{q}^j_{2\,L} \gamma^\mu q^j_{1\,L})\,, \quad \mathcal{O}_2 = (\bar{c}^i_L \gamma_\mu b^j_L)(\bar{q}^j_{2\,L} \gamma^\mu q^i_{1\,L})\,, \tag{3.2}$$

where q_L denotes the left-handed quark. It is the basis (3.2) that is used for the computation of the Wilson coefficients.

However, for the purposes of the present computation we use a different operator basis (cf. [6]), which is obtained after applying a four-dimensional Fierz transformation to $\mathcal{O}_2$. The operators of the new basis are diagonal in the color space and have the form

$$\mathcal{O}_1 = (\bar{c}^i \Gamma_\mu b^i)(\bar{q}_2^j \Gamma^\mu q_1^j)\,, \quad \mathcal{O}_2 = (\bar{q}_2^i \Gamma_\mu b^i)(\bar{c}^j \Gamma^\mu q_1^j)\,, \tag{3.3}$$

with $\Gamma_\mu = \gamma_\mu(1-\gamma_5)/2$. We consider two Cabibbo favoured decay channels, $(q_1, q_2) = (u, d)$ and $(q_1, q_2) = (c, s)$.

The main technical tool for our computation is dimensional regularization ($D = 4 - 2\epsilon$) [49]. The Dirac algebra of γ-matrices is usually defined in $D = 4$ and needs to be properly extended to D-dimensional space time [50–53]. In particular, using Fierz transformations can lead to a non-trivial ϵ dependence [54–57]. With this in mind, an arbitrary change of the operator basis valid in four-dimensional space is not allowed if perturbative corrections of higher order are to be included: the change will require the corresponding change of the set of evanescent operators associated with a given basis. For our computation however the required accuracy is such that one can use the new basis without changing the coefficients $C_{1,2}$. A review of the relevant techniques can be found in, e.g. [58, 59].

The B meson decay rate for the inclusive non-leptonic decays can be computed from the discontinuity of the forward scattering matrix element which is computed in the HQE. The property of unitarity of the S-matrix and the optical theorem lead to an expression for the decay width in the form

$$\begin{aligned}\Gamma(b \to c\bar{q}_1 q_2) &= \frac{1}{2M_B}\langle B(p_B)|\mathrm{Im}\, i \int d^4x\, T\{\mathcal{L}_{\rm eff}(x), \mathcal{L}_{\rm eff}(0)\}|B(p_B)\rangle \\ &= \frac{1}{2M_B}\langle B(p_B)|\mathrm{Im}\,\mathcal{T}|B(p_B)\rangle\,.\end{aligned} \tag{3.4}$$

The HQE of the transition operator up to $1/m_b^3$ is given by

$$\begin{aligned}\mathrm{Im}\,\mathcal{T} = \Gamma^0_{\bar{q}_1 q_2}\Bigg(& C_0\mathcal{O}_0 + C_v\frac{\mathcal{O}_v}{m_b} + C_\pi\frac{\mathcal{O}_\pi}{2m_b^2} + C_G\frac{\mathcal{O}_G}{2m_b^2} \\ & + C_D\frac{\mathcal{O}_D}{4m_b^3} + C_{LS}\frac{\mathcal{O}_{LS}}{4m_b^3} + \sum_{i,q} C^{(q)}_{4F_i}\frac{\mathcal{O}^{(q)}_{4F_i}}{4m_b^3}\Bigg)\,.\end{aligned} \tag{3.5}$$

Here $\Gamma^0_{\bar{q}_1 q_2} = G_F^2 m_b^5 |V_{cb}|^2 |V_{q_1 q_2}|^2/(192\pi^3)$, V_{cb} and $V_{q_1 q_2}$ are the corresponding CKM matrix elements, q stands for a massless quark and

$$\mathcal{O}_0 = \bar{h}_v h_v\,, \tag{3.6}$$

$$\mathcal{O}_v = \bar{h}_v (v\cdot\pi) h_v\,, \tag{3.7}$$

$$\mathcal{O}_\pi = \bar{h}_v \pi_\perp^2 h_v\,, \tag{3.8}$$

$$\mathcal{O}_G = \frac{1}{2}\bar{h}_v[\not{\pi}_\perp, \not{\pi}_\perp]h_v = \frac{1}{2}\bar{h}_v[\gamma^\mu,\gamma^\nu]\pi_{\perp\,\mu}\pi_{\perp\,\nu}h_v\,, \tag{3.9}$$

$$\mathcal{O}_D = \bar{h}_v[\pi_{\perp\,\mu},[\pi_\perp^\mu, v\cdot\pi]]h_v\,, \tag{3.10}$$

$$\mathcal{O}_{LS} = \frac{1}{2}\bar{h}_v[\gamma^\mu,\gamma^\nu]\{\pi_{\perp\,\mu},[\pi_{\perp\,\nu}, v\cdot\pi]\}h_v\,, \tag{3.11}$$

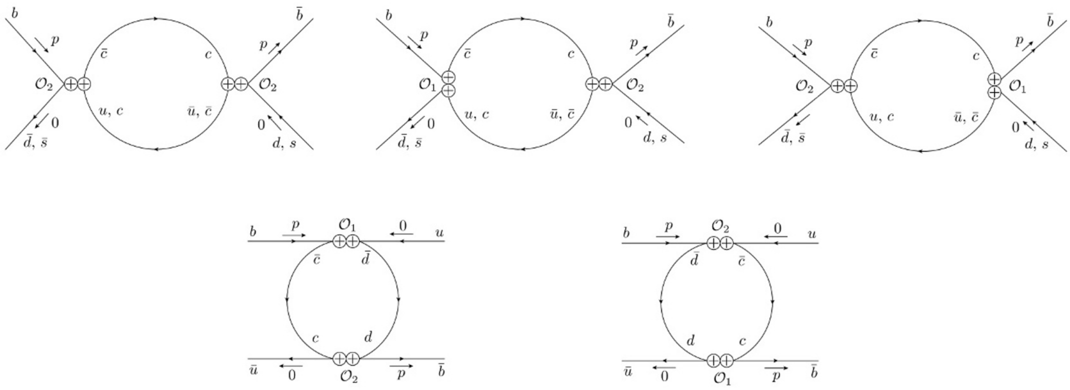

Figure 1. One loop diagrams contributing to the matching coefficients of four-quark operators in the HQE of the forward scattering matrix element of the B meson.

are HQET local operators with $\pi_\mu = iD_\mu = i\partial_\mu + g_s A^a_\mu T^a$ and $\pi^\mu = v^\mu(v\pi) + \pi^\mu_\perp$. The four-quark operators $\mathcal{O}^{(q)}_{4F_i}$ will be defined in section 3.1.

Finally, the QCD spinor b of the bottom quark is replaced by the HQET fermion field h_v. They are related as follows

$$b = e^{-im_b v\cdot x}\left[1+\frac{\not\pi_\perp}{2m_b}-\frac{(v\cdot\pi)\not\pi_\perp}{4m_b^2}+\frac{\not\pi_\perp\not\pi_\perp}{8m_b^2}+\frac{(v\cdot\pi)^2\not\pi_\perp}{8m_b^3}+\frac{\not\pi_\perp\not\pi_\perp\not\pi_\perp}{16m_b^3}+\mathcal{O}(1/m_b^4)\right]h_v\,. \quad (3.12)$$

3.1 Matching of four-fermion operators: computation of $C^{(q)}_{4F_i}$

In this subsection we give some sketch of the matching calculation of four-quark operators relevant for the renormalization of the coefficient C_D. We chose the version of the HQE where the bottom and the charm quarks are integrated out at the same scale $m_c \le \mu \le m_b$ such that only the u, d, s quarks remain as soft (massless) dynamical quarks. As a consequence, the matching coefficients will depend on the mass ratio $r = m_c^2/m_b^2 \sim \mathcal{O}(1)$. We only compute those pieces which are relevant for the renormalization of C_D. Such contributions are diagramatically represented in figure 1.

3.1.1 The channel $b \to c\bar{u}d$

The relevant operators in the HQE are

$$\mathcal{O}^{(d)}_{4F_1} = (\bar{h}_v\Gamma_\mu d)(\bar{d}\Gamma^\mu h_v)\,, \quad (3.13)$$

$$\mathcal{O}^{(d)}_{4F_2} = (\bar{h}_v P_L d)(\bar{d} P_R h_v)\,, \quad (3.14)$$

$$\mathcal{O}^{(u)}_{4F_1} = (\bar{h}_v\Gamma^\sigma\gamma^\mu\Gamma^\rho u)(\bar{u}\Gamma_\sigma\gamma_\mu\Gamma_\rho h_v)\,, \quad (3.15)$$

$$\mathcal{O}^{(u)}_{4F_2} = (\bar{h}_v\Gamma^\sigma\not v\Gamma^\rho u)(\bar{u}\Gamma_\sigma\not v\Gamma_\rho h_v)\,, \quad (3.16)$$

with the matching coefficients in $D = 4 - 2\epsilon$ dimensions

$$\begin{aligned} C^{(d)}_{4F_1} &= -(3C_2^2 + 2C_1C_2(1-\epsilon))\frac{3\cdot 2^{6+4\epsilon}\pi^{5/2+\epsilon}m_b^{-2\epsilon}(1-r)^{2-2\epsilon}(2+r-2\epsilon)}{\Gamma(5/2-\epsilon)} \quad (3.17)\\ &= -(3C_2^2+2C_1C_2)256\pi^2(1-r)^2(2+r) \quad \text{for } \epsilon\to 0\,, \end{aligned}$$

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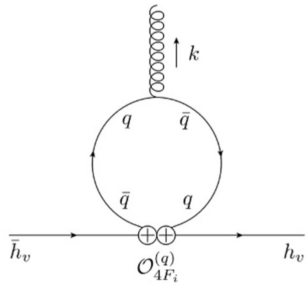

Figure 2. One loop diagrams contributing the renormalization of C_D.

$$\begin{aligned} C^{(d)}_{4F_2} &= -(3C_2^2 + 2C_1C_2(1-\epsilon))\frac{3\cdot 2^{7+4\epsilon}\pi^{5/2+\epsilon}m_b^{-2\epsilon}(1-r)^{2-2\epsilon}(-1+r(-2+\epsilon)+\epsilon)}{\Gamma(5/2-\epsilon)} \\ &= (3C_2^2+2C_1C_2)512\pi^2(1-r)^2(1+2r) \quad \text{for } \epsilon\to 0\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} C^{(u)}_{4F_1} &= C_1C_2\frac{3\cdot 2^{5+4\epsilon}m_b^{-2\epsilon}\pi^{5/2+\epsilon}(1-r)^{3-2\epsilon}}{\Gamma(5/2-\epsilon)} \\ &= C_1C_2 128\pi^2(1-r)^3 \quad \text{for } \epsilon\to 0\,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} C^{(u)}_{4F_2} &= -C_1C_2\frac{3\cdot 2^{6+4\epsilon}\pi^{5/2+\epsilon}m_b^{-2\epsilon}(1-r)^{2-2\epsilon}(-1+r(-2+\epsilon)+\epsilon)}{\Gamma(5/2-\epsilon)} \\ &= C_1C_2 256\pi^2(1-r)^2(1+2r) \quad \text{for } \epsilon\to 0\,. \end{aligned} \tag{3.20}$$

These expressions coincide with the results of ref. [6].

The one-loop matrix elements of these four-fermion operators also contribute to the coefficient C_D (see figure 2). We find that

$$\mathrm{Im}\,\mathcal{T} = \ldots + \Gamma^0_{ud}\frac{1}{4m_b^3}\left(-C^{(d)}_{4F_1}+\frac{1}{2}C^{(d)}_{4F_2}-16C^{(u)}_{4F_1}-4C^{(u)}_{4F_2}\right)\frac{1}{48\pi^2\epsilon}(-k^2)^{-\epsilon}\mathcal{O}_D\,, \tag{3.21}$$

and we can determine the counterterm of C_D in the $\overline{\text{MS}}$ renormalization scheme. We obtain

$$\delta C_D^{\overline{\text{MS}}}(\mu) = \left(C^{(d)}_{4F_1}-\frac{1}{2}C^{(d)}_{4F_2}+16C^{(u)}_{4F_1}+4C^{(u)}_{4F_2}\right)\frac{1}{48\pi^2\epsilon}\mu^{-2\epsilon}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}, \tag{3.22}$$

where $C_D^B = C_D^{\overline{\text{MS}}}(\mu) + \delta C_D^{\overline{\text{MS}}}(\mu)$ and $\bar{\mu}^{-2\epsilon} = \mu^{-2\epsilon}(e^{\gamma_E}/4\pi)^{-\epsilon}$ is the $\overline{\text{MS}}$ renormalization scale.

3.1.2 The channel $b\to c\bar{c}s$

The relevant four-quark operators of the HQE are

$$\mathcal{O}^{(s)}_{4F_1} = (\bar{h}_v\Gamma_\mu s)(\bar{s}\Gamma^\mu h_v)\,, \tag{3.23}$$

$$\mathcal{O}^{(s)}_{4F_2} = (\bar{h}_v P_L s)(\bar{s}P_R h_v)\,, \tag{3.24}$$

with matching coefficients (see figure 1)

$$\begin{aligned} C^{(s)}_{4F_1} &= -(3C_2^2+2C_1C_2(1-\epsilon))\frac{3\cdot 2^{7+4\epsilon}m_b^{-2\epsilon}\pi^{5/2+\epsilon}z^{1-2\epsilon}(1-\epsilon+r(-1+2\epsilon))}{\Gamma(5/2-\epsilon)} \\ &= -(3C_2^2+2C_1C_2)512\pi^2 z(1-r) \quad \text{for } \epsilon\to 0\,, \end{aligned} \tag{3.25}$$

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$$C_{4F_2}^{(s)} = -(3C_2^2 + 2C_1C_2(1-\epsilon))\frac{3\cdot 2^{7+4\epsilon} m_b^{-2\epsilon}\pi^{5/2+\epsilon} z^{1-2\epsilon}(-1-2r+\epsilon)}{\Gamma(5/2-\epsilon)} \tag{3.26}$$
$$= (3C_2^2 + 2C_1C_2) 512\pi^2 z(1+2r) \quad \text{for } \epsilon \to 0\,.$$

Again the one-loop matrix elements of the four-fermion operators also contribute to C_D (see figure 2). We find that

$$\mathrm{Im}\,\mathcal{T} = \ldots + \Gamma_{cs}^0 \frac{1}{4m_b^3}\left(-C_{4F_1}^{(s)} + \frac{1}{2}C_{4F_2}^{(s)}\right)\frac{1}{48\pi^2\epsilon}(-k^2)^{-\epsilon}\mathcal{O}_D\,, \tag{3.27}$$

and we can determine the counterterm of C_D in the $\overline{\mathrm{MS}}$ renormalization scheme, for which we obtain

$$\delta C_D^{\overline{\mathrm{MS}}}(\mu) = \left(C_{4F_1}^{(s)} - \frac{1}{2}C_{4F_2}^{(s)}\right)\frac{1}{48\pi^2\epsilon}\mu^{-2\epsilon}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}, \tag{3.28}$$

where $C_D^B = C_D^{\overline{\mathrm{MS}}}(\mu) + \delta C_D^{\overline{\mathrm{MS}}}(\mu)$ and $\bar{\mu}^{-2\epsilon} = \mu^{-2\epsilon}(e^{\gamma_E}/4\pi)^{-\epsilon}$ is the $\overline{\mathrm{MS}}$ renormalization scale.

3.2 Matching of two-fermion operators: computation of C_i

In this section we describe the matching computation of two-quark operators. The different contributions are diagramatically represented in figure 3. In order to optimize the computation we find expressions for the quark propagator in an external gluon field A.

In the semi-leptonic case the tree level expression for the HQE can be obtained from expanding the external field propagator for the charm or the up quark in powers of the covariant derivative

$$S_q = \frac{i}{\not{Q} + \not{\pi} - m} = \frac{i}{\not{Q} - m}\sum_{\nu=0}^{\infty}\left[i\not{\pi}\frac{i}{\not{Q}-m}\right]^{\nu}, \tag{3.29}$$

which automatically generates the proper ordering of the covariant derivatives. However, in the non-leptonic case the leptonic lines are replaced by quark lines and hence we pick up additional diagrams where gluons are emitted from these quark lines and the simple trick from the semi-leptonic calculation cannot be used here.

Still such contributions can most easily be taken into account by using the expression of the quark propagator in the external gluon field in the Fock-Schwinger gauge $x^\mu A_\mu(x) = 0$ (see, e.g. [60]). This is especially convenient because the expansion of the propagator has then an explicitly gauge invariant form. Another important property of the Fock-Schwinger gauge is that it breaks explicitly the translation invariance of the quark propagator, namely $S_F(x,0) \neq S_F(0,x)$. We obtain

$$S_F(x,0) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} S_F(p)\,, \quad S_F(0,x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\tilde{S}_F(p)\,, \tag{3.30}$$

with explicit expressions in momentum space given by

$$S_F(p) = S_F^{(0)}(p) + \frac{1}{2}[\pi_\rho, \pi_\sigma] S_F^{(0)}(p) i\gamma^\rho S_F^{(0)}(p) i\gamma^\sigma S_F^{(0)}(p) \tag{3.31}$$
$$+\frac{1}{3}([\pi_\sigma,[\pi_\rho,\pi_\lambda]] + [\pi_\rho,[\pi_\sigma,\pi_\lambda]]) S_F^{(0)}(p) i\gamma^\lambda S_F^{(0)}(p) i\gamma^\sigma S_F^{(0)}(p) i\gamma^\rho S_F^{(0)}(p)\,,$$

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$$\tilde{S}_F(p) = S_F^{(0)}(p) + \frac{1}{2}[\pi_\rho, \pi_\sigma] S_F^{(0)}(p) i\gamma^\rho S_F^{(0)}(p) i\gamma^\sigma S_F^{(0)}(p) \\ + \frac{1}{3}([\pi_\lambda, [\pi_\sigma, \pi_\rho]] + [\pi_\sigma, [\pi_\lambda, \pi_\rho]]) S_F^{(0)}(p) i\gamma^\lambda S_F^{(0)}(p) i\gamma^\sigma S_F^{(0)}(p) i\gamma^\rho S_F^{(0)}(p) \,, \tag{3.32}$$

where $S_F^{(0)}(p)$ is the free quark propagator

$$S_F^{(0)}(p) = \frac{i(\not{p} + m)}{p^2 - m^2} \,. \tag{3.33}$$

The expressions $S_F(p)$ and $\tilde{S}_F(p)$ are used for the propagator of the $\bar{q}_1$-quark and q_2-quark respectively to compute the diagrams that do not appear in the semi-leptonic case.

Let's us discuss the peculiarities of each contribution. The computation of the $\mathcal{O}_1 \otimes \mathcal{O}_1$ contribution goes exactly as in the semi-leptonic case. The color structure of the operator $\mathcal{O}_1$ only allows for the radiation of a single gluon from the c-quark in the $\bar{b} S_c b$ line. So we only need to expand the c-quark propagator. The computation is then identical to the case of semi-taonic decays and the corresponding results can be taken from ref. [61].

The computation of $\mathcal{O}_2 \otimes \mathcal{O}_2$ proceeds as the semi-leptonic case as well after replacing $c \to q_2$. The color structure only allows radiation of a single gluon from the q_2-quark in the $b S_{q_2} b$ line ($q_2 = d, s$), so we only need to expand this q_2-quark propagator. In this case one faces the IR divergences due to the gluon emission or the expansion of the massless quark propagator. Within the HQE (and OPE in general) the appearance of such infrared divergence signals the mixing between the local operators that constitute the basis of the expansion. The corresponding local operator develop UV divergences and should be properly renormalized. The well known advantage of using dimensional regularization is that both IR and UV divergences are dealt with simultaneously and a uniform manner. This treatment allows us to retain some vital symmetries of the theory and has technical superiority of simplicity. In fact, it is just this phenomenon of mixing that is the most essential and interesting part of the whole calculation.

The computation of $\mathcal{O}_1 \otimes \mathcal{O}_2$, which is found to be the same that for $\mathcal{O}_2 \otimes \mathcal{O}_1$, differs from the one in the semi-leptonic case. Here the gluon emission or the expansion of the quark propagators from all lines have to be taken into account. Overall, the computation of the coefficient of the ρ_{LS} operator in HQE is infrared safe even for massless quarks, does not require considering mixing with four quark operators, and can be performed in $D = 4$.

4 Results for the Wilson coefficients at order $1/m_Q^3$

Before we give our results for the terms of order $1/m_b^3$, we need to discuss the effects induced by operator mixing. The HQE as any OPE of effective theory gives an example of the general phenomenon of the separation of physics at greatly different scales. Indeed, the hadronic width in the representation

$$\Gamma(b \to c\bar{q}_1 q_2) = \frac{1}{2M_B} \mathrm{Im}\, \langle B(p_B)| i \int d^4x\, T\{\mathcal{L}_{\rm eff}(x), \mathcal{L}_{\rm eff}(0)\} |B(p_B)\rangle \\ = \frac{1}{2M_B} \langle B(p_B)| \mathrm{Im}\, \mathcal{T} |B(p_B)\rangle \,, \tag{4.1}$$

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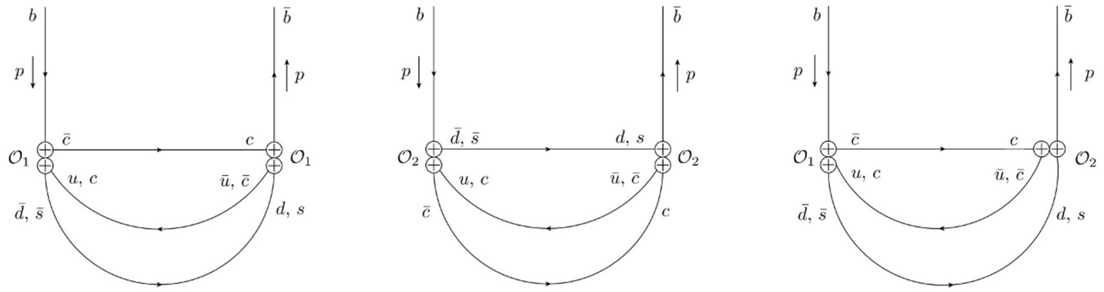

Figure 3. Two loop diagrams contributing to the matching coefficients of two-quark operators in the HQE of the forward scattering matrix element of the B meson.

depends on the heavy quark mass m_b and the infrared scale of QCD $\Lambda_{\rm QCD}$ with $m_b \gg \Lambda_{\rm QCD}$. The HQE in expression (3.5) is organized in such a way that the coefficients are insensitive to $\Lambda_{\rm QCD}$ while the matrix elements of the local operators are independent of the large scale m_b. An explicit naive computation, however, produces at some intermediate stage both IR singularities in the coefficient functions and UV singularities in the matrix elements. The combinatorics of HQE is such that all singularities cancel. Technically the most efficient way to perform computations is to use dimensional regularization for both IR and UV divergences. In such a setup one has to take into account the mixing of local operators at UV renormalization. Thus, an infrared divergence of coefficient functions signals the UV mixing between the local operators that constitute the basis of the expansion. The corresponding local operator develop then UV divergences and should be properly renormalized.

In our computation a naive way of getting the coefficient of the ρ_D operator leads to an IR singularity in the contribution of $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_1 \otimes \mathcal{O}_2$ correlators as one gets the radiation of a soft gluon from the light quark line.

This singularity is canceled by the UV renormalization of the four-quark operator that has the general form (quite symbolically)

$$\langle \bar{b}\gamma s \bar{s}\gamma b\rangle^{\rm ren} = \langle \bar{b}\Gamma s \bar{s}\Gamma b\rangle^{\rm B} + \gamma(\Gamma)\frac{1}{\epsilon}\rho_D \,, \tag{4.2}$$

where the matrix Γ gives the corresponding Dirac structure of the four-quark operator and the quantity $\gamma(\Gamma)$ is the mixing anomalous dimension depending on the Dirac structure Γ. The UV pole in ϵ coming from the operator mixing in eq. (4.2) cancel the IR divergence in the coefficient function for the operator ρ_D.

The HQE of the imaginary part of the transition operator is given by eq. (3.5). However, it is convenient to rewrite it in terms of the local operator $\bar{b}\not{v}b$ defined in full QCD. It can be employed to remove $\mathcal{O}_0$ in the HQE by using the expansion

$$\bar{b}\not{v}b = \mathcal{O}_0 - \tilde{C}_\pi \frac{\mathcal{O}_\pi}{2m_b^2} + \tilde{C}_G \frac{\mathcal{O}_G}{2m_b^2} + \tilde{C}_D \frac{\mathcal{O}_D}{4m_b^3} + \tilde{C}_{LS} \frac{\mathcal{O}_{LS}}{4m_b^3} + \mathcal{O}(\Lambda_{\rm QCD}^4/m_b^4) \,. \tag{4.3}$$

Inserting this expression we get (omitting here and in what it follows the four-quark con-

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tributions)

$$\mathrm{Im}\mathcal{T} = \Gamma^0_{\bar{q}_1 q_2}\Bigg[C_0\bigg(\bar{b}\slashed{v}b + \tilde{C}_\pi\frac{\mathcal{O}_\pi}{2m_b^2} - \tilde{C}_G\frac{\mathcal{O}_G}{2m_b^2} - \tilde{C}_D\frac{\mathcal{O}_D}{4m_b^3} - \tilde{C}_{LS}\frac{\mathcal{O}_{LS}}{4m_b^3}\bigg) + C_v\frac{\mathcal{O}_v}{m_b} + C_\pi\frac{\mathcal{O}_\pi}{2m_b^2} + C_G\frac{\mathcal{O}_G}{2m_b^2} + C_D\frac{\mathcal{O}_D}{4m_b^3} + C_{LS}\frac{\mathcal{O}_{LS}}{4m_b^3}\Bigg], \tag{4.4}$$

which has the advantage that the forward matrix element of the leading term is normalized to all orders.

For the operator $\mathcal{O}_v$ we use the equation of motion

$$\mathcal{O}_v = -\frac{1}{2m_b}(\mathcal{O}_\pi + C_{\mathrm{mag}}(\mu)\mathcal{O}_G) - \frac{1}{8m_b^2}(c_D(\mu)\mathcal{O}_D + c_S(\mu)\mathcal{O}_{LS}) \tag{4.5}$$

to remove it from the expression for $\mathrm{Im}\,\mathcal{T}$

$$\mathrm{Im}\,\mathcal{T} = \Gamma^0_{\bar{q}_1 q_2}\Bigg[C_0\bigg(\bar{b}\slashed{v}b + \frac{C_\pi + C_0\tilde{C}_\pi - C_v}{C_0}\frac{\mathcal{O}_\pi}{2m_b^2}\bigg) + \Big(C_G - C_0\tilde{C}_G - C_vC_{\mathrm{mag}}(\mu)\Big)\frac{\mathcal{O}_G}{2m_b^2} + \bigg(C_D - C_0\tilde{C}_D - \frac{1}{2}C_v c_D(\mu)\bigg)\frac{\mathcal{O}_D}{4m_b^3} + \bigg(C_{LS} - C_0\tilde{C}_{LS} - \frac{1}{2}C_v c_S(\mu)\bigg)\frac{\mathcal{O}_{LS}}{4m_b^3}\Bigg]. \tag{4.6}$$

This is the desired expression for the transition operator, from which we compute the total decay rate

$$\Gamma(b \to c\bar{q}_1 q_2) = \frac{1}{2M_B}\langle B(p_B)|\mathrm{Im}\,\mathcal{T}|B(p_B)\rangle \tag{4.7}$$

in terms of the HQE parameters

$$\langle B(p_B)|\bar{b}\slashed{v}b|B(p_B)\rangle = 2M_B\,, \tag{4.8}$$

$$-\langle B(p_B)|\mathcal{O}_\pi|B(p_B)\rangle = 2M_B\mu_\pi^2\,, \tag{4.9}$$

$$C_{\mathrm{mag}}(\mu)\langle B(p_B)|\mathcal{O}_G|B(p_B)\rangle = 2M_B\mu_G^2\,, \tag{4.10}$$

$$-c_D(\mu)\langle B(p_B)|\mathcal{O}_D|B(p_B)\rangle = 4M_B\rho_D^3\,, \tag{4.11}$$

$$-c_S(\mu)\langle B(p_B)|\mathcal{O}_{LS}|B(p_B)\rangle = 4M_B\rho_{LS}^3\,. \tag{4.12}$$

We obtain

$$\Gamma(b \to c\bar{q}_1 q_2) = \Gamma^0_{\bar{q}_1 q_2}\Bigg[C_0\bigg(1 - \frac{C_\pi + C_0\tilde{C}_\pi - C_v}{C_0}\frac{\mu_\pi^2}{2m_b^2}\bigg) + \bigg(\frac{C_G - C_0\tilde{C}_G}{C_{\mathrm{mag}}(\mu)} - C_v\bigg)\frac{\mu_G^2}{2m_b^2} - \bigg(\frac{C_D - C_0\tilde{C}_D}{c_D(\mu)} - \frac{1}{2}C_v\bigg)\frac{\rho_D^3}{2m_b^3} - \bigg(\frac{C_{LS} - C_0\tilde{C}_{LS}}{c_S(\mu)} - \frac{1}{2}C_v\bigg)\frac{\rho_{LS}^3}{2m_b^3}\Bigg]. \tag{4.13}$$

At leading order we have $c_S = c_D = C_{\mathrm{mag}} = 1$. It is convenient to define new coefficients corresponding to every matrix element

$$\Gamma(b \to c\bar{q}_1 q_2) = \Gamma^0_{\bar{q}_1 q_2}\Bigg[C_0 - C_{\mu_\pi}\frac{\mu_\pi^2}{2m_b^2} + C_{\mu_G}\frac{\mu_G^2}{2m_b^2} - C_{\rho_D}\frac{\rho_D^3}{2m_b^3} - C_{\rho_{LS}}\frac{\rho_{LS}^3}{2m_b^3}\Bigg]. \tag{4.14}$$

We find for the coefficients in the case of $\Gamma(b \to c\bar{u}d)$

$$\begin{aligned}
C_0 &= C_{\mu_\pi} \\
&= (3C_1^2 + 2C_1C_2 + 3C_2^2)(1 - 8r + 8r^3 - r^4 - 12r^2\ln(r))\,, && (4.15) \\
C_v &= (3C_1^2 + 2C_1C_2 + 3C_2^2)(5 - 24r + 24r^2 - 8r^3 + 3r^4 - 12r^2\ln(r))\,, && (4.16) \\
C_{\mu_G} &= C_{\rho_{LS}} \\
&= 3(C_1^2 + C_2^2)(-3 + 8r - 24r^2 + 24r^3 - 5r^4 - 12r^2\ln(r)) \\
&\quad +2C_1C_2(-19 + 56r - 72r^2 + 40r^3 - 5r^4 - 12r^2\ln(r))\,, && (4.17) \\
C_{\rho_D}^{\overline{\rm MS}} &= C_1^2\Big[-77 + 88r - 24r^2 + 8r^3 + 5r^4 - 48\ln(r) - 36r^2\ln(r)\Big] \\
&\quad +\frac{2}{3}C_1C_2\Bigg[-53 + 16r + 144r^2 - 112r^3 + 5r^4 + 96(-1+r)^3\ln(1-r) \\
&\quad -12(4 - 9r^2 + 4r^3)\ln(r) - 48(-1+r)^3\ln\left(\frac{\mu^2}{m_b^2}\right)\Bigg] \\
&\quad +C_2^2\Bigg[-45 + 16r + 72r^2 - 48r^3 + 5r^4 + 96(-1+r)^2(1+r)\ln(1-r) \\
&\quad +12(1-4r)r^2\ln(r) - 48(-1+r)^2(1+r)\ln\left(\frac{\mu^2}{m_b^2}\right)\Bigg]\,, && (4.18)
\end{aligned}$$

and for the case $\Gamma(b \to c\bar{c}s)$

$$\begin{aligned}
C_0 &= C_{\mu_\pi} \\
&= (3C_1^2 + 2C_1C_2 + 3C_2^2)\left[(1 - 14r - 2r^2 - 12r^3)z - 24r^2(-1+r^2)\ln\left(\frac{1+z}{1-z}\right)\right], \\
C_v &= (3C_1^2 + 2C_1C_2 + 3C_2^2)\left[(5 - 38r + 6r^2 + 36r^3)z + 24r^2(1+3r^2)\ln\left(\frac{1+z}{1-z}\right)\right], \\
C_{\mu_G} &= C_{\rho_{LS}} \\
&= -3(C_1^2 + C_2^2)\left[(3 - 10r + 10r^2 + 60r^3)z + 24r^2(-1+5r^2)\ln\left(\frac{1+z}{1-z}\right)\right] \\
&\quad -2C_1C_2\left[(19 - 2r + 58r^2 + 60r^3)z + 24r(-2 - r + 4r^2 + 5r^3)\ln\left(\frac{1+z}{1-z}\right)\right], \\
C_{\rho_D}^{\overline{\rm MS}} &= C_1^2\left[(-77 - 2r + 58r^2 + 60r^3)z + 24(2 - 2r - r^2 + 4r^3 + 5r^4)\ln\left(\frac{1+z}{1-z}\right)\right] \\
&\quad +C_2^2\Bigg[24(-4 + 8r + 7r^2 + 8r^3 + 5r^4)\ln\left(\frac{1+z}{1-z}\right) \\
&\quad +z\left(-45 - 58r + 106r^2 + 60r^3 - 96\ln(r) + 192\ln(z) - 48\ln\left(\frac{\mu^2}{m_b^2}\right)\right)\Bigg] \\
&\quad +\frac{2}{3}C_1C_2\Bigg[24(-6 + 10r - 5r^2 + 20r^3 + 5r^4)\ln\left(\frac{1+z}{1-z}\right) \\
&\quad +z\left(75 - 178r + 250r^2 + 60r^3 - 96\ln(r) + 192\ln(z) - 48\ln\left(\frac{\mu^2}{m_b^2}\right)\right)\Bigg]. && (4.19)
\end{aligned}$$

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Here $r = m_c^2/m_b^2$ and $z = \sqrt{1-4r}$. We note that the equalities $C_0 = C_{\mu_\pi}$ and $C_{\mu_G} = C_{\rho_{LS}}$ are a consequence of reparametrization invariance [62]. We take this here as a check of our calculation. We also note that the results for the coefficient of ρ_D depend on the calculational scheme. This does not only concern the use of the $\overline{\text{MS}}$ scheme, but also the treatment of the Dirac algebra in D dimensions. This is related to the fact that the Fierz-rearrangement in D dimensions generates evanescent operators which result in constants to be taken into account when comparing results [63].

After using the transformation rules (3.20-3.23) in [64], and after proper definition of evanescent operators for the $b \to cud$ channel (see section 4.1), these results are in agreement with [63], where the coefficients were computed in four dimensions. We will comment in more detail on this in the next section.

Note that from these results one readily finds the coefficients of HQE in eq. (3.5) whose computation was described in section 3.2

$$C_\pi = C_{\mu_\pi} - C_0\tilde{C}_\pi + C_v \,, \tag{4.20}$$

$$C_G = C_{\mu_G} + C_0\tilde{C}_G + C_v \,, \tag{4.21}$$

$$C_D^{\overline{\text{MS}}} = C_{\rho_D}^{\overline{\text{MS}}} + C_0\tilde{C}_D + \frac{1}{2}C_v \,, \tag{4.22}$$

$$C_{LS} = C_{\rho_{LS}} + C_0\tilde{C}_{LS} + \frac{1}{2}C_v \,. \tag{4.23}$$

4.1 Comment on the basis of four-quark operators

Our results discussed above are expressed in the operator basis

$$\mathcal{O}_{4F_1}^{(u)} = (\bar{h}_v\Gamma^\sigma\gamma^\mu\Gamma^\rho u)(\bar{u}\Gamma_\sigma\gamma_\mu\Gamma_\rho h_v) = (\bar{h}_v\gamma^\sigma\gamma^\mu\gamma^\rho P_L u)(\bar{u}P_R\gamma_\sigma\gamma_\mu\gamma_\rho h_v)\,, \tag{4.24}$$

$$\begin{aligned}\mathcal{O}_{4F_2}^{(u)} &= (\bar{h}_v\Gamma^\sigma\not{v}\Gamma^\rho u)(\bar{u}\Gamma_\sigma\not{v}\Gamma_\rho h_v) \\ &= (\bar{h}_v\gamma^\sigma\gamma^\rho P_L u)(\bar{u}P_R\gamma_\sigma\gamma_\rho h_v) + 4(\bar{h}_v\gamma^\rho P_L u)(\bar{u}P_R\gamma_\rho h_v) - 4(\bar{h}_v P_L u)(\bar{u}P_R h_v)\,,\end{aligned} \tag{4.25}$$

while one may chose as well the basis

$$\tilde{\mathcal{O}}_{4F_1}^{(u)} = (\bar{h}_v\Gamma_\mu u)(\bar{u}\Gamma^\mu h_v)\,, \tag{4.26}$$

$$\tilde{\mathcal{O}}_{4F_2}^{(u)} = (\bar{h}_v P_L u)(\bar{u}P_R h_v)\,, \tag{4.27}$$

which has been used in ref. [63]. While the two bases are equivalent in $D = 4$, the situation for arbitrary D is more involved. Relating the two bases in D dimensions requires the addition of new operators called evanescent operators. The choice of the evanescent operator is not unique, and a particular recipe reduces to a substitution [6, 59]

$$\gamma_\mu\gamma_\nu\gamma_\alpha P_L \otimes \gamma^\mu\gamma^\nu\gamma^\alpha P_L \to (16 - a\epsilon)\gamma_\alpha P_L \otimes \gamma^\alpha P_L + E_1^{\text{QCD}}\,, \tag{4.28}$$

$$\gamma_\mu\gamma_\nu P_L \otimes \gamma^\mu\gamma^\nu P_R \to (4 - b\epsilon)P_L \otimes P_R + E_2^{\text{QCD}}\,. \tag{4.29}$$

A conventional choice is $a = 4$ and $b = -4$, with $d = 4 - 2\epsilon$. We will call the basis fixed by this choice to be the *canonical* basis of four-quark operators. The evanescent operators are thus defined as

$$E_1^{\rm QCD} = \gamma_\mu\gamma_\nu\gamma_\alpha P_L \otimes \gamma^\mu\gamma^\nu\gamma^\alpha P_L - (16 - a\epsilon)\gamma_\alpha P_L \otimes \gamma^\alpha P_L\,, \tag{4.30}$$

$$E_2^{\rm QCD} = \gamma_\mu\gamma_\nu P_L \otimes \gamma^\mu\gamma^\nu P_R - (4 - b\epsilon) P_L \otimes P_R\,. \tag{4.31}$$

The choice of the evanescent operators $E_{1,2}^{\rm QCD}$ is not unique. This choice is motivated by the requirement of validity of Fierz transformation at one-loop order [6, 51].

Thus the complete operator basis reads

$$\tilde{\mathcal{O}}_{4F_1}^{(u)} = (\bar{h}_v \Gamma_\mu u)(\bar{u}\Gamma^\mu h_v)\,, \tag{4.32}$$

$$\tilde{\mathcal{O}}_{4F_2}^{(u)} = (\bar{h}_v P_L u)(\bar{u} P_R h_v)\,, \tag{4.33}$$

$$E_1^{\rm QCD} = (\bar{h}_v \gamma_\mu\gamma_\nu\gamma_\alpha P_L u)(\bar{u}\gamma^\mu\gamma^\nu\gamma^\alpha P_L h_v) - (16 - a\epsilon)(\bar{h}_v \Gamma_\mu u)(\bar{u}\Gamma^\mu h_v)\,, \tag{4.34}$$

$$E_2^{\rm QCD} = (\bar{h}_v \gamma_\mu\gamma_\nu P_L u)(\bar{u} P_R \gamma^\mu\gamma^\nu h_v) - (4 - b\epsilon)(\bar{h}_v P_L u)(\bar{u} P_R h_v)\,, \tag{4.35}$$

and the rule for the transformation between the two bases is

$$\mathcal{O}_{4F_1}^{(u)} = (16 - a\epsilon)\tilde{\mathcal{O}}_{4F_1}^{(u)} + E_1^{\rm QCD}\,, \tag{4.36}$$

$$\mathcal{O}_{4F_2}^{(u)} = 4\tilde{\mathcal{O}}_{4F_1}^{(u)} - b\epsilon\tilde{\mathcal{O}}_{4F_2}^{(u)} + E_2^{\rm QCD}\,. \tag{4.37}$$

In the new basis the imaginary part of the transition operator becomes

$$\mathrm{Im}\mathcal{T}(b \to c\bar{u}d) = \Gamma^0_{\bar{q}_1 q_2}\left(\ldots + \tilde{C}_{4F_1}^{(u)}\frac{\tilde{\mathcal{O}}_{4F_1}^{(u)}}{4m_b^3} + \tilde{C}_{4F_2}^{(u)}\frac{\tilde{\mathcal{O}}_{4F_2}^{(u)}}{4m_b^3} + C_{E_1}^{(u)}\frac{E_1^{\rm QCD}}{4m_b^3} + C_{E_2}^{(u)}\frac{E_2^{\rm QCD}}{4m_b^3}\right), \tag{4.38}$$

with

$$\tilde{C}_{4F_1}^{(u)} = (16 - a\epsilon)C_{4F_1}^{(u)} + 4C_{4F_2}^{(u)}\,, \tag{4.39}$$

$$\tilde{C}_{4F_2}^{(u)} = -b\epsilon C_{4F_2}^{(u)}\,, \tag{4.40}$$

$$C_{E_1}^{(u)} = C_{4F_1}^{(u)}\,, \tag{4.41}$$

$$C_{E_2}^{(u)} = C_{4F_2}^{(u)}\,. \tag{4.42}$$

The operators $E_{1,2}^{\rm QCD}$ do not contribute to the anomalous dimension of C_{ρ_D}. However, the change of basis produces a shift in the ρ_D coefficient which depends on the choice of the evanescent operators i.e. on a and b. We call the new coefficient $C_{\rho_D}^{\overline{\rm MS}}(a,b)$. The difference between the results obtained in the two bases is

$$C_{\rho_D}^{\overline{\rm MS}}(a,b) - C_{\rho_D}^{\overline{\rm MS}} = \frac{8}{3}C_1 C_2 (1-r)^2 (a(1-r) - b(1+2r)) \tag{4.43}$$

whereas the difference between our results and the ones obtained in ref. [63], where the coefficients are computed in $D = 4$, is

$$C_{\rho_D}^{\overline{\rm MS}}(a,b) - C_{\rho_D}^{\overline{\rm MS},\,D=4} = \frac{8}{3}C_1 C_2 (1-r)^2 (4(2+r) + b(1+2r) - a(1-r))\,. \tag{4.44}$$

Note that for the canonical choice of the evanescent operators the difference vanishes.

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$b \to c\bar{c}s$	C_1^2	C_2^2	$C_1 C_2$
C_0	0.84	0.84	0.56
C_{μ_π}	0.84	0.84	0.56
C_{μ_G}	−4.99	−4.99	−14.6
C_{ρ_D}	44.0	−56.1	−49.5
$C_{\rho_{LS}}$	−4.99	−4.99	−14.6
$C_{4F_1}^{(s)}/(128\pi^2)$	0	−9.35	−6.23
$C_{4F_2}^{(s)}/(128\pi^2)$	0	11.6	7.71

Table 1. Numerical values for the coefficients of $b \to c\bar{c}s$. For illustration we take the numerical values $\mu = m_b = 4.8\,\mathrm{GeV}$ and $m_c = 1.3\,\mathrm{GeV}$.

As we mentioned there is some freedom when choosing the evanescent operators $E_{1,2}^{\mathrm{QCD}}$. The difference in the results due to the different choice of a and b corresponds to a shift in the coefficient

$$C_{\rho_D}^{\overline{\mathrm{MS}}}(a_1,b_1) - C_{\rho_D}^{\overline{\mathrm{MS}}}(a_2,b_2) = -\frac{8}{3}C_1 C_2 (1-r)^2(-(a_1-a_2)(1-r)+(b_1-b_2)(1+2r)) \,. \quad (4.45)$$

When inserting numbers one has to keep in mind, that the coefficient is thus dependent on the scheme. This scheme dependence is compensated by the four-quark operators, which are scheme-dependent quantities.

4.2 Numerical analysis

In this section we give numerical values for phenomenological applications. We will chose the canonical scheme for the evanescent operators such that the results in [63] can be directly compared to our results. We employ the $\overline{\mathrm{MS}}$ scheme for the definition of ρ_D and chose for the scale $\mu = m_b$.

For both channels we have contributions which come from the operators $\mathcal{O}_1$ and $\mathcal{O}_2$, which come with the Wilson coefficients C_1 and C_2, see (3.1). In table 1 we give the numerical values of the coefficients for the transition $b \to c\bar{c}s$. We also list the values of the coefficients for the transition $b \to c\bar{u}d$ in the four-quark operator basis of section 3.1.1 in table 2, and in the canonical basis in table 3.

In order to get an idea about the size of the total contribution of ρ_D to the non-leptonic width we insert values for the Wilson coefficients $C_1(m_b) = -1.121$ and $C_2(m_b) = 0.275$ (note that $C_1(M_W) = -1$ and $C_2(M_W) = 0$ to leading logs). We denote $\langle \mathcal{O}_{4F_i}^{(q)} \rangle \equiv \langle B(p_B)|\mathcal{O}_{4F_i}^{(q)}|B(p_B)\rangle/(2M_B)$ and use the abbreviation $\Gamma_{\bar{q}_1 q_2}$ to refer to $\Gamma(b \to c\bar{q}_1 q_2)$. We obtain

$$\frac{\Gamma_{\bar{c}s}}{\Gamma_{\bar{c}s}^0} = 0.94 - 0.47\frac{\mu_\pi^2}{m_b^2} - 1.07\frac{\mu_G^2}{m_b^2} - 33.2\frac{\rho_D^3}{m_b^3} + 1.07\frac{\rho_{LS}^3}{m_b^3} + 383\frac{\langle \mathcal{O}_{4F_1}^{(s)}\rangle}{m_b^3} - 475\frac{\langle \mathcal{O}_{4F_2}^{(s)}\rangle}{m_b^3} \,, \quad (4.46)$$

$b \to c\bar{u}d$	C_1^2	C_2^2	$C_1 C_2$
C_0	1.75	1.75	1.17
C_{μ_π}	1.75	1.75	1.17
C_{μ_G}	−7.09	−7.09	−30.2
C_{ρ_D}	55.2	−50.3	52.4
$C_{\rho_{LS}}$	−7.09	−7.09	−30.2
$C_{4F_1}^{(d)}/(128\pi^2)$	0	−10.7	−7.12
$C_{4F_2}^{(d)}/(128\pi^2)$	0	11.8	7.88
$C_{4F_1}^{(u)}/(128\pi^2)$	0	0	0.80
$C_{4F_2}^{(u)}/(128\pi^2)$	0	0	1.97

Table 2. Numerical values for the coefficients of $b \to c\bar{u}d$. For illustration we take the numerical values $\mu = m_b = 4.8\,\text{GeV}$ and $m_c = 1.3\,\text{GeV}$.

$b \to c\bar{u}d$	C_1^2	C_2^2	$C_1 C_2$
C_0	1.75	1.75	1.17
C_{μ_π}	1.75	1.75	1.17
C_{μ_G}	−7.09	−7.09	−30.2
C_{ρ_D}	55.2	−50.3	71.4
$C_{\rho_{LS}}$	−7.09	−7.09	−30.2
$C_{4F_1}^{(d)}/(128\pi^2)$	0	−10.7	−7.12
$C_{4F_2}^{(d)}/(128\pi^2)$	0	11.8	7.88
$\tilde{C}_{4F_1}^{(u)}/(128\pi^2)$	0	0	20.6
$\tilde{C}_{4F_2}^{(u)}/(128\pi^2)$	0	0	0
$C_{E_1}^{(u)}/(128\pi^2)$	0	0	0.80
$C_{E_2}^{(u)}/(128\pi^2)$	0	0	1.97

Table 3. Numerical values for the coefficients of $b \to c\bar{u}d$ in the canonical basis ($a = 4$, $b = -4$). For illustration we take the numerical values $\mu = m_b = 4.8\,\text{GeV}$ and $m_c = 1.3\,\text{GeV}$.

$$\frac{\Gamma_{\bar{u}d}}{\Gamma^0_{\bar{u}d}} = 1.98 - 0.99\frac{\mu_\pi^2}{m_b^2} - 0.07\frac{\mu_G^2}{m_b^2} - 24.7\frac{\rho_D^3}{m_b^3} + 0.07\frac{\rho_{LS}^3}{m_b^3} + 438\frac{\langle \mathcal{O}_{4F_1}^{(d)} \rangle}{m_b^3} - 485\frac{\langle \mathcal{O}_{4F_2}^{(d)} \rangle}{m_b^3} - 77.5\frac{\langle \mathcal{O}_{4F_1}^{(u)} \rangle}{m_b^3} - 192\frac{\langle \mathcal{O}_{4F_2}^{(u)} \rangle}{m_b^3}, \tag{4.47}$$

$$\left.\frac{\Gamma_{\bar{u}d}}{\Gamma^0_{\bar{u}d}}\right|_{\text{can. basis}} = 1.98 - 0.99\frac{\mu_\pi^2}{m_b^2} - 0.07\frac{\mu_G^2}{m_b^2} - 21.8\frac{\rho_D^3}{m_b^3} + 0.07\frac{\rho_{LS}^3}{m_b^3} + 438\frac{\langle \mathcal{O}_{4F_1}^{(d)} \rangle}{m_b^3} - 485\frac{\langle \mathcal{O}_{4F_2}^{(d)} \rangle}{m_b^3} - 2\cdot 10^3\frac{\langle \tilde{\mathcal{O}}_{4F_1}^{(u)} \rangle}{m_b^3} - 77.5\frac{\langle E_1^{\text{QCD}} \rangle}{m_b^3} - 192\frac{\langle E_2^{\text{QCD}} \rangle}{m_b^3}, \tag{4.48}$$

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Assuming that $\langle E_1^{\rm QCD}\rangle/m_b^3 = \langle E_1^{\rm QCD}\rangle/m_b^3 = 0$ and $\rho_D^3/m_b^3 \sim \langle \mathcal{O}_{4F_i}^{(q)}\rangle/m_b^3 \sim \Lambda_{\rm QCD}^3/m_b^3$, the Darwin coefficient gives a correction to the tree level values of the coefficients of the four-quark operator of $\sim 7\%$ for $b \to c\bar{c}s$ and of $\sim 1\%$ for $b \to c\bar{u}d$ in the canonical basis (we take the largest coefficient of the four-quark operators to compare).

5 Discussion and conclusions

We have computed the contributions of the Darwin and the spin-orbit term appearing at order $1/m_Q^3$ in the HQE of the non-leptonic width. Although the coefficient of the spin-orbit term is fixed by reparametrization invariance, we have explicitly computed it as a check of our methods.

The most interesting part is the computation of the coefficient of ρ_D, since one has to take into account the mixing with the four-quark operators. The coefficient of the Darwin term turns out to be sizable which was also found in the semi-leptonic case.

In fact, this may become relevant for lifetime differences. To be specific, we will consider the $\mathrm{SU}(3)_{\rm Flavor}$ triplet of ground state B hadrons $\mathcal{B} = (B_u, B_d, B_s)$. It has been noticed already very early [17] that up to and including $1/m_b^2$ the operators appearing in $\mathrm{Im}\,\mathcal{T}$ are $\mathrm{SU}(3)_{\rm flavour}$ singlets and hence a lifetime difference to this order can only emerge from the SU(3) breaking coming from the states, meaning that μ_π and μ_G differ between the three B-meson ground states. In turn, assuming the SU(3) flavour symmetry, no lifetime differences can be induced up to this order. Since the coefficients of the HQE parameters at order $1/m_b^2$ are small, the effect on lifetime differences due to the $\mathrm{SU}(3)_{\rm flavour}$ breaking in μ_π and μ_G is very small.

The situation changes at the order $1/m_b^3$ where four-quark operators appear, involving light quarks. Since the weak hamiltonian is sensitive to the light-quark flavor, the resulting four-quark operators have different matching coefficients. Analyzing the $\mathrm{SU}(3)_{\rm flavour}$ structure of the four-quark operators, we may decompose them into a singlet and an octet contribution with respect to $\mathrm{SU}(3)_{\rm Flavor}$ according to

$$\bar{h}_v \Gamma (\bar{q}\mathbf{1}q)^T \Gamma h_v\,, \qquad \bar{h}_v \Gamma (\bar{q}\mathbf{T}^a q)^T \Gamma h_v\,, \tag{5.1}$$

where $q = (u, d, s)$ is the light quark triplet and $\mathbf{1}$ and $\mathbf{T}^a$ are the generators of $\mathrm{U}(3)_{\rm flavour}$. If we, in addition, use the equation of motion

$$\mathrm{Tr}\{\lambda^a[(iD_\mu), [(iD^\mu), (iD^\nu)]]\} = g \sum_q \bar{q}\gamma^\nu \lambda^a q\,, \tag{5.2}$$

(where λ^a are the Gell-Mann matrices of $\mathrm{SU}(3)_{\rm color}$) we can eliminate ρ_D in favour of four-quark operators, contributing to the $\mathrm{SU}(3)_{\rm Flavor}$ singlet part only. Obviously the mixing of ρ_D can only happen with the $\mathrm{SU}(3)_{\rm Flavor}$ singlet part of the four-quark operators.

Overall, we thus find that we can re-write (4.14) as (schematically)

$$\begin{aligned}\Gamma(b \to c\bar{q}_1 q_2) = \Gamma^0_{\bar{q}_1 q_2}\Bigg[& C_0\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) + \frac{C_{\mu_G}}{2m_b^2}\left(\mu_G^2 - \frac{1}{m_b}\rho_{LS}^3\right) \\ & + \frac{1}{m_b^3}\left(\sum_i C_{T,s,i} T^{4q}_{i,\rm singlet} + \sum_j C_{T,o,j} T^{4q}_{j,\rm octet}\right)\Bigg]\,,\end{aligned} \tag{5.3}$$

where the sums run over the matrix elements of the four-quark operators.

Clearly the octet part of the four-quark operators is the main source for lifetime differences, which is present also if the states are exactly SU(3) symmetric. However, the precision of the lifetime measurements has increased and thus also the SU(3) breaking through the states needs to be taken into account, which means that also the matrix elements $T^{4q}_{i,\text{singlet}}$ will contribute to lifetime differences. This effect may be important, since the complete calculation of these terms shows that the coefficients of these terms are large, even enhanced by phase space factors. However, a quantitative study of their impact on lifetime differences needs estimates of the $\text{SU}(3)_{\text{flavour}}$ breaking in the matrix elements $T^{4q}_{i,\text{singlet}}$ which is beyond the scope of the present paper.

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A Some technical results

Here we collect some technical results used for the computations.

A.1 The decay $b \to c\bar{c}s$

The most complicated part of the computation technically is the decay into two heavy quarks — c-quarks.

We define the most general two-loop integral that can appear as

$$\begin{aligned} &J(n_1,n_2,n_3,n_4,n_5) \\ &\quad\equiv \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{1}{(q_1^2)^{n_1}((p+q_1-q_2)^2-m_c^2)^{n_2}(q_2^2-m_c^2)^{n_3}((q_1+p)^2)^{n_4}((q_2+p)^2)^{n_5}} \\ &\quad= \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}}\,, \end{aligned} \tag{A.1}$$

where $+i0$ prescriptions are assumed in the propagators and $p^2 = m_b^2$. Using the program LiteRed [65, 66], one finds that the amplitude for the relevant diagrams can be expressed as a combination of the following three master integrals

$$\begin{aligned} J(0,1,1,0,0) &= \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{1}{((p+q_1-q_2)^2-m_c^2)(q_2^2-m_c^2)} \\ &= \int \frac{d^Dq_1}{(2\pi)^D}\frac{1}{q_1^2-m_c^2}\int\frac{d^Dq_2}{(2\pi)^D}\frac{1}{q_2^2-m_c^2}\,, \end{aligned} \tag{A.2}$$

$$J(1,1,1,0,0) = \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{1}{q_1^2((p+q_1-q_2)^2-m_c^2)(q_2^2-m_c^2)}$$
$$= \int \frac{d^Dq_1}{(2\pi)^D}\frac{1}{(p-q_1)^2-m_c^2}\int \frac{d^Dq_2}{(2\pi)^D}\frac{1}{(q_1-q_2)^2(q_2^2-m_c^2)}\,, \qquad (A.3)$$
$$J(2,1,1,0,0) = \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{1}{q_1^4((p+q_1-q_2)^2-m_c^2)(q_2^2-m_c^2)}$$
$$= \int \frac{d^Dq_1}{(2\pi)^D}\frac{1}{(p-q_1)^2-m_c^2}\int \frac{d^Dq_2}{(2\pi)^D}\frac{1}{(q_1-q_2)^4(q_2^2-m_c^2)}\,. \qquad (A.4)$$

We are only interested in the imaginary part of the corresponding integrals, related to the discontinuity across the cut. We denote $\bar{J} \equiv \mathrm{Im}\, J$. On the one hand $\bar{J}(0,1,1,0,0) = 0$. On the other hand, we can use that

$$\frac{d}{dm_c^2}J(1,1,1,0,0) = J(1,2,1,0,0) + J(1,1,2,0,0) = 2J(1,2,1,0,0)\,, \qquad (A.5)$$

and the reduction of $J(1,2,1,0,0)$ to a combination of the three master integrals above

$$J(2,1,1,0,0) = -\frac{1}{(-4+D)m_b^2(m_b^2-4m_c^2)}\Big[(-2+D)^2J(0,1,1,0,0)$$
$$+(-3+D)(-8+3D)(m_b^2-2m_c^2)J(1,1,1,0,0)$$
$$+8(-3+D)m_c^2(m_c^2-m_b^2)J(1,2,1,0,0)\Big]\,, \qquad (A.6)$$

in order to express the master integral $J(2,1,1,0,0)$ only in terms of $J(1,1,1,0,0)$ and $J(0,1,1,0,0)$ whose imaginary part is zero

$$J(2,1,1,0,0) = -\frac{1}{(-4+D)m_b^2(m_b^2-4m_c^2)}\Bigg[(-2+D)^2J(0,1,1,0,0)$$
$$+(-3+D)(-8+3D)(m_b^2-2m_c^2)J(1,1,1,0,0)$$
$$+8(-3+D)m_c^2(m_c^2-m_b^2)\frac{1}{2}\frac{d}{dm_c^2}J(1,1,1,0,0)\Bigg]\,. \qquad (A.7)$$

Therefore there is only one master integral we need to compute, which is $J(1,1,1,0,0)$. Note that eq. (A.7) has a pole in $D = 4-2\epsilon$ dimensions. Therefore the $\mathcal{O}(\epsilon)$ expansion of $J(1,1,1,0,0)$ will be needed. We find

$$\bar{J}(1,1,1,0,0) = \frac{2^{-9+6\epsilon}\pi^{-3/2+2\epsilon}\csc(\pi\epsilon)}{\Gamma(3/2-\epsilon)}m_b^{2-4\epsilon}$$
$$\times\Bigg[\frac{r^{2-2\epsilon}(-5+2\epsilon)H_1}{\Gamma(3-\epsilon)} + \frac{r^{2-2\epsilon}(1+4r(1-4\epsilon))H_2}{\Gamma(3-\epsilon)}$$
$$-\frac{\Gamma(1-\epsilon)}{(-1+2\epsilon)\Gamma(3-3\epsilon)}\left(\frac{H_3}{\Gamma(-1+\epsilon)} + \frac{(2-3\epsilon)H_4}{\Gamma(\epsilon)}\right)\Bigg]\,, \qquad (A.8)$$

where

$$H_1 = {}_2F_1(-1/2, 2\epsilon, 3-\epsilon, 4r)\,, \qquad (A.9)$$
$$H_2 = {}_2F_1(1/2, 2\epsilon, 3-\epsilon, 4r)\,,$$

$$H_3 = {}_2F_1(-1/2+\epsilon, -2+3\epsilon, -1+\epsilon, 4r)\,,$$
$$H_4 = {}_2F_1(-1/2+\epsilon, -1+3\epsilon, \epsilon, 4r)\,.$$

A.2 The computation of the decay $b \to c\bar{c}s$ in a scheme with a hard IR regularization

The case we are considering can be used for pedagogical purposes of OPE (HQE). There is only an IR divergence in the original amplitude. One can proceed by regulating IR with a small mass and use four dim to compute the coefficients. The relevant master integral is then defined as follows

$$J(0,1,1,0,2;m_0)|_{d=4} \equiv \int \frac{d^4q_1 d^4q_2}{((p+q_1-q_2)^2-m_c^2)(q_2^2-m_c^2)((q_2+p)^2-m_0^2)^2}\,. \quad \text{(A.10)}$$

Here m_0 is an IR regulator, $p^2 = m_b^2$, and $m_0 \ll m_b$. The HQE for the non-leptonic correlator at the order $1/m_b^3$ has the general form

$$Amp = C_4\mathcal{O}_4 + C_D\mathcal{O}_D\,. \quad \text{(A.11)}$$

For the sake of demonstration we take the only four-quark operator $\mathcal{O}_4 = \bar{b}\gamma_\mu s\bar{s}\gamma_\mu b$. The coefficient C_4 is computed in four dimensions and is finite in the limit $m_0/m_b \to 0$. The expression for the ρ_D coefficient C_D is also obtained in four dimensions but contains an IR singularity in the limit $m_0/m_b \to 0$.

Thus the HQE becomes

$$Amp = C_4\mathcal{O}_4 + \left(C_D^{\text{finite}} + C_4\ln(m_0/m_b)\right)\mathcal{O}_D\,. \quad \text{(A.12)}$$

Now one defines the finite matrix element of the $\bar{b}\gamma_\mu s\bar{s}\gamma_\mu b$ operator as (MS-subtracted)

$$\langle b|(\bar{b}\gamma_\mu s\bar{s}\gamma_\mu b)^{\text{R}}|gb\rangle \sim \left(\frac{1}{\varepsilon} + \ln(m_0/\mu) + c\right) - \frac{1}{\varepsilon}\,. \quad \text{(A.13)}$$

Upon substituting this expression in eq. (A.11) one gets the finite coefficient C_D^{finite}. This is a rough sketch of the procedure used in [63] (for a related discussion, see [6]). In such an approach one needs an expansion of the master integral at the limit of small m_0/m_b.

We have obtained an analytical expression for the required expansion in the form

$$\left[J(0,1,1,0,2;m_0)|_{d=4} - z\ln(m_0/m_b)\right]_{m_0/m_b\to 0} = 2(1-\rho)\ln\left(\frac{1+z}{1-z}\right) + z(1+2\ln(\rho)-4\ln(z))\,,$$

where $r \equiv \rho = m_c^2/m_b^2$ and $z = \sqrt{1-4\rho}$.

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