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Eckehard W. Mielke

Geometrodynamics of Gauge Fields

On the Geometry of Yang-Mills and Gravitational Gauge Theories

Second Edition

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Eckehard W. Mielke
Departamento de Física, CBI Iztapalapa
Universidad Autónoma Metropolitana
Mexico City
México

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Gratefully dedicated to
my parents, Walter and Charlotte Mielke
and to
my teacher, John Archibald Wheeler

Preface to the Second Edition

This revised monograph still aims at a unified geometric foundation of gauge theories of elementary particle physics and gravity. The underlying geometric structure is unfolded in a coordinate-free manner via the modern mathematical notions of fiber bundles, exterior differential forms, and their Clifford-algebra-valued generalizations. In the first part, Maxwell theory is treated as the simplest example, with an emphasis on the more recent measurement of the vector potential 1-form A via electron interference.

By transferring these concepts to local spacetime symmetries, affine generalizations of Einstein's theory of gravity arise in a Riemann–Cartan space with curvature and torsion. In this context, recent accounts on the Einstein–Cartan theory, teleparallelism, as well Yang's gauge approach to gravity are treated in more detail, with emphasis on gravitational instantons. Duality projections of curvature squared models, with their Einsteinian macroscopic "nucleus," are analyzed with respect to the issue of quantization, or at least asymptotic safeness. The Cartan-type geometric structure of BRST quantization with nonpropagating topological ghosts is developed in some detail.

In order to obtain more insight into the open issue of quantizing gravity, Chern–Simons-induced topological three-dimensional gravity, like the Mielke–Baekler model, is analyzed, in which torsion provides a kind of linearization of the vacuum field equations. Moreover, the peculiar feature of Dirac fields in curved 3D space is geometrically related to flexural modes of new materials such as graphene.

Quantized Dirac fields suffer from nonconservation of the axial current, leading to chiral and trace anomalies also in Riemann–Cartan space.

Since the discovery of the Higgs boson, concepts of spontaneous symmetry-breaking in gravity have come again into focus: departing from a topological de Sitter-type gauge theory, some new progress in the constrained BF model with a primordial $SL(5, \mathbb{R})$ gauge group is presented. After a tiny symmetry-breaking and the spontaneous generation of the metric, Einstein's standard general relativity with cosmological constant again emerges as the classical background.

To my colleges and friends Torsten Asselmeyer-Maluga, Dirk Kreimer, Garrett Lisi, Roberto Percacci, Martin Reuter, and Dimitri Vassiliev I am very grateful for suggestions for improvements in preliminary versions of some chapters. Also, I am pleased to acknowledge the undergraduate students Ulises Alcántara, Jesús Ocampo Jaimes, and Luis Daniel Vargas Sánchez for their help with LaTeX.

Mexico City, Mexico Eckehard W. Mielke

Preface to the First Edition

Since the days of Riemann, scientific work in natural philosophy has concentrated on answering questions "… concerning the intrinsic (physical) basis for the metrical relations in space …"[1] thus striving for a unified geometric description of fundamental physical interactions. The present study reports on recent achievements in this endeavor.

It will begin with a coordinate-free presentation of the underlying geometric structure of electromagnetic fields and their nonabelian generalizations by utilizing rather modern mathematical concepts, such as those of fiber bundles and Lie-algebra-valued differential forms. Such nonabelian theories of Yang–Mills type are founded on Weyl's basic principle of gauge invariance and appear to be the most appropriate framework in which to describe phenomena so as the weak and strong interactions in particle physics. In particular, the unification of weak and electromagnetic forces within the Weinberg–Salam model has gained much empirical evidence during the last two decades.

As for macroscopic gravity, it is taken for granted that Einstein's theory of general relativity is the geometric theory that is empirically the most successful. However, concerning attempts of quantization, it is flawed by the conceptual disadvantage that it cannot be molded completely into the scheme that is put forward by Maxwell's theory. Following, however, the suggestions of not only Weyl but Einstein himself, theories of gravity can be worked out that not only incorporate general relativity, but are also invariant with respect to local translations and local Lorentz transformations. Such Poincaré gauge theories are to be located in a Riemann–Cartan spacetime with curvature and torsion, and consequently can be coupled not only to the mass but also canonically to the spin of fundamental particles. The resulting gravitational field equations are of a formal structure that is

[1]"… die Frage nach dem inneren Grunde der Massverhältnisse des Raumes …" (Bernhard Riemann, 10 Juni 1854).

analogous to that of the “spontaneously broken” Yang–Mills–Higgs model. For this reason, they can be solved by means of appropriate duality Ansätze as in the case of the instanton solutions of nonabelian gauge theories. It turns out that the metrical background, concerning macroscopic length measurements, is represented by Einstein spaces. Deviations are to be expected only within microscopic regions. This is especially true for interactions with fundamental spinor fields.

Not considering questions concerning the global topology, it has to be realized that the contortion of spacetime induces principally a nonlinearity into the Dirac equation. This self-interaction of the axial vector-type is equivalent to what was suggested by Heisenberg in his unified field theory of elementary particles. The soliton solutions, which are occurring in such nonlinear models, are investigated with respect to their semiclassical and quantum meaning.

The ultimate goal of all these unifications is to build up a theoretical superstructure, an all-encompassing grand synthesis of all physical interactions. Within the limits of the present study, theoretical models are dealt with that are of unequivocal geometric character. These include the Rainich geometrization of the Einstein–Maxwell system, the nonabelian generalizations of the five-dimensional theory of relativity by Kaluza and Klein, and last but not least, the tensor dominance model of Salam et al. It is above all the Kaluza–Klein model being coupled to Dirac fields that, in contrast to the generally covariant Yang–Mills theory, promises a far deeper understanding of the parity violations occurring in the decay of certain metastable mesons.

Most of today’s outstanding theories of fundamental interactions postulate - following Gell-Mann - the so-called quarks as hypothetical building blocks of matter. In order to guarantee the permanent confinement of these enigmatic archaic forms in the observable hadrons, a geometrodynamical mechanism of confinement common to all geometric models of unification is proposed in a speculative prospect.

The bulk of the present study is a slightly revised and amended version of the author’s habilitation thesis, which was submitted to the Faculty of Science of the Christian-Albrechts-University of Kiel in April 1982.

Apart from Prof. J.A. Wheeler, it is especially Prof. F.W. Hehl to whom I owe innumerable direct and indirect suggestive ideas that have helped me to shape structurally important physicomathematical concepts that have entered this work during the stages of its genesis. I would also like to express my sincere gratitude for the hospitality and the highly stimulating working atmosphere at the International Centre for Theoretical Physics, Trieste, which were extremely helpful and for which I should like to thank Prof. Abdus Salam, the International Atomic Energy Agency, and UNESCO.

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Finally it is Mr. H.J. Schneider whom I wish to thank for the painstaking task of translating the German version into nearly literary English.

Flensburg Eckehard Mielke
December 1985

Contents

Chapter 1
Historical Background

Despite a long history, the gauge-theoretic status of gravitation still remains the single gap in the quantum gauge picture of fundamental interactions. The cornerstones of Einstein's theory of gravitation are the relativity and *equivalence principles.* To be more precise, Einstein's ingenious conception of *general relativity* (GR) (EINSTEIN 1915, 1916, 1950) was basically founded on the principle of general equivalence of all noninertial spacetime frames. This theory of gravity met with a first empirical verification at rather early stages in time. Thus it was already in 1919 during an eclipse of the sun that the predicted value of the deflection of light by the sun could be experimentally corroborated[1](EDDINGTON 1920). This was highly important for the scientific approval of Einstein's theory, since the empirically established value, in agreement with GR, is twice as big as what ensues from the conventional Newtonian theory. Further verifications of the so-called "classical tests" within the field of GR followed and are discussed in great detail by MISNER et al. (1973).

Today, gauge theory provides the theoretical basis for a unified description of all particle interactions. In 1967, WEINBERG and ABDUS SALAM (1968) devised a theory that combined Faraday' and Maxwell's theory of electromagnetism (MAXWELL 1881) and Fermi's theory of weak interactions, being the cause for the β-decay of certain radioactive nuclei into a single scheme. Although the *Weinberg–Salam model* is formulated within the frame of geometric concepts related to GR, it took much more time and sophisticated large-scale experimentation to establish it empirically; cf. SALAM (1980). This theory of combined electroweak interactions has gained further empirical evidence by the detection of the intermediating $W^{\pm}$ and Z^{o} bosons as the result of high-energy experiments that were conducted at CERN (ARNISON 1983).

From a systematic point of view, this increase in positive knowledge is of equal initiating importance with the breakthrough that was achieved by James Clerk Maxwell

[1]"Alle Zweifel sind entschwunden, Endlich ist es nun gefunden: Das Licht, das läuft natürlich krumm Zu Einsteins allergrößtem Ruhm!" (Postcard sent to Einstein by Debye, Weyl, and others at October 11,1919).

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(1831–1879) regarding the idea of a unified interpretation of electric and magnetic phenomena.

The most remarkable characteristic of this theoretical approach within our context[2] is the firm and fundamental conviction that all hypotheses concerning the real physical world can be given—before quantization—in purely geometric terms.

Einstein's convictions are rooted in Descartes's idea of the *res extensa* and certainly in the metaphysics of Spinoza as well. The main theme of the latter philosopher can be summarized as follows: if the world can be comprehended *more geometrico*, then its inherent order has to be geometric in itself. A thorough philosophical analysis of all programs intending to geometrize physics is to be found in GRAVES (1971) and KANITSCHEIDER (1971).

And yet Einstein himself was always aware that the foundation of a theory of general relativity was inherently bound to make use of a priori assumptions. To a certain extent, such conditions had already been expounded philosophically by Kant (see also: HÜBNER 1983).

Nevertheless, it was Einstein's first and foremost goal to work out geometric concepts that would make possible the unification of the divergent theoretical descriptions of nature. At his time, however, physics was far from being advanced enough to achieve such an ultimate goal.

Nowadays, however, one of the most certain assets of science is that the classical fields of physics, which are involved in the effort to describe electromagnetic and gravitational interactions, admit an interpretation in terms of differential geometry. Already for Riemann, who among others provided the mathematical bases for the theory of general relativity, it was conceivable that not only the concept of fields—derived from the principle of action at close distances—but also that of a rigid body would have to be modified in the infinitesimal small:

> It is accordingly pretty well conceivable that the metrical relations of space are not in accordance with the premise of the (Euclidean) geometry in the infinitesimal small. And actually this is exactly what should be assumed in case the phenomena could thus be explained more easily.[3]

CLIFFORD (1982) went even further by putting forward the hypothesis that particles consist of nothing but curved *empty* space:

> I hold in fact
>
> (1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
>
> (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

[2]Cf. (MIELKE 1985).

[3]"Es ist also sehr wohl denkbar, daß die Maßverhältnisse des Raumes im Unendlichkleinen den Voraussetzungen der [Euklidischen] Geometrie nicht gemäß sind, und dies würde man in der Tat annehmen müssen, sobald sich dadurch die Erscheinungen auf einfachere Weise erklären ließen." (RIEMANN, Habilitations-Colloquium vom 10. Juni 1854, p 285).

(3) That this variation of curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

Expressed in more modern terms, this view holds that particles are operationally characterized by the (bundle) curvature and the global topology of the underlying spacelike hypersurface evolving in spacetime. This is exactly what was considered more closely by Weyl.

Provided that space is *multiply connected* in the infinitesimal small, one cannot say any longer, "here *is* charge, but only that this closed surface, which is located in the field, encloses charge" (WEYL 1924, p 57).[4] However, before these visionary ideas could grow into a sound theoretical model of definite physical meaning, it was for Albert Einstein to lay the foundation for a geometric interpretation of (macroscopic) gravity by formulating his theory of general relativity (EINSTEIN & GROSSMANN 1913; EINSTEIN 1915, 1916).

Instead of the absolute space of Newton with its Galileo-invariant inertial frames, GR requires an infinite number of merely locally defined Lorentz-invariant frames of reference.

Each of these frames is valid only in its own spacetime region. Moreover, it is connected with its adjacent system by a kind of guiding field: the metrical connection that was introduced by LEVI- CIVITA. Adopting Mach's principle, for Einstein the spacetime geometry ceases to be a God-given Euclidean participant standing high above the "battles" of matter and energy but itself takes part in this struggle. Certainly it is geometry that rules matter how to move, but it is matter that imposes curvature upon spacetime according the principle of action and reaction.[5] As is well known, this was already envisioned by Gauss and Riemann.

Up to the present, this theory of Einstein's passed all empirical verification procedures (WILL 1993). Therefore, it may serve as a prototype of a physical theory that is not only conceptually satisfying but also empirically "true." According to an experiment first devised by EÖTVÖS (1890), the equivalence of inertial and gravitational mass is measured nowadays with a precision better than 1×10^{-12}; this accuracy can be considered a match to what is achieved in the verification of the theoretical value of the electron's magnetic moment predictable within the framework of quantum electrodynamics (QED, BJORKEN & Drell 1964). For EINSTEIN (1915), "…this implied a true triumph of the methods of differential geometry founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civita."

Encouraged by this success, there were early attempts to incorporate other classical fields, such as the electromagnetic fields, into this geometric framework. The first approaches of WEYL (1918), for instance, and of EINSTEIN & Mayer (1931) were doomed to fail, however, although they are mathematically ingenious. Weyl's

[4]Falls der Raum im Kleinen *mehrfach zusammenhängend* ist, kann man nicht mehr "sagen: hier *ist* Ladung, sondern nur: diese im Felde verlaufende geschlossene Fläche schließt Ladung ein.".

[5]WHEELER'S formulations have been adopted here for our purposes (1968, p 4).

theory, for instance, predicts a variation of the spectral lines of neighboring atoms, a prediction that is certainly not in accordance with empirical evidence.

Thus it was for the important work of RAINICH (1925) to show that gravity and electromagnetism, within the framework of general relativity, always represent an "*already unified field theory*." Apart from a degenerate case (GEROCH 1966), the source-free electromagnetic fields imprint such characteristic "footprints" onto the spacetime geometry that all the necessary information for reconstructing the Maxwell fields can be gathered by analyzing the curvature.

This insight, which was developed further by MISNER & WHEELER (1957), encouraged Wheeler to think of a grand design to extend the visions of Riemann, Clifford, and Einstein into a *geometrodynamics* (WHEELER 1962, 1968). In geometrodynamics as such, particles and fields are not considered foreign entities that are immersed *in* geometry, but are regarded as manifestations of geometry proper.

The ultimate goal of this bold and visionary enterprise has never been put forward more convincingly than by EINSTEIN himself (1949, p 81):

> If one had the field-equation of the total field, one would be compelled to demand that the particles themselves would *everywhere* be describable as singularity-free solutions of the completed field-equations. Only then would the general theory of relativity be a *complete* theory.

In 1955, the year of Einstein's death, numerical studies of the field equations yielded the first geometrodynamic model of a localized massive object: the *geon*. In partial fulfillment of the requirements of Einstein's unification program, a singularity-free static solution of the coupled Einstein–Maxwell system had been found that as far as physics is concerned, may be regarded as a concentration of electromagnetic or even purely gravitational radiation kept together for a finite time by its own gravitational attraction. A first example of a *gravitational soliton*! Nevertheless, it is to be admitted that from the angle of the Rainich geometrization, this electrogravitational "ball-lightning" consists only of curved, empty space. Following a furthergoing conception, charge can be regarded as electric flux lines that are free of singularities and "trapped" within a multiply connected spacelike hypersurface of the spacetime manifold. This so-called "wormhole" model (WHEELER 1955), which can be traced back to the speculations of Clifford, Weyl, and EINSTEIN & ROSEN (1935), admits a purely geometrodynamic interpretation of electric charge. We will return to this highly interesting point of view in the context of modern gauge theories of strong interaction.

In order to represent elementary particles, however, a half-integer spin has to be introduced into the framework of geometrodynamics in a nonartificial way. A consistent description of this "nonclassical two-valuedness" (PAULI) of spin-1/2 fields can be achieved only—as DIRAC (1928) has shown—via bispinor fields that satisfy the familiar Dirac equation. This seems to be beyond the scope of the geometrodynamic program.

Nevertheless, it was already in 1928 that Hermann WEYL (1928, p 88) was ready to point out that it is possible to couple these quantum-mechanical spinor fields naturally to the electromagnetic fields if the partial derivatives ∂_j in the Dirac equation are replaced by covariant derivatives $D_j := \partial_j + iA_j$. In the latter concept, a "minimal" substitution of the electromagnetic vector potential A_j is taken into consideration.

According to this new approach, the Lagrangian density of the total system remains invariant under the *local* transformation $\psi \to e^{i\theta}\psi$ of the phase of the wave function ψ and the simultaneous transformation $A_j \to A_j - \partial_j\theta$ of the four-dimensional vector potential A_j. For the first time, such recalibrations, or "gauge transformations," were applied to the metrical relations in spacetime by WEYL in 1918 in order to achieve a unified theory of gravity and electromagnetism. As mentioned above, this theory was contradictory to empirical observations. It was Schrödinger, however, who analyzed Bohr's quantum orbit of the "electron" in 1923 within the theoretical ramifications of this newly laid down principle. The theoretical impact, however, was fully realized only by LONDON (1927a, 1927b) after the quantum-mechanical wave functions of Schrödinger and de Broglie had been introduced into physics. According to this pioneering concept of *gauge invariance*, "…the electromagnetic field is a necessary accompaniment of the matter-wave field and not of gravity" (WEYL 1929).

This raises the question whether it would be possible to use this principle of "generalized relativity" as WEYL had put it in 1929, too, in order to achieve a coupling of the ψ to the gravitational field. Concerning this highly relevant issue, it is of use to remember that CARTAN had devised a generalization of Einstein's theory of gravity in the years 1922 to 1925. By generalizing Riemannian geometry, Cartan proceeded not only from the curvature but also from the torsion as a possible manifestation of the intrinsic angular momentum of matter in spacetime, even before the spin of the electron had been experimentally detected by UHLENBECK & GOUDSMIT (1925).

As for a consistent formulation of Dirac's theory of the electron within the frame of GR, the resulting theory had to be invariant not only with respect to arbitrary differential transformations of the coordinates but also with respect to (Lorentz) rotations of the four-dimensional orthogonal systems of *local* axes ("tetrads"). These local frames are necessary anyhow in order to represent spinor fields in a curved spacetime. As WEYL could show in 1929, this new *gauge theory of gravity* employs in general a linear (asymmetric) connection; its antisymmetric part, i.e., the contortion, turns out to be proportional to the canonical spin tensor of the Dirac field. As WEYL showed in 1950, this is of great concern for the theory, since it necessitates a nonlinear term in the Dirac equation. This term results from the induction of spacetime torsion through the axial vector current of the Dirac field. This was reflected rudimentarily in 1929 by WEYL.

Through the years, this so-called Einstein–Cartan theory was developed further by UTIYAMA (1956); KIBBLE (1961); SCIAMA (1962). Later, it was particularly HEHL (1970); HEHL et al. (1976) who was working on it. As for a supermultiplet with spin-2 and spin-3/2 particles (gravitons and the hypothetical "gravitinos"), this theory leads directly to the idea of supergravity (VAN NIEUWENHUIZEN 1981). On the other hand, the EC theory admits a Rainich geometrization (KUCHAŘ 1966) of the spinor fields and is therefore to be considered an extension of GR that is compatible with the geometrodynamic program.

As a generalization of the EC theory, *Poincaré gauge* (PG) *theories* of gravity (HEHL 1980, 1981) have been taken into consideration more recently. The guiding idea is to cope with theoretical problems concerning the issues of quantizing gravity and subsequently the renormalization of its divergences. Facing these problems, it is generally considered an advantage to determine the dynamics by a Lagrangian 4-form that contains not only linear, but—in analogy to Maxwell's theory—also quadratic expressions with respect to curvature and torsion. Fortunately, in such a theory the metrical background of vacuum solutions with nontrivial torsion (BAEKLER et al. 1982) is not deformed but is retained within the class of Einstein spaces. This remains valid in generic PG theories, provided that the (electrovac) solutions obey a modified duality relation (MIELKE 1984a, 1984b) for the curvature.

So far, it can be ascertained that within the range of these notions, the coupling of spinor fields of the electron to both electromagnetism and gravity can be rather well understood. As for the representation of protons and neutrons, however, HEISENBERG'S isospin formalism (1932) requires the introduction of two distinct Dirac spinors. Accordingly, WEYL'S principle of gauge invariance necessitates generalized electromagnetic field strengths as a concomitant phenomenon. In the year 1954, YANG & MILLS and roughly at the same time SHAW (1955); see SALAM (1980), presented the corresponding gauge theory with local isospin invariance. Geometrically speaking, in such theories a space of "internal" particle attributes is attached to each point in spacetime on which the (local) invariance group–in this case the group $SU(2)$ of special unitary two-dimensional transformations—can act freely. This can be understood as a transition to a kind of "relativity theory" in higher-dimensional ($n > 4$) spaces and was, to a certain extent, anticipated by the pioneering works of KALUZA (1921) and KLEIN (1926). However, in the *Yang–Mills theories*, the bond between the "internal space" and the spacetime is not as rigid as in the Kaluza–Klein model. This can easily be shown if both theories are compared with each other in the light of the mathematical theory of the fiber bundles (KOBAYASHI & NOMIZU 1963); compare also with the very instructive presentation of BERNSTEIN & PHILLIPS (1981).

The spin-1 fields of the Yang–Mills gauge theories are classically massless, similar to those in Maxwell's theory of electromagnetism. However, some of the fields, which are distinguished by internal degrees of freedom, acquire a nonzero mass through the Higgs–Kibble mechanism of dynamic symmetry-breaking (HIGGS 1964; KIBBLE 1967). In the Weinberg–Salam model (see, e.g., TAYLOR 1979 for a review), this is required in order to obtain a realistic theory of weak interactions that remains renormalizable according to the criteria of perturbative quantum field theory.

Today's classification schemes of particles are based on a (broken) flavor symmetry group containing $SU(3)$. Unfortunately, the lowest-dimensional representation of $SU(3)$, the triplet representation, cannot be associated with any of the known physical states. Nevertheless, the hypothesis (GELLMANN & NE'EMAN 1964) that the corresponding particles, e.g., the *quarks*, constitute the fundamental building blocks of all hadronic matter led to extremely useful interpretations concerning the structure of hadrons, and this within the framework of a naive and nonrelativistic approach. Within this context, the explanation of the anomalous magnetic moment of both the proton and the neutron is especially worth mentioning (KOKKEDEE 1969).

Quarks have to be considered structureless fields with spin 1/2, since it would otherwise be impossible to explain the total spin of baryons and hadronic mesons, which in turn are interpreted as bound states of three or two quarks, respectively. This assumption, however, is already, in a nonrelativistic approach to certain excited baryon states, contradictory to Pauli's exclusion principle. In order to save the quark model, it has been suggested to subject these elementary fields to a so-called parastatistics instead of the Fermi–Dirac statistics. In other words, the quarks have to be discriminated by means of additional *color* degrees of freedom (GREENBERG & NELSON 1977). If the associated exact SU(3) color group is postulated only as a local symmetry, this again leads to a gauge theory of Yang–Mills type, but this time as a model for strong interactions. The corresponding spin-1 gauge fields, the so-called gluons, are supposed to mediate strong interactions as "intermediate bosons."

Furthermore, a saturation of the "colored" quarks has to be procured in order to keep the internal color structure of the observable hadrons concealed (almost in the sense of "hidden" quantum-mechanical parameters). The central problem of *quantum chromodynamics* (QCD) (MARCIANO & PAGELS 1978), i.e., of the quantized version of this gauge theory of color symmetry, is the question how to guarantee the permanent confinement of the quarks and gluons in the observable stable particles. Up to now, this central problem has not been solved convincingly. Ad hoc constructions, for instance, that resort to stringlike or baglike models (CHODOS et al. 1974) or to lattice approximations (WILSON 1974) of gauge theories have some phenomenological relevance but appear to be too artificial if judged by a more scientifically oriented standard that demands a more functional solution of this fundamental issue of the structure of matter.

Moreover, it has to be mentioned in this context that in high-energy experiments, there occur hadronic mesons of a spin larger than one. In particular, there occurs an SU(3) nonet that consists of the massive spin-2^+ mesons f, f′, A_2, $K^*(1420)$. This may be interpreted as an indication that strong interactions contain a spin-2 part, which may even be dominant as in the *tensor dominance model*. This is in accordance with the assumption of tensor forces in nonrelativistic phenomenological models of the nucleus (LANDAU & LIFSHITZ 1965).

For a fundamental theory of particles, however, it is also necessary to begin with principle-guided reflections and theoretically explained premises in order to determine the dynamics of an effective spin-2 interaction. It is for this reason that ISHAM et al. and Wess and Zumino suggested independently to let the f-tensor fields obey nonlinear differential equations, which, in analogy to gravity, are similar to Einstein's field equations.

Einstein's theory of general relativity is invariant with respect to the (infinite-dimensional) group cf coordinate transformations that induce, in terms of physics, a change of the *local inertial* frames. Moreover, as has already been mentioned, it may be regarded as a gauge theory of the Lorentz group or its covering group $SL(2, \mathbb{C})$, respectively. If we apply these findings to *strong gravity*, then it is obvious to tie together the internal symmetries, such as the SU(3) one, with the spacetime symmetries that result from the Lorentz transformations according to the subgroup chain $SL(2, \mathbb{C}) \subset SL(2, \mathbb{C}) \otimes SU(3) \subset SL(6, \mathbb{C})$. If these symmetries are regarded as local ones, we get an SL$(6, \mathbb{C})$ gauge theory of strong interaction (SALAM 1973) that provides a geometric rendering for both the internal degrees of freedom of the quarks and the spin of the fundamental particles. This explanation is based on the Riemannian curvature and Cartan's torsion of a generalized space.

As in the Poincaré gauge theory, the torsion induces nonlinear terms into the Dirac equation, which now, however, account for the "internal" spin, as also for the isotopic spin, hypercharge, strangeness, etc. Thereby, an additional coupling between the fundamental spinor fields, which carry different internal quantum numbers, is achieved.

The resulting *nonlinear spinor equation* (MIELKE 1977) has much in common with the equation that was suggested by HEISENBERG and PAULI in 1957 (HEISENBERG 1967). Thereby, besides the modified Planck length

$$\ell^* := \sqrt{8\pi\hbar G_N/c^3} \simeq 8 \times 10^{-33}\text{cm}, \tag{1.1}$$

a further length $\ell := \ell^*/\sqrt{\kappa}$ is introduced into particle physics that corresponds to the scaled "Newtonian" coupling constant $\mathrm{G_S} = \mathrm{G_N}/\kappa$ of strong gravity. SALAM & STRATHDEE (1977a, 1977b), and at the same time, MIELKE (1977) laid down the hypothesis that the *tensor gluons* occurring in the SL$(6, \mathbb{C})$ gauge theory play a decisive role in an understanding of the problem of quark confinement.

Such models, which are worked out according to the implications of Einstein's theory of gravity, provide solutions that assert that the quarks are confined within baglike configurations (SALAM & STRATHDEE 1978) similar to those in the MIT bag model. In this idealization, the geometric structures dealt with are supposed to be spaces of constant curvature (RIEMANN 1854), i.e., de Sitter-like *microuniverses*, within which the quarks appear to be banished for a full cycle of the evolution of the macroscopic universe (MIELKE 1977). In such a "color geometrodynamics" (MIELKE 1980), solutions of Kerr–Newman–de Sitter type (KERR 1963) are more realistic, since they also account for the total angular momentum and the charge of a composed object.

In astrophysics, these configurations are known to play an important role in the relativistic description of collapsed stars, i.e., *black holes* (WHEELER 1964a, b, 1974). These exact solutions imply a fully relativistic relation for the increment of the total mass as a function of the total angular momentum (spin) and the (generalized) charge of the composed object. On an astrophysical scale, this relation can hardly be proved or checked. In the microcosm, however, e.g., after scaling it down to hadronic dimensions, this relation is surprisingly well covered by part of the baryon spectrum (MIELKE 1977, 1980, 1981).

At first sight, it may seem rather far-fetched to propose that such geometric and topological configurations, which are entirely contradictory to our everyday experience, play such a prominent part in the solution of the problem of quark confinement and therefore for the theoretical explication of the internal structure of matter itself.

Fig. 1.1 Wheeler and the author in front of Lake Plön, Germany, 1985

However, on the occasion of the centennial celebrations of Einstein's birth, it was ADLER (1980) who emphasized the structural similarities between black holes and cosmological solutions of GR on the one hand and the magnetic monopole and instanton configurations within the framework of Yang–Mills theories on the other hand. These ideas are in agreement with the above suggested suppositions. Thus it remains for the future to show whether it is possible to work out a unified geometric interpretation of the fundamental structure of matter by incorporating into a coherent theory the (different) hypotheses that have been suggested by such renowned members of the scientific community as Riemann, Clifford, Einstein, Cartan, Weyl, Kaluza, Yang, Wheeler (Fig. 1.1), Weinberg, and Salam.

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Chapter 2
Geometry of Gauge Fields

The geometrization program of field theories has already established a remarkable tradition in modern physics. So far, this approach has centered on the gravitational fields whose intricate structures have found a sound and convincing consolidation in Einstein's theory of general relativity. The prevalence of his theoretical explanation is accounted for by the universality of this interaction (Eötvös–Dicke experiment; see MTW, p. 1050). Maxwell's theory of electromagnetism as an almost archetypical[1] model of a gauge theory (WEYL 1928) is also in harmony with this geometric paradigm. Later on, it was shown by YANG & MILLS (1954) that a theory invariant with respect to local rotations acting on the internal space of isotopic spin may have a related geometric interpretation. If one were to search ab initio for a nonlinear generalization of Maxwell's theory, three conceptually basic assumptions would have to be clarified:

(i) The idealization of the *spacetime continuum* must be expounded as the precondition of any field theory.
(ii) The notion of the physical field magnitudes that are attached to a point of the spacetime manifold, whether are of electromagnetic, gravitational, internal, or even quantum-mechanical origin, has to be defined globally.
(iii) The principle of *action at close distances* requires the existence of a connection between different fields in order to make possible interaction and, in the wake of it, to ensure measurable physical processes.

It turned out that the precise mathematical framework for such constructions is to be found in the theory of fiber bundles. Roughly speaking, these theories deal with appropriate generalizations of the conventional Cartesian product of the "external" and "internal" spaces in question. It is not by chance that these were first formulated by mathematicians (see, e.g., STEENROD 1951) in order to solve global topological problems. At the latest, it was the study of WU & YANG (1975) that made abundantly

[1]We are adopting here the striking and useful coinage of WEINBERG (1977).

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clear that it is exactly this property of the bundle theory that accounts for its being an ideal organ for the analytical description of interacting fields within the gauge-theoretic concept (Yang 1977).

There exists an extensive literature on fiber bundles. On the one hand, there are works that lead to a profound theoretical consolidation of differential geometry (Kobayashi & Nomizu, Vols. I and II, 1963, 1969; hereinafter referred to as KN I and KN II respectively). On the other hand, there are studies (Lubkin 1963; Trautman 1970; Mayer & Drechsler 1977; Daniel & Viallet 1980) that mainly try to work out a mathematically precise basis for the theory of gauge fields as presented in those treaties, which are of an outspokenly physically oriented nature (see, e.g., the reviews of Abers & Lee 1973; Weinberg 1974; 1977; Taylor 1979; O'Raifeartaigh 1979; Cheng & Li 1984). Consequently, the present study can make use of the elegant and concise calculus of differential forms, and thus develop the geometric aspects of Yang–Mills theories almost exclusively on a deductive basis. Furthermore, we try to establish a general theoretical framework that allows not only the incorporation of the theory of gravity into this concept, but subsequently also that of the extended geometrodynamics. "Gauge invariance …has the character of general relativity… and can certainly only be understood with reference to it" (Weyl 1929). It is therefore advisable to present the gauge theories of particle physics from the beginning in a curved spacetime.

2.1 Differentiable Manifolds

The theory of general relativity as well as gauge theories in their classical, i.e., unquantized, form are based on the concept of a continuous spacetime, which, however, may comprise curvature and possibly a nontrivial topology, too. And this despite the fact that the hypothesis of continuity can no longer be taken as self-evident, as it was, due to historical limitations, in classical mechanics, but has to be modified with respect to the principles of quantum mechanics (compare, e.g., Penrose 1968; Wilson 1976; Heller & Staruszkiewicz 1981; Friedberg & Lee 1984).

At first, the illustrative notion of smooth (curved) surfaces will be generalized into the more abstract notion of a manifold, agreeing with Riemann (1854). The latter is an entity that is *locally* similar to the n-dimensional Euclidean space. For the sake of a more precise explanation, a *topological space* M is postulated, i.e., a set with a notion of neighborhood. This space should be equipped with *coordinates*. To achieve this, we consider injective (reversible) mappings $\mathrm{x} : \mathrm{M} \to \mathbb{R}^{\mathrm{n}}$, whose range extends to an open subset of the usual Euclidean space E^{n}. Such a map together with its domain $\mathrm{U_i}$ is called an n-dimensional chart. By the projection π_i onto the single axes of the coordinate system of $\mathbb{R}^{\mathrm{n}}$, the *local coordinates* $\mathrm{x}_i = \pi_i \circ \mathrm{x}$, $i = 1, \ldots, \mathrm{n}$ (see Fig. 2.1), come into existence.

In general, one chart does not suffice to cover a set completely. (As is well known, at least two charts are necessary for the stereographic projection of the n-dimensional sphere S^{n}). However, it is possible to construct a collection of charts of identical

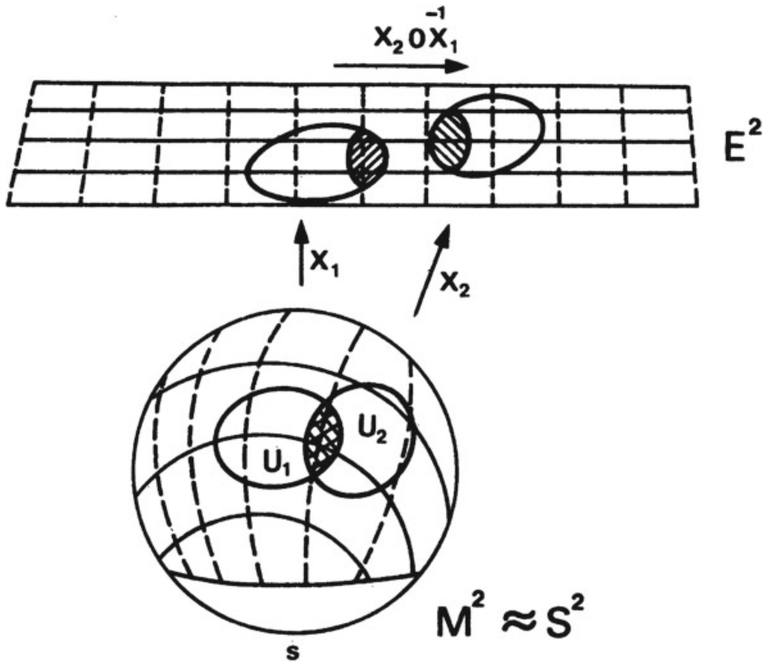

Fig. 2.1 Coordinates of the sphere obtained by stereographic projection

dimensions so that the set-theoretic union of their domains $U_i \subset M$ cover M completely. In order to achieve this, it is reasonable to demand compatibility, which means that the change of coordinates (transition function in the theory of the fiber bundles)

$$x_2 \circ \overset{-1}{x}_1 : \mathbb{R} \to \mathbb{R} \tag{2.1.1}$$

is a diffeomorphism in the area of the intersection of their domains. Such mappings form an infinite-dimensional group with respect to composition, i.e., the group $\mathscr{D}(M)$ of differential coordinate transformations (an analysis of the mathematical complex structure of the Lie group $\mathscr{D}(M)$ is, for instance, to be found in OMORI (1973).

A collection (U_i, x_i) of compatible charts is called a C^∞-atlas of M. (The common assumption is that these transformations are r times continuously differentiable, i.e., of class C^r). This atlas again can be extended unequivocally to a *complete* one, which in turn determines a differential structure of dimension n.

Definition A topological (locally connected) Hausdorff space M endowed with a C^∞-structure of dimension n is termed a *differentiable manifold* (or for the sake of brevity, “manifold”).

It has to be pointed out that the notion of a manifold has been defined here intrinsically, and thus does not make use of the possible embedding into a Euclidean space of a higher-dimensional order. According to a theorem of Janet and Cartan, locally isometric embeddings of analytic Riemannian manifolds into $E^{n(n+1)/2}$ are possible (cf. KN II, p. 354), but the required embedding spaces would cause problems for a physical interpretation.

A global isometric embedding theorem for a compact manifold into Euclidean dimension $N = n(3n + 11)/2$ was proven by NASH (1956); for a noncompact one, the extravagant dimension $N = n(n + 1)(3n + 11)/2$ results; cf. CHEN (2000) for a survey. Later, in the smooth case, FROLOV (2006) considered the embedding of the surface of a rotating Kerr–Newman black hole into $\mathbb{E}^4$.

For the present, local physics without interaction is applied only to the *tangent space* $T_m(M)$ attached to a point $m \in M$. Contrary to the given illustration (see Fig. 2.2), it is to be accentuated that the construction of this space should depend only on the structure of the manifold and avoid an embedding of any kind. In order to achieve this, a curve s(t) on the manifold is chosen that passes through a point $m \in M$. As an auxiliary device, an arbitrary smooth (i.e., C^∞-differentiable) function f is considered. Its derivative

$$e^{\mathrm{f}} := \left.\frac{df(s(t))}{dt}\right|_{t_o=\overset{-1}{s}(m)} = \frac{\partial \mathrm{f}}{\partial x^i} \cdot \frac{dx^i(s(t))}{dt} \tag{2.1.2}$$

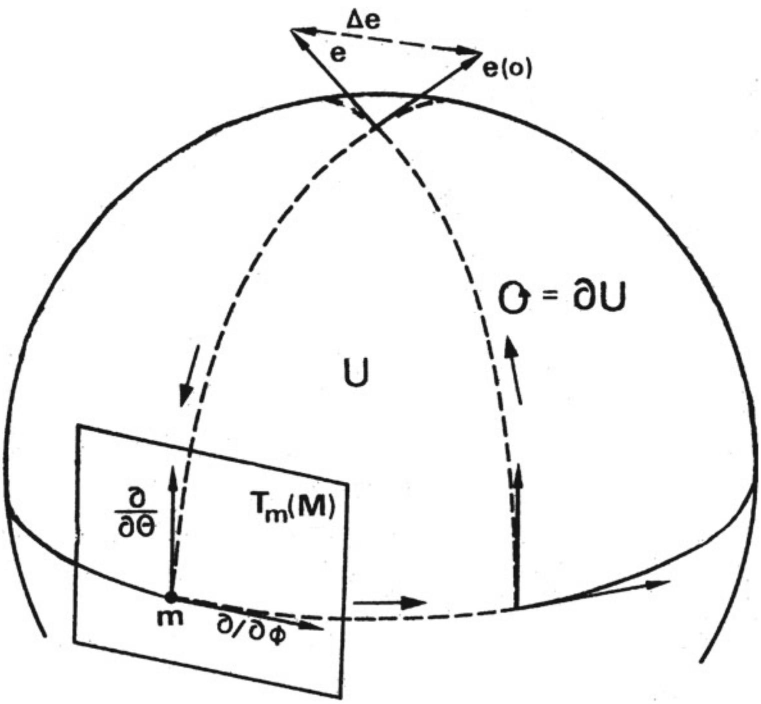

Fig. 2.2 Tangent vectors concerning holonomic coordinates on the sphere

along the direction of the curve, considered a mapping of the algebra of the differentiable functions f to the space $\mathbb{R}$ of real numbers, defines the *tangent vector* e(m) or "velocity vector" at m (see MILNOR & STASHEFF 1974, p. 5). Concerning a given rigid basis of $\mathbb{R}^n$, the n-dimensional tangent space $T_m(M)$ is spanned by the linearly independent vectors $e_\alpha(m), \alpha = 1, \ldots, n$. It follows from (2.1.2) that the partial derivates $\partial_i := \partial/\partial x^i$ constitute a local basis for $T_m(M)$ with respect to a *holonomic* coordinate system. By definition, the relation $[e_i(m), e_j(m)] = [\partial_i, \partial_j] = 0$ is satisfied. Accordingly, the tangent basis for the "comoving" anholonomic frame (Cartan's "repère mobile") can be transcribed as follows:

$$e_\alpha(m) = e^i_{.\alpha}(m)\,\partial_i. \tag{2.1.3}$$

The occurring nonsingular matrices $e^i_\alpha(m)$ depend on the point in question and are consequently called concomitant "n-Beine"[2] or *tetrads* (four dimensions). The latter were introduced by EINSTEIN (1928) in order to formulate a theory of gravity complying with the hypothesis of teleparallelism. By the union of all of the manifold's tangential spaces, the so-called *tangent bundle* is brought into existence:

$$T(M) = \bigcup_{m\in M} T_m(M). \tag{2.1.4}$$

Herein we find a first example of a bundle in terms of a modern mathematical concept, which has to be specified in the following paragraphs of the present study.

2.2 Tensor Fields and Exterior Forms

In addition to the tangent bundle T(M), a so-called *dual* bundle $T^*(M) = \bigcup_{m\in M} T^*_m(M)$ consisting of all possible cotangent spaces $T^*_m(M)$ on the manifold can be constructed. As is to be shown, these are represented by the canonical differential forms of first degree.

Let $\vartheta^\alpha, \alpha = 1, \ldots, n$, be a basis of $T^*_m(M)$. Then the "duality"[3] requires that ϑ^α be orthonormal to a basis of $T_m(M)$ with respect to the natural interior product

$$e_\beta \rfloor \vartheta^\alpha = \delta^\alpha{}_\beta. \tag{2.2.1}$$

With respect to the holonomic coordinate system, a natural basis of $T^*_m(M)$ is given by dx^i. In terms of the rigid basis of E^n, the 1-forms

$$\vartheta = \vartheta^\alpha P_\alpha \quad \in \quad C^\infty(T^*(M)), \tag{2.2.2}$$

[2] Gell-Mann's German term "Vielbeinfeld" is even more to the point.

[3] This duality should not be confused with the duality of exterior forms, as introduced later on.

which are also called Pfaffian forms, can be expressed in general by

$$\vartheta^{\alpha} = \mathrm{E}^{\alpha}_{j}(m)\, dx^{j}. \tag{2.2.3}$$

The latter constitute a so-called *anholonomic* (dual) basis, in as much as the exterior derivative $\mathrm{d}\vartheta^{\alpha}$, in contrast to that of dx^j, is generally different from zero. Following (2.2.1), $\mathrm{E}^{\alpha}_{\cdot\, j}$ can be understood as the dual tetrad field "reciprocal" to e^{i}_{β} (cf. (2.1.3)). These spacetime-dependent matrices are related to each other by

$$E^{\alpha}_{i}\; e^{i}_{\cdot\beta} = \delta^{\alpha}{}_{\beta}. \tag{2.2.4}$$

By repeated tensorial multiplication of tangent and cotangent spaces with themselves, *tensor bundles* of generic, co- and contravariant degrees (p,q) will be obtained:

$$T^{q}_{p}(M) := \bigcup_{m\in M} \otimes^{p}\, T^{*}_{m}(M)\; \otimes^{q}\; T_{m}(M). \tag{2.2.5}$$

Completely symmetric or antisymmetric products are denoted by $\otimes_{\mathrm{s,a}}$ or, even more commonly, by the symbols $\vee$ and $\wedge$, respectively. Note that a tensor is a geometric object that is defined independently from the choice of coordinates. Relative to a basis of the tensor space, a tensor may, however, be locally expanded as follows:

$$T = T^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}(m)\vartheta^{\alpha_1}\otimes\cdots\otimes\vartheta^{\alpha_p}\otimes e_{\beta_1}\otimes\cdots\otimes e_{\beta_q}\in C^{\infty}(T^{q}_{p}(M)). \tag{2.2.6}$$

The quantities $\mathrm{T}^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}(\mathrm{m})$ are called its p covariant and q contravariant components, and $\mathrm{T}^{\mathrm{q}}_{\mathrm{p}}(\mathrm{M})$ may be regarded as a bundle associated with L(M) having the manifold M as a base, the tensor representation $\mathrm{D}^{(\mathrm{p,q})}$ of GL(n, $\mathbb{R}$), i.e., $\rho^{(\mathrm{p,q})}(\mathrm{GL}(\mathrm{n},\mathbb{R}))$ as a structure group and the (Cartesian) product $\otimes^{\mathrm{p}*}\mathbb{R}^{\mathrm{n}}\otimes{}^{\mathrm{q}}\mathbb{R}^{\mathrm{n}}$ as a typical fiber.

The (pseudo-)Riemannian metric on M is one of the most important examples not only for differential geometry, but also for gauge theories in curved spacetime. Formally, this metric can be defined as a covariant symmetric tensor field of degree (2, 0):

$$ds^2 = g_{\alpha\beta}\vartheta^{\alpha}\otimes_s\vartheta^{\beta} =: g_{ij}\,dx^{i}\otimes_s\, dx^{j}. \tag{2.2.7}$$

This "metrical groundform" (WEYL 1923) or square of the line element determines the scale of the manifold. Due to the postulated symmetry of the components $\mathrm{g}_{\mu\nu}$, it is always possible, by a linear transformation of the main axes, to rearrange the metric locally as follows:

$$ds^2 = g_{ij}\,dx^{i}\otimes_s dx^{j} \overset{(*)}{=} -\sum_{i=1}^{s}(\mathring{\vartheta}^{i})^2 + \sum_{j=s+1}^{n}(\mathring{\vartheta}^{j})^2. \tag{2.2.8}$$

If these signatures occur in the same characteristic manner throughout all points of the manifold, it is generally called a *pseudo-Riemannian manifold of signature s*; cf. SAKHAROV (1984).

In the geometric description not only of gauge fields but of gravitational fields as well, those cases are prevalent in which one makes use only of an irreducible subspace of $\mathrm{T}_{\mathrm{p}}^{\mathrm{q}}(\mathrm{M})$, i.e., the space of *completely antisymmetric* covariant tensor fields of degree (p, 0). Usually, $\wedge := \otimes_{\mathrm{a}}$ is an abbreviation for the antisymmetrized tensor product, while cross sections of this special tensor bundle are called (alternating) differential form of degree p:

$$\alpha^{(p)} = \frac{1}{p!} A_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \in C^\infty(\wedge^p T^*(M)). \tag{2.2.9}$$

The mathematically elegant calculus of exterior (alternating) differential forms, essentially developed by Poincaré and E. Cartan, is based on this. As far as we know, this calculus was first applied to physics by MISNER & WHEELER (1957) in order to achieve both a more concise reformulation of Maxwell's theory of electromagnetism and an incorporation of that theory into an "already unified field theory" of electromagnetism and gravitation. The standard reference book concerning gravitation (MTW 1973) gives an instructive account with respect to this mathematical tool. Five years later, HOWE & TUCKER (1978) rewrote the SU(2)-gauge theory in the language of differential forms and especially drew attention to the subtleties of the real Minkowski space.

2.3 Fiber Bundles as an Enlarged Geometric Arena

Atomic spectra can be described quantum-mechanically by the Schrödinger theory, which is based on a detailed knowledge of the *dynamics* of the microscopic system. Subsequently, a deepened interpretation of the atomic and nuclear phenomena was achieved by *group-theoretic* methods (WEYL 1928; WIGNER 1931). In the subnuclear domain of particles, we are almost completely dependent on a classification of particle properties according to such group-theoretic criteria. WIGNER's analysis (1939, 1957) of the representations of the transformation group in flat spacetime, i.e., the *Poincaré group*, is an outstanding example that yields the well-known invariant characterization of particles in terms of *mass* and *spin*. Moreover, the overwhelming number of "excited states" of stable particles that have been discovered recently has certainly received a thorough and satisfying classification by the assumption of "internal symmetries." These have been exemplified by the hypothesis of isotopic-spin invariance HEISENBERG (1932) or, for instance, by the hypothesis of the unitary group SU(3) of GELL- MANN & NEEMAN (1964).

Within the framework of quantum field theory (QFT, BJORKEN & DRELL 1964), the most crucial problem is to find a geometric relation between the spacetime symmetries and the postulated internal symmetries that leads to a natural concept of interaction. It is to be remembered that in QFT, an elementary particle is represented by a complex field with several components, or to be more precise, by an array of rays in a Hilbert space $\mathscr{H}$, which is to be transformed according to an irreducible unitary representation of the Poincaré group. From the geometric point of view, it is assumed that φ at a point m is equal to the value φ(m) in a complex vector space V_m. Additionally, it can be postulated that a vector space of this kind is attached to each point of the spacetime manifold. Similarly to the presentation of a tangent space, it is required, by a global point of view, to take into consideration a *bundle* of such abstract spaces of "particle attributes." In order to realize internal symmetries, the vector spaces V have to be generalized in such a way that they can function as representation spaces for the internal group G. Accordingly, these "internal rotations" are seen as operating pointwise on the spacetime manifold. It is only to be considered that under these circumstances, at least a transformed description of the same physical reality is achieved that is represented by matter fields.

Intuitive concepts, rather, like these have their precise counterpart in the theory of fiber bundles. As has already been mentioned, these are appropriate generalizations of the familiar Cartesian product of the spaces under consideration. In particular, it involves the group manifold associated with the internal symmetries and the spacetime continuum. However, this generalization is qualified to take *globally* nontrivial topological structures into consideration.

Definition A *fiber bundle* (F, M, π) consists of two C^∞-differentiable manifolds F and M and a smooth surjective mapping

$$\pi : F \to M. \tag{2.3.1}$$

For obvious reasons (see Fig. 2.3), F is called the *total space*, M the *base space*, and π the *projection*. The closed submanifold $F_m := \overset{-1}{\pi}(m)$ will be called a *typical fiber*, for in contrast to more general bundles, these are all isomorphic in the case of fiber bundles. Furthermore, (F, M, π) is required to be locally trivial, i.e., each

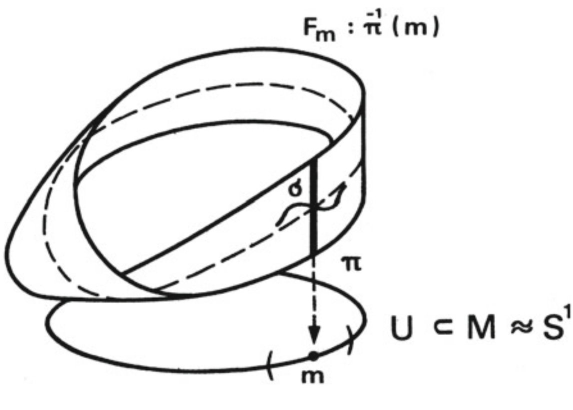

Fig. 2.3 Möbius strip as an example of a globally nontrivial fiber bundle

point m ∈ M has a neighborhood U such that $\overset{-1}{\pi}(U) \approx U \times F_m$ is isomorphic to the product bundle $(U \times F_m, U, \pi_u)$. The mapping

$$\sigma : U \to F \qquad \text{where} \qquad \pi \circ \sigma = \text{id} \tag{2.3.2}$$

defines a local *cross section* through the bundle. In the more physically oriented literature, this is better known as a choice of a "local gauge."

The most striking example of a global nontrivial fiber bundle is the Möbius strip. In this case, the (multiply connected) circle is considered a base space, whereas a one-dimensional real vector space, e.g., the unit interval [0, 1], represents the typical fiber. This fiber is subjected to such a twist in the total space that opposite points of the interval can be identified after a full revolution.

The basic idea of the very concept of gauge invariance, however, is that interacting fields (e.g., gauge fields) at any given point of the spacetime can be varied by local "internal rotations" (e.g., by local isospin transformations), but that this results only in an equivalent description of the same physical reality, as far as matter fields are concerned. In order to heighten the precision of this notion, the internal symmetry group G (usually assumed to be continuous) and the spacetime manifold are considered as a whole and extended into a single *enlarged geometric arena*, the principal fiber bundle.

Definition Provided P and G are manifolds and the structure group G is a Lie group,then the collection $P(M, G, \pi, \delta)$ is called a *principal fiber bundle* if

(i) (P, M, π) is a fiber bundle with typical fiber G;
(ii) (P, G, δ) is a G-manifold (whereby G is acting on P from the right);
(iii) P is *locally* trivial, which means that every point of M has a neighborhood $U \subset M$ together with an isomorphism $\iota : U \times G \to \overset{-1}{\pi}(U)$ for which the following property holds:

$$\iota(m, g_1 g_2) = \iota(m, g_1) \cdot g_2, \qquad m \in U, g \in G. \tag{2.3.3}$$

A manifold is called a *G-manifold* (P, G, δ) if G acts on it as a free transformation group either from the left or from the right:

$$\delta : \left\{ \begin{array}{ccccc|c} G & \times & P & \to & P & \\ \Psi & & \Psi & & \Psi & e \cdot p_\circ = p_\circ \in P \\ g_p & \cdot & p_\circ & = & p & \end{array} \right\}. \tag{2.3.4}$$

This fixes the transformations of the total space P if it is subjected to the action of the symmetry group G. (If the action of G were also a transitive one, i.e., if there always existed a g that would provide a relation $p_2 = g\, p_1$ for given p_1 and p_2, then G considered as a manifold would be isomorphic to the base M.)

These highly formalized constructions, however, yield only an "arena" for the representation of physical fields.

2.4 Associated Bundles and Physical Fields

In order to characterize *matter fields* within the framework of bundle theory, the notion of *vector bundles* being *associated* with P(M, G, π, δ) is also required. For this purpose, the fiber bundles V(M, F; G, P) associated with P(M, G, π, δ) are constructed as follows:

Let the typical fiber F be a manifold on which G acts from the right-hand side, i.e., on the product space P × F, an action from the right-hand side is defined by

$$\delta_g(p, \zeta) := (pg, g^{-1}\zeta) \subset P \times F, \quad g \in G. \tag{2.4.1}$$

The quotient space of P × F with regard to this action of the group G will be denoted by $V = P \times_G F$. The isomorphism $\overset{-1}{\pi}(U) \approx U \times G$ concerning the original domain of a neighborhood U, which results from the very construction of a fiber bundle, will induce the isomorphism $\overset{-1}{\pi}_V(U) \approx U \times F$ in the associated bundle. Provided that $\overset{-1}{\pi}_V(U)$ is an open submanifold of V, then V can be equipped with a differentiable structure as a whole.

Usually, in the cases of physical relevance it is not the structure group itself that occurs, but its linear representation $\rho : G \to GL(N, \mathbb{C})$, i.e., the Lie homomorphism of G into the general linear group of complex N × N matrices. If their representation space, the N-dimensional vector space $\mathbb{C}^N$ over the field of complex numbers, is considered a typical fiber, an associated bundle can be constructed from it,

$$V^\rho := V^\rho(M, \mathbb{C}^N, \rho(G) \subseteq GL(N, \mathbb{C}), P), \tag{2.4.2}$$

which will be referred to as a *complex vector bundle* on M. By taking the product with the complexified[4] cotangent bundle $T^*_{\mathbb{C}}(M)$, further vector bundles will come into existence on the spacetime manifold.

Physical fields on M are then to be considered p-forms of (representation) type ρ, i.e., the cross section

$$\phi^{(p)} := \varphi^p \otimes b \in C^\infty(\wedge^p T_{\mathbb{C}}{}^*(M) \otimes V^\rho), \quad b \in C^\infty(V^\rho). \tag{2.4.3}$$

If this space is endowed with a Hermitian inner product, then such fields are referred to as being of "charged type" (compare Mack 1981).

For $N \geq 1$, these fields form an infinite-dimensional vector space over $\mathbb{C}^N$. In (2.4.3), the bundle coordinates relative to the local basis fields $b = (b_A) : U \subset M \to V^\rho$ are denoted by

$$\varphi^{(p)} = \left\{\varphi^{(p)A} | A = 1, \ldots, N\right\}. \tag{2.4.4}$$

[4]For reasons of consistency, the cotangent bundle has itself to be complexified, since in physics, complex vector bundles are dealt with almost exclusively.

Let the Lie homomorphism $\rho : G \to GL(N, \mathbb{C})$ be explicitly given by the nonsingular complex $N \times N$ matrices $\rho_A{}^B(g(\xi))$, where $\xi^j (j = 1, \ldots, \dim G)$ prescribes a parametrization of the Lie algebra. Then the *infinitesimal operators*

$$L_j := \left[\frac{\partial \rho_A{}^B(g(\xi))}{\partial \xi^j} \Big|_{g=e} \right] \in T_e(\rho(G)) \tag{2.4.5}$$

are constructed by means of the "derived" homomorphism of G. In the case of an identical representation, i.e., for $\rho = id$, the infinitesimal generators I_j form a basis of the Lie algebra $\mathfrak{g}$ of G. As a representation of I_j, the infinitesimal operators inherit the commutation relations from the Lie algebra, i.e.,

$$\left[L_i, L_j \right] = c_{ij}{}^k L_k. \tag{2.4.6}$$

These algebraic relations are determined by the structure constants $c_{ij}{}^k$ of the Lie algebra $\mathfrak{g}$. "Gauge fields," unlike matter fields, are considered as cross sections of a vector bundle V^{Ad} respecting the adjoint representation of G as a structure group. Thus, they can be represented by the Lie-algebra-valued forms

$$\phi^{(p)} = \frac{1}{p!} \phi^j_{\alpha_1 \ldots \alpha_p} L_j \otimes \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \in C^\infty \left(\wedge^p T^*_{\mathbb{C}}(M) \otimes V^{Ad} \right). \tag{2.4.7}$$

All rules of the calculus of exterior forms are valid, except that in the nonabelian case, the wedge product is no longer an alternating operation. In contrast, the commutator

$$\left[\phi^{(p)}, \psi^{(q)} \right] := \phi^{(p)} \wedge \psi^{(q)} - (-1)^{pq} \psi^{(q)} \wedge \phi^{(p)} = (-1)^{pq+1} \left[\psi^{(q)}, \phi^{(p)} \right] \tag{2.4.8}$$

of Lie-algebra-valued forms has this useful alternating property. Accordingly, the commutator of a form of even degree with itself is equal to zero on account of

$$\left[\phi^{(2k)}, \phi^{(2k)} \right] = - \left[\phi^{(2k)}, \phi^{(2k)} \right] = 0. \tag{2.4.9}$$

2.5 Connection and Covariant Derivative

Up to now, these bundle structures are still unconnected in the sense that the internal spaces that are thought of as being attached to the spacetime M cannot be "compared" in a differentiable manner along a given curve $s(t) \subset M$ in the base manifold. In order to make possible such a parallel displacement of the fibers (Levi- Civita 1926; KN I, p. 68), a *connection* is required, i.e., a kind of guiding field.

There are several equivalent definitions concerning the connection in a fiber bundle (compare, e.g., Eguchi et al. 1980). Such a connection is imprinted on a principal fiber bundle $P(M, G, \pi, \delta)$ in a mathematically rather abstract way: Proceed from the cotangent bundle $T_p{}^*(P)$ at the point $p \in P$ (see KN I, Chap. 1). Then decompose it

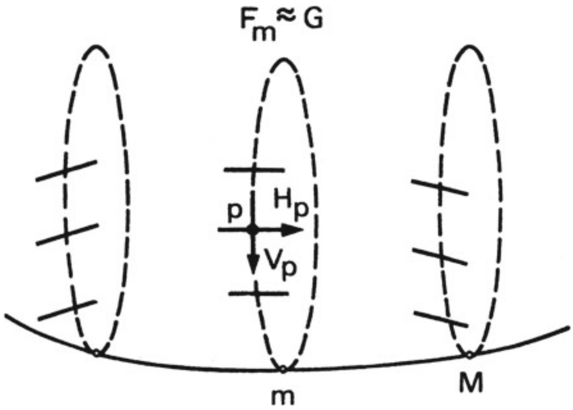

Fig. 2.4 Horizontal and vertical cotangent spaces of a principal fiber bundle

into the direct sum of horizontal and the vertical subspaces, i.e., $T_p{}^*(P) = H_p \oplus V_p$. Then consider an abstract parallel displacement in the vertical subspace (see the straight lines in Fig. 2.4) prescribed by

$$V_p{}^* = T_p{}^*(F_m(M)), \tag{2.5.1}$$

i.e., in the cotangent space of the bundle $F_m(M)$ of typical fibers on M.

This will be achieved by the assertion of a 1-form $\omega(e) \in \mathscr{C}$ with values in the Lie algebra $\mathfrak{g}$ of the structure group G that is subject to the following conditions:

(i) With respect to a right-transformation of the tangent vectors of P, this 1-form transforms itself according to the inverse, adjoint representation of G:

$$\omega(eg) = g^{-1}\omega(e)g, \qquad e \in T(P). \tag{2.5.2}$$

(ii) For vertical elements $e_v = pdg \in V_p$, it will be mapped to the left-invariant element

$$\omega(p\,dg) = g^{-1}dg \qquad \in \quad T_e(G) \approx \mathfrak{g} \tag{2.5.3}$$

of the Lie algebra of G.

For the treatment of physical problems, the following equivalent approach to the concept of connections is more appropriate (Atiyah 1978). In this second approach, a connection in an associated vector bundle $\bigwedge^p T^*_{\mathbb{C}}(M) \otimes V^\rho$ is defined via the *covariant derivative*. This is analogous to the procedure known from classical differential geometry (see Milnor & Stasheff 1974). Consider a linear differential operator

$$D : C^\infty(V^\rho) \longrightarrow C^\infty(T^*_{\mathbb{C}}(M) \otimes V^\rho) \tag{2.5.4}$$

in the vector bundle V^ρ and regard it as a mapping of the space of sections of V^ρ onto those of the product bundle $T^*_{\mathbb{C}}(M) \otimes V^\rho$. This operator will be defined in such a way that it acts similarly to the "absolute differential," which was introduced by LEVI- CIVITA (1926), with respect to the bundle basis b.

To be more precise, this means that the result of the action of D on b can be expressed as a linear transformation of the selfsame basis of V^ρ, i.e.,

$$Db = \omega \otimes b, \quad b \in C^\infty(V^\rho). \tag{2.5.6}$$

(The called-for linearity is by no means as self-evident as it might seem. For a system of paths that generalize geodesics, *nonlinear connections* can be introduced into differential geometry, too (LAUGWITZ 1965, p. 190; cf. GOENNER 1984). Furthermore, for the cross sections of the vector-valued forms (2.4.3), the differential operator D is expected to have a natural *extension* that satisfies the *generalized Leibniz rule*:

$$D(\varphi^{(p)} \otimes b) = d\varphi^{(p)} \otimes b + (-1)^p \varphi^{(p)} \otimes Db. \tag{2.5.7}$$

(Attention should be drawn to the fact that the symbol D is used throughout our presentation, although the occurrence of varying representations ρ of the same group G is by no means excluded in the formulas.) Concerning the adjoint representation in the relevant case of a vector bundle V^{Ad}, (2.5.7) is converted into the more familiar relation

$$D(\varphi^{(p)} \otimes b) = \left\{ d\varphi^p + (-1)^p [\varphi^{(p)}, \omega] \right\} \otimes b. \tag{2.5.8}$$

The image of Db will be referred to as the *(G-)covariant exterior derivative* of b. Since D is a local operator, a global connection can be defined unequivocally by its restriction to a neighborhood U of M. Let b_A, $A = 1, \ldots, N$ be a local basis for the cross section $V^\rho_{\mathbb{U}}$ restricted to U. With respect to this local system of reference, the resulting effect of D on b_A can be expanded as follows:

$$Db_A = \omega_A{}^B b_B, A, B = 1, \ldots, N. \tag{2.5.9}$$

This is identical to (2.5.6) if given in the more abstract notation of matrices. The matrix $[\omega_A{}^B]$ acquires values in $T^*_{\mathbb{C}}(M)$ and will consequently be called a *connection 1-form*, which means that it can be expanded locally as

$$\Gamma := \overset{*}{\sigma}\omega = \Gamma_\alpha{}^j L_j \otimes \vartheta^\alpha, \quad j = 1, \ldots, \dim G, \tag{2.5.10}$$

i.e., by a pullback via the cross section.

The coefficients $\Gamma_\alpha^j(m)$ will turn out to be generalized gauge potentials as they occur in modern physical field theories. With respect to unitary structure groups U(f), these potentials will be denoted, as is usually done, by

$$A_\alpha{}^j(m) := \Gamma_\alpha{}^j(m)_{|G=U(f)} \tag{2.5.11}$$

2.6 Curvature

For differential forms in general, the identity dd $\equiv 0$ holds, which may be regarded as a counterpart to the homological relation $\partial\partial \equiv 0$. Figuratively speaking, this means that the "boundary of a boundary" of a manifold vanishes identically. In general, such a relation would not hold for the twofold *covariant* exterior derivative, for the departure from integrability is accounted for by the curvature of the bundle space, in analogy to the concepts of differential geometry:

$$D\,Db =: \Omega \otimes b. \tag{2.6.1}$$

It follows from

$$\begin{aligned} D\,Db = D(\omega \otimes b) &= d\omega \otimes b - \omega \wedge Db \\ &= (d\omega - \omega \wedge \omega) \otimes b \end{aligned} \tag{2.6.2}$$

that the curvature 2-form Ω satisfies the *second structure equation* of É. CARTAN:

$$\boxed{\Omega = d\omega - \omega \wedge \omega = d\omega - \frac{1}{2}[\omega, \omega]}. \tag{2.6.3}$$

As was to be expected, Ω is a 2-form of type Ad G. Most often, its local presentation

$$\overset{*}{\sigma}\Omega = \frac{1}{2} F_{\alpha\beta}{}^{j} L_j \otimes \vartheta^{\alpha} \wedge \vartheta^{\beta} \tag{2.6.4}$$

is preferred in physical applications. Its components with two spacetime indices and one group index are called the field strengths of the gauge fields in question.

If we are using (2.5.10) in order to notate (2.6.3), it will become obvious that these field strengths are in absolute compliance with the familiar relations

$$F_{\alpha\beta}{}^{j} = \partial_{\alpha}\Gamma_{\beta}{}^{j} - \partial_{\beta}\Gamma_{\alpha}{}^{j} - c_{kl}{}^{j}\Gamma_{\alpha}{}^{k}\Gamma_{\beta}{}^{l}, \tag{2.6.5}$$

which are well known from the Yang–Mills theory. Concerning electromagnetism, which can be formulated in a fiber bundle with abelian structure group G $= U(1)$, the structure constants are identically zero. Consequently, in holonomic coordinates, the well-known relation

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{2.6.6}$$

holds.

The curvature of the bundle is interpreted geometrically, similar to the explanation that is offered in Riemannian geometry. For illustrative purposes, we consider the variation

$$\Delta b = b - b(\circ) = \oint (db)_{\text{hor}}, \tag{2.6.7}$$

which in the local basis b of the bundle results from a parallel displacement of b along an *infinitesimal* closed curve $\circlearrowleft = \partial U \subset M$. The very concept of a connection guarantees that

$$(db)_{\text{hor}} = Db = \omega \otimes b \tag{2.6.8}$$

is valid, since it is only the *horizontal* part of this displacement that matters. This relation together with Stokes's theorem yields

$$\begin{aligned} \Delta b &= \int_{\partial U} \omega \otimes b = \int_U d(\omega \otimes b)_{\text{hor}} = \int_U D(\omega \otimes b) \\ &= \int_U \Omega \otimes b \simeq \Omega \otimes b \end{aligned} \tag{2.6.9}$$

as a measure of the *nonintegrability* of a parallel displacement along the infinitesimal closed curve ∂U. Figure 2.2 shows this for a parallel displacement of a pair of tangent vectors along a path constructed entirely of great circles of the sphere. The so-called holonomy group H(M, m) of a manifold is generated by linear transformations of $T_m(M)$ onto itself. These are generated by displacements of $e(m) \in T_m(M)$ along arbitrary curves that begin and end at m. According to the theorema egregium,

$$K = \lim_{U \to 0} \left(\int_U \Omega \otimes b \right) \Big/ \left(\int_U 1 \otimes b \right) \tag{2.6.10}$$

is the Gaussian or local curvature of a 2-dimensional surface (Laugwitz 1965; cf. Sulanke & Wintgen 1972, p. 242).

Analogously, the *nonintegrability* of the parallel displacement of the bundle basis is measured by the integral (2.6.9). This accounts not only for the Riemannian curvature of the base space M, but also for the “internal” curvature derived from the prescribed connection in the vector bundle that has been dealt with so far. For the covariant derivative, a relation similar to $dd \equiv 0$ occurs at a higher degree of differentiation only:

$$DDDb \equiv 0. \tag{2.6.11}$$

This relation implies the (second) Bianchi identity

$$\boxed{D\Omega \equiv 0} \tag{2.6.12}$$

for the curvature 2-form. The proof is obtained by writing out (2.6.11) explicitly by inserting the structure equation (2.6.3) repeatedly:

$$
\begin{aligned}
DDDb &= D(\Omega \otimes b) = (d\Omega + [\Omega, \omega]) \otimes b \\
&= (d\Omega + \Omega \wedge \omega - \omega \wedge \Omega) \otimes b \\
&= (dd\omega - d\omega \wedge \omega + \omega \wedge d\omega \\
&\qquad + d\omega \wedge \omega - \omega \wedge \omega \wedge \omega \\
&\qquad + \omega \wedge \omega \wedge \omega - \omega \wedge d\omega) \otimes b \equiv 0.
\end{aligned} \tag{2.6.13}
$$

On the other hand, this derivation suggests relating the Bianchi identity to the homological identity $\partial\partial \equiv 0$. A very stimulating discussion of these far-reaching theorems of differential topology are to be found in the standard reference book on gravitation (MTW, Chap. 15).

2.7 Gauge Transformations

Einstein's theory of general relativity is founded firmly on the following basic principle as far as its axiomatic argument is concerned: "Natural laws are to be expressed by equations that are covariant under the group of continuous coordinate transformations" (EINSTEIN 1949, p. 69).

It is not by chance that we have formulated our approach to gauge theories in terms of differential forms: these transform themselves covariantly with respect to the group $\mathscr{D}$(M) of coordinate transformation. Additionally, it has been suggested by the empirical occurrence of internal symmetries that use should be made of the principal fiber bundle P(M, G, π, δ) as an "enlarged geometric arena." Thus it is to be expected that additional transformations of P play a part in gauge theories similar to that of the group $\mathscr{D}$(M) of diffeomorphisms in GR. In particular, those diffeomorphisms

$$G(p) : P \to P, \quad G(p) \in \mathscr{G}_p \tag{2.7.1}$$

of a principal fiber bundle P(M, G, π, δ) should be taken into consideration, which are subject to the following conditions (ATIYAH 1978):

(i) G(p) is equivariant, i.e.,

$$G(g\,p) = g\,G(p), \quad g \in G, \quad p \in P; \tag{2.7.2}$$

(ii) G(p) preserves each fiber $F_m = \overset{-1}{\pi}(m)$, i.e., acts trivially on the base space,

$$\pi \circ G = \pi. \tag{2.7.3}$$

These diffeomorphisms generate *inner automorphisms* of P. In terms of composition, they constitute an infinite-dimensional group, the group $\mathscr{G}_P$ of *local gauge transformations* (which are of the so-called second kind). This group can be identified with the group of smooth cross sections of the product bundle of P and G under the adjoint action of G (Bourguignon & Lawson 1981):

$$\mathscr{G}_p \approx C^\infty(P \times_{\mathrm{Ad}} G). \tag{2.7.4}$$

The exponential mapping exp : $\mathfrak{g} \to$ G from the Lie algebra $\mathfrak{g}$ into the structure group G of P induces a corresponding mapping within $P \times_{\mathrm{Ad}}$ G. As a result, each element of the group of gauge transformations can be expressed locally in the following form:

$$G(p) = \exp i\,\theta^k(m)L_k \;\in\; \mathscr{G}_p. \tag{2.7.5}$$

Here $\theta^k(m) \in C^\infty(M)$ denote real functions on the base space. Gauge transformations within an associate vector bundle $\mathrm{V}^\rho(\mathrm{M}, \mathbb{C}^\mathrm{N}, \mathrm{GL}(\mathrm{N}, \mathbb{C}), \mathrm{P})$, i.e., elements of $\mathscr{G}_\mathrm{V}$, are represented by the same expression to the extent that L^k is to be considered an infinitesimal operator with respect to a representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{N}, \mathbb{C})$.

For simplicity's sake, let us consider a physical system that is determined by a 0-form, i.e., a scalar of representation type ρ:

$$\phi = \varphi \otimes b = {}^{G^{-1}}\varphi \otimes {}^{G}b \qquad \in\; C^\infty(V^\rho). \tag{2.7.6}$$

An equivalent local description of the same system will be obtained if the physical system, represented by the so-called bundle coordinates φ, is subjected to the *active* gauge transformation

$$\varphi \to {}^{G^{-1}}\varphi := G^{-1}\varphi \qquad \in\; C^\infty(M, \mathbb{C}),\, G \in \mathscr{G}_V, \tag{2.7.7}$$

whereas the local basis of the sections suffers from a *passive* transformation:

$$b \to {}^{G}b := Gb \qquad \in\; C^\infty(V^\rho). \tag{2.7.8}$$

It has to be emphasized that the bundle coordinates $\overline{\varphi}$ of the vector bundle constructed on the conjugate ("Dirac adjoint") representation $\overline{\rho}$ transforms according to

$$\overline{\varphi} \to {}^{G^{-1}}\overline{\varphi} := \overline{\varphi}G \qquad \in\; C^\infty(V^{\overline{s}}). \tag{2.7.9}$$

As has already been indicated in (2.7.6), the total effect of these transformations is physically unobservable. Thus the bundle theory provides us automatically with a sensible instruction for a "recalibration" or gauging of the matter fields. This rule is completely equivalent to the formalism developed by Utiyama (1956, 1980).

2.8 Topological Invariants

Solutions of source-free gauge field equations do not have a merely local meaning. Some of them may even have a global extension to the whole base space. For a classification of the configuration spaces of such global solutions, the mathematics of the fiber bundles is mandatory and not to be considered as only instrumental. The characterization will be achieved via invariant polynomials of the curvature of the bundles in question.

Let V(M, $\mathbb{C}^N$, GL(N, $\mathbb{C}$), P) be a complex vector bundle that is associated with the "geometric arena" P(M, G, π, δ) of the particular Yang–Mills-type model. If we consider the determinant of the curvature of the former bundle, we obtain

$$\det\left(1+\frac{i}{2\pi}\Omega\right)=\overset{*}{\pi}(1+\gamma_1+\cdots+\gamma_m) \tag{2.8.1}$$

as an invariant polynomial. As can be shown, it is decomposable into gauge-invariant 2k-forms γ_k on P whose inverse images after the projection on the base space M read as follows:

$$\overset{*}{\pi}(\gamma_k)=\frac{(-1)^k}{(2\pi i)^k k!}\mathrm{Tr}(\Omega\wedge\cdots\wedge\Omega). \tag{2.8.2}$$

The number of curvature 2-forms Ω in the exterior product is k $\leq n/2$. From the Bianchi identity (2.6.12) and its projection onto a 2k-form on M (KN II, Chap. XII), it follows that the γ_k are *closed* exterior forms, i.e.,

$$\begin{aligned}\overset{*}{\pi}(d\gamma_k)&=d\overset{*}{\pi}(\gamma_k)=D\overset{*}{\pi}(\gamma_k)\\&=\frac{(-1)^k}{(2\pi i)^k(k-1)!}\mathrm{Tr}(D\Omega\wedge\Omega\wedge\cdots\wedge\Omega)=0.\end{aligned} \tag{2.8.3}$$

In that case, they are known to determine so-called cohomology classes. The structure of the latter is not determined by the particular choice of the connection ω, but depends solely on the bundle structure of P(M, G, π, δ) or its associated vector bundle V. In other words, the 2k-forms correspond to the *characteristic classes* of V. More precisely (see: KN II, Theorem 3.1), it can be stated that the abstractly defined k^{th} Chern class c_k(V) of a complex vector bundle V is represented by the closed 2k-form γ_k, as given above.

In order to obtain the characteristic classes of a real vector bundle $V^{\mathbb{R}}$ over M with the typical fiber $\mathbb{R}^N$, it will be enlarged to a complex vector bundle V with the typical fiber $\mathbb{C}^N$. The latter arises from the complexification of each fiber of $V^{\mathbb{R}}$. Then the so-called k^{th} Pontryagin class of $V^{\mathbb{R}}$ is defined by

$$p_k(V^{\mathbb{R}})=(-1)^k c_{2k}(V). \tag{2.8.4}$$

In the physically important case of a four-dimensional compact "spacetime" of the Euclidean signature s = 0, we find locally, cf. DANIEL & VIALLET (1980),

$$\overset{*}{\pi}(\gamma_1) = -\frac{1}{2\pi i}\mathrm{Tr}\Omega \quad (= 0) \tag{2.8.5}$$

$$\overset{*}{\pi}(\gamma_2) = -\frac{1}{8\pi^2}\{\mathrm{Tr}(\Omega \wedge \Omega) - \mathrm{Tr}(\Omega) \wedge \mathrm{Tr}(\Omega)\} \tag{2.8.6}$$

$$\left(= -\frac{1}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}{}^{j}F_{\mu\nu j}\sqrt{|g|}d^4x\right)$$

$$\overset{*}{\pi}(\gamma_k) = 0 \quad \text{for} \quad k \geq 3. \tag{2.8.7}$$

It is typical for the principal fiber bundles $P(S^4, SU(f), \pi, \delta)$, which are to be found in Yang–Mills theories, that they have special Lie groups as structure groups, i.e., those for which det G = 1 holds. Inasmuch as the trace of the Lie-algebra-valued curvature 2-form Ω vanishes in such instances, the gauge-invariant 4-form $\mathrm{Tr}(\Omega \wedge \Omega)$ suffices for a complete characterization of the configuration space. Any consideration of the general dynamics of the Yang–Mills gauge fields should therefore incorporate this form into the Lagrangian formalism.

The integration of the second Chern class $\overset{*}{\pi}(\gamma_2)$ over the base space yields a characteristic number,[5] which due to (2.8.1) is termed the Chern index:

$$c_2(M) = -\int_M \overset{*}{\pi}(\gamma_2) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega \wedge \Omega). \tag{2.8.8}$$

In the literature of physics, this topological invariant is often called the "Pontryagin index"; see JACKIW (1977). Concerning bundles with structure group G = SU(f), however, it is mathematically more precise to call it the Chern index, in order to preserve the term Pontryagin index for the classification of real associated vector bundles (MAYER & DRECHSLER 1977). Nevertheless, the denotation of BELAVIN et al. (1975) is correct, since in their paper, the isospin group SU(2) has been enlarged to the real structure group $\widetilde{SO}(4) \approx SU(2) \otimes SU(2)$, due to a special isomorphism (HELGASON 1962).

The actual calculation of this index is firmly based on the fact that the projection of the closed form γ_2 onto M is even an exact one. This is a property that it shares with all exterior forms representing characteristic classes (CHERN & WHITE 1976). In this particular case, the relation

$$\mathrm{Tr}(\Omega \wedge \Omega) = d\,\mathrm{Tr}\left(\omega \wedge \Omega + \frac{1}{3}\omega \wedge \omega \wedge \omega\right) \tag{2.8.9}$$

[5] Excepting the case of the meron solutions, this will usually be an integer; see DE ALFARO et al. (1979)

holds. As for the proof, it has to be remembered first that the exterior derivative d commutes with the linear operator Tr of forming the trace. Then, the right-hand side of (2.8.9) yields the following chain of equations:

$$\begin{aligned}\mathrm{Tr}\left\{d\left(\omega\wedge d\omega-\frac{2}{3}\omega\wedge\omega\wedge\omega\right)\right\} &= \mathrm{Tr}\{d\omega\wedge d\omega-\omega\wedge dd\omega\\ &-\frac{2}{3}d\omega\wedge\omega\wedge\omega+\frac{2}{3}\omega\wedge d\omega\wedge\omega-\frac{2}{3}\omega\wedge\omega\wedge d\omega\}\\ &=\mathrm{Tr}\{d\omega\wedge d\omega-d\omega\wedge\omega\wedge\omega-\omega\wedge\omega\wedge d\omega\}\\ &=\mathrm{Tr}\{(d\omega-\omega\wedge\omega)\wedge(d\omega-\omega\wedge\omega)\}=\mathrm{Tr}(\Omega\wedge\Omega).\end{aligned} \tag{2.8.10}$$

(Since they are under trace, these forms can be treated as ordinary exterior forms, although otherwise, they are to be considered as Lie-algebra-valued differential forms.) It is especially the last step that makes use of the identity

$$\mathrm{Tr}\{\omega\wedge(\omega\wedge\omega\wedge\omega)\}=-\mathrm{Tr}\{(\omega\wedge\omega\wedge\omega)\wedge\omega\}=0. \tag{2.8.11}$$

Thus by the application of Stokes's theorem, the Chern index can be determined by the following integral over the boundary of M:

$$\begin{aligned}c_2(M) &= \frac{1}{8\pi^2}\int_M d\mathrm{Tr}\left(\omega\wedge\Omega+\frac{1}{3}\omega\wedge\omega\wedge\omega\right)\\ &=\frac{1}{8\pi^2}\int_{\partial M}\mathrm{Tr}\left(\omega\wedge d\omega-\frac{2}{3}\omega\wedge\omega\wedge\omega\right).\end{aligned} \tag{2.8.12}$$

For a further evaluation of this latter term, more information concerning the asymptotic behavior of the gauge fields is needed. If the dynamics of a Yang–Mills gauge theory is determined by a Lagrangian 4-form, it can be shown by an analysis of the action integral

$$S_{\mathrm{YM}}=\int_M L_{\mathrm{YM}} \tag{2.8.13}$$

that the field strengths $\mathrm{F}_{\alpha\beta}{}^{\mathrm{j}}$ (components of the curvature form) have to vanish faster than $|\mathrm{x}|^{-2}$ at infinity, since otherwise, the integral (2.8.13) would not exist. In order to guarantee its finiteness, it is sufficient to postulate that the solutions behave asymptotically as "pure" ("fake" according to UTIYAMA 1980) gauge fields:

$$\overset{\infty}{\omega}:=-G^{-1}dG, \tag{2.8.14}$$

i.e., that the relation

$$\omega\sim\overset{\infty}{\omega} \tag{2.8.15}$$

should hold for $|x| \to \infty$. Here $\overset{\infty}{\omega}$ denotes a Maurer–Cartan connection, i.e., a left-invariant $\mathfrak{g}$-valued 1-form that satisfies the equation

$$\begin{aligned} &d\overset{\infty}{\omega} - \overset{\infty}{\omega} \wedge \overset{\infty}{\omega} \\ &= -d(G^{-1}) \wedge dG - G^{-1}ddG - (G^{-1}dG) \wedge (G^{-1}dG) \\ &= -d(G^{-1}) \wedge dG + G^{-1}Gd(G^{-1}) \wedge dG = 0 \end{aligned} \tag{2.8.16}$$

(see KN I, p. 41; however, with opposite sign conventions). Consequently, the curvature, derived from a pure gauge field, has to vanish:

$$\Omega(\overset{\infty}{\omega}) = 0. \tag{2.8.17}$$

If the boundary ∂M is chosen in such a way that only a pure gauge connection $\overset{\infty}{\omega}$ exists there, we get from (2.8.12) the relation

$$c_2(M) = \frac{1}{24\pi^2} \int_{\partial M} \mathrm{Tr}(\overset{\infty}{\omega} \wedge \overset{\infty}{\omega} \wedge \overset{\infty}{\omega}), \tag{2.8.18}$$

in full compliance with (2.8.15); cf. Jackiw (1980). If G = SU(2) serves as a structure group, the 3-dimensional integral reduces itself to the (invariant) Haar integral over the 3-dimensional Lie group SU(2), regarded as a manifold. If ∂M is taken to be S^3 topologically, then it can be shown that $c_2(M)$ determines the "mapping degree" or the *winding number*[6] of the mapping of S^3 in $S^3 \approx SU(2)$. These considerations explain why in the physics literature, the Chern index is regarded as the "quantum number" of the *topological charge*. It is of considerable importance for the classification of the so-called instanton solutions. Before focusing on these configurations, the works of Atiyah & Jones (1978), Atiyah (1979), which deal with further aspects of such global solutions, should be mentioned for those readers who prefer a more mathematically oriented approach.

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[6]Speaking more generally, the n^{th}-homotopy group of the n-dimensional sphere S^n is given by the group $\mathbb{Z}$ of integers, i.e., by $\pi_n(S^n) = \mathbb{Z}$. As such, it determines the winding number of the mapping $S^n \to S^n$.

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Chapter 3
Maxwell and Yang–Mills Theory

3.1 The Lagrangian Formalism

The Lagrangian formalism has to be seen as the point of departure for classical field theories. This approach is also providing the scaffold for (canonical) quantum field theories. Instead of using a local notation, our representation of the formalism will be based upon differential forms which are globally defined on a pseudo-Riemannian manifold of dimension n.

Accordingly, the dynamics of a non-interacting material system is determined by the Lagrangian n-form

$$L = L(\varphi, d\varphi)\eta \tag{3.1.1}$$

which is supposed to depend on the fields φ and their first derivatives only. Such formalism of the "first order" guarantees that the field equations obtained by Hamilton's *principle of least action*

$$\delta \int_M L = 0 \tag{3.1.2}$$

being subjected to the subsidiary condition

$$\delta\varphi = 0 \qquad \text{at the boundary } \partial\text{M of M} \tag{3.1.3}$$

are partial differential equations of at most second order.

The operator $\delta\varphi$ of *small variations* $\varphi + \delta\varphi$ of the fields φ is satisfying the Leibniz rule, similar as a partial derivative, and commutes with the exterior derivative:

$$\delta(d\varphi) := d(\varphi + \delta\varphi) - d\varphi = d\delta\varphi \tag{3.1.4}$$

Let

$$\pi := \frac{\partial L}{\partial(\partial_\alpha \varphi)} \vartheta^\alpha \tag{3.1.5}$$

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be the $(n-1)$-form of the *field momenta* which are *canonically conjugated* to $d\varphi$. It follows that the variation principle yields

$$\begin{aligned}\delta L &= \delta\varphi \wedge {}^*L_\varphi + \delta(d\varphi) \wedge \pi \\ &= \delta\varphi \wedge {}^*L_\varphi + d(\delta\varphi) \wedge \pi \\ &= \delta\varphi \wedge ({}^*L_\varphi + d\pi) + d(\delta\varphi \wedge \pi),\end{aligned} \tag{3.1.6}$$

after δ has been interchanged with the integration. The last term is an exact form. Taking advantage of Stoke's theorem, this contribution can be converted into an integral over the boundary which vanishes due to the subsidiary condition (3.1.3) for the variation:

$$\int_M d(\delta\varphi \wedge \pi) = \int_{\partial M} \delta\varphi \wedge \pi. \tag{3.1.7}$$

Consequently, this term will not contribute to the equations of motion. From the principle of variation we then get the *Euler–Lagrange equations*.

$$\lambda^\varphi := (-1)^s L_\varphi + {}^*d\pi = 0, \tag{3.1.9}$$

where s is the signature of the pseudo-Riemannian manifold.

In local notation, these partial differential equations read:

$$\frac{\partial L}{\partial \varphi} - \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha \varphi)} = 0. \tag{3.1.10}$$

In order to complete the field-theoretical description let us turn to the conservation laws, too. They can be derived explicitly within the limits of our formalism by considering the so-called *Hamiltonian complex*:

$${}^*H := L - d\varphi \wedge \pi \tag{3.1.11}$$

(compare: e.g. Rund & Lovelock 1972).

Its exterior derivative reads

$$d^*H = dL + d\varphi \wedge {}^*L_\varphi + d\varphi \wedge d\pi - d\,d\varphi \wedge \pi \tag{3.1.12}$$

where the last term vanishes identically, due to $dd \equiv 0$. Following the physically sensible assumption that the Lagrangian n-form is translation invariant and consequently does not depend upon the coordinates x^i explicitly, we get the well-known conservation law

$$d\,{}^*H \simeq 0 \tag{3.1.13}$$

of the system's total energy-momentum, provided that the Euler–Lagrange equations (3.1.9) hold. The transcription of the Hamilton-Jacobi formalism of classical field theory into the calculus of differential forms is mathematically equivalent to the introduction of a symplectic structure on the manifold (MTW, p. 125; see also SIMMS & WOODHOUSE 1976).

The formalism of first order suffices completely, since it is possible to describe fields with arbitrary spin by introducing the Harish-Chandra equation (see: WIGHTMAN 1973)

$$\beta \wedge^{*} D\psi = m\psi\eta. \tag{3.1.14}$$

This equation comprises not only the Dirac equation for the description of fundamental fermions but as well the field equation, e.g., for a charged scalar field φ. To see this, it is only necessary to convert the Klein–Gordon equation

$$\Box\varphi + m^2\varphi = 0 \tag{3.1.15}$$

via the substitution $\varphi \to \psi := (\varphi, \Theta^{(1)} := d\varphi)$ into a first order system of the Petiau–Duffin–Kemmer-type. Additionally, attention can be drawn to the fact that the Lagrangian n-form, corresponding to (3.1.14), in the interaction-free case reduces to:

$$\overset{=}{L}_{\text{mat}} = \overset{=}{L}_{\text{mat}}(\overline{\psi}\psi, \overline{\psi}\beta \cdot d\psi)\eta = \overline{\psi}\beta \wedge {}^{*}d\psi - m\overline{\psi}\psi\eta. \tag{3.1.16}$$

3.2 G-Equivalence Principle

As has been shown by the preceding considerations, the Lagrangian formalism of the first order may well serve as a basis for the investigation of the physical implications of the gauge transformations without impairing the generality. With WEYL (1929a), p. 331 we postulate that the fundamental laws of physics have to satisfy as well the principles of *"generalized" relativity*: The Lagrangian n-form or the partial differential equations for the fields should not only be invariant or covariant with respect to the group $\mathscr{D}(M)$ of general coordinate-transformations of the spacetime labels, but as well with respect to the "internal covariance group" $\mathscr{G}$ of gauge transformations. Since the first postulate corresponds to Einstein's equivalence principle (see THORNE et al. 1973 for a detailed discussion), the required gauge invariance has been termed *G-equivalence* principle (HENNIG & NITSCH 1981). Both principles together require a "generalized covariance" of the laws of nature.[1]

The invariance of L_{mat} with respect to $\mathscr{D}$(M) is already guaranteed by the use of differential forms. Therefore, it is sufficient to turn to the question of the gauge invariance of a Lagrangian n-form: for it is a consequence of the transformation

[1] MACK (1981) interpreted this postulate physically as a result of a "Naheinformationsprinzip", regarding it as a generalization of the principle of action-at-close-distances.

properties of φ and $\overline{\varphi}$ with respect to gauge transformations that L_{mat} may solely depend upon $\overline{\varphi}\mathscr{O}\varphi$, where $\mathscr{O}$ denotes a gauge-covariant operator

$$\mathscr{O} \to {}^{G^{-1}}\mathscr{O} = G^{-1}\mathscr{O}G. \tag{3.2.1}$$

For exemplification, let β be such an operator. Due to the occurrence of the exterior derivative, it is then mandatory to supplement the Lagrangian n-form

$$L_{\text{mat}} = L(\overline{\varphi}\varphi, \overline{\varphi}\beta \cdot (d + \omega')\varphi)\eta = L(\overline{\varphi}\varphi, \overline{\varphi}\beta \cdot D'\varphi)\eta \tag{3.2.2}$$

by means of a 1-form ω' in such a way that the term GdG^{-1} resulting from gauge transformations can be compensated. This, indeed, is only to be achieved, if the added 1-form is transforming itself *inhomogeneously*

$$\omega' \to {}^{G^{-1}}\omega' = G^{-1}\omega' G + (dG^{-1})G \tag{3.2.3}$$

concerning the "actively" applied gauge transformation $\mathrm{G} \in \mathscr{G}_{\mathrm{V}}$. For the derivation of (3.2.3) is cogent, if and only if D' is as well satisfying the generalized Leibniz rule as the covariant derivative D.

Here the question arises, whether or not the additional form ω' can really be identified with a connection in an associated vector bundle V^{ρ}. To verify this, the matrix ${}^{\mathrm{G}}\omega$ of the connection relative to a basis ${}^{\mathrm{G}}\mathrm{b}$, which is locally "rotated" by the action of the group element $\mathrm{G} \in \mathscr{G}_{\mathrm{V}}$, is taken into consideration. Henceforth the definition of of the covariant derivative implies

$$D({}^{G}b) = {}^{G}\omega \otimes {}^{G}b. \tag{3.2.4}$$

On the other hand the application of the Leipniz rule yields

$$\begin{aligned} D({}^{G}b) &= dG \otimes b + G\omega \otimes b \\ &= \left\{dGG^{-1} + G\omega G^{-1}\right\} \otimes {}^{G}b. \end{aligned} \tag{3.2.5}$$

Compared with (3.2.4), it shows that the 1-form of the connection transforms inhomogenously with respect to "passive" gauge transformations

$$\omega \to {}^{G}\omega = G\omega G^{-1} + dG\, G^{-1}. \tag{3.2.6}$$

If G ist substituted by G^{-1} exactly the relation (3.2.3) comes into existence, which is to be expected, due to the invariance of the Lagrangian n-form with respect to "active" gauge transformations. Consequently, it can be concluded that the term ω' in the Lagrangian n-form is (gauge-)equivalent to the connection ω in V^{ρ}.

Thus it follows naturally that the "*compensation field*" can be taken to be identical with the connection 1-form ω in the following. Furthermore, it has to be pointed out that a corresponding transformation formula concerning the transition functions, which specify a local change of the cross-section, holds as well for the inverse image or pull-back $\overset{*}{\sigma}_\alpha \omega$ due to the defining properties of a connection (see: KN I, p. 66; DANIEL & VIALLET 1980). In the present study, however, we prefer a *global* formulation of gauge transformations relative to the bundle P, and we are thus in concordance with ATIYAH et al. (1978).

A further important consequence of these considerations is that the covariant exterior derivative D transforms "(gauge-)covariantly" with respect to the following automorphisms of the vector bundle:

$$D \to {}^{G^{-1}}D = G^{-1} D\, G. \tag{3.2.7}$$

This is also the case for the curvature 2-form:

$$\Omega \to {}^{G^{-1}}\Omega = G^{-1} \Omega G. \tag{3.2.8}$$

The transformed connection ${}^{G^{-1}}\omega$ deviates from the original one solely by a gauge-covariant 1-form, i.e., more precisely by

$$\begin{aligned} {}^{\Delta}\omega := {}^{G^{-1}}\omega - \omega &= G^{-1}[\omega, G] - G^{-1} d\, G \\ &= -G^{-1} D G = -\ {}^{G^{-1}}D. \end{aligned} \tag{3.2.9}$$

For a so-called "copy" $\overline{\omega}$ of a connection ω, this expression is to be taken as different from zero, although - according to the definition - its curvature $\overline{\Omega} = \mathrm{G}^{-1}\Omega\mathrm{G}$ is gauge-covariantly related to Ω as in (3.2.8) (see: DESER & DRECHSLER 1979). The transversal or Coulomb condition $d\,{}^*\omega \simeq 0$ does not fix uniquely (large) gauge transformations. In nonabelian gauge theories, Gribov copies are unavoidable (SINGER 1978) and topologically non-trivial. However, a Lorentz-invariant BRST quantization may be free (SLAVNOV 2009) of this Gribov ambiguity.

3.3 Maxwell's Theory in Differential Forms

In the standard framework of a U(1) gauge theory, as well as in the original publications of Maxwell, the electromagnetic four-potential $A = A_i dx^i$, a one-form, is the fundamental variable, whereas Faraday's field strength F is a derived concept and defined by the two-form

$$F := dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j\ . \tag{3.3.1}$$

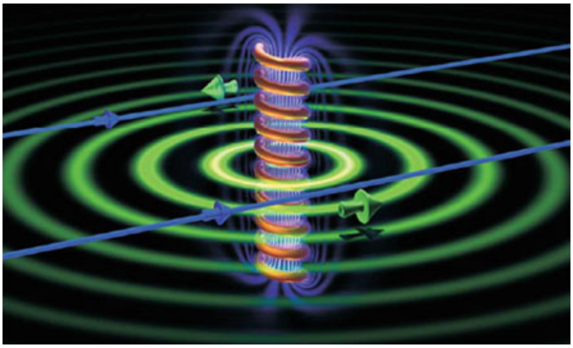

Fig. 3.1 Vector potential **A** outside of a solenoid. The electron beam for the interference is indicated by *blue arrows*

On the other hand, $dd\Phi \equiv 0$ for any exterior p-form Φ, due to the Poincaré lemma. Thus if $dF \equiv 0$, there exists locally a one-form A such that $F = dA$. However, the potential A is not uniquely determined. The field strength F is invariant under the local $U(1)$ *gauge transformation*

$$A \to A + d\theta(x) \quad \Rightarrow \quad F \to dA + dd\theta = dA = F\,. \tag{3.3.2}$$

Although, A is needed for the standard minimal coupling $D \to \mathscr{D} := D + ieA$ to Dirac fields in quantum electrodynamics (QED), there had been a discussion on the physical relevance of the vector potential **A**. Already Franz (1939) suggested that the wave function of a Dirac electron could suffer from the non-integrable phase factor

$$\theta = -\frac{e}{\hbar}\int \mathbf{A}\cdot d\mathbf{l}. \tag{3.3.3}$$

This so-called Aharonov-Bohm effect, cf. Fig. 3.1 has been conclusively confirmed via electron interferometry by Tonomura, cf. Batelaan & Tonomura (2009).

For a Lagrangian formulation, let us depart from the four-form L which consist of the gauge field part and L_{mat} governing the matter field Ψ and its minimal coupling to A:

$$L = L(A, dA) + L_{\text{mat}}(A, dA, \Psi, d\Psi). \tag{3.3.4}$$

When the field equations are at most of second differential order, the Lagrangian can usually be assumed to be of first order in the fields. Stationarity of the action $S = \int L$ leads to the *gauge field equation*

$$\frac{\delta L}{\delta A} := \frac{\partial L}{\partial A} + d\frac{\partial L}{\partial dA} = 0\,, \tag{3.3.5}$$

where the variational derivative of the one-form A is defined in the usual manner. The excitation H is the field momentum conjugated to A, i.e.

$$H := -\frac{\partial L}{\partial dA} = -\frac{\partial L}{\partial F} \qquad \text{and} \qquad j := \frac{\delta L_{\mathrm{mat}}}{\delta A}\,, \tag{3.3.6}$$

whereas the matter current uses the variational derivative.

Then, in the framework of a relativistic U(1) gauge theory, the field equations take the rather elegant form

$$dF \equiv 0, \qquad dH = j\,. \tag{3.3.7}$$

The homogeneous Maxwell equation $dF \equiv 0$ is a Bianchi type identity as consequence of working with the potential A. Since the Poincaré lemma implies that $ddH \equiv 0$, the field equations imposes 'on shell' that the electric current j is conserved:

$$dj \simeq 0\,. \tag{3.3.8}$$

Without a spacetime metric, the only gauge-invariant Lagrangian permitted in four dimensions is the Pontryagin four-form

$$L_{\mathrm{Pontr}} = -\frac{1}{2} F \wedge F = -\frac{1}{2} dC\,, \tag{3.3.9}$$

where $C := A \wedge F = A \wedge dA$ is the abelian Chern–Simons term, known to violate parity P.

Involving the metric via the Hodge dual, there exists the additional four-form

$$L_{\mathrm{Max}} = -\frac{1}{2} F \wedge {}^*F\,, \tag{3.3.10}$$

i.e., the standard Lagrangian[2] of Maxwell's theory in natural units.

The conversion from the exterior into the vector notation can be obtained by the identification

$$\begin{aligned} F &:= E \wedge dt + B \\ &= (E_x dx + E_y dy + E_z dz) \wedge dt \\ &\quad + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy. \end{aligned} \tag{3.3.11}$$

For the extensive quantities, the excitation two-form includes two parts according to

$$H := -\mathfrak{H} \wedge dt + \mathfrak{D} = \frac{1}{2} H_{ij} dx^i \wedge dx^j\,, \tag{3.3.12}$$

whereas the current three form can be decomposed into

---

[2]More generally, one could imagine the existence of topological modified Lagrangians $\overset{\theta}{L} = -(\cos\theta\, F \wedge {}^*F + \sin\theta\, F \wedge F)/2$, where θ is the'vacuum' angle of duality rotations or higher order functionals of the two quadratic invariants.

$$
\begin{aligned}
j &:= -\mathscr{J} \wedge dt + \hat{\rho} \\
&= -(J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \wedge dt \\
&\quad + \rho dx \wedge dy \wedge dz\,.
\end{aligned} \tag{3.3.13}
$$

They have to be supplemented by a constitutive law which, in vacuum, reads

$$
H = {}^*F\,. \tag{3.3.14}
$$

The Hodge dual $*$ depends on the metric g and on the orientation of the manifold, as can be easily inferred from its tensor version:

$$
H_{mn} = \frac{\sqrt{|g|}}{2}\,\varepsilon_{mnkl}\, g^{ki} g^{lj} F_{ij}\,. \tag{3.3.15}
$$

Then the homogeneous Maxwell equation implies the two vector equations

$$
dF \equiv 0 \qquad \begin{cases} \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} \equiv 0 \\ \nabla \cdot \mathbf{B} \equiv 0\,. \end{cases} \tag{3.3.16}
$$

Likewise, the inhomogeneous Maxwell equation[3] incorporates the familiar vector equations

$$
dH = j \quad \begin{cases} \nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{c^2 \partial t} = \mu \mathbf{J} \\ \nabla \cdot \mathbf{E} = \rho/\varepsilon\,. \end{cases} \tag{3.3.17}
$$

In components, Maxwell's Lagrangian four-form can be expressed as

$$
L_{\text{Max}} := -\frac{1}{2} F \wedge^* F = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\eta\,, \tag{3.3.18}
$$

where η is the standard volume four-form. The Pontryagin four-form can be written in components as

$$
L_{\text{Pontr}} := -\frac{1}{2} F \wedge F = \mathbf{E} \cdot \mathbf{B}\eta\,. \tag{3.3.19}
$$

The *canonical* energy-momentum three-form is given by

$$
\begin{aligned}
\Sigma_\alpha &:= e_\alpha \rfloor L - (e_\alpha \rfloor dA) \wedge \frac{\partial L}{\partial (dA)} \\
&= \frac{1}{2}[(e_\alpha \rfloor F) \wedge H - F \wedge (e_\alpha \rfloor H)]
\end{aligned} \tag{3.3.20}
$$

[3] The displacement current $\partial \mathfrak{D}/\partial t$, where $\mathfrak{D} = \varepsilon \mathbf{E}$, was anticipated 1839 by James Mac Cullagh, cf. Darrigol (2010). It later on turned out to be a necessary ingredient for rendering electromagnetism relativistic-invariant.

and turns out to be symmetric, i.e., $\vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0$. For non-vanishing charge current j, a three-form, we obtain from (3.3.7) the differential form version

$$d\Sigma_\alpha \simeq (e_\alpha \rfloor F) \wedge j \tag{3.3.21}$$

of the *Lorentz force*, cf. HEHL et al. (1991).

3.3.1 Constrained BF Scheme for Maxwell Fields

A rather new development in topological field theory is the *BF* (*background field*) formalism, which potentially provides interesting relations to higher-dimensional knots; cf. MIELKE (1977a), CATTANEO & ROSSI (2005).

As an instructive example of such a constraint formalism, let us consider here the abelian case, where the U(1) connection one-form $A = A_i dx^i$ and an auxiliary[4] two-form $B = \frac{1}{2} B_{ij} dx^i \wedge dx^j$ are varied *independently*, slightly reminiscent of the Schwinger formalism (SCHWINGER 1962). In its minimal version, it starts from the Lagrangian four-form

$$L_{\rm BF} = -B \wedge F = -B \wedge dA\,. \tag{3.3.22}$$

Independent variations with respect to A and B lead to $dB \cong 0$ and to the *constraint* of vanishing field strength $F := dA \cong 0$. This implies, however, that such a primordial model has *no local* degrees of freedom.

An additional term quadratic in B leads to

$$\begin{aligned}\tilde{L}_{\rm BF} &:= -B \wedge dA + \frac{1}{2} B \wedge B \\ &\cong -B \wedge dA + dC\,.\end{aligned} \tag{3.3.23}$$

Now, independent variations provide a definition of the field strength F together with the corresponding *Bianchi identity*

$$B \cong dA\,, \qquad dB \cong dF \equiv 0\,, \tag{3.3.24}$$

respectively, in compliance with the Poincaré lemma $dd \equiv 0$. It still defines a topological theory (HOROWITZ 1989), since the "on shell" Lagrangian (3.3.23) differs from (3.3.22) only by a boundary term dC derived from a *Chern–Simons* (CS) three-form

$$C = \frac{1}{2} A \wedge F, \qquad dC = \frac{1}{2} F \wedge F\,, \tag{3.3.25}$$

[4]In four dimensions, B resembles the two-form potential for the gauge-invariant field strength or excitation $H = dB$, the Kalb–Ramond *axion* three-form.

the latter akin to the Pontryagin invariant. As is well known, Bianchi-type identities can be recovered via the variation of the Pontryagin term, e.g., $\delta dC/\delta A = dF \equiv 0$ in the abelian case.

Like Maxwell's original theory, this scheme is invariant under the usual local $\mathrm{U}(1)$ gauge transformations

$$A \to A' = A + d\theta(x). \tag{3.3.26}$$

In addition, the field equations (3.3.24) are invariant under the symmetry

$$A \to \widetilde{A} = A + \psi, \qquad B \to \widetilde{B} = B + d\psi\,, \tag{3.3.27}$$

where $\psi = \psi_i dx^i$ is a one-form or a covector; cf. LUCCHESI et al. (1993). In a BRST quantization, due to the zero mode $s\psi = D\phi$ with $s\phi = 0$, the "topological" symmetry (3.3.27) allows for a nilpotent ghost for ghost structure; cf. CATTANEO et al. (1998) for the Yang–Mills case. Thus, one of the most clear-cut approaches to topological field theory is the *BF* formalism which, in the case of gravity, was anticipated by PLEBAŃSKI already in 1977.

Since *Bianchi identities* do not allow for nontrivial couplings to matter, in *realistic physical models* such as Maxwell's theory and QCD, the constraint formalism has to depart from

$$L_{\text{Max}} = -B \wedge dA + \frac{1}{2} B \wedge {}^*B + L_{\text{matter}}, \tag{3.3.28}$$

where the Lagrangian necessarily involves the *Hodge dual* * depending, however, on the determinant of the metric. Then independent variations of (3.3.28) again provide the definition of the field strength (3.3.24), but as a bonus, from the relation $F = {}^*B$ arises a nontrivial physical field equation

$$-\,dB = d\,{}^*F \cong j, \tag{3.3.29}$$

where $j := \delta L_{\text{matter}}/\delta A$ is the matter current.

However, it should not be overlooked that in a coupling to matter, such a *BF* scheme would leave the minimal coupling prescription, since it generates the current three-form

$$\begin{aligned} j := \frac{\delta L_{\text{matter}}}{\delta A} &= \frac{\partial L_{\text{matter}}}{\partial A} + d\frac{\partial L_{\text{matter}}}{\partial dA} \\ &= \Psi \wedge \frac{\partial L_{\text{matter}}}{\partial D\Psi} + D\frac{\partial L_{\text{matter}}}{\partial B}, \end{aligned} \tag{3.3.30}$$

which "on shell," is conserved classically, i.e., $dj \cong 0$. In general, this includes *Pauli-type terms* generated by the variation of the Lagrangian with respect to dA; cf. (5.2.18) of HEHL et al. (1995). Due to (3.3.24), this additional term is equivalent to that generated by the variation with respect to B, as indicated in (3.3.30).

The quadratic term in (3.3.28) can be generalized to a four-form potential $V(B)$. Then the field equation $dB + j \cong 0$ together with the relation $F \cong \partial V/\partial B$ for the Faraday field strength emerges. In order to invert the latter with respect to B, a nondegenerate Hessian is mandatory. In three dimensions, though, nonabelian BF systems with a cubic term $-B \wedge F + B \wedge B \wedge B/3$ are directly related to Chern–Simons theories departing from the noninvariant three-form $C := A \wedge F - A \wedge A \wedge A/3$; cf. BRODA (2005), BOROWIEC & FRANCAVIGLIA (2005). The resulting field equations $F = -B \wedge B$ together with the Bianchi identity $DB = DF \equiv 0$ correspond, in 3D gravity (MIELKE & MAGGIOLO 2007), to those with an induced cosmological term.

3.4 Yang–Mills Fields

The "Generalized covariance" of the Lagrangian n-form of the matter system necessitates the introduction of a connection 1-form ω (GELL- MANN's principle of "minimal coupling" 1956). With WEYL (1929b), it thus can be ascertained that *gauge fields are a necessary accompaniment to the matter-wave field.*

Speaking a priori, the connection ω and the corresponding gauge potentials $\Gamma_\alpha{}^{\mathrm{j}}(\mathrm{m})$ in its local expansion, respectively, are completely arbitrary. In order to establish a coupling to the matter-wave fields and to make physical interactions possible, it is necessary for ω to acquire a dynamical status. Consequently, an additional Lagrangian n-form is to be considered that is dependent on the connection and again only of its first derivative $d\omega$. The only *gauge-covariant* object of this kind that meets the given postulates is exactly the 2-form Ω of the curvature, according to (3.2.8).

As far as physics is concerned, the restriction to the *four*-dimensional spacetime manifold as a base space is a sensible one. From the exterior calculus, we know that the only scalars that are invariant with reference to $\mathscr{D}(\mathrm{M})$ and $\mathscr{G}_{\mathrm{V}}$, which can be constructed from the curvature 2-form, are

$$\mathscr{F} := \frac{1}{\alpha_g}\mathrm{Tr}\left\{{}^*(\Omega \wedge {}^*\Omega)\right\} = \frac{1}{4\alpha_g} F_{\alpha\beta}{}^j\, F^{\alpha\beta}{}_j \tag{3.4.1}$$

and

$${}^*\mathscr{F} := \frac{1}{\alpha_g}\mathrm{Tr}\left\{{}^*(\Omega \wedge \Omega)\right\} = \frac{1}{4\alpha_g} F_{\alpha\beta}{}^j\, {}^*F^{\alpha\beta}{}_j. \tag{3.4.2}$$

An additional Hodge star operator is implemented in order to convert the obtained 4-forms into 0-forms, i.e., scalar functions. In quantized gauge theories, $\mathscr{F}$ and ${}^*\mathscr{F}$ are known to be even and odd with respect CP transformations (JACKIW 1980). This result is unique in four dimensions and has given rise to speculations concerning the dimensionality of the real world (WEYL 1918, 1924; EDDINGTON 1924). As a substitute for the gauge-coupling constant g (= e in the case of electromagnetism), the dimensionless "fine-structure constant"

$$\alpha_g := g^2/\hbar c \tag{3.4.3}$$

was introduced in (3.4.1) and (3.4.2) for the sake of convenience. Then the Lagrangian 4-form for the gauge potentials yields the general structure

$$L_\omega = L(\mathscr{F}, {}^*\mathscr{F})\eta. \tag{3.4.4}$$

The 2-form Π^Ω of the *field momenta* being *canonically conjugated* to Ω are formally constructed via

$$\delta L_\omega =: \delta\Omega \wedge \Pi^\Omega. \tag{3.4.5}$$

Using (3.4.4), the more explicit form

$$\Pi^\Omega = \frac{2}{\alpha_g}\left(\frac{\partial L}{\partial \mathscr{F}}\,{}^*\Omega + \frac{\partial L}{\partial^* \mathscr{F}}\,\Omega\right) \tag{3.4.6}$$

can be derived.

In order to obtain the corresponding Euler–Lagrange equations, it is to be observed that the variation of the curvature with respect to ω commutes with the covariant derivative:

$$\begin{aligned}\delta\Omega &= \delta(d\omega - \omega\wedge\omega) = \delta d\omega - \delta\omega\wedge\omega - \omega\wedge\delta\omega\\ &= d\delta\omega - [\omega, \delta\omega] = D\delta\omega.\end{aligned} \tag{3.4.7}$$

Consequently, the variation of the Lagrangian 4-form with respect to the gauge fields leads to the interim result

$$\begin{aligned}\delta L_\omega &= \delta\Omega\wedge\Pi^\Omega = D(\delta\omega)\wedge\Pi^\Omega\\ &= -\delta\omega\wedge D\Pi^\Omega + d(\delta\omega\wedge\Pi^\Omega).\end{aligned} \tag{3.4.8}$$

Note that the last term is an exact form that need not be accounted for any further in the variational procedure.

A physical system that is to be considered complete has to comprise not only the matter fields $\varphi(\mathrm{m})$ but also the interacting gauge potentials $\Gamma_\alpha{}^{\mathrm{j}}(\mathrm{m})$. If the dynamics of such a system are given by the addition of the separate Lagrangian 4-forms

$$L = L_{\mathrm{mat}} + L_\omega, \tag{3.4.9}$$

it follows that the gauge potentials have to be determined *self-consistently* by the $(n-1)$-form τ of the matter current as a source. The latter is canonically conjugate to ω and therefore to be defined by

$$\delta L_{\mathrm{mat}} =: \delta\omega\wedge\tau. \tag{3.4.10}$$

Then the dynamics of the gauge fields are determined by the (generalized) *Yang–Mills equations* (YANG & MILLS 1954).

$$\boxed{D\Pi^{\Omega} = \tau}. \tag{3.4.11}$$

Due to (3.4.8) and (3.4.10), these can be obtained as Euler–Lagrange equations from the variation $\delta L/\delta\omega$ of the total system. They are complemented by the Bianchi identity, i.e., by

$$D\Omega \equiv 0. \tag{3.4.12}$$

The gauge invariance of the Yang–Mills theory is closely related to the concept of charge, which is of central importance in physics (see SALAM 1980). As is well known, this concept is an *integral* one essentially constructed from a *locally* conserved current.

It is a consequence of the gauge-theoretic formulation of the dynamical system that the matter current τ is covariantly conserved with respect to the gauge group $\mathscr{G}$. This can be verified by differentiating (3.4.11) and then inserting (3.4.6):

$$\begin{aligned} D\tau &= D\,D\Pi^{\Omega} = [\Omega, \Pi^{\Omega}] \\ &= \frac{2}{\alpha_g}\left\{[\Omega, {}^*\Omega]\frac{\partial L}{\partial\mathscr{F}} + (-1)^s[\Omega,\Omega]\frac{\partial L}{\partial^*\mathscr{F}}\right\} = 0. \end{aligned} \tag{3.4.13}$$

The last step of this proof profits from the fact that an arbitrary Lie-algebra-valued 2-form commutes not only with itself but also with its dual 2-form. More generally, it has been shown by HORNDESKI (1978, 1980) that a field theory of Yang–Mills type is gauge-invariant if and only if its source current τ is covariantly conserved. However, (3.4.13) cannot be extended to a global conservation law in the nonabelian case.

On the other hand, the conservation law

$$\operatorname{div}\iota := (-1)^{1+s}\,{}^*d^*\,\iota = 0 \tag{3.4.14}$$

for the *locally*, non-gauge-covariant, *"internal" current*

$$\iota := \tau + (-1)^{n+s}\,{}^*[\omega, \Pi^{\Omega}] \tag{3.4.15}$$

(cf. BERNSTEIN 1974) follows from the expanded Yang–Mills equation

$$d\,\Pi^{\Omega} + \left[\omega, \Pi^{\Omega}\right] = \tau. \tag{3.4.16}$$

This conservation law guarantees that the internal Yang–Mills charges (i.e., isospin I and hypercharge Y in the case of SU(2) as a structure group), which are defined by

$$Q = Q^{(i)} I_i := \int_{H^3} \iota = \int_{\partial H^3} \Pi^{\Omega}, \tag{3.4.17}$$

are independent of both timelike translations and (global) Lorentz transformations. The occurring integrations have to be carried out over a three-dimensional *spacelike* hypersurface H^3 or its boundary ∂H^3, respectively. Note that ι involves not only the matter current τ but also a gauge current arising from the nonlinear coupling of the gauge fields. The *electric* charge is determined by the GELL- MANN–NISHIJIMA relation (GELL- MANN & NEEMAN 1964).

$$Q_e = \frac{1}{2} Y + I_3. \tag{3.4.18}$$

Originally, the YANG–MILLS theory (1954) was intended to describe a gauge model of interactions (initially of strong interactions) with a local isospin symmetry, i.e., it is based on SU(2) as a structure group. As for the specification of the dynamics, the simplest Lagrangian 4-form was chosen that is compatible with the principle of gauge invariance:

$$L_{\mathrm{YM}} = -\frac{1}{\alpha_g} \mathrm{Tr}(\Omega \wedge {}^*\Omega). \tag{3.4.19}$$

This is, according to (3.4.4), the *linear* functional

$$L(\mathscr{F}, {}^*\mathscr{F}) = -\mathscr{F}. \tag{3.4.20}$$

This choice harmonizes with the theory of electromagnetism as first formulated by Faraday and Maxwell in 1861 (see MAXWELL 1892). It is obvious that then the field momenta, being canonically conjugate, are proportional to the curvature 2-form, i.e.,

$$\Pi^{\Omega} = -\frac{2}{\alpha_g} {}^*\Omega. \tag{3.4.21}$$

In more generalized versions, these theories make use of SU(N) as a structure group, and are then applied not only to the description of weak and electromagnetic interactions (TAYLOR 1976), but also to the formalization of strong interactions in the so-called color gauge theories, as well to “grand unified theories” (GUT); see MARCIANO & PAGELS (1978), CHENG & LI (1984).

Relevant aspects of the physical consequences and problems that arise from the introduction of this hypothesis will be discussed later on. In order to justify our more general formalism, it is only to be mentioned here that in contrast to the Maxwell–Lorentz expression (3.4.19), Lagrangian 4-forms have been discussed that lead to an essentially nonlinear type of dynamics. For instance, it was suggested by MILLS (1979) to substitute (3.4.20) by the functional

$$L_{\text{Mills}}(\mathscr{F}, {}^*\mathscr{F}) = \mathscr{F}\left[1 - \left(1 + \frac{1}{b^2}\mathscr{F}^2\right)^{-1}\right] \tag{3.4.22}$$

in order to achieve a confinement of the color gauge fields (i.e., "gluons," which are thought to saturate the postulated quarks) already within the framework of the unquantized theory. This issue is remotely related to the self-energy problem of point charges. In order to solve the latter problem, BORN & INFELD developed in 1934 a modified theory of electromagnetism based on the related nonlinear Lagrangian

$$L_{\text{BI}}(\mathscr{F}, {}^*\mathscr{F}) = \sqrt{1 - 2\mathscr{F} - ({}^*\mathscr{F})^2} - 1. \tag{3.4.23}$$

Within the gauge-theoretic framework, the choice of an appropriate functional $L(\mathscr{F}, {}^*\mathscr{F})$ is of course a mere empirical one and can be answered only by means of a comparison with the physical phenomena. This comparison is, admittedly, itself biased by other theoretical concepts and assumptions.

3.5 Instantons

In Yang–Mills theories, the gauge fields may be regarded only as phenomena of accompaniment of matter. Nevertheless, vacuum solutions of the Yang–Mills equations have positively a meaning of their own. Within the classical theory, they can be regarded as junction solutions, which instead of the self-consistently generated gauge field, are valid outside of the localization of the matter-wave field. A special role is played by those solutions that satisfy the (generalized) *duality ansatz*

$$\Pi^{\Omega} = \zeta\,\Omega. \tag{3.5.1}$$

We are referring to that special occurrence of general duality rotations; see RAINICH (1925), which transforms the field equations (3.4.11) exactly into the Bianchi identity (3.4.12). This point of departure reduces the field equations (3.4.11), which are of second order with respect to the connection ω, to differential equations of first order.

In order to obtain explicit solutions for the Yang–Mills theory proper, BELAVIN et al. (1975) proceeded as follows. They enlarged the structure group $\mathrm{G} = \mathrm{SU}(2)$ with the infinitesimal generators $\mathrm{I}^{\mathrm{j}} = \mathrm{i}\sigma^{\mathrm{j}}$, thus obtaining the group $\mathrm{SU}(2) \times \mathrm{SU}(2) \approx \mathrm{SO}(4)$ by proceeding from the array of matrices

$$\overset{(-)}{\sigma}{}^{\mu\nu} := \left\{ \begin{array}{l} \sigma^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma_k \\ \sigma^{i0} = (-)\frac{1}{2}\sigma^i \end{array} \middle|\, i, j, k = 1, 2, 3 \right\} \tag{3.5.2}$$

as generators. (The minus sign in parentheses would hold for the conjugate representation.) These operators satisfy the commutation relations of the Lie algebra of SO(4). Consequently, they constitute a representation of its infinitesimal generators. Concerning this tensor representation,

$$\overset{*}{\sigma}\Omega = \frac{1}{4} F_{\alpha\beta\mu\nu}\sigma^{\mu\nu} \otimes \vartheta^{\alpha} \wedge \vartheta^{\beta} \tag{3.5.3}$$

serves as a local presentation of the curvature 2-form. Since these generators are themselves self-dual or anti-self-dual, i.e.,

$$^{*}\sigma^{\mu\nu} = \sigma^{\mu\nu}, \quad ^{*}\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\mu\nu}, \tag{3.5.4}$$

in the Yang–Mills theory proper, the relation (3.5.1) leads more correctly to the *double dual* ansatz

$$^{*}F^{*}_{\alpha\beta\mu\nu} = \overset{+}{(-)} F_{\alpha\beta\mu\nu} \tag{3.5.5}$$

for the components of the field strength; see BELAVIN et al. (1975).

In order to solve this equation, one may tentatively proceed from the Lie-algebra-valued gauge potentials

$$A_{\mu} = i\sigma_{\mu i}\partial^{i} \ln h\,. \tag{3.5.6}$$

The duality ansatz yields the relation

$$\Box h - h^{-1}\partial_i h \partial^i h = h^3 \tag{3.5.7}$$

(see WITTEN 1977) for the remaining scalar function, which depends solely on $|x|$, i.e., $h = h(|x|)$ in the spherically symmetric case. A similar nonlinear equation occurs in the determination of conformal changes of metrics in Riemannian geometry; cf. MIELKE (1977b), GU (1978). The most general solution that has been *explicitly* constructed reads

$$h = \sum_{i=1}^{k+1} \frac{l^2_{(i)}}{(x - x_{(i)})^2}\,, \tag{3.5.8}$$

cf. JACKIW et al. (1977). In a flat space M^4 with Euclidean signature $s = 0$, this is a globally nonsingular solution with Pontryagin index ("winding number") given by

$$p_1(M) = k. \tag{3.5.9}$$

It is commonly referred to as the kth *instanton solution*, since its field strength centers around some point in spacetime and attains its maximum at some "instant" in time in the case of $k = 1$ ('T HOOFT 1977). The configuration as given above depends on $5k + 4$ parameters. From studies of related objects in algebraic topology it is known that the kth instanton depends maximally on 8k-3 gauge-invariant parameters (ATIYAH 1979). Consequently, (3.5.8) cannot be considered to be the most general solution. This raises the further question whether all spherically symmetric solutions of the Yang–Mills equation in a Euclidean space $\mathbb{R}^4 \cup \{\infty\} \approx S^4$, which is conformally compactified, can be obtained from the duality ansatz (3.5.5). For "weakly stable" Yang–Mills fields, this issue was answered affirmatively by BOURGUIGNON & LAWSON (1981).

Now, what is the significance of these classical instanton solutions or pseudoparticle solutions for quantum field theory? According to Feynman's method of quantization by means of path integrals, all physical quantities are derivable from the transition amplitude between the asymptotic states $|-\infty\rangle$ and $\langle\infty|$. This amplitude or *generation functional* for the Green's functions is given by the following functional integral:

$$\langle\infty|-\infty\rangle = Z\int_{\mathcal{M}_\omega} d\omega\mu[\omega]\exp i\int_M (L_\omega + L_\infty). \tag{3.5.10}$$

Its virtue is that the spacetime integration is performed over the *classical* Lagrangian n-form of the gauge fields and over a boundary term that if present, ought to eliminate undetermined phase factors in the transition amplitude. Here Z denotes an (in general infinite) normalization constant. The functional integration extends over the configuration space $\mathcal{M}_\omega$ of all inequivalent gauge fields, i.e., of all possible connection 1-forms ω on P modulo gauge transformations. Consequently, the quotient space $\mathcal{M}_\omega = \mathcal{C}/\mathcal{G}_p$ has to be taken as the configuration space proper. Under not too stringent conditions, this space forms an infinite-dimensional (pseudo-)Riemannian manifold (Babelon & Viallet 1981) and may therefore be regarded as a gauge-theoretic counterpart of Wheeler's notion (1970) of a superspace in geometrodynamics. It can be shown that the "measure" $\mu[\omega]$ for the functional integration, as given by the Faddeev–Popov determinant, is nothing but the Jacobian determinant with respect to the "supermetric" of the configuration space (Babelon & Viallet 1979, 1981). However, it has to be pointed out that $\mu[\omega]$ constitutes a mathematically well-defined integration measure only for a base space M with Euclidean signature. Therefore, it is necessary that the functional integral be continued analytically to a "Euclideanized" spacetime manifold with an "imaginary" time variable. In order to generate physically meaningful assertions by means of the quantum formalism, a so-called Wick rotation has to be applied again. Then it is possible to evaluate the results in the real Minkowski space by means of perturbative expansions. Further details of these important quantum-field-theoretic aspects of Yang–Mills theories are dealt with by Faddeev & Slavnov (1980), Jackiw (1980) in their respective surveys.

For our purposes, it is important to accentuate that the functional integration—in contrast to the formalism of the perturbation theory—subsumes also a summation over different and nontrivial topologies of the configuration space. In the Yang–Mills theories proper, this circumstance is more appropriately accounted for by adding the boundary term

$$L_\infty = (\overset{+}{-})\frac{1}{\alpha_g}\mathrm{Tr}(\Omega\wedge\Omega) \tag{3.5.11}$$

of Chern–Pontryagin type to the classical Lagrangian 4-form (3.4.19). The relation

$$L_{\mathrm{YM}} + L_\infty = (\overset{+}{-})\frac{1}{2\alpha_g}\mathrm{Tr}({}^*\Omega(\overset{-}{+})\Omega)^2 \geq 0 \tag{3.5.12}$$

indicates that the leading action for the functional integration attains extrema for self-dual or anti-self-dual configurations, i.e., for solutions satisfying

$$^*\Omega = \overset{+}{(-)}\Omega\,; \tag{3.5.13}$$

cf. GU (1981). According to the inequality in (3.5.12), which holds only for a Euclidean spacetime, it can be stated that the functional integration (3.5.10) is dominated by *self-dual* instanton solutions.[5] Their stability is guaranteed—similarly to that of the solitons in the strict sense—by an exact conservation of the "topological charge." Moreover, a quantum-mechanical "tunneling" can occur between topologically distinct vacuum sectors of the theory (JACKIW 1977).

3.6 Relation to the Seiberg–Witten Equations

The *duality* of electric and magnetic fields in Maxwell's theory was already known to von Laue; see SOMMERFELD (1910). In 1925, the symmetry of *duality rotations* was realized by RAINICH (1925) and developed further in *geometrodynamics* by MISNER & WHEELER (1957). Then MONTONEN & OLIVE (1977) observed in the context of magnetic monopoles that this generates also a duality of the strong–weak coupling regime of gauge fields, the so-called *S-duality*.

These ideas were taken up by SEIBERG & WITTEN (1994) because of their possible consequences for quark confinement and the Higgs field. From a mathematical perspective, the Donaldson invariants of four-dimensional manifolds should be calculable in terms of classical solutions of a system of gauge equations (DONALDSON 2006) coupled to spinors ψ regarded as a lift from the frame bundle to $\mathrm{Spin}^c(4)$.

Let us begin with the Seiberg–Witten (SW) Lagrangian

$$\begin{aligned} L_{\mathrm{SW}} &= \frac{1}{2}\overline{D_\pm\psi}\wedge {}^*D_\pm\psi \mp i\left(F^\pm - \frac{1}{2}\overline{\psi}\sigma_\pm\psi\right)^2 \\ &= \mp i\mathrm{Tr}\left(F^\pm\wedge F^\pm\right) + \frac{1}{2}\overline{D_\pm\psi}\wedge {}^*D_\pm\psi \\ &\quad \pm i\overline{\psi}\sigma_\pm\psi\wedge F^\pm \mp \frac{i}{4}\overline{\psi}\sigma_\pm\psi\wedge\overline{\psi}\sigma_\pm\psi\,; \end{aligned} \tag{3.6.1}$$

cf. JOST et al. (1996), where here the decomposition $\psi = \psi_L + \psi_R = P_-\psi + P_+\psi$ of the Dirac spinors into left- and right-handed pieces is suppressed. The $U(1)$ gauge part of the SW Lagrangian corresponds to a chiral decomposition.

It can be regarded as a self-dual or anti-self-dual Maxwell (or Yang–Mills) Lagrangian coupled to the convective and polarization parts of the Lagrangian result-

[5] According to DE ALFARO (1979a, b), the so-called meron solutions with half-integer topological charge are supposed to be dominant instead.

ing from a (classical) Gordon decomposition of a Dirac field; cf. HEHL et al. (1991). The squared term of the polarization moment two-form

$$\mathscr{P}_{\pm} := \overline{\psi}\sigma_{\pm}\psi/2m, \tag{3.6.2}$$

as is typical for a four-fermion-type self-interaction, then plays the role of an effective mass term.

The variation with respect to $\overline{\psi}$ and A leads to the *convective*-type spinor equation

$$D_{\pm}{}^{*}D_{\pm}\psi \mp i\left(F^{\pm} - \frac{1}{2}\overline{\psi}\sigma_{\pm}\psi\right)\wedge\sigma_{\pm}\psi = 0 \tag{3.6.3}$$

coupled to the Yang–Mills-type equation

$$D_{\pm}\left(F^{\pm} - \frac{1}{2}\overline{\psi}\sigma_{\pm}\psi\right) = \mp\frac{i}{2}\overline{\psi}\,{}^{*}D_{\pm}\psi = \mp\frac{1}{2}\overline{\psi}\,{}^{*}\gamma\wedge{}^{*}(i\,{}^{*}\gamma\wedge D_{\pm}\psi)\,. \tag{3.6.4}$$

The convective part of the above spinor equation is related to the Lichnerowicz–Weizenböck formula

$$D\,{}^{*}D\psi = \left(\nabla{}^{*}\nabla + \frac{1}{4}R + \frac{1}{2}F^{+}\right)\psi. \tag{3.6.5}$$

Solutions (SAÇLIOĞLU 1999) of this system necessarily satisfy the *Seiberg–Witten equations*

$$i\,{}^{*}\gamma\wedge D_{\pm}\psi = 0 \quad\text{and}\quad F^{\pm} = \frac{1}{2}\overline{\psi}\sigma_{\pm}\psi = m\mathscr{P}_{\pm}\,, \tag{3.6.6}$$

which constitute a kind of *linearization* of the system (3.6.3) and (3.6.4). These equations, in contrast to those of Yang–Mills, cease to be conformally invariant. The solutions, called monopoles also for Euclidean signature, *minimize* the Lagrangian (3.6.1). For a nonnegative scalar curvature R in the Weizenböck formula, all solutions satisfy $\psi = 0$ and then are $U(1)$ instantons.

It is interesting to note that the algebraic SW relation for the gauge field strength $F^{\pm}$ resembles (MIELKE 1984) the modified (double) duality ansatz

$$\Omega^{g}_{\pm} = \frac{1}{2\ell^{2}}\sigma_{\pm}, \tag{3.6.7}$$

later on used for the Riemann–Cartan curvature in the (broken) Poincaré gauge theory of gravity. There, its solutions are known (BAEKLER 1981) to be of anti-de Sitter (AdS) type.

3.7 Higgs Fields

Pure Yang–Mills theories have no direct physical applications. One reason for this is the occurrence of infrared divergences in the quantized theory. Even in the most prominent model for strong interactions, that of quantum chromodynamics (QCD; see Marciano & Pagels 1978), the conceptual simplicity of the theory is marred by the addition of gauge fixing terms. In order to formulate a consistent quantum field theory, even the "ghost" terms of Faddeev–Popov take over a necessary part. Furthermore, it is the principle of gauge invariance that forbids massive terms for the gauge potentials $A_\alpha{}^j$ in the Lagrangian n-form. However, there is only one massless spin-1 particle, the photon, known in nature.

In order to incorporate massive states of short range into the theory, Higgs (1964a, b, 1966) suggested a dynamical mechanism of symmetry breaking. This mechanism was then generalized to the nonabelian case by Kibble (1967), which again made it possible for some of the gauge potentials to be transformed into fields with mass without thereby destroying the gauge invariance of the theory. In this elaborated approach, a non-linearly coupled scalar field is instrumental for this mechanism. Here, we are concerned with the geometric aspects of this theory only.

Following the presentation of Bernstein (1974), we are adopting the first-order formalism of Petiau, Duffin, and Kemmer, according to which not only a set of scalar fields φ^j with f components but also the vector fields $\Theta^{(1)j}$ will be coupled to the gauge potentials. More precisely, these components are the *bundle coordinates* of the 0-form $\phi^{(0)}$ and the 1-form $\Theta^{(1)}$ with values in the adjoint representation of G. In terms of these fields, the Lagrangian 4-form (3.4.19) of the Yang–Mills theory is supplemented as follows:

$$L_{\mathrm{H}} = -\bar{\Theta}^{(1)} \wedge {}^*D\phi^0 + \bar{\Theta}^{(1)} \wedge {}^*\Theta^{(1)} + \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \overline{\varphi}\cdot\varphi\right)^2 \eta. \tag{3.7.1}$$

The form of the dynamics is borrowed from the Landau–Ginzburg theory of superconductivity (Rose-Innes & Rhoderick 1969), which has many important analogies to the Higgs model. The coupling of the gauge potentials $A_\alpha{}^j(m)$ to the Higgs field, which is achieved via the G-covariant derivative, is in perfect harmony with the G-equivalence principle. From the variations and $\delta L_{\mathrm{H}}/\delta\overline{\phi}^{(0)}$ and $\delta L_{\mathrm{H}}/\delta\overline{\Theta}^{(1)}$, the following field equations for the Higgs field are obtained:

$$D^*\Theta^{(1)} = \frac{\lambda}{2}\left(\frac{\mu^2}{\lambda} - \overline{\varphi}\cdot\varphi\right)\varphi\eta, \tag{3.7.2}$$

$$\Theta^{(1)} = D\phi^{(0)}. \tag{3.7.3}$$

Insertion of the form $\Theta^{(1)}$ into (3.7.2) by means of (3.7.3) leads back to a gauge-covariant generalization of a nonlinear Klein–Gordon equation. According to the Lagrangian formalism, these fields couple back to the gauge fields that are dynamically restricted by (3.4.11). The source is given by the 3-form of the canonical current

$$\tau_{\mathrm{H}} = -[\phi^{(0)}, {}^{*}\Theta^{(1)}] \tag{3.7.4}$$

of the scalar field. Then the field equations of the total system, consisting of Yang–Mills and Higgs fields, take on the following form:

$$D^{*}\Theta^{(1)} - \frac{\lambda}{2}\left(\frac{\mu^2}{\lambda} - \overline{\varphi}\cdot\varphi\right)\varphi\eta = 0, \tag{3.7.5}$$

$$D\Pi^{\Omega} + [\phi^{(0)}, {}^{*}\Theta^{(1)}] = 0. \tag{3.7.6}$$

It should be noted that the parameter μ in (3.7.5) complies formally with an imaginary mass of the Klein–Gordon-type field. However, the trivial field configuration $\varphi = 0$ does not correspond to the ground state of this model. In the configuration space, the local maximum occurs just at $\varphi = 0$, whereas the "locus" of local *minima* is found at

$$\overline{\varphi}\cdot\varphi := \varphi^{*j}\varphi_j = \frac{\mu^2}{\lambda}. \tag{3.7.7}$$

This is a consequence of the *nonlinear term* in (3.7.1), which is quartic in $|\varphi|$. Therefore, it is reasonable within this theory to assume the existence of a nontrivial quantum-mechanical state ${}^{\lambda}\varphi$, for which the vacuum expectation value

$$\langle O\,|\,{}^{\lambda}\varphi\,|\,O\rangle = \mu/\sqrt{\lambda} \tag{3.7.8}$$

does not vanish. This is a crucial feature of the model, insofar as the shifted fields $\varphi_{\mathrm{H}} := \varphi - \mu/\sqrt{\lambda}$ along with the gauge fields $\mathrm{A}_{\alpha}{}^{\mathrm{j}}$ acquire a real, i.e., physical, mass that is determined by the curvature of the potential at the bottom, i.e., by $\mathrm{m}_{\mathrm{H}} = \sqrt{2}\mu$, and this for all but one of the gauge field components.

Consequently, the gauge symmetry of the Yang–Mills–Higgs system appears to be *spontaneously* broken by the occurrence of a nontrivial vacuum sector (Fig. 3.2).

From the geometric point of view, this mechanism of symmetry breaking may be construed as follows (TRAUTMAN 1977): the constraint (3.7.8) restricts the Higgs field φ such that its range is an orbit G/H of G in V^{ρ} with respect to a subgroup H, i.e., the minimal Higgs field consists of the cross section

$$\varphi_{\min} : P \to C^{\infty}\left(\overset{\circ}{V}(M, \rho(G/H), \rho(G), P) \subset V^{\rho}\right). \tag{3.7.9}$$

Let $\mathrm{v}_0 \in V^{\rho}$ be a fixed vector for which the cross section $\overset{\circ}{\varphi}_{\min} := \sigma(\mathrm{v}_0) \in \mathrm{C}^{\infty}(\mathrm{V}^{\rho})$ satisfies (3.7.8). Since the orbit has the structure of an equivalence class, it is possible to find a $\mathrm{g} \in \mathrm{G}$ that connects any two vectors v and v_0 from V^{ρ} via the relation $\mathrm{v} = \rho(\mathrm{g})\cdot\mathrm{v}_0$. An invariant subgroup of G, the so-called isotropy group

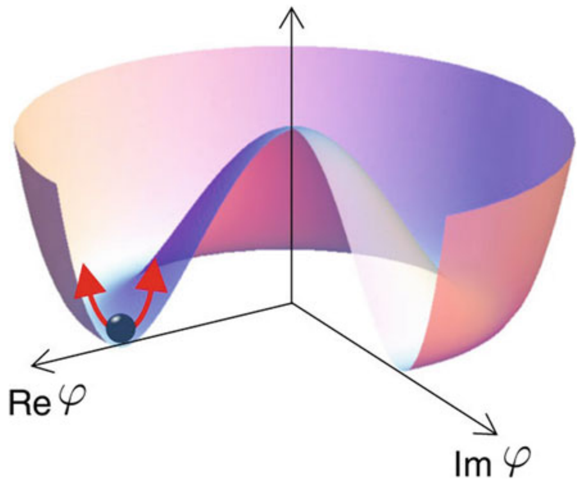

Fig. 3.2 Mexican hat potential in the abelian Higgs model

$$H := \{h \in G | \rho(h) \mathrm{v}_0 = \mathrm{v}_0\}\,, \tag{3.7.10}$$

can be *uniquely* adjoined to the fixed $\overset{\circ}{\varphi}_{\min} = \sigma(\mathrm{v}_0)$. Then

$$Q = \{p \in P | \varphi(p) = \sigma(\mathrm{v}_0)\} \tag{3.7.11}$$

is a subbundle of P(G,M,π, δ) over the same base space; its structure group is H (MADORE 1977, KN I 1963). But what is to be considered the structure of the connection that Q inherits from P? This is answered by the following result:

Proposition *A connection ω in P is reducible to an $\mathfrak{h}$-valued connection ω_{H} in Q if and only if the "Higgs field" φ is covariantly constant with respect to the original connection, i.e., if and only if*

$$D\phi^{(0)} = 0. \tag{3.7.12}$$

Asymptotically, the magnetic monopole solutions found by 'T HOOFT (1974) and POLYAKOV (1974) and the vortex solution of NIELSEN & OLESEN (1973) (cf. also TAYLOR 1976, p. 48) satisfy this condition. There exists an instructive method of generating such solutions that may establish rather good parallels to the gauge theories of gravity that will be developed later on. Let ω_0 be a timelike component of the Yang–Mills connection with regard to an (an-)holonomic frame of reference. The curvature splits into a timelike part

$$\begin{aligned} \Theta^{\perp} := \Omega_0 = d\omega_0 - (\omega \wedge \omega)_0 &= d\omega_0 - \omega_0 \wedge \omega - \omega \wedge \omega_0 \\ &= d\omega_0 - [\omega_0, \omega] = D\omega_0 \end{aligned} \tag{3.7.13}$$

and a remaining purely spacelike curvature 2-form

$$\vec{\Omega} = d\vec{\omega} - (\overrightarrow{\omega \wedge \omega}) = d\vec{\omega} - \vec{\omega} \wedge \vec{\omega}. \tag{3.7.14}$$

In the following, we will restrict ourselves to a frame of reference in which the monopole is at rest, i.e., to static solutions that satisfy

$$d_0{}^*\Omega_0 = 0. \tag{3.7.15}$$

For the time being, let us concentrate on the Yang–Mills equations (3.4.11) in vacuum for which the field momentum is given by (3.4.21). This equation decomposes under the above assumptions into

$$(D^*\Omega)_0 = \vec{D}^*\Omega_0 = \vec{D}^*\Theta^\perp = 0 \tag{3.7.16}$$

and

$$\begin{aligned}(D^*\Omega) &= \vec{D}^*\vec{\Omega} + [\omega_0, \Omega_0] \\ &= \vec{D}^*\vec{\Omega} + [\omega_0, \Theta^\perp] = 0.\end{aligned} \tag{3.7.17}$$

A comparison with the Yang–Mills–Higgs system reveals strikingly that the system (3.7.16) and (3.7.17) reproduces exactly the coupled system (3.7.5) and (3.7.6) in the static case.

As can be shown, it is a precondition for the construction that the component ω_0 of the connection can be identified with the Higgs field ϕ without self-interaction:

$$\omega_0 = \varphi^j L_j \otimes \vartheta_0. \tag{3.7.18}$$

According to this result, first observed by Bogomol'nyi (1976), a solution of the Yang–Mills–Higgs system (3.7.5) and (3.7.6) in the so-called Prasad–Sommerfield limit $\lambda \to 0$ may be obtained by identifying ω_0 in the interaction-free Yang–Mills equation (3.4.11) with a 1-form involving the Higgs field ϕ. The spacelike part of ω provides the static Yang–Mills potential $\overset{*}{\sigma}\vec{\omega}$ of the monopole. After this identification, the resulting pure Yang–Mills equation may be solved by means of the duality ansatz

$${}^*\Omega = (\overset{+}{-})\Omega. \tag{3.7.19}$$

Almost exclusively, such a method provides the starting point for the deduction of monopole-like configurations of the Yang–Mills–Higgs system. For further details, we refer the reader to the papers of Prasad & Sommerfield (1975), Witten (1977), Tchrakian (1981, 1983), Prasad & Rossi (1981a, b). The topological meaning of the magnetic charge quantization (see Strazhev & Tomil'chik 1973) was analyzed in terms of fiber bundles by Quirés et al. (1982), Alvarez (1985).

In this context, it is worthwhile to mention that in the nonlinear σ-model, a variant of the Higgs model, the conditions (3.7.7) for an extremum are introduced as geometric constraints via the method of Lagrange multipliers. The geometric as well as gauge-theoretic aspects of this theory are concisely treated by, e.g., BARBASHOV & NESTERENKO (1980), and in the case of a four-dimensional base manifold, by FELSAGER & LEINAAS (1980) and PERCACCI (1981).

At this stage, it is important to accentuate that the Poincaré gauge field theories of gravity (HEHL 1981)—as will be demonstrated in the next chapter—can also be formulated as a dynamically "broken" theory. Therefore, the formal structures dealt with in this section may provide a prerequisite of prior order for the understanding of these modern gravitational models.

3.8 Translation of Terminologies

Following, e.g., WU & YANG (1975), this chapter will close with a comparison of the notation used in the theory of physical gauge fields and that used in the mathematical theory of fiber bundles.

Gauge Field Terminology	**Fiber Bundle Terminology**
spacetime	base space
space of phase factors	bundle space
symmetry or gauge group	structure group
gauge transformations	inner automorphisms of the bundle
gauge principle	G-equivalence principle
classical fields	cross section of a vector bundle
gauge potential	connection 1-form
gauge field strength	curvature 2-form
electromagnetism	dynamical theory with a connection in a U(1)-bundle
electromagnetism with monopoles	connection in a nontrivial U(1)-bundle over $\mathbb{R}^2 \times S^2$
Dirac's monopole quantization	classification of U(1)-bundles according to the first Chern class
Yang–Mills theory	dynamical theory with a connection in a SU(2)-bundle
instanton number	classification of SU(2)-bundles over S^4 according to the second Chern class

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Chapter 4
Gravitation as a Gauge Theory

Reconsidering current developments in particle physics, clear evidence can be gathered that all efforts converge in the goal of a unified theory concerning all the fundamental physical forces. The most promising approach seems to be founded within the geometric framework of gauge field theories. At very high energies, the gravitational interaction is expected to dominate all other interactions and this despite its diminutive coupling constant given by the Planck length ℓ^*. It may even provide the final regularization of all the divergences that occur in quantum field theory (ISHAM et al. 1971c, 1972). The question then arises whether gravitational interaction can also be formulated in terms of gauge fields.

To be sure, it is commonplace that Einstein's theory of general relativity (GR; EINSTEIN & GROSSMANN 1913; EINSTEIN 1915, 1916, 1955) is already a highly satisfactory theory of gravity. Built on Ricci's tensor calculus (SCHOUTEN 1954), it is generally covariant and so far has not only passed—in contrast to most of its concurring and competing alternatives—all classical tests (see: MTW, pp. 1045ff., WILL 1981), but also has recently been verified with great accuracy for a highly relativistic Keplerian system (see, e.g., STRAUMANN 1981). Facing these facts, one has to think of convincing reasons to legitimate the reformulation of the theory of gravity in terms of gauge fields. First of all, it has to be kept in mind that GR is applicable only to *macroscopic* matter concentrations and to electromagnetic fields. This is implied in the notion of massive structureless test particles and massless (scalar) photons, an axiomatic notion that underlies the foundation of GR (THORNE et al. 1973).

However, from a *microscopic* point of view (HEHL 1985), all tangible matter consists of fermions. In order to obtain a consistent coupling to Dirac's theory of the electron, for instance, a theory that is empirically at least as successful as GR (see BJORKEN & DRELL 1964), it is indispensable to develop a theory of gravity that is invariant with respect not only to general coordinate transformations, but also to local rotations of a pseudo-orthogonal "comoving" frame of reference (Cartan's "repère mobile"). This is a prerequisite for the construction of spinor fields, a fact that was

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pointed out by (WEYL 1919). Due to the postulated orthogonality of the associated tetrads, these transformations have to contain local Lorentz rotations. Moreover, it has to be recognized that special-relativistic fields in flat spacetime—as exemplified by Dirac spinors—not only are Lorentz-invariant, but, speaking mathematically in a more precise manner, transform according to *induced* unitary *representations* of the *Poincaré group* (MACKEY 1968; see also NACHTMANN 1967),[1] which, taken together with the reflections, is the complete invariance group of the flat (affine) Minkowski space of special relativity (SR). Besides the (global) Lorentz transformations, it includes the group of space and time translations.

The method of inducing representations via certain subgroups (in this case via the Lorentz group) into those of the full group may be substantiated by the theory of fiber bundles in a geometrically natural manner (TRAUTMAN 1970). By applying this theory of representations to the Poincaré group, WIGNER showed in 1939 that all fields that are used for a quantum-field-theoretic description are characterized invariantly by *mass and spin* (i.e., helicity in the massless case). "The universal applicability of the mass–spin classification scheme to all known particles establishes the Poincaré group as an unalterable element in any approach to spacetime physics" (HEHL 1980).

If the G-equivalence principle is again adopted as one of the principles of physics, it follows that the dynamical construction is to be invariant merely with regard to *local* Poincaré transformations. This brings about a gauge theory of gravity, as suggested by HEHL (1970), which couples the Poincaré gauge potentials canonically not only to the current of the *energy–momentum* but also to the current of *proper angular momentum* (spin) of material sources. Their geometric counterparts are to be found in both the curvature and the torsion of spacetime (CARTAN 1922). Rather shortly after Einstein outlined his theory of GR, such theoretical concepts were taken into consideration by CARTAN (1923). But it was only after Dirac's relativistic interpretation (DIRAC 1928) of the spin of the electron and under the gradually increasing influence of gauge ideas that these structures were noticed on a larger scale, as, for instance, by WEYL (1929, 1950), UTIYAMA (1956), KIBBLE (1961) and SCIAMA (1962).

The precise gauge-theoretic dimension, as well as the empirical significance of those theories of gravity, was worked out in more detail by HEHL (1980, 1981) using tensor calculus. As shown in the fundamental works of VON DER HEYDE (1976a; 1976b), the dynamics of the gravitational gauge fields can be described by elegant Yang–Mills-type equations. The resulting formalism is flexible enough to include GR or the Einstein–Cartan theory as those subcases that remain physically the most relevant (at least if considered on a macroscopic scale).

With reference to the proper foundation of a gauge theory of gravity, however, there is no absolute agreement among the members of the scientific community. It is the incorporation of a *dynamical geometry* as realized by Einstein via the pseudo-

[1]With reference to MIELKE (1977c), we seize this contextual opportunity in order to correct a printing error. Concerning the action of the induced representation of G on a local cross section, it rather should read

$$T_G: \quad T_g\psi(m) = D\left(g_m^{-1} g g_{g^{-1}m}, m_0\right)\psi(g^{-1}m).$$

Riemannian metric that seems to prevent a direct transfer of the Yang–Mills gauge program. Remarkably enough, there exists, however, a rather close structural analogy between the gauge theories of gravity and the Yang–Mills–*Higgs* models. In this chapter, this analogy will be worked out using almost exclusively the geometric intrinsic calculus of differential forms as advocated, e.g., by TRAUTMAN (1973).

4.1 Affine Frames

In order to present the conceptual foundation of the Poincaré gauge theory as clearly and precisely as possible, it is instructive to enlarge our field of interest only slightly and to start off deductively from the affine group $A(n, \mathbb{R})$ as structure group. As a generalization of the Poincaré group, it consists of the affine transformations of the Euclidean space E^n, regarded as an affine space A^n.

Let $a = (a^\alpha) \in \mathbb{R}^n$ be a row vector. According to its definition, the action of an element of $A(n, \mathbb{R})$ on a vector $x \in A^n$ can expressed as the composition of a general linear transformation and a translation:

$$x \to x' = gx + a, \quad g \in GL(n, \mathbb{R}), \ a \in \mathbb{R}^n. \tag{4.1.1}$$

In the following, it will be convenient to introduce a vector $\tilde{x} := \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$ with $n + 1$ components and to replace the group action (4.1.1) by the abstract transformations of $\tilde{x}$ via the matrices

$$a := \left[\begin{array}{c|c} g & a \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbb{1}_n & a \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right] = a^{\mathrm{T}} \cdot a^{\mathrm{L}} \in A(n, \mathbb{R}). \tag{4.1.2}$$

This faithful representation of $A(n,\mathbb{R})$ by a subgroup of $GL(n + 1, \mathbb{R})$ shows clearly that the affine group decomposes into the *semidirect* product

$$A(n, \mathbb{R}) = \mathbb{R}^n \otimes GL(n, \mathbb{R}) \tag{4.1.3}$$

of the translations $a \in \mathbb{R}^n$ and the linear transformations $g \in GL(n, \mathbb{R})$. The rule for such a semidirect multiplication of group elements is already implied in (4.1.2).

In order to obtain the Poincaré group

$$P := \mathbb{R}^n \otimes 0(1, n - 1) \subset A(n, \mathbb{R}), \tag{4.1.4}$$

the linear transformations are to be restricted to the pseudo-orthogonal subgroup $0(1, n - 1) \subset GL(n, \mathbb{R})$. The invariance of the (flat) metric ground form of the n-dimensional Minkowski space will be thereby guaranteed. For the Lie algebra $\mathfrak{a}(n, \mathbb{R})$ of the affine group, the analogous decomposition

$$\mathfrak{a}(n, \mathbb{R}) = \mathbb{R}^n \circledS \mathfrak{gl}(n, \mathbb{R}) \tag{4.1.5}$$

is effective due to the fact that the Lie algebra of the abelian vector group of translations is isomorphic to $\mathbb{R}^n$. That $\mathfrak{a}(n, \mathbb{R})$ has the structure of a semidirect product is also reflected in the commutation relations:

$$[P_\alpha, P_\beta = 0] \qquad (P_\alpha := \partial/\partial x^\alpha), \tag{4.1.6}$$

$$[P_\alpha, L_{\gamma\delta}] = g_{\alpha\delta} P_\gamma - g_{\alpha\gamma} P\delta, \tag{4.1.7}$$

$$[L_{\alpha\beta}, L_{\gamma\delta}] = c_{\alpha\beta\gamma\delta}{}^{\varepsilon\,\xi} L_{\varepsilon\gamma}, \tag{4.1.8}$$

$$c_{\alpha\beta\delta\gamma}{}^{\varepsilon\zeta} := \delta^{[\varepsilon}_{(\alpha} \delta^{\zeta]}_{\delta} \eta_{\beta\gamma)}. \tag{4.1.9}$$

In these formulas, $L_{\alpha\beta} \in \mathfrak{gl}(n, \mathbb{R})$ and $P_\alpha \in \mathbb{R}^n$ denote the infinitesimal generators of the linear and translational parts of $\mathfrak{a}(n, \mathbb{R})$ with respect to an appropriate parametrization of $A(n, \mathbb{R})$.

For a dynamical formulation, a "geometric arena" is needed. As for the theory of special relativity, the notion of an inertial system is of paramount significance. Correspondingly, for a geometric theory of gravity ruled by the principle of equivalence, the introduction of a more general, local, *frame of reference* serves as the starting point for its foundation. In order to get a precise definition of such a local "inertial" frame, we define it deductively with reference to the notion of a tangent vector e(m) spanning the tangent space $T_m(M)$ at $m \in M$. With the aid of the natural basis of $\mathbb{R}^n$, we obtain the ordered basis

$$e_\alpha(m) = e^i_\alpha \partial_i \tag{4.1.10}$$

for the tangent space. The general linear group $GL(n,\mathbb{R})$ acts on this basis as follows:

$$e_\alpha(m) \to e_\alpha{}'(m) = g_\alpha{}^\beta e_\beta(m), \tag{4.1.11}$$

$$g = [g_\alpha{}^\beta] \in GL(n, \mathbb{R}). \tag{4.1.12}$$

It thus generates a space $L_m(M)$ of *linear frames of reference* at the point m. Let

$$L(M) = \bigcup_{m \in M} L_m(M) \tag{4.1.13}$$

be the bundle resulting from the set-theoretic union of these frames. This bundle may be equipped with a differential structure. In this context, it is important to remember that in a neighborhood $U \subset M$ of a point $m \in M$ with coordinates x^i, a basis vector $e_a(m)$ of $T_m(M)$ may always be written in the way given above. Furthermore, the group $GL(n,\mathbb{R})$ has a natural manifold structure if it is regarded as an open neighborhood of $\mathbb{R}^{n^2}$. Since $e^j_\alpha(m)$ represents a nonsingular $n \times n$ matrix, the collection $(x^i, e^{\cdot j}_\alpha)$ may be regarded as a choice of coordinates of the Cartesian product $U \times GL(n, \mathbb{R})$. These coordinates arise from the original domain $\overset{-1}{\pi}(U)$, for which the projection π of the principal fiber bundle with the structure group GL(n, $\mathbb{R}$) onto M is defined.

On the other hand, this principal fiber bundle can be identified with the bundle

$$L(M) = P(M, GL(n, \mathbb{R}), \pi, \delta) \tag{4.1.14}$$

of linear frames. This is the case if an $\mathbb{R}^n$-valued 1-form ϑ exists that not only is left-invariant, i.e.,

$$\delta_g^* \vartheta = g^{-1} \vartheta, \quad g \in GL(n, \mathbb{R}), \tag{4.1.15}$$

but also has a contact of first order, i.e.,

$$\vartheta(e) = 0 \Leftrightarrow \pi(e) = 0, e \in T_p(P), \tag{4.1.16}$$

with the manifold. The mathematical theory of jets considers also frames that enter into contact of a higher order with the manifold (Kobayashi 1972). Considering this close contact and due to the fact that ϑ has rank n = dim M, it is natural to call it the canonical 1-form (Kobayashi & Nomizu 1963, p. 118) or *soldering 1-form*[2] (Trautman 1979; Kobayashi & Nomizu 1963, p. 140) of the manifold. In local coordinates, it is endowed with the expansion

$$\overset{*}{\sigma} \vartheta^\beta = E_j^\beta dx^j \tag{4.1.17}$$

which has already been employed for generating a basis of the cotangent bundle $T^*(M)$. The tangent space per se is simply the bundle

$$T(M) = V(M, \mathbb{R}^n, GL(n, \mathbb{R}), L(M)) \tag{4.1.18}$$

associated with L(M). Its standard fiber is $\mathbb{R}^n$, and thus T(M) has the same dimension as the manifold.

By comparison of the linear frame bundle L(M) with the *bundle of affine frames* A(M), we get

$$A(M) := P(M, A(n, \mathbb{R}), \pi, \delta) = L(M) \times \mathbb{R}^n. \tag{4.1.19}$$

The possibility of identifying L(M) with the quotient bundle $A(M)/\mathbb{R}^n$ results from the homomorphism

$$\beta : A(n, \mathbb{R}) \to GL(n, \mathbb{R}) = A(n, \mathbb{R})/\mathbb{R}^n. \tag{4.1.20}$$

The left-action of A(n, $\mathbb{R}$) on this fiber bundle is determined by

$$\delta : \begin{cases} A(n, \mathbb{R}) & \times A(M) & \longrightarrow & A(M) \\ \downarrow & \downarrow & & \downarrow \\ (g, a) & (e(m), x) & = (g.e(m), & g^{-1}(x - a)). \end{cases} \tag{4.1.21}$$

[2] Derived from the French “soudure”.

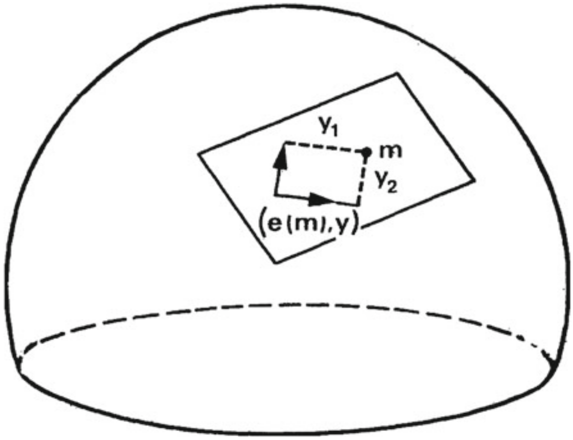

Fig. 4.1 Affine frame of reference at the point m $\in$ M

Accordingly, the affine frame can be considered a linear one, for which the origin of the basis vectors e_α(m) in the tangent space T_m(M) at the point m $\in$ M appears to be shifted (Fig. 4.1).

This relation between the affine and the linear frames results from the fact that the group T $= \mathbb{R}^n$ of translations can be regarded as the quotient space A(n,$\mathbb{R}$)/GL(n,$\mathbb{R}$), i.e., as an affine space in accordance with (4.1.3). Correspondingly, the affine tangent bundle of M is nothing but the bundle

$$T_A(M) = \overset{\circ}{V}(M, \mathbb{R}^n, A(n, \mathbb{R}), A(M)) \subset V^{\text{id}} \tag{4.1.22}$$

on M associated with A(M). In such an affine "geometric arena," translations are represented "merely" intrinsically (NE'EMAN 1978). This is related to the fact mentioned above that the representations of the Poincaré group of physical relevance consist of *induced representations* of the Lorentz group. Here the translational subgroup is realized by translations acting on the coset space T $=$ P$/$L, which is isomorphic to the Minkowski space regarded as an affine space. Related geometric constructions are considered by MÜLLER- HOISSEN (1984) in the case of the inhomogeneous Galilean group.

4.2 Affine Gauge Theory with Torsion

In order to explicate a gauge theory of gravity within this geometric framework, it is again necessary to introduce potentials for the formulation of the dynamics. This is to be accomplished by endowing the bundle A(M) of affine frames with a connection. Let $\overset{\approx}{\omega}$ be the 1-form of such a *generalized affine connection*, and let χ : L(M) $\to 0_m \times$ L(M) $\subset$ A(M) be the natural injection of linear frames into the bundle of affine frames that is induced by the zero vector field 0. Then the inverse image $\overset{*}{\chi}\overset{\approx}{\omega}$ of the connection $\overset{\approx}{\omega}$ is an $\mathfrak{a}(\mathrm{n}, \mathbb{R})$-valued 1-form in L(M) that decomposes into

$$\overset{*}{\chi}\overset{\approx}{\omega} = \omega^{\mathrm{L}} \otimes \omega^{\mathrm{T}}, \tag{4.2.1}$$

or in matrix notation, into

$$\overset{*}{\chi}\overset{\approx}{\omega} = \left[\begin{array}{c|c} \omega^{\mathrm{L}} & \omega^{\mathrm{T}} \\ \hline 0 & 0 \end{array}\right]. \tag{4.2.2}$$

This splitting is a logical consequence of the infinitesimal version of the representation (4.1.2) of A(n,$\mathbb{R}$). In the above formula, ω^{L} denotes a $\mathfrak{gl}(\mathrm{n},\mathbb{R})$-valued 1-form in M, while ω^{T} is an $\mathbb{R}^{\mathrm{n}}$-valued 1-form, i.e., a tensorial form of the representation type $(\mathrm{GL}(\mathrm{n}, \mathrm{R}), \mathbb{R}^{\mathrm{n}})$. The 1-form ω^{L} of type Ad defines a connection in L(M) that is referred to as a *linear connection of the manifold* (KOBAYASHI & NOMIZU 1963, p. 119), which is due to the close contact of L(M) with M. On the other hand, the $\mathfrak{a}(\mathrm{n}, \mathbb{R})$-valued 1-form $\overset{\approx}{\omega}$ that is imprinted in A(M) merely determines a *Cartan connection* with an absolute parallelism in L(M). The decomposition (4.2.1) is true for each of the groups G for which the quotient space G/H forms a weakly reductive structure with respect to a subgroup H $\subset$ G.

A more detailed discussion of these mathematical concepts can be found in the works of KOBAYASHI (1956). In the context of bilocal quantum-mechanical models of strong interactions, Cartan connections have been studied by DRECHSLER (1977) and DRECHSLER & MAYER (1977).

The curvature $\overset{\approx}{\Omega}$ in A(M) can be derived from $\overset{\approx}{\omega}$ by an analogous application of the structure equation. For the following, it is important to have a precise theoretical definition of the affine curvature that is in conformity with the decomposition (4.2.1). The commutator $[\omega^{\mathrm{T}}, \omega^{\mathrm{T}}]$ occurring in the calculation

$$\begin{aligned} \overset{*}{\chi}\overset{\approx}{\Omega} &= \overset{*}{\chi}(d\overset{\approx}{\omega} - \frac{1}{2}[\overset{\approx}{\omega}, \overset{\approx}{\omega}]) \\ &= d\omega^{\mathrm{L}} - \omega^{\mathrm{L}} \wedge \omega^{\mathrm{L}} + d\omega^{\mathrm{T}} - \omega^{\mathrm{T}} \wedge \omega^{\mathrm{L}} - \omega^{\mathrm{L}} \wedge \omega^{\mathrm{T}} + \frac{1}{2}[\omega^{\mathrm{T}}, \omega^{\mathrm{T}}] \\ &= \Omega^{\mathrm{L}} + \Omega^{\mathrm{T}} \end{aligned} \tag{4.2.3}$$

vanishes due to the abelian structure of translations. The term that is additional to the "linear" part of the curvature may for this reason be written as the covariant exterior derivative

$$\Omega^{\mathrm{T}} := D\omega^{\mathrm{T}} = d\omega^{\mathrm{T}} - [\omega^{\mathrm{T}}, \omega^{\mathrm{L}}], \tag{4.2.4}$$

which is induced by ω^{L}. It is of importance to realize that the commutator $[\omega^{\mathrm{T}}, \omega^{\mathrm{L}}]$ takes values in the Lie algebra $\mathfrak{a}(\mathrm{n}, \mathbb{R})$ of the affine group.

A gauge theory with an affine connection should be in accordance with the G-equivalence principle. Following the formalism developed before, this means that the theory has to be invariant with respect to the infinite-dimensional group

$$\mathscr{A}(n,\ \mathbb{R}) := C^\infty\Big(A(M) \times_{\mathrm{Ad}} A(n,\ \mathbb{R})\Big). \tag{4.2.5}$$

The group

$$\mathscr{G}(n,\ \mathbb{R}) := C^\infty(A(M) \times_{\mathrm{Ad}} GL(n,\ \mathbb{R})) \tag{4.2.6}$$

of linear gauge transformations and the group

$$\mathscr{T}(n,\ \mathbb{R}) := C^\infty(A(M) \times_{\mathrm{Ad}} \mathbb{R}^n) \tag{4.2.7}$$

of local translations are subgroups of $\mathscr{A}(n,\ \mathbb{R})$. Taking the cross section in the associated bundle is abbreviated by C^∞, and Ad denotes the adjoint representation with respect to $GL(n, \mathbb{R})$. Due to its construction, the group of local translations $\mathscr{T}(n,\ \mathbb{R})$ is *locally* isomorphic to the group of active diffeomorphisms $\mathit{Diff}\,(n, \mathbb{R})$ of the manifold. The latter group is isomorphic to the group $\mathscr{D}$(M) of diffeomorphisms of M; cf. OGIEVETSKII (1973), STERNBERG (1985). The infinite-dimensional group $\mathit{Diff}\,(n, \mathbb{R})$ contains the $(n + n^2)$-dimensional group $A(n, \mathbb{R})_{\mathrm{H}}$ of *holonomic* affine transformations as a subgroup, generated by the vector fields $P_i = \partial_i := \partial/\partial x^i$ and $L^i{}_j = x^i\partial_j$. Note that differentiable coordinate transformations, which leave exterior forms invariant, are regarded here as passive diffeomorphisms.

Similarly, affine gauge transformations act inhomogeneously on the connection 1-form

$$\stackrel{\approx}{\omega} \to {}^{A^{-1}}\stackrel{\approx}{\omega} = A^{-1}\stackrel{\approx}{\omega}A - A^{-1}dA, \quad A \in \mathscr{A}_p. \tag{4.2.8}$$

Proceeding to the matrix representation (4.1.2) of the affine group, an analogous representation

$$A = \left[\begin{array}{c|c} G & T \\ \hline 0 & 1 \end{array}\right] \in \mathscr{A}_p, \quad G \in \mathscr{G}_p, T \in \mathscr{T}_p \tag{4.2.9}$$

of the gauge transformation is induced. The splitting of $\stackrel{\approx}{\omega}$ allows us to derive the effect of the affine gauge transformations on the linear and translational parts of the affine connection:

$$\omega^{\mathrm{L}} \to {}^{A^{-1}}\omega^{\mathrm{L}} = G^{-1}\omega^{\mathrm{L}}G - G^{-1}dG, \tag{4.2.10}$$

$$\omega^{\mathrm{T}} \to {}^{A^{-1}}\omega^{\mathrm{T}} = G^{-1}\omega^{\mathrm{T}} - G^{-1}DT. \tag{4.2.11}$$

Likewise affected is the transformation formula

$$\stackrel{\approx}{\Omega} \to {}^{A^{-1}}\stackrel{\approx}{\Omega} = A^{-1}\stackrel{\approx}{\Omega}A \tag{4.2.12}$$

of the curvature of the affine bundle. Under the inverse image (4.2.3) with regard to the natural injection χ, this formula decomposes into

$$\Omega^{\mathrm{L}} \to {}^{A^{-1}}\Omega^{\mathrm{L}} = G^{-1}\Omega^{\mathrm{L}}G \tag{4.2.13}$$

and

$$\Omega^{\mathrm{T}} \to {}^{A^{-1}}\Omega^{\mathrm{T}} = G^{-1}(\Omega^{\mathrm{T}} - DDT) = G^{-1}\Omega^{\mathrm{T}} - G^{-1}\Omega^{\mathrm{L}} \otimes T. \qquad (4.2.14)$$

Concerning affine "recalibrations," the linear part shows the typical behavior of a connection with its corresponding curvature.

4.2.1 Affine "Higgs" Mechanism

The latter is not true for the translational remnants, which indicates that they can be regarded only as being represented "purely" after a "spontaneous" breaking of the underlying symmetry has taken place. In order to achieve this, we transfer the geometric construction of the Higgs–Kibble mechanism, which has been discussed before, to the affine case. For our purposes, the local cross section

$$\overset{\approx}{\varphi} : A(M) \to C^{\infty}(\overset{\circ}{V}) \qquad (4.2.15)$$

in the associated affine bundle provides the alleged "*affine Higgs field*." Its range in V^{id} consists of an orbit $\mathbb{R}^{\mathrm{n}}$ of A(n,$\mathbb{R}$). The isotropy group of the zero section $\overset{\circ}{\tilde{\varphi}} = \sigma(0_{\mathrm{m}})$ is H = GL(n, $\mathbb{R}$), i.e., the general linear group. Then the subbundle

$$Q_A = \{p \in A(M) | \overset{\approx}{\varphi}(p) = \sigma(0_m)\} = L(M) \qquad (4.2.16)$$

is isomorphic to the bundle L(M) of linear frames due to (4.1.19). It follows from (4.2.9) that the Higgs field (4.2.15) transforms inhomogeneously with respect to affine gauge transformations, i.e.,

$$\overset{\approx}{\varphi} \to {}^{A^{-1}}\overset{\approx}{\varphi} = G^{-1}(\overset{\approx}{\varphi} - T). \qquad (4.2.17)$$

Let

$$\omega^{\mathrm{T}} = \overset{\approx}{\vartheta} - D\overset{\approx}{\varphi} \qquad (4.2.18)$$

tentatively be a decomposition of the translational part of the connection; cf. IVANENKO & SARDANASHVILY (1983). If this is inserted in (4.2.11), it yields

$$\begin{aligned} {}^{A^{-1}}\omega^{\mathrm{T}} &= {}^{A^{-1}}\overset{\approx}{\vartheta} - G^{-1}DG^{A^{-1}}\overset{\approx}{\varphi} \\ &= G^{-1}\overset{\approx}{\vartheta} - G^{-1}D\overset{\approx}{\varphi}, \end{aligned} \qquad (4.2.19)$$

if use has been made of the covariance of D. This means, however, that the contribution $\overset{\approx}{\vartheta}$ transforms under $\mathscr{A}_{\mathrm{p}}$ like

$$\overset{\approx}{\vartheta} \rightarrow {}^{A^{-1}}\overset{\approx}{\vartheta} = G^{-1}\overset{\approx}{\vartheta}. \tag{4.2.20}$$

Comparing this with (4.1.15), we come to the conclusion that $\overset{\approx}{\vartheta}$ can be identified with the canonical 1-form, up to a multiplication by a scalar function f, i.e.,

$$\overset{\approx}{\vartheta} = f\vartheta. \tag{4.2.21}$$

The covariant derivative

$$\boxed{\Theta := D\vartheta = d\vartheta - [\omega^{\mathrm{L}}, \vartheta]} \tag{4.2.22}$$

of the canonical 1-form is known to represent Cartan's *torsion 2-form* of the underlying spacetime manifold. It can be related to a local notation (cf. HEHL 1980) via the pullback

$$\overset{*}{\sigma}\Theta = \frac{1}{2}T_{\alpha\beta}^{\cdot\cdot c}P_c \otimes \vartheta^\alpha \wedge \vartheta^\beta = \frac{1}{2}T_{ij}^{\cdot\cdot c}P_c \otimes dx^i \wedge dx^j, \tag{4.2.23}$$

where Cartan's torsion tensor is given by

$$T_{\alpha\beta}^{\cdot\cdot c} = 2D_{[\alpha}E^c_{\cdot|\beta]} = 2\left(\partial_{[\alpha}E^c_{\cdot|\beta]} + \Gamma_{[\alpha|d}^{\cdot\cdot c}E^d_{\cdot|\beta]}\right). \tag{4.2.24}$$

Geometrically speaking, the torsion of spacetime may be interpreted as the mismatch, i.e., closure failure, of parallel transported vectors, analogously as in the theory of crystal dislocations (HEHL & KRÖNER 1965; HEHL 1985). From (4.2.20), if applied to $\mathrm{D} = (\overset{*}{\chi}\overset{\approx}{\mathrm{D}})$, it follows that the torsion 2-form transforms like

$$\Theta \rightarrow {}^{A^{-1}}\Theta = G^{-1}\Theta, \tag{4.2.25}$$

i.e., as a vector with respect to the subgroup $\mathscr{G}_{\mathrm{p}}$ of linear gauge transformations. As a result of the Ansatz (4.2.18), the relation

$$\Omega^{\mathrm{T}} = f\Theta + df \wedge \vartheta - \Omega^{\mathrm{L}} \otimes \overset{\approx}{\varphi} \tag{4.2.26}$$

is found for the translational part of the affine curvature. Thus the torsion is only one part of the translational curvature of the generalized affine connection (NORRIS et al. 1980). This result can also be derived from a more formal consideration of Cartan connections on M. For a Higgs–Kibble-type mechanism of "symmetry breaking," a subsidiary condition like

$$\overset{\approx}{D}\overset{\approx}{\varphi} = D\overset{\approx}{\varphi} = 0 \tag{4.2.27}$$

is completely sufficient, since it restricts unequivocally (while for $\mathbb{R}^n$-valued fields $\overset{\approx}{\mathrm{D}} = \mathrm{D}$ it remains valid) the affine connection to the linear $\mathfrak{gl}(n,\mathbb{R})$-valued connection ω^L within the reduced bundle L(M). Disregarding the factor of proportionality, it then follows from (4.2.18) that the torsion remains the only constituent of the translational part of the affine curvature. In this way, the local symmetry group A(n,$\mathbb{R}$) of a gauge theory with affine connection gets broken down to the group GL(n,$\mathbb{R}$).

In order to achieve these interim results, which are of relevance concerning their physical interpretation, we have considered mainly the advanced propositions of TSEYTLIN (1982). The idea of breaking the symmetry via an affine Higgs field can be found already in PILCH (1980). Within the framework of that study, however, the idea is flawed by an unfortunate and unnecessary mixup with the reduction of the group GL(n,$\mathbb{R}$) to the Lorentz group 0(1, n-1). This should rather be carried out in a further step. According to a slightly divergent view (HENNIG & NITSCH 1981), the difference between gauge theories with affine connections and those with linear connections can be traced back to a differing order of "contact" of the frame bundles with the underlying manifold. This has to be analyzed within the framework of the mathematical theory of "jets" (KOBAYASHI 1961).

If in addition to the connection, the spacetime manifold is also endowed with a metric tensor as an independent field, a metric–affine theory of gravitation can be constructed; cf. HEHL & KERLICK (1978). Its classification as a gauge theory of the affine group was particularly well clarified by LORD (1978). In general, the affine connection is not metric-compatible in these models, and the tensor

$$Q_{\kappa\mu\nu} := -D_\kappa \, g_{\mu\nu} \tag{4.2.28}$$

of *nonmetricity* measures this deviation. In a dynamical formulation, the field momentum being canonically conjugate to the general affine connection leads to the notion of *hypermomentum* current. Such models might have considerable impact on high-energy physics. According to a proposal of HEHL et al. (1978, 1989), it is exactly the hypermomentum that is responsible for those band structures in the mass spectrum of hadrons that follow the Regge trajectories. For theoretical reasons, the light-cone structure of spacetime is deformed by the shear part of the nonmetricity tensor, so that violations of causality cannot principally be excluded. This effect, however, can by no means be reason enough to reject such theories in general (HAYASHI 1976), since within dynamical models, it is typical of these defects that they remain restricted to the interior of microscopically "extended" particles (HEHL et al. 1976). In terms of quantum field theory, however, it is possible to postulate only a microscopic causality condition—if any such condition at all.

4.3 Metric Structure of the "Spontaneously Broken" Poincaré Gauge Theory

A space with an affine connection has to be a richer structure than a differential manifold, since it allows parallel displacements of vectors. However, we are still not able to measure distances within this space. This, in turn, is mandatory for determining geometrically the relative location of physical objects in space. (A discussion of the postulate of metricity from an ontological viewpoint is to be found in GRÜNBAUM 1973). According to the special theory of relativity (EINSTEIN 1955), the field-free flat region between such objects is to be characterized as follows: The square of the "distance" between two adjacent points in spacetime is a measurable quantity if an apt coordinate system is provided for. Its amount is to be determined by

$$\begin{aligned}(\Delta s)^2 &= -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= g_{ij}\Delta x^i \Delta x^j .\end{aligned} \tag{4.3.1}$$

Due to the fundamental postulate of the constancy of light propagation in vacuum, this four-dimensional expression is indefinite. Such a Minkowski structure, however, is only locally existent in the presence of gravitational fields, due to curvature of spacetime.

A conceptually central problem of measurability for a generic curved manifold consists in associating a positive length

$$\ell(\mathscr{C}) := \int_{t_0}^{t_1} ds(t) \tag{4.3.2}$$

to a smooth curve $\mathscr{C} \subset$ M that is parametrized by s(t). In order to satisfy

$$\ell(\mathscr{C}_1 + \mathscr{C}_2) = \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2) \tag{4.3.3}$$

for a piecewise smooth curve, the infinitesimal length of the curve ought to be expressed as

$$ds = F(m, \vartheta)dt, \tag{4.3.4}$$

where F(m, ϑ) : M $\times$ T*(M) $\to \mathbb{R}^+$ is a smooth, *positive*, and *homogeneous* fundamental *function*, i.e.,

$$\begin{aligned}&F(m, f\vartheta) = |f|F(m, \vartheta), \\ &F(m, \vartheta) > 0 \quad \text{if} \quad \vartheta \neq 0.\end{aligned} \tag{4.3.5}$$

The homogeneity of first order is at our disposal on account of the postulated invariance of (4.3.4) versus admissible reparametrizations of the curve $\mathscr{C}$. Generally, these requirements lead up to *Finsler spaces* (RUND 1959; MATSUMOTO 1977) with the anholonomic components of the metric defined by

$$g_{\alpha\beta}(m, \vartheta) := \frac{1}{2} \frac{\partial^2 F^2(m, \vartheta)}{\partial \vartheta^\alpha \otimes_s \partial \vartheta^\beta}. \tag{4.3.6}$$

Riemannian spaces are distinguished by their intrinsic geometry as metric spaces with a special Finslerian function. Then the latter is derivable from the quadratic differential form

$$ds^2 = (F(m, \vartheta)dt)^2 = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta = g_{ij}(m)dx^i \otimes_s dx^j; \tag{4.3.7}$$

see, e.g., LAUGWITZ (1965). The angle between two infinitesimal vectors dV and dW can be determined implicitly in such spaces by

$$\cos\phi = \frac{g_{\mu\nu}dV^\mu dW^\nu}{\sqrt{g_{\alpha\beta}dV^\alpha dV^\beta}\sqrt{g_{\alpha\beta}dW^\alpha dW^\beta}}. \tag{4.3.8}$$

Subsequently, a covariant symmetric tensor field of degree (2, 0) is imprinted onto this space, which is now referred to as a *(pseudo-)Riemannian manifold.* According to the axiomatic foundation of GR, the components of this fundamental tensor for a holonomic coordinate system determine not only the metric relations of spacetime. Indeed, the functions $g_{ij} \in C^\infty(M)$ describe "concerning the chosen, arbitrary coordinate system both the metrical relations of the spacetime continuum as well as the gravitational field" (EINSTEIN 1955).

It is for these reasons that the metric seems to play a special role in a gauge approach to gravity, especially if one tries—as has earlier been attempted by, e.g., THIRRING (1978)—to establish a direct relationship between this metric field and the gravitational gauge potentials. In order to work out the true nature and significance of the metric tensor within the framework of a unified gauge-theoretic approach, let us compare the two equivalent formulations of the line element (4.3.7), e.g., written in holonomic and anholonomic coordinates. This yields the relation

$$g_{ij}e_\alpha^{\cdot i}e_\beta^{\cdot j} = g_{\alpha\beta}. \tag{4.3.9}$$

Since ϑ^α forms a basis for the frame bundle L(M), it follows that this Eq. (4.3.9) can be expressed in terms of exterior forms as

$$\vartheta \wedge {}^*\vartheta = \eta. \tag{4.3.10}$$

(For tensor fields, the metric ground form is essential in performing the so-called contraction, while for exterior forms, the metric is concealed in the definition of the Hodge dual *.)

Similarly as in the Higgs–Kibble mechanism of spontaneous symmetry breaking (*SSB*), the tetrad field or the canonical 1-form is *kinematically* restricted by (4.3.9) or (4.3.10), respectively. In the Poincaré gauge theory, however, in contrast to the affine gauge theory in which the group GL(n,$\mathbb{R}$) is the residual structure group after the symmetry breaking, there are only orthogonal tetrads, or *orthogonal* canonical 1-forms ϑ admitted. Since the $e_{\alpha}^{\cdot i}$ have been considered as local bundle coordinates of L(M), the bundle of linear frames is then reduced to the orthogonal frame bundle

$$L^g(M) := P(M, O(s, n-s), \pi, \delta). \tag{4.3.11}$$

In order to ensure that the linear connection ω^{L} within this bundle reduces to a $\mathfrak{O}$(s, n − s)-valued connection ω^{g}, the constraint

$$D\, e_{\alpha}^{\cdot i} = 0 \tag{4.3.12}$$

is a necessary and sufficient condition. For tensor-valued 0-forms $\phi^{(\circ)}$, the exterior covariant derivative D can be replaced by the covariant derivative ∇ in the following manner:

$$D\,\phi^{(\circ)} = \vartheta^{\alpha}\nabla_{\alpha}\phi^{(\circ)}. \tag{4.3.13}$$

It is for this and the relation (4.3.9) that the above-mentioned constraint can be considered equivalent to the postulate

$$D_k\, g_{ij} = E^{\alpha}_{\cdot k}\nabla_{\alpha} g_{ij} = 0 \tag{4.3.14}$$

of the metric compatability of the Riemannian connection ω^{g}. Locally, the latter is usually written as

$$\overset{*}{\sigma}\omega^g = \frac{1}{2}\omega^{\alpha\beta}L_{\alpha\beta} = \frac{1}{2}\Gamma_i^{\cdot\alpha\beta}L_{\alpha\beta}\otimes dx^i, \tag{4.3.15}$$

where the $\Gamma_{\mathrm{i}}^{\cdot\alpha\beta}$ denote the (local) *Ricci rotation coefficients*. Due to the fact that the infinitesimal generators $L_{\alpha\beta}$ become skew after the restriction to the orthogonal subgroup, the constraint (4.3.14) analogously yields $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$, i.e., the skew-symmetry

$$\Gamma_i^{\cdot\alpha\beta} = -\Gamma_i^{\cdot\beta\alpha} \tag{4.3.16}$$

of the rotation coefficients with respect to the last two indices. According to the preceding considerations, these coefficients can be regarded as gauge potentials of the Lorentz group, whereas the (invertible) $e_{\alpha}^{\cdot i}$ may be interpreted as the "soldered" gauge potentials of local translations (Hehl 1980).

Within GR, the condition (4.3.14) is commonly justified by the equivalence principle. If applied here, this means that the measurement of lengths and angles in the neighborhood of a point $m_o \in M$ depends entirely on Minkowski's metric tensor o_{ij} of special relativity, i.e., that

$$g_{ij} \overset{(*)}{=} o_{ij} \tag{4.3.17}$$

holds for orthonormal frames. Deviations from the flatness of the spacetime domain occur, if at all, only at the second differential order,[3] i.e.,

$$\partial_k g_{ij} \overset{(*)}{=} 0. \tag{4.3.18}$$

Its covariant generalization, i.e., (4.3.14), guarantees that the square of the distance, considered an "infinitesimal measuring rod," remains constant with respect to parallel displacements. This requirement, which is physically sensible and experimentally not explicitly disproven, represents, strictly speaking, an a priori element in GR; see HÜBNER (1983).

From our viewpoint, however, it is easily to be recognized that a further mechanism of symmetry breaking of Higgs–Kibble type is at work here. As has already been mentioned, the tetrad field is kinematically restricted by the "metric" subsidiary condition (4.3.9). It is then that the "geometric arena" (4.3.11) of a gravitational gauge theory has only the orthogonal "isotropy group" $\mathrm{H} = \mathrm{O(s, n-s)}$ as residual symmetry. As it turns out, (4.3.14) is nothing but the consistency condition for the reduction of the linear connection ω^{L} to an $\mathfrak{o}(\mathrm{s, n-s})$-valued connection ω^{g} in $\mathrm{L^g(M)}$. This consistency does not, as is implied by HANSON & REGGE (1979), demand the vanishing of the torsion, i.e., $\mathrm{D}\vartheta = 0$. In a Lagrangian formulation of such a *"spontaneously broken" Poincaré gauge theory*, there will occur not only its connection and curvature. Similarly as in the Yang–Mills–*Higgs* models, the metric Higgs field ϑ, i.e., the canonical *orthogonal* 1-form,[4] and the resulting torsion Θ constitute additional elements for the construction of the dynamics.

In the following, we can assume that the orthogonal bundle $\mathrm{L^g(M)}$ is endowed with a connection 1-form ω^{g} compatible with the metric. The curvature 2-form

$$\Omega^g = d\omega^g - \omega^g \wedge \omega^g \tag{4.3.19}$$

satisfying the general definition may locally be written as

$$\overset{*}{\sigma}\Omega^g = \frac{1}{4} R_{\alpha\beta}^{\cdot\cdot cd} L_{cd} \otimes \vartheta^\alpha \wedge \vartheta^\beta. \tag{4.3.20}$$

Here

$$R_{\alpha\,\beta\,c}^{\cdot\cdot\cdot d} = 2\left\{\partial_{[\alpha} \Gamma_{\beta]c}^{\cdot\cdot d} + \Gamma_{[\alpha|h}^{\cdot\cdot d} \Gamma_{|\beta]c}^{\cdot\cdot h}\right\} \tag{4.3.21}$$

are the anholonomic components of the curvature tensor in a Riemann–Cartan space.

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[3]Locally, these requirements are always to be satisfied by introducing the Riemannian normal coordinates; see LAUGWITZ (1965).

[4]As for the nomenclature, the 1-form ϑ, which is restricted by (4.3.10), will not be denoted by ϑ^{g} and thus differs from ω^{g} and Ω^{g}, since we prefer to stick to (4.3.10) or (4.3.9), respectively, as explicit subsidiary conditions for the breaking of symmetry with respect to both the tetrad field and the torsion.

Implicitly the Riemann–Cartan curvature tensor contains torsion-dependent pieces. In order to separate them from the purely Riemannian content, the metric-compatible connection can be decomposed into

$$\omega^{g} = \omega^{\{\}} - K. \tag{4.3.22}$$

Here $\omega^{\{\}}$ denotes the 1-form of the torsion-free Christoffel-type connection for which

$$d\vartheta - [\omega^{\{\}}, \vartheta] = 0 \tag{4.3.23}$$

holds by way of assumption. Using this in the defining relation (4.2.22) of the torsion, we obtain the relation

$$\Theta = [K, \vartheta] \tag{4.3.24}$$

between the 2-form Θ and the 1-form K of the so-called *contortion*, having values in the Lie algebra of the Lorentz group (see also: KOBAYASHI & NOMIZU 1963, p. 159). Locally, the latter may be expanded as

$$\overset{*}{\sigma} K = \frac{1}{2} K_{\alpha}^{\cdot \beta c} L_{\beta c} \otimes \vartheta^{\alpha}. \tag{4.3.25}$$

Consequently, the components of the contortion tensor[5] are antisymmetric with respect to the last two indices:

$$K^{c}_{\cdot \alpha\beta} = -\frac{1}{2}\left(T^{c}_{\cdot \alpha\beta} - T^{\cdot\cdot c}_{\alpha,\beta} + T^{\cdot c}_{\beta \cdot \alpha}\right) = -K^{c}_{\cdot \beta\alpha}. \tag{4.3.26}$$

If all these decompositions are taken together, it leads to the desired splitting

$$\begin{aligned} \Omega^{g} &= \Omega^{\{\}} - dK + K \wedge \omega^{\{\}} + \omega^{\{\}} \wedge K - K \wedge K \\ &= \Omega^{\{\}} - D^{\{\}} K - K \wedge K = \Omega^{\{\}} - DK + K \wedge K \end{aligned} \tag{4.3.27}$$

of the RC-curvature tensor[6] into its Riemannian and non-Riemannian parts. Later, we are going to make use of this decomposition.

In order to complete the transcription into Ricci's tensor calculus—"contaminated by indices," cf. EGUCHI et al. (1980)—it remains to be added that the rotation coefficients are given by

$$\Gamma_{ij}^{\cdot\cdot\kappa} = e_{b}^{\cdot\kappa} \Gamma_{i\alpha}^{\cdot\cdot b} E^{\alpha}_{\cdot j} + e_{\beta}^{\cdot\kappa} \partial_i E^{\beta}_{\cdot j} \tag{4.3.28}$$

with respect to a holonomic basis. This formula was obtained by applying local translations (gauge transformations). It then follows that the decomposition (4.3.22)

[5] In German, *Verdrehungstensor*.

[6] Following SCHOUTEN (1954), a Riemann–Cartan manifold of dimension n *with* torsion is also denoted by U_n, in contrast to a purely Riemannian manifold, which is symbolized by V_n.

takes the form

$$\Gamma_{ij}^{\cdot\cdot\kappa} = \begin{Bmatrix} \kappa \\ ij \end{Bmatrix} - K_{ij}^{\cdot\cdot\kappa}, \tag{4.3.29}$$

where

$$\begin{Bmatrix} \kappa \\ ij \end{Bmatrix} = \frac{1}{2} g^{\kappa l} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{4.3.30}$$

are the more familiar Christoffel symbols of the second kind. They are formed entirely out of the metric tensor. The splitting (4.3.27) has its local counterpart in

$$R_{ijc}^{\cdot\cdot\cdot d} = R_{ijc}^{\{\}\cdot\cdot\cdot d} - 2D_{[i} K_{j]c}^{\cdot\cdot d} + 2K_{[i|c}^{\cdot\cdot h} K_{|j]h}^{\cdot\cdot d}. \tag{4.3.31}$$

In this formula, $R_{ijc}^{\{\}\cdot\cdot\cdot d}$ are the mixed components of the Riemannian curvature tensor, which again is calculated according to (4.3.21) merely using Christoffel symbols. Since it is the curvature tensor of a metric-compatible symmetric connection, its holonomic components $R_{ij\kappa l}^{\{\}}$ have the following symmetries with respect to an interchange of indices:

$$\begin{aligned} R_{(ij)\kappa l}^{\{\}} &= R_{ij(\kappa l)}^{\{\}} = 0 \\ R_{ij\kappa l}^{\{\}} &= R_{\kappa l ij}^{\{\}}. \end{aligned} \tag{4.3.32}$$

In the torsion-free case, the first Bianchi identity reduces to the *algebraic* identity

$$R_{i[j\kappa l]}^{\{\}} \equiv 0 \tag{4.3.33}$$

involving a cyclic permutation of three indices. For later purposes, it is important to write down the contractions of the curvature tensor, which lead successively to the Ricci tensor

$$R_{ij} := R_{\alpha ij}^{\cdot\cdot\cdot\alpha} \tag{4.3.34}$$

and the scalar curvature

$$R := R_{\mu}^{\cdot\mu} = R_{\alpha\mu}^{\cdot\cdot\mu\alpha} = {}^{*}(\Omega^{g} \wedge \vartheta \wedge \vartheta). \tag{4.3.35}$$

After this digression into the tensor calculus of classical differential geometry, we can summarize the following results, which are relevant for a gauge-theoretic approach to gravity.

Starting from a gauge theory with an affine connection, it was the twofold application of a Higgs–Kibble-type mechanism of *SSB* that resulted in the construction of a "broken" Poincaré gauge theory of gravity. Following "hierarchical" procedures, the affine structure group A(n, ℝ) is "spontaneously" broken into the Lorentz group O(1, n-1) as the "residual" symmetry of the theory via the general linear group GL(n, ℝ).

The idea of regarding the metric (more appropriately, the tetrad field) as the gravitational analogue of the Higgs field appears in ISHAM et al. (1971a). With certain modifications, this interpretation was later adopted by TRAUTMAN (1979), TSEYTLIN (1982), WALLNER (1982), and MÜLLER-HOISSEN (1984). Concerning the quantization of gravitational fields, it was also considered by HANSON & REGGE (1979). However, it must be pointed out that our approach to the Poincaré gauge theory of gravity, as it is expounded here, has not yet gained the status of a generally accepted scientific explication. This is made explicitly clear by a comparison with the publications of UTIYAMA (1980) and HEHL (1981).

Thus it is HEHL (1980), above all following VON DER HEYDE (1976a) and NE'EMAN (1978), who favors a gauge-theoretic interpretation of the Poincaré group, in which the generators of local translations are not represented by partial derivatives but in an *already curved* spacetime by *covariant* derivatives. Subsequently, these generators violate the usual commutation relations (4.1.6), and as a result, they change into noncommutative ones. Due to the appearance of the curvature tensor in the commutator

$$[D_\alpha, D_\beta] = R_{\alpha\beta}^{\cdot\cdot cd} L_{cd} - T_{\alpha\beta}^{\cdot\cdot c} D_c, \tag{4.3.36}$$

spacetime-dependent "structure constants" $\mathrm{R}_{\alpha\beta}^{\cdot\cdot\mathrm{cd}}(\mathrm{m})$ occur in the generic case. (The additional torsion-dependent term would disappear in a holonomic formulation.) A deeper understanding of these insights has been established by SOHNIUS (1983) with a new approach involving "soft" gauge algebras.

4.4 Gravitational Field Equations

Independently of the particular gauge-theoretic approach to gravity, there exists a general consensus as to its geometric and dynamical structure. Compared to Yang–Mills theories, the gravitational gauge models have a richer structure in that both the linear connection ω^{L} and the canonical 1-form ϑ represent independent dynamical variables. Arguments have already been advanced that allow us to consider ϑ a "metric Higgs field."

The corresponding gauge field strengths consist of the curvature Ω^{g} of the Riemann–Cartan space and the covariant exterior derivative of ϑ, i.e., the torsion 2-form Θ. Consequently, the Lagrangian n-form depends on the following geometric objects for the generic case:

$$L_g = L(g_{\alpha\beta}; \vartheta, (\omega^g), \Theta, \Omega^g). \tag{4.4.1}$$

However, contrasting with $\vartheta, \Theta, \Omega^{\mathrm{g}}$ (see (4.2.20), (4.2.14), and (4.2.13)), the 1-form ω^{g} does not transform as a vector or covariantly, respectively, with regard to the residual local Lorentz group. Due to this property, the postulate of gauge invariance allows only an indirect dependence of the Lagrangian n-form on ω^{g}, and this only via of the forms Θ and Ω^{g}. The variation of L_{g}, regarded as a total differential, i.e.,

$$\delta L_g =: \delta\vartheta \wedge E + \delta\Theta \wedge \Pi^{\mathrm{T}} + \delta\Omega^g \wedge \Pi^{\mathrm{L}}, \tag{4.4.2}$$

defines the *t*ranslational and (*L*orentz-)rotational canonical gauge field momenta Π^{T} and Π^{L}. Due to the special role of ϑ, there occurs additionally the (n-1)-form E, comprising the energy–momentum current of the gravitational gauge fields.

In order to derive the field equations, the variation with respect to the torsion Θ and the curvature Ω will have to be rewritten with respect to the original dynamical variables ϑ and ω^{g}. Within this variation procedure, the general result may be adopted here with respect to the curvature; a corresponding conversion of the variation for the torsion yields

$$\begin{aligned}\delta\Theta &= \delta(d\vartheta - [\omega^{\mathrm{L}}, \vartheta]) \\ &= d(\delta\vartheta) - [\omega^{\mathrm{L}}, \delta\vartheta] - [\delta\omega^{\mathrm{L}}, \vartheta] \\ &= D(\delta\vartheta) - [\delta\omega^{\mathrm{L}}, \vartheta].\end{aligned} \tag{4.4.3}$$

Our subsequent proceeding is similar to that in the Yang–Mills case. By the Leibniz rule, the total variation results in

$$\begin{aligned}\delta L_g &= \delta\vartheta \wedge E - \delta\vartheta \wedge D\Pi^{\mathrm{T}} + d(\delta\vartheta \wedge \Pi^{\mathrm{T}}) \\ &- [\delta\omega^g, \vartheta] \wedge [\Pi^{\mathrm{T}}_{[\,]}, \vartheta] - \delta\omega^g \wedge D\Pi^{\mathrm{L}} + d(\delta\omega^g \wedge \Pi^{\mathrm{L}}),\end{aligned} \tag{4.4.4}$$

where the "contortional" 1-form $\Pi^{\mathrm{T}}_{[\,]}$, having values in the Lie algebra of the Lorentz group, is derived from the translational gauge field momentum implicitly by $\Pi^{\mathrm{T}} := [\Pi^{\mathrm{T}}_{[\,]}, \vartheta]$. This is in complete analogy to the relation (4.3.24) for the contortion. The exact forms occurring in (4.4.4) in general do not yield any contribution to the resulting field equations, since they generate only constant boundary terms within the action integral

$$S = \int_M (L_g + L_{\mathrm{mat}}) \tag{4.4.5}$$

of the gravitationally coupled system.

The application of Hamilton's principle to the independent variations of S for $\delta\vartheta$ and $\delta\omega^{\mathrm{g}}$ leads to the *Euler–Lagrange* equations

$$\boxed{\begin{aligned} &D\Pi^{\mathrm{T}} - E = \Sigma \\ &D\Pi^{\mathrm{L}} + \Pi^{\mathrm{T}}_{[\,]} = \tau_{\mathrm{s}} \end{aligned}} \tag{4.4.6}$$

(4.4.7)

for the gravitational field; cf. MIELKE (1982).

In terms of Ricci's classical tensor calculus, these equations were first derived by VON DER HEYDE (1976a, b). Later, they were analyzed in detail by HEHL (1980) particularly in the case of a quadratic Poincaré gauge field theory.

The sources on the right-hand side are formally defined by the first-order variations

$$\delta L_{\text{matter}} =: \delta\vartheta \wedge \Sigma + \delta\omega^{g} \wedge \tau_{s} + \delta_{\varphi} \wedge \lambda^{\varphi} \tag{4.4.8}$$

of the material Lagrangian n-form. The external sources that occur have to be identified with the 3-form Σ, the energy–momentum current, and the 3-form τ of the spin current of the matter fields, as was seen by WEYL (1929).

If the Euler–Lagrange equations $\lambda^{\varphi} = \text{o}$ of the matter fields are satisfied, it will turn out that Σ and τ_{s} are the field momenta of matter that are *canonically conjugate* to the translational and rotational degrees of freedom, respectively. In analogy to this, the 3-form E, canonically conjugate to the translations (compare (4.4.2)), will be interpreted as the intrinsic energy–momentum current of the gravitational gauge fields. Analogously, the term $\Pi^{\text{T}}_{[]}$ has to be regarded as the canonical spin current of the gravitational gauge fields. Compared to the Yang–Mills equations, both these additional terms constitute covariant self-interactions of the gauge fields that are on a par with the matter sources. Similar field equations arise in a gauge theory with an affine connection. Only the spin current τ_{s} would have to be replaced by the hypermomentum current Υ of the matter fields in such a generalization (HEHL & ŠIJAČKI 1980).

4.4.1 Bianchi Identities and their Contractions

The general field Eqs. (4.4.6) and (4.4.7) are complemented by the *Bianchi identities*

$$D\Theta \equiv [\vartheta, \Omega^{\text{L}}] \tag{4.4.9}$$

and

$$D\Omega^{g} \equiv 0, \tag{4.4.10}$$

which are relations of the same differential order. Considering not only the definition (4.2.22) of the torsion but also those of the curvature, the first identity can be obtained by calculating the exterior derivative of the torsion,

$$\begin{aligned} d\Theta &= dd\vartheta - d[\omega^{\text{L}}, \vartheta] \\ &= -[d\omega^{\text{L}}, \vartheta] + [\omega^{\text{L}}, d\vartheta] \\ &= -[\Omega^{\text{L}}, \vartheta] - [\omega^{\text{L}} \wedge \omega^{\text{L}}, \vartheta] + [\omega^{\text{L}}, \Theta] + [\omega^{\text{L}}, [\omega^{\text{L}}, \vartheta]] \\ &= [\omega^{\text{L}}, \Theta] + [\vartheta, \Omega^{\text{L}}]. \end{aligned} \tag{4.4.11}$$

This identity holds for a general linear connection. The restriction to a metric-compatible Lorentz connection and to the corresponding orthogonal 1-form ϑ is self-evident. The identity (4.4.10) likewise represents a special case of the general second Bianchi identity, which is valid for the curvature of any principal fiber bundle.

In addition, it can be stated that the conservation law for the current of the total angular momentum has its counterpart in the *contracted* form

$$D[\Theta, \vartheta] \equiv [\vartheta \wedge \vartheta, \Omega^g] \tag{4.4.12}$$

of the first Bianchi identity (4.4.9). In order to prove this, the definition of the torsion is used as well as the already mentioned fact that the commutator of a Lie-algebra-valued 2-form with itself vanishes. This yields

$$\begin{aligned} D[\Theta, \vartheta] &= [D\Theta, \vartheta] + [\Theta, D\vartheta] \\ &= [[\vartheta, \Omega^g], \vartheta] + [\Theta, \Theta] \\ &= \vartheta \wedge \Omega^g \wedge \vartheta - \Omega^g \wedge \vartheta \wedge \vartheta + \vartheta \wedge \vartheta \wedge \Omega^g - \vartheta \wedge \Omega^g \wedge \vartheta \\ &= [\vartheta \wedge \vartheta, \Omega^g]. \end{aligned} \tag{4.4.13}$$

Accordingly, the contraction

$$D(\vartheta \wedge {}^*\Omega^g) = \Theta \wedge {}^*\Omega^g \tag{4.4.14}$$

of the second Bianchi identity (4.4.10) is on a par with the conservation law. Compared with more general gauge models of gravity, it is a distinctive feature of the Einstein–Cartan theory that the conservation laws via the field equations intertwine completely with the contracted Bianchi identities.

A related structural redundancy is inherent in the two field Eqs. (4.4.6) and (4.4.7) of the PG theory, and this despite their being deduced independently from Hamilton's variational principles. In order to prove this, let us consider the "contortional" *antisymmetric part*

$$D\Pi^{\mathrm{T}}_{[\,]} - \mathrm{E}_{[\,]} = \Sigma_{[\,]} \tag{4.4.15}$$

of the first field equation (4.4.6). After covariant differentiation of the second field Eq. (4.4.7), exactly the same dynamical contribution of the translational momentum occurs in

$$D\,D\Pi^{\mathrm{L}} + D\Pi^{\mathrm{T}}_{[\,]} = D\tau_s. \tag{4.4.16}$$

According to the general rule, the twofold covariant derivative of the rotational field momentum Π^{L} can be converted into a commutator with the curvature 2-form Ω^g. Both field equations being valid, a comparison of (4.4.15) and (4.4.16) yields

$$[\Omega^g, \Pi^{\mathrm{L}}] + \mathrm{E}_{[\,]} = D\tau_s - \Sigma_{[\,]}. \tag{4.4.17}$$

Under the sensible assumption that the Euler–Lagrange equations of the gravitationally coupled matter system are satisfied, the second Noether identity forces both sides of (4.4.17) to vanish separately. This argument can also be seen the other way round: the algebraic relation

$$[\Omega^g, \Pi^L] + \mathrm{E}_{[]} = 0 \tag{4.4.18}$$

involving only the gauge fields and the corresponding field momenta ensures the validity of the conservation law.

4.5 Noether Identities

Within the canonical formalism, let us assume that the matter Lagrangian 4-form depends at most on the matter field Ψ and its first derivative:

$$L_{\mathrm{mat}} = L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi)\,. \tag{4.5.1}$$

Then the forms

$$\Sigma_\alpha := \frac{\partial L}{\partial \vartheta^\alpha}\,, \quad \sigma^{\alpha\beta} := 2\,\frac{\partial L}{\partial g_{\alpha\beta}}\,, \quad \text{and} \quad \tau_{\alpha\beta} := \frac{\partial L}{\partial \Gamma^{\alpha\beta}} = I_{\alpha\beta}\Psi \wedge \frac{\partial L}{\partial D\Psi} \tag{4.5.2}$$

are the canonical *energy–momentum*, the Hilbert stress-energy, and the *spin* currents, respectively. Here and in the following, the partial derivative with respect to the antisymmetric connection 1-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ is defined by $\delta L = \delta\Gamma^{\alpha\beta} \wedge (\partial L/\partial\Gamma^{\alpha\beta})$.

The action $S = \int_M L_{\mathrm{mat}}$ for the matter Lagrangian is, by construction, invariant under the group *Diff* (M) of coordinate transformations and local frame rotations. In order to obtain a *covariant Noether identity* from invariance of L under a one-parameter group of *local* translations $\mathscr{T} \subset$ *Diff* (M), we employ the $SO(1,3)$-covariant Lie derivative $\text{Ł}_\xi := \xi\rfloor D + D\xi\rfloor$ on M with respect to an arbitrary vector field ξ. Since $Dg_{\alpha\beta} = 0$, we obtain

$$\begin{aligned}
\text{Ł}_\xi L &= (\text{Ł}_\xi\, g_{\alpha\beta}) \wedge \frac{\partial L}{\partial g_{\alpha\beta}} + (\text{Ł}_\xi \vartheta^\alpha) \wedge \frac{\partial L}{\partial \vartheta^\alpha} \\
&\quad + (\text{Ł}_\xi \Psi) \wedge \frac{\partial L}{\partial \Psi} + (\text{Ł}_\xi D\Psi) \wedge \frac{\partial L}{\partial D\Psi} \\
&= D\Big[(\xi\rfloor\vartheta^\alpha) \wedge \frac{\partial L}{\partial \vartheta^\alpha} + (\xi\rfloor\Psi) \wedge \frac{\partial L}{\partial \Psi} + (\xi\rfloor D\Psi) \wedge \frac{\partial L}{\partial D\Psi}\Big] \\
&\quad - (\xi\rfloor\vartheta^\alpha) D\frac{\partial L}{\partial \vartheta^\alpha} + (\xi\rfloor T^\alpha) \wedge \frac{\partial L}{\partial \vartheta^\alpha} + (\xi\rfloor R_\beta{}^\gamma) \wedge I^\beta{}_\gamma \Psi \wedge \frac{\partial L}{\partial D\Psi} \\
&\quad + (\xi\rfloor D\Psi) \wedge \frac{\delta L}{\delta \Psi} + (-1)^p (\xi\rfloor\Psi) \wedge D\frac{\delta L}{\delta \Psi}\,.
\end{aligned} \tag{4.5.3}$$

Recall that $\xi\rfloor$, formally acting analogously to a derivative of degree -1, obeys the Leibniz rule. Since the Lagrangian L is a 4-form, its Lie derivative reduces to $Ł_\xi L = D\xi\rfloor L$. Comparing the boundary term, we can read off the identity

$$\xi\rfloor L \equiv (\xi\rfloor\vartheta^\alpha)\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\xi\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi}\,. \tag{4.5.4}$$

Incidentally, the left-hand side is just the Bessel–Hagen term occurring in the non-covariant Noether theorem; see HEHL et al. (1991).

If we replace $\xi \to e_\alpha$ by the basis frame, then (4.5.4) yields directly the explicit form of the *canonical energy–momentum current*

$$\Sigma_\alpha = e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi} - (e_\alpha\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi}\,. \tag{4.5.5}$$

The last term vanishes for a 0-form, as exemplified by the Dirac field. From the nondivergence part of (4.5.3) we can read off the *first Noether identity*

$$\begin{aligned} D\Sigma_\alpha &= (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R^{\beta\gamma})\wedge\tau_{\beta\gamma} + F_\alpha \\ &\simeq (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R^{\beta\gamma})\wedge\tau_{\beta\gamma}\,. \end{aligned} \tag{4.5.6}$$

Here, in the case of forms of arbitrary degree,

$$F_\alpha = (e_\alpha\rfloor D\Psi)\,\frac{\delta L}{\delta\Psi} + (-1)^p(e_\alpha\rfloor\Psi)\wedge D\frac{\delta L}{\delta\Psi} \tag{4.5.7}$$

is the Lorentz-type covector force four-form. This is, similarly as in Maxwell's theory, coupled to the electric current three-form j, where the Lorentz force reads in exterior calculus $F_\alpha^{\text{Max}} = (e_\alpha\rfloor F)\wedge j$.

Accordingly, our first line provides the Noether identity in the *strong* form, where no field equations are invoked. *Weak* identities, which are denoted by $\simeq$, hold only if the matter field equation $\delta L/\delta\Psi = 0$ is satisfied.

For the derivation of the Noether identity arising from Lorentz transformations, we apply the "internal variation" of the general Noether procedure:

$$\delta L = \delta g_{\alpha\beta}\wedge\frac{\partial L}{\partial g_{\alpha\beta}} + \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial\vartheta^\alpha} + \delta\Psi\wedge\frac{\partial L}{\partial\Psi} + \delta(D\Psi)\wedge\frac{\partial L}{\partial D\Psi}\,. \tag{8.10}$$

Since $\delta D\Psi = D\delta\Psi + \delta\Gamma_\alpha{}^\beta\wedge I^\alpha{}_\beta\Psi$, this is equivalent to

$$\begin{aligned} \delta L = {}& \delta g_{\alpha\beta}\wedge\frac{\partial L}{\partial g_{\alpha\beta}} + \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial\vartheta^\alpha} \\ & + \delta\Gamma_\alpha{}^\beta\wedge I^\alpha{}_\beta\Psi\wedge\frac{\partial L}{\partial D\Psi} + \delta\Psi\wedge\frac{\delta L}{\delta\Psi} + D\left(\delta\Psi\wedge\frac{\partial L}{\partial D\Psi}\right). \end{aligned} \tag{4.5.8}$$

Under an infinitesimal Lorentz rotation $\varepsilon_\alpha{}^\beta(x) := \Lambda_\alpha{}^\beta(x) - \delta_\alpha^\beta$ of the frames, we have

$$\delta g_{\alpha\beta} = 2\varepsilon_{(\alpha}{}^\gamma\, g_{\beta)\gamma}\,, \qquad \delta\vartheta^\alpha = -\vartheta^\beta\, \varepsilon_\beta{}^\alpha\,, \tag{4.5.9}$$

$$\delta\Gamma_\alpha{}^\beta = D\varepsilon_\alpha{}^\beta, \qquad \delta\Psi = -\varepsilon_\alpha{}^\beta I^\alpha{}_\beta \Psi\,. \tag{4.5.10}$$

Consequently, we get (HEHL et al. 1995)

$$\delta L = -\varepsilon_\alpha{}^\beta \left[-\sigma^{\alpha\gamma}\, g_{\gamma\beta} + \vartheta^\alpha \wedge \frac{\partial L}{\partial \vartheta^\beta} + D\frac{\partial L}{\partial \Gamma_\alpha{}^\beta} + I^\alpha{}_\beta \Psi \wedge \frac{\delta L}{\delta \Psi} \right]. \tag{4.5.11}$$

Thus, due to the antisymmetry of the RC-connection, the *second Noether identity* reads

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} = -I_{\alpha\beta}\Psi \wedge \frac{\delta L}{\delta \Psi} \simeq 0\,. \tag{4.5.12}$$

Again, we distinguish between strong and weak versions. In a first-order formalism, the potentials and field strengths, as well as the Bianchi and Noether identities, are summarized in the following Table:

potential	**field strength**	**Bianchi identity**	**Noether identity**
ϑ^α	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma$	$\overset{\frown}{D}\,\Sigma_\alpha \cong (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma$ $-\frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma}) \wedge \sigma^{\beta\gamma}$
$\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta$ $-\Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta$	$DR_\alpha{}^\beta \equiv 0$	$D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta \cong \sigma^\alpha{}_\beta$

For the first Noether identity, it is convenient to use the covariant exterior derivative $\overset{\frown}{D}$ with respect to the *transposed* connection $\overset{\frown}{\Gamma}{}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta$.

The physical meaning (HEHL et al. 1989) can be extracted by passing to an isolated matter system in special relativity (SR), stipulating that the Euler–Lagrange equation $\delta L/\delta\Psi = 0$ for the matter field Ψ is satisfied. Then torsion and curvature vanish. Accordingly, *global* (or rigid) Poincaré invariance of SR yields, due to Noether's theorem, the differential identities

$$D\Sigma_\alpha \simeq 0\,, \tag{4.5.13}$$

$$D(\tau_{\alpha\beta} + x_{[\alpha} \wedge \Sigma_{\beta]}) \simeq 0\,. \tag{4.5.14}$$

These four-form relations represent the 4 plus 6 *conservation laws* of energy–momentum and (total) angular momentum. The latter, the so-called TETRODE (1928) identity, consists of an intrinsic or *spin* part $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$ and an orbital part $x_{[\alpha} \wedge \Sigma_{\beta]}$, a fact that is familiar from the nontensorial expression in (4.5.14) if we use Cartesian coordinates x^i with $x^\alpha = \delta_i^\alpha\, x^i$.

Since local affine variations of the "soldering" form ϑ implement the action of the diffeomorphism group and conversely, the second Noether identity can also be deduced from the requirement that the matter Lagrangian be invariant with respect to the group $\mathscr{D}$(M) of differentiable coordinate transformations; see SCHWEITZER (1980).

This correspondence in flat spacetime can be considered the most convincing argument in favor of our identification of Σ and τ with the *canonical* currents, which represent cornerstones in any interpretation of the dynamics of matter fields. Further evidence for their importance is Wigner's successful mass–spin classification of elementary particles.

4.5.1 Mass and Spin of the Kerr–AdS Solution

In the anti-de Sitter-type gauge model of gravity proposed by MACDOWELL & MANSOURI (1977), (WISE 2010), see also PAGELS (1984), the translational part of the gauge algebra is *spontaneously broken*.

Recently (MIELKE 2001), the combined energy and spin complex

$$\varepsilon_{\rm RC} := \xi^\alpha\, \Sigma_\alpha + (e_\beta \rfloor\, \widehat{D}\, \xi^\gamma)\, \tau^\beta{}_\gamma\,, \tag{4.5.15}$$

of TRAUTMAN (1973) and HECHT et al. (1992) for Riemann–Cartan (RC) spacetime was generalized. Then in such a spontaneously broken AdS gauge model, the modified complex

$$\widetilde{\varepsilon}_{\rm MA} := \varepsilon_{\rm RC} + \xi \rfloor L \cong -\left(\mathscr{L}_\xi \Gamma_\beta{}^\gamma\right) \wedge H^\beta{}_\gamma + dH_\infty \tag{4.5.16}$$

leads, "on shell," to

$$\hat{M} := M^3/[M^2 + \frac{\Lambda}{3} J^2], \qquad \hat{J} := JM^2/[M^2 + \frac{\Lambda}{3} J^2], \tag{4.5.17}$$

i.e., to the renormalized mass and angular momentum of the Kerr–anti-de Sitter solution, if $\xi = \partial/\partial t$ or $\xi = \partial/\partial\varphi$, respectively, is chosen. Here no "factor of two" discrepancy in the Komar currents occurs.

In the first Casimir operator

$$C_{\rm AdS} = \hat{M}^2 + \frac{\Lambda}{3} \hat{J}^2 \tag{4.5.18}$$

of the AdS group, the cosmological constant features as *regularizer* of the gravitational "charges," which at the end can be put to zero. In the subcase of the Schwarzschild–AdS solution, it is shown that the Komar and Euler parts of the complex each contribute *one-half* of the mass, so that the divergent terms $\pm(\Lambda/3)r^3$ proportional to the cosmological constant Λ exactly cancel each other.

4.6 Gravitational Aharonov–Bohm Effect and Cartan Circuits

The *generalized affine connection*

$$\overset{\approx}{\Gamma} := \Gamma^{(T)\alpha} P_\alpha + \Gamma_\alpha^{(L)\beta} L^\alpha{}_\beta \tag{4.6.1}$$

includes the true translational potential $\Gamma^{(T)\alpha}$ and the $GL(n,\mathbb{R})$-gauge connection $\Gamma_\alpha^{(L)\beta}$, as can be deduced from the Möbius-type five-dimensional representation of the affine gauge group $A(n,\mathbb{R}) := \mathbb{R}^n \rtimes GL(n,\mathbb{R})$.

The Cartan transport may be understood rather directly from the affine point of view: the condition for a parallel transport of an affine vector $\overset{\approx}{\xi}$ around a small closed loop by means of $\overset{\approx}{\Gamma}$ reads

$$\begin{aligned} \overset{\approx}{D}\overset{\approx}{\xi}{}^\alpha &= d\xi^\alpha + \Gamma_\beta^{(L)\alpha} \wedge \xi^\beta + \Gamma^{(T)\alpha} \\ &= D\xi^\alpha + \Gamma^{(T)\alpha} = 0\,. \end{aligned} \tag{4.6.2}$$

Consequently, the parallel transport of (4.6.2) along an affine tangent vector of the Cartan circuit yields

$$\begin{aligned} \overset{\approx}{\text{Ł}}_{\overset{\approx}{y}} \overset{\approx}{D}\overset{\approx}{\xi}{}^\alpha &= \overset{\approx}{y}\rfloor\big(\overset{\approx}{D}\overset{\approx}{D}\overset{\approx}{\xi}{}^\alpha\big) + \overset{\approx}{D}\big(\overset{\approx}{y}\rfloor\overset{\approx}{D}\overset{\approx}{\xi}{}^\alpha\big) \\ &= y\rfloor(DD\xi^\alpha + R^{(T)\alpha}) = 0\,, \end{aligned} \tag{4.6.3}$$

where $\text{Ł}_y := y\rfloor D + Dy\rfloor$ denotes the gauge-covariant Lie derivative and $R^{(T)\alpha}$ the translational curvature. Integration of the first one-form in (4.6.3) along a closed loop parametrized by y yields

$$\begin{aligned} \Delta\xi^\alpha &= -\oint_C y\rfloor(DD\xi^\alpha) = \oint_C y\rfloor R^{(T)\alpha} = \int_S R^{(T)\alpha} \\ &\simeq \frac{1}{2}(T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha\,\xi^\beta) \int_S dy^i \wedge dy^j\,. \end{aligned} \tag{4.6.4}$$

This rather concise derivation via (4.2.26) yields the standard result.

This may also affect the issue of measurability of a connection. Quite generally, quantum interference measurements (Batelaan & Tonomura 2009) depend on the *nonintegrable phase factor*

$$\Phi(A, \gamma) = P \exp\big[(i/\hbar) \oint A\big], \tag{4.6.5}$$

where $A = A_i^J \lambda_J dx^i$ is a Yang–Mills-type connection and P is the principal value of the integral. If the loop γ lies in a field-free region, i.e., one with $U(1)$ field strength $F := dA = 0$ or Yang–Mills curvature $F = dA + A \wedge A = 0$, but encloses a “confined” region with nontrivial “magnetic” flux $F \neq 0$, the potential A can still be measured via the amount of phase shift for closed loops. In a nutshell, this is the meaning of the Aharonov–Bohm effect, actually first described by FRANZ (1939).

In principle, the same would hold true for a gravitationally induced phase factor

$$\Phi(\overset{\approx}{\Gamma}, \gamma) = P \exp\left[(i/\hbar) \oint (\Gamma^{(T)\alpha} P_\alpha + \Gamma_\alpha^{(L)\beta} L^\alpha{}_\beta)\right] \tag{4.6.6}$$

arising from a gauge theory of the affine structure group $A(4, \mathbb{R})$.

For a closed loop enclosing an infinitesimally small surface area S, the *total phase shift* induced by the generalized affine connection $\overset{\approx}{\Gamma}$ is given by

$$\Delta\Phi(\overset{\approx}{\Gamma}, \gamma) \simeq \frac{i}{\hbar} \int_S (R^{(T)\alpha} P_\alpha + R_\alpha^{(L)\beta} L^\alpha{}_\beta). \tag{4.6.7}$$

This total phase shift involves the same contribution from the translational curvature as in the result (4.6.4) obtained from the Cartan circuit.

For a so-called manifield Ψ carrying no $GL(4, \mathbb{R})$-excitations, i.e., no spin, no shear, and no dilation, we need a *closed loop* γ to detect the gravitational analogue of the Aharonov–Bohm effect in a conical space, since outside the (rounded) apex of the cone there is $\Gamma^{(T)\alpha} \overset{*}{=} 0$ locally.

This analogy to Yang–Mills theory would break down, however, if we considered, instead of the true translational potential $\Gamma^{(T)\alpha}$, the coframe ϑ^α soldered to the spacetime manifold; cf. ANANDAN (1993, 1994). Because the coframe is nondegenerate by definition, it could be measured even by a nonclosed loop, showing its essentially classical character.

Since the physical dimension of P_α is $2\pi\hbar/\ell$, the gravitational analogue of Dirac’s quantization condition would be

$$\Phi(\overset{\approx}{\Gamma}, \gamma) = (2\pi\hbar \mathscr{M}_n G/\hbar\ell c^2) = 2\pi n, \tag{4.6.8}$$

i.e., the mass would turn out to be a multiple $\mathscr{M}_n = n M_{\text{Planck}}$ of the (huge) Planck mass.

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Chapter 5
Einstein–Cartan Theory

5.1 Introduction

The difference between Einstein's general relativity and its Cartan extension is analyzed classically as well as within the scenario of *asymptotic safety* of quantum gravity. In particular, we focus on the four-fermion interaction, which distinguishes the Einstein–Cartan theory from its Riemannian limit.

In coupling gravity to Dirac-type spinor fields (WEYL 1929b), it is at times surmised that the Einstein–Cartan (EC) theory (TRAUTMAN 1973a) is superior to standard general relativity (GR), inasmuch as the involved torsion tensor of CARTAN (1924) can accommodate the spin of fundamental fermions of electrons and quarks in gravity.

Classically, however, the effects of spin and torsion cannot be detected by Lageos or Gravity Probe B (WILL 2011) and would be significant only at densities of matter that are very high, but nevertheless smaller than the Planck density, at which quantum-gravitational effects are believed to dominate. It was even claimed (TRAUTMAN 1973b) that EC theory may avert the problem of singularities in cosmology, but for a coupling to Dirac fields the opposite happens (O'CONNELL 1976b).

The Riemann–Cartan (RC) connection $\Gamma^{\alpha\beta} = \Gamma^{\{\}\alpha\beta} - K^{\alpha\beta}$, a one-form, can be split into the unique Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$ of Riemannian geometry and a *contortion* one-form $K_{\alpha\beta} = -K_{\beta\alpha}$, a tensor-valued one-form $K = iK^{\alpha\beta}\sigma_{\alpha\beta}/4$, implicitly related to torsion via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. Recently, it has been stressed by WEINBERG (2005) that this RC connection $\Gamma = \Gamma^{\{\}} - K$ is just a *deformation* (or field redefinition) of the Christoffel connection $\Gamma^{\{\}}$ by the contortion, at least from the (quantum) field-theoretic point of view. Although algebraically complying with the review of HEHL et al. (1995), this argument has been refuted (BLAGOJEVIĆ & HEHL 2013) on the basis of the geometric interpretation (WISE 2010; WESTMAN & ZLOSNIK 2013) of Cartan's torsion within models of (singular) classical defects. Nevertheless, for an agreement with experiments (PROVILLE et al. 2012), a *quantization* of crystal vibrational modes appears necessary.

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It is well known (Hehl & Datta 1971; Mielke & Romero 2006) that EC theory coupled to the Dirac field is effectively GR with an additional *four-fermion* (FF) interaction. However, such *contact* interactions are likewise perturbatively non-renormalizable in $D > 2$ without Chern–Simons (CS) terms (Alves et al. 1999), which was one of the reasons for giving up Fermi's original theory of beta decay.

Since GR with a cosmological constant Λ appears to be asymptotically safe, in the scenario (Reuter & Saueressig 2012) first devised by Weinberg (1979), one may ask (Mielke 2015) what the situation is in EC theory, where Cartan's algebraic equation relates torsion to spin, i.e., to the axial current j_5 in the case of Dirac fields, on dimensional grounds coupled with gravitational strength.

5.2 Dirac Fields in Riemann–Cartan Spacetime

Let us recall that a Dirac field is a bispinor-valued zero-form ψ, for which $\overline{\psi} := \psi^\dagger \gamma_0$ denotes the Dirac adjoint and $D\psi := d\psi + \Gamma \wedge \psi$ is the exterior covariant derivative with respect to the RC connection one-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$, providing a minimal gravitational coupling.

In the manifestly *Hermitian* formulation, the Dirac Lagrangian is given by the four-form

$$L_{\rm D} = L(\gamma, \psi, D\psi) = \frac{i}{2}\left\{\overline{\psi}\,{}^*\gamma \wedge D\psi + \overline{D\psi} \wedge {}^*\gamma\,\psi\right\} - m\,\overline{\psi}\psi\eta\,, \tag{5.2.1}$$

where $\gamma := \gamma_\alpha \vartheta^\alpha$ is the Clifford-algebra-valued coframe satisfying $D\gamma = [\gamma, K] = \gamma_\alpha T^\alpha$, and $T^\alpha := D\vartheta^\alpha$ is the torsion two-form.

Since $L_{\rm D} = \overline{L}_{\rm D} = L_{\rm D}^\dagger$ even in an unholonomic frame, the minimal coupling provides us automatically with the *Hermitian* charge current and standard axial current three-forms

$$j := \overline{\psi}\,{}^*\gamma\psi = j^\mu\,\eta_\mu \quad \text{and} \quad j_5 := \overline{\psi}\,{}^*\gamma\gamma_5\psi = j_5^\mu\,\eta_\mu\,, \tag{5.2.2}$$

respectively, which are familiar from *quantum electrodynamics* (QED) in curved spacetime.

Let us now separate in (5.2.1) the purely Riemannian part from spin–contortion pieces:

$$\begin{aligned} L_{\rm D} &= L(\gamma, \psi, D^{\{\}}\psi) - \frac{i}{2}\overline{\psi}\left({}^*\gamma \wedge K - K \wedge {}^*\gamma\right)\psi \\ &= L(\gamma, \psi, D^{\{\}}\psi) + \frac{1}{4}\mathscr{A} \wedge j_5\,. \end{aligned} \tag{5.2.3}$$

Hence, in an RC spacetime, a massive Dirac spinor feels only the *axial torsion* one-form

$$\mathscr{A} := \frac{1}{4}\,{}^*Tr(\gamma \wedge D\gamma) = {}^*(\vartheta^\alpha \wedge T_\alpha) = \frac{1}{2}T^{[\alpha\beta\gamma]}\eta_{\alpha\beta\gamma} = \mathscr{A}_i\, dx^i\,, \tag{5.2.4}$$

which can be expressed in various equivalent forms.

The *spin current* of the Dirac field is given by the Hermitian three-form

$$\tau_{\alpha\beta} := \frac{\partial L_{\rm D}}{\partial \Gamma^{\alpha\beta}} = \frac{1}{8}\overline{\Psi}\left({}^*\gamma\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\,{}^*\gamma\right)\Psi = \frac{1}{4}\,\eta_{\alpha\beta\gamma\delta}\,\overline{\Psi}\gamma^\delta\gamma_5\Psi\eta^\gamma = \tau_{\alpha\beta\gamma}\eta^\gamma \tag{5.2.5}$$

with *totally antisymmetric* components $\tau_{\alpha\beta\gamma} = \tau_{[\alpha\beta\gamma]}$. Eventually, torsion merely couples to the *spin–energy potential* $\mu_\alpha = \vartheta_\alpha \wedge {}^*j_5/4$, i.e., to a two-form proportional to the axial current j_5 of the Dirac field.

Transitions from EC theory or GR to $\mathrm{GR}_\parallel$ can be *generated* via the Chern–Simons-type term $C_{\rm TT^*} := \vartheta^\alpha \wedge {}^*T_\alpha$, involving the Hodge dual of the torsion. On the material side, a related change of the Dirac Lagrangian can be induced via $L_{\rm D} \to L_{\rm D} + dU$, where $U = \vartheta^\alpha \wedge \mu_\alpha$ features as a superpotential. Then the corresponding boundary term

$$dU = T^\alpha \wedge \mu_\alpha - \vartheta^\alpha \wedge D\mu_\alpha = \vartheta^\alpha \wedge [e_\beta\rfloor(T^\beta \wedge \mu_\alpha) - D\mu_\alpha] \tag{5.2.6}$$

compensates the torsion coupling in (5.2.3) and thereby induces the *relocalization*

$$\begin{aligned} \Sigma_\alpha \to \sigma_\alpha &:= \Sigma_\alpha - D\mu_\alpha + e_\beta\rfloor(T^\beta \wedge \mu_\alpha)\,, \\ \tau_{\alpha\beta} \to \hat{\tau}_{\alpha\beta} &:= \tau_{\alpha\beta} - \vartheta_{[\alpha} \wedge \mu_{\beta]} = 0 \end{aligned} \tag{5.2.7}$$

of the fermionic Noether currents, resembling the *Belinfante–Rosenfeld*-type expressions (R1) and (R2) of Mielke et al. (1989), such that the relocalized spin $\hat{\tau}_{\alpha\beta}$ vanishes. Observe that for Dirac spinors, $U = \vartheta^\alpha \wedge \vartheta_\alpha \wedge {}^*j_5/4$ is trivial “on shell.”

5.3 Classical Einstein–Cartan Theory

Already Eddington (1924) considered the possibility of an asymmetric connection, but it was Cartan (1922) who envisioned a possible significance of torsion for the gravitational coupling to matter with spin. After the discovery of the spin of the electron, Weyl (1929a, b, 1950) accomplished the coupling of the Dirac field to the so-called EC theory by means of gauge-theoretic concepts, resulting in the specification

$$L_{\rm EC} := -\frac{1}{2\kappa}R^{\alpha\beta} \wedge \eta_{\alpha\beta}, \tag{5.3.1}$$

of the Lagrangian 4-form of the gravitational field. It depends on the curvature tensor and the orthogonal tetrads, whereas $\kappa = 8\pi G_{\rm N}$ is the gravitational constant in natural units.

This theory was enlarged by KIBBLE (1961) and SCIAMA (1962) and later by HEHL (1970). (It is for this reason that the EC theory is also referred to as the Einstein–Cartan–Sciama–Kibble theory (KERLICK 1975; NESTER & ISENBERG 1977); however, for the sake of convenience we stick to the abbreviation.) Subsequently, it was rendered in the more elegant calculus of differential forms by TRAUTMAN (1972, 1973a). A detailed survey incorporating a list of further reading is to be found in the well-known work of HEHL et al. (1976).

The Einstein–Cartan (EC) equation (TRAUTMAN 1973a)

$$G_\alpha := \frac{1}{2} R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \kappa\, \Sigma_\alpha\,, \tag{5.3.2}$$

coupled to the canonical energy–momentum current Σ_α of matter, is obtained by varying the matter coupled Lagrangian $L_{\rm EC}$ with respect to the coframe ϑ^α.

It satisfies the contracted Bianchi identity

$$\widehat{D} G_\alpha \equiv \frac{1}{2} (e_\alpha \rfloor R^{\beta\gamma}) \wedge \eta_{\beta\gamma\mu} \wedge T^\mu \tag{5.3.3}$$

with respect to the transposed connection $\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta$. Observe that the three-form (5.3.2) is not covariantly conserved in RC spacetime, which seems to have discouraged Cartan from pursuing such a model further; cf. TRAUTMAN (2006). Only for vanishing torsion does it reduce to the contracted second Bianchi identity

$$D^{\{\}} G_\alpha^{\{\}} \equiv 0 \tag{5.3.4}$$

familiar from GR.

When molded into "Clifforms," *torsion* and *curvature* become Clifford-algebra-valued forms:

$$\Theta := D\gamma \quad \text{and} \quad \Omega^g := d\Gamma + \Gamma \wedge \Gamma. \tag{5.3.5}$$

Consequently, the first and second *Bianchi identities* assume in RC geometry the concise form

$$D\Theta \equiv [\Omega^g, \gamma]\,, \qquad\qquad D\Omega^g \equiv 0\,, \tag{5.3.6}$$

respectively, which involve p-form commutators; cf. MIELKE (2001) for details. Then the *Einstein tensor* can be rewritten as the three-form

$$G := G_\alpha \gamma^\alpha := \frac{1}{2} R^{\mu\nu} \wedge \eta_{\mu\nu\lambda} \gamma^\lambda = -i\, \gamma_5 (\Omega^g \wedge \gamma + \gamma \wedge \Omega^g) = i[\gamma, \gamma_5 \Omega^g]\,. \tag{5.3.7}$$

In view of the *contracted* Bianchi identities

$$D[\gamma\,, \Theta] \equiv 2i[\sigma\,, \Omega^g]\,, \qquad D[\gamma\,, \Omega^g] \equiv [\Theta\,, \Omega^g]\,, \tag{5.3.8}$$

involving the unit-curvature two-form $\sigma := \frac{1}{2}\sigma_{\alpha\beta}\,\vartheta^{\alpha} \wedge \vartheta^{\beta} = \frac{i}{2}\,\gamma \wedge \gamma$, the equivalent Clifform version

$$DG \equiv i[\Theta, \gamma_5 \Omega^g] \tag{5.3.9}$$

of the identity (5.3.3) arises.

Consequently, the automatic conservation of the Einstein three-form holds only for vanishing torsion, i.e., for $\Theta = 0$ in Einstein's standard GR.

By varying (5.3.1) with respect to the linear connection $\Gamma^{\alpha\beta}$, we obtain the second field equation of EC theory, i.e., Cartan's algebraic relation

$$\eta_{\alpha\beta\gamma} \wedge T^{\gamma} = 2\kappa\,\tau_{\alpha\beta} \tag{5.3.10}$$

between torsion and the canonical spin of matter.

Simple supergravity (SUGRA) is essentially equivalent to EC theory coupled to a massless Rarita–Schwinger field Ψ, a spinor-valued one-form; cf. MIELKE & MACIAS (1999) for details.

5.3.1 Effective Einstein Equations

Similarly to the subsequent decomposition of L_{EC}, the EC equations can be decomposed into Riemannian and torsion-dependent parts. Applying again the Cartan equation, an Einstein-type field equation in a Riemannian spacetime comes into existence in which Hilbert's definition

$$\mathrm{T}_{\mu\nu} := \frac{2}{\sqrt{g}}\frac{\delta L}{\delta g_{\mu\nu}} \tag{5.3.11}$$

of a metric energy–momentum tensor is replaced by the "combined" tensor

$$\begin{aligned}\widetilde{T}_{\mu\nu} = T_{\mu\nu} + \ell^{*2}\Big\{&-4\tau_{\mu\cdot[\lambda|}^{\cdot\kappa}\,\tau_{\nu\cdot|\kappa]}^{\cdot\lambda} - 2\tau_{\mu}^{\cdot\kappa\lambda}\tau_{\nu\kappa\lambda} + \tau_{\cdot\cdot\mu}^{\kappa\lambda}\tau_{\kappa\lambda\nu}\\ &+ \frac{1}{2}g_{\mu\nu}(4\tau_{\delta\cdot[\lambda}^{\cdot\kappa}\,\tau_{\cdot\cdot\kappa]}^{\delta\lambda} + \tau^{\delta\kappa\lambda}\tau_{\delta\kappa\lambda})\Big\}.\end{aligned} \tag{5.3.12}$$

Cartan's equation has already been inserted in this expression in order to show that $\widetilde{\mathrm{T}}_{\mu\nu}$ contains additional terms depending quadratically on the spin angular momentum of matter; cf. HEHL et al. (1976). In the instance that $\tau_{\mathrm{s}} = \mathrm{o}$, the EC equation reduces exactly to Einstein's field equation of GR.

These field equations and, indirectly, the dissociation of the Lagrangian 4-form reveal that the additional non-Riemannian terms sacrifice minimal coupling because they induce a *spin–spin contact interaction* into the theory. On account of this additional interaction, the EC theory is effectively equivalent to GR in the pseudo-Riemannian space if outfitted with a nonminimally coupled source. The additional terms in (5.3.12) are proportional to ℓ^{*4} and could thus, with a modified Planck length $\ell^* = 10^{-33}$ cm, play a role on the macroscopic scale only in cosmology; cf., for instance, the early survey of KUCHOWICZ (1975a, b, c). The rather idealized situation of a dust-filled universe, whose corpuscles would carry aligned spins, would not be threatened by a final singularity, since the gravitational collapse of 10^{80} baryons would bounce, according to TRAUTMAN (1973b) and HEHL et al. (1974), at a minimal radius of about 1 cm.

However, for the physically much more justified description of spinning matter via Dirac fields, the collapse predicted by the singularity theorems of Hawking and Penrose (HAWKING & ELLIS 1973) is more likely to be accelerated than deferred in the EC theory. The reason for this, as will be seen in the following, is the total antisymmetry of the canonical angular momentum tensor of spinor fields. This leads to a weak but *attractive* spin–spin contact interaction as first pointed out, on the Lagrangian level, by O'CONNEL (1976a, 1977).

To be more specific, let us decompose the RC curvature two-form

$$R^{\alpha\beta} = R^{\{\}\alpha\beta} - D^{\{\}} K^{\alpha\beta} - K^{\alpha}{}_{\mu} \wedge K^{\mu\beta} \tag{5.3.13}$$

into the Riemannian curvature $R^{\{\}\alpha\beta}$ plus contortion pieces.

Then the *geometric identity*

$$\begin{aligned} R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} \; &\equiv R^{\alpha\beta} \wedge \eta_{\alpha\beta} - K^{\alpha\mu} \wedge K_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta} + K^{\alpha\beta} \wedge T^{\gamma} \wedge \eta_{\alpha\beta\gamma} \\ &\quad + d(K^{\alpha\beta} \wedge \eta_{\alpha\beta}) \\ &= R^{\alpha\beta} \wedge \eta_{\alpha\beta} + T^{\alpha} \wedge {}^*\!\left(- {}^{(1)}T_{\alpha} + 2\, {}^{(2)}T_{\alpha} + \frac{1}{2}\, {}^{(3)}T_{\alpha}\right) + 2dC_{\mathrm{TT}^*} \end{aligned} \tag{5.3.14}$$

relates the Hilbert–Einstein Lagrangian to the EC Lagrangian, and to proper teleparallelism as a consequence. Here ${}^{(i)}T_{\alpha}$ are the three irreducible pieces of the torsion. In particular,

$${}^{(3)}T_{\alpha} = \frac{(-1)^s}{3}\, {}^*\,(\vartheta_{\alpha} \wedge \mathscr{A}) \tag{5.3.15}$$

is the irreducible axial torsion two-form algebraically related to the *axial torsion* one-form (5.2.4).

When only this axial torsion $\mathscr{A}$ enters algebraically, the EC Lagrangian

$$L_{\mathrm{EC}} := -\frac{1}{2\kappa} R^{\alpha\beta} \wedge \eta_{\alpha\beta} = L_{\mathrm{HE}} + \frac{1}{12\kappa} \mathscr{A} \wedge {}^*\mathscr{A} \tag{5.3.16}$$

generalizes[1] the metric Hilbert–Einstein Lagrangian L_{HE} to an RC spacetime with torsion.

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[1]Adding torsion squared terms (DAUM & REUTER 2012, 2013) is not an unambiguous procedure, since the particular combination $T^{\alpha} \wedge {}^*\!\left({}^{(1)}T_{\alpha} - 2\,{}^{(2)}T_{\alpha} - \frac{1}{2}\,{}^{(3)}T_{\alpha}\right)$ of irreducible pieces is related to a nontopological boundary term derived from the dual CS term $C_{\mathrm{TT}^*} := \vartheta^{\alpha} \wedge {}^*T_{\alpha}$. In the space of gravity theories, the term dC_{TT^*} interrelates GR with its teleparallelism equivalent

Likewise, the EC three-form

$$G_\alpha := \frac{1}{2} R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \tag{5.3.17}$$
$$= G_\alpha^{\{\}} + \frac{(-1)^s}{12}\left(e_\alpha \rfloor \mathscr{A} \wedge {}^*\mathscr{A} - \frac{1}{3}\mathscr{A} \wedge e_\alpha \rfloor {}^*\mathscr{A} \right) + \frac{(-1)^s}{6} \vartheta_\alpha \wedge d\mathscr{A}$$

can be decomposed into the Einstein three-form $G_\alpha^{\{\}} = G_\alpha{}^\beta \eta_\beta$ with respect to the Riemannian connection $\Gamma^{\{\}}$ and additional axial torsion pieces (Hehl & Datta 1971; Mielke & Romero 2006).

Due to (5.2.5), in the case of Dirac fields, the Cartan equation (5.3.10) is equivalent to

$$^*\mathscr{A} = \frac{\kappa}{2} j_5 \,, \tag{5.3.18}$$

coupled via the "bare" fundamental length $\ell = \sqrt{\kappa}$. Then "on shell," EC theory coupled to Dirac spinors deviates from GR merely via

$$\Delta\widetilde{L} := \kappa(L_{\text{EC}} - L_{\text{HE}}) \simeq \frac{\kappa^2}{48} j_5 \wedge {}^* j_5 = 4 f^2 j_5 \wedge {}^* j_5 \,, \tag{5.3.19}$$

i.e., by a four-fermion interaction.

5.4 Asymptotic Safety of EC Theory?

Particle physics is based on the Yang–Mills theory, which in the standard model is a *renormalizable* and *asymptotic-free* quantum field theory (QFT). For the weak interactions, as shown by Veltman and 't Hooft (Veltman 2000), an important part is played by the Higgs mechanism, which provides a scalar ghost that cancels divergencies in the propagators of the gauge bosons. Nowadays, these cancellations can be more easily understood as a result of a global BRST symmetry ('t Hooft 2007) that to some extent allows extensions to gravity (Mielke 2008).

Since GR is perturbatively nonrenormalizable, one recurs to *asymptotic safety* (AS). It amounts to the requirement that dimensionless coupling constants remain bounded in the ultraviolet limit $k \to \infty$. Quite generally, the renormalization flow is controlled by an exact functional identity, the Wetterich equation (Wetterich 1993) for the effective action Γ_k, i.e.,

$$k\partial_k \Gamma_k = \frac{1}{2}\text{STr}\left\{ [\Gamma_k^{(2)} + R_k]^{-1} (k\partial_k R_k) \right\}, \tag{5.4.1}$$

(Footnote 1 continued)
(Mielke 1992). Exactly, the above teleparallel "nucleus" leaves its traces in the controversies (Hecht et al. 1996; Ho & Nester 2011) about the well-posedness of the classical Cauchy problem and the particle content of the (broken) Poincaré gauge theory.

where $k\partial_k$ is a scale derivative returning the Euler degree of homogeneous functionals.

For the Hilbert–Einstein Lagrangian

$$L_{\mathrm{HE}} := L_{\mathrm{HE}} + \frac{\Lambda}{\kappa}\,\eta \tag{5.4.2}$$

with cosmological term, the *dimensionless running* coupling constants can be defined by

$$g_{\mathrm{N}} := \kappa k^2\,, \qquad \lambda := \Lambda/k^2\,, \tag{5.4.3}$$

where k is the renormalization group (RG) scale in momentum space and Λ the cosmological constant related to dark energy (DE) of density ρ_Λ; see also BJORKEN (2013). Then, in the case of gravity in 4D, the *renormalization group equations* are

$$k\frac{\partial}{\partial k}g_{\mathrm{N}} = \beta_1(g_{\mathrm{N}},\lambda) = (2+d_{\mathrm{N}})g_{\mathrm{N}}\,, \qquad k\frac{\partial}{\partial k}\lambda = \beta_2(g_{\mathrm{N}},\lambda)\,, \tag{5.4.4}$$

where d_{N} is the anomalous dimension of the running Newton coupling g_{N}.

According to the AS scenario (REUTER & SAUERESSIG 2012), the coupling constants (5.4.3) run into some nontrivial fixed points $g_{\mathrm{N}*}$ and λ_*, depending on the specific truncation of the effective Lagrangian to the celebrated Hilbert–Einstein Lagrangian (5.4.2) without torsion. This can be extended (FALLS et al. 2016) to high-order polynomials R^n of the Ricci scalar, similarly as in the classically bifurcating $f(R)$ models (SCHUNCK et al. 2005), but then the issue of physical ghosts or nonunitarity arises, familiar (KUHFUSS & NITSCH 1986; LEE & NE'EMAN 1990) from Stelle-type higher-derivative models.

Quite generally, the dimensionless product

$$\mu = \frac{4}{3}\kappa\Lambda = \frac{1}{3}(2\kappa)^2\rho_\Lambda \le \frac{4}{3}g_{\mathrm{N}*}\lambda_* \simeq 0.2 \tag{5.4.5}$$

appears to have a rather robust and universal bound independent of the particular truncation.

A topological field theory, however, is empirically much closer to a Gaussian fixed point when an SSB is assumed. Thus, gravity does *not* appear so nonrenormalizable; cf. KREIMER (2008). This could also affect cosmological constraints (KISELEV & TIMOFEEV 2011) on the mass of the standard Higgs boson. The AS scenario yields a rather realistic estimate (SHAPOSHNIKOV & WETTERICH 2010; LITIM et al. 2012) of the Higgs mass, when the mass of the tau lepton is used as input, and it provides an exactly scale-invariant (CONTILLO et al. 2012) scalar power spectrum in inflationary cosmology.

5.4.1 *The Issue of Four-Fermion Interactions*

Interestingly enough, the EC-induced *four-fermion* (FF) interaction (5.3.19) with its tiny "bare" coupling constant

$$f^2 = \frac{1}{192}\kappa^2 = 3 \times 2^{-10}\left(\frac{\mu}{\Lambda}\right)^2 = 2^{-8}\frac{\mu}{\rho_\Lambda} \tag{5.4.6}$$

also scales with the gravitational constant κ, but inversely compared to the Hilbert–Einstein and cosmological terms.

If the renormalization flow starts to the right from a non-Gaussian fixed point, the coupling actually diverges (EICHHORN & GIES 2011) at a finite RG scale. When the contact- or point-like truncation breaks down, a boson-like description of fermion bilinears is mandatory, including the $1/k^4$ dependence in the functional integral. Then, the FF interaction becomes nonlocal (SHAPOSHNIKOV & WETTERICH 2010), and the corresponding dimensionless renormalized running coupling f_*^2 becomes asymptotic safe or even free. In a nonlinear σ model (BAZZOCCHI et al. 2011), nonrenormalizable FF interactions may be instrumental for restoring asymptotic safety.

In view of these problems, the EC theory has been amended (DAUM & REUTER 2012; BENEDETTI & SPEZIALE 2011) by the pseudocurvature scalar term of HOJMAN et al. (1980) (the infamous "Holst" term, cf. MIELKE 2009), or even nonminimally coupled Dirac fields (OBUKHOV & HEHL 2012). Unfortunately, in many of these extensions (POPŁAWSKI 2012, KHRIPLOVICH 2012, KHRIPLOVICH & RUDENKO 2012), a possible running of the gravitational couplings should not be ignored.

Moreover, in QFT, the axial current is not conserved; rather, there arises in RC spacetime the *axial anomaly*

$$\langle dj_5 \rangle = 2im\langle \overline{\psi}\gamma_5\psi\rangle\eta - \frac{1}{96\pi^2}\left[2R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + d\mathscr{A} \wedge d\mathscr{A}\right] \tag{5.4.7}$$

for its vacuum expectation value, which involves the topological Pontryagin term quadratic in the curvature.

One way to avoid such anomalies is to employ curvature constraints like $R_{\alpha\beta} \equiv 0$, typical for teleparallel models (MIELKE 2002). Another approach, inspired by the BF schemes (MIELKE 2012, 2013) of topological quantum field theory (TQFT), is to start from a "minimalist" $SL(5,\mathbb{R})$ gauge model that includes only a "bare" Pontryagin-type four-form as its own counterterm. Then a tiny symmetry-breaking would be mandatory in order to recover the classical metric background of GR.

So far, the search for a theory of quantum gravity (QG) that is free of anomalies and leaves Einstein's GR as a well-established macroscopic "nucleus" has produced rather contradictory partial results, to some extent resembling a "Babylonian confusion," according to NICOLAI (2014).

5.5 Constraints from the Weak Equivalence Principle

In a constraint-type gauge approach, the *weak equivalence principle* (WEP) is, after SSB, anchored in the covariant constancy of the ground state (MIELKE 2011). Thus, cornerstones in Einstein's foundation of general relativity (GR) may surface here as "Mach"-type features of the gravitational Higgs vacuum, resembling (BRANS 1999, NE'EMAN 2006, KAISER 2007) a new "ether," which, however, remains locally Lorentz-invariant, nowadays measured at the unprecedented $\Delta c/c \simeq 10^{-17}$ level (HERRMANN et al. 2009).

In the laboratory, the WEP can be corroborated with a torsion balance (WAGNER et al. 2012) at the 10^{-13} level. Moreover, the upper bound for a violation of the WEP in the proton–antiproton system, taking into account the binding energy of the quarks, is about 10^{-8}; cf. HUGHES (1993).

A neutrino pulse from the supernova SN 1987A lasting for about ten seconds was detected by Kamiokande II and the IMB detectors in Japan and the USA, respectively. Assuming that not only the dominant delayed antineutrinos $\overline{\nu}_e$ but at least one prompt neutrino ν_e was in the pulse, one could infer that the equivalence principle is valid (GUZZO et al. 2002) for electron–neutrinos ν_e up to 10^{-6}.

Moreover, a test of the CPT theorem for the neutral kaon system indirectly yielded an upper bound for the relative difference of the *inertial masses* of the K^0 meson and its antiparticle $\overline{K}^0$ of less than 10^{-18}, when employing (MURAYAMA 2004) new bounds from neutrino oscillations. In the proposed STEP satellite experiment (OVERDUIN et al. 2009), the WEP would be tested macroscopically to an unprecedented accuracy.

Observationally, GR is rather well established for the solar system (WILL 2006) and, more recently, in double pulsars (KRAMER & WEX 2009; SHAO 2014) as well as via large-scale gravitational lensing (REYES et al. 2010) of galaxy clustering.

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Chapter 6
Teleparallelism

According to FEYNMAN (1962/63), "gravity is that field which corresponds to a gauge invariance with respect to displacement transformations." Taking this literally would favor Einstein's teleparallelism equivalent of GR, which has been recast (MIELKE 1992; MIELKE et al. 1996) into a Yang–Mills-type *gauge theory of translations*. On the other hand, Ashtekar's reformulation ASHTEKAR (1986, 1991) of general relativity (GR) is a complex reformulation that essentially projects out the right and left helicity modes of the graviton.

6.1 Chiral Teleparallelism

Originally, these complex variables were developed in the Hamiltonian approach. In the equivalent Lagrangian formulation, this change of variables is more simply induced by a *generating function* (DOLAN 1989; MIELKE 1990) that involves a boundary term $dC_{\rm TT}$ constructed from a *translational Chern–Simons term* (CS) multiplied by the imaginary unit. This is facilitated in the framework of Riemann–Cartan (RC) geometry, where the coframe ϑ^α surfaces as a soldered *translational* gauge potential (MIELKE et al. 1993; TRESGUERRES & MIELKE 2000) having $T^\alpha := D\vartheta^\alpha$ as its corresponding translational field strength, i.e., torsion.

Via a sort of duality rotation, CS-type boundary terms dC transform into a viable gravitational Lagrangian: the standard Hilbert action of general relativity (GR), or, as was already suggested by EINSTEIN (1928), a theory with *teleparallelism* ($\mathrm{GR}_{\|}$). Recently, it was shown (MIELKE 1992) how a complex Yang–Mills-type version of $\mathrm{GR}_{\|}$ arises by a canonical transformation induced via $dC_{\rm TT}$. Moreover, the naturalness of Sparling's energy complex in this approach has been pointed out; cf. DUBOIS-VIOLETTE & MADORE (1987).

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Since the resulting chiral version of gravity is surprisingly close to Yang–Mills, we can adopt a related Hamiltonian formulation in which the Poisson brackets for the constraints are on a par with the local Poincaré algebra. Interestingly enough, the corresponding Gauss constraint of spatial diffeomorphisms is annihilated, on the operator level, by a state vector that is proportional to the exponentiated translational CS term, thus improving on MIELKE (1998, 1999). Classical configurations dominating this action are *torsion instantons*, thus hinting at another close formal parallelism of internal Yang–Mills with chiral $GR_{\|}$. Moreover, this similarity with quantum gauge theories paves the way to a Becchi–Rouet–Stora–Tyutin (BRST) quantization of gravity based on *translational* ghost operators. The correlation function for Wilson loops is known (GUADAGNINI et al. 1990) to yield, to first order, the Gauss self-linking number. For the full expectation value, the generalized Jones polynomial related to the Kauffman bracket of knots (BRÜGMANN et al. 1992) arises once a *framing* has been chosen. In the case of *chiral teleparallelism*, the loops automatically carry an orthonormal frame along their path, a concept that was realized much earlier by CARTAN (1924). Thus such *Cartan circuits* winding around torsion instantons may provide another clue to a field quantization of gravity based on chiral $GR_{\|}$.

One may speculate that teleparallel ("flat") spacetime may reveal, on the Planck scale $\ell := \sqrt{8\pi\hbar G_{\rm N}/c^3} \simeq 10^{-33}$ cm, a topologically rich spectrum of dislocations and knotlike loops.

6.2 Parity-Violating Topological Invariants in Gravity

In the one-dimensional harmonic oscillator model (MIELKE 1998) with q as generalized coordinate, a *canonical transformation* can be induced by a boundary term derived from the *Chern–Simons* (CS)-type term $\mathscr{C} = q^2/2$ as *generating function*. After quantization, the corresponding operator $\mathscr{S} = \exp(-\mathscr{C}) = \exp\left(-q^2/2\right)$ induces a well-known renormalization of the Schrödinger wave function. On the other hand, for diffeomorphism-invariant topological field theories, HOROWITZ (1989) has shown that $\Psi = N \exp\left(i\theta \int \underline{\mathscr{C}}\right)$ is, up to an overall factor, the unique solution of the Hamiltonian constraints. It is worth investigating whether this carries over to gravity:

Since the Poincaré group $P := \mathbb{R}^4 \rtimes SO(1,3)$ is the semidirect product of translations and Lorentz rotations, its gauging leads to the two associated Chern–Simons (CS) three-forms:

$$C_{\rm TT} := \frac{1}{2\ell^2}\left(\vartheta^\alpha \wedge T_\alpha\right) = -\frac{(-1)^{\rm sig}}{2\ell^2}\,{}^*\!\mathscr{A}, \tag{6.2.1}$$

$$C_{\mathrm{RR}} := -\mathrm{Tr}\left(\Gamma \wedge \Omega - \frac{1}{3}\Gamma \wedge \Gamma \wedge \Gamma\right)$$
$$= -\frac{1}{2}\left(\Gamma_{\alpha}{}^{\beta} \wedge R_{\beta}{}^{\alpha} + \frac{1}{3}\Gamma_{\alpha}{}^{\beta} \wedge \Gamma_{\beta}{}^{\gamma} \wedge \Gamma_{\gamma}{}^{\alpha}\right). \tag{6.2.2}$$

They are *translational* and Lorentz-rotational CS terms, where $\mathscr{A} := {}^*(\vartheta_\alpha \wedge T^\alpha) = \mathscr{A}_i dx^i$ is the *axial torsion* one-form. In the first *translational* CS three-form, there necessarily occurs a fundamental length ℓ for dimensional reasons.

In a gauge theory with the linear group $SL(5, \mathbb{R})$ as structure group, containing the de Sitter group SO(1, 4) or SO(2, 3) as a subgroup, both CS three-forms are intimately interrelated: via a Wigner–Inönü-type contraction, the CS decomposition

$$\hat{C} = C_{\mathrm{RR}} - 2C_{\mathrm{TT}} \tag{6.2.3}$$

into linear and translational terms was realized already in HEHL et al. (1995) (see footnote 31), and later by NIEH (2007), MERCURI (2009). In contrast to the metric-free Pontryagin form, in the NY term (6.2.4), a metric $g_{\alpha\beta}$ is needed to raise and lower the indices, for instance in $T_\alpha = g_{\alpha\beta}T^\beta$.

The corresponding boundary or Nieh–Yan term (NIEH & YAN 1982; GUO et al. 1999) can be obtained by exterior differentiation

$$dC_{\mathrm{TT}} \equiv \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right), \tag{6.2.4}$$

whereas the corresponding Pontryagin term

$$dC_{\mathrm{RR}} = -\mathrm{Tr}\ (\Omega \wedge \Omega) = \frac{1}{2}\,R^{\alpha\beta} \wedge R_{\alpha\beta} \tag{6.2.5}$$

is more familiar. Its integration is proportional to the Pontryagin number. Both are *parity violating* (MIELKE 1999).

Whereas the Pontryagin term (6.2.5) is a topological Lagrangian whose variation returns the second Bianchi identity

$$DR_{\alpha\beta} \equiv 0, \tag{6.2.6}$$

the less well known torsion identity (6.2.4) is based, after exterior multiplication by ϑ_α, on the first Bianchi identity

$$DT^\alpha \equiv R_\beta{}^\alpha \wedge \vartheta^\beta \tag{6.2.7}$$

in Riemann–Cartan (RC) geometry.

6.2.1 *From Chern–Simons to Einsteinian Gravity*

The starting point of Ashtekar's formulation of gravity with complex variables is the parity-violating boundary four-form (6.2.4) of NIEH & YAN (1982) (NY).[1] A fundamental length ℓ necessarily enters in order to keep all topological invariants (LI 1999) dimensionless.

By converting one field strength via a duality rotation into its Hodge dual, the NY term suggests two options for a viable gravitational Lagrangian:

1. Hilbert's original choice:

$$L_{\mathrm{HE}} = -\frac{1}{2\kappa} R^{\{\}}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) \tag{6.2.8}$$

of general relativity (GR), where $R^{\{\}}_{\alpha\beta}$ denotes the Riemannian curvature with respect to the Levi-Civita connection $\Gamma^{\{\}}_{\alpha\beta}$.

2. A torsion–square Lagrangian: The NY term, after a duality rotation, suggests another option (KOPCZYŃSKI 1982; NESTER 1988) for a classically viable gravitational Lagrangian: the special torsion–square Lagrangian

$$L_{\|} := -\frac{1}{2\kappa} T^\alpha \wedge {}^*\left({}^{(1)}T_\alpha - 2\, {}^{(2)}T_\alpha - \frac{1}{2}\, {}^{(3)}T_\alpha \right), \tag{6.2.9}$$

involving a specific combination of irreducible torsion components. Here

$$H^{\|}_\alpha := -\partial L_{\|}/\partial T^\alpha = (1/\kappa)\eta_{\alpha\beta\gamma} \wedge K^{\beta\gamma} \tag{6.2.10}$$

is dual to the contortion one-form $K_{\alpha\beta}$, which features in the decomposition $\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} = \Gamma^{\{\}}_{\alpha\beta} - K_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} + e_\alpha \rfloor T_\beta + (e_\alpha \rfloor e_\beta \rfloor T_\gamma) \wedge \vartheta^\gamma$ of the RC connection.

Due to the geometric identity

$$L_{\|} \equiv L_{\mathrm{HE}} + \frac{1}{2\kappa} R_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{2\ell^2}{\kappa} dC_{\mathrm{TT}^*}, \tag{6.2.11}$$

where $C_{\mathrm{TT}^*} := \vartheta^\alpha \wedge {}^*T_\alpha / 2\ell^2$ is a dual CS term, proper *teleparallelism* ($\mathrm{GR}_{\|}$) specified by (6.2.9) is classically *equivalent* to GR up to a boundary term when constrained by the vanishing of RC curvature. In a consistent Lagrangian formulation, the proper constraint $R^{\alpha\beta} = 0$ has to be imposed by subtracting $R^{\alpha\beta} \wedge \lambda_{\alpha\beta}$ from (6.2.9), where the 2-form $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$ is a Lagrange multiplier, see HEHL et al. (1980) and KOPCZYŃSKI (1990). Then the proper teleparallelism Lagrangian reads

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[1]The self-dual formulation of gravity was anticipated already by PLEBAŃSKI (1977), whereas HOJMAN et al. (1980) discussed the pseudoscalar curvature as a *parity-violating* Lagrangian for gravity and noted already 1980 its relationship to a complete divergence, before NIEH & YAN (1982), NIEH (2007).

$$\tilde{L}_{\|} = L_{\|} - R^{\alpha\beta} \wedge \lambda_{\alpha\beta}. \qquad (6.2.12)$$

By varying this Lagrangian independently with respect to ϑ^α, $\Gamma^{\alpha\beta}$, and the multiplier $\lambda_{\alpha\beta}$, we obtain as field equations

$$DH^{\|}_{\alpha} - E^{\|}_{\alpha} = \Sigma_\alpha, \qquad (6.2.13)$$

$$D\lambda_{\alpha\beta} + \vartheta_{[\alpha} \wedge H^{\|}_{\beta]} = \tau_{\alpha\beta}, \qquad (6.2.14)$$

and

$$R^{\alpha\beta} = 0. \qquad (6.2.15)$$

Since the multiplier term in (6.2.12) does not depend on the coframe, the resulting first field Eq. (6.2.13) is the same as that of the unconstrained Lagrangian $L_{\|}$. The integrability condition for the second field equation is identically satisfied, because

$$DD\lambda_{\alpha\beta} = -2R_{[\alpha|}{}^{\gamma} \wedge \lambda_{\gamma|\beta]} = 0 \qquad (6.2.16)$$

in a Weitzenböck spacetime, whereas

$$D(\tau_{\alpha\beta} - \vartheta_{[\alpha} \wedge H^{\|}_{\beta]}) = 0 \qquad (6.2.17)$$

follows from the "weak" Noether identity (see (6.8.12)) for matter and gravitational gauge fields, together with the first field equation. Thus, the only role of the second field equation is to determine (nonuniquely) the Lagrangian multiplier $\lambda_{\alpha\beta}$. For *spinless* matter, a possible solution is $\lambda_{\alpha\beta} = (\gamma/\ell^2)\eta_{\alpha\beta}$, as will be shown below.

We should remark that the Cauchy problem for $\mathrm{GR}_{\|}$ is not completely settled; cf. Ehlers (1981), Kopczyński (1982), Hecht et al. (1991).

6.2.2 *Yang–Mills-Type Formulation of Complex* $\mathrm{GR}_{\|}$

In the self-dual formulation, the Lagrangian of the Einstein–Cartan (EC) theory of gravity (Trautman 1973) plus cosmological term takes the form

$$\overset{(\pm)}{L}_{\mathrm{EC}} := L_{\mathrm{EC}} \pm i dC_{\mathrm{TT}} = -\frac{1}{2\ell^2}\overset{(\pm)}{R}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{\Lambda}{\ell^2}\eta.$$

This *chiral* reformulation (Samuel 1987; Mielke et al. 1996, 1996b) was to some extent anticipated in 1982 in the complex duality ansatz

$$R_{\alpha\beta} = \xi\, {}^*R^{(\star)}_{\alpha\beta} + (\gamma/\ell^2)\left[\vartheta_\alpha \wedge \vartheta_\beta \pm i\, {}^*(\vartheta_\alpha \wedge \vartheta_\beta)\right], \qquad (6.2.18)$$

of BAEKLER et al. (1982); if $\sigma = \pm i\gamma$ is chosen for one of the free parameters there. Later on, this was "recovered" by SOO (1995) for quadratic-curvature Lagrangians.

The NY term is again instrumental (MIELKE 1992) for converting the teleparallel version (6.2.9) of Einstein's GR for the choice $\theta_{\rm T} = \pm i$ into a *chiral gauge theory of translations*:

$$\overset{(\pm)}{L}_{\parallel} := L_{\parallel} \pm i\frac{2\ell^2}{\kappa} dC_{\rm TT} = L_{\rm HE} - L_{\rm EC} \pm i\frac{2\ell^2}{\kappa} d\overset{(\mp)}{C}_{\rm TT}. \tag{6.2.19}$$

The deviation of chiral $GR_{\parallel}$ from the Hilbert–Einstein action turns out to be a boundary term derived from the *chiral* CS term

$$\overset{(\pm)}{C}_{\rm TT} := \frac{1}{2\ell^2}\left(\vartheta^\alpha \wedge \overset{(\pm)}{T}_\alpha\right), \tag{6.2.20}$$

where $\overset{(\pm)}{T}{}^\alpha := \frac{1}{2}\,(T^\alpha \pm i\,{}^*T^\alpha)$ denotes the self-dual or anti-self-dual torsion. The resulting complex field momenta

$$\overset{(\pm)}{\Pi}_\alpha = -\partial \overset{(\pm)}{L}/\partial T^\alpha = H^{\parallel}_\alpha \mp (i/\kappa) T_\alpha \tag{6.2.21}$$

satisfy the algebraic relation

$$\overset{(+)}{\Pi}{}^\beta \wedge \overset{(-)}{\Pi}_\beta = H^\beta \wedge H_\beta + \ell^{-4}\, T^\beta \wedge T_\beta = 0. \tag{6.2.22}$$

Consequently, the field equation

$$\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_\alpha \mp \frac{i}{4}\ell^2 e_\alpha \rfloor \left(\overset{(+)}{\Pi}{}^\beta \wedge \overset{(-)}{\Pi}_\beta\right) = \Sigma_\alpha, \tag{6.2.23}$$

for the complex translational momentum of $GR_{\parallel}$ takes on the particularly *concise Yang–Mills-type form*

$$\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_\alpha = \Sigma_\alpha. \tag{6.2.24}$$

Moreover, the complexified quadratic PG Lagrangian

$$\begin{aligned}\overset{(\pm)}{L} = &-\frac{i}{4}\ell^2\left[\overset{(+)}{\Pi}{}^\alpha - \overset{(-)}{\Pi}{}^\alpha\right] \wedge \overset{(\pm)}{\Pi}_\alpha \\ &-\frac{i}{4}\left[\overset{(+)}{\Pi}{}^{\alpha\beta} - \overset{(-)}{\Pi}{}^{\alpha\beta}\right] \wedge \left(\overset{(\pm)}{\Pi}_{\alpha\beta} \mp \frac{i}{\ell^2}\vartheta_\alpha \wedge \vartheta_\beta - \frac{a_0}{2\ell^2}\eta_{\alpha\beta}\right)\end{aligned} \tag{6.2.25}$$

reduces to the $\mathrm{GR}_{\|}$ Lagrangian

$$\overset{(\pm)}{L}_{\|} = \mp\frac{i}{4}\ell^2\,\overset{(\pm)}{\Pi}{}^{\alpha}\wedge\overset{(\pm)}{\Pi}_{\alpha}, \qquad \overline{\overset{(+)}{L}_{\|}} = \overset{(-)}{L}_{\|}, \tag{6.2.26}$$

which is *purely quadratic* in the new translational momentum. To some extent, we have thus realized Trautman's and Thirring's "dream" (cf. Trautman 1973; Thirring 1980) of rewriting Einstein's equations in a Maxwell-type form. Observe that the Hodge star is absorbed in the definition of the complex field momenta. Matter with spin could formally be incorporated in our scheme by admitting (Mielke et al. 1989) an additional piece ${}^{(\tau)}\lambda_{\alpha\beta}$ in the ansatz for the Lagrangian multiplier, provided it satisfies $D\,{}^{(\tau)}\lambda_{\alpha\beta} = \tau_{\alpha\beta}$.

The complex translational field momenta $\overset{(\pm)}{\Pi}{}^{\alpha}$ replace the role of the torsion if we use the Sen connection

$$\overset{(\pm)}{D}\vartheta^{\alpha} = \pm\frac{i}{2}\ell^2\,g^{\alpha\beta}\,\overset{(\pm)}{\Pi}_{\beta}, \tag{6.2.27}$$

which is not metric-compatible. The equivalent presentations of the $\mathrm{GR}_{\|}$ Lagrangian

$$\overset{(\pm)}{L}_{\|} = -\frac{1}{2}\Big[\vartheta^{\alpha}\wedge\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_{\alpha} + d(\vartheta^{\alpha}\wedge\overset{(\pm)}{\Pi}_{\alpha})\Big] = -\frac{1}{2}\overset{(\pm)}{D}\vartheta^{\alpha}\wedge\overset{(\pm)}{\Pi}_{\alpha}, \tag{6.2.28}$$

allow us to rederive the field Eq. (6.2.24) via $\delta\overset{(\pm)}{L}_{\|}/\delta\vartheta^{\alpha}$ from a variational principle.

In order to solve the vacuum equation (6.2.24) for the coframe and the complex momenta, the *integrability conditions* have to be satisfied. In a teleparallel Riemann–Cartan spacetime with $R_{\alpha\beta} = 0$ and in view of the first Noether identity (6.8.11), these conditions read in vacuum[2]

$$DD\vartheta^{\alpha} = R_{\beta}{}^{\alpha}\wedge\vartheta^{\beta} = 0,\ \overset{(\pm)}{D}\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_{\alpha} = -\overset{(\pm)}{R}_{\alpha}{}^{\beta}\wedge\overset{(\pm)}{\Pi}_{\beta} = \pm\frac{i}{2}\ell^2\,(e_{\alpha}\rfloor\overset{(\pm)}{\Pi}{}^{\beta})\wedge\Sigma_{\beta}\cong 0. \tag{6.2.29}$$

Is it then possible to generate all exact vacuum solutions of Einstein's equations by means of prolongation methods? At least for the complex Ernst equation, a similar type of approach is possible for generating "almost all" stationary axisymmetric solutions (Harrison 1983; Plebanski 1975).

As a consequence of this, the complexified Lagrangian (6.2.19) becomes *quadratic* in these new field momenta, cf. (6.2.26), resembling the chiral version

$$\overset{(\pm)}{L}_{\mathrm{YM}} = \mp(1/4)\mathrm{Tr}\left(\overset{(\pm)}{F}\wedge\overset{(\pm)}{F}\right) \tag{6.2.30}$$

[2] In the presence of matter, the first Noether identity enters the game. For a resolution of (6.2.24) in terms of the momenta $\overset{(\pm)}{\Pi}_{\alpha}$, it would be convenient to have a *relocalized* energy–momentum current $\overset{(\pm)}{\Sigma}_{\alpha}$ for which $\overset{(\pm)}{D}\overset{(\pm)}{\Sigma}_{\alpha}\cong 0$ holds.

of Yang–Mills theory. The gravitational field equations of chiral $GR_{\|}$ in vacuum read

$$\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_{\alpha} = 0, \tag{6.2.31}$$

where the *modified* covariant exterior derivative

$$\overset{(\pm)}{D}\overset{(\pm)}{\Pi}_{\alpha} := D\overset{(\pm)}{\Pi}_{\alpha} \pm \frac{i}{2}\ell^2\big(e_\alpha\rfloor\overset{(\mp)}{\Pi}{}^{\beta}\big)\wedge\overset{(\pm)}{\Pi}_{\beta}, \tag{6.2.32}$$

contains the linear connection $\Gamma_\alpha{}^\beta$, which is combined with the complex translational momentum $\overset{(\pm)}{\Pi}_{\alpha}$ to the complex connection

$$\overset{(\pm)}{\Gamma}_{\alpha}{}^{\beta} := \Gamma_\alpha{}^\beta + i\,\overset{(\mp)}{A}_{\alpha}{}^{\beta}, \qquad \overset{(\pm)}{A}_{\alpha}{}^{\beta} := \pm\frac{1}{2}\ell^2\, e_\alpha\rfloor\overset{(\pm)}{\Pi}{}^{\beta}, \tag{6.2.33}$$

and it is of the type introduced by Sen (1982) in a rather ad hoc way.[3] The tensor-valued 1-form $\overset{(\mp)}{A}_{\alpha}{}^{\beta}$ deforms our original Riemann–Cartan connection $\Gamma_\alpha{}^\beta$ similarly to the way in which the contortion deforms the Levi-Civita connection. From the viewpoint of quantum field theory, the deformation (6.2.33) of the connection could be regarded as the classical counterpart of a *field redefinition* (Mielke & Hehl 1991) of the "bare" connection.

They are formally those of Yang–Mills for the *translational* gauge field momenta "living" on a *nondynamical RC background* fixed by the teleparallelism constraint $R^{\alpha\beta} = 0$. For consistency, this constraint ought to be enforced via Lagrange multipliers $\lambda_{\alpha\beta}$ in the *teleparallel* Lagrangian (6.2.12) with constraints.

6.3 Energy–Momentum Complex

It is a consequence of the first field equation of the (broken) PG theory that the *energy–momentum complex*

$$\check{E}_\alpha := E_\alpha + \Gamma_\alpha{}^\beta\wedge H_\beta + \Sigma_\alpha \cong dH_\alpha, \tag{6.3.1}$$

due to the Poincaré lemma $dd \equiv 0$, is weakly closed. In the subcase of $GR_{\|}$, related expressions have been given before; cf. Møller (1961), Hayashi & Bregman (1973), Thirring (1978), Hehl et al. (1980). In the teleparallelism limit $\underline{\Gamma}{}^{\alpha\beta}_{\|} \to 0$, the "superpotential" H_α is the *Møller* or *Freud complex* (Freud 1939)

[3]The *transposed* connection, which has the property (Mielke et al. 1989) that $\widehat{D}\eta_\alpha := D\eta_\alpha - (e_\alpha\rfloor T^\beta)\wedge\eta_\beta \equiv 0$, may be regarded as a special real version of our Sen-type connection, for which $\overset{(\pm)}{D}\eta_\alpha = \pm(i/2)\ell^2\eta_{\alpha\beta}\wedge\overset{(\pm)}{\Pi}{}^{\beta}$ holds.

$$H_\alpha^{\|} = \frac{1}{\ell^2}\left[\vartheta^\beta \wedge {}^*(d\vartheta_\beta \wedge \vartheta_\alpha) - \frac{1}{2}\vartheta_\alpha \wedge {}^*(d\vartheta^\beta \wedge \vartheta_\beta)\right] = \frac{1}{2\ell^2}\,\Gamma^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \tag{6.3.2}$$

of $\mathrm{GR}_\|$ or GR, respectively. On the basis of (6.3.1), it has been observed by ANDRADE et al. (2000) that E_α is a covariant conserved gravitational energy, i.e., that $DE_\alpha = DDH_\alpha \equiv -R_\alpha{}^\beta \wedge H_\beta = 0$ due to the vanishing of the Ricci identity in a teleparallel spacetime.

It also should be noticed that the complex conjugate three-form

$$\overset{(\pm)}{S}{}_\alpha := \overset{(\pm)}{\Gamma}{}_\alpha{}^\beta \wedge \overset{(\pm)}{\Pi}{}_\beta + \Sigma_\alpha \cong d\overset{(\pm)}{\Pi}{}_\alpha, \qquad \overline{\overset{(\pm)}{S}}{}_\alpha = \overset{(\mp)}{S}{}_\alpha \tag{6.3.3}$$

is the corresponding energy–momentum complex of chiral $\mathrm{GR}_\|$; see MIELKE (1992). Its real part turns out to be related to the so-called *Sparling form*; cf. DUBOIS-VIOLETTE & MADORE (1987).

However, in spacetimes with Killing symmetries, energy and angular-momentum expressions suffer from the well-known factor two problem. One way out is to amend the equivalent Hilbert–Einstein Lagrangian by an additional Euler term; cf. MIELKE (2001).

6.4 Hamiltonian Formulation of Complex $\mathrm{GR}_\|$

For the canonical analysis, let us assume that the spacetime manifold admits a slicing into a family of spacelike three-dimensional hypersurfaces Σ_t that are parametrized by the coordinate time $x^0 = t$. Then there exists the future-directed *timelike* vector field

$$n := \partial_t - N^A \partial/\partial x^A, \qquad A = 1, 2, 3, \tag{6.4.1}$$

which is uniquely linked to the hypersurface orthogonal 1-form $u := -N^2\, dt$, $\iff du \wedge u = 0$ via $u = g(n,\ .)$. The lapse function N and the shift vector N^A, which are familiar from the ADM formalism, are arbitrary; cf. Fig. 6.1

The *normal* part of a p-form Ψ is defined by

$${}^\perp\Psi := dt \wedge \Psi_\perp; \qquad \Psi_\perp := n\rfloor\Psi, \tag{6.4.2}$$

whereas the part *tangential* to the hypersurface Σ_t is given by

$$\underline{\Psi} := n\rfloor(dt \wedge \Psi) = (1\ -\ {}^\perp)\Psi, \qquad n\rfloor\underline{\Psi} \equiv 0. \tag{6.4.3}$$

The decomposition operators "⊥" and "_" form a complete set of projection operators. (For more details on this very concise (3+1) decomposition of exterior forms in Riemann–Cartan spacetime, see WALLNER 1990; MIELKE & WALLNER 1988.)

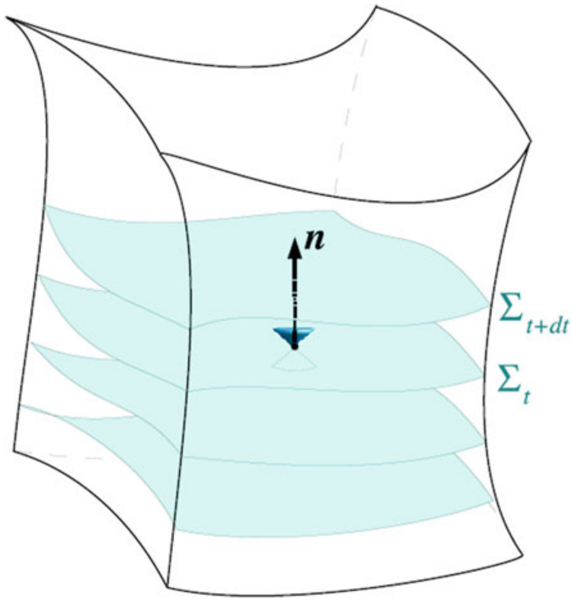

Fig. 6.1 Foliation by a family of spacelike hypersurfaces

Let us consider a Lagrangian $L = L(\vartheta^\alpha,\ \Psi,\ D\Psi)$ which is of first differential order in the (collection of) fields Ψ. In concordance with the classical prescription, the corresponding Hamiltonian 3-form is given by

$$\mathscr{H} = \mathscr{K} - \underline{\overset{\bullet}{\Psi}} \wedge \left(\frac{\partial \mathscr{K}}{\partial \underline{\overset{\bullet}{\Psi}}}\right), \tag{6.4.4}$$

where $\mathscr{K} = L_\perp := n\rfloor L$ denotes the normal part of the Lagrangian. Since the "time derivative" $^\bullet$ is more precisely given by the Lie derivative $\ell_n := dn\rfloor + n\rfloor d$ along the timelike vector field n, we have

$$\mathscr{H} = n\rfloor L - \ell_n \underline{\Psi} \wedge \underline{\left(\frac{\partial L}{\partial d\Psi}\right)}. \tag{6.4.5}$$

Now compare this with the canonical energy–momentum 3-form

$$\begin{aligned} \Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} = e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi) \wedge \frac{\partial L}{\partial D\Psi} - (e_\alpha\rfloor \Psi) \wedge \frac{\partial L}{\partial \Psi} \\ - (e_\alpha\rfloor T^\beta) \wedge \frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor R_\beta{}^\gamma) \wedge \frac{\partial L}{\partial R_\beta{}^\gamma}. \end{aligned} \tag{6.4.6}$$

For the intended comparison, we drop the contribution from possible "Pauli terms" and transvect the resulting expression with the four-vector $n^\alpha := n\rfloor\vartheta^\alpha$ of *lapse and shift*, for which the useful relation $n^\alpha\, e_\alpha\rfloor\Psi = n\rfloor\Psi$ can be derived. For the projection of the anholonomic index of the truncated energy–momentum current (6.4.6) in the normal direction, we get the expression

$$n^\alpha \Sigma_\alpha = n\rfloor L - (n\rfloor D\Psi) \wedge \frac{\partial L}{\partial D\Psi} - (n\rfloor \Psi) \wedge \frac{\partial L}{\partial \Psi}. \tag{6.4.7}$$

In introducing the gauge-covariant Lie derivative $\text{Ł}_n = Dn\rfloor + n\rfloor D$, separating a boundary term, and using the variational derivative (6.8.2), the following relation, which holds 'strongly' or 'weakly', respectively, is obtained:

$$\begin{aligned} n^\alpha \Sigma_\alpha &= n\rfloor L - (\text{Ł}_n \Psi) \wedge \frac{\partial L}{\partial D\Psi} - (n\rfloor \Psi) \wedge \frac{\delta L}{\delta \Psi} + D\Big[(n\rfloor \Psi) \wedge \frac{\partial L}{\partial D\Psi}\Big] \\ &\cong n\rfloor L - \big(\ell_n \Psi + \Gamma_\perp{}^{\alpha\beta}\rho(I_{\alpha\beta})\Psi\big) \wedge \frac{\partial L}{\partial D\Psi} + D\Big[(n\rfloor \Psi) \wedge \frac{\partial L}{\partial D\Psi}\Big]. \end{aligned} \tag{6.4.8}$$

We are going to show that the *tangential part* of this energy–momentum expression is related to the Hamiltonian (6.4.5). Indeed, using $\underline{n\rfloor\Psi} = n\rfloor(dt \wedge n\rfloor\Psi) = n\rfloor\Psi$ and the fact that the boundary term is a Lorentz scalar, we obtain

$$\mathscr{H} \cong n^\alpha \underline{\Sigma}_\alpha + \Gamma_\perp{}^{\alpha\beta}\rho(I_{\alpha\beta})\underline{\Psi} \wedge \underline{\frac{\partial L}{\partial D\Psi}} - \underline{d}\Big[(n\rfloor\Psi) \wedge \underline{\frac{\partial L}{\partial D\Psi}}\Big]. \tag{6.4.9}$$

If Ψ represents a collection of gauge fields, the partial derivatives in the truncated definitions

$$\Sigma_\alpha := \frac{\partial L}{\partial \vartheta^\alpha}, \qquad \tau_{\alpha\beta} := \frac{\partial L}{\partial \Gamma^{\alpha\beta}} = \rho(I_{\alpha\beta})\Psi \wedge \frac{\partial L}{\partial D\Psi} \tag{6.4.10}$$

of the 3-forms of energy–momentum and spin are to be replaced by *variational derivatives*, and the sum over all individual Ψ's has to be performed. Quite generally, the Hamiltonian then acquires the following form:

$$\begin{aligned} \mathscr{H} &\cong n^\alpha \underline{\frac{\delta L}{\delta \vartheta^\alpha}} + \Gamma_\perp{}^{\alpha\beta} \underline{\frac{\delta L}{\delta \Gamma^{\alpha\beta}}} \\ &\quad - \underline{d}\Big[(n\rfloor\Psi) \wedge \underline{\frac{\partial L}{\partial D\Psi}} + n^\alpha \underline{\frac{\partial L}{\partial T^\alpha}} + \Gamma_\perp{}^{\alpha\beta} \underline{\frac{\partial L}{\partial R^{\alpha\beta}}}\Big]. \end{aligned} \tag{6.4.11}$$

Neglecting the boundary term, the gravitational Hamiltonian of general PG theory takes the form

$$\mathscr{H} \cong n^\alpha \mathscr{G}_\alpha + \Gamma_\perp{}^{\alpha\beta} \mathscr{G}_{\alpha\beta}, \tag{6.4.12}$$

where the normal vector $n^\alpha := n\rfloor\vartheta^\alpha$ comprises *lapse and shift* and $\Gamma_\perp{}^{\alpha\beta} := n\rfloor\Gamma^{\alpha\beta}$ is the normal part of the Lorentz connection and $\mathscr{G}_\alpha$ and $\mathscr{G}_{\alpha\beta}$ represent the tangential generators.

6.4.1 Poisson Brackets

Their *Poisson brackets* (BAEKLER & MIELKE 1988) at "equal times" read

$$\{\mathscr{G}_\alpha(t,\mathbf{x}),\ \mathscr{G}_\beta(t,\mathbf{y})\} = \left(-T_{\alpha\beta}{}^\gamma \mathscr{G}_\gamma + R_{\alpha\beta\gamma}{}^\delta \mathscr{G}^\gamma{}_\delta\right)\cdot\delta(\mathbf{x}-\mathbf{y}), \tag{6.4.13}$$

$$\{\mathscr{G}^\alpha{}_\beta(t,\mathbf{x}),\ \mathscr{G}_\gamma(t,\mathbf{y})\} = \delta^\gamma_\beta \mathscr{G}_\beta \delta(\mathbf{x}-\mathbf{y}), \tag{6.4.14}$$

$$\{\mathscr{G}^\alpha{}_\beta(t,\mathbf{x}),\ \mathscr{G}^\gamma{}_\delta(t,\mathbf{y})\} = (\delta^\alpha_\delta \mathscr{G}^\gamma{}_\beta - \delta^\gamma_\beta \mathscr{G}^\alpha{}_\delta)\cdot\delta(\mathbf{x}-\mathbf{y}). \tag{6.4.15}$$

This Poisson bracket structure is on a par with the *local* affine algebra:

$$\left[D_\alpha, D_\beta\right] = -T_{\alpha\beta}{}^\gamma(x)\, D_\gamma + R_{\alpha\beta\gamma}{}^\delta(x)\, L^\gamma{}_\delta, \tag{6.4.16}$$

$$\left[L^\alpha{}_\beta, D_\gamma\right] = \delta^\alpha_\gamma\, D_\beta, \tag{6.4.17}$$

$$\left[L^\alpha{}_\beta,\ L^\gamma{}_\delta\right] = \delta^\alpha_\delta L^\gamma{}_\beta - \delta^\gamma_\beta L^\alpha{}_\delta, \tag{6.4.18}$$

where $D_\alpha := e_\alpha \rfloor D$ are the anholonomic components of the exterior covariant derivative. Equations (6.4.16)–(6.4.18) constitute a *soft gauge algebra* (NE'EMAN et al. 1980; SOHNIUS 1983), since the group-theoretic "structure constants" are the spacetime-dependent components of torsion and curvature.

In the Lagrange multiplier formulation (6.2.12) of teleparallelism, there would arise the extra tangential term $\underline{D\lambda_{\alpha\beta}}$ in the *Lorentz* generator $\mathscr{G}_{\alpha\beta}$, where $\underline{\Psi} := \Psi - dt\wedge n\rfloor\Psi$ in the notation of differential forms. However, as suggested by the geometric identity (6.2.11), the Lorentz constraints are identically satisfied by $\underline{\lambda}_{\alpha\beta} = (1/2\ell^2)\,\underline{{}^*(\vartheta_\alpha\wedge\vartheta_\beta)} = -(1/N\ell^2) n_{[\alpha}\,{}^*\underline{\vartheta}_{\beta]}$. This is another manifestation of the dynamical equivalence of GR and $\mathrm{GR}_\parallel$ up to a canonical transformation induced by the boundary term $d\overset{(\mp)}{C}{}_{\mathrm{TT}}$.

After this trivialization of the Lorentz constraints, the Hamiltonian ($\alpha=\hat{0}$) and diffeomorphism constraints ($\alpha = B$) of complexified $\mathrm{GR}_\parallel$, in terms of a complex Sen-type connection (MIELKE 1992) are

$$\overset{(\pm)}{\mathscr{G}}{}_\alpha := \underline{\overset{(\pm)}{D}}\,\underline{\overset{(\pm)}{\Pi}}{}_\alpha = \underline{\overset{(\pm)}{D}}\,{}^*\underline{\overset{(\pm)}{\mathscr{A}}}{}_\alpha = 0, \tag{6.4.19}$$

generalizing the Gauss constraint of Maxwell's theory to a gauge theory of translations. Only *first class* constraints remain after this gauge fixing, according to MALUF (1994). Here the tangential Ashtekar-type connection $\underline{\overset{(\pm)}{\mathscr{A}}}{}_\alpha := {}^*\underline{\overset{(\pm)}{\Pi}}{}_\alpha$ is three-dual to the translational field momentum, and canonically conjugate to the triad densities ${}^*\underline{\vartheta}_\alpha$, where we note that the spatial Hodge dual is involutive, i.e., $\underline{*}\,\underline{*} = +1$. Moreover, by adopting tetrads in the temporal gauge $\underline{\vartheta}^{\hat{0}} = 0$ of SCHWINGER (1963) and the gauge $\underline{\overset{(\pm)}{\mathscr{A}}}{}_{\hat{0}} = d\overset{(\pm)}{\phi}$ of NESTER (1989), the Hamiltonian constraint

$$\mathscr{H}_\parallel = \underline{\overset{(\pm)}{D}}\,{}^*\underline{\overset{(\pm)}{\mathscr{A}}}{}_{\hat{0}} = 0, \tag{6.4.20}$$

i.e., (6.4.19) for $\alpha = \hat{0}$, vanishes identically.

6.4.2 Chern–Simons Solutions of the Chiral Teleparallelism Constraints

According to this canonical decomposition of forms (Mielke 1992), the tangential parts of the basic one-forms, i.e., the *triad densities* $\underline{{}^*\vartheta}_\alpha$ and the tangential part of the self-dual or anti-self-dual connection $\overset{(\pm)}{\underline{\mathscr{A}}}_\alpha := {}^*\overset{(\pm)}{\underline{\Pi}}_\alpha$, the three-dual of the chiral momenta $\overset{(\pm)}{\Pi}_\alpha$, become the generalized coordinates q and momenta $\overset{\pm}{p}$ of the bosonic sector.

In the transition to quantum gravity, in contrast to GR (Kodama 1990), for $GR_\parallel$ the *Schrödinger representation*

$$q: \quad \underline{{}^*\vartheta}^\beta\, \Psi_\parallel(\vartheta) = \underline{{}^*\vartheta}^\beta\, \Psi_\parallel(\vartheta), \tag{6.4.21}$$

$$\overset{\pm}{p}: \quad {}^*\overset{(\pm)}{\underline{\Pi}}_\alpha\, \Psi_\parallel(\vartheta) = -i\ell^2 \frac{\delta}{\delta\, \underline{{}^*\vartheta}_\alpha} \Psi_\parallel(\vartheta), \tag{6.4.22}$$

prevails (Mielke 2002), where the complex field “momenta” ${}^*\overset{(\pm)}{\underline{\Pi}}_\alpha$ become differential operators, whereas the triad densities $\underline{{}^*\vartheta}^\beta$ remain generalized coordinates q.

For the remaining Gauss constraint (6.4.8) with $\alpha = B$, let us try the state vector

$$\Psi_\parallel(\vartheta) = \exp\left(-\int_{M_3} \overset{(\pm)}{\underline{C}}_{\rm TT}\right) = \exp\left(\frac{-1}{2\ell^2}\int_{M_3}\left[\underline{{}^*\vartheta}_B \wedge {}^*\overset{(\pm)}{\underline{T}}{}^B\right]\right) \tag{6.4.23}$$

as a solution, with the integration variables hidden in the 3-forms and a sign inverse to that of Mielke (1998, 1999). It involves the tangential complexified *translational* Chern–Simons term $\overset{(\pm)}{\underline{C}}_{\rm TT}$ in terms of the triads or triad densities and the tangential part of the self-dual or anti-self-dual torsion.

When the tangential complexified *translational* Chern–Simons term (6.2.20) is rewritten in terms of the triad densities $\underline{{}^*\vartheta}^B$ with $B = 1, 2, 3$ and the tangential part of the self-dual or anti-self-dual torsion, the chiral version of Eq. (16) of Mercuri (2008) for “large”-gauge transformations is anticipated (Mielke 2002). Then the momentum operator (6.4.22) returns the chiral torsion $\overset{(\pm)}{\underline{T}}{}^B$ as a prefactor:

$$-\ell^2 \frac{\delta}{\delta\, \underline{{}^*\vartheta}_B} \Psi_\parallel(\vartheta) = {}^*\overset{(\pm)}{\underline{T}}{}^B\, \Psi_\parallel(\vartheta). \tag{6.4.24}$$

Since the corresponding connection $\underline{\Gamma}_\parallel^{\alpha\beta} \overset{*}{=} 0$ is of pure gauge type, the state vector (6.4.23) depends merely on the triads as sole dynamical variables. Consequently, the operator version of the Sen covariant tangential diffeomorphism constraint (6.4.19) simplifies to

$$-\ell^2 \overset{(\pm)}{\underline{D}}\,{}^{*}\!\left(\frac{\delta}{\delta\,{}^{*}\!\underline{\vartheta}_B}\right) = \overset{(\pm)}{\underline{D}}\overset{(\pm)}{\underline{T}}{}^B \equiv \overset{(\pm)}{\underline{R}}{}_C{}^B \wedge \overset{(\pm)}{\underline{\vartheta}}{}^C = 0, \tag{6.4.25}$$

where the rule $\Psi \wedge {}^*\Phi = \Phi \wedge {}^*\Psi$ for differential forms of the same degree has been applied. Observe that the first Bianchi identity (6.2.7) gets truncated to

$$\overset{(\pm)}{\underline{D}}\overset{(\pm)}{\underline{T}}{}^B = 0 \tag{6.4.26}$$

in a teleparallel space with zero RC curvature or at least for configurations of self-dual or anti-self-dual tangential RC curvature.

Thus, similarly to what occurs in topological field theory (Horowitz 1989) with a flat connection, in teleparallelism, (6.4.23) appears to be the unique solution.

Unlike the Kodama (1990) approach, our formalism has the advantage that the CS term for constructing the state vector is gauge-invariant. In the former, the choice of the state vector

$$\Psi_\Lambda(A) = \exp\left(\frac{3}{\Lambda}\int_{M_3} \overset{(\pm)}{\underline{C}}{}_{\mathrm{RR}}{}^{\{\}}\right), \tag{6.4.27}$$

borrowed from $SU(2)$ Chern–Simons field theory (Witten 1989; Guadagnini 1993; Cotta-Ramusino et al. 1990), is known to solve the Hamiltonian constraint $\mathscr{H}_\Lambda\,\Psi_\Lambda(A) = 0$ of GR with cosmological constant in the nonperturbative loop approach (Brügmann et al. 1992; Griego 1996).

Moreover, the translational CS term $\overset{(\pm)}{\underline{C}}{}_{\mathrm{TT}}$ does not depend on any cosmological constant Λ in a singular manner, and in spite of the appearance of the Planck length ℓ, it is dimensionless and nonsingular even for $\ell \to 0$, due to $dim\,\underline{\vartheta}^B = [\ell]$.

Since the coframe is nondegenerate, the largest contribution would come from configurations with $\overset{(\mp)}{\underline{T}}{}^B = 0$, i.e., *self-dual or anti-self dual* tangential torsion. Consequently, Wilson-type solutions (6.4.23) of the corresponding quantum Gauss constraint are dominated by *self-dual* torsion solutions satisfying $\overset{(\pm)}{\underline{T}} = 0$.

6.4.3 Torsion Instantons

Such exact instanton-type solutions for teleparallelism have been found: for spherical symmetry, one was given by Obukhov et al. (1997) in a study of chiral anomalies (Mielke & Kreimer 1998, 1999, Kreimer & Mielke 2001; Mielke & Macías 1999), whereas a related axial solution has been studied by Nakamichi et al. (1991) for self-dual topological gravity.

Exact *torsion instantons* "live" on a conformally compactified Euclidean space $\mathbb{R}^4 \cup \infty = S^4$ with the spherically symmetric metric

$$ds^2 = h^2\, dr^2 + f^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{6.4.28}$$

where (r, ψ, θ, ϕ) are the standard hyperspherical coordinates that parameterize the unit three-sphere S^3. The function f carries physical dimension [length]. This space is *parallelizable* such that the RC curvature vanishes but then is endowed (OBUKHOV et al. 1997) with the nontrivial spacelike torsion

$$T^A = \frac{1}{f}\left(df \wedge \vartheta^A - 2\eta^{0A\beta\gamma}\vartheta_\beta \wedge \vartheta_\gamma \right) = -\frac{2}{f}\eta^A{}_{\mu\nu}\vartheta^\mu \wedge \vartheta^\nu = \pm {}^*T^A, \quad T^{\hat{0}} = 0, \tag{6.4.29}$$

which is self-dual or anti-self-dual, provided that $df = \pm 2\, hdr$. Here the self-dual generalized Levi-Civita symbol $\eta^A_{\mu\nu} := \eta_0{}^A{}_{\mu\nu} + \delta^A_\mu \delta^0_\nu - \delta^A_\nu \delta^0_\mu$ of 'T HOOFT (1991), cf. BAEKLER et al. (1982), MINKOWSKI (1986), is instrumental. Substituting this into the translational Chern–Simons term is particularly simple (CHANDIA & ZANELLI 1997) in the zero-connection gauge $\underline{\Gamma}^{\alpha\beta}_{\|} \overset{*}{=} 0$, where

$$\ell^2 \underline{C}_{\rm TT} = \frac{1}{2}\underline{\vartheta}^\alpha \wedge d\underline{\vartheta}_\alpha = \frac{1}{2}\underline{\vartheta}^A \wedge d\underline{\vartheta}_A = 3\, \underline{\vartheta}^{\hat{1}} \wedge \underline{\vartheta}^{\hat{2}} \wedge \underline{\vartheta}^{\hat{3}} \tag{6.4.30}$$

results.

Applying Stokes's theorem and integrating over the boundary three-sphere at radial infinity $r \to \infty$ yields

$$n_{\rm NY} := \int_{R^4} dC_{\rm TT} = \int_{S^3_\infty} \underline{C}_{\rm TT} = 3{\rm Vol}(S^3)k = 6\pi^2 k. \tag{6.4.31}$$

One can deduce (LI 1999) that k is the winding or instanton number of Pontryagin, in compliance with the CS decomposition (6.2.3). If torsion is self-dual or anti-self-dual, i.e., $\overset{(\mp)}{\underline{T}}{}^B = 0$, then integration over the chiral CS term $\overset{(\pm)}{\underline{C}}_{\rm TT}$ yields the same value or zero, respectively.[4]

As an example, one can imagine

$$f = \frac{ar^2}{r^2 + c^2} \tag{6.4.32}$$

for an instanton-type behavior in 4D Euclidean space. The axially symmetric solution given earlier by NAKAMICHI et al. (1991) also is teleparallel with self-dual torsion, whereas the instanton of D'AURIA & REGGE (1982) has vanishing translational CS term.

Thus one can surmise that the state vector $\Psi_{\|}(\vartheta) \propto e^{-C}$ as a solution of the quantum Gauss constraint (6.4.25) is dominated by such *torsion instantons*.

[4]Interestingly, in the gauge $\underline{\vartheta}^{\hat{0}} = 3\kappa d\theta_{\rm L} = hdr = \pm df/2$, such instantons are solutions to the topological Eq. (6.7.12), due to $T^{\hat{0}} = 0$.

6.5 Wilson Loops and Links

In QFT, a useful tool beyond perturbation theory are Wilson loops, when the four-dimensional spacetime lattice is exhausted via the enclosed plaquettes; cf. KAKU (1993).

From the *holonomy* of a Lie-algebra-valued connection one-form $A = A^I_\mu \lambda_I dx^\mu \in \mathscr{C}$, one can construct a *gauge-invariant* operator, or character, along the loop with coordinates $\gamma^\mu(s)$. Then the Wilson loop can be written as

$$W(A, \gamma) = \mathrm{Tr}\mathbf{P}\exp \oint A = \mathrm{Tr}\mathbf{P}\exp \oint ds \dot{\gamma}^i(s)\, A_i{}^J(\gamma(s))\lambda_J, \tag{6.5.1}$$

where **P** denotes path-ordered exponentials of line integrals along knots parametrized by s. We disregard here the self-linking problem or go over to *framed loops* introduced in the original work of WITTEN (1989), which are equivalent to twisted bands.

In lattice gauge theories such nonintegrable phase factors form an (overcomplete) basis for the infinite-dimensional group $\mathscr{G}$ of gauge transformations in spacetime. The state vector $\Psi(A)$ for the gauge potential can be transformed via the Wilson loop into a loop-dependent state:

$$\Psi(\gamma) = \int_{\mathscr{C}/\mathscr{G}} \mathscr{D}A\ W(A, \gamma)\Psi(A). \tag{6.5.2}$$

The integration is performed over a coset space $\mathscr{C}/\mathscr{G}$ in order to factor out gauge-equivalent connections. This *loop transform* may be regarded as a generalization of the usual Fourier transform.

In topological CS field theory (GUADAGNINI et al. 1990; OOGURI & VAFA 2000), it is known that the first-order part of the correlation function or expectation value

$$\begin{aligned} \langle 0|W(\gamma)|0\rangle^{(1)} &= \int_{\mathscr{C}/\mathscr{G}} \mathscr{D}A\ e^{iC}\, W(A, \gamma) \\ &= \frac{1}{4\pi} \oint ds \oint dt \dot{\gamma}^\mu(s)\, \dot{\gamma}^\nu(t) \eta_{0\mu\nu\kappa} \frac{\gamma^\kappa(s) - \gamma^\kappa(t)}{|\gamma(s) - \gamma(t)|^3} \end{aligned} \tag{6.5.3}$$

of the Wilson loop is the *self-linking number* $G(\gamma)$ of Gauss. It is interesting to see how this link invariant of the nineteenth century turns into one of the key ingredients of CS theories (COTTA-RAMUSINO et al. 1990; BRÜGMANN et al. 1992).

Moreover, the full correlation function

$$\langle 0|W(\gamma)|0\rangle e^{\Lambda\, G(\gamma)}\, V(\gamma), \tag{6.5.4}$$

associated with a link L satisfies the skein relation (JONES 1985) of a generalization of the Jones polynomial $V(\gamma)$. If we could formally transform these results of lattice gauge theory to quantum (super)gravity, cf. ARMAND-UGON et al. (1996), then the state vector $|\Psi\rangle$ of the universe would be characterized by knot invariants, such as the KAUFFMAN bracket (1990).

A comprehensive exposition of this subject can be found in WU (1992) as well as in the monograph of GUADAGNINI (1993). These line integrals can be calculated either numerically or by expanding the generalized Jones polynomial mentioned above, as demonstrated in a paper by ALVAREZ & LABASTIDA (1995).

6.6 Topology of Cartan Circuits?

The teleparallel version of GR is a gauge theory of translations with the abelian group $\mathbb{R}^4$ as structure group. Then, instead of Wilson-type loops, we encounter *Cartan circuits* (CARTAN 1924) with dislocations at the Planck scale. Due to the condition $R_{\alpha\beta} = 0$ of "flat" RC spacetime, this is a parallel transport of "sliding" without "rolling." In spaces with topological defects, the phase could be affected by the *gravitational* Aharanov–Bohm effect (MACÍAS et al. 1996). Since these Cartan loops carry triads along, they are inherently *framed* (BRÜGMANN et al. 1992; GUADAGNINI 1993) via one of the orthonormal legs $n^\alpha = n\rfloor\vartheta^\alpha$. In view of (6.4.25), it is a conjecture that the degeneracy problem (WITTEN 1989) of the Wilson-type loops in the Ashtekar approach is automatically avoided for $\text{GR}_\parallel$, and *diffeomorphism invariance* is retained.

It is well known from differential geometry that the Euler number $(1/16\pi^2)$ $\int R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta}$ and the Pontryagin integral $(1/8\pi^2) \int R_\alpha{}^\beta \wedge R_\beta{}^\alpha$ bear an intimate relationship to topology. However, in $\text{GR}_\parallel$, both vanish due to the teleparallelism constraint $R_{\alpha\beta} = 0$. Thus one can expect (MIELKE 1999) that gravitationally induced chiral anomalies in the coupling to Dirac fields are also absent. On the other hand, dynamical torsion perse is generally considered to be unrelated to topology, due to the following argument: For a real constant ϵ with $0 \leq \epsilon \leq 1$, the RC connection can be deformed in a continuous way into its Riemannian piece according to $\Gamma_{\alpha\beta}(\epsilon) = \Gamma^{\{\}}_{\alpha\beta} - \epsilon K_{\alpha\beta}$ such that $\Gamma_{\alpha\beta}(0) = \Gamma^{\{\}}_{\alpha\beta}$. Provided a topological quantity depends on the connection $\Gamma_{\alpha\beta}$, it must be invariant under this continuous deformation, and consequently, a continuous torsion two-form drops out; cf. MILNOR & STASHEFF (1974), as well as WU & ZEE (1984).

A different situation arises when torsion is *discontinuous* or has singularities (HANSON & REGGE 1979) as in the three-dimensional case of classical crystal dislocations (HEHL & KRÖNER 1965): a well-known example is the case of a pure vector torsion $T := e_\alpha\rfloor T^\alpha = d\sigma$ derived from a scalar field via a gradient (GREGORASH & PAPINI 1981). (Such a scalar field may measure the vorticity of the "coupled superfluid" and may be related field-theoretically to the Higgs phase or the conformal structure of spacetime.) The integration over a closed contour in the four-dimensional manifold, which may enclose the singularity line of the dislocation N times, yields

$$\int T = \int d\sigma = 2\pi N, \tag{6.6.1}$$

provided the scalar field is not single-valued, as, for instance, in the case of $\sigma = \arcsin(|x|)$.

For our spherically symmetric *torsion instanton* (6.4.28, 6.4.29), we expect that a nontrivial winding number will occur as a result of the homotopy group $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, the 3D hypersurface admits the intriguing mapping $\ell \underline{\vartheta}^A \to \underline{\Gamma}^{\star A} := \frac{1}{2}\eta^{ABC}\,\underline{\Gamma}_{BC}$ of the triads to the corresponding three-dual of the $\mathfrak{su}(2)$-valued connection (Baekler et al. 1992). Since this implies a mapping $\underline{C}_{\mathrm{TT}} \to \underline{C}_{\mathrm{RR}} + \Lambda' \underline{\eta}$ to the associated Chern–Simons term with "induced" cosmological constant Λ, we expect that the chiral translational CS term $\overset{(\pm)}{\underline{C}}_{\mathrm{TT}}$ will account for the same knot invariants as the Ashtekar connection, but in a more natural manner.

Wilson-type loops usually live on simply connected spacetime. However, on the small scale of "quantum gravity," the underlying spacetime manifold $\mathbb{R} \times W^3$ may have a foamlike structure, with a hypersurface that is not simply connected, such as the three-dimensional sphere with N wormholes (Mielke 1977; Nicolai & Niedermaier 1989) attached:

$$W^N := S^3 \#^N (S^1 \times S^2). \tag{6.6.2}$$

Thus, it would be interesting to see applications of Cartan circuits to spacetimes with *knot wormholes* (Mielke 1977) with their proper energy spectrum (Moffatt 1990), for which generalizations of torus knots (Labastida & Marino 2001) and links would occur, which cannot be shrunk to a point.

After reviewing the group-theoretic origins of the two parity-violating topological terms of Pontryagin and NY, we shall analyze the modifications of the gravitational gauge equations by such θ-terms. Then the topological amendment (6.7.2) provides an intriguing relation (6.7.12) for axial torsion $\mathscr{A}$, independent of RC curvature. This result has repercussions on teleparallelism constrained by (6.2.12), where the path-integral-type CS solution (6.4.23) of the quantum constraints are dominated by torsion instantons.

In classical EC theory, the net axial current production dj_5 seems (Chang & Soo 2003; Kaul 2008) to establish a link to the NY term via the Cartan relation (6.7.7). However, a careful analysis of the *axial and trace anomalies* (Kreimer & Mielke 2001; Mielke 2006) in gravity does not support this, but rather provides a relation to the *scale-invariant* Pontryagin term, including a $U(1)$-type four-form $d\mathscr{A} \wedge d\mathscr{A}$ involving the axial torsion. Since torsion instantons are characterized via (6.4.31) by the instanton number k, they would ultimately induce a periodic θ-vacuum of quantum gravity, similar to what occurs in Yang–Mills theory; cf. Mielke & Romero (2006).

6.7 Topologically Modified Einstein–Cartan Theory

Let us generalize the EC Lagrangian by including, besides the Pontryagin term, a dynamical coupling to the NY term, and in addition, liberate possible θ-angles to scalar fields.

This amounts to considering the topologically modified gravitational Lagrangian

$$L := L_{\mathrm{EC}} + L_\theta + L_{\mathrm{D}}, \tag{6.7.1}$$

where the θ-type boundary term

$$L_\theta = \theta_{\mathrm{T}} dC_{\mathrm{TT}} + \theta_{\mathrm{L}} dC_{\mathrm{RR}} \tag{6.7.2}$$

is a linear superposition[5] of the topological Nieh–Yan term and the Pontryagin four-forms. We recover parity or CP invariance (MIELKE et al. 1999) in the case that the θ-angles are axionlike pseudoscalars (MIELKE & ROMERO 2006).

In the translational field momentum,

$$H_\alpha := -\frac{\partial L}{\partial T^\alpha} = -\frac{\theta_{\mathrm{T}}}{\ell^2} T_\alpha, \tag{6.7.3}$$

there is only one torsion term, whereas the rotational field momenta

$$H_{\alpha\beta} := -\frac{\partial L}{\partial R^{\alpha\beta}} = \frac{1}{2\kappa}\eta_{\alpha\beta} - \frac{\theta_{\mathrm{T}}}{2\ell^2}\vartheta_\alpha \wedge \vartheta_\beta - \theta_{\mathrm{L}} R_{\alpha\beta} \tag{6.7.4}$$

of the EC theory is amended by two contributions from θ terms.

Due to this topological modification, the first gauge field equation reduces to

$$D\theta_{\mathrm{T}} \wedge T_\alpha + \theta_{\mathrm{T}} DT_\alpha + \ell^2 E_\alpha = -\ell^2 \Sigma_\alpha. \tag{6.7.5}$$

Using the second Bianchi identity (6.2.6) for the curvature, the second gauge field equation reduces to

$$\begin{aligned} \frac{1}{2\kappa} T^\gamma \wedge \eta_{\alpha\beta\gamma} - \frac{D\theta_{\mathrm{T}}}{2\ell^2} \wedge \vartheta_\alpha \wedge \vartheta_\beta + D\theta_{\mathrm{L}} \wedge R_{\alpha\beta} &= \vartheta_{[\alpha} \wedge \mu_{\beta]} \\ &= \frac{1}{4}\vartheta_\alpha \wedge \vartheta_\beta \wedge {}^* j_5. \end{aligned} \tag{6.7.6}$$

This is a generalization of the Cartan equation

$$C_{\mathrm{TT}} \cong \frac{\kappa}{4\ell^2} j_5 \tag{6.7.7}$$

[5]The translational angle $\theta_{\mathrm{T}} = 2/\gamma$ is at times identified (FREIDEL et al. 2005) with the inverse Barbero–Immirzi parameter γ. Such θ-terms and the canonical transformation induced by the translational Chern–Simons term dC_{TT} were considered earlier by MIELKE (1992).

for constant values of θ.

Note that the angular momentum part $\vartheta_{[\beta} \wedge H_{\alpha]}$ of the gauge fields induced by the NY term cancels identically against one part of $DH_{\alpha\beta}$. In the case of Dirac fields (MIELKE 2004), the *spin–energy potential* μ_α, a two-form, is related to the axial current three-form $j_5 = \overline{\psi}\,{}^*\gamma\gamma_5\psi$ via

$$\mu_\alpha = \frac{1}{4}\vartheta_\alpha \wedge {}^*j_5, \tag{6.7.8}$$

where the one-form $\gamma := \gamma_\alpha \vartheta^\alpha$ is Clifford-algebra-valued. At first sight, it appears that (6.7.6) for $D\theta_{\rm L} \neq 0$ provides torsion with a dynamical coupling to RC curvature. However, in view of the first Bianchi identity (6.2.7), this is not quite true: by contracting (6.7.6) with the coframe ϑ^α, it converts into

$$T^\gamma \wedge \eta_{\gamma\beta} + \kappa D\theta_{\rm L} \wedge DT_\beta = 0\,. \tag{6.7.9}$$

This is a first-order equation only for torsion, even in the presence of Dirac fields, since the antisymmetric piece of its spin–energy potential vanishes automatically, i.e., $\mu_\alpha \wedge \vartheta^\alpha = 0$, in view of (6.7.8).

Equivalently, it can be rewritten as

$$\kappa D\theta_{\rm L} \wedge DT_\beta = -T \wedge \eta_\beta = D\eta_\beta, \tag{6.7.10}$$

where the vector torsion one-form $T = e_\alpha \rfloor T^\alpha$ enters as an intermediate source. Due to the Poincaré lemma $DD\theta = dd\theta \equiv 0$ for a (pseudo) scalar field, Eq. (6.7.10) has the exact torsion solution

$$\kappa d\theta_{\rm L} \wedge T_\beta = -\eta_\beta \tag{6.7.11}$$

as a first integral. After a contraction with ϑ^β, the dual of the axial torsion one-form $\mathscr{A} := {}^*(\vartheta^\alpha \wedge T_\alpha)$, i.e., the translational CS term, turns out to be related to the volume four-form η via

$$\kappa\ell^2 d\theta_{\rm L} \wedge C_{\rm TT} = 2\eta. \tag{6.7.12}$$

This topological result is independent of the RC curvature. There occurs, however, a coupling to a kinetic term arising from the pseudoscalar field $\theta_{\rm L}$ rescaling the Pontryagin term (6.2.5).

Information on other irreducible torsion components can be obtained from (6.7.5) or from (6.7.6) after covariant differentiation, with the result that

$$DT^\gamma \wedge \eta_{\alpha\beta\gamma} + \kappa D\theta_{\rm T} \wedge T_{[\alpha} \wedge \vartheta_{\beta]}/\ell^2 = 2\kappa\left(T_{[\alpha} \wedge \mu_{\beta]} - \vartheta_{[\alpha} \wedge D\mu_{\beta]}\right). \tag{6.7.13}$$

Second-order derivatives of the axionlike field drop out, again due to the Poincaré lemma $DD\theta = dd\theta \equiv 0$, and a quadratic torsion term vanishes identically, i.e., $T^{[\gamma} \wedge T^{\mu]}\eta_{\alpha\beta\gamma\mu} = 0$. Again we end up with a first-order equation for torsion, where, however, the spin-energy potential μ_α (of Dirac fields) remains as a source.

6.8 Dynamics of Quadratic Poincaré Gauge Theory

In Poincaré gauge (PG) theory, the total action of interacting matter and gravitational gauge fields reads

$$W = \int \Big[L(\vartheta^\alpha, T^\alpha, R^{\alpha\beta}, \Psi, D\Psi) + L_g(\vartheta^\alpha, T^\alpha, R^{\alpha\beta}) \Big]. \qquad (6.8.1)$$

It is a functional of a (nonminimally) coupled matter field Ψ, which, in general, may be a p-form, and of the geometric variables ϑ^α and $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$. Their *independent* variations yield the following *field equations*:

$$\frac{\delta L}{\delta \Psi} = \frac{\partial L}{\partial \Psi} - (-1)^p\, D \frac{\partial L}{\partial D\Psi} = 0, \qquad \text{(MATTER)} \qquad (6.8.2)$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \qquad \text{(FIRST)} \qquad (6.8.3)$$

$$DH_{\alpha\beta} - E_{\alpha\beta} = \tau_{\alpha\beta}. \qquad \text{(SECOND)} \qquad (6.8.4)$$

Here the *gauge field momenta* are defined by

$$H_\alpha := -\frac{\partial L_g}{\partial d\vartheta^\alpha} = -\frac{\partial L_g}{\partial T^\alpha}, \qquad \text{and} \qquad H_{\alpha\beta} := -\frac{\partial L_g}{\partial d\Gamma^{\alpha\beta}} = -\frac{\partial L_g}{\partial R^{\alpha\beta}}. \qquad (6.8.5)$$

Part of the sources for the Yang–Mills-type divergence terms "DH" are the material *energy–momentum current*

$$\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} = \frac{\partial L}{\partial \vartheta^\alpha} + D \frac{\partial L}{\partial T^\alpha} \qquad (6.8.6)$$

and *spin current*

$$\tau_{\alpha\beta} := \frac{\delta L}{\delta \Gamma^{\alpha\beta}} = \rho(I_{\alpha\beta})\Psi \wedge \frac{\partial L}{\partial (D\Psi)} + \vartheta_{[\alpha} \wedge \frac{\partial L}{\partial T^{\beta]}} + D \frac{\partial L}{\partial R^{\alpha\beta}}, \qquad (6.8.7)$$

respectively. Our definitions take care of optional "Pauli-type terms" in the matter Lagrangian L. In addition, the 3-forms of energy–momentum

$$E_\alpha := \frac{\partial L_g}{\partial \vartheta^\alpha} = e_\alpha \rfloor L_g + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}, \qquad (6.8.8)$$

and the spin current

$$E_{\alpha\beta} := -\vartheta_{[\alpha} \wedge H_{\beta]} \qquad (6.8.9)$$

of the gravitational gauge fields themselves occur in (6.8.3) and (6.8.4), respectively. Intuitively, this is due to the universality of gravitational interactions. Observe that the antisymmetric piece of the energy–momentum current (6.8.8), formed with the aid of the coframe ϑ^α, in general gives us back the gauge Lagrangian L_g amended by Yang–Mills-type terms:

$$\vartheta^\alpha \wedge E_\alpha = 4L_g + 2T^\alpha \wedge H_\alpha + 2R^{\alpha\beta} \wedge H_{\alpha\beta}. \tag{6.8.10}$$

From local Poincaré invariance one "weakly" obtains the *first* and the *second Noether identities*

$$D\Sigma_\alpha \cong (e_\alpha \rfloor T^\gamma) \wedge \Sigma_\gamma + (e_\alpha \rfloor R^{\gamma\delta}) \wedge \tau_{\gamma\delta} \tag{6.8.11}$$

and

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} \cong 0, \tag{6.8.12}$$

respectively, provided that the matter field equation $\delta L/\delta\Psi = 0$ is satisfied. Note that in the differential identity (6.8.11) for the energy–momentum current there occur, on the right-hand side, Lorentz-type force densities of the general structure *field strength* $\times$ *current*. The first term corresponds to a translational force of Peach–Koehler type, and the second term is known as Mathisson–Papapetrou force (Hehl 1985).

If we discard a possible "bare" cosmological term, the gauge gravitational Lagrangian, which is at most *quadratic* in the irreducible pieces ${}^{(i)}T^\alpha$ and ${}^{(j)}R^{\alpha\beta}$ of torsion and curvature, respectively, can be written (Mielke 1984; Hehl 1985), using Euler's theorem on homogeneous functions, as

$$L_{\mathrm{qPG}} = -\frac{1}{2}T^\alpha \wedge H_\alpha - \frac{1}{2}R^{\alpha\beta} \wedge \left(H_{\alpha\beta} - \frac{a_0}{2\ell^2}\eta_{\alpha\beta}\right), \tag{6.8.13}$$

where the translational field momentum is given by

$$H_\alpha = -\frac{1}{\ell^2}\,{}^*\left(\sum_{i=1}^{3} a_i\,{}^{(i)}T_\alpha\right), \tag{6.8.14}$$

and the Lorentz–rotational field momentum by

$$H_{\alpha\beta} = -\frac{a_0}{2\ell^2}\,\eta_{\alpha\beta} - \frac{1}{\kappa}\,{}^*\left(\sum_{j=1}^{6} b_j\,{}^{(j)}R_{\alpha\beta}\right). \tag{6.8.15}$$

The gravitational coupling constant is absorbed in the Planck length $\ell \simeq 10^{-32}$ cm, whereas a_0, a_i $(i = 1, 2, 3)$, κ, and b_j $(j = 1, \ldots, 6)$ are dimensionless coupling constants.

The zeroth field equation arising from the variation of the metric $g_{\alpha\beta}$ is omitted here because it is known to be *redundant* "on shell," i.e., once the matter equation is satisfied.

In metric–affine extensions, there are not 6 but 11 irreducible pieces: five more quadratic terms have been proposed by Esser (1996) in an interesting decomposition. However, they may be partially related to the irreducible components of the topological Pontryagin and Euler invariants. For instance, the Ricci squared term (27) of Vassiliev (2005) appears to be part of the Euler invariant. From the corresponding second Noether identity there arises the generalized Bach–Lanczos identity, (A.3.7) of Hehl et al. (1995), which relates some of the a priori independent quadratic curvature pieces in the first of the two vacuum field equations as equivalent terms.

6.9 CP Violation in Quantum Gravity?

According to Treder (1975), Einstein indirectly discovered in 1925 the particle–antiparticle symmetry in physics. As an instructive example, let us consider, with Einstein, the Reissner–Nordstrøm metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (6.9.1)$$

which is an exact solution of the coupled Einstein–Maxwell equations. Applying a charge conjugation $C : Q \to Q' = -Q$, one obtains the same gravitational field. This, however, has to originate from exactly the same mass. Since the Einstein–Maxwell equations are invariant under CPT, charge conjugation C has to be accompanied by an additional time reversal T (particles moving backward in time $\hat{=}$ antiparticles) and by a space reflection P. Thus there is a particle–antiparticle symmetry already inherent in classical GR coupled to the Maxwell field.

However, a CP violation cannot be ruled out in the high-energy region of quantum gravity. In order to exhibit its possible origin, let us consider, for example, a qPG Lagrangian that is purely quadratic in torsion and curvature, i.e., one that is specified by the choice $a_0 = 0$, $a_i = 1$, and $b_j = 1$. In order to obtain its complexified version, we add, similarly as before, two Chern–Simons-type boundary terms. But in this more general setting, the boundary terms will be multiplied by two constant *complex* parameters θ_{T} and θ_{L}:

$$\begin{aligned} \overset{\theta}{L} = {} & \frac{1}{2\ell}\Big[T^\alpha \wedge {}^*T_\alpha + \theta_{\mathrm{T}}\,(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta)\Big] \\ & + \frac{1}{2}\Big[R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} + \theta_{\mathrm{L}}\, R^{\alpha\beta} \wedge R_{\alpha\beta}\Big]. \end{aligned} \quad (6.9.2)$$

Note that all 4-forms involving the Hodge star, as for example the volume form

$$\eta = {}^*1 = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\, \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta, \quad (6.9.3)$$

are invariant under space reflections $P : \vartheta'^A = -\vartheta^A$, where $A = 1, 2, 3$, due to the occurrence of the determinant of the metric in $\eta_{\alpha\beta\gamma\delta}$. In contradistinction to these dual forms, the θ-terms violate parity P; cf. Hojman et al. (1980). If the θ's are real, these terms will also violate CP. On the other hand, in the standard process of generating Ashtekar's variables, the θ's are purely imaginary, thus preserving CP invariance. According to a different strand of ideas (Ashtekar et al. 1989; Seriu & Kodama 1990), it is not indispensable to go over to complex variables, but then one would have to face CP violation.

The term $R_\alpha{}^\beta \wedge R_\beta{}^\alpha$ is known to yield, after integration over spacetime, a topological winding number, the *Pontryagin index*

$$p = \frac{1}{8\pi^2} \int R_\alpha{}^\beta \wedge R_\beta{}^\alpha . \tag{6.9.4}$$

Observe that no metric is involved here. Consequently, this formula should hold for "spacetime" manifolds of any signature and provide us with a genuine topological invariant. On the other hand, the translational θ_{T} term is not directly connected with the topology of spacetime. Nevertheless, for certain nontrivial configurations, this term could still acquire some topological significance, provided it is dynamically induced by the Pontryagin term via field equations. Configurations in which torsion is dynamically related to boundary terms in the curvature are Taub–NUT solutions with torsion, for which a *Witten-type effect* (Mielke 1985) has already been demonstrated: the θ_{L} term induces a "rotation" in the parameter plane spanned by the Schwarzschild mass M and the NUT parameter N ("dual mass").

The Yang–Mills type terms quadratically in the curvature are surmised to dominate in the high-energy region, and one would expect a CP violation by a real θ_{L} term in *quantum gravity* (Ashtekar 1988). On the other hand, the low-energy region is governed by the Hilbert–Einstein Lagrangian, i.e., in our teleparallel formulation by specific torsion-squared terms, supplemented by the θ_{T} terms. Because of the teleparallelism condition $R^{\alpha\beta} = 0$, one would naively expect that the Pontryagin term does not contribute in this case. However, teleparallelism holds only classically. Even above the Planck scale, we can at most require $\langle 0|R^{\alpha\beta}|0\rangle = 0$, but we could be forced to admit small curvature fluctuations with induced CP violation.

The generating function for our new complex variables is *necessarily* imaginary. If we go over from the (classical) Hamiltonian formulation to quantum theory, the complexification of the action will induce a nonunitary transformation of the corresponding operators such that in a Schrödinger representation, the states become *renormalized*, and the measure for the momentum representation becomes a nonlocal function (Fukuyama & Kamimura 1990). In gravity, this may have far-reaching consequences for the corresponding Wheeler–DeWitt equation (Wheeler 1968; Mielke 1977; Kiefer 1994) for the wave function of the universe.

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Chapter 7
Yang's Theory of Gravity

7.1 Introduction

It was not only the sixtieth anniversary of the Yang–Mills (YM) equation (YANG & MILLS 1954), but also the fortieth anniversary of Yang's theory of gravity (YANG 1974) that was commemorated in 2014. The historical route to $SU(2)$ gauge theory is laid out beautifully by MILLS (1989). Ramifications (MIELKE & HEHL 1988) are mainly due to the paper of SCHRÖDINGER (1932), in which the compact "Clifform" formula (MIELKE 2001) for the Riemannian curvature anticipated, to some extent, the concept of *gauge* curvature. In a letter to the author reprinted in MIELKE & MAGGIOLO (2005), the late Bob Mills confessed that he remained "still very much puzzled by gauge fields."

Here we will concentrate on the gravitational aspect: YANG proposed 1974 to generalize Einstein's general relativity (GR) by an *affine* gauge theory with a YM-type action. In fact, curvature-squared models had been considered before, first by WEYL in 1919 and then later by STEPHENSON (1958), HIGGS (1959), KILMISTER & NEWMAN (1961), as well as STELLE (1977); cf. SCHIMMING & SCHMIDT (1990) for more details.

Its scale invariance qualifies Yang's theory as an archetype of a fundamental theory of (quantum) gravity in the high-energy limit. In this chapter, we investigate its nonperturbative classical limit, corresponding to the most probable extremal "trajectories" in the Feynman path integral. In turns out that these are classical configurations with anti-self-double-dual curvature.

It was emphasized already by WEYL (1929) that curvature-squared models of gravity are scale-invariant, and therefore may convert into a fundamental theory of *quantum gravity* (QG) in the high-energy limit (HEHL et al. 1989), without invoking extra dimensions or supersymmetry; cf. KIBBLE & STELLE (1986). From the work of STELLE (1977), we know that the curvature-squared gravity in Riemannian spacetime is perturbatively *renormalizable* but plagued with ghosts (LEE & NE'EMAN 1990). However, by absorbing the quadratic Weyl curvature part of (7.3.1) into the

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Wess–Zumino action, these negative-metric states may be removed dynamically (Hamada 2000) and unitarity restored.

On the other hand, the path integral quantization a la Feynman (1988) eventually leads to a dimensional parameter, in this case Newton's constant $\kappa = 8\pi G_{\mathrm{N}}$ (in natural units), pertinent to the emerging Hilbert–Einstein Lagrangian for the classical background.

7.2 Dual Conformal Structure

Similar to the SO(4) gauge-invariant Yang–Mills theory, the dual representation[1] permits the construction of a double-dual curvature form

$$ {}^*\Omega^{g(\star)} := \frac{1}{4}{}^*R_{\alpha\beta}^{(\star)cd} L_{cd} \otimes \vartheta^\alpha \wedge \vartheta^\beta , \tag{7.2.2} $$

whereby the star operator $(\star)$ applies only to the Lie algebra. Independent of the signature of the base space, the repetition of this double-dual operation leads back to the original curvature form, i.e., it is *involutive*. We already encountered such terminology in studying the pseudoparticle solutions (Belavin et al. 1975) of Yang–Mills theories with SO(4) as structure group. Then, the decomposition

$$ \Omega^g = \overset{+}{\Omega}{}^g + \overset{-}{\Omega}{}^g \tag{7.2.3} $$

into the self-double-dual or anti-self-double-dual curvature 2-forms

$$ \overset{\pm}{\Omega}{}^g := \frac{1}{2}\left(\Omega^g \pm {}^*\Omega^{g(\star)}\right), \tag{7.2.4} $$

respectively, is a dissection into real eigenspaces.

Let Ω_{C} be the 2-form of a generalized curvature tensor (Nomizu 1972) that is locally equivalent to the *conformal* Weyl tensor

$$ C^{ab}_{..cd} = R^{ab}_{..cd} - 2R^{[a\cdot}_{\cdot[c}\delta^{b]}_{d]} + \frac{R}{6}\delta^a_{[c}\delta^b_{d]}. \tag{7.2.5} $$

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[1]The bispinor representation of the covering group $SL(2,\mathbb{C}) \approx \widetilde{SO}(3,1)$ is inapt for this purpose, since its infinitesimal generators $\sigma_{\alpha\beta} = \frac{1}{2}[\gamma_\alpha, \gamma_\beta]$ are neither self-dual nor anti-self-dual. As a direct sum of the spinor and its conjugate representation, they rather satisfy the relation $\sigma^{(\star)}_{\alpha\beta} = \gamma^5\sigma_{\alpha\beta}$, whereby the matrix $\gamma^5 := \frac{1}{4!}\varepsilon_{\mathrm{abcd}}\gamma^{\mathrm{a}}\gamma^{\mathrm{b}}\gamma^{\mathrm{c}}\gamma^{\mathrm{d}}$ is, within the Clifford algebra, an axial Lorentz scalar. However, (Bjorken & Drell 1965) the following generators are self-dual:

$$ L^{(\star)}_{ab} := \frac{1}{2}\varepsilon_{abcd}L^{cd} = L_{ab}. \tag{7.2.1} $$

.

Then the following decomposition is valid:

$$\Omega^g = \Omega_C + \bar{\Omega}^g - \frac{1}{6}\vartheta \wedge \vartheta R. \qquad (7.2.6)$$

This decomposition is even irreducible (DEBEVER 1956) if we restrict ourselves to a Riemannian manifold.

According to the "remarkable" relation

$$\bar{\Omega}^g = \vartheta \wedge {}^*(\vartheta \wedge \Omega^g) - \frac{1}{4}\vartheta \wedge \vartheta R, \qquad (7.2.7)$$

which was found first by LANCZOS (1938), or in components

$$\bar{R}_{..cd}{}^{ab} = 2\not{R}^{[a}_{\cdot[c}\ \delta^{b]}_{d]}; \qquad \not{R}^a_{\cdot b} := R^a_{\cdot b} - \frac{1}{4}\delta^a_b R, \qquad (7.2.8)$$

the anti-self-double-dual part spans the orthogonal complement to Weyl's curvature 2-form (NOMIZU 1972) in the space of the trace-free generalized curvature forms, for which

$$\mathrm{Tr}(\vartheta \wedge \vartheta \wedge \Omega_C) = \mathrm{Tr}(\vartheta \wedge \vartheta \wedge \bar{\Omega}^g) = 0 \qquad (7.2.9)$$

is valid.

In a four-dimensional Riemannian manifold with the Euclidean signature $s = 0$, both the Hodge star operating on differential forms and the Lie dual ($\star$) are involutive, a fact that can be explained by the special isomorphism

$$\mathfrak{so}(4) \approx \mathfrak{so}(3) \times \mathfrak{so}(3) \approx \mathfrak{su}(2) \times \mathfrak{su}(2). \qquad (7.2.10)$$

(HELGASON 1962, p. 204). This, however, even allows it to refine the decomposition (7.2.6) and to split Weyl's curvature 2-form into

$$\Omega_C = \overset{(+)}{\Omega}_C + \overset{(-)}{\Omega}_C, \qquad \overset{(\pm)}{\Omega}_C := \frac{1}{2}\left(\Omega_C \pm \Omega_C{}^{(*)}\right) \qquad (7.2.11)$$

(SINGER & THORPE 1969). Since the Weyl tensor and the duality operator, when applied to a 2-form, are invariant under conformal changes of metrics

$$g_{ij} \to \bar{g}_{ij} = \phi^L g_{ij}, \qquad (7.2.12)$$

cf. MIELKE (1977a), it clearly follows that the condition

$$\overset{(\mp)}{\Omega}_C = 0 \qquad (7.2.13)$$

determines a *self-dual or anti-self-dual conformal structure* on an orientable four-dimensional Riemannian manifold. A classification of the corresponding eigenspaces is to be found in CHAO-HAO et al. (1978) and XIN (1980).

7.3 SKY Gravity

When written in differential forms of Cartan, the so-called Stephenson–Kilmister–Yang (SKY) Lagrangian (STEPHENSON 1958; KILMISTER & NEWMAN 1961), (YANG 1974) is given by the purely quadratic 4-form

$$L_{\mathrm{SKY}} = \mathrm{Tr}(\Omega^g \wedge {}^*\Omega^g) = -\frac{1}{2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta}\,, \tag{7.3.1}$$

where $R_{\alpha\beta}$ is the curvature two-form. In WEYL (1919), in the concluding chapter of that famous work it was pointed out that from a gauge-theoretic point of view, (7.3.1) has to be appreciated as a rather natural choice. Nevertheless, and especially with regard to an improvement of the tensor dominance model (ISHAM et al. 1971) of strong interaction, this model is of sufficient relevance to justify a further analysis. Classically, the short-time initial value problem of Yang's *vacuum* equations

$$D^*\Omega^g = 0, \tag{7.3.2}$$

or

$$D\,{}^*R_{\alpha\beta} = 0, \tag{7.3.3}$$

in anholonomic components is well posed (GUILFOYLE & NOLAN 1998), in contrast to some other quadratic curvature Lagrangians including boundary terms; cf. (BAEKLER & HEHL 2011).

These follow from varying the Lagrangian with respect to the connection one-form $\Gamma^{\alpha\beta} := \Gamma_i{}^{\alpha\beta} dx^i$. In his purely affine approach, Yang did not consider the canonical *energy–momentum* current. More explicitly, this was left to STEPHENSON (1958), who added to the vacuum equations the energy–momentum current

$$E_\alpha := \partial L_{\mathrm{SKY}}/\partial \vartheta^\alpha = \frac{1}{2}\left(e_\alpha \rfloor R^{\mu\nu} \wedge {}^*R_{\mu\nu} - R^{\mu\nu} \wedge e_\alpha \rfloor {}^*R_{\mu\nu}\right) = 0 \tag{7.3.4}$$

of the gravitational gauge fields themselves. They arise readily via variation of (7.3.1) with respect to the coframe $\vartheta^\alpha = E_j{}^\alpha\, dx^j$ dual to the local anholonomic frame e_α.

In fact, the expression (7.3.4) generalizes the current three-form

$$G_\alpha^{\{\}} := \frac{1}{2} R^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma}\,, \qquad G_{ij} := \mathrm{Ric}_{ij}^{\{\}} - \frac{1}{2} g_{ij}, \tag{7.3.5}$$

which is dual to the usual Einstein tensor G_{ij} in Riemannian spacetime. In exterior form notation, the symmetric Ricci tensor is the holonomic version of the zero-form $\mathrm{Ric}_{\alpha\beta} := (-1)^s\,{}^*(R_{(\alpha}{}^{\delta} \wedge \eta_{\delta|\beta)})$.

Macroscopically, the matter coupling[2] to the spin current three-form $\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge \mu_{\beta]}$ seems to disfavor such models, a criticism which has already raised by FAIRCHILD JR (1976). However, at short distances or very high energies, spinfoam approaches (ORITI & TLAS 2006) to QG may provide a natural extension.

In Riemannian spacetime, with its Levi–Civita connection $\Gamma^{\{\}\alpha\beta} \propto \partial g$, these equations are, respectively, of third and second order (THOMPSON 1975) in the metric. However, all the metric information can be encoded into the "decent" equations (7.3.7) and (7.3.8) for classical configurations with self-double-dual or anti-self-double-dual curvature.

7.3.1 *Double-Dual SKY Gravity*

In order to exhibit more clearly the intriguing instanton or pseudoparticle content of Yang's theory, the SKY Lagrangian is supplemented by the topological Euler term (7.7.49) as a boundary term, i.e.,

$$\begin{aligned} L^{(\star)}_{\mathrm{SKY}} &= -\frac{1}{2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} - \frac{(-1)^s}{2} R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \\ &= -\frac{1}{4}\left(R_{\alpha\beta} + {}^*R^{(\star)}_{\alpha\beta}\right) \wedge {}^*\left(R^{\alpha\beta} + {}^*R^{\alpha\beta(\star)}\right) \\ &= \frac{1}{2}\mathrm{Tr}(\bar{\Omega}^g \wedge {}^\star\bar{\Omega}^g)\,. \end{aligned} \tag{7.3.6}$$

In this partially topological Lagrangian, we distinguish between the Hodge dual * and the Lie dual $^{(\star)}$ of the curvature $R_{\alpha\beta}$ in a space(time) of signature s.

It is obvious from the equivalent binomial form of the Lagrangian that there arises a branching of SKY gravity into two exact subspaces with respect to *double duality*: anti-self-dual solutions

$$R_{\alpha\beta} = -\,{}^*R^{(\star)}_{\alpha\beta}, \tag{7.3.7}$$

i.e., Einstein spaces (MIELKE 1981), *annihilate* the corresponding partially topological action, whereas the self-dual spaces

$$R_{\alpha\beta} = {}^*R^{(\star)}_{\alpha\beta}, \tag{7.3.8}$$

i.e., so-called THOMPSON (1975) spaces, are extremals in the Lagrangian functional, i.e., they lead to $L^{(\star)}_{\mathrm{SKY}} = 2L_{\mathrm{SKY}}$. Both second-order duality conditions satisfy Yang's

[2]For Dirac fields ψ, the *spin-energy potential* $\mu_\alpha = \vartheta_\alpha \wedge {}^*j_5/4$ is dual (MIELKE 2004) to the axial current $j_5 := \overline{\psi}\,{}^*\gamma\gamma_5\psi$.

equation (7.3.3) due to the Bianchi identity (7.7.42) for the curvature, as well as its Lie dual, due to $D\eta_{\alpha\beta} = 0$ in a Riemannian spacetime. More on this "vacuum degeneracy" can be found in VASSILIEV (2002), where such a dual reformulation has been called Yang–Mielke theory of gravity.

In accordance with our general procedure, these equations are to be satisfied by the double-duality ansatz

$$^{*}\Omega^{g(*)} = \zeta\,\Omega^{g}, \qquad (7.3.9)$$

whereby only $\zeta = \pm 1$ (and zero) are admitted.

On behalf of the decomposition (7.2.7) of the curvature tensor, the eigenspaces respecting duality in a Riemannian spacetime can be characterized as follows (SINGER & THORPE 1969):

$\zeta = +1$: *self-double-dual curvature*

$$\overline{\Omega}^{\{\}} = 0 \Longleftrightarrow \Omega_{\mathrm{C}}^{\{\}} = 0 \quad \text{and} \quad R^{\{\}} = 0. \qquad (7.3.10)$$

$\zeta = -1$: *anti-self-double-dual curvature*

$$\overset{+}{\Omega}{}^{\{\}} = 0 \Longleftrightarrow \not{R}_{ij}^{\{\}} = 0 \Longleftrightarrow R_{ij}^{\{\}} = \Lambda' g_{ij}. \qquad (7.3.11)$$

Subsequently, it follows (MIELKE 1981; BAEKLER et al. 1982) that Yang's theory, in the case of anti-self-double-dual curvature, reduces to Einstein's GR with an *arbitrary* cosmological term, while the self-double-dual case reproduces NORDSTRÖM's vacuum theory (1913) as reformulated in 1914 by EINSTEIN & FOKKER; cf. MTW, p. 429.

Concentrating on topological terms such as those of Pontryagin and Euler, related self-dual modifications are advocated as *topological 4D self-dual gravity* by NAKAMICHI et al. (1991). There, self-dual and anti-self-dual solutions "live" on Einstein spaces as well. The addition of the Pontryagin term with respect to the Riemannian curvature $R_{\alpha\beta}^{\{\}}$ and the axial torsion one-form $\mathscr{A} := {}^{*}(\vartheta_\alpha \wedge T^\alpha)$ is motivated by the *axial anomaly* $\langle dj_5\rangle = 2i\,{}^{*}m\langle\overline{\psi}\gamma_5\psi\rangle - \left(R_{\alpha\beta}^{\{\}} \wedge R^{\{\}\alpha\beta} + \frac{1}{2}d\mathscr{A}\wedge d\mathscr{A}\right)/48\pi^2$ in the coupling to Dirac fields ψ.

7.4 The Path Integral Dominated by Einstein Spaces

The path integral formulation of Feynman, as later exemplified in the case of the H-atom (DURU & KLEINERT 1979), implies a reconciliation of quantum theory with classical physics, the latter providing the most probable contribution to all possible amplitudes. In the gravitational context, Einstein spaces are emerging as the most probable classical background when properly chosen boundary conditions are imposed for the SKY Lagrangian (7.3.6) modified by the topological Euler term.

Analogous to the internal YM case, where the path integral is dominated by instanton solutions with finite action—cf. Chap. 16 of the textbook by CHENG & LI (1984) for details—in the path integral approach to quantum gravity, the quantum-mechanical transition amplitude

$$\int \mathscr{D}\Gamma \exp\left[\int L^{(\star)}_{\mathrm{SKY}}/\hbar\right] \\ = \int \mathscr{D}\Gamma \exp\left[-\int \left(R_{\alpha\beta} + {}^*R^{(\star)}_{\alpha\beta}\right) \wedge {}^*\left(R^{\alpha\beta} + {}^*R^{\alpha\beta(\star)}\right)/4\hbar\right] \tag{7.4.1}$$

arises, where the four-dimensional integration over d^4x is already implied in our four-form notation.

After a Wick rotation, in an imaginary "spacetime" with Euclidean signature, cf. MIELKE (1984a), the evaluation of the path integral is dominated by anti-self-double-dual *instanton*-type configurations (CHAO-HAO et al. 1978; JACKIW 2005) near the classical ones, i.e., Einstein spaces. Accordingly, they are much more probable than the "spurious" Thomson spaces, which are heavily "damped" due to $L^{(\star)}_{\mathrm{SKY}} = 2L_{\mathrm{SKY}}$ in this case. (This "vacuum degeneracy" can be lifted (MIELKE 1984b; ZHYTNIKOV 1994) via a modified duality ${}^*R_{\alpha\beta} = \theta^{\star}_{\mathrm{L}} R^{(\star)}_{\alpha\beta} + (\theta^{*}_{\mathrm{T}}\Lambda/6)\ \eta_{\alpha\beta}$, which, however, explicitly breaks scale invariance. For torsionless configurations, only Einstein's GR surfaces, consistently coupled to the Belinfante–Rosenfeld (MIELKE et al. 1989; MIELKE & MAGGIOLO 2005) symmetrized energy–momentum current $\sigma_\alpha := \Sigma_\alpha - D^{\{\}}\mu_\alpha$.)

As is well known, the Riemannian curvature 2-form can be decomposed into

$$R_{\alpha\beta} \equiv C_{\alpha\beta} + \frac{1}{2} Ri\not{c}_{\alpha\beta} - \frac{R}{12}\vartheta_\alpha \wedge \vartheta_\beta, \tag{7.4.2}$$

with the conformal Weyl curvature, the trace-free Ricci curvature, and its remaining trace with $R := e_\alpha \rfloor e_\beta \rfloor R^{\alpha\beta}$ as irreducible pieces. Then the double-dual curvature inherits, following GÉHÉNIAU & DEBEVER (1956), WALLNER (1983), MC CREA (1987), HAMMON & NORRIS (1993), the decomposition

$${}^*R^{(\star)}_{\alpha\beta} = -C_{\alpha\beta} + \frac{1}{2} Ri\not{c}_{\alpha\beta} + \frac{R}{12}\vartheta_\alpha \wedge \vartheta_\beta. \tag{7.4.3}$$

Accordingly, the double-self-dual Thompson spaces (7.3.8) are characterized by vanishing Weyl and scalar curvatures, i.e.,

$$C_{\alpha\beta} = 0 \quad \text{and} \quad R = 0, \tag{7.4.4}$$

such that exact solutions are conformally flat,

$$g_{ij} = \Omega^2 o^{\mathrm{Mink}}_{ij}, \tag{7.4.5}$$

with o_{ij}^{Mink} denoting the (pseudo)orthogonal Minkowski metric. Earlier, EINSTEIN & FOKKER (1914) had considered this in their covariant reformulation of NORDSTRÖM's scalar theory of 1913. In the path integral approach to QG, however, such a classical background is exponentially suppressed and therefore less probable.

Rather, the dominating part comes from the anti-self-double-dual configurations (7.3.7), which are equivalent to

$$\not{Ric}_{\alpha\beta} := \mathrm{Ric}_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} = 0. \tag{7.4.6}$$

A model with vanishing trace-free Ricci tensor in vacuum was tentatively considered by Einstein and Grossmann in their *Entwurftheorie* and was to some extent resurrected by WILCZEK (1998).

Due to the contracted Bianchi identity $D^{\{\}} G_{\alpha}^{\{\}} \equiv 0$ in Riemannian spacetime, the curvature scalar is constant, i.e., more precisely, $R = -4\Lambda$. Thus, (7.4.6) is equivalent to Einstein's equations

$$G_{\alpha} - \Lambda\, \eta_{\alpha} = 0 \tag{7.4.7}$$

with cosmological constant $\Lambda \simeq 1/\kappa$. It is crucial to note that the latter carries physical dimension of mass squared, thereby realizing the "dimensional transmutation" necessary for inducing a mass gap.

7.5 Graviton Spectrum

Let us consider the conformally covariant d'Alembertian

$$\Box^{c} := \Box - \frac{1}{6} R, \quad \Box := (1/\sqrt{g}) \partial_i (\sqrt{g} g^{ij} \partial_j). \tag{7.5.1}$$

Then for conformally flat spacetimes,

$$\Box^{c} g_{ij} = \Box^{c} (\Omega^2 o_{ij}^{\mathrm{Mink}}) = \Omega^6 \Box^{c} o_{ij}^{\mathrm{Mink}} = 0 \tag{7.5.2}$$

holds. This tells us that gravitons are *exactly massless* propagating modes on a classical Einstein–Fokker background due to $R = 0$ in double-dual Thompson spaces.

On the other hand, the linearized Einstein equations for $h_{ij} := g_{ij} - g_{ij}^{\mathrm{dS}}$ in a de Sitter background g_{ij}^{dS} with a positive cosmological constant are

$$\left(\Box - \frac{2}{3}\Lambda\right) h_{ij} = 0 \tag{7.5.3}$$

when the harmonic or de Donder gauge

$$\partial^i h_{ij} - \partial_j h = 0 \tag{7.5.4}$$

is imposed on the tensor modes.

This means that the lowest-order graviton propagator (ALLEN 1986) of the partially "topological" SKY gravity will exhibit a real *mass gap*

$$\Delta m = \sqrt{2\Lambda/3} > 0 \tag{7.5.5}$$

in an expanding universe with a de Sitter background, due to the induced cosmological constant occurring in the dominating branch (7.4.7).

Massive models of gravity (HINTERBICHLER 2012) are commonly plagued by the van Dam–Veltman–Zakharov (vDVZ) discontinuity (VAN DAM & VELTMAN 1970; ZAKHAROV 1970; IWASAKI 1970), and thus disfavored by observations of gravitational light bending. However, in the de Sitter background, the construction of two-point functions is still a subject of debate (BORCHERS & BUCHHOLZ 1999; YOUSSEF 2013) for massless scalar fields. Even the notion of masslessness is not unambiguous, since one may require conformal invariance instead; cf. MIELKE (1977b). According to KOGAN et al. (2001), the vDVZ discontinuity can be avoided for gravitons in de Sitter space if m/H is kept finite in the massless limit. For gravitons, the mass of the corresponding helicity-two states are below the recent observational limits (GOLDHABER & NIETO 2010) of $m \leq 7 \times 10^{-32}$ eV, but larger than the Hubble scale of $H \simeq 10^{-33}$ eV.

7.6 Mass Gap in Yang–Mielke Theory of Gravity?

In Yang's gravitational theory, via a double duality decent, the mass gap (7.5.5) is induced via the tiny cosmological constant $\Lambda_{\text{obs}} > 0$ of our expanding universe. Invoking a Mach-type principle, a "dimensional transmutation" arises due to a degeneracy of the trace-free Ricci model, in which R is an arbitrary constant of dimension mass squared.

Comparing this with the quantization of the YM field, its dimensionless coupling constant $1/g^2$ is 'traded" for a dimensional one. This "dimensional transmutation" à la Sidney Coleman is absent in the classical theory in four dimensions; cf. FEYNMAN (1981). One of the Clay Millennium Problems (JAFFE & WITTEN 2006) requires the conclusive demonstration that there exist a mass gap in the spectrum of YM fields, i.e., "the question of the mass of the **b** quantum" according to the original formulation of YANG & MILLS (1954). Classically, the YM theory is scale-invariant, whereas its divergences due to field quantization appear to be beneficial (FADDEEV 2002) for the generation of such a dimensional parameter.

Originally, Yang and Mills formulated their model as an $SU(2)$ gauge theory with some hindsight (Yang 2012) to Einstein's gravity. Its gravitational analogue (Yang 1974) is an $SO(1,3)$ gauge theory, whose covering gauge group, after a Wick rotation to Euclidean space in the path integral quantization, is locally isomorphic to $\overline{SO}(4) \cong SU(2) \otimes SU(2)$. This doubling of the (internal) unitary structure group $SU(2)$ was used by Belavin et al. (1975) to explicitly construct *instanton* or "pseudoparticle" solutions of the YM theory. This construction is at the root of the *double* duality used here for the space(time) curvature in the gravitational case.

No solution to the Clay problem is presented here. It is merely indicated how such a "dimensional transmutation" may occur in QG in view of the above-mentioned group-theoretic correspondence. Conversely, in 2007, Faddeev & Niemi considered "the tantalizing possibility that long distance Einstein gravity metamorphoses into a renormalizable Yang–Mills theory at short distances."

Moreover, according to an indirect argument of Kholodenko (2011), a de Sitter-type gravitational background may assist the generation of a mass gap for internal YM fields as well as for chiral fermions (Flachi & Fukushima 2014) in QCD. However, the analogy presented here focuses more on *duality transformations* (Jaffe & Witten 2006) within the path integral formulation of double-dual SKY gravity.

In the scenario (Daum & Reuter 2013; Mielke 2013a) of asymptotic safety, the gravitational constant κ, as well as Λ, runs toward an ultraviolet fixed point. Consequently, similarly (Gogokhia 2014) to the case of QCD, dual SKY gravity may exhibit a "renormalized" or asymptotically safe mass gap.

On the other hand, in generalizing Yang's theory to a topological $SL(5,\mathbb{R})$ gauge model of gravity, it is challenging that a (spontaneous) *symmetry breaking* should occur at the tiny *dimensionless scale* of $\kappa\Lambda_{\text{obs}} \simeq 10^{-123}$, as suggested by observations; cf. Mielke (2012, 2013b).

7.7 Field Redefinition Scheme of Renormalization

In perturbative quantum gravity (Alvarez 1989), there arise counterterms ΔL of higher order in the curvature. According to 't Hooft (1974), Dietz & Rollnick (1975), these terms can be simulated, already on the classical level, by a first-order *field redefinition* (FR)

$$g_{ij} \to \widetilde{g}_{ij} = g_{ij} + a\mathrm{Ric}_{ij} + b g_{ij}\,\mathrm{Ric}_k{}^k \tag{7.7.1}$$

of the metric. In exterior form notation, the symmetric Ricci tensor is the holonomic version of the zero-form $\mathrm{Ric}_{\alpha\beta} := {}^*(R_{(\alpha}{}^{\delta} \wedge \eta_{\delta|\beta)})$.

More generally, in a gauge framework based on the (broken) Poincaré group, the independent variables are the one-forms ϑ^α and $\Gamma^{\alpha\beta}$. In applying this to a Yang–Mills-type formulation of gravitational interactions, such an FR is, in general, nonlinear (Mielke & Hehl 1991; Mielke 2006) and dictated by the appropriate form degree and the correct physical dimension as follows:

$$\vartheta^{\alpha} \quad \rightarrow \quad \widetilde{\vartheta}^{\alpha} := \vartheta^{\alpha} + \ell^2\, e_{\beta}\rfloor\, {}^*H^{\alpha\beta}\,, \tag{7.7.2}$$

$$\Gamma_{\alpha}{}^{\beta} \quad \rightarrow \quad \widetilde{\Gamma}_{\alpha}{}^{\beta} := \Gamma_{\alpha}{}^{\beta} + \ell^2\, e_{\alpha}\rfloor\, {}^*H^{\beta}\,. \tag{7.7.3}$$

Here the field momenta H_{α} and $H_{\alpha\beta}$ are understood as arising from a generating n-form G as part of some effective gauge Lagrangian L_{eff} that includes the counterterms from the sought after renormalization. Observe also that a *fundamental length* ℓ squared necessarily occurs for dimensional reasons.

The dual unit two-form then gets deformed according to

$$\eta_{\alpha\beta} \quad \rightarrow \quad \widetilde{\eta}_{\alpha\beta} = \eta_{\alpha\beta} + \ell^2 \eta_{\alpha\beta\gamma} \wedge e_{\mu}\rfloor\, {}^*H^{\mu\gamma} + \frac{\ell^4}{2}\eta_{\alpha\beta\gamma\delta}(e_{\mu}\rfloor\, {}^*H^{\gamma\mu}) \wedge (e_{\nu}\rfloor\, {}^*H^{\delta\nu}). \tag{7.7.4}$$

In our dynamical approach, the $(n-2)$-forms H_{α} and $H_{\alpha\beta}$ will be gauge field momenta canonically conjugate to the coframe and the Lorentz connection, respectively. Due to the *semidirect product* structure of the Poincaré group $P := \mathbb{R}^4 \rtimes SO(1,3)$, the gauge field momenta contribute to the gauge potentials via $H^{\alpha\beta} \rightarrow \vartheta^{\alpha}$ and $H^{\alpha} \rightarrow \Gamma^{\alpha\beta}$ in the FR (7.7.2, 7.7.3) just in an *intertwined* manner.

In the field redefinition (7.7.3) of the connection, we could have included, similarly as for Yang–Mills fields, a term proportional to ${}^*DH_{\alpha}{}^{\beta}$. However, "on shell," i.e., when the second vacuum field equation is satisfied, this is equivalent to ${}^*\left(H^{\beta} \wedge \vartheta_{\alpha}\right) \equiv e_{\alpha}\rfloor\, {}^*H^{\beta}$ due to an algebraic identity. In the FR (7.7.2) of the coframe, the same situation arises, with the modification that the "on shell" term ${}^*DH^{\alpha} \cong {}^*E^{\alpha}$ is of second order in the field strength and therefore would be equivalent to a higher-order generation functional $\widetilde{G}$. When coupling to matter, FRs have to be handled with care, because they may induce violations of the macroscopic principle of equivalence; cf. Brans (1988).

7.7.1 Legendre Transformation

For exhibiting physically equivalent gauge field Lagrangians via a Legendre transformation, let us depart from the Hilbert–Einstein Lagrangian of GR or the Einstein–Cartan Lagrangian

$$L_{\text{EC}} = -\frac{1}{2\kappa^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} \tag{7.7.5}$$

as our prime field-theoretic "nucleus."

Let us compare this with the more general Lagrangian

$$\widetilde{L} = -\sum_{k=0}^{K} (1/2k)\, R^{\alpha\beta} \wedge \overset{(2k)}{H}{}_{\alpha\beta} + L_{\theta}\,, \tag{7.7.6}$$

which is quadratic, quartic, etc. in the curvature. The first term in this expansion corresponds to the SKY Lagrangian quadratic in the curvature. The gauge field momenta $\widetilde{H}_{\alpha\beta} := -\partial\widetilde{L}/\partial R^{\alpha\beta}$ can be expanded as $\widetilde{H}_{\alpha\beta} = \overset{(2)}{H}_{\alpha\beta} + \overset{(4)}{H}_{\alpha\beta} + \cdots$. (For the time being, the field momentum $\widetilde{H}_\alpha := -\partial\widetilde{L}/\partial T^\alpha$ conjugate to the torsion $T^\alpha := D\vartheta^\alpha$ will be set to zero.)

Then we can infer the resulting *Yang–Mills-type field equations*

$$- \widetilde{E}_\alpha := -e_\alpha \rfloor \widetilde{L} - (e_\alpha \rfloor R^{\beta\gamma}) \wedge \widetilde{H}_{\beta\gamma} = \Sigma_\alpha \,, \tag{7.7.7}$$

$$D\widetilde{H}_{\alpha\beta} = \tau_{\alpha\beta} \,. \tag{7.7.8}$$

However, the *Legendre transformation* (Jakubiec & Kijowski 1988)

$$\widetilde{L} \;\rightarrow\; L = -\frac{1}{2}\left(R^{\alpha\beta} \wedge \widetilde{H}_{\alpha\beta} - \widetilde{L}\right) \tag{7.7.9}$$

provides physically equivalent gravitational dynamics. (The overall factor 1/2 is chosen such as to render the EC Lagrangian invariant.) The new rotational gauge field momenta

$$H_{\alpha\beta} := -\frac{\partial L}{\partial R^{\alpha\beta}} = \widetilde{H}_{\alpha\beta} + \frac{1}{2} R^{\mu\nu} \wedge (\partial\widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta}) \tag{7.7.10}$$

will depend on the *Hessian* (n-4) form

$$\widetilde{H}_{\alpha\beta\mu\nu} := \frac{\partial^2\widetilde{L}}{\partial R^{\mu\nu}\partial R^{\alpha\beta}} = -\frac{\partial\widetilde{H}_{\mu\nu}}{\partial R^{\alpha\beta}} \tag{7.7.11}$$

of the higher-order Lagrangian that we started with.

7.7.2 Vanishing Hessian: GR as a Stable Fixed Point

Since classical GR or EC theory tends to surface from different higher-order models, we have an infinite ambiguity in such a "renormalization" program; cf. Kaku (1993), p. 210. However, we can improve this by showing that EC theory is a *stable* fixed point of the quadratic SKY gravity.

For a *fixed point* of the transformation (7.7.9), the Hessian $\widetilde{H}_{\alpha\beta\mu\nu}$ obviously has to vanish. This condition, i.e., $\partial\widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta} = 0$, can be readily solved.

If parity-violating terms such as $\theta_{\mathrm{T}}\, R^{\alpha\beta} \wedge \vartheta_\alpha \wedge \vartheta_\beta$ arising from the Nieh–Yan term (7.7.46) are admitted, then the relation

$$\widetilde{\eta}_{\alpha\beta} := \frac{\theta_{\mathrm{T}}^*}{2}\eta_{\alpha\beta} - \frac{\theta_{\mathrm{T}}}{2}\,\vartheta_\alpha \wedge \vartheta_\beta - \ell^2\,\widetilde{H}_{\alpha\beta} = 0 \tag{7.7.12}$$

can be regarded as a singular FR derived from (7.7.2) via some appropriate effective Lagrangian $\widetilde{L}$.

Accordingly, $\eta_{\alpha\beta}$ and $\widetilde{H}_{\alpha\beta}$ interchange their roles as generalized coordinates and momenta, respectively. If we had started from $\widetilde{L} = L_{\rm EC}$, then we would be led back to $L = L_{\rm EC}$ for the choice $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = 0$. In the case $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = i$, this leads to a *chiral* formulation of gravity. Thus, the EC Lagrangian or its chiral version remains *"stable"* under FR, provided that it is embedded in a class of gravitational Lagrangians for which $L_{\rm EC}$ is located at some local functional minimum.

More recently, the asymptotic freedom of Einstein's GR with such "on-shell" redundant higher curvature terms has been analyzed by SLOVICK (2013).

Our gauge framework clearly exhibits the coupling to fundamental matter, such as to the Dirac field. However, as has been stressed by DIETZ & ROLLNICK (1975), an FR of the *coframe* may ruin the nice features of the Dirac Lagrangian, which in GR and its RC extensions, has to be formulated in terms of ϑ^α only in a *multiplicative* manner: otherwise, dangerous derivative couplings would occur in the Dirac equation, and the positivity of energy could get lost during this procedure.

On the other hand, in the transformation to Ashtekar's complex variables, the coframe is kept fixed, whereas the connection is subject to the *complex* field redefinition $\Gamma_\alpha{}^\beta \;\rightarrow\; \overset{(\pm)}{\Gamma}{}_\alpha{}^\beta := \Gamma_\alpha{}^\beta \mp (i\ell^2/2)\, e_\alpha \rfloor \overset{(\mp)}{H}{}^\beta$, induced by the translational Chern–Simons term $idC_{\rm TT}$. The resulting Sen-type connection still couples *minimally* to the Dirac field, but poses the issue of *reality conditions*.

Appendix: Clifford-Algebra-Valued Exterior Forms

More concise formulations employ Clifford-algebra-valued differential forms, or "Cliffordms" for short (MIELKE 2001), as described here.

Dual Basis

In a topologically trivial frame bundle, the *Hodge dual* * of exterior forms is defined such that the normalization

$$^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta) = \eta^{\alpha\beta\gamma\delta}, \quad \text{where} \quad \eta_{\alpha\beta\gamma\delta} := +\delta^{0123}_{\alpha\beta\gamma\delta}, \tag{7.7.13}$$

holds. Applied to p-forms, it is almost involutive, i.e., $^{**}\Phi = (-1)^{p(4-p)+s}\Phi$. For spacetimes in which $s := \text{sig} = 1$ holds, it induces an *almost complex structure*, and in the case of two-forms, it is *conformally invariant* (BRANS 1974; ATIYAH et al. 1978).

The volume four-form

$$\eta := \eta_{\alpha\beta\gamma\delta}\, \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta/4! \tag{7.7.14}$$

makes it possible to generate the so-called η, or dual, basis $\{\eta,\ \eta_\alpha,\ \eta_{\alpha\beta},\ \eta_{\alpha\beta\gamma}\ \eta_{\alpha\beta\gamma\delta}\}$ of exterior forms via consecutive interior products: $\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = \eta_{\alpha\beta\gamma}\, \vartheta^\gamma = e_\beta \rfloor e_\alpha \rfloor \eta = {}^*(\vartheta_\alpha \wedge \vartheta_\beta) = \frac{1}{2}\eta_{\alpha\beta\gamma\delta}\, \vartheta^\gamma \wedge \vartheta^\delta$, and $\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$. Anholonomic indices are lowered by $o_{\alpha\beta} = e^i{}_\alpha\, e^j{}_\beta\, g_{ij}$, where $o_{\alpha\beta}$ denotes the Minkowski metric.

Distances in space(time) are measured by

$$ds^2 = o_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta = g_{ij} dx^i \otimes dx^j\,, \quad o_{\alpha\beta} := \mathrm{diag}\{1, \underbrace{-1, \cdots, -1}_{\mathrm{sig}}\}, \tag{7.7.15}$$

where $o_{\alpha\beta}$ is *constant* in an anholonomic frame with signature sig. On a four-dimensional manifold, the Hodge dual of a p-form $\Phi = \frac{1}{p!}\Phi_{i_1\cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is defined by

$${}^*\Phi := \frac{1}{(4-p!)p!}\sqrt{\det g_{ij}}\,\varepsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{4-p}}\Phi_{i_1\cdots i_p}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{4-p}}, \tag{7.7.16}$$

so that it is almost involutive, i.e., ${}^{**}\Phi = (-1)^{p(4-p)+\mathrm{sig}}\Phi$.

Clifforms and Chiral Transformations

In the familiar Pauli representation, the 16 matrices $\{\mathbf{1}_4, \gamma_\alpha, \sigma_{\alpha\beta}, \gamma_5, \gamma_5\gamma_\alpha\}$, where $\sigma_{\alpha\beta} := \frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$ are the Lorentz generators, and $\gamma_5 = -i\,\gamma_{\hat 0}\,\gamma_{\hat 1}\,\gamma_{\hat 2}\,\gamma_{\hat 3}$ with $\gamma_5^2 = (-1)^{\mathrm{sig}+1}\mathbf{1}_4$ constitute a basis of the *Clifford algebra* in four dimensions with the defining relation

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2o_{\alpha\beta}\,\mathbf{1}_4\,. \tag{7.7.17}$$

They are normalized by the traces $\mathrm{Tr}(\gamma_\alpha\ \gamma_\beta) = 4\, o_{\alpha\beta}$ and $\mathrm{Tr}(\sigma_{\alpha\beta}\ \sigma^{\gamma\delta}) = 8\,\delta^\gamma_{[\alpha}\ \delta^\delta_{\beta]}$, where $[\alpha\ \beta] = \frac{1}{2}(\alpha\beta - \beta\alpha)$ denotes the antisymmetrization of indices. For the unit two-form $\sigma := \frac{1}{2}\sigma_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta = \frac{i}{2}\,\gamma \wedge \gamma$ of dimension [length^2], the Hodge dual and the Lie dual turn out to be the same, i.e.,

$${}^*\sigma = \frac{1}{2}\sigma_{\alpha\beta}\, {}^*(\vartheta^\alpha \wedge \vartheta^\beta) = \frac{1}{2}\sigma_{\alpha\beta}\, \eta^{\alpha\beta} =: \sigma^{(\star)} = i\,\gamma_5\,\sigma\,. \tag{7.7.18}$$

The Hodge dual of the basic Clifform

$$\gamma := \gamma^\alpha\, \vartheta_\alpha \tag{7.7.19}$$

leads to the associated three-form

$${}^*\gamma = \gamma^\alpha\, \eta_\alpha = \frac{i}{6}\,\gamma_5\, \gamma \wedge \gamma \wedge \gamma\,, \tag{7.7.20}$$

whereas for the *Lie* (or right) dual $\sigma^{(\star)}_{\alpha\beta} := \frac{1}{2}\,\sigma^{\gamma\delta}\,\eta_{\alpha\beta\gamma\delta} = i\gamma_5\,\sigma_{\alpha\beta}$, we obtain the associated two-forms

$$\sigma := \frac{1}{2}\sigma_{\alpha\beta}\,\vartheta^{\alpha} \wedge \vartheta^{\beta} = \frac{i}{2}\,\gamma \wedge \gamma\,, \qquad {}^*\sigma = \frac{1}{2}\sigma_{\alpha\beta}\,\eta^{\alpha\beta} =: \sigma^{(\star)} = i\,\gamma_5\,\sigma\,. \quad (7.7.21)$$

In orthonormal frames, the Hodge dual * and the Lie dual $^{(\star)}$ are identical operations for σ. Moreover, this allows one to *reconstruct* the Hodge dual and therefore a conformal equivalence class of spacetime metrics from the Lie dual as defined by Kähler; cf. Trautman (1999): for the metric-free two-form σ, we can build the Lie dual $\sigma^{(\star)}$ solely by multiplication by γ_5, which here is regarded as just an anticommuting element of the Clifford algebra. This Lie dual is antiinvolutive:

$$\sigma^{(\star)} := i\,\gamma_5\,\sigma\,, \qquad \sigma^{(\star\star)} = i^2\,\gamma_5^2\,\sigma = -\sigma\,. \quad (7.7.22)$$

Since the Clifform relation

$$[\gamma, \sigma^{(\star)}] = \gamma \wedge i\gamma_5\sigma - i\gamma_5\sigma \wedge \gamma = -i\gamma_5(\gamma \wedge \sigma + \sigma \wedge \gamma) = 2\gamma_5\,\gamma \wedge \gamma \wedge \gamma \quad (7.7.23)$$

relates this to $12i\,{}^*\gamma$, the Hodge dual[3] for the basis of Clifforms has been recovered. This allows us to identify

$$\gamma_5 := (i/4!)\,{}^*(\gamma \wedge \gamma \wedge \gamma \wedge \gamma) \quad (7.7.24)$$

with a zero-form that appears metric-free.

Also, the *self-dual or anti-self-dual* combination

$$\sigma_{\pm} := (\sigma \pm i\,{}^*\sigma)/2 = \frac{1}{2}(1 \mp \gamma_5)\,\sigma, \quad \text{with} \quad i\,{}^*\sigma_{\pm} = \pm\sigma_{\pm} \quad (7.7.25)$$

occurs, originally being due to Debever (1964) and Brans (1974), but at times is referred to as the *Plebański two-form* (Plebański 1975,1977). Our Clifford representation involves explicitly the *chirality projector* $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$, obeying $P_{\pm}\,P_{\pm} = P_{\pm}$.

In four dimensions, the Hodge dual applied to two-forms is conformally invariant (Atiyah et al. 1978) under the Weyl rescaling $\gamma \to \widetilde{\gamma} = e^{\varphi}\gamma$, where φ can be viewed as the *dilaton*-type field. Conversely, an initially metric-free *involutive* star operation $^\#$ on *arbitrary* two-forms allows us to *reconstruct* a spacetime metric h, which is,

[3] For Minkowski signature (sign = 1), the Hodge dual satisfies $^{**} = -1$, and therefore, $i\,^*$ is used at times instead, in order to have an *involutive* duality operator.

however, merely conformally related to g. This decomposition is invariant under the *chiral transformation*

$$\gamma \to \gamma^\theta = e^{i\gamma^5\theta}\gamma e^{-i\gamma^5\theta} \tag{7.7.26}$$

of the coframe, where θ denotes the so-called theta angle; cf. 'T HOOFT (1991).

Clifford-Algebra-Valued Torsion and Curvature

The Riemann–Cartan (RC) geometry is a basis for the Einstein–Cartan (EC) theory of gravity. In terms of the Clifford-algebra-valued *connection* $\Gamma := \frac{i}{4}\Gamma^{\alpha\beta}\,\sigma_{\alpha\beta}$, the $\overline{SO}_\circ(1,3) \cong SL(2,C)$-covariant exterior derivative

$$D = d + [\Gamma, \quad] \tag{7.7.27}$$

of p-forms employs the algebra-valued *form commutator* $[\Psi, \Phi] := \Psi \wedge \Phi - (-1)^{pq}\,\Phi \wedge \Psi$.

The Clifford-algebra-valued coframe and *connection*

$$\gamma := \vartheta^\alpha \gamma_\alpha\,, \qquad \Gamma := \frac{i}{4}\Gamma^{\alpha\beta}\,\sigma_{\alpha\beta} = \Gamma^{\{\}} - K\,, \tag{7.7.28}$$

involves $K := \frac{i}{4}K^{\alpha\beta}\,\sigma_{\alpha\beta}$, the contortion one-form.

In view of Cartan's structure equations, differentiation of the basic variables leads to the Clifford-algebra-valued *torsion* and *curvature* two-forms

$$\begin{aligned}\Theta := D\gamma = d\gamma + [\Gamma, \gamma] &= (d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta)\gamma_\alpha \\ &= T^\alpha\gamma_\alpha = \frac{1}{2}\,T_{ij}{}^\alpha\gamma_\alpha\, dx^i \wedge dx^j\,,\end{aligned} \tag{7.7.29}$$

$$\begin{aligned}\Omega^g := d\Gamma + \Gamma \wedge \Gamma &= \frac{i}{4}(d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta)\,\sigma^\alpha{}_\beta \\ &= \frac{i}{4}R^{\alpha\beta}\,\sigma_{\alpha\beta} = \frac{i}{8}\,R_{ij}{}^{\alpha\beta}\,\sigma_{\alpha\beta}\, dx^i \wedge dx^j\,,\end{aligned} \tag{7.7.30}$$

respectively.

This compact "Clifform" formula for the curvature is due to SCHRÖDINGER (1932), who to some extent anticipated the concept of *gauge* curvature developed much later by YANG & MILLS (1954). It admits a generalization to (singular) distribution-valued forms (TAUB 1980).

The *Lie dual* of the curvature involves a permutation of the Lie algebra indices, i.e.,

$$\Omega^{(\star)} = \frac{1}{4} R^{(\star)}_{\alpha\beta} \sigma^{\alpha\beta} , \qquad R^{(\star)}_{\alpha\beta} := \frac{1}{2} \eta_{\alpha\beta\gamma\delta} R^{\gamma\delta} . \tag{7.7.31}$$

Our "unit" curvature two-form is not covariantly constant, i.e.,

$$D\sigma = \frac{i}{2} [\Theta, \gamma] \tag{7.7.32}$$

in RC spacetime with nonvanishing torsion.

The torsion two-form can be irreducibly decomposed into the trace part ${}^{(2)}\Theta := \frac{1}{3} \gamma \wedge T$, the axial torsion ${}^{(3)}\Theta := -\frac{1}{3} {}^{*}(\gamma \wedge \mathscr{A})$, and the tensor torsion ${}^{(1)}\Theta := \Theta - {}^{(2)}\Theta - {}^{(3)}\Theta$, where the one-forms of the trace and axial vector torsion, respectively, are defined by

$$T := \frac{1}{4} \operatorname{Tr} \left(\check{\gamma} \rfloor \Theta \right) = e_\alpha \rfloor T^\alpha , \qquad \mathscr{A} := \frac{1}{4} {}^{*}\mathrm{Tr}(\gamma \wedge \Theta) = {}^{*}(\vartheta_\alpha \wedge T^\alpha) . \tag{7.7.33}$$

Under a Weyl rescaling, the axial torsion remains invariant, i.e., $\widetilde{\mathscr{A}} = \mathscr{A}$. Both the trace and the axial torsion pick up a gradient term under a conformal or chiral transformation:

$$\widetilde{T} = T - 3 d\varphi , \qquad \mathscr{A}^{\theta} = \mathscr{A} - i d\theta . \tag{7.7.34}$$

The Hilbert–de Donder- and Lorentz-type *gauge conditions* on the coframe or connection are the following four-form conditions:

$$d\, {}^{*}\gamma = 0 , \qquad d\, {}^{*}\Gamma = 0 \tag{7.7.35}$$

involving the Hodge dual.

Bianchi Identities

The Ricci identity for a p-form Ψ reads

$$DD\Phi = [\Omega^g, \Phi] , \tag{7.7.36}$$

whereas the first and second *Bianchi identities* assume in RC geometry the form

$$D\Theta \equiv [\Omega^g, \gamma] , \qquad D\Omega^g \equiv 0 , \tag{7.7.37}$$

respectively; cf. MIELKE (2001) for further details.

The Cliffom version of the *Einstein tensor* can be rewritten as the three-form

$$G := G_\alpha \gamma^\alpha := \frac{1}{2} R^{\mu\nu} \wedge \eta_{\mu\nu\lambda} \gamma^\lambda = -i \gamma_5 (\Omega^g \wedge \gamma + \gamma \wedge \Omega^g) = i[\gamma, \gamma_5 \Omega^g] . \tag{7.7.38}$$

In view of the *contracted* Bianchi identities

$$D[\gamma\,,\Theta]\equiv 2i[\sigma\,,\Omega^g]\,,\qquad D[\gamma\,,\Omega^g]\equiv[\Theta\,,\Omega^g]\,,\qquad DG\equiv i[\Theta,\gamma_5\Omega^g]\,, \tag{7.7.39}$$

the automatic conservation of the Einstein three-form holds only for vanishing torsion, i.e., for $\Theta=0$ in Einstein's standard GR.

The *Lie dual* of Lorentz-algebra-valued forms such as contortion and curvature is defined by

$$K^{(\star)}:=\eta_{\alpha\beta\gamma}\wedge K^{\beta\gamma}\gamma^\alpha\,, \tag{7.7.40}$$

$$\Omega^{g(\star)}:=\frac{i}{8}R_{\alpha\beta}\,\eta^{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}=-\frac{1}{4}R^{\alpha\beta}\gamma_5\sigma_{\alpha\beta}=i\gamma_5\Omega^g\,. \tag{7.7.41}$$

The latter satisfies the second Bianchi identity

$$DR^{(\star)}_{\alpha\beta}\equiv 0, \tag{7.7.42}$$

i.e., $D\Omega^{g(\star)}\equiv 0$, provided that $D\eta_{\alpha\beta\gamma\delta}=0$ holds as in spaces with vanishing Weyl covector.

In four dimensions, it is useful to consider also the self-dual and anti-self-dual torsion and curvature two-forms

$$\Theta^{\pm}:=\frac{1}{2}\left(\Theta\pm{}^*\Theta\right)\,,\qquad \Omega^{g\pm}:=\frac{1}{2}\left(\Omega^g\pm{}^*\Omega^g\right)\,,\qquad \Omega^{(\pm)}:=\frac{1}{2}\left(\Omega^g\pm\Omega^{g(\star)}\right)\,, \tag{7.7.43}$$

defined respectively in terms of the Hodge and Lie duals.

Topological Terms in Four-Dimensional Manifolds

In a RC spacetime, the translational and Lorentz-rotational *Chern–Simons terms* read

$$C_{\rm TT}:=\frac{1}{8\ell^2}{\rm Tr}\,(\gamma\wedge\Theta)=\frac{1}{2\ell^2}\,\vartheta^\alpha\wedge T_\alpha=-\frac{(-1)^{\rm sig}}{2\ell^2}\,{}^*\mathscr{A}\,, \tag{7.7.44}$$

$$C_{\rm RR}:={\rm Tr}\left(\Gamma\wedge\Omega^g-\frac{1}{3}\Gamma\wedge\Gamma\wedge\Gamma\right)\,, \tag{7.7.45}$$

where $\mathscr{A}:={}^*(\vartheta_\alpha\wedge T^\alpha)=\mathscr{A}_i dx^i$ is the *axial torsion* one-form. The translational Chern–Simons term is not Weyl-invariant, cf. (3.14.9) of HEHL et al. (1995), due to the occurrence of a fundamental length ℓ. The Clifford algebra approach has the advantage that we can employ the trace in the definition (7.7.44), whereas the usual translational generators P_α commute and do not have a nondegenerate Cartan–Killing metric.

The Lagrangians corresponding to the Bianchi identities (7.7.37) are the boundary terms

$$L_{\rm NY} := dC_{\rm TT} = \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right), \tag{7.7.46}$$

and

$$\begin{aligned} L_{\rm Pontr} :=& dC_{\rm RR} = \frac{1}{2}R_\alpha{}^\beta \wedge R_\beta{}^\alpha = -\frac{1}{2}R^{\{\}}_{\alpha\beta}\wedge R^{\{\}\alpha\beta} \\ & -\frac{1}{12}d\left[{}^*\mathscr{A}\wedge R^{\{\}} - \frac{1}{3}\mathscr{A}\wedge d\mathscr{A} + \frac{1}{9}{}^*\mathscr{A}\wedge^*(\mathscr{A}\wedge{}^*\mathscr{A})\right]. \end{aligned} \tag{7.7.47}$$

Up to normalizations, the four-forms (7.7.46) and (7.7.47) are known as NIEH & YAN (1982) and gravitational Pontryagin terms, respectively. The latter contains, provided only axial torsion is present, a term proportional to the Riemannian curvature scalar $R^{\{\}} := (-1)^{\rm sig+1}\,{}^*(R^{\{\}\alpha\beta}\wedge\eta_{\beta\alpha})$ and the axial torsion piece $d\mathscr{A}\wedge d\mathscr{A}$ familiar from the axial anomaly, with a relative factor of 9.

On the other hand, there exists a boundary term involving the dual torsion that induces a transition of Einstein's to its teleparallelism equivalent $\rm GR_{\|}$, i.e.,

$$\begin{aligned} 2\ell^2\, dC_{\rm TT^*} &:= 2d(\vartheta^\alpha\wedge{}^*T_\alpha) \\ &= R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} - R^{\alpha\beta}\wedge\eta_{\alpha\beta} \\ &\quad - T^{\alpha\,*}\left[T_\alpha - \vartheta_\alpha\wedge(e_\beta\rfloor T^\beta) - \frac{1}{2}\,e_\alpha\rfloor(T^\beta\wedge\vartheta_\beta)\right] \\ &= R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} - R^{\alpha\beta}\wedge\eta_{\alpha\beta} - T^\alpha\wedge{}^*\!\left(-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right), \end{aligned} \tag{7.7.48}$$

where ${}^{(i)}T_\alpha$ are the three irreducible torsion pieces.

The topological Euler term

$$\begin{aligned} L_{\rm Euler} &:= (-1)^{\rm sig+1}{\rm Tr}\{\Omega\wedge\Omega^{(\star)}\} = \frac{(-1)^{\rm sig}}{2}R^{\alpha\beta}\wedge R^{(\star)}_{\alpha\beta} \\ &= dC_{\rm RR^{(\star)}} = \frac{(-1)^{\rm sig}}{2}d\left(\Gamma_{\alpha\beta}\wedge R^{\alpha\beta(\star)} - \frac{1}{3}\Gamma_\alpha{}^{\beta(\star)}\wedge\Gamma_\beta{}^\gamma\wedge\Gamma_\gamma{}^\alpha\right) \\ &\equiv \frac{1}{2}R_{\alpha\beta}\wedge{}^*R^{\alpha\beta} - 2{\rm Ric}_{\alpha\beta}\wedge{}^*{\rm Ric}^{\alpha\beta} + \frac{1}{2}{\rm Ric}_\alpha{}^\alpha\wedge{}^*{\rm Ric}_\beta{}^\beta \\ &\equiv -L_{\rm SKY} \\ &\quad -2\left({\rm Ric}_{\alpha\beta}\wedge{}^*{\rm Ric}^{\alpha\beta} - \frac{1}{4}{\rm Ric}_\alpha{}^\alpha\wedge{}^*{\rm Ric}_\beta{}^\beta\right) \end{aligned} \tag{7.7.49}$$

has, in view of the *Lanczos identity*, an equivalent representation in terms of Yang's Lagrangian $L_{\rm SKY}$ as well as a Ricci-squared term and a scalar-curvature-squared term. The expression in terms of the symmetric Ricci tensor, i.e., the zero-form ${\rm Ric}_{\alpha\beta} := (-1)^{\rm sig}\,{}^*(R_{(\alpha}{}^\delta\wedge\eta_{\delta|\beta)})$, is also known as the Gauss–Bonnet term.

Due to the algebraic *Lanczos identity* (Lanczos 1938), the *double-dual curvature*

$$^{*}R^{(\star)}_{\alpha\beta} \equiv (-1)^{\mathrm{sig}} R_{\alpha\beta} + e_{[\alpha}\rfloor G_{\beta]} + \frac{1}{4} R\eta_{\alpha\beta} + (-1)^{\mathrm{sig}+1} D_{[\alpha} T_{\beta]} \qquad (7.7.50)$$

can be written in terms of a contraction of the Einstein–Cartan (EC) three-form $G_\alpha := R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}/2$ and the curvature scalar R. The latter is the zero-form

$$R := e_\beta \rfloor e_\alpha \rfloor R^{\alpha\beta}, \qquad ^{*}R^{\alpha\beta} \wedge \vartheta_\alpha \wedge \vartheta_\beta \equiv R^{\alpha\beta} \wedge \eta_{\alpha\beta} \equiv -R\eta\,, \qquad (7.7.51)$$

which constitutes one irreducible piece

$$^{(6)}R^{\alpha\beta} = -\frac{1}{12} R\vartheta^\alpha \wedge \vartheta^\beta \qquad (7.7.52)$$

of the curvature. Likewise, the EC three-form

$$\begin{aligned} G &:= \frac{1}{2} R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \gamma^\alpha \qquad (7.7.53) \\ &= G^{\{\}} + \frac{(-1)^{\mathrm{sig}}}{12} \left(e_\alpha \rfloor \mathscr{A} \wedge {}^{*}\mathscr{A} - \frac{1}{3} \mathscr{A} \wedge e_\alpha \rfloor {}^{*}\mathscr{A} \right) \gamma^\alpha + \frac{(-1)^{\mathrm{sig}}}{6} \gamma \wedge d\mathscr{A} \end{aligned}$$

decomposes into the Einstein three-form $G^{\{\}} = G^{\{\}\beta}_{\alpha} \eta_\beta \gamma^\alpha$ with respect to the Riemannian connection $\Gamma^{\{\}}$ and axial torsion pieces, provided that the vector and tensor torsion are absent; see Mielke & Romero (2006) for details.

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Chapter 8
BRST Quantization of Gravity

8.1 Introduction

On the macroscopic level, Einstein's general relativity (GR) has passed every test "with flying colors"; see WILL (2014, 2006) for recent reviews. However, Einstein's theory has thus far resisted every attempt at quantization, e.g., it is known to be perturbatively nonrenormalizable, partially due to its dimensional coupling constant.[1] String theory, or brane scenarios with extra dimensions, has been proposed as a rescue, though some of these scenarios imply deviations of standard gravity in the submillimeter range. Recent torsion balance experiments (KAPNER et al. 2007) have probed the inverse square law and found no deviation even below the hypothetical dark energy (DE) scale of $\lambda_{\rm DE} = (\hbar c/\rho_{\rm DE})^{1/4} \simeq 85\ \mu$m. Thus this "window" of possibly new gravitational physics seems to be closing.

In 1974, YANG proposed an affine gauge theory gravity that due to its scale invariance can be regarded as a rather promising fundamental theory of (quantum) gravity in the high-energy limit (HEHL et al. 1989), without invoking extra dimensions or supersymmetry (KIBBLE & STELLE 1986). Moreover, from the work of STELLE (1977), we know that the curvature-squared gravity in Riemannian spacetime is perturbatively *renormalizable* but unfortunately plagued with *physical* ghosts, i.e., negative residues in the graviton propagator (LEE & NE'EMAN 1990). This finding has diminished the initial interest (FAIRCHILD 1976, 1977) in such models.

Much more promising and elegant is to start from a purely topological classical action, proportional to the gravitational Pontryagin (or Euler) invariant and then quantize this model by nilpotent *Becchi–Rouet–Stora–Tyutin* (BRST) transformations generated by s. Such a topological action is not only completely metric-free, but also conformally invariant (WITTEN 1988a; LABASTIDA & PERNICI 1988), and

[1] It has to be kept in mind that Newton's gravitational constant G is one of the less precisely known constants of physics. In order to improve this situation, there are plans (ALEXEEV et al. 2001) to measure the gravitational attraction of two bodies in a spaceship (Project SEE), where the larger body will function as a shepherd for the movement of the test mass, similarly as in the rings of Saturn.

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it can provide a consistent *topological quantum field theory* (TQFT), as shown by WITTEN (1988b), BIRMINGHAM et al. (1991). Later, it was realized by BAULIEU & SINGER (1988) that in the Yang–Mills case, Witten's action is a gauge-fixed version of the classical topological action through the standard BRST quantization procedure. Then the total Lagrangian consists of a d-exact part, as exemplified by the Pontryagin invariant of the relevant structure group, as well as an s-exact piece accounting for the chosen gauge fixing. Modulo an exact form, the full Lagrangian turns out to be BRST-invariant.

In the case of gravity, we can obtain a rather realistic gravitational "background" dynamics if we complete the topological action by *gauge constraints*, enforcing the Lorentz condition on the linear connection and *double duality* on the curvature. Then the resulting model depends on the metric of spacetime only via the s-exact term. This conforms with the general expectation that it is ultimately the process of quantization that necessarily induces a physical scale into a primordial topological and conformally invariant model. According to a lucid essay of FADDEEV (1996), quantization amounts to a *stable deformation* of the classical Poisson algebra with a dimensional transmutation due to the physical dimension $[\hbar] = [p][q]$ of Planck's (reduced) constant $\hbar$.

In this chapter, a BRST quantization of topological gravity (WITTEN 1988a) is developed, following essentially BAULIEU & SINGER (1988) and BAULIEU (1985), DE CARVALHO & BAULIEU (1992), BAULIEU & TANZINI (2002). Moreover, its classical limit, corresponding to the most probable extremal "trajectories" in the Feynman path integral, is analyzed. In the case of gravity, these are classical configurations with either self-dual or anti-self-dual curvature. In order to lift this "vacuum degeneracy," a modified double duality constraint is considered that explicitly breaks scale invariance. For torsionless configurations or algebraic constraints on their coupling constants, only Einstein's GR, consistently coupled to the symmetrized energy–momentum current of matter fields, surfaces as a low-energy (long-range) effective theory, thereby satisfying all macroscopic tests. Extending MIELKE (2008), the focus is on the breaking of the duality symmetry accompanied by a scale violation. This symmetry-breaking is achieved by replacing double duality by a modified constraint induced via a four-parameter boundary term.

8.2 Topological BRST Transformations of Gauge Fields

In the BRST formalism, the infinitesimal gauge transformations are converted, via ghosts, into operator transformations. Let $C := \frac{i}{4} C^{\alpha\beta}\,\sigma_{\alpha\beta}$ denote the zero-form of the usual Faddeev–Popov ghost, (MIELKE & MAGGIOLO 2003; VAN HOLTEN 2005), $\Psi := \frac{i}{4}\Psi_j^{\alpha\beta}\sigma_{\alpha\beta}dx^j$ the topological ghost one-form, and $\Phi := \frac{i}{4}\Phi^{\alpha\beta}\sigma_{\alpha\beta}$ the corresponding ghost of the topological ghost. All are Lie-algebra-valued due to the appearance of the generator $\sigma_{\alpha\beta}$ of the linear (or Lorentz) group.

Then the global BRST transformations generated by the zero-form s take the form

$$\begin{aligned} s\Gamma &= \Psi - DC\,, \\ sC &= \Phi - \frac{1}{2}[C, C]\,, \\ s\Omega^g &= -D\Psi - [C, \Omega^g]\,, \\ s\Psi &= -D\Phi - [C, \Psi]\,, \\ s\Phi &= -[C, \Phi]\,. \end{aligned} \tag{8.2.1}$$

This is consistent with the interpretation of Γ and Ω^g as connection one-form and gauge two-form, respectively, under infinitesimal gauge transformations. The topological ghost Ψ complements the gauge fields in a reducible manner, and due to its own gauge invariance, it is supplemented by its associated ghost Φ such that the net number of degrees of freedom carried by the ghosts C, Ψ, and Φ is equal to that of the connection Γ. By construction, these BRST transformations are nilpotent for all variables, i.e., $s^2 = 0$.

A rather elegant geometric interpretation was obtained by Baulieu & Singer (1988) by introducing the *graded*[2] connection and curvature forms

$$\widetilde{\Gamma} := \Gamma \oplus C\,, \qquad \widetilde{\Omega}^g := \Omega^g \oplus \Psi \oplus \Phi. \tag{8.2.2}$$

Then the corresponding *graded* Cartan-type structure equation and second Bianchi identity for the graded curvature, i.e.,

$$\begin{aligned} (d \oplus s)\widetilde{\Gamma} + \frac{1}{2}[\widetilde{\Gamma}, \widetilde{\Gamma}] &= \widetilde{\Omega}^g\,, \\ (d \oplus s)\widetilde{\Omega}^g + [\widetilde{\Gamma}, \widetilde{\Omega}^g] &\equiv 0, \end{aligned} \tag{8.2.3}$$

are satisfied. This graded Cartan-type formalism constitutes an ordinary de Rham cohomology (Baulieu 1987) and comprises all the BRST transformations (8.2.1). The latter can be recovered by an expansion in the ghost number g and collecting those terms with the same form degree p and ghost number g.

Moreover, a straightforward proof of the nilpotency $ss = 0$ of the BRST operator s follows simply from $(d \oplus s)(d \oplus s) \equiv 0$ as a result of the graded Bianchi identity (8.2.3), the anticommutation of the graded commutator $[s, d] := sd + ds = 0$, and the Poincaré lemma $dd \equiv 0$ for the exterior derivative.

In order to implement the gauge constraints, one can employ the antifield formalism, where the Lorentz-algebra-valued antighosts $\overline{C}$, $\overline{\chi}$, and $\overline{\Phi}$ obey the following BRST transformation rules:

---

[2]The grading permits using the direct sum $\oplus$ of exterior forms carrying different form degree p and ghost number g, such that the graded commutator is more generally defined by $[\Psi, \Phi] := \Psi \wedge \Phi - (-1)^{(p_1+g_1)(p_2+g_2)} \Phi \wedge \Psi$, (Chang & Soo 1992).

$$
\begin{aligned}
s\overline{C} &= \mathbb{B}\,, & s\mathbb{B} &= 0\,,\\
s\overline{\chi} &= \beta\,, & s\beta &= 0\,,\\
s\overline{\Phi} &= \overline{\eta}\,, & s\overline{\eta} &= 0\,.
\end{aligned} \tag{8.2.4}
$$

By construction, s is nilpotent for the antighosts, since the Lagrange multiplier $\overline{\eta}$, the self-dual two-forms $\mathbb{B}$, and β are auxiliary fields introduced as trivial pairs. A symmetric ghost/antighost spectrum of an extended BRST invariance could be obtained via a field redefinition (BRAGA & GODINHO 2000). (For metric-affine gravity, an antifield formalism has been developed in GRONWALD (1998) without, however, resorting to topological ghosts.)

By introducing a BRST gauge field $\alpha = \alpha_i dx^i$ with ghost number -1 and a commuting ghost λ of α, one can promote (DE CARVALHO & BAULIEU 1992) the global BRST transformations (8.2.1) into *local* ones, where

$$
s_{\rm loc}(\alpha+\lambda) = -d\lambda\,, \qquad s_{\rm loc}\lambda = 0 \tag{8.2.5}
$$

satisfies the algebra

$$
(d \oplus s_{\rm loc})(\alpha+\lambda) = d\alpha. \tag{8.2.6}
$$

The cohomology (8.2.3) of the BRST transformation remains unchanged by this promotion, which likewise, can be generated via the field redefinitions $C \to (\alpha + \lambda)C$, $\Psi \to (\alpha+\lambda)\Psi$ as well as $\Phi \to (\alpha+\lambda)^2\Phi$ of the ghosts. Thus, local BRST invariance of an action puts no more restrictions on its form than the usual global one: the gauge field α is present only to compensate for the enlargement of the symmetry, from global to local, but it cannot propagate, due to its nonvanishing ghost number.

8.3 BRST Quantization of Translations

In the affine or (broken) Poincaré gauge theory, the coframe $\gamma = \vartheta^\alpha \gamma_\alpha$ (locally equivalent to the familiar tetrads) is usually "soldered" (TRESGUERRES & MIELKE 2000) to the base manifold. Then the topological structure equations (8.2.3) for the linear connection get amended by the corresponding *graded first Cartan structure equation* and the first Bianchi identity

$$
\begin{aligned}
(d \oplus s)\gamma + [\widetilde{\Gamma}, \gamma] &= \widetilde{\Theta}\,,\\
(d \oplus s)\widetilde{\Theta} + [\widetilde{\Gamma}, \widetilde{\Theta}] &\equiv [\widetilde{\Omega}^g, \gamma]\,,
\end{aligned} \tag{8.3.1}
$$

respectively. In the graded torsion two-form $\widetilde{\Theta} := \Theta \oplus \psi$, there occurs the translational "Clifform" $\psi := \psi_j^\alpha \gamma_\alpha dx^j$ of the corresponding topological ghost.

In an expansion in the ghost number, we can recover the Clifform definition of the torsion and deduce the BRST transformations

$$s\gamma = \psi - [C, \gamma] \tag{8.3.2}$$

of the coframe and the corresponding "unit"-curvature two-form

$$s\sigma := \frac{i}{2}s(\gamma \wedge \gamma) = \frac{i}{2}\left([\psi, \gamma] - [[C, \gamma], \gamma]\right) = \frac{i}{2}[\psi, \gamma] - [C, \sigma]\,. \tag{8.3.3}$$

Eventually, this needs to be amended by the BRST transformations of the antifields, e.g.,

$$s\bar{c} = b\,, \quad sb = 0\,. \tag{8.3.4}$$

8.3.1 Diffeomorphisms

Holonomically, spacetime *diffeomorphisms* can be taken into account of by generalizing the BRST transformations s via $s \to \widetilde{s} = s + \text{Ł}_\zeta$ involving the covariant Lie derivative $\text{Ł}_\zeta := \zeta\rfloor D - D\zeta\rfloor$ along an anticommuting ghost vector field $\zeta = \zeta^i \partial_i$. The definition involves the interior product $\rfloor$ and a sign difference in the definition of Ł_ζ, since ζ has ghost number one. Its BRST transformation is $s\zeta = \phi + \text{Ł}_\zeta \zeta$, such that the BRST algebra remains intact (Baulieu & Tanzini 2002; Chang & Soo 1992), up to a redefinition of all graded fields by means of a similarity transformation generated by the formal exponential

$$\exp(\zeta\rfloor) := \mathbf{1} + \zeta\rfloor + \frac{1}{2!}\zeta\rfloor\zeta\rfloor + \frac{1}{3!}\zeta\rfloor\zeta\rfloor\zeta\rfloor + \cdots\,. \tag{8.3.5}$$

Its action on a p-form, reduces to a finite series with up to $p + 1$ terms. In effect, the graded curvature and torsion in the cohomologies (8.2.3) and (8.3.1) are replaced by $\exp(\zeta\rfloor)\widetilde{\Omega}^g$ and $\exp(\zeta\rfloor)(\widetilde{\Theta} \oplus \phi)$, respectively, where $\phi := \phi^\alpha \gamma_\alpha$ are the ghosts of the translational ghosts. Due to the notion of "horizontality" in curved spacetime (Baulieu 1987), the exterior derivative d suffers from the similarity transformation

$$d \quad \to \quad \exp(\zeta\rfloor)d\exp(-\zeta\rfloor)\,. \tag{8.3.6}$$

In undoing the "soldering" of γ, and really gauging the translational part $\mathbb{R}^4$ of the affine group (Mielke et al. 1993; Hehl et al. 1995), there would arise a dimensionless *translational* connection $\Gamma^{(\mathrm{T})} = (\gamma - D\xi)/\ell$ and corresponding curvature $\Omega^{(\mathrm{T})} = D\Gamma^{(\mathrm{T})} = (\Theta - DD\xi)/\ell$. Then the translational connection $\Gamma^{(\mathrm{T})}$ should be graded as well, e.g., by the substitution $\gamma \to \widetilde{\gamma} := \ell(\Gamma^{(\mathrm{T})} \oplus c)$ in the first structure equation.

The "quartet" of scalars $\xi := \xi^\alpha \gamma_\alpha$ corresponds to the "generalized radius vectors" of Cartan and lives on the coset space $A(4, \mathbb{R})/GL(4, \mathbb{R}) \approx \mathbb{R}^4$ of the affine group. Quite recently, similar scalars where introduced by 't Hooft (2009), who observed that his "alternative" metric

$$ds^2 = \frac{1}{4}\mathrm{Tr}\{\gamma \otimes \gamma\} \stackrel{*}{=} o_{\alpha\beta} D\xi^\alpha \otimes D\xi^\beta \tag{8.3.7}$$

naturally arises in the affine gauge theory after, locally, the gauge $\Gamma^{(\mathrm{T})} \stackrel{*}{=} 0$ is imposed on the true translational connection.

8.4 Topological Gravity Action

Let us adopt ideas of WITTEN (1988b) for a topological Yang–Mills theory (TYM) and replace the internal $SU(N)$ group by the linear group $SL(4, \mathbb{R})$ of the tangent space embracing the Lorentz group SO(1, 3) (or SO(4) in Euclidean space with signature sig = 0) as subgroup. Then one starts from the gravitational Pontryagin four-form

$$\begin{aligned} L_{\mathrm{Pontr}} := dC_{\mathrm{RR}} &= \mathrm{Tr}\{\Omega^g \wedge \Omega^g\} = \frac{1}{2} R_\alpha{}^\beta \wedge R_\beta{}^\alpha \\ &= \frac{1}{2} d\big(\Gamma_\alpha{}^\beta \wedge R_\beta{}^\alpha - \frac{1}{3}\Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha\big)\,, \end{aligned} \tag{8.4.1}$$

which is locally a d-exact form violating, however, parity P. This topological Lagrangian is completely metric-free[3] and invariant under the topological BRST transformations s modulo an exact form as well, i.e.,

$$sL_{\mathrm{Pontr}} = -2d\mathrm{Tr}\{\Psi \wedge \Omega^g\} = -d(\Psi_\alpha{}^\beta \wedge R_\beta{}^\alpha)\,. \tag{8.4.2}$$

On top of the standard BRST transformations of Yang–Mills fields, BAULIEU & SINGER (1988) employed an arbitrary field redefinition (FR) of the gauge fields in order to introduce a *topological ghost* Ψ. Thereby the field content is enlarged in a *reducible* manner and needs to be constrained afterward. In the gravitational case, the one-form Ψ has values in the Lie algebra of the linear group.

This purely topological action can be amended by any s-exact four-form $s\{\cdots\}$, provided s is a nilpotent BRST transformation, i.e., one with $s^2 = 0$, without affecting the nice properties of the topological action. Following again BAULIEU & SINGER (1988), we may choose the Lorentz-type conditions $d\,{}^*\Gamma = 0$, $D\,{}^*\Psi = 0$ on the connection and the topological ghost, respectively, as well as the self-duality or anti-self-duality condition $\Omega^{g(\pm)} = 0$ on the curvature as *gauge constraints* consistently implemented via the Faddeev–Popov-type Lagrangian

[3] For the CP-invariant Euler term, a similar result would hold, i.e., $sL_{\mathrm{Euler}} = (-1)^{\mathrm{sig}+1} d\left(\Psi^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta}\right)$. However, the latter is only partially metric-free, since it involves the signature sig of the metric implicitly in the definition of the Lie dual ${}^{(\star)}$ (BLAU & THOMPSON 1991), and therefore appears less well qualified as a starting point.

$$L_{\rm FP} := -\,s{\rm Tr}\left\{\overline{\chi}\wedge{}^*\overset{\pm}{\Omega}{}^g + \overline{\Phi}D\,{}^*\Psi + \frac{1}{2}\rho\overline{\chi}\wedge{}^*\beta + \overline{C}d\,{}^*\Gamma + \frac{1}{2}\overline{C}\wedge{}^*\mathbb{B}\right\}$$
$$= \frac{1}{2}s\left[\overline{\chi}^{\alpha\beta}\wedge{}^*R^{(\pm)}_{\alpha\beta} + \overline{\Phi}_{\alpha\beta}D\,{}^*\Psi^{\alpha\beta} + \frac{1}{2}\rho\overline{\chi}^{\alpha\beta}\wedge{}^*b_{\alpha\beta} + \widetilde{\Psi}_{\rm F}\right]. \tag{8.4.3}$$

The trace is over the generators of the Lie algebra. Such a constraint, at the same time, constricts the linear group $SL(4,\mathbb{R})$ to the Lorentz (or 4D orthogonal) group as a subgroup. For comparison, its component form is also given. In order to comply with the four-form character of a Lagrangian, the auxiliary fields $\overline{\chi}$ and β have to be self-dual two-forms. In this framework, the so-called fermionic constraint is $\widetilde{\Psi}_{\rm F} := \overline{C}_{\alpha\beta}(d\,{}^*\Gamma^{\alpha\beta} + \frac{1}{2}{}^*\mathbb{B}^{\alpha\beta})$.

In the following, we consider the full topological gravity Lagrangian

$$L_{\rm TG} = dC_{\rm RR} + L_{\rm FP} = dC_{\rm RR} + s\{\cdots\} \tag{8.4.4}$$

and are going to demonstrate that it is BRST-invariant and classically equivalent to the double self-dual or anti-self-dual version of SKY gravity. Hence the constraint four-form $\{\cdots\}$ is able to induce a dependence on the metric that in is not present in the primordial topological action. The physics will be independent of any parameters that are introduced into the theory via BRST-exact terms, except for an overall factor in the resulting partition function and a possible "background" dependence (ALFARO & DAMGAARD 1990).

Performing the BRST transformation s in the gauge-fixing Lagrangian (8.4.3), we obtain, after a long but straightforward calculation using the algebra specified before, the result

$$L_{\rm FP} = -\,{\rm Tr}\Big\{\beta\wedge{}^*\overset{\pm}{\Omega}{}^g - \overline{\chi}\wedge{}^*D\Psi^{(\pm)} - \overline{\chi}\wedge{}^*\left[C,\overset{\pm}{\Omega}{}^g\right]$$
$$+\,\overline{\eta}D\,{}^*\Psi + \overline{\Phi}\,D\,{}^*D\Phi + \overline{\Phi}\,[\Psi,{}^*\,\Psi] + \mathbb{B}\,d\,{}^*\Gamma$$
$$-\,\overline{C}\,d\,{}^*\Psi + \overline{C}\,d\,{}^*(DC) + \frac{1}{2}\rho\,\beta\wedge{}^*\beta + \frac{1}{2}\mathbb{B}\wedge{}^*\mathbb{B}\Big\}. \tag{8.4.5}$$

The variation with respect to the auxiliary field $\mathbb{B}$ yields the Lorentz-type condition $d\,{}^*\Gamma = {}^*\mathbb{B}$ on the linear connection. Moreover, for vanishing real *gauge parameter* $\rho = 0$, the equation of motion for the auxiliary Lagrange-multiplier-type two-form β enforces the self-*double-duality* or anti-self-*double-duality* condition

$$\overset{\pm}{\Omega}{}^g := \frac{1}{2}\left(\Omega^g \pm {}^*\Omega^{g(\star)}\right) = 0 \tag{8.4.6}$$

on the curvature two-form Ω^g, where we distinguish between the Hodge dual * and the Lie dual $^{(\star)}$ in a space(time) of signature sig.

8.4.1 Effective Self-dual SKY Gravity

In the case of the choice $\rho = 1$ of the real gauge parameter, the two-form β is present in two terms, but then can be eliminated by a Gaussian integration (in Euclidean space) such that up to gauge-fixing terms, the SKY Lagrangian remains supplemented by the topological Euler term as a boundary term, i.e.,

$$\begin{aligned} L_{\rm SKY}^{(\star)} &= \frac{1}{2}\mathrm{Tr}\left(\overset{\pm}{\Omega}{}^g \wedge {}^*\overset{\pm}{\Omega}{}^g\right) \\ &= -\frac{1}{2}R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} \mp \frac{(-1)^{\rm sig}}{2}R^{\alpha\beta} \wedge R_{\alpha\beta}^{(\star)} \\ &= -\frac{1}{4}\left(R_{\alpha\beta} \pm {}^*R_{\alpha\beta}^{(\star)}\right) \wedge {}^*\left(R^{\alpha\beta} \pm {}^*R^{\alpha\beta(\star)}\right) . \end{aligned} \tag{8.4.7}$$

Due to the second Bianchi identity $D\Omega^g \equiv 0$, the dual field equation

$$D\,{}^*\overset{\pm}{\Omega}{}^g = 0 \tag{8.4.8}$$

is equivalent to Yang's original equation. The corresponding canonical energy–momentum current of the gravitational gauge fields, i.e.,

$$E_\alpha = \frac{1}{2}\mathrm{Tr}\left(\overset{\pm}{\Omega}{}^g \wedge e_\alpha\rfloor{}^*\overset{\pm}{\Omega}{}^g - e_\alpha\rfloor\overset{\pm}{\Omega}{}^g \wedge {}^*\overset{\pm}{\Omega}{}^g\right) = 0, \tag{8.4.9}$$

is zero, which is equivalent to the vanishing of the metric stress–energy tensor, i.e., $T_{\mu\nu}:=2\,{}^*(\partial L/\partial g_{\mu\nu}) = 0$. In turn, this implies that the BRST quantization is independent of the metric "background."

It is rather obvious from the equivalent binomial form of the effective SKY Lagrangian that anti-self-dual solutions (Mielke 1981; Benn et al. 1981; Mielke & Maggiolo 2005) as well as self-dual spaces, i.e.,

$$R_{\alpha\beta} = \mp\,{}^*R_{\alpha\beta}^{(\star)} \tag{8.4.10}$$

are extrema. The sign difference in the curvature constraints resemble the two roots of a quadratic equation.

Concentrating on topological terms such as those of Pontryagin and Euler, related self-dual modifications have also been advocated as *topological 4D self-dual gravity* by Nakamichi et al. (1991) with the emphasis on gravitational instantons with an additional duality constraint on torsion. Conformal gravitational instantons living on Einstein spaces with the additional constraint of anti-self-dual Weyl curvature (Torre 1990; Myers & Periwal 1991) and their deformations can be classified topologically (Perry & Teo 1993). The consideration of the Pontryagin term with respect to the Riemannian curvature $R^{\{\}}_{\alpha\beta}$ and the *axial torsion* one-form $\mathscr{A} := {}^*(\vartheta_\alpha \wedge T^\alpha)$ is rather well motivated by the *axial anomaly* in RC spacetime

$$\langle dj_5 \rangle = 2i\, {}^*m \langle \overline{\psi} \gamma_5 \psi \rangle - \frac{1}{48\pi^2} \left(R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + \frac{1}{2} d\mathscr{A} \wedge d\mathscr{A} \right) , \tag{8.4.11}$$

where ψ is a Dirac spinor field.

8.5 Symmetry Breaking Via Duality Rotations

In order to lift the vacuum degeneracy of self-dual SKY gravity, we may use the freedom in the choice of the constraint four-form and impose instead the gauge constraint (8.4.3) modified via

$$\Delta L_{\mathrm{FP}} = s \mathrm{Tr} \left\{ \frac{\theta^*_{\mathrm{T}} i}{4\ell^2}\, \overline{\chi} \wedge {}^*\sigma - \overline{c} d\, {}^*\gamma \right\} \tag{8.5.1}$$

involving the "unit"-curvature two-form

$$\sigma := \frac{i}{2} \gamma \wedge \gamma . \tag{8.5.2}$$

In addition, the coframe $\gamma := \vartheta^\alpha \gamma_\alpha$ is constrained by the Hilbert–de Donder gauge condition $d^*\gamma = 0$ complying with the BRST algebra of translations.

Then for $\rho = 0$ there arises from $\widetilde{L}_{\mathrm{FP}} := L_{\mathrm{FP}} + \Delta L_{\mathrm{FP}}$ the *modified double duality* constraint

$$\overset{\pm}{\Omega}{}^g = i \frac{\theta^*_{\mathrm{T}}}{4\ell^2}\, \sigma \tag{8.5.3}$$

on the curvature two-form Ω^g, which is real due to the extra imaginary unit i. However, since σ is automatically double-(anti-)self-dual, i.e.,

$$^*\sigma^{(\star)} = (-1)^{\mathrm{sig}} \sigma , \tag{8.5.4}$$

only one sign for the dual curvature $\overset{\pm}{\Omega}{}^g$ leads to a self-consistent constraint. This depends on the signature: in Minkowski spacetime, e.g., a modified anti-self-double-dual curvature arises.

The choice $\rho = 1$ of the real gauge parameter would leave the two-form β present in two terms, but it can again be eliminated by a formal Gaussian integration in Euclidean space such that up to gauge-fixing terms, the Einstein–Cartan Lagrangian, the SKY Lagrangian, and an induced cosmological term remain supplemented by the topological Euler four-form, i.e.,

$$\widetilde{L}^{(\star)}_{\rm SKY} = \frac{1}{2}{\rm Tr}\left[\left(\overset{\pm}{\Omega}{}^{g} - i\,\frac{\theta^{*}_{\rm T}}{4\ell^{2}}\sigma\right) \wedge {}^{*}\left(\overset{\pm}{\Omega}{}^{g} - i\,\frac{\theta^{*}_{\rm T}}{4\ell^{2}}\sigma\right)\right] \tag{8.5.5}$$
$$= -\frac{\theta^{*}_{\rm T}}{2\ell^{2}}R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \theta^{*}_{\rm T}\Lambda_{\rm eff}\,\eta - \frac{1}{2}R^{\alpha\beta} \wedge {}^{*}R_{\alpha\beta} \mp \frac{(-1)^{\rm sig}}{2}R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta}\,.$$

Related Lagrangians[4] with broken scale invariance were considered before in the Poincaré gauge framework in 4D or as a reduction of five-dimensional de Sitter models (MacDowell & Mansouri 1977). In our topological BRST formalism, they arise as gauge constraints.

The quadratic form (8.5.5) of the Lagrangian, related to the expression (9.8) of Mielke (1984a) in components, again suggests an important link to the path integral approach to quantum gravity. Then *instanton*-type configurations (Gu et al. 1978; Atiyah et al. 1978) near the classical ones, i.e., Einstein spaces, are more probable then the "spurious" Thomson spaces, in concordance with what one would expect naively. For the modified duality (8.5.3) accompanied by a breaking of scale invariance, the transition amplitude peaks at classical Einstein spaces *only*. Alternatively, in a four-dimensional Yang–Mills theory gauging the de Sitter group (MacDowell & Mansouri 1977; Aouane et al. 2007), scale invariance would get spontaneously broken by a pseudo-Goldstone-type "radius vector" (Pagels 1984; Tresguerres & Mielke 2000), odd under CP transformations, in order to recover the Hilbert–Einstein action plus the Euler term.

More generally, following Rainich (1925), Misner & Wheeler (1957), Mielke (1987), one could consider the *double duality rotation*

$$\Omega^{g(\theta)} := \sqrt{2}\left(\Omega^{g}\sin\theta + {}^{*}\Omega^{g(\star)}\cos\theta\right) \tag{8.5.6}$$

of the curvature, which for $\theta = \pi/2$ specializes to the self-dual curvature $\overset{+}{\Omega}{}^{g}$, and for $\theta = 3\pi/2$ to the anti-self-dual curvature $\overset{-}{\Omega}{}^{g}$. Since the "unit"-curvature two-form σ is either self-dual or anti-self-double-dual, depending on the signature, cf. (8.5.4), the additional term (8.5.1) in the constraint again would induce a breaking of the *dual symmetry* (8.5.6). Thereby, the vacuum degeneracy of Yang's equations is lifted.

8.6 Generalized Double Duality

Before analyzing its consequences for the metric "background," let us turn to a further generalization that will include the constraint (8.5.3) as a special case. Such a generalization can be more concisely derived in a gravitational gauge framework

[4] The instanton solutions of Yang's theory of gravity, classified as early as (1981) by Mielke, are a special case of the Ansatz (8.6.4) for the choice $\theta_{\rm L} = \theta_{\rm T} = 0$ and $\theta^{\star}_{\rm L} = \mp(-1)^{\rm sig}$. Interestingly enough, it can be regarded as a *field redefinition* (FR) of the linear connection Γ such that (8.6.4) is induced; see Mielke (2006b) for details. Such an FR was applied in Obukhov & Hehl (1996) to Euler- and Pontryagin-type terms. However, such deformations change the latter four-forms to no longer being d-exact terms, thus preventing a topological interpretation.

with torsion, where the most general parity-invariant *quadratic* Poincaré gauge (qPG) Lagrangian reads

$$L_{\rm qPG} = \frac{\Lambda}{\ell^2}\,\eta - \frac{a_0}{4\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{1}{2} T^\alpha \wedge H_\alpha - \frac{1}{2} R^{\alpha\beta} \wedge H_{\alpha\beta}\,. \tag{8.6.1}$$

Here

$$H_\alpha := -\,\partial L_{\rm qPG}/\partial T^\alpha = \frac{1}{\ell^2}\,{}^*\!\left(\sum_{M=1}^{3} a_{(M)}\,{}^{(M)}T_\alpha\right), \tag{8.6.2}$$

and

$$H_{\alpha\beta} := -\,\partial L_{\rm qPG}/\partial R^{\alpha\beta} = \frac{a_0}{2\ell^2}\,\eta_{\alpha\beta} + {}^*\!\left(\sum_{N=1}^{6} b_{(N)}\,{}^{(N)}R_{\alpha\beta}\right) \tag{8.6.3}$$

are the translational and rotational field momenta, respectively. The fundamental length ℓ fixes the relative strength of the rotational and translational interaction parts of the gravitational Lagrangian $L_{\rm qPG}$. In the field momenta, each of the three irreducible torsion pieces and six irreducible curvature pieces contribute to the Lagrangian with individual dimensionless weights $a_{(M)}$ and $b_{(N)}$, respectively. The volume form η accounts for the possible occurrence of a "bare" cosmological constant Λ.

In such a more general setting, one could impose (Mielke 1984b, 2006b) the *generalized double duality Ansatz* (DD)

$$\begin{aligned} H_{\alpha\beta}(**) &= \theta_{\rm L}\, R_{\alpha\beta} + \theta^\star_{\rm L}\, R^{(\star)}_{\alpha\beta} + \frac{\theta^*_{\rm T}}{2\ell^2}\,\eta_{\alpha\beta} - \frac{\theta_{\rm T}}{2\ell^2}\,\vartheta_\alpha \wedge \vartheta_\beta \\ &\simeq -\partial L_\theta/\partial R^{\alpha\beta}\,, \end{aligned} \tag{8.6.4}$$

for the rotational field momenta (Mielke 1992; Zhytnikov 1994), where $\theta_{\rm T}$, $\theta^*_{\rm T}$, $\theta_{\rm L}$, and $\theta^\star_{\rm L}$ are dimensionless constants. The constraint (8.5.3) corresponds to the SKY Lagrangian with $a_0 = 0$, $b_{(N)} = 1$, and the choice $\theta^\star_{\rm L} = \mp(-1)^{\rm sig}$ as well as $\theta_{\rm L} = \theta_{\rm T} = 0$. The left-hand side of (8.6.4) can be, as indicated, related to the θ-type boundary term

$$L_\theta := \theta_{\rm T}\, dC_{\rm TT} + \theta^*_{\rm T}\, dC_{\rm TT^*} + \theta_{\rm L}\, dC_{\rm RR} + \theta^\star_{\rm L}(-1)^{\rm sig+1} dC_{\rm RR^\star} = dC_\theta\,. \tag{8.6.5}$$

The latter is a linear superposition of the topological Nieh–Yan term, the term $dC_{\rm TT^*} := d(\vartheta^\alpha \wedge {}^*T_\alpha)/2\ell^2$ inducing the teleparallelism equivalence, as well as the Pontryagin and Euler four-forms. The DD Ansatz (8.6.4) could be implemented as the BRST gauge constraint $\frac{1}{2} s\overline{\chi}^{\alpha\beta} \wedge (H_{\alpha\beta} + \partial L_\theta/\partial R^{\alpha\beta})$ generalizing the corresponding terms in the Faddeev–Popov Lagrangian $\widetilde{L}_{\rm FP}$ modified by (8.5.1).

The metric consequences can be derived by inserting the duality Ansatz (8.6.4) into the two *nonlinear* gauge field equations (HEHL et al. 1995),

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{8.6.6}$$

$$DH_{\alpha\beta} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta}\,, \tag{8.6.7}$$

where due to the universality of gravity, there occur the three-form of the *energy–momentum*

$$E_\alpha := \partial L/\partial\vartheta^\alpha = e_\alpha \rfloor L + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma} \tag{8.6.8}$$

and the *angular momentum current* $\vartheta_{[\beta} \wedge H_{\alpha]}$ of the gravitational gauge fields.

The DD Ansatz maps the second field equation (8.6.7) into the second Bianchi identity for the RC curvature, provided the translational gauge field momenta H_α satisfy the algebraic equation

$$\frac{\theta_{\mathrm{T}}^*}{2\ell^2}\eta_{\alpha\beta\gamma}T^\gamma + \frac{\theta_{\mathrm{T}}}{\ell^2}\vartheta_{[\alpha} \wedge T_{\beta]} + \vartheta_{[\alpha} \wedge H_{\beta]}(**) = \tau_{\alpha\beta}\,. \tag{8.6.9}$$

The covariant derivative D of the "unit"-curvature pieces proportional to $\eta_{\alpha\beta}$ and $\vartheta_\alpha \wedge \vartheta_\beta$ in the DD Ansatz is responsible for the two additional torsion terms. For spinless matter, i.e., $\tau_{\alpha\beta} = 0$, one can algebraically resolve (8.6.9) using Eq. (A.1.26) of HEHL et al. (1995) for the translational momenta $H_\alpha(**)$ constrained by double duality, with the result that

$$\begin{aligned} H_\alpha(**) &= \frac{\theta_{\mathrm{T}}^*}{2\ell^2}\, K^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} - \frac{\theta_{\mathrm{T}}}{\ell^2}T_\alpha = \frac{1}{\ell^2}\left[\theta_{\mathrm{T}}^*\, K_\alpha^{(\star)} - \theta_{\mathrm{T}} T_\alpha\right] \\ &= \frac{\theta_{\mathrm{T}}^*}{\ell^2}\, {}^*\!\left[T_\alpha - \vartheta_\alpha \wedge (e_\beta \rfloor T^\beta) - \frac{1}{2}\, e_\alpha \rfloor (T^\beta \wedge \vartheta_\beta)\right] - \frac{\theta_{\mathrm{T}}}{\ell^2}T_\alpha \\ &\simeq -\partial L_\theta/\partial T^\alpha\,. \end{aligned} \tag{8.6.10}$$

Here the identity $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$ has been employed in the conversion from torsion to contortion, and vice versa. In vacuum, this implies the vanishing of some of the three irreducible pieces of torsion or algebraic constraints on its coupling parameters (MCCREA 1987, 1995). For the general case with spin, see MIELKE (1984b).

The insertion of the duality Ansatz (8.6.4) and (8.6.10) into the first field equation (8.6.6) implies important information (MIELKE 1984a; ZHYTNIKOV 1994; MIELKE 2006b) on the Riemannian background, but requires rather involved algebraic manipulations, even with computer algebra like REDUCE.

A calculational shortcut occurs on the Lagrangian level, provided the constraints are compatible with the variational principle. After inserting (8.6.4) and (8.6.10) into (8.6.1), we obtain

$$
\begin{aligned}
L_{\text{qPG}}(**) &= \frac{\Lambda}{\ell^2}\,\eta - \frac{a_0}{4\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{1}{2} T^\alpha \wedge H_\alpha(**) - \frac{1}{2} R^{\alpha\beta} \wedge H_{\alpha\beta}(**) \\
&= \frac{\Lambda}{\ell^2}\,\eta - \frac{\theta_{\text{T}}^* + a_0}{4\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{1}{2\ell^2} T^\alpha \wedge \left[\theta_{\text{T}}^*\, K_\alpha^{(\star)} - \theta_{\text{T}} T_\alpha\right] \\
&\quad - \frac{1}{2} R^{\alpha\beta} \wedge \left[\theta_{\text{L}}\, R_{\alpha\beta} + \theta_{\text{L}}^\star\, R_{\alpha\beta}^{(\star)} - \frac{\theta_{\text{T}}}{2\ell^2} \vartheta_\alpha \wedge \vartheta_\beta\right]. \qquad (8.6.11)
\end{aligned}
$$

Since the explicit torsion terms drop out due to the Nieh–Yan relation and the teleparallelism identity, we are left, up to an exact form, with an *effective* Hilbert–Einstein Lagrangian[5]

$$
L_{\text{eff}} := L_{\text{qPG}}(**) - dC_\theta = -\frac{\theta_{\text{T}}^*}{2\ell^2} R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{\theta_{\text{T}}^* \Lambda_{\text{eff}}}{\ell^2}\,\eta\,. \qquad (8.6.12)
$$

The contraction $H_{\alpha\beta}(**) \wedge \vartheta^\alpha \wedge \vartheta^\beta$ of the DD Ansatz (8.6.4) together with the linear expansion (8.6.3) implies, for $\theta_{\text{L}} \neq 0$, that the pseudoscalar curvature vanishes for consistency, i.e., $R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta = 0$. Then the scalar curvature

$$
R := e_\beta \rfloor e_\alpha \rfloor R^{\alpha\beta} = \frac{6(\theta_{\text{T}}^* - a_0)}{\ell^2(\theta_{\text{L}}^\star - b_6)} \qquad (8.6.13)
$$

is constrained to be a constant such that an *effective* “cosmological” constant

$$
\Lambda_{\text{eff}} = \frac{1}{\theta_{\text{T}}^*}\left[\Lambda - \frac{1}{4}(\theta_{\text{T}}^* - a_0) R\right] = \frac{\Lambda}{\theta_{\text{T}}^*} - \Lambda_\theta \qquad (8.6.14)
$$

of partially *topological origin* (Mielke 1984b, 1987; McCrea 1995) is induced. Observe that the “bare” cosmological constant Λ gets subtractively *renormalized* by θ-terms induced via boundary terms. Thus one may speculate that some topological mechanism is responsible for the necessary fine-tuning of the “bare” constant Λ to the tiny cosmological constant of the present epoch of our universe.

After inserting the double duality Ansatz (8.6.4), we are left with Eq. (5.8.29) of Hehl et al. (1995), i.e.,

$$
-E_\alpha = \frac{\theta_{\text{T}}^*}{2} R^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma} - \theta_{\text{T}}^* \Lambda_{\text{eff}}\, \eta_\alpha = \ell^2\, \widehat{\Sigma}_\alpha\,. \qquad (8.6.15)
$$

[5]There occurs an interesting modification in the case that $\theta_{\text{L}} = 0$, since then the pseudoscalar curvature four-form $\theta_{\text{T}} R^{\alpha\beta} \wedge \vartheta_\alpha \wedge \vartheta_\beta / 4\ell^2$ needs to be subtracted from (8.6.12), which would induce (Mielke 1992) a partially chiral reformulation of Einsteinian gravity à la Ashtekar. However, this would violate parity P and even CP if θ_{T} remained real, (Mielke 2001).

8.6.1 Classical GR by Constraining Torsion to Vanish

The same *Riemannian "background"* can be obtained by imposing the constraint of vanishing torsion $T^\alpha \simeq 0$ via Lagrange multipliers in the gravitational gauge field equations, cf. MIELKE & MAGGIOLO (2003). This alternative method eventually amounts to a degenerate variational principle for the metric: to be consistent, the constraint of vanishing torsion is implemented via Lagrange multipliers,

$$\widetilde{L}_{\mathrm{qPG}} = L_{\mathrm{qPG}} + \lambda_\alpha \wedge T^\alpha \, . \tag{8.6.16}$$

Then obviously, $T^\alpha = 0$ will emerge by varying the Lagrange multiplier, and the second field equation (8.6.7) amounts to an algebraic equation for the two-form λ_α. After its resolution, it converts the first field equation (8.6.6) into Eq. (5.8.25) of HEHL et al. (1995), i.e.,

$$2D\left(e^\beta \rfloor DH_{\alpha\beta} - \frac{1}{4}\vartheta_\alpha \wedge e^\gamma \rfloor e^\delta \rfloor DH_{\gamma\delta}\right) - E_\alpha = \Sigma_\alpha - D\mu_\alpha \, , \tag{8.6.17}$$

which, generically, is of third order in the Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$ of Riemannian spacetime, i.e., of fourth order in the metric, with all the implications comprehensively reviewed in SCHIMMING & SCHMIDT (2004).

In the case of Dirac spinors, the *spin-energy potential* $\mu_\alpha = \frac{1}{4}\vartheta_\alpha \wedge {}^*j_5$ is dual to the axial current $j_5 := \overline{\psi}\, {}^*\gamma\gamma_5\psi$; cf. MIELKE (2004). Since λ_α involves μ_α, (8.6.17) can likewise be obtained from the variation $\mu_\alpha \wedge \delta T^\alpha$.

Let us now consider again the *double-duality Ansatz* (8.6.4) for the rotational field momenta where the θ's are dimensionless "vacuum angles" related to the individual coupling constants in the θ-type boundary term (8.6.5): since $DR_{\alpha\beta} \equiv 0$, $DR^{(\star)}_{\alpha\beta} \equiv 0$, $D\eta_{\alpha\beta} = 0$, and $D(\vartheta_\alpha \wedge \vartheta_\beta) = 2T_{[\alpha} \wedge \vartheta_{\beta]} = 0$ in a Riemannian spacetime, the higher-derivative Cotton-type three-form in (8.6.17) drops out completely. Moreover, the Lie dual $R^{(\star)}_{\alpha\beta}$ of the curvature does not contribute in (8.6.8), due to the Bach–Lanczos identity (A.3.7) of HEHL et al. (1995) for Riemannian spacetimes.

Then we are again left with (8.6.15) or, finally, with Einstein's field equations

$$G^{\{\}}_\alpha - \Lambda_{\mathrm{eff}}\, \eta_\alpha = \kappa_{\mathrm{eff}}\, \widehat{\Sigma}_\alpha \, . \tag{8.6.18}$$

Here $G^{\{\}} := R^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma}\gamma^\alpha/2$ is the Einstein three-form dual to the usual Einstein tensor $G_{ij} := \mathrm{Ric}^{\{\}}_{ij} - \frac{1}{2}g_{ij}$ written in terms of the Riemannian connection $\Gamma^{\{\}}$. Matter fields would act as source via the *symmetric* Belinfante–Rosenfeld current three-form

$$\widehat{\Sigma}_\alpha := \Sigma_\alpha - D^{\{\}}\mu_\alpha , \tag{8.6.19}$$

whereas the effective gravitational coupling constant $\kappa_{\mathrm{eff}} = \ell^2/\theta^*_{\mathrm{T}}$ depends on a "bare" length scale ℓ "renormalized" by the vacuum angle θ^*_{T}.

In order to maintain macroscopic correspondence, the coupling of the *effective Einstein equation* (8.6.18) to the symmetrized energy–momentum current $\widehat{\Sigma}_\alpha$ of matter requires, at least locally, the macroscopic Newtonian value $\kappa_{\text{eff}} = 8\pi G_{\text{N}}/c^4$ of the gravitational coupling constant. In the coupling to fundamental Dirac fields, again a BRST formulation (Grensing 2002) is feasible.

Observe that our constraint method provides a simple rendering of the *Lovelock theorem*, stating that the Einstein equations with cosmological constant are the only second-order partial differential equations for the metric; cf. Rund & Lovelock (1972).

8.6.2 *Effective Einsteinian "Background"*

Thus by modifying the gauge constraint, the double duality relation (8.5.3) surfaces, which eliminates the "vacuum ambiguity" for the exact instanton solutions of SKY gravity. The latter emerges as a special case from the generalized DD Ansatz (8.6.4) mapping the gravitational field excitations $H_{\alpha\beta}$ to those generated from the four-parameter boundary term (8.6.5). As an intermediate step, we have considered an at most quadratic Lagrangian L_{qPG} with ten[6] parameters $a_{(M)}$ and $b_{(N)}$.

The classically allowed set is still the subject of investigation: In a pioneering work, Sezgin & van Nieuwenhuizen (1980), using spin-projection operators, determined the propagating modes and the particle content. By performing a mode decomposition based on a *flat* Minkowskian background, a three-parameter class of unitary qPG Lagrangians has been found (Kuhfuss & Nitsch 1986). However, these earlier works effectively depart from a linearization of the gravitational gauge fields. Then problems with the Cauchy formulation, shock waves, and the positivity of the gravitational energy may arise (Hecht et al. 1991; Hecht et al. 1996). A more recent Hamiltonian analysis (Yo & Nester 2002; Deser et al. 2014) has revealed that due to nonlinear effects entering the Poisson brackets, a *bifurcation* in the constraint chain or a field activation may occur. Thus, in passing from the strong to the weak field regime, the status of presumably viable parameters can switch.

Nevertheless, the 10-parameter Lagrangian (8.6.1) serves as a convenient means to facilitate the derivation of the metric consequences of the generalized double-dual constraint (8.6.4), resulting in Einstein spaces as the only classical Riemannian "background."

Due to the explicit appearance of a length scale in the Ansatz (8.6.4), it is suggestive to associate this with a (spontaneous) *symmetry breaking* $\langle\varphi\rangle \propto 1/\ell$ of the scale or Weyl invariance of the original Lagrangian L_{qPG}, for instance in a model (Hehl et al. 1989) dynamically coupled to the dilaton field φ; cf. Dereli & Tucker (2002). In a Riemann–Cartan–Weyl spacetime, by alleviating the induced algebraic torsion constraint (8.6.10), generalizations of Einstein's equations with a coupling to axial

[6] One has to keep in mind that due to the Bach–Lanczos identity, the parameters $b_{(N)}$ are not all independent; cf. Eq. (A.3.7) of Hehl et al. (1995).

torsion as well as a derivative coupling to the Weyl covector can arise (Kaul 2006). When these covectors are induced by an axion a and dilaton φ as respective potentials, a cancellation of the axial torsion part in the chiral anomaly (8.4.11) can be achieved (Mielke & Romero 2006; Mielke 2006a). Similarly to the case of strings, both may even combine into a single complex scalar, the *axidilaton* $\Phi = a + i f_\varphi \exp(-\varphi/f_\varphi)$; cf. Mielke & Schunck (2001).

Quite generally, an *induced* cosmological constant

$$\Lambda_\theta = \frac{3(\theta_{\mathrm{T}}^{*} - a_0)^2}{2\ell^2(\theta_{\mathrm{L}}^{\star} - b_6)\theta_{\mathrm{T}}^{*}} \tag{8.6.20}$$

of partially topological origin (Mielke 1984b) is unavoidable, leading to an interesting (anti-) de Sitter "background," resembling, to some extent, the intriguing AdS/CFT correspondence and/or a coupling of the Euler or Gauss–Bonnet term to a hypothetical scalar field (Cognola et al. 2007).

Thus, there is still a valid avenue to a consistent quantization based on a topological version of self-double-dual SKY gravity or its modifications, departing, in a gauge-covariant approach, from a d-exact topological term. Due to the nilpotency of the corresponding BRST charges (Mielke & Maggiolo 2003), the s-exact term can easily account for the necessary gauge constraints such as (8.6.4), implying standard Einsteinian gravity for the classical "background." This, to some extent, provides a partial answer to the issue already raised in 1963 by Feynman et al. (1995), whether Einstein's GR, in view of its force-free geometric concepts, needs to be quantized at all or whether curved spacetime can be left as an arena for quantized (topological) fields to play out. Possible observables (Aouane et al. 2007) include field polynomials in the Pontryagin, Euler, and Nieh–Yan invariant (Kreimer & Mielke 2001; Nieh 2007) constructed from torsion and curvature two-forms, including their *grading* via the topological ghosts as well, or even from superconnections (Ne'eman 1998). Relations to the Donaldson invariants, lucidly reviewed by Atiyah (1990), need to be seen as well as the unique and quite singular role of *four dimensions* in topology.

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Chapter 9
Gravitational Instantons

Most of the known vacuum solutions of Einstein's field equations, as for instance classified by PETROV (1964), HARRISON (1980), and recently by, e.g., STEPHANI et al. (2009) are providing the metric background for *nontrivial* torsion solutions. In order to derive these exact solutions explicitly, it is then sufficient to solve the duality ansatz with the symmetry of the looked after configurations given in advance.

9.1 Exact Solutions

A particularly simple case is SKY gravity in vacuum, since then, a complete classification of the double-dual subspaces is available from the preceding chapter. For anti-self-double-dual solutions of the Yang–Stephenson equation, these are Einstein spaces. Spherical symmetry provides Birkhoff's theorem in its more precise form, cf. NEVILLE (1980), RIEGERT (1984), and BOUCHER et al. (1984) that such solutions—after due application of differentiable coordinate transformations—can be traced back first to the Schwarzschild–de Sitter (SdS) space or Weyl–Trefftz metric:

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^{n-2} \tag{9.1.1}$$

with

$$e^\nu = e^{-\lambda} = 1 - \left(\frac{2\mu}{r}\right)^{n-3} - \frac{2\Lambda_{\text{eff.}}}{(n-1)(n-2)} r^2 \tag{9.1.2}$$
$$\mu = \frac{M}{m^{*2}}; \quad d\Omega^2 := d\vartheta^2 + \sin^2\vartheta d\phi^2$$

(here generalized to an n-dimensional manifold) and second for $\Lambda_{\text{eff}} > 0$ to the Nariai metric

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$$ds^2 = \frac{n-2}{2\Lambda_{\text{eff}} t^2}(-dt^2 + dr^2) + \frac{(n-2)(n-3)}{2\Lambda_{\text{eff.}}} d\Omega^{n-2} \tag{9.1.3}$$

(Nariai 1950, 1951). This second and less well known line element is isometric to the canonical metric for the Cartesian product of both a two- and (n-2)-dimensional space of *constant* curvature. Contrary to possible misperceptions, this "cosmological" configuration—exactly like the SdS metric—is a static solution of Einstein's field equations. On the other hand, the Nariai solution also satisfies the coupled Einstein–Maxwell system with a cosmological term and is then considered to be a configuration in which homologous electromagnetic fields are "trapped" within the metric background of the product topology (Percacci 1979).

Anti-self-double-dual configurations lead to the theory of Nordstrøm, whose most general spherically symmetric solution is conformally flat,

$$ds^2 = \left(\frac{N(r \pm t)}{r}\right)^2 (-dt^2 + dr^2 + r^2 d\Omega^2), \tag{9.1.4}$$

with $\mathrm{N} = \mathrm{N}(\mathrm{r} \pm \mathrm{t})$ denoting an arbitrary harmonic function of r and t; cf. Ni (1975).

In a pseudo-Riemannian space, these metrics exhaust all possible *spherically symmetric* solutions of SKY gravity (Baekler & Yasskin 1984; Baekler et al. 1982). The fact that it is especially the Schwarzschild metric that is contained in this solution manifold was recognized already by Pauli (1919a, b) while he was investigating the scale-invariant theory of Weyl (1919) mentioned earlier. In real spacetime, however, the Schwarzschild solution (9.1.1, 2) with $\Lambda_{\text{eff}} = 0$ is of paramount significance for the description of the external (in GR also with respect to the internal) gravitational field of a macroscopic mass distribution, as for instance that of a star; cf. Scheel & Thorne (2014).

But what role do these solutions play in the more *microscopically* founded Poincaré gauge theory? According to the canonical common opinion, such theories are normally not thought to contribute to a realistic description of physical reality unless they have undergone a satisfactory quantization. Again we apply Feynman's method of quantization by means of path integrals; see, e.g., De Witt (1972). Then it can be expected that solutions with duality properties play a dominant role, at least as *gravitational analogues* of the *pseudoparticle* solutions of Yang–Mills theory. Following the nomenclature of Gibbons & Hawking (1979), these are given by nonsingular, complete, and positive definite metrics that satisfy Einstein's equations (cosmological term included). Since the qPG theory is closer to the Yang–Mills theories, and this not only as far as its conceptual base is concerned but also with respect to its dynamics, it seems to be far more appropriate to define those nonsingular configurations as gravitational instantons, which, in a Euclideanized spacetime with signature $\mathrm{s} = 0$, satisfy duality Ansätze. This concept, however, is not necessarily contradictory to the findings of the Cambridge school, since we know that we are bound to Einstein spaces as a metric background, as far as vacuum solutions are concerned.

In order to show, for instance, that the SdS metric in a Euclidean spacetime is a nonsingular one, we first pass to a different coordinate system. Let r^* be the "tortoise" coordinate (WHEELER 1955) defined by the differential form

$$dr^* := e^{(\lambda-\nu)/2} dr. \tag{9.1.5}$$

Using the implicitly given Kruskal–Szekeres coordinates (MTW, p. 831)

$$u^2 - v^2 = e^{r*/4\mu} \tag{9.1.6}$$

and

$$(u+v)/(u-v) = e^{t/2\mu}, \tag{9.1.7}$$

the line element (9.1.1) transforms into

$$ds^2 = 16\mu^2 e^{\nu - r^{*/2\mu}}(-dv^2 + du^2) + r^2 d\Omega^2, \tag{9.1.8}$$

whereby r and r^* are considered to be functions of u and v. On the surfaces $u^2 - \nu^2 = 1$, that is, for $r = r^* = o$, this geometry becomes singular in real spacetime. Here, exactly as in the case with the instantons of Yang–Mills theory, this can be avoided by making use of a purely "imaginary" coordinate $\tilde{v} := iv$. The sphere S^2 given by $e^\nu = 0$ deteriorates to a rotational axis, while the imaginary time axis $\tau :=$ it represents an angle variable about this axis in Euclidean spacetime. (Corresponding nonsingular cross sections are to be found in the complexified (SCHIFFER et al. 1973) KERR metric (2008)). It is especially the Euclidean version (EGUCHI & FREUND 1976), of the de Sitter cosmos that can be considered the most convincing analogue to the instanton solution of BELAVIN et al. (1975), since it grows asymptotically "flat" with respect to all four dimensional directions.

There are other exact solutions of Einstein's field equations that can be considered as gravitational instantons. Provided that only simple self-duality, equivalent to certain deformations of a flat twistor space (PENROSE 1976), of the curvature tensor is considered, a condition that leads to real metrics in a space with 2signature $s = 0$, it is well known (HAWKING & POPE 1978) that the universal covering space of the K_3-surface represents the only compact manifold that is allowed by a metric with self-dual curvature. Gravitational multi-instantons can be generated by a generalization of the Taub–NUT spaces (TAUB 1951, NEWMAN et al. 1963). Further details concerning these solutions are to be found in EGUCHI et al. (1980), a work that also presents a good introduction to the characteristic classes, which are of utmost importance for questions of global topology. Further gravitational instantons of Petrov type D have been studied by LAPEDES & PERRY (1981).

9.1.1 Solutions with Torsion

Spherically symmetric solutions of the qPG theory with nontrivial torsion, for a spacetime with Minkowski signature, were first found by BAEKLER (1980, 1981). Since these nontrivial torsion solutions with duality properties "live" on Einstein spaces as metric background, the only information that needs to be added is the structure of the torsion 2-form being coded into the duality ansatz. In order to decode this information, we have to recall the splitting of the metric-compatible Riemann–Cartan connection ω^g into a Christoffel-type connection $\omega^{\{\}}$ and the remaining contortion 1-form K. This reflects itself in the corresponding decomposition of the curvature 2-form in an RC spacetime. Its Riemannian part may be decomposed further into

$$\Omega^{\{\}} = \Omega_{\mathrm{C}}^{\{\}} + \overline{\Omega}^{\{\}} - \frac{1}{6} R^{\{\}} \vartheta \wedge \vartheta, \tag{9.1.9}$$

where $\Omega_{\mathrm{C}}^{\{\}}$ denotes Weyl's conformal curvature 2-form in a Riemannian spacetime. Concerning the qPG Lagrangian 4-form with quasilinear field momenta and with the choice $\mathrm{f_i} = \mathrm{o}$ and $\zeta = -1$, as discussed by BAEKLER et al. (1982), the duality ansatz used for the reduction of the field equations is simplified to

$$\overset{+}{\Omega}{}^{g} = \frac{\gamma\kappa}{\ell^{*2}} \vartheta \wedge \vartheta. \tag{9.1.10}$$

As a consequence of these algebraic relations, the Riemann–Cartan spaces are of constant scalar curvature

$$R = {}^{*}(\Omega^{g} \wedge \vartheta \wedge \vartheta) = -\frac{6\kappa\gamma}{\ell^{*2}} = 4\Lambda_{\mathrm{eff}}. \tag{9.1.11}$$

On the other hand, the metric background manifold has to be an Einstein space which, in vacuum, may be characterized in the following equivalent way:

$$\overline{\Omega}^{\{\}} = 0. \tag{9.1.12}$$

Its Riemannian background is likewise of constant scalar curvature

$$R^{\{\}} = -\frac{6\kappa\gamma}{\ell^{*2}}. \tag{9.1.13}$$

If (9.1.9) and (9.1.12) are taken together, the constant-curvature pieces drop out, and the double-duality constraint (9.1.10) on the solution spaces reduces to

$$(\overset{+}{D}K) - K \wedge K = \overset{+}{\Omega}_{\mathrm{C}}{}^{\{\}}. \tag{9.1.14}$$

As a result, the remaining contortion 1-form K is determined solely by Weyl's conformal curvature 2-form of the background manifold via a nonlinear differential equation involving double duality. In the derivation of (9.1.14), use is made of the fact that $K \wedge K$ is already self-double-dual.

Exact solutions can now be derived (for details, see MIELKE 1984, BAEKLER 1985) from the remaining Eq. (9.1.14), after specifying the symmetry of the background and independently those symmetries of the torsion. For our purposes, however, it suffices to present only some of the most important examples in the case of spherical symmetry. Then, the background manifold is given by the 4-dimensional Schwarzschild–de Sitter line element (3.9.1,2), whereas the spherical reflection-invariant torsion tensor can be written as

$$\left[\mathrm{T}_{\alpha\beta}{}^{\gamma}\right] = \begin{bmatrix} f & -h & \cdot & \cdot \\ \cdot & \cdot & -k & \cdot \\ \cdot & \cdot & \cdot & -k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -g \\ \cdot & \cdot & g & \cdot \end{bmatrix} \begin{matrix} 01 \\ 02 \\ 03 \\ 23 \\ 31 \\ 12 \end{matrix} \qquad (9.1.15)$$
$$\quad\; 0 \quad 1 \quad 2 \quad 3$$

in a (bivector)·(vector) representation (MTW, p. 360). Subsequent to a tedious calculation (BAEKLER et al. 1982) that will not be reproduced here, the radial torsional functions turn out to be

$$f = -h = g = k = -\frac{\mu}{r^2} e^{-\nu/2}. \qquad (9.1.16)$$

This solution, found first by BAEKLER (1981) while analyzing the original qPG equations, was generalized to a *charged* configuration by LEE (1983). It is highly important to note that the resulting torsion (9.1.15), after passing to a *pure* de Sitter cosmos, vanishes without Schwarzschild's source term, i.e., for $\mu = 0$. This is in complete analogy to the properties of certain instanton solutions of the coupled Einstein–Yang–Mills system. As could be shown by CHARAP & DUFF (1977a, b), the latter configurations can exist only on a *curved* spacetime, but not on a flat Minkowski space.

On the other hand, there is a further solution, found by BAEKLER et al. (1982), that exists only in a spacetime of constant curvature, i.e., one for which the mass parameter *mu* vanishes in the metric ground form. Here the radial functions occurring in the matrix representation (9.1.15) take the form

$$f = -\frac{1}{r}\left(1 + \frac{\kappa}{2\ell^{*2}} r^2\right) e^{-\nu/2}, \; g = \frac{1}{r} e^{\nu/2}$$
$$h = \underset{(+)}{-} \frac{1}{\ell*} \frac{\sqrt{-\kappa}}{3}, \quad k = \underset{(-)}{+} \frac{1}{2\ell*} \sqrt{-3\kappa}. \qquad (9.1.17)$$

For a real solution, $\kappa < 0$ is necessary. In this case, torsion can exist only in an anti-de Sitter cosmos. Spherically symmetric solutions based on the Nariai metric (9.1.3) have not yet been considered. Among the vacuum solutions of qPG theories are known not only those with a spherically symmetric metric background (Baekler et al. 1983) but also some with cylindrical (McCrea 1983) and axial symmetry (McCrea 1984). The latter is a torsional extension of the Taub–NUT solution. A solution with purely axial torsion was constructed by Lenzen (1984).

Moreover, *electrovac solutions*, i.e., solutions of the qPG theory coupled to electromagnetism have been found (Baekler & Hehl 1984, Baekler 1985; cf. Mielke 1984), which likewise exhibit double duality properties. Some anticipating results aiming prospectively in this direction may be found in the works of Benn et al. (1980), Baekler et al. (1982), Mielke (1981b, 1985) and Rüdiger (1984).

9.1.2 Instantons with Torsion

In an imaginary "spacetime" with Euclidean signature s = 0, some of the vacuum solutions should yield *gravitational instantons* not only in SKY gravity, but also in the full qPG theory as well. However, a transfer of the exact torsion solutions to the Euclidean domain is usually prohibited. A *direct* correspondence of spacetimes with Minkowski signature to those with a positive definite signature arises only for those qPG solutions in which the torsion components remain real after application of Weyl's trick, i.e., the formal substitution $t \to -i\tau$ of the timelike variable. It follows from the matrix representation (9.1.15) that the anholonomic components $f = T_{\hat{0}\hat{1}}{}^{\cdot\cdot\hat{0}}$ and $g = T_{\hat{1}\hat{2}}{}^{\cdot\cdot\hat{2}} = T_{\hat{1}\hat{3}}{}^{\cdot\cdot\hat{3}}$ are the only ones that are then allowed to be nonzero. A solution with one nonzero torsion component was first derived by Benn et al. (1981) on a real spacetime; after it has been transferred to a "Euclideanized" Schwarzschild–de Sitter space given by

$$\begin{aligned} ds^2_{\text{Sds}} &= \left(1 - \frac{2\mu}{r} + \frac{\kappa}{4\ell^{*2}}\right) d\tau^2 \\ &\quad + \left(1 - \frac{2\mu}{r} + \frac{\kappa}{4\ell^{*2}}\right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned} \tag{9.1.18}$$

it reads

$$\begin{aligned} f &= \frac{2\mu}{r^2}\left(1 - \frac{2\mu}{r} + \frac{\kappa}{4\ell^{*2}}\right)^{-1/2} \\ g &= h = k = 0 \end{aligned} \tag{9.1.19}$$

(cf. Mielke 1984). For $\mu = 0$, it degenerates to a de Sitter-type torsionless instanton, which nevertheless has to be regarded as a pseudoparticle solution of the full qPG theory. Topologically, the base manifold is S^4. Within the framework of the Einstein–Cartan theory, it was TSEYTLIN (1982) who obtained a torsional instanton.

As already mentioned, the criteria for gravitational instantons are stated differently by different schools; cf. MIELKE (1981a). But within the context of the qPG theory, we prefer to distinguish gravitational instantons by both the *double-duality* properties and the invariant characteristics of the global topology.

9.2 Topological Invariants on Manifolds

Accordingly, we have to turn to the question whether and to what degree gravitational instantons are globally characterized by topological invariants. Again we restrict ourselves to a manifold of dimension $n = 4$. The previous classification of the source-free Yang–Mills gauge fields, especially by means of the Pontryagin–Chern index, can be transferred to the gravitational case insofar as the tangent bundle T(M) is exactly the bundle associated to L(M). Furthermore, the *real* Euclidean space $\mathbb{R}^n$ occurs as the typical fiber of T(M), and thus it is allowed to characterize topologically the space of *linear* connections ω^L (i.e., those defined in L(M)) by the *Pontryagin index*

$$p_1(M^4) = \frac{1}{2(2\pi)^2}\int_{M^4} \mathrm{Tr}(\Omega^L \wedge \Omega^L)$$
$$= \frac{(-1)^{s/2}}{(4\pi)^2}\int_{M^4} R_{\alpha\beta\gamma\delta}{}^{*}R^{\alpha\beta\gamma\delta}\sqrt{|g|}d^4x \qquad (9.2.1)$$

(KN II, p. 312). This index is known, for instance, as soon as there exists a smooth self-transversal immersion of the oriented compact manifold M^4 into the Euclidean space R^6. For then

$$p_1(M^4) = -3\tau \qquad (9.2.2)$$

is valid, and τ provides the number of isolated triple points, i.e., the Hirzebruch signature. These threefold self-intersections are isolated due to self-transversality (WHITE 1975). Since torsion can be switched off continuously, it does not change the Pontryagin index of a manifold (WU & ZEE 1984).

"Broken" Poincaré gauge theories of gravity do not reside on the bundle L(M) but on the bundle $L^g(M)$ of *orthogonal* frames. Furthermore, our formal transfer of the Higgs–Kibble mechanism onto gravity demanded the introduction of *metric-compatible* connections ω^g. Connections of this kind are additionally to be characterized by the Euler classes (KN II, p. 314). Regarding oriented compact (pseudo-) Riemann–Cartan manifolds of dimension $n = 4$, these classes are represented by the closed *Euler form*

$$L_{\text{Euler}}, = \overset{*}{\pi}(\gamma_2^g) = \frac{1}{(4\pi)^2}\text{Tr}(\Omega^g \wedge \Omega^{g(*)})$$
$$= \frac{(-1)^{s/4}}{32\pi^2} R^{\alpha\beta\gamma\delta *} R^*_{\alpha\beta\gamma\delta}\sqrt{|g|}d^4x. \tag{9.2.3}$$

Under the above-mentioned conditions, it is the integration of the inverse image of the form γ_2^g over the basis manifold, which provides for a topological invariant. It is exactly given by the *Euler–Poincaré characteristic*

$$\chi(M) := \sum_{p=0}^{n}(-1)^p b_p = \int_M \overset{*}{\pi}(\gamma_2^g) \tag{9.2.4}$$

according to the generalized Gauss–Bonnet theorem (Chern 1944, 1963). In algebraic topology, however, this invariant can be determined in the stated way from the rank of the pth homology group $H_P(M, \mathbb{R})$, i.e., from the so-called Betti numbers $b := \dim H_p(M, \mathbb{R})$, which were introduced by Riemann. Similarly as in the case of the Chern index, it can be shown that the Euler form, projected to the base, is even an exact one. An elementary proof of this is to be found in Nieh (1980). The generalizations given above for spaces with indefinite metrics can be found in the works of Levine & Zund (1970), which themselves are indebted to an earlier extension of Chern (1963). The first-mentioned paper contains also sufficient conditions for the vanishing of $\chi(M)$. In particular, it is known that a compact even-dimensional manifold admits a pseudo-Riemannian metric of signature $s = 1$ ("Lorentz metric") if and only if the Euler characteristic vanishes (cf. Sulanke & Wintgen 1972, p. 242).

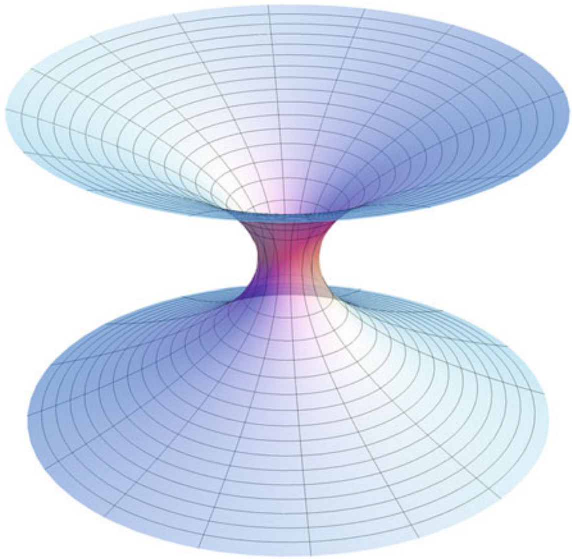
In the case of *noncompact* manifolds, a boundary term has to be taken into consideration as well:

$$\chi(M) = \int\limits_M \overset{*}{\pi}(\gamma_2^g) - \frac{1}{(4\pi)^2}\int\limits_{\partial M}\left\{\alpha \wedge \Omega^{g(\star)} + \frac{2}{3}\alpha \wedge (\alpha \wedge \alpha)^{(\star)}\right\}. \tag{9.2.5}$$

This additional term, however, is then dependent on the second fundamental form $\alpha : \omega^g - \overset{\infty}{\omega}{}^g$ of the embedding of the boundary ∂M in M. Although it is exactly this situation that is likely to be the generic one (Gibbons & Hawking 1979), it is impossible to enlarge upon it here.

The same is true with regard not only to the important relation of the Euler number to the Pontryagin index (cf. Taub 1976) but also with respect to their relation to other *topological invariants* and to the evaluation of their numerical size, as for instance in the case of Einstein spaces (Eguchi et al. 1980; Matsushita 1981). A necessary condition for the proof of (9.2.4) is also the metric compatibility of the connection ω^g. With reference to interesting analogies between gravity and superconductivity, this has also been accentuated via counterexamples by Milnor & Stasheff (1974, p. 312) and by Hanson & Regge (1979). However, here the restriction to a (torsion-free) Levi-Civita connection is not necessary.

Fig. 9.1 Wormhole embedding



In contrast to the case of the Chem–Pontryagin index, the Euler number (9.2.4) is solely determined by the topology of the base manifold, and accordingly, the Gauss–Bonnet theorem can be proved as well by a purely topological–combinatorial proof (PALAIS 1978). The first statement becomes more transparent if the Euler number is introduced axiomatically (cf. for instance OSBORN 1975). Within this foundation, the connected sum M#N of two manifolds of the same dimension—which do not have to be distinct—is an important geometric concept. For the construction of this sum, an open ball B^n is to be cut out of M and N in order to identify the then respectively existing boundary spheres $\partial B^n = S^{n-1}$ with each other via a diffeomorphism that inverts the orientation. Since this is equivalent to adding a handle or "wormhole" $W^n := S^1 \times S^{n-1}$ (WHEELER 1955), the manifold $M\#(S^1 \times S^{n-1})$ comes into existence. The embedding of Scharzschild's exterior solution is a segment of paraboloid of revolution calculated in 1916 by FLAMM (2015); see Fig. 9.1.

The Euler–Poincaré characteristic is the sole invariant that satisfies the topological relations

$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(S^n), \tag{9.2.6}$$

$$\chi(M \times N) = \chi(M) \cdot \chi(N), \tag{9.2.7}$$

$$\chi(M^2) = 2(g-1), \tag{9.2.8}$$

$$\chi(\mathbb{C}P^2) = 3 \tag{9.2.9}$$

for manifolds of *even* dimensions. Here g is supposed to be the genus or "handle number" of a surface (cf. SEIFERT & THREFALL 1934), and $\mathbb{C}P^2$ denotes the complex projective two-space in E^2. For spaces with uneven dimensions,

$$\chi(M^{2k+1}) = 0 \tag{9.2.10}$$

is valid in any case. Although there is no field-theoretic analogue to the concept of the electric charge within the framework of general relativity (WHEELER 1963), it is these properties of the Euler number that suggest it as the topological–gravitational "charge" in a Riemann–Cartan space. Something that is comparable to a related interpretation of the Chern index of instanton solutions in the Yang–Mills theory.

It was already in 1973, in the standard reference book on gravitation (MTW, p. 381), that the central issue was raised whether such topological invariants are of some physical meaning. In the case of the Poincaré gauge theory, the duality Ansatz, induces an Euler-type term on the level of the Lagrangian 4-form. The Euler–Poincaré characteristic is henceforth a necessary element of the "reduced" action, concerning solutions with duality properties. This has an important consequence if we are considering, e.g., the subcase of SKY gravity. From the inequality

$$\begin{aligned}&(1+\zeta^2)\int_{M^4} L_{\mathrm{SKY}} - (-1)^s \zeta (4\pi)^2 \cdot \chi(M^4)\\ &= \frac{1}{4}\int_{M^4} (\Omega^g - \zeta^* \Omega^{g(\star)}) \wedge {}^*(\Omega^g - \zeta^* \Omega^{g(\star)}) \geq 0,\end{aligned} \tag{9.2.11}$$

which is valid for spaces with positive definite metrics (Euclidean gravity), it follows that the gravitational action is minimized by the self-double-dual solutions for which the Euler number is zero. "Euclideanized" de Sitter universes (EGUCHI & FREUND 1976) as connected complete spaces of constant curvature are isometric to the sphere S^4 (or to the *real projective* space in the case of nonorientability). Consequently, the Pontryagin index vanishes here, while the Euler number is "merely" 2 (or 1 respectively). It is for this reason that Yang's gravitational action is not minimized by the de Sitter metric (9.1.1, 2), having $\mu = 0$, but for instance by the flat torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$. Solutions with anti-self-double-dual curvature exist globally only on manifolds with vanishing Euler number, and this on account of (9.2.11).

9.3 Quantum Meaning of Gravitational Instantons

The meaning of such topologically characterized gravitational instantons has to be looked for in the domain of the quantized theory. We are thinking not only of those methods that have been developed in the context of the Hawking effect (HAWKING 1975, 1976), i.e., the quantum field-theoretic treatment of nongravitational fields in a curved, metric background (DE WITT 1975); cf. also MIELKE (1977b). But the central issue is the quantization of the gravitational field itself. There are mainly two methodical approaches that have to be distinguished: first, the *canonical* quantization in "superspace" ("*quantum geometrodynamics*"; cf. also ISHAM 1976), a method that has been especially fostered by Wheeler (1962, 1968, 1970), and second the so-called *covariant* procedures (DE WITT 1972; cf. HAMAMOTO 1983). Concerning the calculation of the scattering matrix in terms of asymptotic states, the latter depart from Feymann's method of quantization via path integrals. In the case of gravity, it can be shown (CATENACCI & MARTELLINI 1984), similarly as in the case of Yang–Mills theories, that the occurring functional measure of Faddeev–Popov type is nothing but the volume element of the extended "super-space" $\mathscr{S}_*(\mathrm{M}) := \mathscr{M}/\tilde{\mathscr{D}}$ of all Riemannian metrics modulo the diffeomorphisms that respect isometries as fixed points.

For a quantized *quadratic* Poincaré gauge theory (qPG), it is then to be expected that the contributions of the gravitational instantons dominate the transition amplitudes in the "Euclideanized" version. Configurations with nontrivial topology (HAWKING 1978) and nonvanishing torsion may be important. As for the compensation of divergences occurring in the "one loop" or WKB approximation, it is necessary already in Einstein's GR to supplement the conventional Einstein–Hilbert Lagrangian 4-form with counterterms that are quadratic in the curvature. STELLE (1977) were able to show that a model that from the outset pays attention to Yang–Mills-like terms in the gravitational Lagrangian 4-form is renormalizable in each order of the perturbative expansion. However, in such a modified SKY gravity, physical "ghosts" generally come into existence. These can be suppressed in the special cases of the qPG theory that were worked out by SEZGIN & VAN NIEUWENHUIZEN (1980). On the other hand, such tensorial states, having a negative norm in a Hilbert space, are innocuous, since they do not necessarily cause a violation of Froissart's unitarity condition (boundedness condition) in the scattering cross sections (SALAM & STRATHDEE 1978; SMOLIN 1984; TOMBOULIS 2015).

9.3.1 Euler Term and Induced Wormhole Configurations

Let us recall that the metric-dependent Euler–Poincaré invariant

$$
\begin{aligned}
dC_{\mathrm{RR}^{(\star)}} &:= \frac{1}{2}\, d\left(\Gamma_{\alpha\beta} \wedge R^{\alpha\beta(\star)} - \frac{1}{3}\Gamma_{\alpha}{}^{\beta(\star)} \wedge \Gamma_{\beta}{}^{\gamma} \wedge \Gamma_{\gamma}{}^{\alpha}\right) \\
&\equiv -L_{\mathrm{SKY}} - 2\mathrm{Ric}_{\alpha\beta} \wedge {}^{*}\mathrm{Ric}^{\alpha\beta} + \frac{1}{2}\mathrm{Ric}_{\alpha}{}^{\alpha} \wedge {}^{*}\mathrm{Ric}_{\beta}{}^{\beta} \qquad (9.3.1)
\end{aligned}
$$

has an equivalent representation in terms of Weyl's quadratic curvature Lagrangian

$$
L_{\mathrm{SKY}} := -\frac{1}{2}\, R_{\alpha\beta} \wedge {}^{*}R^{\alpha\beta}, \qquad (9.3.2)
$$

amended by Ricci-squared and curvature scalar-squared terms. This is known as the *Gauss–Bonnet* (GB) theorem. In the realm of gravity, topological ideas date back to Riemann, Clifford, and Weyl. They found a rather concrete realization in the wormholes of Wheeler, characterized by the Betti number related to the integrated

Euler–Poincaré topological invariant (9.3.1). The two-sphere part S^2 of these $\mathbb{R} \times S^1 \times S^2$ topological configurations may even be knotted (Mielke 1977a), and within BF theory (Cotta-Ramusino & Martellini 1994), it can be characterized by the Alexander knot invariant. A possible *topology change* on the Planck scale has been recently analyzed via the Ricci flow of Hamilton and Perelman; cf. Dzhunushaliev (2013).

On the other hand, *four dimensions*, according to the results of Donaldson, are topologically special, since there are uncountably many nonisomorphic smooth or piecewise linear structures (Milnor 2011) on a noncompact $\mathbf{R}^4$. However, manifolds in such an "exotic" set are homeomorphic and thus will have the same Euler characteristic. On the other hand, the Euler number does not characterize "exotic" smooth manifolds, and one has resort to the intersection form in terms of the first Chern class; cf. Asselmeyer- Maluga & Brans (2007). Thus its contribution will not affect the path-integral approach (Asselmeyer-Maluga 2016) to quantum gravity (QG). One-loop ultraviolet divergencies in perturbative QG are proportional to a GB term in the trace anomaly; cf. Bern et al. (2015).

Moreover, in the reduction of the BF model, the emergent Euler invariant necessarily gets multiplied by the *inverse* of the symmetry-breaking constant μ, possibly a huge parameter. However, in view of the observed positive value of the cosmological constant, the presence of a primordial "gas of wormholes" would move the contribution from the Euler invariant toward a tiny number close to zero (Kawai & Okada 2012), similarly as in quintessence scenarios, where the cosmological term is associated with the energy of a scalar field rolling down a runaway potential. This may solve the naturalness problems in the standard model and the current cosmology without introducing "new physics" such as supersymmetry or extra dimensions.

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Chapter 10
Three-Dimensional Gravity

Since the first three-dimensional (3D) model of Staruszkiewicz (1963), the topological model of gravity of Deser, Jackiw, and Templeton (DJT) (Deser et al. 1982a, b; Deser & Jackiw 1984; Deser et al. 1984) continues to attract considerable theoretical interest. In order to obtain a nontrivial vacuum theory, DJT added a metric Chern–Simons (CS) term for the Riemannian curvature to the 3D Hilbert–Einstein Lagrangian, regarded as the high-temperature limit (for Euclidean signature) or dimensional reduction of the four-dimensional theory. The topological term is supposed to come from the θ-vacuum of 4D physics. The nice and intrinsic feature of the DJT model is that the CS term induces a mass for the "graviton" without breaking infinitesimal gauge invariance or invoking the Higgs mechanism. The discovery (Baekler et al. 1992) of anti-de Sitter (AdS) and black hole solutions (Bañados et al. 1992) has added further interest in 3D gravitational models as a "laboratory" to study geometric, dynamical, and statistical properties. In part, this stems from the fact that spacetime in 3D is Ricci-flat, and the dynamical properties cannot be attributed to the metric. The dynamical properties must be induced by, e.g., topological terms instead.

From a gauge-theoretic point of view, however, it appears much more natural to formulate a dimensionally reduced gravitational theory in a Riemann–Cartan (RC) space(time) with torsion (Hehl et al. 1995), thereby going over to what is conventionally called a *first-order formalism*. This happens in simple supergravity (Achucarro & Townsend 1986; Mielke & Macias 1999), where torsion enters as an auxiliary field facilitating local supersymmetric transformations (Deser & Zumino 1976).

There are other reasons for studying the dynamical aspects of topological gravity in three dimensions: some problems in 4D gravity reduce to an effective 3D theory, such as the high-temperature behavior of 4D theories or some membrane models of extended relativistic systems. Moreover, some formal aspects of black hole thermodynamics are effectively reduced to problems in 3D (Carlip 1995, 2005). On the other hand, the continuum theory of lattice defects in crystal physics is similar to gravity with torsion in 3D, where such defects are modeled by connections in

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the orthonormal frame bundle, and the Chern–Simons-type free-energy integral by *Riemann–Cartan* (RC) spaces with constant torsion (DERELI & VERÇIN 1991).

To this end, the trivial dynamics of the *Einstein–Cartan* (EC) Lagrangian

$$L_{\rm EC} = -\frac{\chi}{\ell}\vartheta^{\alpha} \wedge R^{\star}_{\alpha} \tag{10.0.1}$$

in 3D is generalized by adding Chern–Simons (CS) -type terms, an approach similar to Witten's (WITTEN 1988). By gauging the Poincaré group, we arrive at the MIELKE–BAEKLER (MB) model (1991), which is at most *linear* in the field strengths. A slight modification arises by the introduction of a "mixed" Chern–Simons-type term $C_{\rm TL}$, which simulates to some extent Einstein's theory in 3D, allowing us to build an almost completely topological theory (MIELKE & MAGGIOLO 2003).

In four dimensions, the symmetry under *duality rotations* has a long history. For Maxwell's theory it was known to RAINICH since 1925 and developed further in the context *geometrodynamics* by MISNER & WHEELER (1957), MIELKE (1987). In analogy, the general 3D Poincaré gauge field equations can be simplified by modified *duality rotations* intertwining the 3D field momenta. As a result, the field momenta exhibit the strong/weak coupling duality inverting the contortional coupling constant $\varepsilon \to 1/\varepsilon$.

These encouraging findings imply the following dynamics: First, a general Proca equation constrained by a Hilbert–de Donder-type condition is an exact result of the *S*-duality Ansatz. In particular, for RC spaces with constant axial torsion, an induced cosmological constant surfaces as an important consequence. An equivalence relation for the energy–momentum complex of the MB model induced by the mixed CS term thereby becomes more transparent. In contrast to other topologically massive models (DESER & TEKIN 2002), in which there is an unavoidable conflict in the choice of the coupling constant of the Einstein action in order to avoid ghosts and tachyon excitations, our *S*-duality approach leads to a ghost-free positive energy without inducing a negative residue in the propagator.

Outside of quantum gravity, the continuum theory of lattice defects in crystal physics can be regarded as "analogue gravity" including Cartan's torsion in 3D (LAZAR & HEHL 2010). Recently, flexural modes of graphene have also been considered as membranes with a "gravitational" metric (KERNER & NAUMIS 2012) or coframe induced from its embedding into three-dimensional spacetime.

10.1 Chern–Simons Gravity with Torsion in 3D

In three spacetime dimensions, the basic gravitational variables in the RC formalism are the one-forms of the coframe and the Lie dual of the (Lorentz) rotational connection $\Gamma^{\beta\gamma} = \Gamma_j{}^{\beta\gamma} dx^j$, i.e.,

$$\vartheta^\alpha = E_i{}^\alpha dx^i \quad \text{and} \quad \Gamma^\star_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}. \tag{10.1.1}$$

Since the coframe ϑ_α has physical dimensions [length] and the connection $\Gamma^\star_\alpha$ is dimensionless, one can use ϑ_α/ℓ and $\Gamma^\star_\alpha$ as *dimensionless* gauge potentials of local translations and local rotations, respectively, even though the coframe is "soldered" to the base manifold (TRESGUERRES & MIELKE 2000).

The related field strengths are the two-forms of torsion

$$T^\alpha := d\vartheta^\alpha - (-1)^s\,\eta^{\alpha\beta} \wedge \Gamma^\star_\beta \tag{10.1.2}$$

and curvature

$$R^\star_\alpha = \frac{1}{2}\eta_{\alpha\beta\gamma}R^{\beta\gamma} := d\Gamma^\star_\alpha + \frac{(-1)^s}{2}\eta_\alpha^{\beta\gamma}\Gamma^\star_\beta \wedge \Gamma^\star_\gamma, \tag{10.1.3}$$

respectively.

In 3D, the two *Bianchi identities* of the RC geometry can be rewritten as

$$DT^\alpha \equiv (-1)^s\,\eta^{\alpha\beta} \wedge R^\star_\beta \tag{10.1.4}$$

and

$$DR^\star_\alpha \equiv 0\,. \tag{10.1.5}$$

The corresponding Chern–Simons three-forms of gauge type $C = \mathrm{Tr}\{A \wedge F\}$ are the *translational* CS-type term

$$C_{\mathrm{T}} := \frac{1}{2\ell^2}\vartheta^\alpha \wedge T_\alpha = -\frac{(-1)^s}{\ell^2}\eta^\alpha \wedge K^\star_\alpha, \tag{10.1.6}$$

and the (Lorentz) rotational term (CHERN & SIMONS 1971; DESER et al. 1982b; WITTEN 1988) involving the curvature:

$$C_{\mathrm{L}} := (-1)^s\,\Gamma^{\star\alpha} \wedge R^\star_\alpha - \frac{1}{3!}\,\eta_{\alpha\beta\gamma}\,\Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \Gamma^{\star\gamma}. \tag{10.1.7}$$

The variational derivatives

$$\frac{\delta C_{\mathrm{T}}}{\delta\vartheta^\alpha} = \frac{1}{\ell^2}\,T_\alpha \quad \text{and} \quad \frac{\delta C_{\mathrm{T}}}{\delta\Gamma^{\star\alpha}} = \frac{(-1)^s}{\ell^2}\,\eta_\alpha\,, \tag{10.1.8}$$

as well as

$$\frac{\delta C_{\mathrm{L}}}{\delta\vartheta^\alpha} = 0 \quad \text{and} \quad \frac{\delta C_{\mathrm{L}}}{\delta\Gamma^{\star\alpha}} = (-1)^s\,2R^\star_\alpha, \tag{10.1.9}$$

are uniquely related to the torsion T_α, the curvature $R^\star_\alpha$, and a cosmological term η_α, respectively (HEHL et al. 1991).

Table 10.1 Geometric objects and fields

	Objects	p-form	Components	n = 4	3	2
ϑ^α	Vector	1	n^2	16	9	1
$\Gamma^\star_\alpha$	Vector	1	n^2	16	9	1
T^α	Vector	2	$n^2(n-1)/2$	24	9	2
$R^{\alpha\beta}$	Bivector	2	$n^2(n-1)^2/4$	36	9	1
Σ_α	Vector	$n-1$	n^2	16	9	4
$\tau_{\alpha\beta}$	Bivector	$n-1$	$n^2(n-1)/2$	24	9	2
η_α	Vector	$n-1$	n^2	16	9	4

It is pertinent to 3D with torsion that there exists the "mixed" topological term

$$C_{\mathrm{TL}} := \frac{1}{\ell}\left(\Gamma^{\star\alpha} \wedge T_\alpha - \frac{(-1)^s}{2}\,\eta_{\alpha\beta\gamma}\,\Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \vartheta^\gamma\right). \tag{10.1.10}$$

Its variations lead to

$$\frac{\delta C_{\mathrm{TL}}}{\delta\vartheta^\alpha} = \frac{1}{\ell}R^\star_\alpha \quad \text{and} \quad \frac{\delta C_{\mathrm{TL}}}{\delta\Gamma^{\star\alpha}} = \frac{1}{\ell}T_\alpha\,, \tag{10.1.11}$$

i.e., with a response of gauge two-forms opposite the character of the field.

Table 10.1 summarizes the number of components of the basic variables and their components in various dimensions: Observe that only for $n = 3$ do all fields have the same number of components and that all decompose under SO(3) or the lower-dimensional Lorentz group SO(1, 2) as a $9 = 5 \otimes 3 \otimes 1$ multiplet after bivectors are converted into vectors via the Lie dual; a linear combination of all variables paves the way to a better understanding of the geometry in 3D.

10.1.1 Noether Theorem in 3D Gravity

In 3D with vanishing nonmetricity, independent variations of a gauge-invariant Lagrangian with respect to the coframe and connection lead to

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{10.1.12}$$

and

$$DH^\star_\alpha - S_\alpha = \tau^\star_\alpha, \tag{10.1.13}$$

respectively. In 3D, the one-forms

$$H_\alpha := -\frac{\partial L}{\partial T^\alpha}, \quad H^\star_\alpha := -\frac{(-1)^s}{2}\frac{\partial L}{\partial R^{\star\alpha}}, \tag{10.1.14}$$

are the translational and rotational field momenta (or the excitation $H = -\partial L/\partial F$ as in Maxwell's theory) with dimensions [1/length] and dimensionless, respectively. The canonical energy–momentum two-form of the gauge fields is given by

$$E_\alpha := \frac{\partial L}{\partial \vartheta^\alpha} = e_\alpha \rfloor L + (e_\alpha \rfloor T^\beta) \wedge H_\beta + 2(-1)^s (e_\alpha \rfloor R^{\star\beta}) \wedge H^\star_\beta, \tag{10.1.15}$$

where the covariant right-hand side follows again from the Noether theorem; cf. HEHL et al. (1995). The gauge spin two-form is defined as

$$S_\alpha := \frac{(-1)^s}{2}\frac{\partial L}{\partial \Gamma^{\star\alpha}} = \frac{1}{2}\eta_{\alpha\beta} \wedge H^\beta. \tag{10.1.16}$$

(In the particular case of an antisymmetric field $H_\alpha = H_{[\alpha\beta]}\vartheta^\beta$, this is dual to the translational field strength, i.e., $S_\alpha = -{}^*H_\alpha$.) The sources for the gravitational field are the material energy–momentum and the spin current

$$\Sigma_\alpha := \frac{\delta L_{\rm MAT}}{\delta \vartheta^\alpha}, \qquad \tau^\star_\alpha := \frac{(-1)^s}{2}\frac{\delta L_{\rm MAT}}{\delta \Gamma^{\star\alpha}}, \tag{10.1.17}$$

respectively, which are both two-forms in three dimensions.

10.2 Topological Mielke–Baekler Model

In 3D, the Einstein–Cartan (EC) Lagrangian

$$L_{\rm EC} := -\frac{\chi}{\ell}\vartheta^\alpha \wedge R^\star_\alpha \equiv -\chi\, C_{\rm TL} - \frac{\chi}{\ell}\, d(\Gamma^\star_\alpha \wedge \vartheta^\alpha) \tag{10.2.1}$$

merely gives rise to locally trivial dynamics. This can be traced back to its equivalence to a "mixed" Chern–Simons-type term $C_{\rm TL}$ plus a total divergence.

Let us now add further CS-type terms: by gauging the Poincaré group $\mathbb{R}^3 \rtimes SO(1,2)$, we arrive at the MIELKE & BAEKLER (1991) (MB) model, which is at most *linear* in the field strengths. This is slightly modified here by replacing $L_{\rm EC}$ with the "mixed" Chern–Simons-type term $C_{\rm TL}$, which to some extent simulates, in 3D, Einstein's theory with "cosmological" term, as is indicated above.

Allowing for arbitrary "vacuum angles" $\theta_{\rm T}$, $\theta_{\rm L}$, and $\theta_{\rm TL} = -\chi$, the most general *topological* gravity Lagrangian in 3D, in first-order formalism, takes the form

$$L_{\rm MB}(\vartheta^\alpha, \Gamma^\star_\alpha) = \theta_{\rm T} C_{\rm T} + \theta_{\rm L} C_{\rm L} + \theta_{\rm TL} C_{\rm TL}\,, \tag{10.2.2}$$

where $C_{\rm T}$, $C_{\rm L}$, and $C_{\rm TL}$ are respectively the translational, rotational, and "mixed" CS three-forms given above. The three-form (10.2.2) is the topological Lagrangian of the Mielke–Baekler (MB) *mix* model (MIELKE & MAGGIOLO 2003). Since the translational term $C_{\rm T}$ is covariant, it appears that the MB model is only semi-topological, as is reflected in the number of propagating modes.

Consequently, varying the Lagrangian (10.2.2) with respect to ϑ^α and $\Gamma^{\star\alpha}$ yields the covariant field equations

$$-\theta_{\rm TL}\, R^\star_\alpha - \frac{1}{\ell}\theta_{\rm T}\, T_\alpha = \ell\, \Sigma_\alpha \tag{10.2.3}$$

and

$$-(-1)^s\theta_{\rm TL}\, T_\alpha - \frac{1}{2\ell}\theta_{\rm T}\, \eta_\alpha - \theta_{\rm L}\ell\, R^\star_\alpha = \ell\, \tau^\star_\alpha\,; \tag{10.2.4}$$

cf. Eq. (6.9) of BAEKLER et al. (1992). Observe that the translational CS term proportional to $\theta_{\rm T}$ induces in the second field equation a constant term proportional to $\theta_{\rm T}/2\ell$, resembling a cosmological constant familiar from 4D gravity.

When including matter couplings and combining the field equations (10.2.4) and (10.2.3), we explicitly obtain

$$T_\alpha - \frac{\varepsilon}{\ell}\eta_\alpha = \frac{2}{A}\ell\left(\theta_{\rm TL}\tau^\star_\alpha - \theta_{\rm L}\ell\,\Sigma_\alpha\right) \tag{10.2.5}$$

for the torsion, and

$$R^\star_\alpha - \frac{\rho}{\ell^2}\eta_\alpha = \frac{2}{A}\left(\theta_{\rm TL}\ell\,\Sigma_\alpha - \theta_{\rm T}\tau^\star_\alpha\right) \tag{10.2.6}$$

for the RC curvature.

In *vacuum*, torsion and RC curvature are constrained by

$$T_\alpha = \frac{\varepsilon}{\ell}\eta_\alpha\,, \qquad R^\star_\alpha = \frac{\rho}{\ell^2}\eta_\alpha, \tag{10.2.7}$$

where the contortional constants $\varepsilon = \theta_{\rm TL}\theta_{\rm T}/A$ and $\rho = -\theta_{\rm T}^2/A$ are related to the vacuum angles. A singular case can be excluded by assuming that the constant $A := -(-1)^s\theta_{\rm TL}^2 + 2\theta_{\rm T}\theta_{\rm L} \neq 0$ is nonvanishing.

10.2.1 "Prolongation" of Anti-de Sitter to Black Hole Solutions

In order to study vacuum solutions, it is convenient to consider the decomposition

$$\Gamma^\star_\alpha = \Gamma^{\{\}\star}_\alpha - K^\star_\alpha \tag{10.2.8}$$

of the connection into Riemannian and contortional pieces. This implies the identity

$$R_\alpha^\star \equiv R_\alpha^{\{\}\star} - DK_\alpha^\star + \frac{1}{2}\eta_{\alpha\beta\gamma}K^{\star\beta} \wedge K^{\star\gamma} \tag{10.2.9}$$

for the RC curvature. From the relation $K_\alpha^\star = -{}^*T_\alpha/2 = -(-1)^s(\varepsilon/2\ell)\vartheta_\alpha$ for the contortion and the definition $T^\alpha := D\vartheta^\alpha$, it can be inferred that the Riemannian part of the curvature is also constant, i.e.,

$$R_\alpha^{\{\}\star} = -\Lambda_{\text{eff}}\ \eta_\alpha\ , \tag{10.2.10}$$

where

$$\Lambda_{\text{eff}} = -[4\rho - (1+2(-1)^s)\varepsilon^2]/4\ell^2 = \theta_{\text{T}}^2\,[(9+2(-1)^s)\theta_{\text{TL}}^2 + 8\theta_{\text{T}}\theta_{\text{L}}]/(2A\ell)^2 \tag{10.2.11}$$

is the effective cosmological constant, which is induced by the topological terms in our gauge Lagrangian (10.2.2) as confirmed by EXCALC/REDUCE calculations (SCHRÜFER 2004).

In principle, we can have a nonzero effective cosmological constant even for $\varepsilon = 0$, i.e., in a purely Riemannian spacetime. Alternatively, for $\rho = 0$, i.e., in the limit of vanishing RC curvature, there exists a nontrivial “parallelizing” torsion, resembling the “squashed” seven-sphere construction of ENGLERT et al. (1983), in higher dimensions.

Inasmuch as the three-dimensional “image” of a cosmological term of either sign is already *induced* by the Chern–Simons terms in the Lagrangian, one can disregard a “bare” cosmological term and is still able to simulate cosmological models in 3D. In our topological model, however, the translational CS term proportional to θ_{T} is indispensable for obtaining a nontrivial result.

For vanishing torsion, the three-dimensional Einstein equations with effective cosmological term Λ_{eff} has the anti-de Sitter (AdS) metric

$$ds^2 = -(1 - \Lambda_{\text{eff}}\,r^2)dt^2 + (1 - \Lambda_{\text{eff}}\,r^2)^{-1}dr^2 + r^2\,d\phi^2 \tag{10.2.12}$$

as an exact solution. BAEKLER et al. (1992) were the first to recognize this for the MB model.

From (10.2.12), by appropriate identifications of the boundaries, the vacuum solution

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2\,[d\phi + N^\phi(r)dt]^2 \tag{10.2.13}$$

can be obtained (BAÑADOS et al. 1992, 1993), where the lapse squared and shift are given by

$$N^2(r) = -M - \Lambda_{\text{eff}}\,r^2 + \frac{J^2}{4r^2}\ , \qquad N^\phi(r) = -\frac{J}{2r^2}\ , \tag{10.2.14}$$

respectively. Observe that the shift is proportional to the angular momentum J of the solution, which allows for $J^2 \leq M^2$ to interpret this configuration as a *rotating black hole* with mass M (MENOTTI & SEMINARA 2000). For the (unconventional) normalization $M = -1$ and $J = 0$, it reduces to the AdS metric.

A check via EXCALC confirms that the rotating black hole solution (10.2.13) has *constant Riemannian* curvature (10.2.10) and is therefore nowhere singular. Then the construction of configurations with constant axial torsion and RC curvature (10.3.6) rest essentially on a "prolongation" $\Gamma_\alpha^{\{\}\star} \to \Gamma_\alpha^\star = \Gamma_\alpha^{\{\}\star} - K_\alpha^\star$ of the Riemannian to an RC connection and an inversion of (10.2.9); cf. BAEKLER (1991). Provided the effective cosmological constant Λ_{eff} is related to ε and ρ via (10.2.10), the black hole configuration (10.2.13) with (10.2.14) is also an exact solution of the 3D topological gauge model. Generalizations to spaces with nonmetricity have been considered by TRESGUERRES (1992).

10.3 S-Duality in 3D

The previous "prolongations" suggest that one may consider a continuous deformation [or a field redefinition (FR)] of the (Lorentz) rotational connection by adding a tensor-valued one-form, similarly as in Eq. (3.11.1) of HEHL et al. (1995). In 3D, the particular deformation

$$\widetilde{\Gamma}_\alpha^\star = \Gamma_\alpha^\star - (-1)^s \frac{\varepsilon}{2\ell} \vartheta_\alpha, \tag{10.3.1}$$

where ε is a continuous parameter, involves the Lie dual $\Gamma_\alpha^\star = \frac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}$ of the connection. This FR implies

$$\widetilde{T}_\alpha = T_\alpha - \frac{\varepsilon}{\ell}\eta_\alpha, \qquad \widetilde{R}_\alpha^\star = R_\alpha^\star - (-1)^s \frac{\varepsilon}{2\ell} T_\alpha + (-1)^s \frac{\varepsilon^2}{4\ell^2}\eta_\alpha \tag{10.3.2}$$

for the deformed torsion and curvature, respectively. Two special subcases can arise: Riemannian spacetime with $\widetilde{T}_\alpha = 0$ and deformed *teleparallelism* in the local gauge $\widetilde{\Gamma}_\alpha^\star \stackrel{*}{=} 0$, equivalent to the covariant constraint of vanishing modified RC curvature, i.e., $\widetilde{R}_\alpha^\star = 0$. In the latter case, coframe and connection are locally *Lie dual* to each other, i.e.,

$$\Gamma_\alpha^\star = (-1)^s \frac{\varepsilon}{2\ell} \vartheta_\alpha, \tag{10.3.3}$$

which implies the corresponding duality

$$S_\alpha = \frac{\ell}{\varepsilon} E_\alpha \tag{10.3.4}$$

of gauge spin and covariant energy–momentum. This induces a complete *symmetry under duality rotations* in the two vacuum field equations (10.1.12) and (10.1.13), provided that, additionally,

$$H^{\star}_{\alpha} = \frac{\ell}{\varepsilon} H_{\alpha} \tag{10.3.5}$$

holds.

Observe the inversion of the parameter ε, i.e., a small deformation ε will induce a large rotational momentum proportional to $1/\varepsilon$ and conversely, resembling, to some extent, strong/weak duality.

In fact, for internal Yang–Mills theory, MONTONEN & OLIVE (1977) and OLIVE (1996) discovered a duality of the *strong/weak* coupling regime of gauge fields, the so-called *S-duality*. For Chern–Simons (super)gravity, such aspects have also been discussed in DESER & MCCARTHY (1990), GARCIA- COMPEAN et al. (2001). In Yang's theory of gravity, a related (double) duality Ansatz has been analyzed in 4D as well (MIELKE 1981; ZHYTNIKOV 1994; MIELKE & MAGGIOLO 2005), see ELLWANGER (2005) for related ideas. In 3D, the intertwining mapping (10.3.5) between the translational/rotational pair of field momenta arises as a novel feature.

The seemingly trivial case of a completely *flat* deformed spacetime, i.e., $\widetilde{T}_{\alpha} = 0$ and $\widetilde{R}^{\star}_{\alpha} = 0$, corresponds to configurations with constant axial torsion and constant RC curvature

$$T_{\alpha} = \frac{\varepsilon}{\ell}\, \eta_{\alpha}, \qquad R^{\star}_{\alpha} = \frac{\rho}{\ell^2}\, \eta_{\alpha}, \tag{10.3.6}$$

as originally envisioned by CARTAN (1924). Here $\rho = (-1)^{\text{sign}} \varepsilon^2/4$ depends quadratically on the deformation parameter ε. A visualization of Cartan's spiral "staircase" should not ignore that $R^{\star}_{\alpha} = (-1)^{\text{sign}} \varepsilon\, T_{\alpha}/4\ell$ induces, for $\varepsilon \neq 0$, a constant-curvature background; cf. GARCIA et al. (2003).

10.3.1 Modified S-Duality in 3D

More generally, one can consider, as in MIELKE & MAGGIOLO (2003), the modified *S*-duality Ansatz

$$H^{\star}_{\alpha} = \delta \ell H_{\alpha} + \frac{\gamma}{\ell} \vartheta_{\alpha} \tag{10.3.7}$$

in 3D, which "breaks" the dual symmetry (10.3.5) of the field momenta in order to allow for a coupling to the translational CS term (10.1.6). Here δ and γ are *dimensionless* constants that depend on the corresponding model, and ℓ denotes a fundamental length that guarantees dimensional consistency.[1] Inserting the Ansatz (10.3.7) into the second field equation (10.1.13), we obtain the *first*-order equation

$$\delta \ell D H_{\alpha} + \frac{\gamma}{\ell} T_{\alpha} - \frac{1}{2} \eta_{\alpha\beta} \wedge H^{\beta} = \tau^{\star}_{\alpha} \tag{10.3.8}$$

[1] A feasible additional term proportional to $\Gamma^{\star}_{\alpha}$ is not considered here due to its lack of gauge covariance.

for the translational momenta H_α. Observe that our intertwining Ansatz (10.3.7) has achieved a decoupling of the two Poincaré gauge equations (10.1.12) and (10.1.13), with the exception that the Lie dual of the material spin now becomes the source.

A vacuum solution of (10.3.8) is

$$H_\alpha = -\frac{\gamma\varepsilon}{\ell^2(1+\delta\varepsilon)}\vartheta_\alpha \quad \Leftrightarrow \quad T_\alpha = \frac{\varepsilon}{\ell}\eta_\alpha, \tag{10.3.9}$$

provided the RC spaces have *constant* axial torsion, similar to case of the spiral "staircase" of Cartan.

Due to (10.3.7), we obtain

$$H^\star_\alpha = \frac{\gamma}{\ell(1+\delta\varepsilon)}\vartheta_\alpha = -\frac{\ell}{\varepsilon}H_\alpha, \tag{10.3.10}$$

which can be view as a different branch of the strong/weak duality inverting the coupling constant $\varepsilon \to 1/\varepsilon$, but with opposite sign. Applying a kind of "mirror symmetry" $\varepsilon \to -\varepsilon$ in (10.3.9) and (10.3.10), we see that the limit $\gamma = (1-\delta\varepsilon) \to 0$ would lead us back to the original duality (10.3.5).

On the other hand, the modified S-duality relation (10.3.7) converts the energy–momentum (10.1.15) of the gauge fields into

$$E_\alpha = e_\alpha \rfloor L + \left[(e_\alpha \rfloor T^\beta) + (-1)^s 2\delta\ell(e_\alpha \rfloor R^{\star\beta})\right] \wedge H_\beta + (-1)^s\frac{2\gamma}{\ell}(e_\alpha \rfloor R^{\star\beta}) \wedge \vartheta_\beta, \tag{10.3.11}$$

where the last term corresponds to the field energy of an induced EC Lagrangian (10.2.1). For $\gamma \neq 0$, the vacuum solution (10.3.9) yields

$$E_\alpha = e_\alpha \rfloor L(\star) - \frac{2\gamma\varepsilon^2}{\ell^3(1+\delta\varepsilon)}\eta_\alpha + (-1)^s\frac{2\gamma}{\ell(1+\delta\varepsilon)}\left[(e_\alpha \rfloor R^{\star\beta}) \wedge \vartheta_\beta\right], \tag{10.3.12}$$

where an implicit reduction of the Lagrangian is understood. Observe that a cosmological term proportional to η_α is induced by the modified S-duality. In particular, in the limit $\varepsilon \to 0$ of *weak* axial torsion coupling, the translational field momenta (10.3.9) will vanish and

$$E_\alpha = e_\alpha \rfloor L(\star) + (-1)^s\frac{2\gamma}{\ell}\left[(e_\alpha \rfloor R^{\star\beta}) \wedge \vartheta_\beta\right] \tag{10.3.13}$$

remains as a field energy, whereas the rotational momenta (10.3.10) become unity, i.e., $H^\star_\alpha = \gamma\vartheta_\alpha/\ell$, similar to the case of the EC action. Conversely, in the limit $\varepsilon \to \infty$ of *strong* axial torsion coupling, the translational momenta (10.3.9) become $H_\alpha = -\gamma\vartheta_\alpha/\delta\ell^2$, and the rotational momenta turn out to be $H^\star_\alpha = 0$. This implies

$$E_\alpha = e_\alpha \rfloor L(\star) - \frac{2\gamma}{\delta\ell^2} T_\alpha, \tag{10.3.14}$$

as expected for spacetimes with teleparallelism.

10.3.2 Toward Integrability in 3D

Starting from (10.3.8), one can attempt to derive more general exact solutions by projecting out the axial part via ϑ^α and taking the Hodge dual. We obtain

$$^*D(\vartheta^\alpha \wedge H_\alpha) - {}^*(T^\alpha \wedge H_\alpha) = \frac{\gamma}{\delta\ell^2}\, {}^*(\vartheta^\alpha \wedge T_\alpha) + \frac{1}{\delta\ell}\, {}^*(\eta^\alpha \wedge H_\alpha)\,. \tag{10.3.15}$$

By defining the one-form

$$H = {}^*(\vartheta^\alpha \wedge H_\alpha), \tag{10.3.16}$$

as well as the following axial zero-forms

$$t = {}^*(T^\alpha \wedge H_\alpha), \qquad \mathscr{A} := {}^*(\vartheta^\alpha \wedge T_\alpha), \qquad h = {}^*(\eta^\alpha \wedge H_\alpha), \tag{10.3.17}$$

a basis $\{H, dt, d\mathscr{A}, dh\}$ of one-forms can be constructed that, for nonzero torsion and H, is "overcomplete" in 3D. However, (10.3.15) has the *integrability condition*

$$\Box H - (-1)^s\, {}^*D\, {}^*DH = dt + \frac{\gamma}{\delta\ell^2} d\mathscr{A} + \frac{1}{\delta\ell} dh\,, \tag{10.3.18}$$

where

$$\Box := (-1)^{pn+s}\left[\, {}^*D\, {}^*D + (-1)^n\, D\, {}^*D\, {}^*\right] \tag{10.3.19}$$

is the gauge-invariant d'Alembert operator in n dimensions.

For $H = 0$, the three one-forms on the right-hand side become linearly dependent, thereby sending us back to the exact solution (10.3.9). However, for $H_\alpha \neq 0$, an exact metrical solution

$$ds^2 = (-1)^s dt^2 + d\mathscr{A}^2 + dh^2 \tag{10.3.20}$$

can be proposed in which one coordinate "leg," for instance $d\mathscr{A}$, needs to be constrained by (10.3.18). Although a complete integrability (MIELKE et al. 1993) as in the case of 2D Poincaré gauge models is not available here, a new avenue for implicitly deriving exact solutions in 3D is opening up.

10.3.3 Nonlinear Gravitons

As a first step toward analyzing the particle content in 3D, we derived a general expression for the gauge-covariant d'Alembertian of the coframe ϑ^α, departing from the modified S-duality (10.3.7).

Employing the Poincaré gauge field equations (10.1.12) and (10.1.13), as well as the modified S-duality relation (10.3.7), we get

$$\Box\vartheta_\alpha = \frac{\ell}{\gamma}\,{}^*D^*\left[\tau^\star_\alpha - S_\alpha - \delta\ell(\Sigma_\alpha + E_\alpha)\right] + \frac{\ell}{\gamma}D^*\left(D\,{}^*H^\star_\alpha - \delta\ell D\,{}^*H_\alpha\right). \tag{10.3.21}$$

The covariant subsidiary condition

$$D^*\vartheta_\alpha := \frac{\ell}{\gamma}\left[D\,{}^*H^\star_\alpha - \delta\ell D\,{}^*H_\alpha\right] \tag{10.3.22}$$

generalizes the Hilbert–de Donder or *transversality condition* $d^*\vartheta_\alpha = 0$ for the coframe.

In vacuum, we arrive at

$$\Box\vartheta_\alpha + \frac{\ell}{\gamma}\,{}^*D^*\left(\delta\ell E_\alpha + S_\alpha\right) = 0, \tag{10.3.23}$$

jointly with (10.3.22). Hence, the nonlinear Klein–Gordon or Proca-type equation (10.3.21) and the general subsidiary condition (10.3.22) for the coframe are both *exact* consequences of the modified S-duality (10.3.7).

In principle, one could now determine the different propagating modes in 3D gravity and distinguish physical particles, physical ghost with negative residue of the propagator, or tachyons with complex poles. However, the right-hand side may contain higher-order derivatives, depending on the model. In general, one would expect the occurrence of nonlinear gravitons, akin to those of Penrose (1976), as well as "caustics." Fortunately, in special cases, (10.3.22) and (10.3.23) admit a complete classification.

10.3.4 Effective Proca Equation

To this end, we will employ the relation $\eta_\alpha = {}^*\vartheta_\alpha$ for the η_α basis and iterate the algebraic relations (10.3.6) for the torsion and RC curvature, as well as the first Bianchi identity (10.1.4). Then the Proca-type equation

$$\left[\Box + (-1)^s m^2\right]\vartheta^\alpha \cong 0, \tag{10.3.24}$$

where $m = \varepsilon/\ell$, results in vacuum and the gauge-covariant Hilbert–de Donder or *transversality condition*

$$D\,{}^*\vartheta^\alpha \cong 0\,, \tag{10.3.25}$$

for the 3D coframe.

Consequently, in the MB model, the count of the propagating degrees of freedom is as follows: In 3D, the coframe $\vartheta^\alpha = E_i{}^\alpha dx^i$ has $3 \times 3 = 9$ components, of which 3 are pure gauge due to the $SO(1,2)$ Lorentz transformation $\vartheta'^\alpha = \Lambda_\beta{}^\alpha(x)\vartheta^\beta$. The Hilbert–de Donder transversality condition (10.3.25) amounts to three further constraints. The remaining three modes are the two in the DJT model,[2] spin-2 degrees of freedom of a *massive* graviton and a massive scalar mode, as in BAEKLER et al. (1992), DESER et al. (1982b), MIELKE & MAGGIOLO (2003, 2007).

The mass and the induced cosmological constant

$$m = \frac{\theta_{\rm T}\theta_{\rm TL}}{2(\theta_{\rm TL}^2 + \theta_{\rm T}\theta_{\rm L})\ell}\,, \qquad \Lambda_{\rm ind} = \frac{\theta_{\rm T}^2[-(-1)^s 3\theta_{\rm TL}^2 + 8\theta_{\rm T}\theta_{\rm L}]}{4[(-1)^s\theta_{\rm TL}^2 - 2\theta_{\rm T}\theta_{\rm L}]^2\ell^2}\,, \tag{10.3.26}$$

respectively, depend on the vacuum angles θ, but are always real. Since the mass of the Proca equation in an AdS background can be tuned to a cosmological constant for certain combinations of the vacuum angles, the propagating modes can be partially massless, for instance the conformal mode.

10.3.5 Energy–Momentum and Spin Complexes

The nature of real propagating modes from physical ghosts can be distinguished by determining their mass and spin.

To this end, we define the *energy–momentum and spin complexes*

$$\check{\mathscr{E}}_\alpha := dH_\alpha \cong \Sigma_\alpha + E_\alpha - (-1)^s \eta_{\alpha\beta\gamma}\Gamma^{\star\beta}\wedge H^\gamma \tag{10.3.27}$$

and

$$\check{\mathscr{J}}_\alpha^\star := dH_\alpha^\star \cong \tau_\alpha^\star + E_\alpha^\star - (-1)^s \eta_{\alpha\beta\gamma}\Gamma^{\star\beta}\wedge H^{\star\gamma}\,, \tag{10.3.28}$$

respectively, such that these gauge-dependent complexes are "on shell" related to canonical currents (MIELKE & WALLNER 1988) of the Poincaré gauge theory. Observe that the gauge field momenta function here as *superpotentials*. The energy–momentum and spin vectors are then obtained by integration over a two-dimensional spacelike surface, respectively

[2] In MIELKE & MAGGIOLO (2003), it is shown that the remaining three degrees of freedom cannot be *diffeomorphisms* preserving the covariance of the Hilbert–de Donder condition.

$$P_\alpha = \int \check{\mathscr{E}}_\alpha, \qquad J_\alpha = \int \check{\mathscr{J}}^\star_\alpha. \tag{10.3.29}$$

In the case of our topological model (10.2.2), the definitions (10.1.14) lead to

$$H_\alpha = -\frac{\theta_{\rm T}}{2\ell^2}\vartheta_\alpha - \frac{\theta_{\rm TL}}{\ell}\Gamma^\star_\alpha, \qquad H^\star_\alpha = -\frac{\theta_{\rm L}}{2}\Gamma^\star_\alpha, \tag{10.3.30}$$

for the translational and rotational momenta. Consequently, by applying the exterior derivative, we obtain the energy–momentum complex

$$\check{\mathscr{E}}_\alpha = -\frac{\theta_{\rm T}\theta_{\rm TL}}{2\ell^2}\left(\frac{1}{\theta_{\rm TL}}d\vartheta_\alpha - \frac{4\ell}{\theta_{\rm T}\theta_{\rm L}}\check{\mathscr{J}}^\star_\alpha\right), \tag{10.3.31}$$

where the additional term

$$\check{\mathscr{J}}^\star_\alpha = -\frac{\theta_{\rm L}}{2}d\Gamma^\star_\alpha \tag{10.3.32}$$

arises from the "mixed" topological model (10.2.2).

On the other hand, the analogous expression for the original MB model is

$$\check{\mathscr{E}}^{\rm MB}_\alpha = -(-1)^s\frac{\theta_{\rm T}\theta_{\rm L}}{2\chi\ell}\left(d\Gamma^\star_\alpha + \frac{2}{\theta_{\rm L}}\check{\mathscr{J}}^{\rm MB\star}_\alpha\right), \tag{10.3.33}$$

where the spin current

$$\check{\mathscr{J}}^{\rm MB\star}_\alpha = (-1)^s\frac{\chi}{2\ell}d\vartheta_\alpha - \frac{\theta_{\rm L}}{2}d\Gamma^\star_\alpha \tag{10.3.34}$$

contributes to the mass with an "anyon"-type g-factor of $2/\theta_{\rm L}$; cf. JACKIW & NAIR (1991). Consequently, these two energy–momentum complexes are related via

$$\check{\mathscr{E}}^{\rm MB}_\alpha - \check{\mathscr{E}}_\alpha = \frac{\theta_{\rm TL}}{\ell}d\Gamma^\star_\alpha, \tag{10.3.35}$$

where $\Gamma^\star_\alpha$ surfaces as a superpotential. Moreover, this relation is valid even in topological models with a "bare" cosmological constant Λ. These energy complexes differ at most by an exact differential form, and thus would establish a cohomology. In other words, the energy states of the two models fall into the same equivalence class.

The connection $\Gamma^\star_\alpha$ emerges as the superpotential of the *Freud complex*, which upon integration of the timelike field component in (10.3.29) yields the correct ADM mass (MIELKE & WALLNER 1988). Thus, the numerical coefficient in (10.3.33) has to be constrained to $\theta_{\rm T}\theta_{\rm L}/2\chi = 1$, or at least to be positive in order to avoid negative energies, or physical ghosts with a negative residue in the graviton propagator. However, in our framework, this can always be achieved by an appropriate choice of $\theta_{\rm T} \neq 0$, the vacuum angle of the translational Chern–Simons term $C_{\rm T}$. In any case, a nonzero θ_{T} has made possible the equivalence of the field equations (10.2.3) and (10.2.4) to the Proca equation (10.3.24) for the coframe.

In the original DJT model (DESER et al. 1982b) without torsion, there is an unavoidable conflict in the choice of the sign of the constant χ in the Einstein action (10.2.1), the realness of the mass, and the positivity of the energy. Thus ghosts cannot be avoided (DESER et al. 2005). In contradistinction, the inclusion of the torsion in the MB model not only avoids unphysical modes, but due to a lower derivative order, also facilitates an exact equivalence proof of a Proca system for the coframe.

10.3.6 Central Charges in Topological Gravity

Equivalently, we could also begin from the CS-type Lagrangian

$$\widetilde{C}_{\mathrm{L}} = (-1)^s\, \widetilde{\Gamma}^{\star\alpha} \wedge \widetilde{R}^{\star}_{\alpha} - \frac{1}{3!}\,\eta_{\alpha\beta\gamma}\, \widetilde{\Gamma}^{\star\alpha} \wedge \widetilde{\Gamma}^{\star\beta} \wedge \widetilde{\Gamma}^{\star\gamma}\,, \tag{10.3.36}$$

where the deformation (10.3.1) of the (Lorentz) rotational connection and the corresponding one (10.3.2) for the curvature are considered a starting point.

Then, for *different* continuous parameters $\varepsilon \neq \tilde{\varepsilon}$, we obtain

$$\theta_{\mathrm{L}} C_{\mathrm{L}} - \tilde{\theta}_{\mathrm{L}} \widetilde{C}_{\mathrm{L}} = -\frac{\chi}{\ell}\vartheta^\alpha \wedge R^\star_\alpha - \frac{\Lambda}{\ell}\eta + \frac{\theta_{\mathrm{T}}}{2\ell}\vartheta^\alpha \wedge T_\alpha + \Delta\theta_{\mathrm{L}} C_{\mathrm{L}} + \frac{\Delta\theta_{\mathrm{TL}}}{\ell} d(\Gamma^{\star\alpha} \wedge \vartheta_\alpha) \tag{10.3.37}$$

for the deformation of the CS term, provided that the coupling constants are restricted to

$$\chi = \theta_{\mathrm{L}}\varepsilon - \tilde{\theta}_{\mathrm{L}}\tilde{\varepsilon}, \quad \Lambda = \frac{(-1)^s}{4\ell^2}(\theta_{\mathrm{L}}\varepsilon^3 - \tilde{\theta}_{\mathrm{L}}\tilde{\varepsilon}^3), \quad \theta_{\mathrm{T}} = \frac{(-1)^s}{2}(\theta_{\mathrm{L}}\varepsilon^2 - \tilde{\theta}_{\mathrm{L}}\tilde{\varepsilon}^2), \tag{10.3.38}$$

$$\Delta\theta_{\mathrm{L}} = \theta_{\mathrm{L}} - \tilde{\theta}_{\mathrm{L}}, \quad \Delta\theta_{\mathrm{TL}} = \frac{1}{2}(\theta_{\mathrm{L}}\varepsilon - \tilde{\theta}_{\mathrm{L}}\tilde{\varepsilon}). \tag{10.3.39}$$

Let us rewrite this in the form

$$\theta_{\mathrm{L}} C_{\mathrm{L}} - \tilde{\theta}_{\mathrm{L}} \widetilde{C}_{\mathrm{L}} = L_{\mathrm{MB}} + \frac{\Delta\theta_{\mathrm{TL}}}{\ell} d(\Gamma^{\star\alpha} \wedge \vartheta_\alpha)\,, \tag{10.3.40}$$

in order to make the relation to the MB Lagrangian of BAEKLER et al. (1992) transparent. Consequently, the constants θ_{L} and $\tilde{\theta}_{\mathrm{L}}$ give rise to the two central charges

$$c = 12 \cdot 4\pi\theta_{\mathrm{L}}, \qquad \tilde{c} = 12 \cdot 4\pi\tilde{\theta}_{\mathrm{L}}, \tag{10.3.41}$$

in the Virasoro algebra of a purely topological theory with all the implications analyzed by BLAGOJEVIĆ & VASILIĆ (2003, 2005), CACCIATORI et al. (2006).

10.3.7 Coupling to the Symmetric Cotton Tensor

The classical correspondence of the MB model to 3D gravity in Riemannian spacetime arises by reconsidering the constraint of vanishing torsion, consistently implemented by Lagrange multipliers:

$$\widetilde{L} = L + \lambda_\alpha \wedge T^\alpha . \tag{10.3.42}$$

After varying (10.3.42) with respect to the Lagrange multiplier one-form λ_α, one recovers the constraint $T^\alpha = 0$, and the second field equation (10.1.13) amounts to an algebraic equation for λ_α. Employing the algebraic identity (A.1.26) of HEHL et al. (1995), we see that the first field equation (10.1.12) is converted into (5.8.25), which in 3D reads

$$C_\alpha - E_\alpha = \Sigma_\alpha - D^{\{\}}\mu_\alpha , \tag{10.3.43}$$

where

$$C_l = E_l{}^\alpha\, {}^*\left\{ D^{\{\}} \left[2\left(e^\beta \rfloor D^{\{\}} H_{[\alpha\beta]}\right) + \frac{1}{2}\vartheta_\alpha \wedge \left(e^\beta \rfloor e^\gamma \rfloor D^{\{\}} H_{[\beta\gamma]}\right)\right]\right\} \tag{10.3.44}$$

is the one-form associated with the symmetric Cotton tensor $C^{kl} = C^{lk}$. In general, this is a third-order equation in the Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$, i.e., of fourth order in the metric.

Due to $T_\alpha = D\vartheta_\alpha = 0$ in Riemannian spacetime, implying $H_\alpha = 0$ in 3D, the modified S-duality leads to $DH^\star_\alpha = \delta\ell DH_\alpha + (\gamma/\ell)D\vartheta_\alpha \equiv 0$, and the two-form $D^{\{\}}H_{[\alpha\beta]}$ in the higher-derivative Cotton-type one-form (10.3.44) drops out. Then we are left with (10.3.11), i.e.,

$$-E_\alpha = -e_\alpha \rfloor L(\star) - (-1)^s \frac{2\gamma}{\ell}(e_\alpha \rfloor R^{\{\}\star\beta}) \wedge \vartheta_\beta = \ell\, \widetilde{\Sigma}_\alpha \tag{10.3.45}$$

in Riemannian spacetime. Since $G_\alpha = R^{\{\}\star}_\alpha$ is the Einstein current two-form in 3D, we arrive at Einstein's equations

$$G^{\{\}}_\alpha - \tilde{\Lambda}\eta_\alpha = \ell\, \widetilde{\Sigma}_\alpha , \tag{10.3.46}$$

with an induced cosmological constant for the Riemannian background with the *symmetric* Belinfante–Rosenfeld two-form $\widetilde{\Sigma}_\alpha := \Sigma_\alpha - D^{\{\}}\mu_\alpha$ as source.

Comparing (10.3.45) with (10.3.46), we see that the induced cosmological term reads

$$\tilde{\Lambda} = (-1)^s\, {}^*L(\star) + \frac{1}{3\ell}\left[4\gamma + (-1)^s\ell\right] {}^*\left(R^{\{\}\star\alpha} \wedge \vartheta_\alpha\right), \tag{10.3.47}$$

where again a similar reduction of the Lagrangian L is understood. Is this an interesting hint for the origin of the "dark energy" (Perlmutter 2003) in the real 4D universe, induced via CS-type boundary terms dC in 4D?

10.4 Graphene and Emergent Gravity

Recently, graphene (Novoselov et al. 2005) as a new material has attracted considerable attention because its charge carriers can be described by massless Dirac fields, (Ferkous & Bounames 2004), whereas the flexural models of the 2D membrane of graphene have been tentatively considered as membranes, (Kerner & Naumis 2012), evolving in a $(2+1)$-dimensional curved, but conformally flat, spacetime (Iorio 2011). There are also indications of dislocations (de Juan et al. 2010) related to singular torsion. Moreover, the elastic deformation of corrugated membranes of graphene looks like an extrinsic curvature effect but could as well be described more aptly by emergent teleparallelism (Zubkov & Volovik 2015). Ripples in graphene can be regarded as the 3D analogue of "gravitational waves".

A related topological framework with a coupling to Dirac fields in 3D was considered before by Lemke & Mielke (1993). In principle, it seems be possible to enlarge the dynamical framework of the MB-type theory by including Weyl spaces (Iorio & Lambiase 2014), or even supersymmetry (Ezawa 2008), as a new "playground" for the topological ideas developed here.

10.5 Dirac Equation in 3D

Electrons in the graphene sheet are 2+1 Dirac fermions. When massless, the energy spectrum is $E = pc_{\mathrm{F}}$, where c_{F} is the effective speed of fermions in the 2D carbon material.

In general, the spinors that will be used in 3D depend on the signature of space(time): in the Euclidean case, the covering group of the rotation group SO(3) is isomorphic to the unitary group $SU(2)$. Since an element of $SU(2)$ can be parameterized by three numbers, the most convenient basis of the Lie algebra are the familiar Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10.5.1}$$

These matrices satisfy the following Lie algebra:

$$\left[\sigma^\alpha, \sigma^\beta\right] = 2i\eta^{\alpha\beta\gamma}\sigma_\gamma. \tag{10.5.2}$$

On the other hand, for Lorentzian signature $s = 1$, the covering group of SO(1,2) is isomorphic to the real group $SL(2, \mathbb{R})$, or as well to the symplectic algebra $\mathfrak{sp}(2, \mathbb{R})$. In both cases, the generators may be realized by the matrices

$$\gamma_0 = i\sigma^2, \quad \gamma_1 = \sigma^1, \quad \gamma_2 = \sigma^3. \tag{10.5.3}$$

These real matrices satisfy

$$\gamma_\alpha \gamma_\beta = g_{\alpha\beta}\mathbf{1} + \eta_{\alpha\beta\nu}\gamma^\nu, \tag{10.5.4}$$

thus providing a realization of the Clifford algebra $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2g_{\alpha\beta}$ in 3D.

In RC spacetime, the *minimally coupled* manifestly Hermitian Dirac Lagrangian takes the two equivalent forms

$$\begin{aligned} L_{\mathrm{D}} &= \frac{i}{2}\left(\overline{\psi}\,{}^*\gamma \wedge D\psi + D\overline{\psi} \wedge {}^*\gamma\,\psi\right) - m\overline{\psi}\psi\eta \\ &= \overline{\psi}\left(i^*\gamma \wedge D - m\eta\right)\psi + d\left(\frac{i}{2}\overline{\psi}\,{}^*\gamma\psi\right), \end{aligned} \tag{10.5.5}$$

where $D\psi := d\psi - \frac{1}{2}\gamma_\alpha \wedge \Gamma^{\star\alpha}\psi$ denotes the covariant spinor derivative in 3D.

The covariant Dirac field equation

$$i^*\gamma \wedge D\psi - m\psi\eta - \frac{i}{2}(D^*\gamma)\psi = 0 \tag{10.5.6}$$

is obtained by varying L_{D} with respect to $\overline{\Psi}$. Since

$$D^*\gamma = {}^*\gamma \wedge T\,, \qquad T := e_\alpha \rfloor T^\alpha, \tag{10.5.7}$$

one can go over to the new covariant derivative

$$\hat{D} := D - \frac{1}{2}T\,, \tag{10.5.8}$$

involving the trace torsion T. Then we obtain

$$i^*\gamma \wedge \hat{D}\psi - m\eta\psi = 0, \tag{10.5.9}$$

which reveals that the coupling to RC geometry is nonminimal on the level of the covariant Dirac equation.

The curvature associated with the deformed covariant derivative $\hat{D}$ is

$$\hat{R} = R - \frac{1}{2}dT = \frac{i}{4}R^{\alpha\beta}\sigma_{\alpha\beta} - \frac{1}{2}dT \tag{10.5.10}$$

and the corresponding Ricci identity simplifies to

$$[\hat{D}, \hat{D}] = \hat{R} + T^{\alpha}\hat{D}_{\alpha}\,. \tag{10.5.11}$$

Then the integrability condition for the matter equation (10.5.9) is a Klein–Gordon equation

$$\hat{D}^{*}\hat{D}\psi - m^{2}\eta\psi + i^{*}\sigma \wedge \left(\hat{R} + T^{\alpha}\hat{D}_{\alpha}\right)\psi = 0\,, \tag{10.5.12}$$

supplemented by Pauli-like interaction terms. The connection $\Gamma^{\star\alpha}$ still contains contorsional pieces, such as the axial torsion $\mathscr{A} := {}^{*}(\vartheta^{\alpha} \wedge T_{\alpha})$, a zero-form in 3D. However, 2D fermions do not couple to axial torsion; cf. DE JUAN et al. (2010). Materials that support screw dislocations, including axial torsion, are, e.g., graphites. Cosmological solutions in a minimally coupled Dirac–MB model are discussed in SERT & ADAK (2013).

10.6 Topological Massive Photons in 3D

A topological extension of electrodynamics in 3D, sometimes called CS electrodynamics, goes as follows: If A denotes the $U(1)$ connection one-form, then Faraday's field strength reads $F := dA$, i.e., it remains an exact two-form. In 3D, the Maxwell Lagrangian can be supplemented by an abelian Chern–Simons term, with the result that

$$L_{A} = \frac{1}{2}(F \wedge {}^{*}F + m_{\text{photon}}F \wedge A). \tag{10.6.1}$$

However, this three-form is gauge-invariant only modulo an exact form $d(A \wedge {}^{*}F)/2$. Nevertheless, the sourceless field equation

$$d^{*}F - m_{\text{photon}}F = 0 \tag{10.6.2}$$

remains $U(1)$ gauge-invariant. Since $dF \equiv 0$ is still the abelian Bianchi identity, the integrability condition reads

$$\Box F - m_{\text{photon}}^{2}F = 0, \tag{10.6.3}$$

where $\Box := d^{*}d^{*} - {}^{*}d^{*}d$ is the d'Alembertian in $2 + 1$ dimensions. Since (10.6.3) is a Klein–Gordon equation for the field strength F, we conclude that the photons become massive. This is induced by the CS term when inversely coupled, for dimensional reasons, via the Compton wavelength $\lambda = \hbar/cm_{\text{photon}}$ for massive photons.

This construction can be generalized to topological massive Yang–Mills theory (YILDIRIM 2015), where a mass gap is induced via a CS term.

10.7 Membranes with Torsion Defects

As an example of a spacetime with torsion and/or curvature *defects* (DE JUAN et al. 2010) or singularities (Fig. 10.1), let us consider a *planar graphene* solution within the "mixed" MB model governed by the two Einstein–Cartan-type field equations (10.2.5) and (10.2.6).

Let us assume that the 2D membrane of a corrugated graphene is evolving in an *intrinsic* three-dimensional spacetime, suppressing for the moment the embedding of a real graphene into *flat* 4D Minkowski spacetime. Then we may adopt the convention that x^α and y^α are spacelike orthogonal vectors that span the (x, y)-plane perpendicular to the time coordinate t, which itself is orthogonal to the world sheet of the graphene. The corresponding one-forms (MIELKE & KREIMER 1998) are denoted by capital letters, i.e.,

$$X := x_\alpha\, \vartheta^\alpha\,, \qquad Y := y_\alpha\, \vartheta^\alpha\,. \tag{10.7.1}$$

Moreover, the vector n^α is a timelike unit vector normal to the hypersurface with $n^\alpha\, n_\alpha = s$, the signature s of our 3D spacetime. Following SOLENG (1992), ANANDAN (1994), and BAKKE et al. (2009), we assume that the two-forms Σ_α and $\tau^\star_\alpha$ of the energy–momentum and spin current, respectively, vanish outside the graphene sheet, whereas "inside," they are *constant*, i.e.,

$$\Sigma_\alpha = \varepsilon\, x_\alpha\, X \wedge Y\,, \qquad \tau^\star_\alpha = \sigma\, y_\alpha\, X \wedge Y\,, \tag{10.7.2}$$

which satisfy

$$\vartheta^\alpha \wedge \Sigma_\alpha = 0\,, \qquad \vartheta^\alpha \wedge \tau^\star_\alpha = 0 \tag{10.7.3}$$

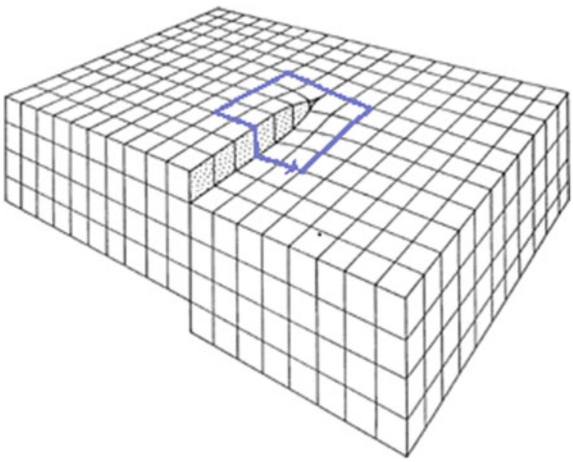

Fig. 10.1 "Screw" dislocation with singular torsion in a cubic lattice. The Cartan circuit is indicated in *blue*; cf. LAZAR & HEHL (2010)

by construction. The constant parameters ε and σ of this *spinning string*-type Ansatz are related to the exterior vacuum solution by appropriate matching conditions. For the related solution with *conical singularities* and torsion of TOD (1994), it turns out that ε and σ are formally *delta distributions* (TAUB 1980) at the idealized location of the defect; cf. Fig. 10.2. From the specification (10.7.1) of the one-forms X and Y, it can easily be inferred that the only nonzero components are $\Sigma_{\hat{0}} \neq 0$ and $\tau_{\hat{1}\hat{2}} = -\tau_{\hat{2}\hat{1}} \neq 0$.

Due to the identities (10.7.3), contractions of the second field equation (10.2.6) with x^α and y^α reveal that $x^{[\alpha} y^{\beta]} R_{\alpha\beta} = R_{\hat{1}\hat{2}} = -R_{\hat{2}\hat{1}} \neq 0$ are the only nonvanishing components of the RC curvature. From its covariant expression

$$R^{\alpha\beta} = \varepsilon \ell^2 \, x^{[\alpha} y^{\beta]} \, X \wedge Y, \tag{10.7.4}$$

there follows the identity

$$R_\beta{}^\alpha \wedge \vartheta^\beta = \frac{\ell^2}{2} \varepsilon (x^\alpha \, Y \wedge X \wedge Y - y^\alpha \, X \wedge X \wedge Y) = 0\,. \tag{10.7.5}$$

We recall that $N^\alpha = n \rfloor \vartheta^\alpha$ is the lapse and shift vector in the (2+1) decomposition à la ADM to see that the corresponding coframe and connection can now be obtained explicitly by applying a finite boost to the usual conical metric of a "cosmic string"-type defect:

$$\begin{aligned} \vartheta^{\hat{0}} &= dt + \ell^2 \sigma \rho^{*2} [1 - \cos(\rho/\rho^*)] d\phi \\ \vartheta^{\hat{1}} &= d\rho\,, \qquad \vartheta^{\hat{2}} = \rho^* \sin(\rho/\rho^*) d\phi\,, \\ \Gamma^{\hat{1}\hat{2}} &= \cos(\rho/\rho^*) d\phi = -\Gamma^{\hat{2}\hat{1}}\,. \end{aligned} \tag{10.7.6}$$

From the Cartan-type relation (10.2.5) and the identities (10.7.3), we can infer that in 3D, the *axial* torsion

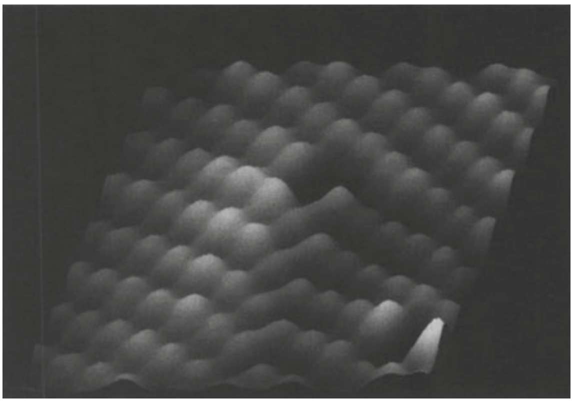

Fig. 10.2 "Screw" dislocation in atomic resolution in inverted perspective viewed by means of a scanning tunnel microscope

$$\mathscr{A} := {}^*(\vartheta^\alpha \wedge T_\alpha) = -(-1)^s \frac{\varepsilon}{\ell^2} \tag{10.7.7}$$

of such a membrane defect is a constant pseudoscalar.

Thus, from the embedding of 3D into 4D, there arises no contribution to the Pointryagin-type term $d(\mathscr{A} \wedge d\mathscr{A})$ from the axial torsion. Moreover, the Nieh–Yan term dC_{T} proportional to $d\,{}^*\mathscr{A}$ *vanishes identically* for this example of a spinning cosmic string exhibiting a *torsion line defect*.

10.8 Supergravity

Fundamental interactions like QCD are rather successfully formulated in terms of Yang–Mills theories with large gauge groups, stipulating that symmetry breaking is prevailing. The idea of supersymmetry or supergravity, anticipated to some extent already by WEYL (1931), goes in the same direction but so far lacks empirical support in particle physics. There are, however, interesting, thus far speculative, directions (ABREU et al. 2015; ALVAREZ et al. 2015) in the realm of condensed matter.

Supergravity (DESER & ZUMINO 1976; FREEDMAN 1994) with one supersymmetry generator, i.e., $\mathscr{N} = 1$, represents the simplest consistent coupling of a Rarita–Schwinger (RS) spin-3/2 field (RARITA & SCHWINGER 1941) to gravity.

In writing the Rarita–Schwinger-type spinor-valued one-form[3]

$$\Psi = \Psi_i dx^i = \Psi_\alpha \vartheta^\alpha \tag{10.8.1}$$

holonomically, it becomes clear that it does not depend on the coframe, inasmuch as $\Psi_\alpha := e_\alpha \rfloor \Psi$ involves the tetrad that is inverse to the frame in the anholonomic formulation. However, in 3D we adhere to the conventions that the holonomic indices run over $i, j, k, \ldots = 0, 1, 2$, whereas $\alpha, \beta, \ldots = \hat{0}, \hat{1}, \hat{2}$ for the anholonomic indices. In addition, the coframe basis ϑ^α converts into one Clifford-algebra-valued one-form

$$\gamma = \gamma_\alpha \vartheta^\alpha . \tag{10.8.2}$$

Then Ψ will become real two-component spinors, with the Dirac adjoint defined by $\overline{\Psi} := \Psi^\dagger \gamma^0$.

---

[3]In four dimensions (4D), the RS field $\Psi := \Psi_\alpha \vartheta^\alpha$ entering (10.8.1) is a *Majorana-spinor*-valued one-form. As is well known (VAN NIEUWENHUIZEN 1981), it satisfies the *Majorana condition*, i.e., $\Psi = C\overline{\Psi}^{\mathrm{T}}$, where C is the charge conjugation matrix given by $C = -i\gamma_0$ satisfying $C^\dagger = C^{-1}$, $C^{\mathrm{T}} = -C$ and $C^{-1}\gamma^\alpha C = -(\gamma^\alpha)^{\mathrm{T}}$. Consequently,

$$\overline{\Psi} \wedge \Psi = 0\,, \quad \overline{\Psi} \wedge \gamma_5 \gamma^\alpha \Psi = 0\,, \quad \overline{\Psi} \wedge \gamma_5 \Psi = 0.$$

For the real Majorana representation, all γ^α are purely imaginary, and the components of the gravitino vector–spinor consequently are all real (MIELKE & MACIAS 1999).

10.9 Rarita–Schwinger Lagrangian in 3D

The corresponding *manifestly Hermitian* RS-type Lagrangian three-form of HOWE & TUCKER (1978) reads

$$L_{\rm RS} = \frac{i}{4}\left(\overline{\Psi} \wedge D\Psi - \Psi \wedge \overline{D\Psi}\right) + \frac{i}{4}m\overline{\Psi} \wedge \gamma \wedge \Psi, \tag{10.9.1}$$

including a mass term. Here minimal coupling to gravity is achieved via

$$D\Psi = d\Psi - \frac{1}{2}\gamma_\alpha \Gamma^{\star\alpha} \wedge \Psi, \tag{10.9.2}$$

which is nothing more than the gauge covariant derivative of a spinor-valued one-form Ψ.

The generalization of $L_{\rm RS}$ analyzed by MIELKE & MAGGIOLO (2012) exists only in 3D. In general, the energy–momentum current two-form Σ_α of matter is given by the variational derivative

$$\Sigma_\alpha := \frac{\delta L_\Psi}{\delta \vartheta^\alpha} = \frac{\partial L_\Psi}{\partial \vartheta_\alpha} + D\frac{\partial L_\Psi}{\partial T^\alpha}, \tag{10.9.3}$$

where the second term accounts for the possibility of a nonminimal coupling to torsion via Pauli-type terms; cf. Eq. (5.1.8) of HEHL et al. (1995).

Without Pauli terms, the energy–momentum current two-form of matter can be rewritten as

$$\begin{aligned}\Sigma_\alpha := {} & e_\alpha \rfloor L_\Psi - (e_\alpha \rfloor \Psi) \wedge \frac{\partial L_\Psi}{\partial \Psi} - (e_\alpha \rfloor \overline{\Psi}) \wedge \frac{\partial L_\Psi}{\partial \overline{\Psi}} \\ & - (e_\alpha \rfloor D\Psi) \wedge \frac{\partial L_\Psi}{\partial D\Psi} - (e_\alpha \rfloor D\overline{\Psi}) \wedge \frac{\partial L_\Psi}{\partial D\overline{\Psi}};\end{aligned} \tag{10.9.4}$$

see (5.4.11) of HEHL et al. (1995) for details. This is often more convenient, since it involves only partial derivatives of the matter fields and avoids the intricate treatment of a possible dependence of the matter Lagrangian on the Hodge dual. Since the kinetic terms in the Rarita–Schwinger Lagrangian $L_{\rm RS}$ do not depend explicitly on the coframe ϑ^α, they provide no contribution to the energy–momentum current, with the result that

$$\Sigma_\alpha = -\frac{i}{4}m\overline{\Psi} \wedge \gamma_\alpha \Psi = -(-1)^s 2m\tau^\star_\alpha\,, \tag{10.9.5}$$

where the 3-dual of the spin current is given by the chain of definitions

$$\tau^\star_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma}\tau^{\beta\gamma} = \frac{(-1)^s}{2}\frac{\delta L_\Psi}{\delta \Gamma^\star_\alpha} = \frac{(-1)^s}{2}\frac{i}{4}\overline{\Psi} \wedge \gamma_\alpha \Psi\,. \tag{10.9.6}$$

Thus for the pure RS Lagrangian in 3D, the energy–momentum current turns out to be proportional to its dual spin.

10.10 Topological Supersymmetry in 3D

The first-order topological Lagrangian

$$L = L_{\infty}(\vartheta^{\alpha}, \Gamma_{\alpha}^{\star}, \Psi) = L_{\mathrm{MB}} + L_{\Psi} \tag{10.10.1}$$

can be verified to be supersymmetric under certain constraints: the variation of its independent variables $(\vartheta^{\alpha}, \Gamma_{\alpha}^{\star}, \Psi)$ yields

$$\delta L = \delta\vartheta^{\alpha} \wedge \frac{\delta L}{\delta\vartheta^{\alpha}} + \delta\Gamma_{\alpha}^{\star} \wedge \frac{\delta L}{\delta\Gamma_{\alpha}^{\star}} + \delta\overline{\Psi} \wedge \frac{\delta L}{\delta\overline{\Psi}}, \tag{10.10.2}$$

where for convenience, only the Dirac adjoint $\overline{\Psi}$ is varied for.

The supersymmetric (SUSY) transformation of Deser et al. (1984) reads, in exterior form notation,

$$\begin{aligned} \delta_{\mathrm{susy}}\vartheta^{\alpha} &= i\overline{\sigma}\,\Psi\gamma^{\alpha}, \qquad \delta_{\mathrm{susy}}\Gamma_{\alpha}^{\star} = i\overline{\sigma}\,\gamma_{\alpha}^{*}D\Psi + ic\overline{\sigma}\,(\gamma_{\alpha}\Psi + e_{\alpha}\rfloor^{*}\Psi), \\ \delta_{\mathrm{susy}}\Psi &= 2D\sigma + c\gamma\sigma, \end{aligned} \tag{10.10.3}$$

where σ stands in for a spinor-valued zero-form and c is a real constant. Let us probe the SUSY invariance of the Lagrangian (10.10.1) by inserting this into (10.10.2):

$$\delta_{\mathrm{susy}}L = i\overline{\sigma}\,\Psi\gamma^{\alpha} \wedge \frac{\delta L}{\delta\vartheta^{\alpha}} + \delta_{\mathrm{susy}}\Gamma_{\alpha}^{\star} \wedge \frac{\delta L}{\delta\Gamma_{\alpha}^{\star}} + \left(2\overline{D\sigma} + c\overline{\sigma}\gamma\right) \wedge \frac{\delta L}{\delta\overline{\Psi}}, \tag{10.10.4}$$

where $\overline{c\gamma\sigma} = c\overline{\sigma}\gamma$ holds for the Dirac adjoint.

In the following, let us assume that the second field equation $\delta L/\delta\Gamma_{\alpha}^{\star} \cong 0$ satisfies, "on shell," Eq. (10.2.4) of the "mixed" MB model. Then the SUSY transformation reduces to

$$\delta_{\mathrm{susy}}L \cong \overline{\sigma}\left(i\gamma^{\alpha}\Psi \wedge \frac{\delta L}{\delta\vartheta^{\alpha}} - 2D\frac{\delta L}{\delta\overline{\Psi}} + c\gamma \wedge \frac{\delta L}{\delta\overline{\Psi}}\right) + 2d\left(\overline{\sigma} \wedge \frac{\delta L}{\delta\Psi}\right). \tag{10.10.5}$$

For later convenience, the Rarita–Schwinger equation

$$\frac{2}{i}\frac{\delta L}{\delta\Psi} = D\Psi + \frac{1}{2}m\gamma \wedge \Psi \cong 0 \tag{10.10.6}$$

is allowed here to be massive. Moreover, in (10.10.5), the term in parentheses following from the supersymmetric transformations explicitly reads

$$
\begin{aligned}
i\gamma^\alpha \Psi \wedge \frac{\delta L}{\delta \vartheta^\alpha} + c\gamma \wedge \frac{\delta L}{\delta \overline{\Psi}} - 2D\frac{\delta L}{\delta \overline{\Psi}} & \\
\cong i\gamma^\alpha \Psi \wedge \left(\frac{\theta_{\mathrm{TL}}}{\ell} R^\star_\alpha + \frac{\theta_{\mathrm{T}}}{\ell^2} T_\alpha + \Sigma_\alpha\right) & \\
+ c\gamma \wedge \left(\frac{i}{2} D\Psi + \frac{i}{4} m\gamma \wedge \Psi\right) - D\left(iD\Psi + \frac{i}{2} m\gamma \wedge \Psi\right) & \\
= i\gamma^\alpha \Psi \wedge \left(\frac{\theta_{\mathrm{TL}}}{\ell} R^\star_\alpha + \frac{\theta_{\mathrm{T}}}{\ell^2} T_\alpha\right) + \gamma^\alpha \Psi \wedge \left(\frac{1}{4} m \overline{\Psi} \gamma_\alpha \Psi\right) & \\
+ c\gamma \wedge \left(\frac{i}{2} D\Psi + \frac{i}{4} m\gamma \wedge \Psi\right) - iR^\star_\alpha \gamma^\alpha \Psi - \frac{i}{2} m T_\alpha \gamma^\alpha \Psi & \\
+ \frac{i}{2} m\gamma \wedge D\Psi. & \qquad (10.10.7)
\end{aligned}
$$

By a Fierz rearrangement, i.e.,

$$\gamma^\alpha \Psi \wedge \overline{\Psi} \gamma_\alpha \Psi = 0, \qquad (10.10.8)$$

terms arising from the energy–momentum current Σ_α, or likewise from the dual spin $\tau^\star_\alpha$, vanish. Moreover, we require $c = -m$, in order to eliminate kinetic terms like $\gamma \wedge D\Psi$. Then, using the 3D formula

$$\gamma \wedge \gamma = -2\gamma^\alpha \eta_\alpha \qquad (10.10.9)$$

of HOWE & TUCKER (1978), we find from (10.10.7) the requirement

$$i\left[\left(\frac{\theta_{\mathrm{TL}}}{\ell} - 1\right) R^\star_\alpha + \left(\frac{\theta_{\mathrm{T}}}{\ell^2} - \frac{m}{2}\right) T_\alpha + \frac{m^2}{2} \eta_\alpha\right] \wedge \gamma^\alpha \Psi = 0, \qquad (10.10.10)$$

so that our Lagrangian becomes supersymmetric.

At first sight, it appears that there is no cosmological constant to compensate a similar one arising from the RS mass. However, one should compare the expression in brackets with the second field equation (10.2.4) *inserted*, which indeed involves a cosmological term induced by the translational CS term proportional to θ_{T}. Then

$$
\begin{aligned}
i\Big[\left(\theta_{\mathrm{L}} + \frac{\theta_{\mathrm{TL}}}{\ell} - 1\right) R^\star_\alpha + \left((-1)^s \frac{\theta_{\mathrm{TL}}}{\ell} + \frac{\theta_{\mathrm{T}}}{\ell^2} - \frac{m}{2}\right) T_\alpha + \frac{1}{2}\left(\frac{\theta_{\mathrm{T}}}{\ell^2} + m^2\right) \eta_\alpha & \\
+ \tau^\star_\alpha\Big] \wedge \gamma^\alpha \Psi \simeq 0\,, & \qquad (10.10.11)
\end{aligned}
$$

results, although the dual spin $\tau^\star_\alpha$ of the RS field will not contribute, again due to the Fierz rearrangement (10.10.8). This finally leads to the mass-dependent "on shell" conditions

$$\theta_{\rm T} \simeq -m^2\ell^2,\ \theta_{\rm TL} \simeq \frac{(-1)^s}{2} m(2m+1)\ell,\ \theta_{\rm L} \simeq 1 - \frac{\theta_{\rm TL}}{\ell} = 1 - \frac{(-1)^s}{2} m(2m+1) \tag{10.10.12}$$

for the coupling constants of the bosonic part of L_∞. Consequently, massless RS spinors do not require a translational or a “mixed” CS term in order to acquire supersymmetry.

In Mielke & Maggiolo (2012), there were preliminary attempts to generalize the peculiar dynamical symmetry of Baekler et al. (1992), identified before as S-duality, to a supersymmetric version via the Ansatz

$$\vartheta_\alpha = (-1)^s \ell\, \Gamma^\star_\alpha + \overline{\sigma}\, \gamma_\alpha \Psi\ . \tag{10.10.13}$$

Here σ is again a spinor-valued zero-form, and ℓ a fundamental length.

More recently, Alvarez et al. (2015) considered an $\mathcal{N} = 2$ supersymmetry without the RS field.

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Chapter 11
Spinor Bundles

The intended unification of electromagnetic, weak, strong, and gravitational interactions within the geometric concept of gauge invariance is still flawed by one essential shortcoming that cannot be disregarded. For it is still lacking an exact geometric description of the constituents of (tangible) matter sources! Here we are following Weyl's assertion, which is still valid nowadays, that the electromagnetic field (and consequently all the different gauge fields) "is a necessary accompaniment of the matter-wave field" (WEYL 1929b, p. 331).

Matter, which not only surrounds us in daily life but also constitutes our very selves, is known to extend itself and thus to take in a "material" part of space. According to the empirically asserted relation between the intrinsic angular momentum (spin) and the so-called Fermi–Dirac statistics, the material core has to be described quantum-mechanically by wave functions with half-integer spin. Moreover, all actual models of particle physics (KOKKEDEE 1969; PATI & SALAM 1973, 1974; TAYLOR 1979; MARCIANO & PAGELS 1978; BARUT 1980) agree that only fermions, such as electrons, protons, neutrinos, muons, and colored quarks, are to be considered fundamental building blocks of matter and that all other phenomena are taken to be accounted for by (quantized) gauge fields. Therefore, it is of basic interest to incorporate DIRAC'S relativistic theory of the electron (1928) into the gauge-theoretic framework. Then it is equally important to investigate the exact physical meaning of the geometric and topological structures thereby introduced. One has to bear in mind, however, that a precise dividing line cannot be drawn between matter and gauge fields proper. It is for this reason that (WEYL 1924, p. 609) considered "a dynamical theory of matter to be the most promising one: matter as a field-inducing agent, the field as an extensive medium that transfers interactions from matter to matter."[1] In this context, it is to be kept in mind that the constituents of subnuclear

[1](WEYL 1924, p. 609), "eine dynamische Theorie der Materie am aussichtsreichsten: *die Materie ein felderregendes Agens, das Feld ein extensives Medium, das die Wirkungen von Körper zu Körper überträgt*."

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matter can be characterized operationally only by the fields with which they are able to interact.

11.1 Global Spinor Fields

In relativistic quantum mechanics (Bjorken & Drell 1964), fermions with spin 1/2 are described by bispinor representations $D^{(-\frac{1}{2},\frac{3}{2})} \otimes D^{(\frac{1}{2},\frac{3}{2})}(\widetilde{SO}_\circ(1,3))$ of the proper Lorentz group $SO_\circ(1,3)$ (Gel'fand et al. 1963).[2] These representations are not unitary, although the fields transform according to unitarily *induced* representations of the (restricted) Poincaré group $P_\circ := \mathbb{R}^4 \rtimes SO_\circ(1,3)$ of flat spacetime (Mackey 1968; Trautman 1970; Mielke 1977c; Kaźmierczak 2010). Despite counterclaims, spinorial polyfield representations of the linear group do exist (Hehl et al. 1978). Unitary representations are infinite-dimensional again, this time consisting of the direct sum of fermion fields with an unlimited increase in spin: $\psi_\infty := \psi_{1/2} \otimes \psi_{5/2} \otimes \psi_{9/2} \otimes \ldots$.

This description of the spinor fields is, however, insufficient to achieve a coupling to generally covariant field theories, since a rudimentary relevance of this description is maintained only in the local tangent space, the reason being that such a local representation does not allow the incorporation of a nontrivial topology of the background manifold and of the configuration space, which is of increasing importance to the formalism of gauge theories. This is particularly true for fermions if one keeps in mind that their half-integer spin originates in a "nonclassical two-valuedness" according to Pauli; cf. Wheeler (1968, p. 75).

The structural reason for this has to be seen in the global topology of the (pseudo-orthogonal) structure group $\mathrm{SO}(s, n-s)$ of the tangent bundle of an n-dimensional manifold[3] of signature s. Provided $s \neq 0$, this topological space has two connected components, which are, however, not simply connected according to homotopy theory. Concerning the connected component $SO_\circ(s, n-s)$ of the unit group element, its connectivity relations are marked by the first homotopy group (Helgason 2001, p. 346):

$$\pi_1(SO_\circ(s, n-s), e) = \begin{cases} \mathbb{Z}_2 & n = 2k \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n = 4k \\ \mathbb{Z}_4 & n = 2(2k+1). \end{cases} \tag{11.1.1}$$

However, it is possible to turn to a simply connected universal covering group $\widetilde{SO}_\circ(s, n-s)$ (=: Spin(n) for $s = 0$), whose irreducible representations provide the desired spinor representations.

[2] Fermions with a higher half-integer spin are not to be considered here.

[3] In order to make possible a generalization of the Dirac equation in the higher-dimensional spaces of the Kaluza–Klein theory, the formalism is again developed for arbitrary dimensions (Brauer & Weyl 1935). For this purpose, the work of Kerner (1980) is rather useful.

This transition is achieved by a group homomorphism $\Lambda : \tilde{\mathrm{G}} \to \mathrm{G}$, which is based on the following explicit construction: Let $C(s, n-s)$ denote the so-called Clifford algebra, which is spanned by *generalized* complex Dirac matrices γ_α. These matrices obey the algebraic relation

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2 \mathrm{o}_{\alpha\beta} \mathbb{1} \tag{11.1.2}$$

and admit realizations by $N \times N$ matrices for which $N = 2^{[n/2]}$ corresponds to the lowest-dimensional faithful complex representation. Here [ ] denotes the next lower integer. In the four-dimensional world, the so-called Pauli realization of the Dirac matrices is commonly used, as is, for instance, done by BJORKEN & DRELL (1964):

$$\gamma^0 := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^k := \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \sigma^k : \text{Pauli matrices.} \tag{11.1.3}$$

Let the transformations of the Euclidean vector space $\mathbf{E}^n$ with coordinates x^α be determined by the (Lorentz) rotations $\mathrm{g} \in SO(s, n-s)$ on behalf of the group action

$$x^\alpha \to x'^\alpha = g^\alpha_\beta \ \ x^\beta, [g^\alpha_\beta] \in SO(s, n-s), \ \ x^\alpha \in \mathbf{E}^n. \tag{11.1.4}$$

These points may be related to a complex $N \times N$ matrix X by means of the one-to-one mapping

$$\mathrm{X} := \imath x^\alpha \gamma_\alpha \longleftrightarrow x^\alpha = -\frac{\imath}{N} \mathrm{Tr}(\gamma^\alpha X). \tag{11.1.5}$$

(Concerning a real vector bundle V on the manifold, this construction can be extended to a 2^N-dimensional bundle, the so-called Clifford bundle CV. For details, see ATIYAH et al. (1964)). This complex vector space is specified as follows. The determinant

$$\det \mathrm{X} = -\mathrm{o}_{\alpha\beta} x^\alpha x^\beta, \tag{11.1.6}$$

as well as the square

$$\mathrm{XX} = -\mathrm{o}_{\alpha\beta} x^\alpha x^\beta \mathbb{1} \tag{11.1.7}$$

of such a matrix, leads back directly to the generalized Minkowski distance squared of signature s (CARTAN 1966). In order to keep this distance invariant, the covering group is bound to act on the space X by means of a *similarity transformation*

$$\mathrm{X} \longrightarrow \mathrm{X}' = SXS^{-1}, \ \ S \in \widetilde{SO}_\circ(s, n-s). \tag{11.1.8}$$

A comparison of (11.1.5) and (11.1.4) indicates that the covering homomorphism is determined by

$$\Lambda : \begin{cases} \widetilde{SO}_\circ(s, n-s) \longrightarrow SO_\circ(s, n-s), \\ \qquad \ \ \Downarrow \qquad\qquad\qquad \Downarrow \\ \qquad \ \ S \qquad\quad \longrightarrow g^\alpha{}_\beta = -\frac{i}{N} \mathrm{Tr}(\gamma^\alpha S i \gamma_\beta S^{-1}), \end{cases} \tag{11.1.9}$$

due to the geometric construction. For instance, the covering group of the proper orthochronous Lorentz group $SO_\circ(1,3)$ is represented by $\widetilde{SO}_\circ(1,3) \approx SL(2,\mathbb{C})$ on account of a special isomorphism (Helgason 2001, p. 353). Furthermore, the infinitesimal generators of the group $\widetilde{SO}_\circ(s, n-s)$ are

$$\sigma^{\alpha\beta} := \frac{i}{4}[\gamma^\alpha, \gamma^\beta], \quad \Lambda(\sigma^{\alpha\beta}) = L^{\alpha\beta}, \tag{11.1.10}$$

which may be derived by a symbolic "inversion" of the homomorphism (11.1.9).

As was stressed initially, it is globally insufficient to characterize the spinor fields as induced representations of the "covering" Poincaré group $P_0 := \mathbb{R}^n \otimes \widetilde{SO}_\circ (s, n-s)$ in a curved spacetime. It is necessary to obtain from a more general "geometric arena" the bundle $L^g(M)$ of orthogonal frames concerning the foundation of the "broken" Poincaré gauge theory of gravity. We resume this here but make use of the principal fiber bundle

$$L^g_\circ(M) := P(M, SO_\circ(s, n-s), \pi, \delta) \tag{11.1.11}$$

having merely the proper[4] Lorentz group $SO_\circ(s, n-s)$ as structure group. However, the topological structure of this group necessitates the consideration of an additional principal fiber bundle with a corresponding covering group as structure group. Then a *spin structure* is imposed on a base space M as follows: Besides $L^g_\circ(M)$, the "covering" principal fiber bundle

$$\widetilde{L}(M) := P(M, \widetilde{SO}_\circ(s, n-2), \tilde{\pi}, \tilde{\delta}) \ni \tilde{p} \tag{11.1.12}$$

is introduced (Milnor 1963) together with the bundle map $f : \tilde{L}(M) \to L^g_\circ(M)$, which must be compatible not only with the right action of the group in both spaces but also with the group homomorphism (11.1.8), i.e.,

$$f(\tilde{p}S) = f(\tilde{p})\Lambda(S), \, S \in \widetilde{SO}_\circ(s, n-2). \tag{11.1.13}$$

A spin structure over M is equivalent to a double covering of $L^g_\circ(M)$. This spin structure is identical to a second one that is represented by $(\tilde{L}'(M), f')$ if the isomorphism between the spinor bundle leads by composition with f to the second bundle mapping f'. Additionally, a *spin manifold* is understood as a Riemannian manifold that can be oriented and for which a spin structure exists in the tangent bundle (Trautman 2008).

This raises a question concerning the premises under which such a spin structure is *globally* legitimate. In mathematically abstract terms, the solution runs as follows: In order to imprint a global spin structure on the $SO_\circ(s, n-s)$ principal fiber bundle, it is necessary and sufficient that the Stiefel–Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$ of the base space M be zero. Here the g^{th} Stiefel–Whitney class is considered as the

[4] This restriction is to be explained later on.

characteristic class of the g^{th} relative cohomology group $H^g(M, \mathbb{Z}_2)$ of the manifold (MILNOR & STASHEFF 1974). The number of inequivalent spin structures is thereby defined by the dimension $\tilde{s} = \dim H^1(M, \mathbb{Z}_2)$ of the first relative cohomology group (MILNOR 1963). Accordingly, there are exactly $\tilde{s} = 2^\nu$ spin structures possible (WHEELER 1968, p. 72) if we are concerned with a spacetime for which the spacelike hypersurface consists of the connected sum of ν handlebodies ("wormholes") $W^3 := S^1 \times S^2$, i.e., for

$$M = \mathbb{R} \times S^3 \#^\nu W^3. \tag{11.1.14}$$

On the other hand, it is known that these multiply connected spaces indeed admit *harmonic* electric fields as solutions of Maxwell's source-free equations but no corresponding spinor solutions of Weyl's equation of the neutrino field; cf. KLAUDER & WHEELER (1957).

While introducing the spin structure, we have deliberately restricted ourselves to the bundle of the orthogonal frames with the connected component $SO_\circ(s, n-s)$ as structure group, since the question of existence of spin structures for the $SO_\circ(s, n-s)$ bundle has to be considered in analogy to the problem of the orientability of $O(s, n-s)$ bundles. The last bundle can be oriented if $w_1 = 0$ is valid, while the number of distinguishable orientations has to be processed out of the dimension of $H^\circ(M, \mathbb{Z}_2)$ (MILNOR 1963). The notion of spin manifold is also of enormous mathematical importance in relation to the construction of an exotic smooth involution on the 7-dimensional sphere S^7 (MILNOR 1965). Attention can be drawn to other topologically oriented works that likewise profit from the use of spin structures (ANDERSON et al. 1966; ASSELMEYER-MALUGA & BRANS 2015).

However, it was GEROCH (1968, 1970) who pointed out an intuitive and in the context of Poincaré gauge theories more convenient premise for the existence of spin structures. Concerning the legitimation of a spin structure on a noncompact spacetime manifold M of signature $s \neq 0$, it is accordingly necessary and sufficient that there exist a global field of (orthonormal) tetrads, or stated otherwise, a system of "orientation-entanglement relations" (WHEELER 1968, p. 69; MTW, p. 1148). This means, for instance, in the 4-dimensional case that

$$\tilde{L}(M^4) = M^4 \times SL(2, \mathbb{C}) \tag{11.1.15}$$

is, viewed from this angle, a trivial bundle and that the possibility of nontrivial spin structures on $T(M)$ is dependent only on the mapping f (ISHAM 1978).

As far as the theory of representations is concerned, it is to be understood that the spin of fermions is to be related to the "external" invariance group of spacetime; in order to incorporate "internal" local symmetries, such as, for instance, the isospin invariance or Gell-Mann's $SU(3)$-group as well into these global constructions, the idea of *generalized* spin structures has been analyzed by AVIS & ISHAM (1980).

Following this intricate terminology of a "spin-carrying geometric arena," we can proceed analogously to our preparatory observations to describe the physically relevant spinor *fields* of matter. Our point of departure is the introduction of a complex vector bundle

$$V^{\bar{s}} := V(M, \mathbb{C}^N, \rho^{\bar{s}}(\widetilde{SO}_\circ(s, n-s)) \subset GL(N, \mathbb{C}), \tilde{L}(M)) \tag{11.1.16}$$

associated to the spin bundle L̃(M). For this reason, it is also called an *associated spinor bundle*. Within this general setup, we consider the fundamental (complex) representation D^{F} of dimension $N = 2^{[n/2]}$ of the structure group $\widetilde{SO}_\circ(s, n-s)$, which contains the physical Lorentz group $\widetilde{SO}_\circ(1, 3)$ as a subgroup. Then the restriction of D^{F} to $\widetilde{SO}_\circ(1, 3)$ yields the following product representation:

$$D^{\mathrm{F}}_{|\widetilde{SO}_\circ(1,3)} = \underbrace{D^{\frac{1}{2}} \times \cdots \times D^{\frac{1}{2}}}_{(n-4)/4}. \tag{11.1.17}$$

Here

$$D^{\frac{1}{2}} := D^{(-\frac{1}{2},\frac{3}{2})} \oplus D^{(\frac{1}{2},\frac{3}{2})} \left(\widetilde{SO}_\circ(1, 3)\right) \tag{11.1.18}$$

denotes the bispinor representation of the covering group $SL(2, \mathbb{C}) \approx \widetilde{SO}_\circ(1, 3)$ of the proper Lorentz group, which itself is the direct sum $\left(-\frac{1}{2}, \frac{3}{2}\right) \oplus \left(\frac{1}{2}, \frac{3}{2}\right)$ of spin-$\frac{1}{2}$ representations labeled according to GEL'FAND et al. (1963). This formulation has the advantage that it can be extended to a representation of the general Lorentz group $\tilde{O}(1, 3)$ if required. Concerning the spinor basis $\tilde{b}_{(1)}$, $i = 1, \ldots, N$, their cross sections form the space of *Dirac spinors*:

$$\Psi = \psi \otimes \tilde{b} \in C^\infty(V^{\frac{1}{2}}). \tag{11.1.19}$$

According to the construction, this means a differential form of degree zero with values in the field $\mathbb{C}^N$ of complex numbers. In generalized Kaluza–Klein models, these n-dimensional Dirac spinors need to be restricted to the four-dimensional physical world. Due to the construction of D^{F}, one finds a "tower" of spin$-\frac{1}{2}$ fermion fields (see, e.g., KALINOWSKI 1984):

$$\hat{\Psi}_{|\widetilde{SO}_\circ(1,3)} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{[\frac{n}{4}]} \end{bmatrix}. \tag{11.1.20}$$

The G-equivalence principle demands that the coordinates ψ of the spinor bundle transform according to

$$\psi \to^{S^{-1}} \psi := S^{-1}\psi \tag{11.1.21}$$

with respect to an element

$$S(p) \in \mathcal{G}_S \approx C^\infty(\tilde{L}(M) \times_{\mathrm{Ad}} \widetilde{SO}_\circ(s, n-s))$$
$$\Lambda(S_{(p)}) = G(p) \in \mathcal{G}_p \tag{11.1.22}$$

of the group $\mathcal{G}_S$ of (local) spin gauge transformations. This group results from $\mathcal{G}_p$, a subgroup of the affine gauge group, by a formal inversion of the covering homomorphism (11.1.9) onto the pseudo-orthogonal group with local gauge transformations. These gauge transformations can be expanded in terms of the infinitesimal generators of the structure group $\widetilde{SO}_\circ(s, n-s)$; due to (11.1.10), this results in the following equivalent representation:

$$\begin{aligned} S(p) &= \exp \imath\ \theta_{\alpha\beta}(m)\Lambda^{-1}(L^{\alpha\beta}) = \exp \imath\ \theta_{\alpha\beta}(m)\sigma^{\alpha\beta} \\ &= \exp\left(-\frac{1}{4}\theta_{\alpha\beta}(m)[\gamma^\alpha, \gamma^\beta]\right). \end{aligned} \tag{11.1.23}$$

See also BJORKEN & DRELL (1964). This again determines the action of the spin transformations on Dirac's γ-matrices

$$S^{-1}(p)\gamma_\alpha S(p) = \gamma_\beta G_\alpha{}^\beta(p), \quad [G_\alpha{}^\beta(p)] \in \mathcal{G}_p. \tag{11.1.24}$$

The spin bundle $\tilde{L}(M)$ can, analogously to the bundle $L^g(M)$ of the orthonormal frames, be endowed with a right-invariant 1-form γ. It takes on values within the Clifford algebra and thus differs from the already familiar canonical 1-form ϑ. To be more precise, the form

$$\gamma = E^\alpha_{\cdot j}(m)\gamma_\alpha \otimes dx^j = \gamma_j dx^j \in C^\infty(T^*_{\mathbb{C}}(M) \times CV) \tag{11.1.25}$$

has to be considered as a cross section in the product bundle of the (complexified) cotangent bundle and the Clifford bundle CV. Moreover, on account of (11.1.25), γ appears to be “soldered” to the spin manifold. In order to be able to reckon with this Clifford-algebra-valued form, corresponding to the calculus of exterior forms, it is only to be kept in mind that the formation of the dual obeys the rule

$$^*(\underbrace{\gamma \wedge \cdots \wedge \gamma}_{p}) = \frac{p!}{(n-p)!}\gamma^{n+1}\underbrace{\gamma \wedge \cdots \wedge \gamma}_{n-p}\ . \tag{11.1.26}$$

The matrix γ^{n+1} that is thereby implicitly defined is an axial Lorentz scalar and can be understood as an element of a higher-dimensional Clifford algebra as well, due to

$$(\gamma^{n+1})^2 = \mp\mathbb{1}. \tag{11.1.27}$$

If one realizes the Dirac matrices in the "Cartan basis," then γ^{n+1} can be regarded as a Casimir operator, since it commutes with all elements of the Clifford algebra. The symmetrized product of the 1-form γ with itself and the formation of traces, due to (11.1.2), leads necessarily to the pseudo-Riemannian line element

$$\frac{1}{N}\mathrm{Tr}(\gamma \otimes_s \gamma) = ds^2 = g_{ij}(m)dx^i \otimes_s dx^j. \tag{11.1.28}$$

This can be looked upon as a generally covariant generalization (SCHRÖDINGER 1932) of the well-known anticommutation relations (11.1.2). For the formulation of the dynamics of the Dirac theory in curved spacetime in the next section, it is convenient to make use of the Dirac adjont spinor, which is defined by

$$\overline{\psi} := \psi^+\gamma^0, \tag{11.1.29}$$

as already noticed by BARGMANN (1932). Following the BJORKEN & DRELL conventions (1964), in addition the following rules are to be obeyed:

$$\overline{\sigma}^{\alpha\beta} := \gamma^0\sigma^{\alpha\beta+}\gamma^0 = \sigma^{\alpha\beta}, \quad \overline{\gamma}^{\alpha} := \gamma^0\gamma^{\alpha+}\gamma^0 = \gamma^{\alpha}. \tag{11.1.30}$$

The introduction of a $\widetilde{SO}_\circ(s, n-s)$-valued 1-form $\tilde{\omega}$ of the connection in the associated spinor bundle (11.1.16) enables us to define the gauge-covariant *spinor derivative*

$$D\psi = d\psi + i\widetilde{\omega}\psi, \tag{11.1.31}$$

which is necessary for the formulation of an interacting gauge-invariant theory.[5] The *spinor connection* $\tilde{\omega}$ was introduced by FOCK (1929); WEYL (1929b) and SCHRÖDINGER (1932), and was given its axiomatic foundation by LUEHR & ROSENBAUM (1974, 1984). Due to the covering homomorphism (11.1.9) or (11.1.10), the spinor connection is closely related via

$$\Lambda(i\widetilde{\omega}) = \omega^{\mathrm{g}} \tag{11.1.32}$$

to the metric-compatible connection ω^{g} in the bundle $L^{\mathrm{g}}(M)$ of orthogonal frames. This implies the following form of the local expansion:

$$\overset{*}{\sigma}\tilde{\omega} = \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta} = \frac{1}{2}\Gamma_i^{\cdot\alpha\beta}\sigma_{\alpha\beta} \otimes dx^i =: \Gamma_i dx^i. \tag{11.1.33}$$

The local components of the spin connection, i.e., Ricci's rotation coefficients $\Gamma_i^{\cdot\alpha\beta}$, can be generated from the holonomic connection coefficients $\Gamma_{ij}^{\cdot k}$ by means of translational gauge transformations, as shown in Chap. 4. These "tetrad gauge trans-

[5] In compliance with our previous conventions, we denote the covariant derivative in a vector bundle by the same symbol D, since it is always possible to infer from the field in question which representation of the structure group or its covering group is meant.

formations" cannot always be covered by a spinor gauge transformation (11.1.36) of $\tilde{\omega}$ in an unequivocal way. It is necessarily dependent on the topological properties of the bundle mapping f of the spin structure and can occasion globally different Lagrangian n-forms (ISHAM 1978).

With respect to the group $\mathscr{G}_S$ of local spin gauge transformations, these spinors transform according to the rules

$$\psi \longrightarrow {}^{S^{-1}}\psi = S^{-1}\psi \tag{11.1.34}$$

and

$$\overline{\psi} \longrightarrow {}^{S^{-1}}\overline{\psi} = \overline{\psi}S, \tag{11.1.35}$$

which are to be expected for cross sections of vector bundles, while more geometric objects, i.e., the connection $\tilde{\omega}$ and the covariant derivative D, turn, respectively, into

$$\tilde{\omega} \longrightarrow {}^{S}\tilde{\omega} = S\tilde{\omega}S^{-1} + (dS)S^{-1} \tag{11.1.36}$$

and,

$$D \longrightarrow {}^{S^{-1}}D = S^{-1}DS. \tag{11.1.37}$$

The transformation of the Clifford-algebra-valued "soldering form"

$$\gamma \longrightarrow {}^{S^{-1}}\gamma = S^{-1}\gamma S \tag{11.1.38}$$

can also be fitted into this canon.

If we had started from a spinor representation of the Poincaré group $\mathbb{R}^n \otimes SO_\circ(s, n-s)$, it would have been possible to extend this to a theory that is invariant with respect to "broken" *affine* spinor gauge transformations. That this is also possible with regard to the 1-form γ is shown via a comparison of the affine transformation rule with (11.1.24). And yet all these global considerations point to a Riemann–*Cartan* space as the appropriate geometric arena for spinor fields. And in fact, it will be shown that a (pseudo-) Riemannian manifold is too narrow a frame for the description of gravitationally coupled spinor fields.

11.2 Covariant Dirac Equation

A generalization of the DIRAC equation (1928) in curved spacetime was advanced at a rather early stage. Especially, the basic works of TETRODE (1928), WIGNER (1929), WEYL 1929b, FOCK 1929, SCHRÖDINGER (1932) can be mentioned here. It was Wigner, effectively influenced by EINSTEIN'S theory of teleparallelism (1928), who strived for an extension by inserting the tetrad fields $e_\alpha^{\cdot i}(m)$, i.e., the local bundle coordinates of $L^g(M)$, in the special-relativistic Dirac equation. However, it was TETRODE

(1928) and subsequently SCHRÖDINGER (1932) who pointed out that it was possible to start off from the generally covariant version (11.1.28) of the commutation relations (11.1.2) of the Dirac matrices γ^{α}. Obviously, it is our coordinate-independent formulation with the help of the Clifford-algebra-valued 1-form (11.1.25) that explains that these locally differing formulations describe globally the same mathematical structure. WEYL'S fundamental works of 1929 on the generally covariant formulation of Dirac's theory of the electron already make use of matrix-valued differential forms in full scale. However, his notation has grown unfamiliar by now. Although there were already hints by TETRODE (1928) concerning the necessity of formulating the Dirac equation not only as generally covariant, but *also* as gauge-invariant with respect to local spin transformations by means of introducing a covariant spinor derivative, this was recognized in full only by WEYL (1929a, b) and FOCK (1929) simultaneously.

Due to these initial works and the modern gauge-theoretic concepts that have been established in the meantime, it is not difficult to determine the precise form into which Dirac's theory has to be moulded. To this end, the gauge-invariant Lagrangian formalism has to be transferred to the case of a spin structure group. According to (11.1.19), the spinor field ψ is a 0-form. In order to construct an n-form out of this by means of the gauge-covariant spinor derivative D and the Clifford-algebra-valued 1-form γ, there is only the following way to achieve this within a first-order formalism:

$$L_{\mathrm{D}} = \frac{i}{2}\left(\overline{\psi}\gamma \wedge {}^{*}D\psi + (\overline{D\psi}) \wedge {}^{*}\gamma\psi\right) - m\overline{\psi}\psi\eta. \tag{11.2.1}$$

Since it will turn out to be useful later, we have chosen an invariant Lagrangian n-form that is symmetric with respect to the formation of the Dirac adjoint spinor; see WIGNER (1928); TRAUMAN (1972); ISHAM (1978). The invariance of (11.2.1) with respect to the group $\mathcal{D}(\mathrm{M})$ of differentiable coordinate transformations is again guaranteed by the occurrence of exterior differential forms. In addition, the transformation rules (11.1.34)–(11.1.38) guarantee that the G-equivalence principle is satisfied, in complete agreement with the idea of gauge invariance. It would be possible, of course, to construct spinor models containing derivatives of higher order, disregarding possible causality violations. However, (11.2.1) is the sole Lagrangian n-form that is of *first* order in ψ and is reducible to the special-relativistic Dirac theory in the interaction-free case.

By variation of (11.2.1) for $\delta L_{\mathrm{D}}/\delta\overline{\psi}$ or for $\delta L_{\mathrm{D}}/\delta\psi$, respectively, the generally covariant Dirac equation

$$i\gamma \wedge {}^{*}D\psi - \frac{i}{2}(D \wedge {}^{*}\gamma)\psi - m\psi\eta = 0 \tag{11.2.2}$$

and the corresponding adjoint equation

$$i(\overline{D\psi}) \wedge {}^{*}\gamma + \frac{i}{2}\overline{\psi}D \wedge {}^{*}\gamma + m\overline{\psi}\eta = 0, \tag{11.2.3}$$

which are gauge-covariant with respect to the structure group $\widetilde{SO}_{\circ}(s, n-s)$, are obtained. As can already be noticed here, these equations, which are valid in a Riemann–Cartan background space, contain an additional term that vanishes only for a covariant constant γ, i.e., for $D\gamma = 0$. However, the latter condition is globally admissible in a *flat* spacetime only.

Before we turn to an additional dynamic aspect of the generally covariant model given by (11.2.1), we have to consider the dynamical feedback of spinors on gauge fields in general. In order to work this out, it has to be kept in mind that the Dirac-adjoint connection

$$\bar{\tilde{\omega}} = \gamma^0 \tilde{\omega}^+ \gamma^0 = \tilde{\omega} \tag{11.2.4}$$

occurring in (11.2.1) cannot be distinguished—due to (11.1.30)—from a common spinor connection. Then the following equivalent description of the Lagrangian n-form of a Dirac field can be given:

$$L_{\mathrm{D}} = \frac{i}{2}(\overline{\psi}\gamma \wedge {}^*d\psi + d\overline{\psi} \wedge {}^*\gamma\psi) - m\overline{\psi}\psi\eta - \frac{1}{2}\overline{\psi}[\tilde{\omega}, {}^*\gamma]\psi, \tag{11.2.5}$$

using the rules concerning exterior forms. According to the definitions $\tau := \delta \mathrm{L}/\delta\omega$ of the canonically conjugate matter currents, the Dirac field is, under all conditions, retroactive not only on the gauge fields of Yang–Mills type by means of the 1-form $\tau = i\overline{\psi}\gamma\psi$ of its charge current, but also on the (Lorentz) rotational gauge fields of gravity by means of its spin current

$$\tau_s = i(\overline{\psi}^* \gamma \psi). \tag{11.2.6}$$

In the derivation of this well-known result, use is made of the fact that the covering homomorphism (11.1.9) is absorbed by Dirac's bilinear product. In case that the Euler–Lagrange equations (11.2.2) and (11.2.3) of the variational problem (11.2.1) are satisfied, it can be shown that the covariant conservation laws[6] are valid with respect to the current τ or τ_s, and this according to whether the "internal" Yang–Mills connection or the "external" spinor connection is used for the construction.

In order to get *locally* conserved currents, we have to add terms to τ_s depending on the gauge potentials. In the limiting case of interaction-free Dirac fields, even

[6] A corresponding conservation law is also valid for the axial vector current $j_{n+1} := i\overline{\psi}^*\gamma\gamma^{n+1}\psi$ of a Dirac field *without* mass. However, a quantum-theoretic treatment by means of a functional integral, due to an anomaly, results in (cf., for instance, Jackiw 1977)

$$\langle -\infty | D\tau_A | \infty \rangle^{\overset{(+)}{-}} = \frac{1}{(4.)8\pi^2}\mathrm{Tr}(\Omega^g \wedge \Omega^{g(*)}).$$

Thus in quantum field theory, the conservation of the axial current is violated by topological contributions of the Pontryagin class, and this in dependence on the asymptotic helicity states concerning γ^{n+1}. This is the physical content of the celebrated *Atiyah–Singer index theorem* (Römer 1981a, b; Eguchi et al. 1980).

$$d\tau_s = 0 \tag{11.2.7}$$

is valid, a fact that permits the introduction of the invariant scalar product

$$(\psi, \chi) := \int_{H^{n-1}} \tau_s(\psi, \chi) = i \int_{H^{n-1}} \psi^* \gamma \chi \tag{11.2.8}$$

into the solution space of the Dirac equation that will be spanned by ψ. For spacelike hypersurfaces $\mathrm{H}^{\mathrm{n}-1}$, this is not only independent of time, but even *positive definite*, and it reduces itself subsequently to

$$(\psi, \chi) = i \int_{H^{n-1}} \psi^+ \chi d^{n-1} x. \tag{11.2.9}$$

Nevertheless, one could not interpret τ_s^0 as the probability density in a non quantized theory, since the energy density of the *linear* Dirac field is not bounded from below. Thus, such a semiclassical model would necessarily imply the "instability and rapid decay of all matter" (JOST 1965, p. 39).

11.3 Nonlinear Heisenberg–Pauli–Weyl Spinor Equation

The introduction of spinor fields as cross sections in the associated spinor bundle with the group $\widetilde{SO}_\circ(s, n-s)$ or the covering group of the Poincaré group as a structure group, respectively, starts from a Riemann–Cartan base manifold together with its frame. In order to show that the spinor fields even necessarily induce a torsion of the spacetime, we decompose the (metric-compatible) spin connection

$$\widetilde{\omega} = \widetilde{\omega}^{\{\}} - \widetilde{K}, \quad \Lambda(i\widetilde{K}) = K \tag{11.3.1}$$

into the Christoffel-like connection $\tilde{\omega}^{\{\}}$ and the spin contortion that is projected by the covering homomorphism (11.1.9) onto the contortion, and insert this into the Lagrangian n-form (11.2.1). This splitting procures a further, equivalent, description of Dirac's Lagrangian n-form:

$$\begin{aligned} L_{\mathrm{D}} = & \frac{i}{2}\left(\overline{\psi}\gamma \wedge {}^*D^{\{\}}\psi + (\overline{D^{\{\}}\psi}) \wedge {}^*\gamma\psi\right) - m\overline{\psi}\psi\eta \\ & + \frac{1}{2}\overline{\psi}[\widetilde{K}, {}^*\gamma]\psi =: L_{\mathrm{D}}^{\{\}} + L_{\mathrm{NL}}. \end{aligned} \tag{11.3.2}$$

Here $\mathrm{D}^{\{\}}$ denotes—similarly as in HEHL et al. (1976)—the covariant derivative with respect to a merely *(pseudo-) Riemannian* spinor connection. It was WEYL who in his work in 1950, noted that an additional term, depending on the contortion, necessarily occurs in RC spacetime. Considering the resolution of the relation $\Theta = [K, \vartheta]$ for the contortion, this additional term, which would not be discernible in the special-relativistic formulation, can also attain the form

$$L_{\mathrm{NL}} = \frac{1}{2}\overline{\psi}[\widetilde{K}, {}^*\gamma]\psi = -3i^*[\Theta, \vartheta]\wedge(\overline{\psi}^*\gamma\psi) \\ = -3\tau_s \wedge [\Theta, \vartheta]. \tag{11.3.3}$$

For the time being, it is to be taken for granted that the Dirac field interacts with the Einstein–Cartan theory of gravity, as was proposed by WEYL (1929a, b, 1950). Then it follows that the modified torsion $[\Theta, \vartheta]$ couples *algebraically* to the canonical spin current of the Dirac field such that Cartan's torsion equation reads

$$[\Theta, \vartheta] = \ell^{*2}\, \tau_s. \tag{11.3.4}$$

This relation can again be used to eliminate the torsion-dependent term (11.3.3) in (11.3.2). With the result that a spin–spin interaction corresponding to a quadratic axial vector coupling necessarily occurs in the Lagrangian n-form (11.3.2) of the Dirac field:

$$L_{\mathrm{NL}} = -3\ell^{*2}\tau_s \wedge {}^*\tau_s = 3\ell^{*2}\overline{\psi}^*\gamma\psi \wedge {}^*(\overline{\psi}^*\gamma\psi) \\ = 3\ell^{*2}\overline{\psi}\gamma^{n+1}\gamma\psi \wedge {}^*(\overline{\psi}\gamma^{n+1}\gamma\psi). \tag{11.3.5}$$

This physical effect, however, is not necessarily restricted to a coupling of the Dirac field to the Einstein–Cartan theory. In a generic Poincaré gauge theory, torsion acquires a more dynamical status according to the second field equation and may propagate. However, the *algebraic* relation of the antisymmetric torsion to the spin current τ_s of the Dirac field is maintained for those dynamic field configurations that satisfy double-duality conditions even in the presence of external sources. In case of quasilinear translational gauge field momenta, the resulting equation can be solved for the antisymmetric torsion $[\Theta, \vartheta]$. This leads back to an algebraic relation like (11.3.4), however, with a differing constant of proportionality corresponding to the model in question.

As an important result, we thus can ascertain:

Spacetime that is twisted ("subject to torsional stress") by the half-integer spin of a particle induces a *quadratic self-interaction* of the spinor fields, and this in a geometrically natural way.

Equivalently (NESTER 1977), this means also that our interpretation draws us back onto the (pseudo-) Riemannian spacetime of GR apart from *"nonminimal"* coupled sources in the field equations. Thus the spinor equation

$$\boxed{i\gamma \wedge {}^*D^{\{\}}\psi - 6\ell^{*2}\gamma^{n+1}\gamma \wedge {}^*(\overline{\psi}\gamma^{n+1}\gamma\psi)\psi - m\psi\eta = 0}, \tag{11.3.6}$$

which is valid in a Riemannian spacetime, is obtained by the variation of (11.3.2) for $\delta L_D/\delta\overline{\psi}$. This equation will be referred to as the *nonlinear Heisenberg–Pauli–Weyl spinor equation* in the following and reads in a more familiar local representation with $D = \vartheta^\alpha \nabla_\alpha$ taken into considerations, as follows:

$$\left\{ i\gamma^\kappa \nabla_\kappa^{\{\}} - \frac{6}{n^2}\ell^{*2}\gamma^{n+1}\gamma^\mu \overline{\psi}\gamma^{n+1}\gamma_\mu \psi - m \right\} \psi = 0. \tag{11.3.7}$$

The fact that the coupling[7] of Dirac's Lagrangian n-form to the Einstein–Cartan theory produces a quartic self-interaction, in contrast to the purely Riemannian ("metric") theory, was discovered by WEYL.

In a strict sense, Weyl noticed this occurrence of the conceptually important intrinsic *nonlinearity* of the Dirac fields in an embryonic form in his famous work of 1929 (note his additional term m'; WEYL 1929b, p. 329). Only a little later, it was IVANENKO (1938, 1957) who, for quantum-theoretic reasons, pointed out the possibility of nonlinear self-interactions.

Facing the permanently expanding spectrum of experimentally observed excited states of hadrons and leptons, Heisenberg and Pauli felt seriously compelled to think of a universal nonlinear equation to describe the basic structure of matter. This equation is constructed on spinors and is meant to do justice to the fundamental symmetries that are given by the Lorentz transformations, the dilations, the combined space reflection and charge conjugation PC, and the time reflection T. To be added is the nearly exact independence of the nuclear forces from the charge. In order to represent at least the proton and the neutron in his isospin formalism (HEISENBERG 1932), charge independence necessitates the invariance concerning an additional "internal" SU(2) group. In order to obtain a nontrivial mass spectrum as well out of a single fundamental equation for the isotopic doublet $\psi := \begin{pmatrix} \psi_P \\ \psi_N \end{pmatrix}$, this equation had to be of an essentially nonlinear character: all these postulates lead to an equation of (11.3.7)-type but possibly without a mass term (HEISENBERG 1957).

The initial geometric arguments for an introduction of a fundamental nonlinearity into the Dirac equation that were put forward by Weyl have been taken up and refined by quite a number of scientists, including GÜRSEY (1957), RODICHEV (1961), FINKELSTEIN (1960), FINKELSTEIN & RAMSAY (1962), PERES (1962), BRAUNSS (1964, 1965), HEHL & DATTA (1971), DATTA (1971), DÜRR (1973) and lastly HEHL (1974). The proposition of "internal" symmetries of the particles implies the transition to multicomponent bispinor fields; and nonlinear coupled spinor models of this kind have been investigated, apart from HEISENBERG (1967) in his unified field theory of elementary particles, by FINKELSTEIN (1961a, b); TAKAHASHI (1979a) and RAÑADA & RAÑADA (1983). More generally, it can be shown that an extension of the "internal" local SU(f) symmetry, taken together with the "external" Lorentz gauge group $SL(2, \mathbb{C})$ of Weyl to a gauge theory with $\overline{\mathrm{G}} = SL(2\mathrm{f}, \mathbb{C})$ as a structure group, results in a $\overline{\mathrm{G}}$-invariant nonlinear Heisenberg–Pauli–Weyl spinor equation (MIELKE 1977b, 1981a). The nonlinear coupling of the single spinor fields is thereby induced by a generalized "isotorsion" of the total space. And yet it is still a highly speculative

[7] At an earlier stage, this was described as a "mixed" theory, since the affine connection and the tetrad field occur as independent variables in the variation procedure.

question whether the fundamental Dirac fields that occur can be identified either with the proton and the neutron, as in Heisenberg's isospin formalism, or with the hypothetical, additionally "color"-carrying quarks—see, for instance, TAKAHASHI (1979a), RAÑADA & RAÑADA (1984)—or with the absolutely stable fermions, such as proton, electron, or neutrino, as was proposed, for instance, by BARUT (1980).

11.4 Solitons

As it is, the physical description of nature rests throughout on linear mathematical models that have been developed along with the historical evolution of mathematics and philosophy of science. However, it has to be admitted that the assumption of a basic linearity is nothing but a model-like approximation if we are ready to accept that the world of phenomena is in general of a *non*linear character. Exempted from this rule seems to be quantum mechanics, since the principle *of linear superposition of* the state functions necessarily belongs to the axiomatic foundation of the theory.

In the physics of the continuum, which is to be described by means of wave equations, an initially localizable wave packet is known to dissociate in a weakly dissipative medium due to the supposition of linearity. A substantially different effect, however, commonly occurs in the solution manifold of nonlinear wave equations: solitary waves maintain the shape they are in, and this not only in the case of disturbances but even after collisions. More than 150 years ago, this phenomenon was observed empirically by SCOTT-RUSSEL (1844) concerning one-dimensional waves propagating in a channel. Theoretically, the existence of such *soliton* solutions was first asserted by KORTEWEG & DE VRIES in 1895. They drew this conclusion while reflecting on the equations of hydrodynamics. The dispersion of these solitons "proper," which occur only in a 2-dimensional spacetime[8] with this absolute stability, is prevented by topological conservation laws. Roughly speaking, the theoretical explanation for this phenomenon is comparable with that which explains why a knot in a closed ribbon is unknottable (in this context see also FINKELSTEIN & RUBINSTEIN 1968).

Such an absolute stability of localized solutions of a nonlinear wave equation is no longer guaranteed, however, in systems that are physically more realistic and not completely integrable, lacking conservation laws originating from global topology (MAKHANKOV 1978). However, the spontaneous dispersion of such "solitons" may possibly be prevented by a quantization of the total charge (KUMAR et al. 1979). It has already been suggested, in 1951 by FINKELSTEIN et al., to amplify the stability of *spinor solitons* by means of *charge quantization*. With reference to a classical spinor model for the confinement of quarks, this idea was again resumed by FRIEDBERG & LEE (1977).

[8]The Heisenberg–Pauli–Weyl spinor equation within two dimensions serves as the starting point for the quantum-field-theoretic THIRRING model (1958). This equation has solutions not only exhibiting quark confinement (CHANG et al. 1975; HORTAÇSU 1977), but also with particle characteristics (YAMAMOTO 1977; RAÑADA & RAÑADA 1984).

Facing the structural richness of the nonlinear models, it is not astonishing that there have been considerations concerning the construction of a classical model of extended particles at very early stages in time. The main concern of these early investigations was the construction of a model that made it possible to concentrate the energy, the charge, and other essential physical characteristics of extended particles in a spacelike domain *without* having the notorious *field singularities* of linear theories.

First attempts concerning such "unitarian"[9] field theories are closely connected with the work of MIE (1912, 1913). By giving up the linearity of Maxwell's equations, the door was open not only for a forthcoming field theory of the electron but also for that of matter. Thus it was possible for Weyl to answer the question, "What is matter?" as follows: matter is an energy knot[10] of the (electromagnetic) field with quantized physical properties. "Concerning the energy or the inertial mass of a compounded body …" Weyl sees in the latter the reason "why we have to put the insoluble energy of its finally material elementary parts and the soluble energy of their mutual binding as opposites."[11]

Mie's concept of electrodynamics, however, had the decisive disadvantage that it was not gauge-invariant. This deficiency was coped with by BORN (1934) and BORN & INFELD (1934, 1935), while they were developing a nonlinear concept of electrodynamics that does full justice to the demands of an up-to-date standard concerning gauge invariance, if one proceeds on to a nonabelian structure group. In a specific model, the electric field strength is determined by the nonsingular function

$$E_r = \sqrt{\alpha} r_\circ^2 \left[1 + \left(\frac{r}{r_\circ} \right)^4 \right]^{-1/2} \sim \sqrt{\alpha} \frac{1}{r^2} \qquad (11.4.1)$$

and thus replaces Coulomb's $1/r^2$-dependence on the distance $r := |\vec{\mathbf{x}}|$ from the center of the charge. This result implies the germ of the soliton concept in its *relativistic, 4-dimensional* shape.

Under these aspects, nonlinear coupled scalar fields were investigated by ROSEN (1939), MENIUS JR & ROSEN (1942) and ROSE & FURRY (1961), G. ROSEN (1965). However, the issue of stability of localized solutions remains an open problem despite the work of DERRICK & KAY–KONG (1968). The existence of stable particle-like solutions has been proved for a specific model (ANDERSON 1971), which results formally from "squaring" the Heisenberg–Pauli–Weyl spinor equation (11.3.7); cf. VÁZQUEZ (1977), DEPPERT & MIELKE (1979). As for particle physics, spinor models are of basic relevance as well with respect to the possible construction of bosons

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[9] As far as we know, this term occurs for the first time in KALUZA (1921), although in a differing context. Theoretically, it was coined by BORN & INFELD (1934, 1935) and, as far as a quantum-theoretic model is concerned, by FINKELSTEIN (1949).

[10] According to the highly speculative notions of JEHLE (1977, 1981), this should be asserted almost literally.

[11] …ferner der Grund, warum wir an der Energie oder trägen Masse eines zusammengesetzten Körpers die nicht auflösbare Energie seiner letzten materiellen Elementarbestandteile der auflösbaren Energie ihrer wechselseitigen Bindung gegenüberstellen." (WEYL 1924, p. 592).

out of fermions. For a class of nonlinear spinor theories comprising (11.3.7), localized solutions not only have been constructed by means of numerical methods but also have been scrutinized for their physical interpretation by FINKELSTEIN et al. in 1951 and 1956. The stability of such particle-like solutions is still much under debate; cf. ALVAREZ & SOLER (1983), MATHIEU & MORRIS (1985).

11.4.1 Soliton-Type Solutions of the Nonlinear Dirac Equation

Within the framework of our more geometrically oriented notions, a possible role of gravitation[12] in the physics of particles should not be discarded. This is why we resume these studies here, however with a consideration of the spacetime curvature (MIELKE 1981b). It is our intention to analyze a spinor system that is bound by a nonlinear self-interaction in a static spherically symmetric curved background, in order to demonstrate the main semiclassical properties of such spinor solitons. Although mathematically more complicated, we are going to consider the dynamical model that is given by (11.3.7) exclusively in the physically(!) relevant case of $n = 4$ dimensions. If the spinors are identical ones, the following equivalent representation for the self-interaction (11.3.5) between the axial vector currents is valid (FINKELSTEIN et al. 1956):

$$\begin{aligned}(\overline{\psi} i\gamma^5\gamma_\mu\psi)(\overline{\psi} i\gamma^5\gamma^\mu\psi) &= (\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)\\ &= (\overline{\psi}\,\psi)^2 - (\overline{\psi}\gamma^5\psi)^2.\end{aligned} \tag{11.4.2}$$

It is therefore justified to choose the nonlinear Dirac equation

$$\left\{ i\gamma^k\nabla_k^{\{\}} + \frac{3\varepsilon}{8}\ell^2(\overline{\psi}\psi - a\overline{\psi}\gamma^5\psi\gamma^5) - m \right\}\psi = 0 \tag{11.4.3}$$

with an algebraically simplified self-interaction, instead of (11.3.7), as a basis for the following analysis. On the other hand, the constants $\varepsilon = \pm 1$, ℓ, and a are to be understood as arbitrary ones in the nonlinear term.

Within the framework of the following explicit calculations, we shall therefore concentrate on a one-particle system. The spinor fields within this system are bound by a static spherically symmetric external field exactly as is assumed in the case of the

[12] Hermann Weyl's still valid admonitory statement is to be remembered here:

> The formulation of Dirac's theory of the electron in the frame of general relativity has to its credit one feature that should be appreciated even by the atomic physicist who feels safe in ignoring the role of gravitation in the building up of the elementary particles: its explanation of the quantum-mechanical principle of "gauge invariance" that connects Dirac's ψ with the electromagnetic potentials

(WEYL 1950).

hydrogen atom (Bjorken & Drell 1964). A spherically symmetric gravitational field occurs here, possibly even a very strong one, instead of the Coulomb potential. As is well known, such a metric background can be described in isotropic coordinates by the line element

$$\begin{aligned} ds^2 &= g_{kl} dx^k \otimes_s dx^l \\ &= e^{\nu} dt^2 - e^{\mu}(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\phi^2). \end{aligned} \tag{11.4.4}$$

Concerning *static* spinor fields, the functions $\nu = \nu(\rho)$ and $\mu = \mu(\rho)$ depend only on the dimensionless radial coordinate

$$\rho := \frac{\sqrt{2\pi}}{\ell^*} r, \tag{11.4.5}$$

and the metric becomes static itself. Thereby the spherical coordinates r $:= |\vec{\mathbf{x}}|$, ϑ, and ϕ are related to the Cartesian coordinates $\vec{\mathbf{x}}$ in the familiar manner. The search for explicit solutions of the generally covariant Dirac equation on this metric background can be simplified by conformally relating the components g_{kl} of the metric with those of a different one by means of

$$\overline{g}_{kl} \longleftarrow g_{kl} = e^{\mu} \overline{g}_{kl}. \tag{11.4.6}$$

For the tetrads, the corresponding relation

$$\overline{E}^{\alpha}_{\cdot k} \longleftarrow E^{\alpha}_{\cdot k} = e^{\mu/2} \overline{E}^{\alpha}_{\cdot k} \tag{11.4.7}$$

can be asserted. Concerning the change of metric (11.4.6), the Christoffel symbols are related by

$$\left\{ \begin{matrix} \alpha \\ kl \end{matrix} \right\} = \overline{\left\{ \begin{matrix} \alpha \\ kl \end{matrix} \right\}} + \frac{1}{2} \left(\delta^{\alpha}_{k} \partial_l + \delta^{\alpha}_{l} \partial_k - g_{kl} \partial^{\alpha} \right) \mu \tag{11.4.8}$$

(see, for instance, Mielke 1977a). Moreover, if we are considering the relation between the Christoffel symbols and the torsion-free rotation coefficients $\Gamma_{\kappa}^{\{\}\cdot\alpha\beta}$, the local components of the spinor connection (11.1.33) assume the form

$$\Gamma_k = \overline{\Gamma}_k - \frac{1}{4} \delta_k^{[\alpha} \partial^{\beta]} \mu \sigma_{\alpha\beta}. \tag{11.4.9}$$

As an interesting interim result, one can write down the nonlinear *Dirac equation* (11.4.3) in a *conformally related* (pseudo-) Riemannian spacetime as

$$\left\{ i\overline{\gamma}^k(\partial_k + i\overline{\Gamma}_k + \frac{3}{4}\partial_k\mu) + \frac{3\varepsilon}{8}\ell^2 e^{\mu/2}(\overline{\psi}\,\psi - \mathrm{a}\overline{\psi}\gamma^5\psi\gamma^5) - e^{\mu/2}m \right\}\psi = 0. \tag{11.4.10}$$

As a noteworthy example, we may apply this result to the de Sitter space with radius $R_\circ$. For this purpose, the de Sitter space is to be covered with the so-called horospherical coordinate system $(\lambda, \mathbf{y})$. Concerning these coordinates, the line element of the de Sitter space reads

$$ds^2 = \frac{1}{\lambda^2}\left(R_\circ^2 d\lambda^2 - \mathrm{o}_{ij}dy^i \otimes_s dy^j\right). \tag{11.4.11}$$

The *linear* Dirac equation

$$\left\{ i\gamma^k\partial_k - \frac{3i}{2R_\circ\lambda}\gamma^0 - \frac{m}{\lambda}\right\}\psi = 0, \tag{11.4.12}$$

which results from (11.4.10) by dropping cubic terms, is completely identical to the spinor equation that emerges from the method of the *induced representation* of the de Sitter group $SO_\circ(1,4)$ (see MIELKE (1977c), Eqs. (1.23) and (4.5) in the case of higher spins). Quantum-field-theoretic aspects of nonlinear Dirac equations within such spaces of maximum symmetry have been analyzed by BÖRNER & DÜRR (1970).

For the sake of completeness, the conformally related component $\overline{\Gamma}_k$ of the spinor connection is given for the general background space (11.4.4). A comparison with the results that have been obtained in a similar case by BRILL & WHEELER (1957, Eq. (30)) indicates that

$$\begin{aligned} \overline{\Gamma}_0 &= \frac{i}{4}e^{(\nu-\mu)/2}\partial_r(\nu-\mu)\gamma_0\gamma_1 \\ \overline{\Gamma}_1 &= 0, \quad \overline{\Gamma}_2 = \frac{i}{2}\gamma_2\gamma_1 \\ \overline{\Gamma}_3 &= \frac{i}{2}(\sin\vartheta\,\gamma_3\gamma_1 + \cos\vartheta\,\gamma_3\gamma_2) \end{aligned} \tag{11.4.13}$$

is valid. If this is inserted in (11.4.10), then

$$\left\{ i\gamma^0\partial_0 - e^{(\nu-\mu)/2}\left[i\vec{\gamma}\cdot\vec{\partial} + i\gamma^1\partial_r\left(\frac{\mu}{2}+\frac{\nu}{4}\right)\right] + \frac{3\varepsilon}{8}\ell^2 e^{\nu/2}(\overline{\psi}\psi - \mathrm{a}\overline{\psi}\gamma^5\psi\gamma^5) - e^{\nu/2}m\right\}\psi = 0 \tag{11.4.14}$$

follows. For the construction of solutions, it is rather convenient to think of the flat spacelike Dirac operator $i\vec{\gamma}\cdot\vec{\partial}$ as being written again in Cartesian coordinates. It is remarkable that the curved metric background occurs in the conformal spinor

Eq. (11.4.14) only in a multiplicative way except for the third term, whose contribution, however, can be absorbed by the additional implementation of the factor $\exp(-\mu/2 - \nu/4)$, as stated, into the separation ansatz

$$\psi = \frac{4}{\ell}\left(\frac{2\pi m}{3}\right)^{1/2} e^{-\mu/2-\nu/4-i\omega mt} \begin{bmatrix} iH(\rho) & x_\kappa^m \\ F(\rho) & \frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|} x_\kappa^m \end{bmatrix}. \tag{11.4.15}$$

Following the notation of Rose (1961), the spin-weighted spherical harmonics χ_κ^m of parity $\mathrm{P} = (-1)^l$ are given by

$$\chi_\kappa^m = \sum_{\overline{m}=\pm 1/2} C\left(l\frac{1}{2}j; m-\overline{m}, \overline{m}\right) Y_l^{m-\overline{m}}(\vartheta, \phi)\chi^{\overline{m}}. \tag{11.4.16}$$

Here the parameter κ is related to the quantum numbers j and l of the total angular momentum and the orbital angular momentum, respectively, by

$$\kappa = \mp(j + \frac{1}{2}) \quad \text{for} \quad j = l \pm \frac{1}{2}. \tag{11.4.17}$$

Here $\mathrm{C}(l\frac{1}{2}\mathrm{j}; \mathrm{m}\text{-}\overline{\mathrm{m}}, \overline{\mathrm{m}})$ denote the Clebsch–Gordan coefficients of the rotation group SU(2), and $\mathrm{Y}_l^m(\vartheta, \phi)$ are the spherical harmonics. Concerning the operators $\vec{J}$ and $\vec{L}$ of the total spin and the orbital angular momentum, these spinors satisfy the eigenvalue equations

$$\vec{J}^2\chi_\kappa^m = j(j+1)\chi_\kappa^m \tag{11.4.18}$$

and

$$\vec{\sigma}\cdot\vec{L}\chi_\kappa^m = -(\kappa+1)\chi_\kappa^m. \tag{11.4.19}$$

The helicity acts as a projection operator

$$\frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\chi_\kappa^m = -\chi_{-\kappa}^m \tag{11.4.20}$$

onto these states. The action of the spacelike part of the Dirac operator on the bispinor is also known:

$$i\gamma^0\vec{\gamma}\cdot\vec{\partial}\begin{bmatrix} iH(\rho) & \chi_\kappa^m \\ -F(\rho) & \chi_{-\kappa}^m \end{bmatrix} = \frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\left(\partial_r - \frac{\vec{\sigma}\cdot\vec{L}}{r}\right)\begin{bmatrix} iF(\rho) & \chi_{-\kappa}^m \\ H(\rho) & \chi_\kappa^m \end{bmatrix}. \tag{11.4.21}$$

What remains is the determination of the explicit shape of the self-interaction after the enactment of the ansatz (11.4.15). Similarly as in FINKELSTEIN et al. (1951, 1956), the formulas

$$\frac{3}{8}\ell^2\overline{\psi}\psi = 4\pi m e^{-\mu-\nu/2}(H^2 - F^2)\left|Y^{|\kappa|-1}_{|\kappa|-1}(\vartheta,\phi)\right|^2 \tag{11.4.22}$$

and

$$\frac{3}{8}\ell^2\overline{\psi}\gamma^5\psi = 4\pi m e^{-\mu-\nu/2}\frac{2}{3}HF\left|Y^{|\kappa|-1}_{|\kappa|-1}(\vartheta,\phi)\right|^2 \tag{11.4.23}$$

come into existence. In order to make possible the separation of the partial differential equations (11.4.14) into angularly and radially dependent parts, it is necessary that the self-interaction "potentials" (11.4.22) and (11.4.23) are spherically symmetric. It follows from the properties of the spherical harmonics that this is the case for $|\kappa| = 1$ only. It has to be considered, however, that then the quantum numbers for the total spin and the orbital angular momentum, like that of the magnetic moment, are restricted to the values $j = \frac{1}{2}$, $l = 0, 1$, and $\mathrm{m} = \pm\frac{1}{2}$, respectively. States of higher quantum numbers would occur if the ansatz (11.4.15) were directly inserted into the action functional $\int L_{\mathrm{D}'}$ belonging to (11.4.3). After integrating over the angular variables (FINKELSTEIN et al. 1956), only the radially dependent Euler–Lagrange equation remains to be solved. This averaging procedure over the angular distribution of the soliton solutions would make it possible to activate states of higher excitations. Since both procedures lead only to equivalent results for $|\kappa| = 1$, we restrict ourselves here to the investigation of the "ground state" of a soliton with spin.

If the separation ansatz (11.4.15) is inserted into the nonlinear Dirac equation, the following system of first-order partial differential equations for both of the radial functions is generated:

$$\partial_{\rho^*}H + \frac{1+\kappa}{\rho}e^{(\nu-\mu)/2}H = \frac{1}{\beta}\left[\omega + e^{\nu/2} - \varepsilon e^{-\mu}\left(H^2 - F^2 - \frac{2}{3}\mathrm{a}H^2\right)\right]F, \tag{11.4.24}$$

$$\partial_{\rho^*}F + \frac{1-\kappa}{\rho}e^{(\nu-\mu)/2}F = \frac{1}{\beta}\left[-\omega + e^{\nu/2} - \varepsilon e^{-\mu}\left(H^2 - F^2 + \frac{2}{3}\mathrm{a}F^2\right)\right]H. \tag{11.4.25}$$

Although it is not absolutely necessary in our case, it is frequently convenient to proceed to the tortoise coordinate of WHEELER (1955), which is implicitly defined by the Pfaffian form

$$d\rho^* := e^{(\mu-\nu)/2}d\rho. \tag{11.4.26}$$

Concerning background metrics with horizons as given, for instance, by the Schwarzschild solution, an utmost slow tending of ρ^* to the "coordinate singularity" is achieved. It is then to be considered as being displaced to negative infinity. In our calculations, the contributions of the "naked" fermion mass m to the so-called Planck mass M^* of the (possibly "strong") gravitational field is absorbed in the coefficients

$$\beta = \frac{M^*}{2m}, \quad M^* := \frac{\sqrt{8\pi}\hbar}{c\ell^*}. \tag{11.4.27}$$

Already in the case of flat spacetime, a computer-guided analysis is necessary. To some extent, this has already been done, not only by FINKELSTEIN et al. (1951, 1956), but as well by SOLER (1970), RAÑADA (1978), TAKAHASHI (1979a, b), and RAÑADA & RAÑADA (1984). And it has resulted in very interesting findings.

Here it may be sufficient to work out only some of the characteristic properties of *localized* and regular solutions. For a soliton with total spin $j = \frac{1}{2}$, i.e., for $\kappa = -1$, let us therefore consider the *asymptotic* solutions for the flat Minkowski space in a neighborhood of spacelike infinity. For such solutions, the quadratic terms can be neglected, and we get asymptotically the following linear system:

$$H' \simeq \frac{1+\omega}{\beta} F, \tag{11.4.28}$$

$$F' + \frac{2}{\rho} F \simeq \frac{1-\omega}{\beta} H. \tag{11.4.29}$$

Out of both equations, in which the derivative has to be taken with respect to ρ, results the differential equation

$$F'' + \frac{2}{\rho} F' - \frac{2}{\rho^2} F \simeq \frac{1-\omega^2}{\beta^2} F \tag{11.4.30}$$

with a regular singularity at the origin. The method of Frobenius allows to seek

$$F \sim C_\infty \left(\frac{1}{\overline{\rho}} + \frac{1}{\overline{\rho}^2} \right) e^{-\overline{\rho}} \tag{11.4.31}$$

and

$$H \sim C_\infty \sqrt{\frac{1+\omega}{1-\omega}} \frac{1}{\overline{\rho}} e^{-\overline{\rho}} \tag{11.4.32}$$

as approximate solutions. For the sake of convenience, we have introduced

$$\overline{\rho} := \frac{1}{\beta} \sqrt{1-\omega^2} \rho \tag{11.4.33}$$

as our new radial variable.

For a soliton with total spin $j = \frac{1}{2}$ and $\kappa = -1$, the radial functions $H(\rho)$ in the ansatz (11.4.15) account for a spherically symmetric distribution with $l = 0$, while according to (11.4.17), the quantum number $l = 1$ of angular momentum can be ascribed to $F(\rho)$. Let us compare this with the situation in a nonlocal version of the

$|\varphi|^6$-model of ROSEN (1965). With reference to an expansion in terms of spherical harmonics $Y_l^m(\vartheta, \phi)$, the occurring *exact* radial solutions

$$f^l(\rho) = \overline{\rho}^l(1 + \overline{e}^{4l+2})^{-1/2} \tag{11.4.34}$$

(MIELKE 1978, 1979) also depend on the quantum number l of orbital angular momentum. A comparison with (11.4.31) and (11.4.32) suggests that we consider the combinations

$$F(\rho) \simeq C_0 f^1(\overline{\rho}) \frac{1}{\cosh \overline{\rho}} \tag{11.4.35}$$

and

$$H(\rho) \simeq C_0 \sqrt{\frac{1+\omega}{1-\omega}} f^0(\overline{\rho}) \frac{1}{\cosh \overline{\rho}} \tag{11.4.36}$$

as fit accommodations to the asymptotic solutions of the radial Dirac equation, the more so, since these functions are a satisfactory approximation of the system even at the origin, provided that the initial increase of $F(\rho)$ is determined to be

$$C_0 = (1-\omega)\sqrt{\frac{-2}{\varepsilon(\omega+1)}} \tag{11.4.37}$$

The qualitative concordance of our approximations with the solutions of minimum energy (RAÑADA 1978; see also GARCIA & RAÑADA 1980), which were obtained by numerical methods, is clearly documented in Fig. 11.1. Our trial functions could

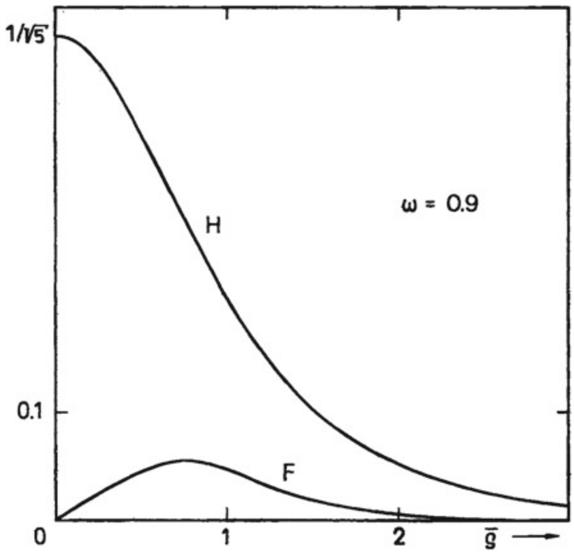

Fig. 11.1 Approximate radial solutions of a nonlinear Dirac equation

serve as a point of departure for a determination of the "energy eigenvalues," i.e., the minima of the field energy of a spinor soliton, according to the Ritz procedure of approximation (FLÜGGE 1971). The obligatory exponential decrease of the radial functions is sufficient for the localization of the resulting spinor solitons. This is clearly shown in the case that the parameter ω appearing in the phase of the ansatz (11.4.15) satisfies $|\omega| < 1$. On the other hand, it is known (VÁZQUEZ 1977) that bounded and square-integrable radial solutions *do not* exist for $|\omega| > 1$.

For stationary configurations with a localized field distribution centered at the origin, it seems to be sensible to define the effective *soliton radius* by

$$r_s \equiv \langle|\vec{x}|^2\rangle_\psi^{1/2} := \left(\frac{\int_{H^3} |\vec{x}|^2 \overline{\psi}^* \gamma\psi}{\int_{H^3} \overline{\psi}^* \gamma\psi}\right)^{1/2}. \tag{11.4.38}$$

Concerning multisoliton systems, it is possible to equate the nonlinear superposition of two such solitons in its effect on the physical action of a Yukawa-type potential $|\psi| \sim e^{-m_\pi r}$ between the centers of charge of the solitons. This is the case if the mutual distance of these centers is larger than the sum of both radii, which is to be calculated according to (11.4.38); cf. ROSEN & ROSENSTOCK (1952). This appears to be a very interesting property of the classical nonlinear field theory, since the occurrence of such potentials is usually considered as a "virtual" exchange of π-mesons in particle physics. In our case, their mass would have to be related via

$$m_\pi = \sqrt{1-\omega^2}\ m \tag{11.4.39}$$

to the "bare" fermion mass m and the parameter ω. With SALAM (1977), this can be explained with the immersion of the "naked" spinor waves into the self-interacting phase within the solitons, and this in the sense of an "Archimedes-type" effect.

The total mass m_s of a spinor soliton is obtained by the integration

$$m_s = \frac{1}{c}\int_{H^3} \Sigma_{\mathrm{D}}^0 \tag{11.4.40}$$

of the canonical energy momentum current Σ_{D} over a spacelike hypersurface H^3. Here the term that is dependent on the Lagrangian 4-form L_{D} can be omitted if ψ is a solution of the Dirac equation (JOST 1965; BRILL & WHEELER 1957). Nevertheless, the definition (11.4.40) indirectly contains contributions of nonlinear self-interacting potentials for a spinor soliton. Due to (11.4.15), it is typical that the soliton mass (11.4.40) is inversely proportional to the "fundamental" length ℓ^* and that m_s would diverge for a vanishing coupling constant of the self-interaction.

11.4.2 Mass Spectrum

Following the preceding studies of FINKELSTEIN et al. (1951, 1956) and SOLER (1970), which were mainly concerned with a model (11.4.3) with $a = 0$, RAÑADA (1978) focused his interest on the "mass spectrum" of the Heisenberg–Pauli–Weyl equation (11.3.7), i.e., on the Eq. (11.4.3) with $a = 1$ but with the opposite sign $\varepsilon = +1$. For solitons corresponding to an $S_{1/2}$-bound state with normalized charge $Q = e$, only nodeless radial solutions whose total rest mass shows an infinite spectrum of local minima have been found. Most interesting is the fact that the three lowest "eigenvalues" can be related rather well to the rest masses of exited nucleon states if a corresponding scaling has been carried out in advance. GARCIA & RAÑADA (1980), while considering an additional coupling to a pseudoscalar meson field, worked out these findings to a classical model of nucleons that accounts approximately for their empirically provable properties such as charge, spin, magnetic moment, and the electric or magnetic acting radii.

According to today's generally accepted notions, however, hadrons are thought to be built up by more fundamental *pointlike* fermions, the so-called quarks (GELL- MANN & NE'EMAN 1964; cf. KOKKEDEE 1969). Since up to now, these hypothetical construction elements have never occurred as free particles, it is necessary for an acceptable theory of strong interactions to guarantee their permanent *confinement* in hadrons. In quantum chromodynamics (QCD, MARCIANO & PAGELS 1978; JOOS 1979), this is thought to be achieved by a confining phase, i.e., as if the quarks were enclosed by a "bag" (CHODOS et al. 1974). Most remarkably, the underlying geometric ideas refer back to EINSTEIN (1919) and DIRAC (1962). Another solution of this problem can be obtained within the framework of nonlinear spinor theories if one starts off from *several* coupled Dirac fields. Let us assume that each of these fields represents one quark. It is to be expected, then, due to the nonlinear interactions of each with itself and among one another, that as a "ground state," a bound state of a multisoliton system comes into existence, which then could be considered a semiclassical model of a hadron. This possibility has been analyzed for a related scalar case (DEPPERT & MIELKE 1979). For nonlinear spinor fields, such a binding occurs only for two or three quarks if one follows the results of RAÑADA & RAÑADA (1984). It remains to be seen whether such a mechanism is really sufficient to bind quarks *permanently* in hadrons. This again ought to be dependent as well on the precise form of the nonlinear interaction (WERLE 1977).

Following the preparatory work of MIELKE (1978, 1980) on scalar models, a first but important attempt (MIELKE 1981b) was carried out to construct *exact* solutions of a nonlinear Dirac equation in *curved* spacetime. The self-interacting potential in (11.4.3) is chosen such that $a = 0$, resulting in a radial system of equations. In order to solve this in closed form, we are postulating a rather special spherically symmetric shape for the metric background (11.4.4), which, unfortunately, does not satisfy the field equations of Einstein.

For a self-consistent procedure, it would have been necessary to take into consideration the back-coupling of the spinor field onto the metric background via the canonical energy–momentum current. In the end, such a procedure would necessitate the study of the spinorial versions (Brill & Wheeler 1957) of those *geon* constructions that were carried out for the coupled Einstein–Maxwell system by Wheeler (1955, 1962). In the model case of a nonlinear scalar field, such *gravitational solitons* have been constructed with numerical methods (Mielke & Scherzer 1981); see also Schunck & Mielke (2003) and the references therein.

An axially symmetric background metric of Kerr–Newman type (Kerr 1963; Newman et al. 1965) seems to be more appropriate in general, and this on account of the nonvanishing intrinsic angular momentum and the charge of the spinor soliton. But it is already in the linear case that the separation of the Dirac equation in terms of the variables of the Kerr metric turns out to be a tedious undertaking; see Teukolsky (1972, 1973); Chandrasekhar (1976) and also Joutei & Chakrabarti (1979), which leads to explicit solutions only in particular cases (Einstein & Finkelstein 1977). Thus the consideration of torsion-induced nonlinearities within the Dirac equation (Inomata 1978) as well as that of the back-coupling of the spinor fields onto the Riemann(–Cartan) spacetime (Brill & Wheeler 1957; Hamilton & Das 1977) is a rather ambitious mathematical program. In addition, it may pave the way to a better understanding not only of the instanton and monopole solutions of nonabelian gauge theories (Joutei & Chakrabarti 1979) but also of the "internal" structure of the fermions and consequently of matter as a whole.

11.5 Quantum-Theoretic Meaning of Nonlinear Classical Field Theories?

There is no way to ascribe a direct physical meaning to the unquantized "free" Dirac field, since its field energy (11.4.40) is not bounded from below. Without imposing a second quantization, this would mean an instability resulting in the rapid decay of all matter (Jost 1965, p. 39). Under these circumstances, it is of interest to define the quantum meaning of the soliton solutions of nonlinear spinor theories.

Besides the difficult but physically most important question concerning the stability of these four-dimensional "solitons," it is important to affirm that the properties of classical fields entering decisively not only in the Wentzel–Kramers–Brillouin (WKB) approximation methods but also in Feynman's method of quantization via path integrals. There is a crucial difference according to whether one starts from the vacuum or from a soliton configuration as a ground state, since the classical solutions are always regarded as approximations of zeroth order to the fully quantized theory. It appears to be impossible to reach the "soliton sector" of the quantized theory, which corresponds to a large coupling constant, by means of perturbation theory (Jackiw 1977; Callan & Gross 1975), since the classical soliton configuration is typically singular for a vanishing coupling constant of the self-interaction.[13] The application of quantum field theory, in which ψ is quantized by means of local anticommutation relations, would cause divergences in the theory due to the singular commutator functions and this apart from the fact that such distribution-valued spinor operators have no classical limit at all. It is for this reason that FINKELSTEIN et al. (1951) proposed to quantize the total charge of a spinor soliton by the requirement

$$Q := ie \int_{H^3} \overline{\psi}^* \gamma \psi = ke, \quad k = 0, 1, 2, \dots . \tag{11.5.1}$$

At first sight, this procedure seems to be rather unsophisticated. However, it has to be acknowledged that it is equivalent to the *old Bohr–Sommerfeld quantization condition* concerning field theories with an infinite number of degrees of freedom and that it is even exactly equivalent to the canonical formalism in many cases; cf. JACKIW (1977).

Attention is still focused on the semiclassical nonlinear Dirac fields for another reason. Usually, it is taken for granted that their temporal evolution violates one of the most fundamental principles in particle physics, namely the Pauli principle. Under the condition, however, that the solitary wave packets, which are thought to represent fermions, are normalized to the quantum of charge $Q = e$ and that their Cauchy initial conditions satisfy the Pauli principle at the beginning of the observation, the orthogonality of the fields is guaranteed until the end, since the scalar product (11.2.8) is an invariant under temporal evolution (RAÑADA 1986). Spinor solitons that are constructed in the described way as well as the charged scalar solitons of MORRIS (1980) represent semiclassical fermions and are, in a sense, akin to DIRAC's model of the extended electrons of 1962. Tempted by these ideas, the author of the present study felt motivated to conceive a nonlinear relativistic quantum theory of *extended* particles (MIELKE 1981c) influenced by DE BROGLIE's *theory of the double solutions* (1960).

For the time being, we have explained the occurrence of nonlinear self-interactions within Dirac's theory in a very fundamental way while tracing it back onto the contortion of the spacetime geometry. But it was IVANENKO (1958), in his profound essay, who pointed out that these "primordial" nonlinearities in quantum field theory are on par with the self-couplings of the fields that are "induced" by vacuum fluctuations. A known example of the latter mechanism is given by the quantum-theoretic treatment (HEISENBERG & EULER 1936) of the theory of pair creation. The quantum-theoretic generation of "virtual" electron–positron pairs can thereby be completely absorbed by classical electrodynamics if only the dynamics are modified by an effective nonlinear Lagrangian 4-form of BORN–INFELD type (1934). Another example is to be seen in quantum electrodynamics itself (BJORKEN & DRELL 1964). The coupled Maxwell–Dirac system leads to the nonlinear and nonlocal Dirac equation

[13]This parameter appears as fundamental length ℓ in both (11.4.3) and the Ansatz (11.4.15).

$$i\gamma \wedge {}^*d\psi(x) + e^2 \int\limits_{y\in M} \overline{\psi}(y)\gamma\psi(y)D_{\mathrm{F}}(y-x) \wedge^* \gamma\psi(x) = m\psi(x)\eta \qquad (11.5.2)$$

after the elimination of the photon field (BARUT & KRAUS 1977) while the amplitudes are not yet subjected to field ("second") quantization. Here $D_{\mathrm{F}}(x)$ denotes the causal Green's function of the electromagnetic field. In the limiting case of an infinite heavy photon (vector bosons), the above *nonlocal* interaction of action-at-a-distance type (FINKELSTEIN et al. 1956) would again change into the spin–spin contact interaction of Fermi type that is induced by the torsion of the spacetime.

A further reason for IVANENKO (1957, 1958) to consider the necessity of nonlinearities in the field equations was the related possibility of introducing a "smallest" *fundamental length* ℓ into the theory. This was meant to cope with the problem of divergences in quantum field theory. VAN DER MERWE (1979) found a modified behavior of the two-point function at short interaction distances of the field operators in a more recent and interesting study. It cannot be excluded therefore, corresponding to the choice of the quantization or regularization method, that the length that occurs in (11.3.7) as a coupling constant increases its "bare" value of the Planck length $\ell^* = 8.1 \times 10^{-33}$ cm to the far larger amount of the Compton wavelength $\ell = 10^{-13}$ cm of the proton, and this by way of renormalization. This would mean that the nonlinearity that is induced by gravity nevertheless provides physically relevant, if not even dominant, contributions to a fully quantized theory, and this despite its small amount, which is characterized by ℓ^*. It is for this reason that HEISENBERG's unified field theory (1967), which starts from $\ell = 10^{-13}$ cm, can refrain from using a "bare" mass term occurring in (11.3.7) without renouncing the possibility of deducing a spectrum of field energies that is of the scale of observable hadron masses. In order to evade the veto of the perturbation-theoretic nonrenormalizability as the theoretical correlate of the occurring four-fermion interactions, Heisenberg developed quantizations rules in Hilbert spaces with an indefinite metric. Since these are problematic in their nature, STUMPF (1980, 1981) prefers functional quantization methods instead, and this also with reference to the confinement problem of the hypothetical quarks.

Nevertheless, HEISENBERG (1974) remained convinced of the physically fundamental meaning of his nonlinear spinor model of matter till the end. He did so on account of the surmised equivalences between the indefinite metric theories and the definite but *nonlocal* field theories. Possibly this is based on a sound reason. Following considerations of EGUCHI (1977), nonlinear spinor theories are equivalent to a renormalizable model in which the fermions interact with collective boson states, according to perturbation theory. What is even more, an SU(f) gauge-invariant nonlinear spinor model (MIELKE 1977b, 1981a) would thus become equivalent to quantum chromodynamics (QCD; MARCIANO & PAGELS 1978), today's most prominent theory of strong interactions.

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Chapter 12
Chiral Anomalies

Anomalies can be viewed as a breaking of some Noether symmetry through the effects of the vacuum. In relativistic quantum field theory (QFT), such a (classical) symmetry is *broken* by field quantization; cf. Holstein (1993, 2014), Van Holten (2005) for rather recent reviews. This has important implications for such physical processes as the decay of the neutral π-meson, cf. Bell & Jackiw (1969), and induced instanton effects (Napsuciale et al. 2002), and it underlies the postulation of the *axion* in quantum chromodynamics (QCD).

12.1 Anomalies for Pedestrians

In quantum electrodynamics (QED), Schwinger (1951) demonstrated that the charge current j can be conserved, i.e., $\langle dj \rangle = 0$, whereas the conservation of the axial current j_5 is broken, $\langle dj_5 \rangle \neq 0$, in the most common convention.

In the Fujikawa (1979) approach, the right-hand side can be obtained from considering the point-split current $j_5(x;\varepsilon) := \overline{\psi}(x)\gamma_5{}^*\gamma\psi(x+\varepsilon)$, where ε is an infinitesimal four-vector in spacetime. Such an expression can be rendered invariant by quantum-field-theoretically dressing it with a path-ordered exponential

$$\overline{\psi}(x)\gamma_5{}^*\gamma\psi(x+\varepsilon) \to \overline{\psi}(x)\gamma_5{}^*\gamma\psi(x+\varepsilon)\mathbf{P}\exp\left\{i\int_x^{x+\varepsilon} A\right\} . \tag{12.1.1}$$

The variation $\delta/\delta A$ of the current $j_5(x;\varepsilon)$ is compensated by the variation of the exponential. Since the parallel transport from $x^i \to x^i + \varepsilon^i$ along the infinitesimal line element can be expanded perturbatively, it is clear that the net effect of this approach is just the standard result

$$\langle dj_5(x) \rangle = 2im\langle P \rangle - (1/96\pi^2) F \wedge F \tag{12.1.2}$$

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for massive fermions, where $P = \overline{\psi}\gamma_5\psi$ is the pseudoscalar current and $F := dA$ is the gauge field strength. Further details of the path integral formulation were developed, e.g., in ALFARO et al. (1988, 1989), URRUTIA & VERGARA (1992) with extension of the regularized Jacobian, as well as in the light-cone gauge (GAMBOA et al. 1997) of the Schwinger model.

There is an intuitive physical interpretation of this result: the additional Chern–Simons (CS) term $C := A \wedge dA$ corresponds to the spin or helicity of the photon, with its spacelike part $\mathbf{A} \cdot \mathbf{B}$ known as magnetic helicity (JACKIW & PI 2000). Since the axial current j_5 is proportional to the spin of a fermion, the deformed current

$$\widetilde{j_5} := j_5 + (1/96\pi^2) A \wedge dA \tag{12.1.3}$$

includes the spin of the photon, lacking, however, gauge invariance. The chiral anomaly can then be understood as the "conservation law"

$$\langle d\widetilde{j_5}\rangle = 0\,, \tag{12.1.4}$$

such that in QFT, "the flow of electronic spin drags some photon spin and vice versa" (WIDOM & SRIVASTAVA 1988).

Anomalies were studied also in Yang–Mills-type gauge models of gravity with Einsteinian instanton solutions (MIELKE & RINCON 2005). Then, the equivalence principle requires a coupling of gravity not only to the energy–momentum current of matter, but also to the spin current. Here we will focus on the intricate inter-relation between the *chiral anomaly* and the spin or helicity of the gravitational gauge field and extend it to post-Riemannian spacetimes with torsion.

12.2 Dirac Fields in Riemann–Cartan Spacetime

In our notation, a Dirac field is a bispinor-valued zero-form ψ for which $\overline{\psi} := \psi^\dagger\gamma_0$ denotes the Dirac adjoint. The minimal coupling to the gauge (electromagnetic) potential $A = A_i dx^i$ is accounted for via $\mathscr{D} := D + iA\wedge$, where $D\psi := d\psi + \Gamma \wedge \psi$ is the exterior covariant derivative with respect to the Riemann–Cartan (RC) connection one-form $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i$.

The Dirac Lagrangian is given by the manifestly *Hermitian* four-form

$$L_{\rm D} = L(\gamma, \psi, \mathscr{D}\psi) = \frac{i}{2}\left\{\overline{\psi}\,{}^*\gamma \wedge \mathscr{D}\psi + \overline{\mathscr{D}\psi} \wedge {}^*\gamma\,\psi\right\} - m\,\overline{\psi}\psi\eta\,, \tag{12.2.1}$$

where $\gamma := \gamma_\alpha \vartheta^\alpha$ is the Clifford-algebra-valued coframe.

The Dirac equation and its adjoint can be obtained by varying $L_{\rm D}$ independently with respect to $\overline{\psi}$ and ψ. Making use of the torsion $\Theta := D\gamma$ and of the properties of the Hodge dual, the Dirac equation assumes the form

$$i\,{}^*\gamma \wedge \left(\mathscr{D} + \frac{i}{4}\,m\gamma - \frac{1}{2}\,T\right)\psi = 0\,, \tag{12.2.2}$$

where $T := \frac{1}{4}\,Tr\left(\check{\gamma}\rfloor\Theta\right) = e_\alpha\rfloor T^\alpha$ is the one-form of the trace (or vector) torsion. However, the covariant derivative D also contains torsion.

In order to separate out the purely Riemannian piece from torsion terms, the Riemann–Cartan connection $\Gamma = \Gamma^{\{\}} - K$ is decomposed into the Riemannian (or Christoffel) connection $\Gamma^{\{\}}$ and the *contortion* one-form $K = \frac{i}{4}K^{\alpha\beta}\,\sigma_{\alpha\beta}$, satisfying $D\gamma = [\gamma, K] = \gamma_\alpha T^\alpha$. Accordingly, the Dirac Lagrangian (12.2.1) splits (MIELKE 2004) into a Riemannian piece and a spin-contortion piece:

$$\begin{aligned} L_{\mathrm{D}} &= L(\gamma, \psi, D^{\{\}}\psi) - \frac{i}{2}\overline{\psi}\left({}^*\gamma \wedge K - K \wedge {}^*\gamma\right)\psi + A \wedge j \\ &= L(\gamma, \psi, D^{\{\}}\psi) + \frac{1}{4}\,\mathscr{A} \wedge j_5 + A \wedge j \\ &= L(\gamma, \psi, D^{\{\}}\psi) - T^\alpha \wedge \mu_\alpha + A \wedge j\,. \end{aligned} \tag{12.2.3}$$

The covariant derivative with respect to the Riemannian connection $\Gamma^{\{\}}$ satisfies $D^{\{\}}\gamma = 0$. Hence, in an RC spacetime, a Dirac spinor feels only the *axial torsion*,

$$\mathscr{A} := \frac{1}{4}\,{}^*Tr(\gamma \wedge D\gamma) = {}^*(\vartheta^\alpha \wedge T_\alpha) = \frac{1}{2}T^{[\alpha\beta\gamma]}\eta_{\alpha\beta\gamma} = \mathscr{A}_i\,dx^i\,, \tag{12.2.4}$$

which is invariant under Weyl rescalings and *chiral transformations* $\gamma \to \gamma^\beta = e^{i\gamma^5\beta}\gamma e^{-i\gamma^5\beta}$ of the coframe, but odd under parity $P : \vartheta^B \to -\vartheta^B$, where $B = 1, 2, 3$.

12.2.1 Classical Axial Anomaly and Spin

Similarly as in QED, the gravitationally coupled Dirac Lagrangian $L_{\mathrm{D}} = \overline{L}_{\mathrm{D}} = L_{\mathrm{D}}^\dagger$ is *Hermitian* as required, even in an anholonomic frame. Then minimal coupling provides us automatically with the following *charge and axial currents*, respectively:

$$j = \overline{\psi}\,{}^*\gamma\psi = j^\mu\,\eta_\mu\,, \qquad j_5 := \overline{\psi}\gamma_5\,{}^*\gamma\psi = \frac{1}{3}\overline{\psi}\sigma \wedge \gamma\psi = j_5^\mu\,\eta_\mu\,. \tag{12.2.5}$$

From the Dirac equation (12.2.2) and its adjoint one can readily deduce that $dj \simeq 0$ "on shell," whereas for the axial current, we obtain the well-known "classical axial anomaly"

$$dj_5 = 2imP = 2im\overline{\psi}\gamma_5\psi\eta \tag{12.2.6}$$

for *massive* Dirac fields (ITZYKSON & ZUBER 1980). The same holds in an RC spacetime. If we restore chiral symmetry in the limit $m \to 0$, this leads to the classical conservation law $dj_5 = 0$ of the axial current for massless Weyl spinors, or since $dj \simeq 0$, equivalently for the *chiral current*

$$j_\pm := \frac{1}{2}\overline{\psi}(1 \pm \gamma_5)\,{}^*\gamma\psi = \overline{\psi}_{\mathrm{L,R}}\,{}^*\gamma\psi_{\mathrm{L,R}}. \tag{12.2.7}$$

As mentioned in the introduction for "pedestrians," the axial current has an intriguing relationship to the (dynamical) *spin current* of the Dirac field canonically defined by the Hermitian three-form

$$\begin{aligned}\tau_{\alpha\beta} &:= \frac{\partial L_{\mathrm{D}}}{\partial \Gamma^{\alpha\beta}} = \frac{1}{8}\overline{\psi}\left({}^*\gamma\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\,{}^*\gamma\right)\psi \\ &= \frac{1}{4}\,\eta_{\alpha\beta\gamma\delta}\,\overline{\psi}\gamma^\delta\gamma_5\psi\eta^\gamma = \tau_{\alpha\beta\gamma}\eta^\gamma = \vartheta_{[\alpha}\wedge\mu_{\beta]}\,.\end{aligned} \tag{12.2.8}$$

In components, the spin tensor $\tau_{\alpha\beta\gamma} = \tau_{[\alpha\beta\gamma]}$ is *totally antisymmetric*. Equivalently, in (12.2.3), torsion merely couples to the two-form

$$\mu_\alpha = \frac{1}{4}\vartheta_\alpha \wedge {}^*j_5. \tag{12.2.9}$$

Consequently, we obtain the remarkable result that the vector one-form

$$\mu := e_\alpha \rfloor \mu^\alpha = \frac{3}{4}\,{}^*j_5 \tag{12.2.10}$$

of the *Dirac spin* is dual to the axial current j_5.

In the correspondence to the axion (Mielke & Romero 2006), it is tentatively assumed that the dimensionless pseudoscalar θ serves as a potential for the axial torsion via $\mathscr{A} = 2d\theta$. Then there arises in (12.2.3) a derivative coupling of the would-be axion $a = \theta f_a$ to two fermions via the CPT-invariant term

$$L_{a\psi\psi} = \frac{1}{2}d\theta \wedge j_5 = \frac{1}{2f_a}da \wedge \overline{\psi}\,{}^*\gamma\gamma_5\psi, \tag{12.2.11}$$

exactly as in the usual formulation, where the axial current j_5 is the Noether current associated with a spontaneously broken Peccei–Quinn symmetry $U(1)_{\mathrm{PQ}}$; cf. Kim & Carosi (2010).

12.2.2 Axial Current in the Einstein–Cartan Theory

The Einstein–Cartan (EC) theory of a gravitationally coupled spin-1/2 Dirac field provides a *dynamical* understanding of the axial anomaly on a semiclassical level: the Lagrangian reads

$$\mathrm{L} = \frac{\mathrm{i}}{2\ell^2}\,\mathrm{Tr}\left(\Omega \wedge {}^*\sigma\right) + \mathrm{L_D} = \frac{1}{2\ell^2}\,\mathrm{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} + \mathrm{L_D}\,, \tag{12.2.12}$$

where $\eta^{\alpha\beta} := {}^*(\vartheta^\alpha \wedge \vartheta^\beta)$ is dual to the unit two-form.

In EC theory, Cartan's algebraic relation between torsion and spin implies the following relation between the *axial current* j_5 of the Dirac field and the translational Chern–Simons (CS) term (12.4.9), or equivalently, for the axial torsion one-form:

$$C_{\mathrm{TT}} \cong \frac{1}{4}\, j_5\,, \qquad \mathscr{A} = 2\ell^2\, {}^*C_{\mathrm{TT}} = (\ell^2/2)\overline{\psi}\gamma_5\gamma\psi\,. \tag{12.2.13}$$

Thus in EC theory, the net axial current production

$$dj_5 \cong 4dC_{\mathrm{TT}} = \frac{2}{\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) \tag{12.2.14}$$

establishes a link to the four-form of NIEH & YAN (1982) for *massive* fields.

This result holds on the level of first quantization. Since the Hamiltonian of the semiclassical Dirac field is not bounded from below, one has to pass to second quantization, where the vacuum expectation value $\langle dj_5\rangle$ of the axial current picks up anomalous terms.

Restoring chiral invariance for the Dirac fields, the limit $m \to 0$, implies that the NY four-form tends to zero "on shell," i.e., $dC_{\mathrm{TT}} \cong (1/4)dj_5 \to 0$. This is consistent with the fact that a Weyl spinor does not couple to torsion at all, because then the axial torsion $\mathscr{A}$ becomes a *lightlike* covector, i.e.,

$$\mathscr{A}_\alpha \mathscr{A}^\alpha \eta = \mathscr{A} \wedge {}^*\mathscr{A} \cong (\ell^4/4)\, {}^*j_5 \wedge j_5 = 0. \tag{12.2.15}$$

Here we implicitly assume that the light-cone structure of the axial covector ${}^*j_5$ is not spoiled by quantum corrections, i.e., that no "Lorentz anomaly" occurs as in $n = 4k+2$ dimensions (LEUTWYLER 1986).

12.3 Chiral Anomaly in Quantum Field Theory

Let us recall a couple of distinguished features of the axial anomaly: Most prominent is its relation to the Atiyah–Singer index theorem (ATIYAH 1998). But also from the viewpoint of perturbative QFT, the chiral anomaly has some features that signal its conceptual importance. For all topological field theories and topological effects like the anomaly, there is the remarkable fact that it does not renormalize: higher-order loop corrections do not alter its one-loop value. This very fact guarantees that the anomaly can be given a topological interpretation. For the anomaly, this is the ADLER & BARDEEN (1969) theorem, while other topological field theories are carefully designed to have vanishing beta functions, for example. Another feature is its finiteness: in any approach, the chiral anomaly as a topological invariant is a finite quantity.

Now, to approach the anomaly in the context of spacetime with torsion, let us first switch off the Riemannian curvature and concentrate on the penultimate term in the decomposed Dirac Lagrangian (12.2.3).

Then this term can be regarded as an *external* axial covector $\mathscr{A}$ coupled to the axial current j_5 of the Dirac field in an *initially flat* spacetime. By applying the result (11–225) of ITZYKSON & ZUBER (1980), we find that only the term $d\mathscr{A} \wedge d\mathscr{A}$ arises in the axial anomaly, but *not* the NY-type term $d\,{}^*\mathscr{A} \sim dC_{\mathrm{TT}}$ as was claimed by CHANDIA & ZANELLI (1997). After switching on the Yang–Mills gluon field G as well as the curved spacetime of Riemannian geometry, we finally obtain

$$\langle dj_5 \rangle = 2im\langle \bar{\psi}\gamma_5\psi\rangle\eta - \frac{1}{4\pi^2}Tr(G \wedge G) - \frac{1}{96\pi^2}\left[2R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + d\mathscr{A} \wedge d\mathscr{A}\right] \tag{12.3.1}$$

for the vacuum expectation value of the *axial anomaly*

This result of KREIMER & MIELKE (2001) is based on diagrammatic techniques and the Pauli–Villars regularization scheme or other kinematic renormalizations (KREIMER & PANZER 2013). In this respect, it is a typical perturbative result, and in agreement with YAJIMA (1996), WIESENDANGER (1996), no NY term arises in the anomaly. Thus only the Weyl invariant term $d\mathscr{A} \wedge d\mathscr{A} = -2\mathscr{E} \cdot \mathscr{B}\eta$ for the axial torsion contributes to the axial anomaly, resembling the $U(1)$ part $F \wedge F = dA \wedge dA$ of the Pontryagin term (12.4.8). Torsion terms like $d\mathscr{A} \wedge d\mathscr{A}$ and $d\,{}^*\mathscr{A} \wedge {}^*(d\,{}^*\mathscr{A}) = 4\ell^4 V_{\mathrm{NY}} \wedge {}^*V_{\mathrm{NY}}$ have been considered previously, as part of the Lagrangian, in order to make the axial torsion propagating. Due to the geometric identity (12.4.10) for the NY term $d\,{}^*\mathscr{A} = 2\ell^2 dC_{\mathrm{TT}} = 2\ell^2 V_{\mathrm{NY}}$, the second term is really quartic in torsion and not scale-invariant.

A rescaling of the tetrad has been proposed. However, one should not ignore the presence of renormalization conditions and the generation of a scale on renormalization. Rescaling the tetrad would ultimately change the wave function renormalization Z-factor, which would creep into the definition of the NY term. This is in sharp contrast to proper topological invariants at the quantum level, which remain unchanged under renormalization.

With no renormalization condition available for the NY term, and other methods obtaining it as zero, one can only conclude that the response function of QFT to a gauge variation (this is the anomaly) delivers no NY term.

12.3.1 Chiral Anomaly in SUGRA

Simple supergravity consists of a consistent coupling of the EC theory to the Rarita–Schwinger spinor-valued one-form $\Psi = \Psi_i dx^i$; cf. URRUTIA & VERGARA (1991), MIELKE & MACIAS (1999) for more details.

The anomaly for the corresponding *axial current* $J_5 := i\overline{\Psi} \wedge \gamma \wedge \Psi$ is -21 times the anomaly for Dirac fields, whereas for the corresponding supersymmetric Yang–Mills anomaly, one obtains 3 times the Dirac result:

Spin	Gravitational	YM anomaly
1/2	1	1
3/2	−21	3

Depending on the asymptotic helicity states, there occur contributions of topological origin of Riemannian Pontryagin or Euler type, respectively. The role of spinors for the index theorem and in the 4D Donaldson invariants via the Seiberg–Witten equation was recently reviewed by ATIYAH (1998). Six-dimensional supergravity free of gauge and gravitational anomalies was studied by ERLER (1994).

12.3.2 Comparison with the Heat Kernel Method

In the heat kernel approach, there exists for small $t \to +0$ the asymptotic expansion

$$K(t, x, \not{D}^2) = (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} K_k(x, \not{D}^2) \tag{12.3.2}$$

of the kernel in n dimensions, where the usual Feynman "dagger" convention $\not{A} := \check{\gamma} \rfloor A = \gamma^\alpha e_\alpha \rfloor A = (-1)^{s+1}\, {}^*\left[{}^*\gamma \wedge A\right]$ for one-forms is used.

The squared Dirac operator

$$\begin{aligned} \not{D}^2 &= -\frac{1}{2}\gamma^\alpha\gamma^\beta \left(\{D^{\{\}}_\alpha, D^{\{\}}_\beta\} + [D^{\{\}}_\alpha, D^{\{\}}_\beta]\right) - 2im\not{D}^{\{\}} \\ &\quad - \frac{i}{4}\gamma_5(\not{D}^{\{\}}\not{\mathscr{A}}) + \frac{1}{2}\gamma_5\sigma^{\alpha\beta}\mathscr{A}_\alpha D^{\{\}}_\beta + m^2 - \frac{1}{2}m\gamma_5\not{\mathscr{A}} - \frac{1}{16}\not{\mathscr{A}}\not{\mathscr{A}} \\ &\cong -\Box - \frac{1}{8}\sigma^{\alpha\beta} R^{\{\}}_{\alpha\beta\mu\nu}\sigma^{\mu\nu} \\ &\quad - \frac{i}{4}\gamma_5(\not{D}^{\{\}}\not{\mathscr{A}}) + \frac{1}{2}\gamma_5\sigma^{\alpha\beta}\mathscr{A}_\alpha D^{\{\}}_\beta - \frac{1}{16}\mathscr{A}_\alpha\mathscr{A}^\alpha - m^2 \end{aligned} \tag{12.3.3}$$

has been explicitly calculated by YAJIMA (1996), OBUKHOV et al. (1997), and the terms additional to the generally covariant Riemannian d'Alembertian operator $\Box := \partial_i\left(\sqrt{|\,g\,|}g^{ij}\partial_j\right)/\sqrt{|\,g\,|}$ can be explicitly identified (MIELKE & KREIMER 1998). Not unexpectedly, besides the familiar Riemannian curvature scalar, only the axial torsion (12.2.4) contributes to the squared Dirac operator for *massive* spinor fields.

The coefficients $K_k(x, \not{D}^2)$ are completely determined by the form of the second-order differential operator $\not{D}^2$, which is positive for Euclidean signature diag $o_{\alpha\beta} =$

$(-1, \ldots, -1)$. For odd $k = 1, 3, \ldots$, these coefficients are zero, while the first nontrivial terms (YAJIMA 1996), which potentially could contribute to the axial anomaly, read

$$Tr(\gamma_5 K_2) = -d\,{}^*\mathscr{A}\,,$$
$$Tr(\gamma_5 K_4) = \frac{1}{6}\left[Tr\left(R^{\{\}} \wedge R^{\{\}}\right) - \frac{1}{4}d\mathscr{A} \wedge d\mathscr{A} + d\mathscr{K}\right], \qquad (12.3.4)$$

where the higher-order term $d\mathscr{K} = d\,{}^*\breve{D} \wedge^* \breve{D}\,{}^*\mathscr{A}$ involves the covariant derivative $\breve{D} = D^{\{\}} + i\mathscr{A}\gamma_5/4$ modified by the axial torsion.

However, there is an essential difference in the physical dimensionality of the terms K_2 and K_4. Whereas in $n = 4$ dimensions, the Pontryagin-type term K_4 is dimensionless and thus for $k = 4$ gets multiplied by $t^{(k-4)/2} = 1$, the term $K_2 \sim d\,{}^*\mathscr{A} = 2\ell^2 dC_{\rm TT}$ carries dimensions. Since a *massive* Dirac spinor has canonical dimension $[length]^{-3/2}$, it scales as $\psi \sim m^{3/2}$. Moreover, the term $t = 1/M^2$ is related to the regulator mass $M \to \infty$ in the FUJIKAWA (1979) method. Then the second-order term in the heat kernel expansion scales as

$$-K_2/t = (2\ell^2/t)dC_{\rm TT} \cong (\ell^2/2t)dj_5 = (im\ell^2/t)\overline{\psi}\gamma_5\psi \sim \ell^2 M^2 m^4 \to 0. \qquad (12.3.5)$$

If we assume in the renormalization procedure that the fundamental length ℓ does not scale (no running gravitational coupling constant), then the second-order term in the heat kernel expansion will tend to zero in the chiral limit $m \to 0$. In the case $m \neq 0$, this term diverges, and the Fujikawa regulator method $M \to \infty$ cannot be applied. To rescale the coframe by $\vartheta^\alpha \to \widetilde{\vartheta}^\alpha = M\vartheta^\alpha$ does not help, since that would also change the dimension of the Dirac field in order to retain the physical dimension $[\hbar]$ of the Dirac action.

Thus the NY term $dC_{\rm TT}$ does not contribute to the *chiral anomaly* in four dimensions, neither classically nor in QFT. On would surmise that in $(n = 2)$-dimensional models, only the term $d\,{}^*\mathscr{A}$ survives in the heat kernel expansion, since it then has the correct dimensions. However, it is well known, cf. HEHL et al. (1995), that in 2D the axial torsion $\mathscr{A}$ vanishes identically. Moreover, gravitational anomalies (LEUTWYLER 1986), specifically the Einstein anomaly and the Weyl anomaly, are fully determined by means of dispersion relations (BERTLMANN & KOHLPRATH 2001).

Let us stress the interrelationship between the scale and chiral invariances: The renormalized conformal (or trace) anomaly (DESER & SCHWIMMER 1993)

$$\langle \vartheta^\alpha \wedge \sigma_\alpha \rangle = -\frac{1}{3\pi^2}\left[Tr(G \wedge {}^*G) + \frac{1}{24}\left(2R^{\alpha\beta\{\}} \wedge R^{\{\}(\star)}_{\alpha\beta} + d\mathscr{A} \wedge {}^*d\mathscr{A}\right)\right], \qquad (12.3.6)$$

where $\sigma_\alpha := \Sigma_\alpha - D\mu_\alpha$ is the energy–momentum current Belinfante symmetrized via the spin energy (12.2.9), receives, in addition to the Riemannian Euler term, a kinetic contribution of the Maxwell type from the axial torsion $\mathscr{A}$. The coefficients are similar to those in (12.3.1), due to the fact that chiral and trace anomalies constitute a supermultiplet (Yang 2004). More recently, the thermal response of the chiral magnetic effect has been studied (Kimura & Nishioka 2012). The role of the QCD trace anomaly on the cosmological constant problem has been analyzed by Schützhold (2002).

12.4 Hamiltonian Interpretation of Anomalies

In the canonical formulation à la Ashtekar (1988), the translational NY term dC_{TT} plays via

$$\overset{(\pm)}{V}_{\mathrm{EC}} := V_{\mathrm{EC}} \pm i dC_{\mathrm{TT}} = \pm\frac{1}{2\ell^2} Tr\ \{(1 \mp \gamma_5)\, \Omega \wedge \sigma\} = -\frac{1}{2\ell^2} \overset{(\pm)}{R}{}^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{\Lambda}{\ell^2}\eta \tag{12.4.1}$$

the role of the *generating functional* (Mielke 1992) for chiral, i.e., self-dual or anti-self-dual variables $\overset{(\pm)}{\Gamma}$ in EC theory as well as in simple supergravity (Mielke & Macias 1999).

The appearance of the Riemannian Pontryagin term $dC^{\{\}}_{\mathrm{RR}}$ in the anomaly (12.3.1) could pose problems for the canonical approach to gravity, since the anomaly does not renormalize. In the presence of gravitational instantons, which due to the necessary condition $\Lambda \neq 0$ could even be the dominating configurations, one gets a net production of chiral zero modes, and global symmetry is broken.

One could argue that this is a perturbative effect. In the Wilson-type loop approach to gravity, the tangential complexified CS term $\underline{C}^{\{\}}_{\mathrm{RR}}$ is known (Kodama 1990) to solve the Hamiltonian constraint

$$\mathscr{H}_\Lambda\, \Psi(\overset{(\pm)}{\underline{\Gamma}}) = 0 \tag{12.4.2}$$

of gravity with cosmological constant, where the complex Ashtekar variable $\overset{(\pm)}{\underline{\Gamma}}$ is the tangential part of the self-dual or anti-self-dual spin connection one-form. Since this solution is intrinsically nonperturbative, no anomaly should occur. In the lattice gauge approach, this is indeed the case, but the problem of fermion doubling (Smalley 1986) appears to be another manifestation of the anomaly.

It is instructive to look at the problem from a Hamiltonian point of view, since the canonical formalism of chiral gravity is closely related to the $\mathfrak{su}(2)$ CS gauge theory on a three-dimensional hypersurface, with $\mathscr{C} := \Gamma/\mathscr{G}$ of inequivalent gauge connections as configuration space.

Gauge anomalies are related to the global topology and have the common feature (Nelson & Alvarez–Gaumé 1985) that the *Gauss constraint*

$$\mathscr{G}^A \cong 0, \quad A = 1, 2, 3 \tag{12.4.3}$$

can no longer be implemented on the physical states (FADDEEV 1984). The reason is that the anomalous Ward identity

$$Ł_n \mathscr{G}^A \cong n\rfloor(D\underline{\tau}^A)\,, \tag{12.4.4}$$

where $Ł_n := n\rfloor D + Dn\rfloor$ is the gauge-covariant Lie derivative along the normal direction, relates the time evolution of the Gauss constraint to the conservation law for the matter current τ^A on the spacelike hypersurface (JIANG 1991). Only when the individual contributions to the anomaly cancel each other can a gauge theory be consistently quantized. In the EC formulation of the gravitationally coupled Dirac field, it is the canonical spin $\tau^A := (1/2)\eta^{0A\beta\gamma}\tau_{\beta\gamma}$ that appears on the right-hand side of the Gauss constraint. Since spin is via (12.2.8) and (12.2.10) related to the axial current j_5, it is precisely the *chiral anomaly* that prevents the Gauss constraint from remaining a proper constraint under time evolution. This result confronts the Ashtekar approach based on loop variables, and thus notions of parallel transport, with the chiral anomaly (MIELKE & KREIMER 1999). The teleparallelism equivalent of chiral gravity, where Wilson loops are replaced by Cartan circuits, may avoid some of these obstacles; see MIELKE (2002).

Appendix: Gravitational Chern–Simons and Pontryagin Terms

When the constant Dirac matrices γ_α satisfying $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2o_{\alpha\beta}$ saturate the index of the orthonormal coframe one-form $\vartheta^\alpha = E_j{}^\alpha dx^j$ and its Hodge dual $\eta^\alpha := {}^*\vartheta^\alpha$, we obtain a basis of Clifford-algebra-valued exterior forms via

$$\gamma := \gamma_\alpha\vartheta^\alpha\,, \qquad {}^*\gamma = \gamma^\alpha\eta_\alpha\,. \tag{12.4.5}$$

In terms of the Clifford-algebra-valued *connection* $\Gamma := \frac{i}{4}\Gamma_i{}^{\alpha\beta}\,\sigma_{\alpha\beta}dx^i$, the $SL(2,\mathbb{C})$-covariant exterior derivative is given by $D = d + \Gamma\wedge$, where $\sigma_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$ are the Lorentz generators entering in the two-form $\sigma := \frac{i}{2}\gamma\wedge\gamma = \frac{1}{2}\sigma_{\alpha\beta}\,\vartheta^\alpha\wedge\vartheta^\beta$.

Differentiation of these basic variables leads to the Clifford-algebra-valued *torsion* and *curvature* two-forms

$$\Theta := D\gamma = T^\alpha\gamma_\alpha\,, \qquad \Omega := d\Gamma + \Gamma\wedge\Gamma = \frac{i}{4}R^{\alpha\beta}\,\sigma_{\alpha\beta} \tag{12.4.6}$$

in RC geometry, respectively. The *Chern–Simons term* for the Lorentz connection reads

$$C_{\mathrm{RR}} := -\mathrm{Tr}\left(\Gamma\wedge\Omega - \frac{1}{3}\Gamma\wedge\Gamma\wedge\Gamma\right)\,. \tag{12.4.7}$$

The corresponding *Pontryagin topological term* can be obtained by exterior differentiation:

$$\begin{aligned} dC_{\rm RR} &= -{\rm Tr}\;(\Omega\wedge\Omega) \\ &= \frac{1}{2}R^{\{\}}_{\alpha\beta}\wedge R^{\{\}\alpha\beta} \\ &+ \frac{1}{12}d\left[{}^*\mathcal{A}\wedge R^{\{\}} - \frac{1}{3}\mathcal{A}\wedge d\mathcal{A} + \frac{1}{9}\,{}^*\mathcal{A}\wedge^*(\mathcal{A}\wedge{}^*\mathcal{A})\right]. \end{aligned} \quad (12.4.8)$$

The latter contains (MIELKE & ROMERO 2006), among others, a term proportional to the curvature scalar $R := {}^*(R^{\alpha\beta}\wedge\eta_{\beta\alpha})$ and the axial torsion piece $d\mathscr{A}\wedge d\mathscr{A}$ of the axial anomaly with a relative factor 9, as required by the supersymmetric path integral (MAVROMATOS 1988).

Since the coframe is the "soldered" translational part (TRESGUERRES & MIELKE 2000) of the Cartan connection, a related *translational* CS term arises:

$$C_{\rm TT} := \frac{1}{8\ell^2}{\rm Tr}\,(\gamma\wedge\Theta) = \frac{1}{2\ell^2}\,\vartheta^\alpha\wedge T_\alpha\,. \quad (12.4.9)$$

By exterior differentiation we obtain the four-form of NIEH & YAN (1982):

$$dC_{\rm TT} = \frac{1}{8\ell^2}{\rm Tr}\,(\Theta\wedge\Theta - 4i\,\Omega\wedge\sigma) = \frac{1}{2\ell^2}\left(T^\alpha\wedge T_\alpha + R_{\alpha\beta}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right)\,. \quad (12.4.10)$$

It is crucial to note that a fundamental length ℓ necessarily occurs here for dimensional reasons. This can also be understood by a de Sitter-type gauge approach, in which the $\mathfrak{sl}(5,\mathbb{R})$-valued connection $\hat{\Gamma} = \Gamma + (\vartheta^\alpha L^4{}_\alpha + \vartheta_\beta L^\beta{}_4)/\ell$ is expanded into the dimensionless linear connection Γ plus the coframe $\vartheta^\alpha = E_i{}^\alpha\, dx^i$, which carries canonical dimension $[length]$. The corresponding Pontryagin term $\hat{C}_{\rm RR}$ splits via

$$\hat{C}_{\rm RR} = C_{\rm RR} - 2C_{\rm TT} \quad (12.4.11)$$

into a linear term and a translational CS term; see footnote 31 of HEHL et al. (1995).

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Chapter 13
Topological SL(5, $\mathbb{R}$) Gauge-Invariant Action

13.1 Introduction

Symmetry plays a predominant role in our perception, although most of the real patterns we observe are far from symmetric. However, in modern particle physics it is much more economical in the mathematical formulation to start from highly symmetric and hence idealized configurations than to try to model the real world directly. A prime example are fundamental interactions, which are rather successfully formulated in terms of *Yang–Mills theories* with large gauge groups with the stipulation that a *symmetry-breaking* occurs spontaneously via the Higgs mechanism (Higgs 2007). The idea of supersymmetry or supergravity, anticipated to some extent already by Weyl (1931), goes in the same direction but so far lacks empirical support. In the case of gravity, a more modest strand of ideas is to enlarge the usual Lorentz or Poincaré group $\mathbb{R}^4 \ltimes SO(1,3)$ to a metalinear group SL(5, $\mathbb{R}$) as a gauge group, but to stick to *four dimensions* (4D). A pure connection formalism arises, in contradistinction to the more common (anti-)de Sitter gauge theories (MacDowell & Mansouri 1977) of gravity.

Observationally, Einstein's general relativity (GR) is rather well established for the solar system, in double pulsars (Wex & Kramer 2009), and, more recently, via a large-scale gravitational lensing of galaxy clustering (Reyes et al. 2010). Conceptually, in GR, the metric plays a double role: measuring macroscopic distances in spacetime and as a gravitational (super)potential for the Christoffel connection. This dichotomy, not seen in Yang–Mills theories, seems to be one of the main obstacles of quantizing gravity. Eddington suggested already in 1923 that the *connection* should be regarded as the basic field and the metric merely as a derived concept.

E.W. Mielke, *Geometrodynamics of Gauge Fields*,
Mathematical Physics Studies, DOI 10.1007/978-3-319-29734-7_13

Our approach departs from a topological action[1] that appears to be renormalizable or even finite. After symmetry-breaking, GR emerges for the macroscopic spacetime.

13.2 Modified BF Scheme

One of the most clear-cut approaches to topological field theory is the BF formalism, which, in the case of gravity, was anticipated by PLEBAŃSKI (1977).

As an instructive example of such a constrained formalism, let us consider the abelian case, where the $U(1)$ connection one-form $A = A_i dx^i$ and an auxiliary two-form $B = \frac{1}{2} B_{ij} dx^i \wedge dx^j$ are varied *independently*, reminiscent of the SCHWINGER formalism (1962). In its primordial version, it starts from the Lagrangian four-form

$$L_{\rm BF} = -B \wedge F = -B \wedge dA. \qquad (13.2.1)$$

Independent variations with respect to A and B lead to $dB = 0$ and to the *constraint* of vanishing field strength $F := dA = 0$. This implies that such a model has *no local* degrees of freedom. An additional term quadratic in B leads to

$$\tilde{L}_{\rm BF} := -B \wedge dA + \frac{1}{2} B \wedge B \cong -B \wedge dA + dC. \qquad (13.2.2)$$

Now, independent variations provide a definition of the field strength F together with the corresponding *Bianchi identity*

$$B \cong dA, \qquad dB \cong dF \equiv 0, \qquad (13.2.3)$$

respectively, in compliance with the Poincaré lemma $dd \equiv 0$. It still defines a topological theory (HOROWITZ 1989), since "on shell," the Lagrangian (13.2.2) differs from (13.2.1) only by a boundary term dC derived from a CS type three-form

$$C = \frac{1}{2} A \wedge F, \qquad dC = \frac{1}{2} F \wedge F, \qquad (13.2.4)$$

the latter being the Pontryagin invariant. As is well known, Bianchi-type identities can be recovered via the variation of the Pontryagin term, e.g., $\delta dC/\delta A = dF \equiv 0$ in the abelian case.

In addition to the usual U(1) gauge transformations

$$A \to A' = A + d\theta(x), \qquad (13.2.5)$$

[1] In the realm of gravity, topological ideas date back to Riemann, Clifford, and Weyl, who found a rather concrete realization in the wormholes (MIELKE 1977) of Wheeler characterized by the Betti number related to the Euler–Poincaré invariant.

the field Eq. (13.2.3) are invariant under the symmetry

$$A \to \widetilde{A} = A + \sigma, \qquad B \to \widetilde{B} = B + d\sigma, \tag{13.2.6}$$

where $\sigma = \sigma_i dx^i$ is a one-form or a covector; cf. LUCCHESI et al. (1993). Since Bianchi *identities* do not allow for nontrivial couplings to matter, in *realistic physical models* such as Maxwell's theory or QCD, the constraint formalism departs from

$$L_{\text{Max}} = -B \wedge dA + \frac{1}{2} B \wedge {}^*B + L_{\text{matter}}, \tag{13.2.7}$$

where, however, the Lagrangian necessarily involves the *Hodge dual* * depending on the determinant of the metric. Then independent variations of (13.2.7) again provide the definition of the field strength (13.2.3), but as a bonus, from the relation $F = {}^*B$ there arises a nontrivial physical field equation

$$-dB = d\,{}^*F \cong j, \tag{13.2.8}$$

where $j := \delta L_{\text{matter}}/\delta A$ is the matter current.

In the case of gravity, it suffices to use instead the *Lie dual*. Then, a "semi-topological" BF scheme based on the metric-free structure group $SL(5, \mathbb{R})$ arises, for which a "breaking" of the intermediate de Sitter gauge symmetry down to Einstein's GR with cosmological constant can be induced.

13.3 Metalinear Group Versus de Sitter Group

In general, the metalinear group $SL(n+1, \mathbb{R})$ with $\det g = 1$ is generated by $n(n+2)$ trace-free generators

$$\not{L}^A{}_B := L^A{}_B - \frac{1}{n}\delta^A_B L^C{}_C. \tag{13.3.1}$$

Their commutation relations

$$[\not{L}^A{}_B, \not{L}^C{}_D] = \delta^A_D \not{L}^C{}_B - \delta^C_B \not{L}^A{}_D \tag{13.3.2}$$

comprise those of a $\mathfrak{gl}(n, \mathbb{R})$ subalgebra

$$[L^\alpha{}_\beta, L^\gamma{}_\delta] = \delta^\alpha_\delta L^\gamma{}_\beta - \delta^\gamma_\beta L^\alpha{}_\delta, \tag{13.3.3}$$

where $\alpha, \ldots, \delta = 0, \ldots, n-1$.

The generators $\ell P_\alpha := \not{L}^{n+1}{}_\alpha = L^{n+1}{}_\alpha$ and $\ell P^\beta_* := \not{L}^\beta{}_{n+1} = L^\beta{}_{n+1}$ transform like those of the translations, i.e.

$$[P_\alpha\,, P_\beta] = 0\,, \qquad [P_*^\alpha\,, P_*^\beta] = 0\,, \tag{13.3.4}$$

$$[L^\alpha{}_\beta\,, P_\gamma] = \delta^\alpha_\gamma\, P_\beta\,, \qquad [L^\alpha{}_\beta\,, P_*^\gamma] = -\delta^\gamma_\beta\, P_*^\alpha\,. \tag{13.3.5}$$

This reveals that P_*^α is a vector and P_β is a covector with respect to the $\mathrm{GL}(n, \mathbb{R})$ subgroup. However, they generate only *pseudotranslations*, since, similarly as for the de Sitter groups (Sobreiro et al. 2012), Eq. (13.3.2) implies that

$$[P_\alpha\,, P_*^\beta] = \frac{1}{\ell^2} L^\beta{}_\alpha\,. \tag{13.3.6}$$

According to Kobayashi (1972), the Lie algebra of metalinear groups is isomorphic to the semisimple *graded affine* group $\mathrm{A}_*(n, \mathbb{R})$ of the same rank:

$$\mathrm{SL}(n+1, \mathbb{R}) \approx \mathrm{A}_*(n, \mathbb{R}) := \mathbb{R}^n \otimes \mathrm{GL}(n, \mathbb{R}) \otimes \mathbb{R}^n_*\,. \tag{13.3.7}$$

(Pseudo)orthogonal groups like $\mathrm{SO}(n+1)$ involve a Cartan–Killing metric with components $\hat{g}_{AB}$. Then a basis of the Lie generators

$$\mathring{L}_{AB} := \hat{g}_{[A|C}\not{L}^{C}{}_{|B]} = -\mathring{L}_{BA}$$

is only $n(n+1)/2$-dimensional. The same would apply to the respective connections with values in the adjoint representations. Equivalently, they can be rewritten in terms of Clifford-algebra-valued differential forms (Mielke 2001). A generalization to massive supergravity (Chamseddine 1978) is not intended here.

13.4 Graded Affine Versus Cartan Connection

Gauge transformations induced by the structure group $\mathrm{SL}(n+1, \mathbb{R})$ are more easily described by its graded version $\mathrm{A}_*(n, \mathbb{R})$. Using a matrix representation, they read

$$A(x) := \left\{ \begin{matrix} \Lambda(x) & \tau(x) \\ \tau_*(x) & 1 \end{matrix} \right\}, \tag{13.4.1}$$

where $A(x) \in \mathrm{A}_*(n,\ R)$ and the $\Lambda(x) \in \mathrm{GL}(n, \mathbb{R})$ are linear gauge transformations, while $\tau(x) := \exp[\xi^\alpha P_\alpha] \in \mathscr{T}(n, R)$ and $\tau_*(x) := \exp[\xi^*_\beta P^\beta_*] \in \mathscr{T}_*(n, R)$ represent local (pseudo)translations. Generalizing the affine gauge approach, we introduce the Lie-algebra-valued one-form

$$\hat{\Gamma} = \begin{pmatrix} \Gamma^{(L)} & \Gamma^{(T)} \\ \Gamma^{(T)}_* & 0 \end{pmatrix} = \begin{pmatrix} \Gamma_\alpha^{(L)\beta} L^\alpha_\beta & \Gamma^{(T)\alpha} \ell P_\alpha \\ \Gamma_\beta^{*(T)} & \ell P^\beta_* \end{pmatrix} \tag{13.4.2}$$

of a *pseudoaffine* connection compatible with the grading; cf. MIELKE (2011a) for more details. In accordance with the Yang–Mills framework, it transforms inhomogeneously under the graded affine gauge transformations:

$$\hat{\Gamma} \overset{A^{-1}(x)}{\longrightarrow} \hat{\Gamma}' = A^{-1}(x)\,\hat{\Gamma}\,A(x) + A^{-1}(x)dA(x). \tag{13.4.3}$$

For the two abelian cosets of pseudotranslations, we require that

$$\tau\tau_* = -\Lambda\tau_*\Lambda^{-1}\tau.$$

For $\ell \to \infty$, this holds automatically due to the degeneracy of the commutation relation (13.3.6) as well as the trace in this limit. The graded affine *curvature* two-form decomposes into

$$\hat{R} := d\hat{\Gamma} + \hat{\Gamma}\wedge\hat{\Gamma} = \begin{pmatrix} R^{(L)} + \Gamma^{(L)}\wedge\Gamma_*^{(L)} & d\Gamma^{(T)} + \Gamma^{(L)}\wedge\Gamma^{(T)} \\ d\Gamma_*^{(T)} + \Gamma_*^{(T)}\wedge\Gamma^{(L)} & 0 \end{pmatrix}, \tag{13.4.4}$$

where the exterior product of the Lie-algebra-valued forms is understood in terms of the adjoint representation $\mathrm{Ad}A(B) = [A, B]$. The curvature

$$R^{(L)} := d\Gamma^{(L)} + \Gamma^{(L)}\wedge\Gamma^{(L)}$$

is associated with the general linear subgroup $\mathrm{GL}(n, \mathbb{R})$, whereas

$$R^{(T)} := d\Gamma^{(T)} + \Gamma^{(L)}\wedge\Gamma^{(T)} = (T^\beta - R_\alpha{}^\beta\xi^\alpha)\ell P_\beta \tag{13.4.5}$$

is the translational part related later on to torsion (HEHL et al. 1995). As required, the curvature transforms covariantly under the graded affine gauge group:

$$\hat{R} \overset{A^{-1}(x)}{\longrightarrow} \hat{R}' = A^{-1}(x)\,\hat{R}\,A(x)\,. \tag{13.4.6}$$

The inhomogeneous transformation law (13.4.3) for the graded affine connection $\hat{\Gamma}$ can be split into that of the linear connection $\Gamma^{(L)}$, which acquires the conventional transformation rule

$$\Gamma^{(L)} \overset{A^{-1}(x)}{\longrightarrow} \Gamma^{(L)\prime} = \Lambda^{-1}(x)\,\Gamma^{(L)}\,\Lambda(x) + \Lambda^{-1}(x)d\Lambda(x) \tag{13.4.7}$$

of a Yang–Mills-type connection for the gauge group $\mathrm{GL}(n, \mathbb{R})$. The remaining translational pieces of the connection (13.4.2) transform as

$$\begin{aligned} \Gamma^{(T)} &\overset{A^{-1}(x)}{\longrightarrow} \Gamma^{(T)\prime} = \Lambda^{-1}(x)\left[\Gamma^{(T)} + D\tau(x)\right], \\ \Gamma_*^{(T)} &\overset{A^{-1}(x)}{\longrightarrow} \Gamma_*^{(T)\prime} = \left[\Gamma_*^{(T)} + D\tau_*(x)\right]\Lambda(x); \end{aligned} \tag{13.4.8}$$

cf. Mielke (2012) for details. In spite of the occurrence of the covariant exterior derivative $D\tau(x) := d\tau(x) + \Gamma^{(L)}\,\tau(x)$, the translational parts $\Gamma^{(T)}$ and $\Gamma_*^{(T)}$ do not transform as covectors, as is required for a coframe.

As a remedy, let us introduce the Möbius-type zero-forms

$$\hat{\xi} = \begin{pmatrix} \xi \\ 1 \end{pmatrix} = \begin{pmatrix} \xi^\alpha P_\alpha \\ 1 \end{pmatrix}, \qquad \hat{\xi}_* = (\xi_*, 1) = (\xi^*_\beta P^\beta_*, 1) \tag{13.4.9}$$

transforming as an affine vector or covector, respectively. Then, the "shifted" one-forms

$$\vartheta := \Gamma^{(T)} + D\xi := \vartheta^\alpha\, P_\alpha\,, \qquad \theta := \Gamma_*^{(T)} + D\xi_* := \theta_\beta\, P^\beta_*\,, \tag{13.4.10}$$

transform as vector- or covector-valued one-forms, respectively, under the graded $\mathrm{A}_*(n, \mathbb{R})$, as required:

$$\vartheta \xrightarrow{\Lambda^{-1}(x)} \vartheta' = \Lambda^{-1}(x)\,\vartheta\,, \qquad \theta \xrightarrow{\Lambda^{-1}(x)} \theta' = \theta\,\Lambda^{-1}(x)\,. \tag{13.4.11}$$

Since $\xi = \xi^\alpha P_\alpha$ and $\xi_* = \xi^*_\beta\, P^\beta_*$ acquire their values in the coset space

$$\mathrm{A}_*(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{R}) \approx \mathbb{R}^n \otimes \mathbb{R}^n_*, \tag{13.4.12}$$

they can be regarded as "generalized Higgs fields," which according to (13.4.11), "hide" the action of the local pseudotranslations $\mathscr{T}(n, R) \otimes \mathscr{T}_*(n, R)$ on ϑ and θ.

Moreover, after the constraints

$$D\xi \overset{*}{=} 0\,, \qquad D\xi_* \overset{*}{=} 0\,, \tag{13.4.13}$$

are imposed, the translational connections $\Gamma^{(T)}$ and $\Gamma_*^{(T)}$ become *soldered* to the spacetime manifold, and the translational parts of the graded affine gauge group get "spontaneously broken" with the gauge action of pseudotranslations partially "eaten up"; cf. Mielke (1987).

If we postulate even stronger constraints $\xi = \xi^* = 0$ of "zero section" vector fields, the pseudoaffine connection $\hat{\Gamma}$ reduces to the graded *Cartan connection*

$$\overset{=}{\Gamma} = \begin{pmatrix} \Gamma^{(L)} & \vartheta \\ \theta & 0 \end{pmatrix}, \tag{13.4.14}$$

which is no longer a connection in the usual sense; cf. Kobayashi (1972).

In the reduction to the (pseudo)orthogonal de Sitter groups, θ and ϑ become mutually transposed one-forms, i.e., $\theta_\alpha = g_{\alpha\beta}\vartheta^\beta$.

When $D\xi^\alpha \overset{*}{=} \vartheta^\alpha$, a Cartan transport of a radius vector ξ^α occurs via "rolling without sliding"; cf. Wise (2010) for his hamster ball analogy in Fig. 13.1.

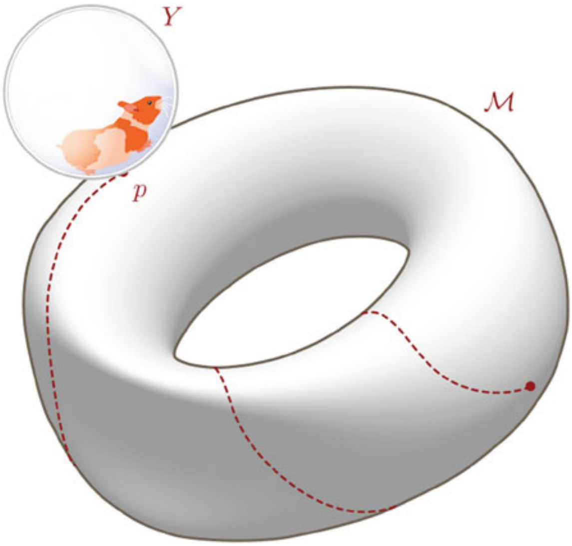

Fig. 13.1 Hamster ball rolling over a torus type manifold

13.5 Gravity from Spontaneous Symmetry-Breaking

In view of the canonical dimension 1/[length] of the generators P_α and P_*^β of pseudo-translations, the $\mathrm{SL}(5,\mathbb{R})$-valued gauge connection of interest here decomposes into

$$\hat{\Gamma} = \Gamma_\alpha{}^\beta\, L^\alpha{}_\beta + \vartheta^\alpha\, P_\alpha + \theta_\beta\, P_*^\beta\,, \tag{13.5.1}$$

i.e., it becomes a Cartan connection, where $\vartheta^\alpha = \ell\,\Gamma_{\hat{4}}{}^\alpha$ and $\theta_\beta = \ell\,\Gamma_\beta{}^{\hat{4}}$ are two independent coframes, i.e., "tetrads." Then the corresponding curvature takes the form

$$\begin{aligned} \hat{R} &:= R_A{}^B \not{L}^A{}_B := \Big[d\Gamma_A{}^B - \Gamma_A{}^C \wedge \Gamma_C{}^B\Big] \not{L}^A{}_B \\ &= \big[R_\alpha{}^\beta - \frac{\Lambda}{3}\theta_\alpha \wedge \vartheta^\beta\big] L^\alpha{}_\beta + T^\beta\, P_\beta + D\theta_\alpha\, P_*^\alpha, \end{aligned} \tag{13.5.2}$$

where $\Lambda = 3/\ell^2$ will later be identified with the cosmological constant.

In a constrained formalism, the Lie-algebra-valued two-form

$$\hat{B} := B_A{}^B \not{L}^A{}_B = (b_\alpha P_*^\alpha + \hat{b}^\alpha P_\alpha)\ell^2 + B_\alpha{}^\beta L^\alpha{}_\beta \tag{13.5.3}$$

functions as a Lagrange multiplier, and the $\mathrm{SL}(5,\mathbb{R})$-invariant BF Lagrangian reads

$$\tilde{L}_{\mathrm{SL}(5,\mathbb{R})} = -\mathrm{Tr}\{\hat{B} \wedge \hat{R}\} - d\hat{C}\,. \tag{13.5.4}$$

Similarly as in the abelian case, the inclusion of the topological Pontryagin four-form $d\hat{C}$ provides us, after variation with respect to ${\Gamma_A}^B$, with the Bianchi identity

$$\hat{D}{B_A}^B \cong \hat{D}{\hat{R}_A}^B \equiv 0. \tag{13.5.5}$$

The variation with respect to $\hat{B}$ would lead to $\hat{R} = 0$, a physically too strong constraint. Thus, we need to "weaken" the strength (SUÉ & MIELKE 1989; SUÉ 1991) of our dynamics by amending the Lagrangian via a term quadratic in $\hat{B}$, namely,

$$\tilde{L}_{\rm SSB} = \tilde{L}_{\rm SL(5,\mathbb{R})} + \frac{1}{2}\eta_{ABCDE}\, B^{AB} \wedge B^{CD} \Phi^E \,. \tag{13.5.6}$$

Here, the antisymmetric unit tensor $\eta_{ABCDE} := \sqrt{\hat{g}}\varepsilon_{ABCDE}$ is constructed from the *metric-free* Levi-Civita symbol ε_{ABCDE}, being invariant under the five-dimensional structure group SL$(5, \mathbb{R})$, together with the determinant $\hat{g}$ of the invertible group metric $\hat{g}_{AB}$. The latter is also needed to raise and lower the Lie algebra indices, e.g., in $B^{AB} := \hat{g}^{AC}{B_C}^B$. Thus, the additional quadratic term provides a *dynamical symmetry-breaking* (SB), which reduces the primordial SL$(5, \mathbb{R})$ to the special orthogonal group SO(5) or, depending on the signature, to the de Sitter or anti-de Sitter group SO(1, 4) or SO(2, 3), respectively. Moreover, the five-index structure of η_{ABCDE} has forced us to complement the construction by a quintet of pseudoscalars $\Phi^A = \{\phi^\alpha, \phi^{\hat{4}}\}$, our *gravitational Higgs field*, in order to be invariant.

13.5.1 Effective MacDowell–Mansouri Theory

Let us now assume that the SO(5) gauge-invariant constraints

$$\hat{g}_{AB}\Phi^A\Phi^B = \mu^2\,, \qquad \Phi_E\hat{D}\Phi^E = 0\,, \tag{13.5.7}$$

arise from a minimum of a Higgs-type potential $U(\Phi^B\Phi_B)$; cf. PAGELS (1984) and Fig. 13.2. After *spontaneous symmetry-breaking* (SSB), cf. MIELKE (2011a), it is assumed that its formal vacuum expectation value

$$\langle\Phi^E\rangle = \Phi_0^E = (0, 0, 0, 0, \mu)^T \tag{13.5.8}$$

takes a constant[2] as its ground state.

Then, as a consequence of both steps, the quadratic term in B of

$$\tilde{L}_{\rm SSB} = -{\rm Tr}\{\hat{B} \wedge \hat{R}\} + \frac{\mu}{2}\eta_{\alpha\beta\gamma\delta}B^{\alpha\beta} \wedge B^{\gamma\delta} - dC_{\rm RR} + 2dC_{\rm TT} \tag{13.5.9}$$

[2]The geometric consequences of a "shifted" gravitational Higgs field are analyzed in (MIELKE 2011b).

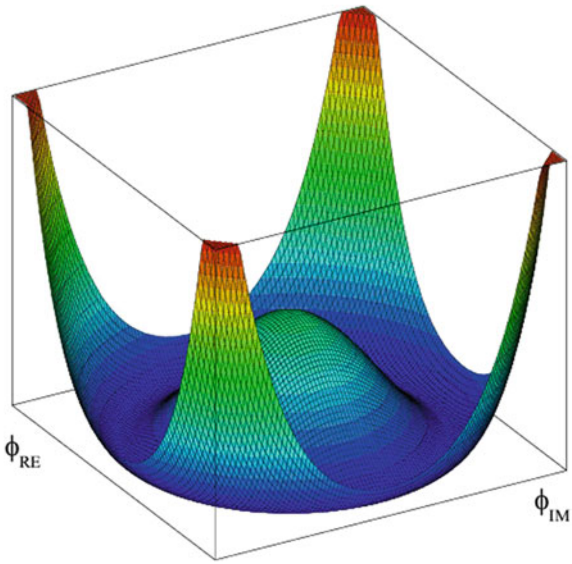

Fig. 13.2 "Mexican hat"-type potential in an abelian Higgs model where $\Phi = \Re\Phi + i\Im\Phi$ is invariant under U(1) gauge transformations. [Adapted from Wikimedia commons, © Gonis.]

is invariant only under a gauge subgroup, in our case the local Lorentz group SO(1, 3). Likewise, the SL(5, $\mathbb{R}$)-invariant topological Pontryagin term decomposes as

$$d\hat{C} = dC_{\mathrm{RR}} - 2dC_{\mathrm{TT}}, \tag{13.5.10}$$

where the resulting Pontryagin term dC_{RR} serves merely to reproduce the Bianchi identity (13.5.5) for the curvature, now in terms of the Lorentz connection $\overset{\circ}{\Gamma}{}^{\alpha\beta} = -\overset{\circ}{\Gamma}{}^{\beta\alpha}$. The variation with respect to the Lorentz-valued two-form $B^{\alpha\beta}$ leads to

$$B_{\alpha\beta} \cong \frac{1}{\mu}\eta_{\alpha\beta\gamma\delta}\left[R^{\gamma\delta} - \frac{\Lambda}{3}\,\theta^{\gamma} \wedge \vartheta^{\delta}\right] \tag{13.5.11}$$

as an equation of motion. Thus, for $\mu \neq 0$, our Lagrangian is "on shell" equivalent to the *deformed* Euler four-form

$$\tilde{L}_{\mathrm{SSB}} + d\hat{C} \cong \frac{1}{2\mu}\eta_{\alpha\beta\gamma\delta}\left[R^{\alpha\beta} - \frac{\Lambda}{3}\vartheta^{\alpha} \wedge \vartheta^{\beta}\right] \wedge \left[R^{\gamma\delta} - \frac{\Lambda}{3}\vartheta^{\gamma} \wedge \vartheta^{\delta}\right]. \tag{13.5.12}$$

It is reminiscent of WEYL'S (1929), cf. SIEROKA (2010), quadratic curvature Lagrangian L_{SKY}, cf. also HIGGS (1959) and YANG (1974), since it is at most quadratic in the Riemann–Cartan (RC) curvature; cf. MIELKE & MAGGIOLO (2005).

For a nondegenerate coframe $\vartheta^\alpha = g^{\alpha\beta}\theta_\beta$ and vanishing torsion, the covariant constancy

$$Dg_{\alpha\beta} = 0, \tag{13.5.13}$$

of the induced spacetime metric is easily deduced (Mielke 2010).

13.5.2 Induced Hilbert–Einstein Action

The effective Lagrangian (13.5.12) resembles that of MacDowell & Mansouri (1977), *except* that the coupling constant μ is *induced* via SSB; see also Castro (2002). It shares the feature that after expansion, the celebrated Einstein–Cartan (EC) Lagrangian

$$\tilde{L}_{\mathrm{SSB}} \cong -\frac{1}{2\kappa} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{\Lambda}{\kappa}\eta \tag{13.5.14}$$

with a cosmological term emerges, supplemented by topological Euler and Pontryagin four-forms (here suppressed). Einstein's gravitational constant $\kappa = 8\pi G_{\mathrm{N}} = 8\pi \ell^2_{\mathrm{Planck}}$ and the density $\rho_\Lambda = \Lambda/\kappa = \Omega_\Lambda \rho_{\mathrm{crit}} < \rho_{\mathrm{crit}}$ corresponding to the cosmological constant Λ are written in natural units. Our expansion of (13.5.12) implies that these constants are related via $\kappa = \mu\ell^2/4$ and $\Lambda = 12\kappa/\mu\ell^4 = 3/\ell^2$ to the symmetry-breaking scale

$$\mu = \frac{4}{3}\kappa\Lambda = \frac{1}{3}(2\kappa)^2 \rho_\Lambda\,. \tag{13.5.15}$$

The symmetry-breaking parameter $\mu \simeq 10^{-123}$ is observationally an extremely small number related to constant dark energy (Frieman et al. 2008). Thus GR with a small cosmological constant Λ, apparently Einstein's "biggest blunder" (Weinberg 2005), turns out to be a tiny *deformation* of a *topological field theory* via a symmetry-breaking $\hat{B} \wedge \hat{B}$ term, as anticipated by Smolin (2000) in the restricted case of the de Sitter group. Thus, in the BF scheme, the "worst fine tuning in physics" (McCarthy & Pagels 1986) converts into a "marble" of symmetry-breaking with gravity as an *emergent* phenomenon; cf. Bjørken (2010). Via (13.5.15), Newton's gravitational constant G_{N} has its origin in SSB, as suggested already by Englert et al. (1975) and Minkowski (1977) in the context of conformal gravity (Nieh 1982; Kaganovich 1989).

13.6 Induced Spacetime Metric in 4D

According to PAGELS (1984), the metric should surface as a *composite* Higgs field via the vacuum expectation value

$$\begin{aligned} ds^2 &:= \frac{\ell^2}{\mu^2}\langle \hat{D}\Phi^A \otimes_{\rm s} \hat{D}\Phi_A\rangle = \frac{\ell^2}{\mu^2}\hat{D}\Phi_0^A \otimes_{\rm s} \hat{D}\Phi_{A0} \\ &= \ell^2 \Gamma_4{}^\alpha \otimes_{\rm s} \Gamma_\alpha{}^4 = (\vartheta^\alpha - D\xi^\alpha) \otimes_{\rm s} \; (\theta_\alpha - D\xi^*_\alpha) \\ &\stackrel{*}{=} \; \vartheta^\alpha \otimes_{\rm s} \theta_\alpha = g_{ij} dx^i \otimes_{\rm s} dx^j \,. \end{aligned} \tag{13.6.1}$$

Since our $\mathrm{SL}(5,\mathbb{R})$ gauge model is broken into two steps from $O(5)$ down to the Lorentz group $\mathrm{SO}(1,3)$ as an exact subgroup, a local holonomic spacetime metric g_{ij} is then induced. In the last step, we assumed that the translational connections are "soldered" via (13.4.13) to the base manifold. Moreover, since the "premetric" $\Gamma_4{}^\alpha \otimes_{\rm s} \Gamma_\alpha{}^4$ is dimensionless, we had to multiply the vacuum expectation (13.6.1) of the derivative of the Higgs field by a *huge* dimensional factor $\ell^2/\mu^2 = (\kappa/4\Lambda_{\rm H})$. This suggests that the induced metric applies only to macroscopic distances, as anticipated in the emerging Einstein equation.

A canonical analysis confirms that the corresponding helicity states of gravitons are strictly massless in accordance with recent observational limits (GOLDHABER & NIETO 2010). Thus classically, a consistent and physically viable scheme emerges.

13.7 Renormalizability of Topological Gravity?

The standard model of particle physics based on the Yang–Mills theory is a *renormalizable and asymptotically free* quantum field theory (QFT) as shown by Veltman and 't Hooft (VELTMAN 2000). For the weak interactions, an important part is played by the Higgs mechanism, which provides a scalar ghost that cancels divergencies in the propagators of the gauge bosons. Nowadays, these cancellations can be more easily understood as a result of a global BRST symmetry ('T HOOFT 2007).

In the case of gravity, we expect that our original topological field theory (13.5.4) based on an $\mathfrak{sl}(5,\mathbb{R})$-valued connection is renormalizable and asymptotically free (FRADKIN & TSEYTLIN 1982) before symmetry-breaking. This can be traced back to a vector supersymmetry (LUCCHESI et al. 1993; CONSTANTINIDIS et al. 2002) in the BRST quantization, generalizing (13.2.6). Moreover, the chiral anomaly (MIELKE 2006) is proportional to the gravitational Pontryagin term $d\hat{C}$. In order to remove it via a counterterm, merely a shift of the coupling of the corresponding term in the already modified Lagrangian (13.5.4) is needed. This would amount simply to a *field redefinition* of $\hat{B}$.

As a result of an SSB, in our semitopological BF scheme, the deformed *Euler* term (13.5.12) emerges as an effective quadratic curvature Lagrangian, inheriting a dimensionless coupling constant μ. Loop corrections could imply a running of the dimensional coupling constants κ and Λ in the spontaneously broken model and therefore also of μ. The gravitational Higgs mechanism may protect, to some extent, the finiteness or renormalizability of the original *BF* scheme as a topological QFT.

After symmetry-breaking, we may neglect the Euler and Pontryagin terms and define for the resulting Hilbert–Einstein truncation with cosmological term the *dimensionless* running coupling constants

$$g_{\mathrm{N}} := \kappa k^2 , \qquad \lambda := \Lambda / k^2 , \tag{13.7.1}$$

where k is the renormalization scale in momentum space. Asymptotic safeness amounts to the requirement that dimensionless coupling constants remain bounded in the ultraviolet limit $k \to \infty$. In our 4D case, this is controlled by the *renormalization group equations*

$$k \frac{\partial}{\partial k} g_{\mathrm{N}} = \beta_1(g_{\mathrm{N}}, \lambda) = (2 + d_{\mathrm{N}}) g_{\mathrm{N}} , \qquad k \frac{\partial}{\partial k} \lambda = \beta_2(g_{\mathrm{N}}, \lambda) , \tag{13.7.2}$$

where d_{N} is the anomalous dimension of g_{N}.

According to the *asymptotic safety scenario* (Niedermaier 2010), they run into some nontrivial fixed points $g_{\mathrm{N}*}$ and λ_*, depending on the specific truncation of the effective Lagrangian (13.5.12) to the celebrated Hilbert–Einstein Lagrangian (13.5.14) without torsion. Quite generally, the product with the universal bound (Niedermaier & Reuter 2006)

$$\mu_* \leq \frac{4}{3} g_{\mathrm{N}*} \lambda_* \simeq 0.2 \tag{13.7.3}$$

appears to be rather robust and independent of the truncation. In our case, it would arise from the SSB of a topological *BF* theory. Thus, gravity appears to be *not* so non-renormalizable; cf. Kreimer (2008). This could also affect cosmological constraints (Kiselev & Timofeev 2011) on the mass of the standard Higgs boson.

13.7.1 The Issue of Chiral Anomalies

In *quantum chromodynamics* (QCD), the accepted theory of the strong interactions, the neutral pion decay into two photons is induced primarily via the global chiral anomaly. Precision measurements of this quantity may serve as a rather stringent test (Bernstein & Holstein 2013) of the validity of QCD itself.

It is unclear how this anomaly affects the AS scenario in view of its possible nonlocality: an example is the GR truncation modified (Daum & Reuter 2012) via the infamous "Holst" term, related to a scale-dependent topological term. The latter was proposed much earlier by various "forerunners" such as Hojman et al.; cf. footnote 1 of MIELKE (2009).

Since the reduction of our "semitopological" BF theory proceeds via the QCD-inspired model of PAGELS (1984), related chiral anomalies may occur that may spoil unitarity in the quantization.

However, in topological models there is a remedy obtained by absorbing the Pontryagin invariant of the anomaly into a counterterm: in the already modified Lagrangian of the "cosmological" BF scheme, merely a shift of the coupling of the corresponding term $d\hat{C}$ is needed. "On shell," this simply amounts to a *field redefinition* of the background field $\hat{B}$.

In lattice gauge theory, instead of the anomaly, typically the equivalent problem of *fermion doubling* arises. It is not clear how this affects the causal dynamical triangulations (LAIHO & COUMBE 2011), (REUTER & SAUERESSIG 2011) in QG, where the spectral dimension (DUNNE 2012) runs from $3/4$ at short distances to four at large distances. A suggestion of SMALLEY in 1986 to use half-integer lattices may not always be applicable.

In the first place, what needs to be seen is how matter couplings, apart from the mere surface terms of FAIRBAIRN & PEREZ (2008), can be incorporated into the BF scheme including fermions (HEHL et al. 1991; MIELKE 2011b).

However, the gravitons primordially present in the $SL(5, \mathbb{R})$ gauge connection could create high energetic gamma pairs that themselves create electron–positron pairs from the vacuum. Similarly, above a larger threshold, gravitons would create W^+ and W^- gauge bosons or even "heavy photons" Z, again via pair creation. These are known to decay into quarks and antiquarks. Thus there appears to be a viable, but so far rather speculative, road from a fluctuating "wormhole gas" to light fermionic matter (EICHHORN 2012) during the expansion of the universe.

13.8 Outlook: Mach-Type Higgs Vacuum?

In the limit $\mu \to 0$ of $SL(5, \mathbb{R})$ symmetry restoration, concepts like macroscopic distances, causal order, or even black holes lose their meaning (with the exception of the volume in the related model of WILCZEK (1998)). Only after an SSB of our topological field theory does a line element ds emerge as a necessary means for measuring macroscopic distances by clocks and rods. Moreover, the *weak equivalence principle* (OVERDUIN et al. 2009) is, after SSB, anchored in the covariant constancy of the ground state (MIELKE 2011b). Thus, cornerstones in the foundation of a standard GR surface here as "Mach"-type features of the gravitational Higgs vacuum, resembling a new 'ether," cf. BRANS (1999), NE'EMAN (2006), KAISER (2007), which, however, remains locally Lorentz-invariant.

An observationally tiny SSB of a primordial metric-free and scale-invariant BF theory seems to be sufficient to generate the feeble gravity we are acquainted with, at the same time incorporating a viable model of *dark energy* (DE) that may even slowly decay (ZEE 2004). The inverse *dimensionless* coupling constants $1/\alpha :=$

$\hbar c/e^2 \simeq 137$ of Sommerfeld and $1/\mu \simeq 10^{123}$ of gravity seem to be on a par except for representing opposite extremes in Dirac's large numbers hypothesis.

In the intermediate step, a Weyl-type theory of gravity surfaces from a Higgs-type SSB of the "Yang–Mielke" theory (MIELKE 1981) after BRST quantization (MIELKE 2008). Since double-anti-dual curvature solutions reside on Einstein spaces, this theory admits gravitational instantons as well; cf. CHEN & TEO (2011).

Appendix: Topological Invariants

Let us recall that the $\mathfrak{sl}(5,\mathbb{R})$-valued Chern–Simons term

$$\hat{C} := -\frac{1}{2}\left(\Gamma_A{}^B \wedge d\Gamma_B{}^A - \frac{2}{3}\Gamma_A{}^B \wedge \Gamma_B{}^C \wedge \Gamma_C{}^A\right) \tag{13.8.1}$$

contains a translational CS three-form $C_{\rm TT}$ that gives rise to the *parity-violating* (MIELKE 2009) boundary four-form

$$dC_{\rm TT} = \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) \tag{13.8.2}$$

of Nieh and Yan (NY) (NIEH 2007). On the other hand, the $\mathfrak{gl}(4,\mathbb{R})$-valued Pontryagin four-form

$$dC_{\rm RR} = -\frac{1}{2}R_\alpha{}^\beta \wedge R_\beta{}^\alpha \tag{13.8.3}$$

is a topological invariant whose variation returns the second Bianchi identity

$$DR_\alpha{}^\beta \equiv 0\,, \tag{13.8.4}$$

whereas the torsion identity (13.8.2) is based on the first Bianchi identity

$$DT^\alpha \equiv R_\beta{}^\alpha \wedge \vartheta^\beta \tag{13.8.5}$$

in Riemann–Cartan (RC) geometry. In contrast to the metric-free Pontryagin four-form (13.8.3), in the NY term a metric $g_{\alpha\beta}$ is needed to raise and lower the indices, for instance in $T_\alpha = g_{\alpha\beta}T^\beta$. Moreover, a fundamental length ℓ necessarily enters into (13.8.2) in order to keep all topological invariants dimensionless.

The topological Euler–Poincaré invariant

$$\begin{aligned} dC_{\rm RR^{(\star)}} &:= \frac{1}{2}\,d\left(\Gamma_{\alpha\beta} \wedge R^{\alpha\beta(\star)} - \frac{1}{3}\Gamma_\alpha{}^{\beta(\star)} \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha\right) \\ &\equiv -L_{\rm SKY} - 2{\rm Ric}_{\alpha\beta} \wedge {}^*{\rm Ric}^{\alpha\beta} + \frac{1}{2}{\rm Ric}_\alpha{}^\alpha \wedge {}^*{\rm Ric}_\beta{}^\beta \end{aligned} \tag{13.8.6}$$

has an equivalent representation in terms of Weyl's quadratic curvature Lagrangian

$$L_{\rm SKY} := -\frac{1}{2}\, R_{\alpha\beta} \wedge {}^*R^{\alpha\beta}, \qquad (13.8.7)$$

amended by Ricci-squared and curvature scalar-squared terms. The second expression involving the symmetric Ricci tensor, i.e., the zero-form $\mathrm{Ric}_{\alpha\beta} := (-1)^{\rm sig}\,{}^*(R_{(\alpha}{}^{\delta} \wedge \eta_{\delta|\beta)})$, is known as the *Gauss-Bonnet identity*.

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Chapter 14
Geometrodynamics and Its Extensions

Ultimately, our description of fundamental interactions aims at the superior and more encompassing question whether a monistic theory of both the intermediating forces and the matter fields themselves can be designed that builds essentially on geometric concepts only. WEYL (1931, p. 51) comments somewhat aloofly:

> The forces of gravitation had been recognized as a result of the metric structure by Einstein's theory; a physical entity had been "geometrized". Thus it is comprehensible that for the sake of a unified conception of the world, the attempt had been made to geometrize all of physics.[1]

As for Kant, space and time are nothing but necessary forms of our perception, so to speak an "arena" in which the physical events of alien fields and particles display their being. However, it was John Archibald Wheeler who resumed the hypotheses of RIEMANN (1854), CLIFFORD (1882) and EINSTEIN (1919), which were far ahead of their time, in order to suggest the following.

> Is curved empty geometry a kind of 'magic' building material out of which everything in the physical world is made (cf. WHEELER 1966; WHEELER & BRILL 1963):
>
> (1) slow curvature in one region of space describes a gravitational field;
> (2) a rippled geometry with a different type of curvature somewhere else describes an electromagnetic field;
> (3) a knotted-up region of high curvature describes a concentration of charge and mass–energy that moves like a particle?

Wheeler has abundantly documented his ideas concerning such a *geometrodynamics* (GMD) in two of his works (WHEELER 1962, 1969). In which the point of

[1]Durch die Einsteinsche Theorie waren die Kräfte der Gravitation als ein Ausfluß der metrischen Struktur erkannt worden; eine physikalische Entität war 'geometrisiert' worden. Es ist verständlich, daß nun um der Einheitlichkeit des Weltbildes willen versucht wurde, die gesamte Physik zu geometrisieren.

E.W. Mielke, *Geometrodynamics of Gauge Fields*,
Mathematical Physics Studies, DOI 10.1007/978-3-319-29734-7_14

departure is RAINICH's geometrically interesting characterization (1925) of the coupled Einstein–Maxwell system. Further geometric concepts, which include the *geon* (WHEELER 1955) and *superspace*[2] (WHEELER 1970), have attracted attention in more recent studies of gravitational coupled solitons (KODAMA 1978; MIELKE & SCHERZER 1981; FRIEDMAN & WITT 1983), as well as in investigations concerning the configuration spaces of gauge theories (BABELON & VIALLET 1981).

Although this model of unification has, up to now, not yielded concrete physical insights and although the ontological dimension of this theory has caused serious criticism (KANITSCHEIDER 1971; GRÜNBAUM 1973), it seems nevertheless too early to declare the downfall of this paradigm (STACHEL 1974). Within the range of this context, it is thus our conviction that the hypotheses concerning a geometric and topological foundation of fundamental physics designed by Riemann, Clifford, Einstein, Weyl, and Wheeler are on the right track and that the only shortcoming of such a geometrization program has to be seen in a possibly too narrow framework (cf. SIEROKA 2010).

As has been shown in the preceding chapters, the general concept of fiber bundles turns out to be a far more appropriate organon, and all the more so because it is the total of their topological properties that is achieving ever increasing relevance for a better understanding of the quantum-theoretical microcosm. There is sound reason to hope that it is possible to gain a unified description of all fundamental physical interactions within this overarching (differential-)*geometric frame*.

As it is, Yang–Mills, gravitational, and fundamental spinor fields find a natural geometric description by means of gauge-theoretic concepts. Under the dynamical aspect, one usually contents oneself with the purely additive combination of the actions or Lagrangian n-forms in question. Such a procedure does not cause a *unification* in the proper sense of the word. Perhaps the most convincing approach, as far as unification is concerned, has to be seen in the idea of looking on the spaces of the "internal" symmetry of Yang–Mills gauge fields as additional dimensions of a higher-dimensional physical spacetime manifold.

Concerning the case of electromagnetism, it was shown by KALUZA (1921) and KLEIN (1926) that this procedure leads to a theory that for the case of absent fermions is physically indistinguishable from the coupled Einstein–Maxwell system.

Since such speculative extensions imply the germ of a generalization to non-abelian structure groups, they have enjoyed a remarkable renaissance, cf. SALAM & STRATHDEE (1982), and it is only logical to document the basic principles.

However, this is not the only way of coping with the problem of the unification of classical fields, which means especially the combination of Maxwell's theory of electromagnetism with Einstein's GR without introducing new speculative elements into those macroscopically well established theories. The Einstein–Maxwell equations have been reformulated by RAINICH (1925) in such a way that the desired information concerning the electromagnetic and gravitational fields can almost uniquely be

[2] The quotient space $\mathscr{S} := \mathscr{M}/\mathscr{D}$ of all Riemannian metrics on a 3-dimensional manifold modulo the group $\mathscr{D}$ of the diffeomorphisms of M^3 is called the "superspace". Accordingly, it has a completely different meaning from what it has in the context of supersymmetry and supergravity.

read off from the curvature of the spacetime. This geometric reformulation was clarified and extended by MISNER & WHEELER (1957). Most importantly, it accentuates that the duality rotations of Rainich, which are rooted in the structure of spacetime, are to be considered a *second* kind of gauge transformation.

Other extensions of GR begin with an *asymmetric*, affine connection, as is the case with CARTAN (1922) and EDDINGTON (1923). It has already been clarified to what extent this assumption leads to the Einstein–Cartan theory, or, respectively, to the Poincaré gauge theory coupled to the *mass* and the *spin* content of matter. In harmony with the Rainich concept, this means a geometrization of the proper angular momentum of a particle by the contortion (KUCHAŘ 1966).

Additionally, one can renounce the symmetry postulate as far as the metric tensor is concerned. This renunciation would provide further degrees of freedom and the possibility of identifying the antisymmetric part of a second-rank tensor with the components F_{ij} of the electromagnetic field strengths. Such a theory was developed not only by EINSTEIN & KAUFMAN (1955) but by SCHRÖDINGER (1950) as well. Alternatively, it is possible, too, to start from a Hermitian metric tensor (EINSTEIN 1948). Einstein expounded his final version of this unified theory of gravity and electricity in the appendix within the body of his lecture "The Meaning of Relativity" (EINSTEIN 1955). A more detailed analysis can be found in HLAVATY (1958), and attention may also be drawn to the sophisticated appreciation of this theory by SCIAMA (1961).

Besides many inapt attempts (see, for instance, VOLLENDORF 1976), there have more recently appeared a growing number of variants of asymmetric theories (MOFFAT 1976, 1977) and references therein) in which an additional Maxwell-like term proportional to $g_{[ij]}g^{[ij]}$ occurs side by side with Hilbert's Lagrangian 4-form, thus deviating from Einstein's original approach. For such a theory there exists an exact solution of Reissner–Nordström type (see Sect. 14.4), which is regular at its origin (MOFFAT & BOAL 1975). Thus far, this model complies with Einstein's geometric conception of classical particles (EINSTEIN & ROSEN 1935). However, for a physical interpretation, these models are still not advanced enough. This is why MOFFAT (1979a), in a more recent study, has given up the identification $\mathrm{g}_{[ij]} := \ell^{*2}\mathrm{F}_{ij}$. Instead, the antisymmetric part of the metric is associated with a new gravitational interaction of long range that does not influence the equations of motion. Moffat thereby accounts for a critical objection of CALLAWAY (1953) against Einstein's asymmetric theory. In an extension of the Hermitian field theory, TREDER (1980, see also 1978), has suggested to relate the purely imaginary antisymmetric part of the metric groundform to the dual of the gluon field strength. This is only one of the attempts to construct a geometric mechanism of quark confinement based on a unification of colored and gravitational gauge models (GREENBERG & NELSON 1977).

It has to be noted that all the physical models that have been presented so far, including the generally covariant Yang–Mills theories, are dependent on a *local isotropy* of the metric structure of the tangent space. However, this assumption is not mandatory for a measurement of macroscopic distances. The metric tensor could be dependent not only on the point $m \in M$ but also on locally distinguished directions, as in the case of a Finsler geometry. Since these mathematical structures are equivalent to those notions that appertain to *nonlocal theories* of extended particles with "internal" (and possibly broken) symmetry, Finslerian extensions of the theory of relativity are discussed again (see, for instance, HORVATH & MOOR 1952; ASANOV 1979, 1981, 1984; IKEDA 1978, 1981a,1981b; ISHIKAWA 1980, 1981). Due to the shortage of space and time, these ideas cannot be discussed in more detail in the present study. The same is true for the system of notions in which the usual space-time coordinates are combined with anticommuting Grassmann-valued additional coordinates to a "superspace" (SALAM & STRATHDEE 1978) in order to generate a unified interpretation of bosonic and fermionic degrees of freedom. Analogously, the gauging of a hypothetical supermultiplet consisting of spin-2 and spin-3/2 particles leads to the theory of *supergravity* (VAN NIEUWENHUIZEN 1981) or to *hypergravity* (ARAGONE & DESER 1979) if spin-5/2 fields are considered instead.

There still remains to be discussed a further complex of theories that belongs to the current speculative extensions of general relativity. That is the tensor dominance model of strong interactions of ISHAM et al. (1971). The reason is not so much the fact that this model is structurally related to GR, but that it offers reasonable results regarding a sensible interpretation of the rotational gauge degrees of freedom of the Poincaré gauge theory. This would have important consequences for the possible existence of a geometric mechanism of quark confinement and hence for an understanding of the structure of hadronic matter.

14.1 Rainich Geometrization of Electromagnetic Fields

If we are ready to accept that spinor fields, which are necessary to describe the "matter core," have a special status, thus differing widely from the suppositions underlying supersymmetric theories, all that has to be done in order to procure a monistic theory of physical interactions boils down to the unification of the Yang–Mills gauge fields with gravitational fields. From a historical point of view, such beginnings have been restricted to classical fields of long range, e.g., to gravitational potentials and electromagnetic fields. Facing the enormous success in describing as well the strong and weak interactions by means of gauge fields, it is our intention to work out the formalism of the Rainich geometrization as far as possible for the general nonabelian case.

On the level of the action functional, the dynamical coupling of the internal gauge fields to gravity is conventionally rendered by means of a simple addition of the corresponding generally covariant Lagrangian 4-forms, which means that

$$L = L_g + L_\omega + L_{\text{mat}} \tag{14.1.1}$$

has to be valid for the total system. This is the case, although mixing terms such as

$$L_{\omega-g} = \text{Tr}\left\{\wedge^*(\vartheta \wedge \Omega^g) \wedge^* (\vartheta \wedge \Omega)\right\} \tag{14.1.2}$$

meet all criteria of gauge invariance according to HORNDESKI (1976, 1978a, b) and hence would have to be permitted also.

Concerning the geometrization procedure that is to be constructed in the following, we shall concentrate on the physically relevant case, that of a coupled Einstein–Maxwell or Einstein–Yang–Mills system. Instead of (14.1.1), we are therefore considering only the system

$$L_{\mathrm{GMD.}} = \frac{1}{\ell^{*2}} L_{\mathrm{W.}} + L_{\mathrm{YM}} + L_{\mathrm{mat}} \tag{14.1.3}$$

in a Riemann–Cartan spacetime.

The field equations, being derived from well-known variation principles, read

$$D^{*}\Omega = \tau_e, \tag{14.1.4}$$

$$D\Omega \equiv 0, \tag{14.1.5}$$

$$\frac{1}{2}{}^{*}(\vartheta \wedge {}^{*}\Omega^{g}) = \ell^{*2}(\Sigma_{\mathrm{YM}} + \Sigma_{\mathrm{mat}}), \tag{14.1.6}$$

and

$$D(\vartheta \wedge {}^{*}\Omega^{g}) \equiv \Theta \wedge {}^{*}\Omega^{g}. \tag{14.1.7}$$

They are supplemented by the Bianchi identities.

If material sources are present, there are still to be considered not only Cartan's equation concerning the torsion but also the contracted Bianchi identity. However, spin-1 gauge fields do not induce a contortion of the spacetime (HEHL et al. 1976).

Within the geometric concept of unification, the central issue is the following one: To what extent are the field strengths Ω of electromagnetic or Yang–Mills-type potentials codified in the curvature Ω^{g} of the spacetime geometry, or conversely, can the codification that is implied in the curvature be restored in terms of gauge field strengths?

This highly important question was propounded by RAINICH (1925) and later on by MISNER & WHEELER (1957) while constructing a dynamical geometry that had to comprise electromagnetic fields as well. In order to answer this question, the mathematical structure of the canonical energy–momentum current Σ_{ω} of the Yang–Mills fields has first to be looked at. This energy–momentum current is rendered by

$$\begin{aligned}\Sigma_{\mathrm{YM}} &= \frac{(-1)^{s}}{2\alpha_{g}} \mathrm{Tr}\left\{\Omega \wedge {}^{*}(\vartheta \wedge {}^{*}\Omega) + {}^{*}\vartheta \wedge {}^{*}(\Omega \wedge {}^{*}\Omega)\right\} \\ &= \frac{(-1)^{s}}{4\alpha_{g}} \mathrm{Tr}\left\{\Omega \wedge {}^{*}(\vartheta \wedge {}^{*}\Omega) - {}^{*}\Omega \wedge {}^{*}(\vartheta \wedge \Omega)\right\}.\end{aligned} \tag{14.1.8}$$

In the component notation of Minkowski, the four-dimensional stress–energy tensor is symmetric and can thus be molded into the form

$$\begin{aligned}\Sigma_{\mu\nu}^{\mathrm{YM}} &= \frac{1}{\alpha_g}\mathrm{Tr}(F_{\mu\alpha}F_{\nu}^{\cdot\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \\ &= \frac{1}{2\alpha_g}\mathrm{Tr}(F_{\mu\alpha}F_{\nu}^{\cdot\alpha} - (-1)^s\ {}^*F_{\mu\alpha}{}^*F_{\nu}^{\cdot\alpha}),\end{aligned} \tag{14.1.9}$$

which was already well known to von Laue (SOMMERFELD 1910). Notoriously, this energy–momentum current is traceless. The contraction that is formed according to the rule

$$\begin{aligned}&\mathrm{Tr}(\Sigma_{\mathrm{YM}} \wedge {}^*\vartheta) = \mathrm{Tr}(\vartheta \wedge {}^*\Sigma_{\mathrm{YM}}) \\ &= \frac{1}{2\alpha_g}\mathrm{Tr}\left\{\vartheta \wedge \Omega \wedge {}^*(\vartheta \wedge {}^*\Omega) + \vartheta \wedge {}^*\vartheta \wedge {}^*(\Omega \wedge {}^*\Omega)\right\} = 0\end{aligned} \tag{14.1.10}$$

vanishes, as can be shown by an exchange of one of the 1-forms ϑ.

This energy–momentum current, however, is the sole source of Einstein's field equations (14.1.6) concerning a spacetime that otherwise is matter-free, i.e., for $\Sigma_{\mathrm{mat}} = 0$.

A corresponding contraction of these field equations shows immediately that the scalar curvature

$$\boxed{\mathrm{R} = {}^*(\Omega^g \wedge \vartheta \wedge \vartheta) = 0} \tag{14.1.11}$$

vanishes for a coupled Einstein–Yang–Mills system. Since massless spin-1 gauge fields do not generate torsion, this result holds even in a RC spacetime.

The energy–momentum tensor (14.1.9), or the form (14.1.8), of the Yang–Mills field transforms not only covariantly with respect to gauge transformations but is distinguished also by a remarkable invariance with respect to *duality rotations* (RAINICH 1925, 1950; MISNER & WHEELER 1957) defined by

$$\Omega \to e^{*\delta}\Omega := \Omega\cos\delta + (-1)^z\ {}^*\Omega\sin\delta. \tag{14.1.12}$$

The angle of complexion δ, which marks the duality transformation, acts additively,

$$e^{*\delta} \circ e^{*\varepsilon} = e^{*\varepsilon} \circ e^{*\delta} = e^{*(\delta+\varepsilon)}, \tag{14.1.13}$$

with respect to composition. Apart from a phase factor, a rotation by the angle $\delta = \pi/2$ is exactly equal to the operation of taking the Hodge dual[3]

[3]For metric spaces with an odd signature s, the resulting imaginary unit is at times incorporated into the definition of the Hodge star operator in order to ensure that it acts involutively on 2-forms. Nevertheless, the hyperbolic functions, which have been used not only by PERCACCI (1979) but erroneously also by MIELKE (1981), seem to be inapt, since they do not yield the special case (14.1.14) for real angles δ.

$$e^{*\pi/2} = (-1)^z *, \quad z := \frac{s-1}{2}. \tag{14.1.14}$$

(In the case of electromagnetism, this duality rotation is also known as a Lamor transformation). The *invariance* of Σ_{YM} with respect to duality rotations can easily be proved by writing out and applying the elementary functional relation $\cos^2\delta + \sin^2\delta = 1$ for trigonometric functions.

In passing, it should be pointed out here that the general duality Ansatz for the reduction of the field equations of the Poincaré gauge theory represents a modification of (14.1.12), since it relates the canonical field momentum, which depends on the rotational gauge field strengths, to the dual curvature by means of a double-duality rotation.

The duality rotation can be exploited in order to transform the curvature 2-form of the gauge fields into a 2-form

$$\xi := e^{*(-\delta)}\Omega \tag{14.1.15}$$

with desired and defined qualities. Those gauge-invariant 4-forms or scalars that occur in the general Lagrangian formalism are not invariant with respect to the duality rotation (14.1.15), but transform into

$$\Omega \wedge {}^*\Omega \to \xi \wedge {}^*\xi = \Omega \wedge {}^*\Omega \cos 2\delta - (-1)^z \Omega \wedge \Omega \sin 2\delta \tag{14.1.16}$$

and

$$\Omega \wedge \Omega \to \xi \wedge \xi = \Omega \wedge \Omega \cos 2\delta - (-1)^z \Omega \wedge {}^*\Omega \sin 2\delta. \tag{14.1.17}$$

It is our assumption that ξ, as a field of reference, represents a purely "electric" gauge field strength. The 3-dimensional vectors of the "electric" and "magnetic" part of the Yang–Mills field strengths are defined by

$$\vec{E} = \{E_j := F_{j0} | j = 1, 2, 3\} \tag{14.1.18}$$

and

$$\vec{B} = \{B_j := \frac{1}{2}\epsilon_{ijk} F^{jk} | i, j, k = 1, 2, 3\} \tag{14.1.19}$$

with regard to a Gaussian frame of reference. It then follows from our above-mentioned assumption that

$$\xi \wedge \xi = 0 \quad (= 2\,\vec{E} \cdot \vec{B} \text{ in 3-dimensional vector notation}) \tag{14.1.20}$$

is valid for an *extremal* field. However, null fields, e.g., pure radiation fields for which additionally

$$\xi \wedge \xi = \xi \wedge {}^*\xi = 0 \tag{14.1.21}$$

holds, shall be excluded for the present, since otherwise, the angle of complexion would remain undetermined. (Geometrodynamics in the case of null fields has already been analyzed in detail by GEROCH (1966).) Equation (14.1.17) yields the angle δ of the applied duality rotation implicitly by

$$\tan 2\delta = (-1)^z \frac{\mathrm{Tr}^*(\Omega \wedge \Omega)}{\mathrm{Tr}^*(\Omega \wedge {}^*\Omega)}. \tag{14.1.22}$$

(In order to carry out the division, it is necessary to form scalars by first applying the duality operator and then the trace operation.) After eliminating the trigonometric functions, the Yang–Mills-like invariant (14.1.16) of the extremal field ξ is calculated to be

$$\xi \wedge {}^*\xi = \pm \frac{\Omega \wedge {}^*\Omega \ \mathrm{Tr}^*(\Omega \wedge {}^*\Omega) - (-1)^{2z}\, \Omega \wedge \Omega \ \mathrm{Tr}^*(\Omega \wedge \Omega)}{\sqrt{(\mathrm{Tr}^*(\Omega \wedge {}^*\Omega))^2 + (-1)^{2z}\, (\mathrm{Tr}^*(\Omega \wedge \Omega))^2}}. \tag{14.1.23}$$

Thus, the original Yang–Mills curvature Ω is also *algebraically* transformed into an extremal, purely "electric" field strength ξ. However, the original curvature form can be regenerated out of this field of reference by means of inverting (14.1.15). This yields

$$\Omega = e^{*\delta}\xi. \tag{14.1.24}$$

The invariance

$$\Sigma_{\mathrm{YM}}(\Omega) = \Sigma_{\mathrm{YM}}(\xi) \tag{14.1.25}$$

of the energy–momentum current of the gauge fields with respect to duality rotations taken together with the postulated extremal property (14.1.20) of ξ, however, makes it possible to state a simple form for the "square" of the 4-dimensional stress–energy tensor. The calculation shows that

$$\begin{aligned} \Sigma_{\mathrm{YM}}(\Omega) \wedge \Sigma_{\mathrm{YM}}(\Omega) &= \frac{1}{4\alpha_g} \mathrm{Tr}\left\{ -\xi \wedge \xi \wedge (\vartheta \wedge {}^*\xi) \wedge (\vartheta \wedge {}^*\xi) \right. \\ &\left. + \vartheta \wedge \vartheta \wedge {}^*(\xi \wedge {}^*\xi)^2 - \vartheta \wedge {}^*\xi \wedge {}^*(\vartheta \wedge {}^*\xi) \wedge {}^*(\xi \wedge {}^*\xi) \right\} \\ &= \vartheta \wedge \vartheta \wedge {}^*(\Sigma_{\mathrm{YM}}(\xi) \wedge {}^*\Sigma_{\mathrm{YM}}(\xi)) \end{aligned} \tag{14.1.26}$$

is a multiple of the unit 2-form. However, this property of the energy–momentum current induces the algebraic relation

$$(\vartheta \wedge {}^*\Omega^g) \wedge (\vartheta \wedge {}^*\Omega^g) = \vartheta \wedge \vartheta \wedge {}^*(\vartheta \wedge {}^*\Omega^g \wedge {}^*(\vartheta \wedge {}^*\Omega^g)) \tag{14.1.27}$$

in vacuum due to Einstein's field equations. Most remarkably, this relation involves solely the 2-form of the spacetime curvature, which is equivalent to the condition

$$\boxed{R^{\{\}}_{i\alpha} R^{\{\}\alpha}_{\cdot j} = \frac{1}{4} g_{ij} R^{\{\}}_{\alpha\beta} R^{\{\}\alpha\beta}} \tag{14.1.28}$$

for the holonomically written components $R^{\{\}}_{ij}$ or the Ricci tensor. Combined with the requirement (14.1.11) of a vanishing scalar curvature, these relations represent the *algebraic Rainich conditions* of geometrodynamics (RAINICH 1925; MISNER & WHEELER 1957).

Differential Rainich conditions can be drawn not only from the effects of the duality transformation (14.1.12) on the Yang–Mills equations (14.1.4) but also from the Bianchi identity (14.1.5). By inserting (14.1.24) into these equations, we get

$$\begin{aligned} D^*\Omega &= (D^*\xi - (-1)^z \xi \wedge D\delta)\cos\delta \\ &\quad - (-1)^z (D\xi + (-1)^z\, {}^*\xi \wedge D\delta)\sin\delta = \tau_e \end{aligned} \tag{14.1.29}$$

and

$$\begin{aligned} D\Omega &= (D\xi + (-1)^z\, {}^*\xi \wedge D\delta)\cos\delta \\ &\quad + (-1)^z (D\, {}^*\xi - (-1)^z \xi \wedge D\delta)\sin\delta = 0. \end{aligned} \tag{14.1.30}$$

In the following, it is assumed that the current of purely electric charges comes into existence via the "chiral" transformation

$$\tau_e = \overline{\tau}_e \cos\delta - (-1)^z\, \overline{\tau}_p \sin\delta \tag{14.1.31}$$

out of the matter currents $\bar{\tau}_e$ and $\bar{\tau}_p$ of a theory (SCHWINGER 1968; JULIA & ZEE 1975) with hypothetical dually charged particles, the so-called *dyons*.

Then the combination of (14.1.29) and (14.1.30) results in the following equations for the "extremal" field of reference:

$$D\, {}^*\xi - (-1)^z \xi \wedge D\delta = \overline{\tau}_e, \tag{14.1.32}$$

$$D\, \xi + (-1)^z\, {}^*\xi \wedge D\delta = \overline{\tau}_p. \tag{14.1.33}$$

With a covariantly constant angle of complexion δ, e.g., for

$$D\delta = d\delta = 0, \tag{14.1.34}$$

we thus manage to solve formally the specific dichotomy between the field equations (14.1.4) with sources and the source-free Bianchi identity (14.1.5), which is typical for gauge theories, since now we are also taking into consideration the nonvanishing current $\bar{\tau}_p$ of the additional magnetic charge carried by the dyons.

If one starts the other way round with DIRAC's theory of monopoles (1931, cf. MIELKE 1985), and generalizes it to a theory of dually charged particles having nonvanishing currents $\bar{\tau}_e$ and $\bar{\tau}_p$, then the equations (14.1.32) and (14.1.33) can be transformed back into the original Maxwell equations by means of a *constant* duality rotation resulting from (14.1.31). The source-free Yang–Mills equations are invariant even with respect to constant duality transformations. This shows that a theory operating with dually charged particles is empirically indistinguishable from traditional electrodynamics, which is defined by the choice $\delta = 0$ of a "dual gauge" (STRAZHEV & TOMIL'CHIK 1973). Historically, it corresponds to the understanding that an electron carries solely *electric* charge.

In a curved spacetime, however, nonlinear effects that are induced by gravity are unavoidable; cf. PARKER (1975). Therefore, let us stick to the general field equations for the extremal field strengths in vacuum and let us ask for their geometric meaning. After the exterior multiplication of (14.1.32) by $\xi \wedge \vartheta$ and (14.1.33) by ${}^*\xi \wedge \vartheta$ and the subsequent addition and forming of the trace, we obtain the relation

$$\begin{aligned}(-1)^z \operatorname{Tr}\{\vartheta \wedge {}^*\xi \wedge {}^*\xi - \vartheta \wedge \xi \wedge \xi\} \wedge d\delta &= \operatorname{Tr}\{\xi \wedge \vartheta \wedge D^*\xi + {}^*\xi \wedge \vartheta \wedge D\xi\} \\ = \frac{1}{2} D \operatorname{Tr}\left\{{}^*(\xi \wedge {}^*(\vartheta \wedge {}^*\xi) - {}^*\xi \wedge {}^*(\vartheta \wedge \xi))\right\} \\ = (-1)^s 2\alpha_g D\Sigma_{\mathrm{YM}} = \frac{\alpha_g}{\ell^{*2}} D(\vartheta \wedge {}^*\Omega^g) \equiv 2\frac{\alpha_g}{\ell^{*2}} \Theta \wedge {}^*\Omega^g.\end{aligned} \tag{14.1.35}$$

It is obvious that the left side of this equation is zero, due to the extremal property (14.1.20) of ξ, while the right side is equal to the Bianchi identity (14.1.7) via Einstein's field equations (14.1.6). The coupled Einstein–Yang system, as has already been mentioned, does not produce any torsion, e.g., $\Theta = 0$, and (14.1.35) is thus only an identity.

Conversely, the multiplication of (14.1.32) by ${}^*\xi \wedge \vartheta$ and also that of (14.1.33) by $\xi \wedge \vartheta$ and the subsequent addition results in the total relation

$$\begin{aligned}&(-1)^z \operatorname{Tr}\{{}^*\xi \wedge \vartheta \wedge \xi - \xi \wedge \vartheta \wedge {}^*\xi\} \wedge d\delta \\ &= \operatorname{Tr}\{{}^*\xi \wedge \vartheta \wedge D\,{}^*\xi + \xi \wedge \vartheta \wedge D\xi\} \\ &= -\frac{1}{2} D\operatorname{Tr}\{{}^*\xi \wedge \vartheta \wedge {}^*\xi - \xi \wedge \vartheta \wedge \xi\}.\end{aligned} \tag{14.1.36}$$

Corresponding to the second notation of the energy–momentum current (14.1.8) of the gauge fields, Eq. (14.1.36) means simply

$$(-1)^z \Sigma_{\mathrm{YM}}(\xi) \wedge d\delta = D\Sigma_{\mathrm{YM}}(\xi). \tag{14.1.37}$$

In order to obtain the solvability of this equation for the 4-dimensional gradient $d\delta$, Eq. (14.1.37) has to be contracted by ${}^*\Sigma_{\mathrm{YM}}(\xi)$. On account of the invariance (14.1.25) of the energy–momentum current with respect to duality rotations and also on account of the validity of Einstein's vacuum field equations (14.1.6), we then have the relation

$$
\begin{aligned}
d\delta &= (-1)^{-z}\, {}^*\Sigma_{\mathrm{YM}}(\Omega) \wedge D\Sigma_{\mathrm{YM}}(\Omega) \big/ {}^*({}^*\Sigma_{\mathrm{YM}} \wedge \Sigma_{\mathrm{YM}}) \\
&= (-1)^{-z}\, {}^*(\vartheta \wedge {}^*\Omega^g) \wedge D(\vartheta \wedge {}^*\Omega^g) \big/ {}^*(\vartheta \wedge {}^*\Omega^g \wedge {}^*(\vartheta \wedge {}^*\Omega^g)). \quad (14.1.38)
\end{aligned}
$$

This means that the 4-dimensional gradient of the angle δ of the complexion is solely determined by the components $R^{\{\}}_{ij}$ of the Ricci tensor. This peculiar result, which was found by RAINICH (1925), reads in local notation (MISNER & WHEELER 1957)

$$
\boxed{\partial_\kappa \delta = \sqrt{|g|}\,\varepsilon_{\kappa\lambda\rho\sigma}\; R^{\{\}\sigma}{}_{.\,\nu} \nabla^\rho R^{\{\}\nu\lambda} \big/ (R^{\{\}}_{\alpha\beta} R^{\{\}\alpha\beta}).} \quad (14.1.39)
$$

In a simply connected spacetime manifold, the angle δ of complexion can be determined, by way of integration, to be

$$
\delta = \int_{c_1} d\delta + \delta_0. \quad (14.1.40)
$$

Due to the identity $dd\delta \equiv 0$, this follows in an unequivocal way from the exactness of the form $d\delta$. For manifolds with Betti number $b_1 = \dim H^1(M) = \nu$, e.g., for those that exhibit ν-fold global connectivity, however,

$$
\oint d\delta = 2\pi b_1 = 2\pi\nu \quad (14.1.41)
$$

is to be regarded as an absolutely necessary result (MISNER & WHEELER 1957; see also GEROCH 1966).

Thus our procedure, which is only slightly beyond the Rainich program, yields the following concluding theorem.

Theorem *The coupled Einstein–Yang–Mills system implies* necessarily *not only the algebraic conditions (14.1.11) and (14.1.28), but also the differential relation (14.1.39) of the Rainich geometry.*

Conversely, the question arises whether such a geometry provides all the required information on the gauge field strengths.

On account of the loss of information in the forming of traces, this seems to be guaranteed only for the abelian theory, e.g., for gravitationally coupled electromagnetic fields. Concerning this case, MISNER & WHEELER (1957) showed that an "extremal Maxwell root" of the Ricci tensor can be determined via

$$
\xi_{\mu\nu}\xi_{\rho\sigma} = -\frac{1}{2}\overline{R}^{\{\}}_{\mu\nu\rho\sigma} - \frac{1}{2}\frac{\overline{R}^{\{\}}_{\mu\nu\gamma\delta}\overline{R}^{\{\}\,\gamma\delta}_{\rho\sigma}}{R^{\{\}}_{\alpha\beta}R^{\{\}\alpha\beta}}. \quad (14.1.42)
$$

It has to be kept in mind that the Ricci tensor is present in (14.1.42) not only by $R^{\{\}}_{\alpha\beta}$ but also via the components of the anti-self-double-dual curvature tensor indicated by an "overline". Consequently, the "extremal Maxwell root" satisfies Einstein's vacuum field equations on account of the algebraic Rainich conditions (14.1.11) and (14.1.28). Furthermore, the electromagnetic field strengths that have been constructed according to (14.1.24) then satisfy Maxwell's equations (14.1.4) and (14.1.5) if the angle of complexion δ has been determined geometrically by way of integration of (14.1.39).

Encouraged by this remarkable result, WHEELER (1962, 1969) was inclined not only to look on Maxwell's theory of electromagnetism coupled to gravity as an already unified theory, but to advance a similar geometrization of other fundamental physical fields and furthermore, to look for the origin of central characteristics of particles such as mass and charge (SALAM 1980) in the structure of the spacetime topology; see also MIELKE (1977).

Nevertheless, it it is clear that the "already unified field theory" or "geometrodynamics" of Rainich, Misner, and Wheeler has left behind some unsolved problems.

As has been hinted at, the case of null fields, e.g., pure radiation fields, needs to be treated separately (NORDTVEDT & PAGELS 1962; GEROCH 1966). Moreover, the geometrization of the Einstein–Maxwell system, which originally is linear in the spacetime curvature, seems to lead back to quadratic Yang–Mills-like theories of gravity. Similar to the field equation of SKY gravity, the Rainich conditions (14.1.28) turn out to be nonlinear in the partial derivatives of second order in terms of the metric. On the other hand, the anti-self-double-dual solutions of the SKY gravity satisfy the algebraic Rainich conditions (MIELKE 1981). Nevertheless, the integration of the Rainich geometry into the general framework of the qPG-theory seems to be intricate, cf. Gomes (2015). This is at least the implication of an attempt of SHARP (1959) to develop a Lagrangian formalism of geometrodynamics.

It is naturally of equal importance to examine as well to what extent other fields can be incorporated into this scheme of geometrization. That it is possible for a scalar field that is coupled to GR has been shown by KUCHAŘ (1963); see also BRILL (1964). Thereby, the covariant generalization of the Klein–Gordon equation can be derived from Einstein's field equations via the contracted Bianchi identity. Within the frame of the Einstein–Cartan theory, the geometrization of the electromagnetic fields has been extended to the case of *massive* vector bosons by BIČÁK (1966). Torsion serves as an additional vehicle to make geometrically determinable not only the counterpart of the electromagnetic field strengths, but also the vector field itself. The Rainich geometrization of spinor fields proceeds analogously. According to the Cartan equation, the canonical spin tensor is proportional to the modified torsion in the Einstein–Cartan theory. Conversely, it is then possible to determine the spinor field on the basis of a given geometry by means of this algebraic relation; see KUCHAŘ (1966), and also the highly informative study by INOMATA (1971). An interesting biframe approach has been advanced by HAMMON & NORRIS 1990. The relation to Killing–Yano tensors has been studied by FERRANDO & SÁEZ (2003).

To conclude, we note that it is impossible to regain the complete structure of a Yang–Mills field by means of the Rainich geometry. Nonabelian gauge fields remain "hidden," and it is exactly this deficiency that displays illustratively the limits of what can be achieved within the Rainich program.

14.2 Nonabelian Kaluza–Klein Theories

Another geometrically attractive possibility for a unified description of fundamental interactions is based on the idea, due to Kaluza and Klein, that spacetime has more than four dimensions. In this framework, one starts from a theory of gravity (or supergravity) in $n = 4 + K$ dimensions, which is then shown or assumed to exhibit spontaneous compactification to four dimensions, such that the remaining K coordinates parametrize a compact "internal" manifold B that is sufficiently small and undetectable by present-day experiments.

Mathematically, in all those theories of electroweak and strong interactions that are constructed on the basis of Yang–Mills-type gauge potentials,[4] the "internal" fibers are only loosely tied to the spacetime manifold. This is decisively different from the case of gauge theories of gravity, since in those models, the spacetime continuum, via the soldering 1-form ϑ, is rather closely interlocked with the locally isomorphic tangent bundle. It is for this reason that gravitational gauge theories grant a more direct geometric interpretation. Since in these theories the mass of particles is accountable for the dynamically induced curvature of the spacetime and furthermore, its spin for the contortion, the question arises naturally whether other characteristics of particles, such as electric charge, isospin, and hypercharge, have a similar geometric origin. A possible response is to unite the abstract "internal" space on which the symmetry group G acts with the curved spacetime manifold. For this purpose, both of them have to be embedded in apt higher-dimensional spaces. Ultimately, this would mean to put the internal quantum numbers of particles on a par with energy and momentum by means of geometric concepts (Salam 1980).

As far as the case of electromagnetism is concerned, Kaluza (1921) and Klein (1926, 1928) proved that this is possible after proceeding to 5-dimensional[5] manifolds. Later, DeWitt (1964) and other scientists (Kerner 1968; Cho & Jang 1975; Mansouri & Chang 1976; Rayski 1977; Witten 1981; Orzalesi 1981; Mecklenburg 1980, 1984) extended this method of unification to Yang–Mills fields with an arbitrary nonabelian structure group. Bergmann (1942) draws parallels not only to the projective theories of relativity, e.g., the Jordan–Thiry theory, cf. Schmutzer (1968), but also to the unified field theories of Einstein & Mayer (1931).

[4] Interestingly enough, London (1927) deliberately made use of the 5-dimensional Kaluza–Klein spaces, too, while applying for the first time Weyl's principle of gauge invariance to Schrödinger's theory of quantum mechanics.

[5] Wilhelm Busch, alluding to the would-be poet Balduin Bählamm:

> "Und zieht als freier Musensohn
> In die Poetendimension,
> Die fünfte, da die vierte jetzt
> Von Geistern ohnehin besetzt".

In order to adumbrate the foundation of Kaluza–Klein theories, it is of convenience to make use of the calculus of exterior forms in higher dimensions, as has already been done by THIRRING (1972), KALINOWSKI (1981b), STRAUMANN (1986). The starting point for our considerations has to be seen in a higher-dimensional manifold M^{4+K} that is endowed with a bundle $\hat{L}(M^{4+K})$ of linear frames and is furthermore equipped with a canonical 1-form $\hat{\vartheta}$ associated with $\hat{L}(M^{4+K})$. Only later does one suppose that the ground state of the higher-dimensional system, because of some dynamical mechanism, cf. CREMMER & SCHERK (1976), is partially compactified such that M^{4+K} takes on the form $\hat{E} \approx M^4 \times B$, where B is a (non)compact K-dimensional homogeneous space, G/H. Here G denotes a semisimple Lie group of dimension K = dim G, and H an isotropic subgroup. As shown by COQUEREAUX & JADCZYK (1983), the "geometric arena" is not the fiber bundle $P(M^4, G, \pi, \delta)$ but $P(M^4, N/H, \pi, \delta)$, where N denotes the normalizer of H in G. The extended space $\hat{E}$, which is locally isomorphic to $M^4 \times G/H$, is globally the associated bundle $\hat{E}(M^4, F = G/H, N/H, P)$ with N/H as structure group.

The graviton and a multiplet of Yang–Mills-type vector bosons should be included among the zero-mass fields of the four-dimensional effective theory. Following LUCIANI (1978), cf. SALAM & STRATHDEE (1982), this can be achieved by parameterizing the 1-form $\hat{\vartheta}$ ("Vielbeinfield") in matrix notation as follows:

$$\hat{\vartheta} = \hat{\vartheta}(m, g) = \underbrace{\left[\begin{array}{c|c} \vartheta & \hat{\ell}\omega \\ \hline 0 & \overset{\circ}{\vartheta} \end{array}\right]}_{4 \quad K} \begin{array}{l} \}4 \\ \}K \end{array} . \tag{14.2.1}$$

Here ϑ denotes the canonical 1-form that is soldered to the 4-dimensional spacetime, and $\omega = \omega^j{}_\alpha L_j \otimes \vartheta^\alpha$ is a 1-form with values in the Lie algebra of G or N/H, respectively. After the reduction of the dimensions, it will locally represent Yang–Mills-type gauge fields. The 1-form $\overset{\circ}{\vartheta}$ is defined on the cotangent bundle $T_e{}^*(G)$ of the group G, being parametrized by the coordinates ξ^j. In its expansion

$$\overset{\circ}{\vartheta} = \varphi_j d\xi^j, \tag{14.2.2}$$

the coefficients φ_j may be regarded as a multiplet of (Higgs-type) scalar fields transforming with respect to the adjoint representation of G. This is due to the fact that the (holonomic) basis $d\xi^j$ is "dual" to the infinitesimal generators I_j of G. Moreover, as a frame "dual" to the Lie vector field $V = V^j I_j$, $\overset{\circ}{\vartheta}$ satisfies the Maurer–Cartan equation, i.e.,

$$\overset{\circ}{d}\overset{\circ}{\vartheta} = \overset{\circ}{\vartheta} \wedge \overset{\circ}{\vartheta}. \tag{14.2.3}$$

A connection $\overset{\circ}{\omega}$ on G arises naturally via the Ansatz

$$\overset{\circ}{\omega} = b\overset{\circ}{\vartheta}. \tag{14.2.4}$$

The *internal* torsion $\overset{\circ}{\Theta}$ and curvature $\overset{\circ}{\Omega}$, respectively, are obtained from Cartan's structure equations and yield the following results:

$$\overset{\circ}{\Theta} := \overset{\circ}{d}\overset{\circ}{\vartheta} - [\overset{\circ}{\omega}, \overset{\circ}{\vartheta}] = (1-2b)\overset{\circ}{\vartheta} \wedge \overset{\circ}{\vartheta}, \tag{14.2.5}$$

$$\overset{\circ}{\Omega} := \overset{\circ}{d}\overset{\circ}{\omega} - \overset{\circ}{\omega} \wedge \overset{\circ}{\omega} = b(1-b)\overset{\circ}{\vartheta} \wedge \overset{\circ}{\vartheta}. \tag{14.2.6}$$

Essentially, two cases are to be distinguished (WU & ZEE 1984).

$b = 1/2$: torsionless connection in a symmetric coset space B of constant curvature $\overset{\circ}{R} = 1/4$.

$b = 0, 1$: Flat connection *with* parallelizing *torsion*.

Analogously, the metric of the (4 + K)-dimensional extended space is rendered by

$$d\hat{s}^2 = \frac{1}{4+K}\mathrm{Tr}(\hat{\vartheta} \otimes_s \hat{\vartheta}^{\mathrm{T}}) = \hat{g}_{AB}\hat{\vartheta}^A \otimes_s \hat{\vartheta}^B; \quad A, B, = 0, \ldots, 3+K$$
$$= \frac{1}{4+K}\mathrm{Tr}\left[\begin{array}{c|c} \vartheta \otimes \vartheta + \hat{\ell}^{\,2}\omega \otimes \omega & \hat{\ell}\omega \otimes \overset{\circ}{\vartheta} \\ \hline \hat{\ell}\overset{\circ}{\vartheta} \otimes \omega & \overset{\circ}{\vartheta} \otimes \overset{\circ}{\vartheta} \end{array}\right]. \tag{14.2.7}$$

This is similar to the Ansatz that was selected by KALUZA (1921) for the abelian case and which was generalized by KERNER (1968) to nonabelian symmetry groups G. The term $\mathrm{Tr}(\overset{\circ}{\vartheta} \otimes \overset{\circ}{\vartheta})$ just gives the internal line element

$$d\overset{\circ}{s}{}^2 = \overset{\circ}{g}_{ij} d\xi^i \otimes_s d\xi^j, \tag{14.2.8}$$

where $\overset{\circ}{g}_{ij}$ denotes the Cartan–Killing metric of G. As far as *simple* Lie groups are concerned, $\overset{\circ}{g}_{ij}$ is the Riemannian or pseudo-Riemannian metric of G regarded as a group manifold.[6] It should be noted that the group metric $\overset{\circ}{g}_{ij}$ is involved not only in the terms $\mathrm{Tr}(\overset{\circ}{\vartheta} \otimes \overset{\circ}{\vartheta})$ and $\mathrm{Tr}(\omega \otimes \overset{\circ}{\vartheta})$ but also in $\mathrm{Tr}(\omega \otimes \omega)$, since ω takes values in the adjoint representation of G.

For a different choice of coordinates, the squared line element (14.2.7) takes on a diagonal block form, thus representing a fiber metric (TRAUTMAN 1970) of $\hat{E}$. For classical, i.e., nonquantized models, the total signature of the line element $d\hat{s}^2$ has to be $s = 1$, as will be shown in the following (see, however, SAKHAROV 1984).

In analogy to the gravitational case, a linear metric-compatible connection

[6] More precisely, the group manifold of G is the symmetric space $G \times G/G_D$, where G_D is the diagonal subgroup of $G \times G$.

$$\hat{\omega} = \left[\begin{array}{c|c} \omega^g & -\alpha \\ \hline \alpha^{\mathrm{T}} & \overset{\circ}{\omega} \end{array}\right] =: \omega^g \oplus \overset{\circ}{\omega} - \hat{\alpha}; \quad \hat{\alpha} = \left[\begin{array}{c|c} 0 & \alpha \\ \hline -\alpha^{\mathrm{T}} & 0 \end{array}\right] \tag{14.2.9}$$

may be imprinted on the bundle $\hat{L}(\hat{E})$ of the linear frames. Here ω^g denotes the gravitational connection in $L(M^4)$, $\overset{\circ}{\omega}$ a connection in the group manifold, and $\hat{\alpha}$ a "contortion"-type 1-form *mediating* between the space of "internal" symmetries and the "exterior" spacetime. (If we regard our construction as a local embedding of spacetime into a (4 + K)-dimensional manifold, α is simply the so-called *second fundamental form*.) The postulate of metric compatability of the associated covariant derivative $\hat{D}$ implies that

$$\hat{\omega}^{AB} = -\hat{\omega}^{BA}, \quad A, B = 0, \ldots, 3 + K \tag{14.2.10}$$

is valid. Connections possessing a nonvanishing (4 + K)-dimensional torsion and nonmectricity tensor as well have been considered by KOPCZYNSKI (1980).

Accordingly, the higher-dimensional Riemann–Cartan curvature 2-form $\hat{\Omega}$, given by a generalization of the structure equation, is dissociating into

$$\hat{\Omega} = \hat{d}\hat{\omega} - \hat{\omega} - \hat{\omega} \wedge \hat{\omega} = \left[\begin{array}{c|c} \Omega^g + \alpha \wedge \alpha^{\mathrm{T}} & -D\alpha \\ \hline (\overset{\circ}{D}\alpha)^{\mathrm{T}} & \overset{\circ}{\Omega} + \alpha^{\mathrm{T}} \wedge \alpha \end{array}\right]. \tag{14.2.11}$$

In our matrix notation, the (4 + K)-dimensional exterior derivative $\hat{\mathrm{d}}$ is given by

$$\hat{d} := \left[\begin{array}{c|c} d & 0 \\ \hline 0 & \overset{\circ}{d} \end{array}\right], \tag{14.2.12}$$

whereas the covariant derivative

$$\overset{(\circ)}{D}\alpha := \overset{(\circ)}{d}\alpha - \omega^g \wedge \alpha - \alpha \wedge \overset{\circ}{\omega} \tag{14.2.13}$$

of the second fundamental form α with respect to spacetime or group coordinates depends on the external connection ω^g and on the connection $\overset{\circ}{\omega}$ in the group manifold as well. Regarded as an embedding condition, the matrix presentation (14.2.11) of our result comprises both the *equations of Gauss* and those of *Codazzi and Mainardi* (cf. SULANKE & WINTGEN 1972).

The higher-dimensional manifold has in general a torsion that is to be determined by means of the covariant exterior derivative of the canonical 1-form $\hat{\vartheta}$ in correspondence with the structure equation, i.e.,

$$\hat{\Theta} := \hat{D}\hat{\vartheta} = \hat{d}\hat{\vartheta} - [\hat{\omega}, \hat{\vartheta}]$$
$$= \left[\begin{array}{c|c} \Theta - \hat{\ell}\,\omega \wedge \alpha^{\mathrm{T}} & \hat{\ell}(d\omega - \omega^{g} \wedge \omega - \omega \wedge \overset{\circ}{\omega}) + \vartheta \wedge \alpha + \alpha \wedge \overset{\circ}{\vartheta} \\ \hline -\alpha^{\mathrm{T}} \wedge \vartheta - \overset{\circ}{\vartheta} \wedge \alpha^{\mathrm{T}} & \overset{\circ}{\Theta} - \hat{\ell}\,\alpha^{\mathrm{T}} \wedge \omega \end{array}\right]. \tag{14.2.14}$$

Relevant for the following is the "intermediating torsion" piece

$$\Theta^{\perp} := \pi(\hat{\Theta} - [\hat{\alpha}, \hat{\vartheta}]) = \left[\begin{array}{c|c} \Theta & \hat{\ell}\Omega' \\ \hline 0 & \pi\overset{\circ}{\Theta} = 0 \end{array}\right], \tag{14.2.15}$$

which after projection to its spacetime components measures the "twist" of the internal space B relative to the four-dimensional spacetime M^4 in the bundle $\hat{E}$. A similar result is stated by Kalinowski (1981a). The relation (14.2.15) contains a Yang–Mills-type curvature 2-form that is modified by a term yielding an additional gravitational coupling. For obvious reasons, it is therefore to be suggested to render the 2-form of the Yang–Mills field strengths in the presence of gravitational fields by

$$\Omega' := \pi(d\omega - \omega \wedge \overset{\circ}{\omega} - \omega^{g} \wedge \omega) = \Omega - \omega^{g} \wedge \omega. \tag{14.2.16}$$

Conventionally, the principle of "minimal" coupling to gravity is not applied in full with respect to massless gauge fields, since the additional gravitational coupling breaks the gauge covariance of (14.2.15), or the gauge invariance of the Yang–Mills-like Lagrangian 4-form that is formed from it (Hehl et al. 1976).

With the dissociation (14.2.11) of the higher-dimensional curvature 2-form $\hat{\Omega}$ as well as with the relation (14.2.15) between the projected "intermediating" contortion $\hat{\alpha}$ and the 2-form Ω' of the Yang–Mills field strengths, we have obtained the decisive tools that we need to prove that the interacting Einstein–Yang–Mills system can be generated out of a (4 + K)-dimensional Lagrangian form of Einstein–Hilbert–Weyl type. Here we closely follow Einstein (1928) and proceed analogously to the proof concerning the equivalence of the theory of teleparallelism and GR. Then the relation

$$\hat{L}_{\mathrm{W.}} = \frac{1}{2}\mathrm{Tr}(\hat{\Omega} \wedge \underbrace{\hat{\vartheta} \wedge \ldots \wedge \hat{\vartheta}}_{2+\mathrm{K\ termms}})$$
$$= \hat{L}^{g}_{\mathrm{W.}} + \hat{L}^{G}_{\mathrm{W.}} + \ell^{*2}\hat{L}_{||} - \frac{1}{2}d\mathrm{Tr}(\hat{\alpha} \wedge \hat{\vartheta} \wedge \cdots \wedge \hat{\vartheta}) \tag{14.2.17}$$

holds for the Lagrangian (4 + K)-form of Weyl type, where

$$\hat{L}_{||} = -\frac{1}{2\ell^{*2}}\mathrm{Tr}[\hat{\alpha} \wedge \hat{\alpha}, \quad \hat{\vartheta} \wedge \cdots \wedge \hat{\vartheta}] \tag{14.2.18}$$

represents the analogue of the Lagrangian 4-form of the theory of teleparallelism. After projecting onto the 4-dimensional world, the form (14.2.18) is exactly altered to the gravitational coupled generalization

$$\pi \hat{L}_{||} = -\frac{\hat{\ell}^2}{\ell^{*2}} \mathrm{Tr}(\Omega' \wedge {}^*\Omega' \wedge \underbrace{\overset{\circ}{\vartheta} \wedge \cdots \wedge \overset{\circ}{\vartheta}}_{\text{K terms}})$$
$$= -\frac{1}{\alpha_g} \mathrm{Tr}(\Omega' \wedge {}^*\Omega') \sqrt{\det \overset{\circ}{g}} = L'_{\mathrm{YM}}\, d_\mu(G) \tag{14.2.19}$$

of the Yang–Mills-type Lagrangian (4 + K)-form, provided the relation (14.2.9) for the intermediating torsion is inserted in (14.2.18) and if

$$\hat{\ell} = (\overset{+}{-})\ell^* / \sqrt{\alpha_g} = (\overset{+}{-})\ell^* \frac{\sqrt{\hbar c}}{g} \tag{14.2.20}$$

is additionally—here following KLEIN (1955)—taken as Planck's length with the modification by the gauge-coupling constant $\alpha_g := g^2/\hbar c$. The term $\hat{L}^G_{\mathrm{W}}$ depends essentially on the internal curvature $\overset{\circ}{\Omega}$. By projection to the spacetime manifold and subsequent integration over the group manifold, this term provides only a constant-volume form that corresponds to a "cosmological" constant of microscopic origin, whose value is determined by the scalar curvature $\overset{\circ}{R}$ and by the density $\sqrt{\det \overset{\circ}{g}}$ of the group volume of the Lie group G, which is here considered a manifold.

Generally, all the terms encountered in (14.2.17) still depend on the internal coordinates ξ^j of B. This is due to the fact that the (4 + K)-dimensional metric tensor (14.2.7) carries an infinite number of new degrees of freedom corresponding to the propagation of excitations in the extra dimensions. In order to extract an effective four-dimensional theory for the "low-energy" phenomena, a preferably spontaneous compactification[7] of the additional dimensions is assumed, i.e., the emergence of a classical solution with the feature that

(i) $M^{4+K} \approx M^4 \times B$ locally and
(ii) $R_4 \gg R_K$ for the corresponding characteristic radii.

Then the metric (14.2.7) can be expanded via

$$\hat{g}_{AB}(x, \xi^j) = \sum_{n=-\infty}^{+\infty} g^{(n)}_{AB}(x) Y_n(\xi^j) \tag{14.2.21}$$

[7] In a quantized Kaluza–Klein theory, this reduction of the extra dimensions and an accompanying "spontaneous" symmetry breaking of the general (4 + K)-dimensional covariance group $\mathscr{D}(M^{4+K})$ is possibly brought about by vacuum fluctuations. Analogously to the Casimir effect in quantum electrodynamics, an attractive force between the "boundaries" of the additional dimensions ensues, which tends to diminish the characteristic radius $R_\circ$ of B to the order of the Planck length ℓ^* (TANAKA 1981; CHODOS 1984).

into a complete set of harmonic functions $Y_n(B)$ on $B = G/H$; cf. SALAM & STRATHDEE (1982). In the case of the original Kaluza–Klein theory, in which a closing up of the five-dimensional world to a cylinder $M^4 \times S^1$ of an exceedingly small radius $R_\circ$ is taken for granted, Eq. (14.2.21) degenerates to a Fourier expansion on the circle S^1. The coefficients in the expansion (14.2.21) are fields of short range on M^4, their mass scale being of the order of the Planck mass

$$M^* = \sqrt{\hbar c/G_{\mathrm{N}}} \simeq 10^{19}\mathrm{GeV}/c^2. \tag{14.2.22}$$

Hence only the so-called zero modes, which should include the graviton and a multiplet of Yang–Mills-type gauge potentials, are expected to be relevant for the low-energy sector.

Under these assumptions, the main result of the Kaluza–Klein construction can be stated as follows:

Consider (14.2.1) a parametrization of the canonical 1-form of a (4 + K)-dimensional Riemann–Cartan space of signature s = 1. Then the higher-dimensional Einstein–Hilbert-type Lagrangian (4 + K)-form after projection to four-dimensional components and restriction to the zero modes yields an effective dynamics that correspond to a coupled Einstein–Yang–Mills system with an induced cosmological term

$$\int_G \pi \hat{L}_{\mathrm{E.}} = \frac{1}{\ell^{*2}} L_{\mathrm{W.}} + L'_{\mathrm{YM}} + \frac{1}{2\ell^{*2}} \vartheta \wedge \vartheta \wedge \vartheta \wedge \vartheta \int_G \overset{\circ}{R} d\mu(G). \tag{14.2.23}$$

This important result was first deduced by (CHO 1975) in the general nonabelian case. (In order to show the analogy to the equivalence proof of the theory of teleparallelism and Einstein's GR, we have here resorted to a rather schematic presentation.) More detailed calculations in components can be found in KERNER (1968), CHO & JANG (1975), and also in MANSOURI & CHANG (1976), DUFF (1994) as well as BERGQVIST & LANKINEN (2005). More mathematically oriented presentations within the framework of the fiber bundle formalism are given by KOPCZYNSKI (1980) and KERNER (1983), while HERMANN (1978) in this context focuses on the theory of Riemannian submersions.

It has to be accentuated that the choice of the (4 + K)-dimensional Lagrangian form is of crucial significance for the proof of equivalence of the Kaluza–Klein theory and the coupled Einstein–Yang–Mills system. The fact that the Hilbert–Weyl-type action functional (14.2.17) can be projected onto the celebrated Lagrangian 4-form of GR and the 4-form of a Yang–Mills-type gauge theory is evaluated as one decisive criterion for the distinction of the Einstein–Cartan theory within the framework of Poincaré gauge theory (see, e.g., KOPCZYNSKI 1980). However, this argument cannot be considered a cogent one (cf. KATANAYEV & VOLOVICH 1985; HSU & YEUNG 1985a, b; ZWIEBACH 1985). Provided we had analogously to the quasilinear case of the Poincaré gauge theory chosen a quadratic Lagrangian (4 + K)-form as a starting point for our construction of a unified theory, the result would have been that, after the restriction to four dimensions, there would also have been quartic terms besides the quadratic ones with respect to the Yang–Mills field strengths. Then those essentially nonlinear gauge theories of Born–Infeld type[8] are recovered geometrically. Concerning the issue of quark confinement, it is exactly such dynamical structures that are considered appropriate for gauge fields with "internal" symmetries according to a proposal of MILLS (1979).

14.2.1 Standard Model Compactification

A realistic unified field theory should account for the local $SU(3)$ color symmetry of strong interactions and the $SU(2) \times U(1)$ gauge group of the Weinberg–Salam model of combined electroweak interactions. Thus, in a Kaluza–Klein approach, the internal space B should admit at least

$$G = SU(3) \times SU(2) \times U(1) \tag{14.2.24}$$

as a group of motions. This is the case for

$$B = \mathbb{C}P^2 \times S^2 \times S^1, \tag{14.2.25}$$

where $\mathbb{C}P^2 = SU(3)/U(2)$ denotes the complex projective two-space and S^n the n-dimensional sphere. The manifold (14.2.25) is not the only example of a seven-dimensional homogeneous space

$$G/H = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}, \tag{14.2.26}$$

the classification of which was initiated by WITTEN (1981) and then completed by RANDJBAR- DAEMI et al. (1984b).

For the realization of a metric ground state with compactified extra dimensions, a difficulty arises due to the fact that the Cartesian product $M^4 \times B$ of the Minkowski spacetime and a compact internal space B is, unlike $M^4 \times S^1$, in general not an Einstein space. In order to remedy this deficiency in, e.g., n = 11 dimensions, it is necessary to "squash" the (round) sphere $S^7 = SO(8)/SO(7)$ by partial rescalings (CASTELLANI et al. 1984) to the coset space

$$\widetilde{S}^{\,7} := \frac{SO(5) \times SO(3)}{SO(3) \times SO(3)}. \tag{14.2.27}$$

As observed by WITTEN (1981), K = 4 + 2 + 1 = 7 of (14.2.25) represents the minimum dimensionality for an internal manifold B admitting G as a transformation

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[8]In order to get exactly the Born–Infeld action functional, a Kaluza–Klein-type Lagrangian (4 + K)-form had to be considered that depends on the square root of the (4 + K)-dimensional scalar curvature (CHANG et al. 1976).

group. If supersymmetry is imposed as well as general covariance, there exist higher-dimensional models in which all fermionic and bosonic fields, together with the metric, belong to a single representation. Most remarkably, $n = 4 + 7 = 11$ is the *maximal* dimension of a theory of simple ($\mathscr{N} = 1$) supergravity, cf. DEWITT & STORA (1984), that contains the necessary fermion fields of spin $\bar{s} \leq 3/2$ without the bosonic fields exceeding the spin or helicity $\bar{s} = 2$ of the graviton. The bosonic part of such supersymmetric theories contains necessarily a completely antisymmetric tensor potential A_{IJK}, a remnant of the internal torsion, which may induce a compactification of M^{4+K}. In the case of the FREUND & RUBIN class (1980) of solutions, the underlying vacuum geometry factors into the product $M^{11} = (\mathrm{AdS})^4 \times S^7$ consisting of a four-dimensional anti-de Sitter spacetime and a seven-dimensional sphere. However, the scalar curvatures

$$^{(4)}R = -\frac{8}{3} R_\circ^{-2}, \quad ^{(7)}\overset{\circ}{R} = \frac{7}{3} R_\circ^{-2} \tag{14.2.28}$$

of *both* spaces are extremely large and consequently yield an admittedly very unphysical model of the real world. For the "squashed" seven-sphere $\tilde{S}^7$, there exists a solution (ENGLERT et al. 1983) with induced torsion that is Ricci-flat but breaks the supersymmetry "spontaneously;" cf. MIELKE (1986).

14.3 Fermion Spectrum from Higher-Dimensional Models

Nonabelian Kaluza–Klein theories provide us with a geometrically elegant but locally equivalent representation of the coupled Einstein–Yang–Mills system. However, in order to account for matter, spinor fields have to be brought into this geometric framework. In this respect, a possible nontrivial global topology of the $(4 + K)$-dimensional total space, if realized (DE WITT 1964), could be of fundamental significance for generalized concepts of interactions. It was KLEIN (1926), see also EINSTEIN & BERGMANN (1938), who utilized the topology $M^5 = \mathbb{R}^4 \times S^1$ of a cylinder for the originally 5-dimensional theory in order to get a physically acceptable interpretation of the charge quantization. The nexus between the theory of homology, the global existence of Riemannian metrics, and the charge quantization on topologically nontrivial 5-dimensional spaces that turns up here has been analyzed in more detail by MILLER (1980). Under these conditions, the charge conjugation

$$\varphi \to \varphi^{\mathrm{C}} := \overline{\varphi} \tag{14.3.1}$$

of a Yukawa-type scalar field φ with values in the adjoint representation of G, which is thought to be coupled to a Dirac field, can be interpreted as a reflection of the fifth, spacelike, coordinate. Concerning its geometric and topological meaning, the observed CPT invariance of all interactions is thus interpreted as an invariance with respect to the inversion of all spacelike coordinates (RAYSKI 1965a, b). A related deduction of the $SU(2)$ transformations of the isotopic spin out of reflections in the pseudo-Euclidean space $E_{2,4}$ of conformal symmetries was pointed out by Budini et al. (1979).

A revisiting and deepening scrutiny of the investigation of the Dirac equation in higher-dimensional Kaluza–Klein spaces, which was initiated by Rayski (1965a, b) and Thirring (1972), broadens our understanding of the adumbrated catalogue of questions. According to the general spinor formalism, the (4 + K)-dimensional analogue

$$\hat{L}_{\mathrm{D.}} = \frac{i}{2}(\bar{\hat{\psi}}\hat{\gamma} \wedge {}^*\hat{D}\hat{\psi} + (\overline{\hat{D}\hat{\psi}}) \wedge {}^*\hat{\gamma}\hat{\psi}) - \hat{m}\bar{\hat{\psi}}\hat{\psi}\eta \tag{14.3.2}$$

of the (symmetrized) Lagrangian n-form of the Dirac field can serve as a starting point. Employing the dissociation (14.2.9) of the connection 1-form $\hat{\omega}$, one obtains the equivalent representation

$$\hat{L}_{\mathrm{D.}} = \frac{i}{2}(\bar{\hat{\psi}}\hat{\gamma} \wedge {}^*\hat{D}^{g+\circ}\hat{\psi} + (\overline{\hat{D}^{g+\circ}\hat{\psi}}) \wedge {}^*\hat{\gamma}\hat{\psi}) - \hat{m}\bar{\hat{\psi}}\hat{\psi}\eta$$
$$+ \frac{1}{2}\bar{\hat{\psi}}[\hat{\alpha}, {}^*\hat{\gamma}]\hat{\psi}. \tag{14.3.3}$$

After projection onto four-dimensional components, the relation (14.2.9) can be applied for the contribution that depends on the "intermediating" contortion $\hat{\alpha}$. Furthermore, on a "compactified" space $M^4 \times B$, the following expansion of the spinor field $\hat{\psi}$ with $2^{[2+\mathrm{K}/2]}$ components is adopted:

$$\hat{\psi} = \sum_{F=1}^{\dim \rho(G)} (\psi_-^F(x)\chi_{\rho F}^-(\xi) + \psi_+^F(x)\chi_{\rho F}^+(\xi)). \tag{14.3.4}$$

Here the spinorial harmonics $\chi_{\rho F}^{\pm}(\xi)$ furnish a representation ρ of the symmetry group G, which are the eigenstates

$$\mathring{\gamma}^{K+1}\chi_{\rho F}^{\pm}(\xi) = \pm\chi_{\rho F}^{\pm}(\xi) \tag{14.3.5}$$

of the internal helicity operator $\mathring{\gamma}^{K+1}$. After projection π to four-dimensional components and integration over the group manifold, the Lagrangian (4 + K)-form (14.3.3) yields the 4-form

$$\int_G \pi\hat{L}_{\mathrm{D.}} = \frac{i}{2}\left(\overline{\psi}\gamma \wedge {}^*D^{g+\omega}\psi + (\overline{D^{g+\omega}\psi}) \wedge {}^*\gamma\psi\right)$$
$$+ \frac{\hat{\ell}}{2}\overline{\psi}\gamma \wedge \gamma \wedge {}^*\Omega\psi - \widehat{m}\overline{\psi}\psi\eta, \tag{14.3.6}$$

where

$$D^{g+\omega} := d + i\widetilde{\omega}^{g} \otimes i\omega \tag{14.3.7}$$

denotes a "combined" covariant derivative.

The variation of (14.3.6) with respect to $\bar{\psi}$ leads to a Dirac equation that is not only "minimally" coupled to gravity and the Yang–Mills fields, being mediated by the (spinor) connections $\tilde{\omega}^{g}$ and ω, respectively, but that in addition, inherits a gauge covariant term $\gamma \wedge \gamma \wedge {}^{*}\Omega\psi$ of electric dipole type. This term in the Lagrangian 4-form reads in tensor notation

$$L_{\text{anom.}} = \frac{\widehat{\ell}}{2}\overline{\psi}\gamma \wedge \gamma \wedge {}^{*}\Omega\psi = \frac{\ell^{*}}{4\sqrt{\alpha_g}}\overline{\psi}F_{\mu\nu}\gamma^{5}[\gamma^{\mu}, \gamma^{\nu}]\psi\eta. \tag{14.3.8}$$

It is dual to the corresponding magnetic dipole term being invoked within phenomenological theories of nuclear *form factors* (BJORKEN & DRELL 1964). This additional term not only gives rise to an anomalous "electric" moment for particles of spin 1/2 but also destroys the invariance of the equation with regard to the time and space reflections T and P. However, due to its coupling strength proportional to the Planck length ℓ^{*}, this results only in a *superweak* violation of the T or CP invariance.

In order to bring about the order of magnitude that has been measured by Fitch, Cronin, and Christensen concerning kaon decays, a Kaluza–Klein theory had to be considered—following THIRRING (1972)—that has a strongly interacting tensor component of short range. This suggestion resembles that on which the f–g theory of gravity is founded. The modification of the Dirac equation that is caused by the Kaluza–Klein theory may as well apply to the particle model of BARUT (1980), in which the absolutely stable particles p, e, and ν_e are regarded as the only primordial entities. Within the framework of that model, strong interactions between these absolutely stable particles are supposed to originate from the "naked" anomalous and allegedly very large *magnetic moments* of these fundamental building blocks. Following THIRRING (1972), such a magnetic dipole term, which is similar to the form (14.3.8), is introduced into quantum electrodynamics by the 5-dimensional unification of gravity and electromagnetism. In general, this also depends on the dimension of the group G (KALINOWSKI 1981b). According to today's canonical wisdom (CHENG & LI 1984), the matter core is composed of quarks and leptons, which are themselves grouped into fermion families. Since these building blocks are very light compared to the Planck mass M^{*}, which sets the scale in the Kaluza–Klein approach, they have to be represented in a first approximation by massless spinor fields. Consider the (4 + K)-dimensional Dirac equation

$$i\widehat{\not{D}}\widehat{\psi} := i\widehat{\gamma} \wedge {}^{*}\widehat{D}\widehat{\psi} = i\gamma \wedge {}^{*}D\widehat{\psi} + i\not{D}^{(int)}\widehat{\psi} = 0, \tag{14.3.9}$$

which follows from the Lagrangian (4 + K)-form in the case $\hat{\mathrm{m}} = 0$. This equation may be split into the usual four-dimensional part and a contribution resulting from the extra dimension as given above. The "internal" operator

$$\not{D}^{(int)} = \overset{\circ}{\gamma} \wedge {}^*\overset{\circ}{D} = \overset{\circ}{\gamma} \wedge {}^*(\overset{\circ}{d} + i\overset{\circ}{\tilde{\omega}}) \tag{14.3.10}$$

depends via the covariant derivative $\overset{\circ}{D}$ on the spinor connection $\overset{\circ}{\tilde{\omega}}$ and therefore acts only on the internal space B. Provided B is compact, $i\not{D}^{(\text{int})}$ has a discrete spectrum of eigenvalues that are zero or of the order of M^*. These eigenvalues constitute the mass matrix of the four-dimensional effective theory (Rayski 1965a, b). Since particle masses of the order of the Planck mass have not been observed, quarks and leptons have to be represented by the zero modes before the internal symmetry gets broken hierarchically.

Moreover, these fermions occur in nature only with one type of chirality (helicity), i.e., the angular momentum component being projected in the direction of propagation. Consequently, it has to be required for the effective four-dimensional theory that the left-handed or right-handed Weyl spinors, i.e., four-component Dirac spinors obeying $\gamma^5\psi = (\mp)\psi$, transform according to inequivalent representations of the symmetry group.

In a Kaluza–Klein-type approach, the chirality depends rather crucially on the dimension of the total space. For $n = 4 + K$, the spinor fields transform according to the fundamental representation of the covering tangent group $\widetilde{SO}(1, 3 + K)$. Inequivalent irreducible representations of this group are distinguished by the two eigenvalues of the n-dimensional chirality operator $\hat{\gamma}^{\,n+1}$. Under the subgroup $\widetilde{SO}(1, 3) \otimes \widetilde{SO}(K)$, such a representation branches according to

$$\begin{aligned} \psi_{\text{L}}: \quad & \gamma^5 = +1, \quad \overset{\circ}{\gamma}{}^{K+1} = +\widehat{\gamma}^{\,n+1} \\ \psi_{\text{R}}: \quad & \gamma^5 = -1, \quad \overset{\circ}{\gamma}{}^{K+1} = -\widehat{\gamma}^{\,n+1} \end{aligned} \tag{14.3.11}$$

into left- and right-handed four-dimensional Weyl spinors.

Then the following cases have to be distinguished:

(i) $n = 2p + 1$: The transformation $\hat{\gamma}^{\,A} \to -\hat{\gamma}^{\,A}$ does not affect the infinitesimal generators $\hat{\sigma}^{\,AB}$ but $\hat{\gamma}^{\,n+1} \to -\hat{\gamma}^{\,n+1}$. Since G-transformations are coordinate transformations of the n-dimensional space, spinors with opposite eigenvalues of $\hat{\gamma}^{\,n+1}$ have the same quantum numbers with respect to the internal group G.

(ii) $n = 4p$: This implies that

$$(\widehat{\gamma}^{\,n+1})^2 = -1 \quad \Rightarrow \quad \widehat{\gamma}^{\,n+1}\widehat{\psi} = \mp i\widehat{\psi}. \tag{14.3.12}$$

Since CPT invariance, which involves complex conjugation, reverses the chirality of the spinor fields, an equal number of left- and right-handed fermions must exist.

(iii) $n = 4p + 2$: Then it follows that

$$(\widehat{\gamma}^{\,n+1})^2 = +1 \quad \Rightarrow \quad \widehat{\gamma}^{\,n+1}\widehat{\psi} = \mp\widehat{\psi}. \tag{14.3.13}$$

Here a CPT transformation maps particles of given helicity into particles of the same helicity. Therefore, it is possible to construct theories that are asymmetric with respect to chirality.

Then it remains to reveal the conditions under which $i\not{D}^{(\mathrm{int})}$ has zero eigenmodes. This can be read off from the squared Dirac operator, which, in the generic case of an n-dimensional spin manifold, may be defined by

$$(i\not{D})^2 := -\mathrm{Tr}\left\{\gamma \wedge {}^*D^*(\gamma \wedge {}^*D)\right\}. \tag{14.3.14}$$

Since $D^* = {}^*D$ holds for a metric-compatible (spinor) connection,

$$(i\not{D})^2 = (-1)^{3n-1+s}\ \mathrm{Tr}\{\gamma \wedge \gamma \wedge {}^*(D\,D) - \gamma \wedge (D\gamma) \wedge {}^*D\} \tag{14.3.15}$$

is valid. Using the formal identity

$$D\,D = \frac{1}{2}\{D, D\} + \frac{1}{2}[D, D] = \frac{1}{2}\{D, D\} + \frac{1}{2}\widetilde{\Omega} \tag{14.3.16}$$

and furthermore the definition

$$\widetilde{\Theta} := D\gamma = \frac{1}{2}T_{\alpha\beta}^{\cdot\cdot c}\ \gamma_c \otimes \vartheta^\alpha \wedge \vartheta^\beta \tag{14.3.17}$$

of a Clifford-algebra-valued torsion 2-form, the concise expression

$$\begin{aligned}\not{D}^2 &= \mathrm{Tr}\left(\frac{1}{2}\gamma \wedge \gamma \wedge {}^*\{D, D\} + \frac{1}{2}\gamma \wedge \gamma \wedge {}^*\widetilde{\Omega} - \gamma \wedge \widetilde{\Theta} \wedge {}^*D\right) \\ &= (\Box^g + \frac{1}{4}R)\eta - \vartheta \wedge \Theta \wedge {}^*D \end{aligned}\tag{14.3.18}$$

may be found for the squared Dirac operator; cf. SCHRÖDINGER (1932), WU (1984). Let us return to the analysis of the spinor modes on the internal space. For a compact homogeneous space $B = G/H$, the scalar curvature $\overset{\circ}{R}$ is positive. Consequently, for vanishing torsion, i.e., $\overset{\circ}{\Theta} = 0$, Eq. (14.3.18) is the sum of a nonnegative and a positive operator, which, according to a theorem of Lichnerowicz (DE WITT 1964), has only nonzero eigenvalues.

In order to circumvent this *no-go theorem*, the following modifications of the geometric arena have been tentatively taken into consideration:

(a) Introduction of additional nongravitational gauge fields having a nontrivial Pontryagin index (RANDJBAR- DAEMI et al. 1984a).
(b) Noncompact internal spaces B with finite volume (WETTERICH 1984).
(c) Connections $\overset{\circ}{\omega}$ with parallelizing torsion. There exist massless, but no chiral, fermion modes (WU 1984).
(d) Riemann–Cartan spaces with nonvanishing *internal* nonmetricity tensor.

According to Kaluza, any Ansatz striving for universal validity is threatened by the sphinx of modern physics, quantum theory.[9] This is particularly true for the quantization of Einstein's theory of gravity (in higher dimensions), which is nonrenormalizable in $n > 2$ dimensions according to perturbation theory. For supergravity with its local symmetry between bosons and fermions it has been shown, however, that he contributions to the scattering matrix are finite at least up to second order in the perturbative expansion. Moreover, simple ($\mathscr{N} = 1$) supergravity in $n = 11$ dimensions leads to an effective four-dimensional theory that includes all fundamental spinors as well as their gauge interactions with (14.2.24) as structure group.

However, chiral fermions can be obtained in a compactified eleven-dimensional supergravity only by supplementing additional gauge fields. Due to quantum-theoretic anomalies (ALVAREZ- GAUME & WITTEN 1984), (ALVAREZ- GAUME & GINSPARG 1985), nonconserved axial currents would arise in these gravitational models that would spoil general covariance. In order to avoid such anomalies, GREEN & SCHWARZ (1984) have suggested certain modifications of supergravity in ten dimensions, such that particles correspond to the vibrations of hypothetical strings. These *superstring theories* with $G = \mathrm{SO}(32)$ or the exceptional group $\mathrm{E}_8 \times \mathrm{E}_8$ as structure group seem to provide a unification of all physical interactions in $n = 10 + \dim G = 506$ dimensions (DUFF et al. 1985) and a consistent and *finite* quantum field theory, but predict unseen particles.

14.4 Kerr–Newman Solution

Independent of particular models of unification is the search for vacuum solutions of the coupled Einstein–Yang–Mills system. The desired solutions—in harmony with the notion of an extended agent (WEYL 1924)—ought to make possible the description of an exterior, field-loaded space of a "pointlike particle" that is characterized by the mass M, the z-component[10] $J_3\hbar$ of the angular momentum, and the total angular momentum $\sqrt{J(J+1)}\hbar$. Furthermore, it may carry the generalized "internal" charge

$$\bar{Q}^{(i)} = Q^{(i)} \cos\delta + P^{(i)} \sin\delta, \quad Q^{(\circ)} \equiv Q_e. \tag{14.4.1}$$

Here $Q^{(i)}$ and $P^{(i)}$ denote the "electric"- and "magnetic"-type charges of a "dually" charged source that is coupled to the gauge fields. In the abelian case of the Einstein–Maxwell system, such an exact solution of the source-free system is known to exist for the exterior space of the particle. The potential of such a dyon is given by

[9]"Überhaupt droht ja jedem universelle Geltung heischenden Ansatz die Sphinx der modernen Physik, die Quantentheorie" (KALUZA 1921, p. 972).

[10]In referring to subsequent interpretations, we shall make use of the familiar eigenvalues of the rotation group. As for our still purely *classical* considerations, $J_3 \mathrel{\hat{=}} J$ remains valid.

$$A^{(\circ)} = \overset{*}{\sigma}\omega^{(\circ)} = -\frac{r^2}{\rho^2}\left[dt - J_3\frac{\hbar}{Mc}\sin^2\vartheta\, d\phi\right]Q^{(\circ)}\cos\delta$$
$$-\frac{\cos\vartheta}{\rho^2}\left[\left(r^2 + J(J+1)\frac{\hbar^2}{M^2c^2}\right)d\phi - J_3\frac{\hbar}{M}dt\right]P^{(\circ)}\sin\delta. \qquad (14.4.2)$$

It solves the coupled Einstein–Maxwell equations (14.1.4) and (14.1.6) within an axially symmetric background, which in turn is determined by the *Kerr–Newman metric* (KERR 1963; NEWMAN et al. 1965)

$$ds^2 = -\frac{\Delta}{\rho^2}\left[cdt - J_3\frac{\hbar}{M}\sin^2\vartheta\, d\phi\right]^2 + \frac{\rho^2}{\Delta}dr + \rho^2 d\vartheta^2$$
$$+\frac{\sin^2\vartheta}{\rho^2}\left\{\left[r^2 + J(J+1)\frac{\hbar^2}{M^2c^2}\right]d\phi - J_3\frac{\hbar}{M}dt\right\}^2. \qquad (14.4.3)$$

In this presentation, use is made of the generalized stationary Boyer–Lindquist coordinates (MTW, p. 878), for which

$$\rho^2 := r^2 + \frac{J(J+1)\hbar^2}{M^2c^2}\cos^2\vartheta \qquad (14.4.4)$$

represents a Schwarzschild-type radial coordinate, whereas

$$\Delta := r^2 - \frac{2M\hbar}{M^{*2}c}r + \frac{J(J+1)\hbar^2}{M^2c^2} + \frac{\alpha_e\hbar^2}{M^{*2}c^2}\mathrm{Tr}\overline{Q}^2 \qquad (14.4.5)$$

substitutes the function r^2e^ν in the Schwarzschild metric. The generalization of exterior Kerr–Newman solutions (NEWMAN et al. 1965) to that of a particle with electric *and* magnetic charge is fully justified on account of the invariance of the electromagnetic energy–momentum current versus duality rotations. The analysis of the separability of the generally covariant Klein–Gordon equation offers an elegant possibility to formulate an extension of the above-cited solution for a space with a nonvanishing cosmological constant: the Kerr–Newman–de Sitter solution. A profound discussion of the interesting global topological structures of this configuration is to be found in CARTER (1968, 1973).

It is especially important to point out that the spherical event horizon, which is typical for black holes and which acts like a semipermeable membrane (see, for instance, WHEELER 1974), is located at the distance

$$r_{\underset{(-)}{+}} = \frac{\hbar}{M^{*2}c}\left(M \underset{(-)}{+} \sqrt{M^2 - J(J+1)\frac{M^{*4}}{M^2} - \alpha_e M^{*2}\mathrm{Tr}\overline{Q}^2}\right) \qquad (14.4.6)$$

from the center. This horizon occurs only for a real square root in (14.4.6). Otherwise, this configuration could possibly describe a “naked” singularity. In the spacelike asymptotic infinity, not only does this geometric configuration act like a particle of the mass M, generalized charge $\overline{Q}^{(\circ)}$, and (classical) angular momentum component $J_3\hbar$ with respect to the axis of rotation, but it also has the magnetic moment $\mu_m = \frac{Q_e}{M} J$, whose gyromagnetic ratio is exactly equal to the familiar one of the special-relativistic Dirac theory.

During interactions of this gravitational configuration with matter and exterior fields, the surface area of the horizon

$$S_{\mathrm{A}} := 4\pi \left(r_+^2 + \frac{J(J+1)\hbar^2}{M^2c^2} \right)$$
$$= \frac{8\pi\hbar^2}{M^{*4}c^2} \left(M^2 + \frac{\alpha_e}{2} M^{*2} \mathrm{Tr}\overline{Q}^2 + M\sqrt{M^2 - J(J+1)\frac{M^{*4}}{M^2} - \alpha_e M^{*2} \mathrm{Tr}\overline{Q}^2} \right) \tag{14.4.7}$$

is never decreasing; the *irreducible mass* defined by

$$M_{\mathrm{ir}}{}^2 := \frac{S_{\mathrm{A}}}{16\pi} \frac{M^{*4}c^2}{\hbar^2} \tag{14.4.8}$$

is even constant for reversible processes, in compliance with the "second law" of black hole dynamics.

The total mass M of a black hole depends on its geometric remnant mass M_{ir}, on the stored electomagnetic field energy, and on its rotational energy. However, the dependence of M on the characteristic values M_{ir}, $\overline{Q}^{(\circ)}$, and J is nonlinear. More precisely, the resolution of (14.4.7) for M shows that the relation

$$M^2 = \left(M_{\mathrm{ir}} + \frac{\alpha_e M^{*2}}{4M_{\mathrm{ir}}} \mathrm{Tr}\bar{Q}^2 \right)^2 + \frac{1}{4} J(J+1) \frac{M^{*4}}{M_{\mathrm{ir}}{}^2}, \tag{14.4.9}$$

which was found by CHRISTODOULOU & RUFFINI (1971), is valid.

The Kerr–Newman solution changes into that of the Einstein–Maxwell system for the spherically symmetric exterior space of a charged scalar particle, which was formulated by REISSNER (1916) and NORDSTRÖM (1918) if $J \hat{=} J_3 = 0$ holds.

The essential singularity that occurs at the origin of the coordinate system can be avoided by an alteration of the topological structure of the spacelike hypersurface. For this purpose—as was done similarly in the EINSTEIN–ROSEN model of 1935—two otherwise separated model universes are to be connected by a "bridge". If the asymptotically flat areas of these worlds are identified with each other, a wormhole model comes into existence (WHEELER 1955; FULLER & WHEELER 1962). In this model, a pair of opposite charges that are locally separated are connected by nonsingular, electric flux lines. As can be inferred from the Kruskal system of coordinates, the narrowest neck of the throat of the Reissner–Nordström geometry exhibits an oscillatory character (GRAVES & BRILL 1960) that prevents a possible causality violation over the bridge. The transfer of these topological notions to configurations with nonvanishing angular momentum yields still richer geometric structures (CARTER 1968, 1973; COHEN 1971; KREISEL 1980). In passing to the Euclidean "spacetime" with the signature $s = 0$, the Reissner–Nordstrøm solution shows all characteristics of an instanton solution of the gravitationally coupled Maxwell equations. In contrast to the Julia–Zee dyon, this configuration is equal to a dually charged pseudoparticle for which not only the magnetic charge but also the *electric* charge has been quantized (DUFF & MADORE 1978).

Generalizations of the Kerr–Newman solution for the nonabelian case of the Einstein–Yang–Mills system are known for anti-de Sitter spaces (WINSTANLEY 2009). For a theory, however, with $SU(2)$ as a structure group, a trivial Ansatz may be made in order to distinguish the third component of the isospin while all other gauge fields are taken to be zero:

$$A = \overset{*}{\sigma}\omega = A^{(\circ)}\lambda_3. \tag{14.4.10}$$

The generalized Gell-Mann matrix λ_3, which is identical to the Pauli matrix σ_3 for $SU(2)$, commutes trivially with itself, with the result that the Yang–Mills equations are exactly reduced to those of the electromagnetic field. For this reason, the whole solution structure (14.4.2) can be adopted (YASSKIN 1975).

Additionally it may be mentioned that in Moffat's version of the asymmetric field theory in the case of a *conventional* coupling to the electric field, the following modification of the metric ground form of the Reissner–Nordstrøm solution has been obtained:

$$ds^2 = -\left(1 - \left(\frac{r_\circ}{r}\right)^4\right)\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2\,d\Omega^2. \tag{14.4.11}$$

Here Δ is to be substituted by the term (14.4.5) with $J = 0$. Apart from the horizon given by (14.4.6), this spacetime geometry has obviously still a further one at $r = r_\circ$, which seems to be sufficient, according to MOFFAT (1979a, b), to keep test particles away from the essential singularity at $r = 0$.

14.4.1 NUT Solution with Dual Mass

There is a further solution of the Einstein–Maxwell system given by NEWMAN et al. (1963), see also MISNER (1963), which should be documented here. In a Schwarzschild-like system of coordinates, this so-called charged NUT solution is given by the metric

$$ds^2 = -\frac{\overline{\Delta}}{\overline{\rho}^2}\left(\frac{4N\hbar}{Mc}\sin^2\left(\frac{\vartheta}{2}\right)d\phi + c\,dt\right)^2 + \frac{\overline{\rho}^2}{\overline{\Delta}}(dr^2 + \overline{\Delta}\,d\Omega^2), \tag{14.4.12}$$

in which the radial function

$$\overline{\Delta} := r^2 - \frac{2M\hbar}{M^{*2}c}r - \frac{N^2\hbar^2}{M^2c^2} + \frac{\alpha_e\hbar^2}{M^{*2}c^2}\mathrm{Tr}\overline{Q}^2 \tag{14.4.13}$$

and the modified radial coordinate

$$\overline{\rho}^2 := r^2 + \frac{N^2\hbar^2}{M^2c^2} \tag{14.4.14}$$

depend on an additional parameter N, thus differing from the Reissner–Nordström solution.

The electromagnetic potential is given in the form

$$\begin{aligned} A = &-\frac{1}{\overline{\rho}^2}(\overline{Q}\cos\delta + N\overline{Q}\sin\delta)c\,dt \\ &+ \frac{1-\cos\vartheta}{\overline{\rho}^2}\overline{Q}\left[-\overline{\rho}^2\sin\delta + 2r\frac{N\hbar}{Mc}\cos\delta\right]d\phi. \end{aligned} \tag{14.4.15}$$

To an observer located at asymptotic infinity, this geometric configuration looks like a particle of mass M with an electric charge $Q = -\overline{Q}\cos\delta$ and a magnetic monopole strength $P = -\overline{Q}\sin\delta$. If the surface area S_A of the event horizon, which is defined analogously to (14.4.6), is identified with the irreducible mass M_{ir} of this structure via

$$S_A := 4\pi r_+^2 = 16\pi \ M_{ir}^2\frac{\hbar^2}{M^{*4}c^2}, \tag{14.4.16}$$

the total mass M is given by the formula

$$M^2/M_{ir}^2 = \frac{\left[1 + \frac{1}{4}(\alpha_e^2\mathrm{Tr}\bar{Q}^2 - 2N^2)(M^*/M_{ir})^4\right]^2}{1 - \frac{1}{4}N^2(M^*/M_{ir})^4}, \tag{14.4.17}$$

which suggests that one should interpret N as the "effective binding energy" of the system (MC-GUIRE & RUFFINI 1975). Despite a large "naked" irreducible mass M_{ir} and a large dual charge $\overline{Q}$, the charge–mass relation and the total mass M of the object will remain in physically tolerable limits. For two-body solutions with the same NUT parameter N, the location of the singularities could be limited to the line connecting both dyons. Facing the amount of "binding energy" that is then to be expected ("Archimedes effect;"cf. SALAM 1977), such relativistic structures could possibly open up a new path to a better understanding of a (classical) quark–antiquark system. Taking the analogy with magnetic monopoles and dyons a step further, RAMASWAMY & SEN (1981) regarded the NUT parameter N as a "*dual mass*," the reason being that a duality rotation in the mass–angular-momentum plane, i.e., in the (M-N)-plane, transforms these parameters into each other. As shown by BAEKLER & HEHL (1984), nontrivial torsion solutions of the Poincaré gauge theory exist on such a metric background. In contrast to Einstein's theory, the NUT parameter N occurring in these configurations is no longer a freely specifiable integration constant, but is necessarily fixed by the coupling constant $1/\kappa$ of the Lorentz "rotons".

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Chapter 15
Color Geometrodynamics

According to empirical evidence, the forces acting between nucleons are stronger than electromagnetic forces by the factor $1/\alpha_e$. On the other hand, all experiments hint at the fact that they have only a short range and that they decrease exponentially for distances of the size

$$\ell := 2\pi\hbar/(M_P c) = \ell^*/\sqrt{\kappa} \simeq 1.3 \times 10^{-13}\,\text{cm} \tag{15.0.1}$$

of the Compton wavelength of the proton. This is accounted for by the Yukawa potential

$$V_{\text{Yu}}(r) \sim \frac{1}{r} e^{-r m_\pi c/\hbar}, \tag{15.0.2}$$

which in phenomenological theories of nuclear forces, see e.g., Bjorken & Drell (1964), is explained by a virtual exchange of scalar π-mesons of mass $m_\pi = 2\, m_e/\alpha_e$.

Following the familiar and up-to-date extensions of the "naive" quark model, which has been applied successfully to the classification of particles (Kokkedee 1969), the hadrons consist of two or three quarks that are kept together by gluons, e.g., by the exchange of vector bosons. The latter are the particles that, after quantization, are associated with spin-1 gauge fields of a Yang–Mills gauge theory with an exact $SU(3)^c$ structure group. The introduction of these additional color degrees of freedom (Greenberg & Nelson 1977) of the quarks is necessary, since a violation of the Pauli principle has to be avoided with respect to the construction of some energetically low-lying baryon states. With quantum chromodynamics (QCD, Marciano & Pagels 1978), a promising theory of strong interactions has been established, although it is exactly the interpretation of the binding forces *between* the nucleons in the nuclei that remains problematic.

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15.1 Tensor Dominance of Strong Interaction

In such theories, however, a phenomenological fact remains unreflected that not only that the interaction between nucleons by the exchange of virtual particles has spin-1 characteristics, but that *tensor forces* occur generally (see LANDAU & LIFSHITZ 1966, p. 460). This notion is fostered also by the occurrence of a nonet of spin-2 mesons, the so-called f-mesons, in scattering experiments. Independently of the supposition whether these objects are composite or elementary, a Lorentz-invariant description of the effective fields is based first on a linear spin-2 field equation of Pauli–Fierz type.

Each *self-consistent* description, however, of interacting massless spin-2 fields in the Minkowski space, which is originally postulated to be a flat one, leads under conditions that have been put forward and straightened out by FANG & FRONSDAL (1979) to a theory that is exactly as nonlinear as that of Einstein's GR. It is for these reasons that ZUMINO (1970) and ISHAM et al. (1971c) developed a new concept that runs as follows: The tensor part of strong interactions should be described by an *effective* Einstein–Hilbert Lagrangian 4-form with a coupling constant that has been scaled down to hadronic dimensions. Within this unification model, the tensor part of strong interactions is coupled dynamically to gravity either by both the symmetric tensor fields f_{ij} and g_{ij} or via the canonical 1-forms ϑ^f and ϑ^g as follows:

$$L_{f-g} = \frac{1}{\ell^2} L_{\mathrm{W}}(f) + \frac{1}{\ell^{*2}} L_{\mathrm{W}}(g) + L_{fg} + \frac{\Lambda}{\ell^{*2}} \vartheta^g \wedge \vartheta^g \wedge \vartheta^g \wedge \vartheta^g \\ + L_{\mathrm{m}}(\text{hadrons, f}) + \mathrm{L}_{\mathrm{m}}(\text{leptons, g}). \tag{15.1.1}$$

This model is referred to as *two-tensor theory of gravity*.

The requirement that the f-mesons, in contrast to the gravitons, aquire a mass necessitates the introduction of an additional mixing term L_{fg}. Were it not for this term, the whole universe, according to the implications of this theory, would consist of two noncommunicating worlds: the first leptonic with gravitational attraction, and the second hadronic with a coupling due to strong tensor forces. It is for this reason that, similar to the mixing scheme concerning photons and ρ-mesons (KROLL et al. 1967), a bridge between these worlds is built by adding the term L_{fg}. The procedure also provides the necessary mass for the f-meson and this most preferably via the Higgs–Kibble mechanism of spontaneous symmetry breaking. Generally, this mixing term is understood to have only a polynomial dependence on ϑ^{f} and ϑ^{g} of degree a or $b - 4$,

$$L_{fg} = P(\vartheta^f, \vartheta^g)\eta, \tag{15.1.2}$$

but that it has no derivatives in these fields. It is to be accentuated that the resulting model of interaction (15.1.1) belongs to the class of the nonpolynomial field theories that have a built-in cutoff parameter concerning the quantum-field-theoretic ultraviolet divergences (ISHAM et al. 1971a, 1972a).

Let $\sum^f$ and $\sum^g$ be the canonical energy–momentum currents that are generated additionally by the mixing term and that are defined by

$$\delta L_{fg} =: \delta\vartheta^f \wedge \Sigma^f + \delta\vartheta^g \wedge \Sigma^g. \tag{15.1.3}$$

On account of the assumed polynomial dependence, these currents are related to one another by the so-called f-g relation (GÜRSES 1981)

$$\frac{1}{a}\vartheta^f \wedge \Sigma^f + \frac{1}{b}\vartheta^g \wedge \Sigma^g = 2\, L_{fg}. \tag{15.1.4}$$

The choice of L_{fg} that had been originally favored by ISHAM et al. (1971b) is equivalent to the covariantly written Pauli–Fierz mass term

$$L'_{fg} = \frac{m_f^2}{\ell^2} |\det\, g_{ij}|^u| \;\; \det\, f_{ij}|^{1/2-u} (\vartheta^f - \vartheta^g) \wedge (\vartheta^f - \vartheta^g) \wedge \vartheta^g \wedge \vartheta^g \tag{15.1.5}$$

for spin-2 fields with the parameter u set to zero. An interaction term of a more cosmological origin is discussed in detail in ISHAM et al. (1971b). It should be noted that (15.1.5) has also an additional cosmological term, as far as the g-metric is concerned. This cosmological term could be absorbed by the part of (15.1.1) that is proportional to Λ. Other mixing terms have been suggested; cf. MCKELLAR (1977).

Considering the suggested polynomial nature of the coupling term (15.1.2), the following equations result from the independent variation of (15.1.1) for ϑ^{g} and ϑ^{f}:

$$\frac{1}{2}\,\vartheta^g \wedge {}^*\Omega^g + \Lambda^*\vartheta_g = \ell^{*2}\left\{\Sigma_{\mathrm{m}}(\text{leptons}) + \Sigma^{\mathrm{g}}\right\}, \tag{15.1.6}$$

$$\frac{1}{2}\,\vartheta^f \wedge^* \Omega^f = \ell^2\left\{\Sigma_{\mathrm{m}}(\text{hadrons}) + \Sigma^{\mathrm{f}}\right\}. \tag{15.1.7}$$

Both equations are formally identical to Einstein's field equations. Without demanding this explicitly for the Lagrangian 4-form (15.1.1), it can be ascertained that the field Eq. (15.1.7) for the f-metric contains a "cosmological" contribution, too. This becomes obvious if we make use of the relation (15.1.4) while solving for $\sum^f$. The resulting field equation

$$\begin{aligned}&\frac{1}{2}\,{}^*(\vartheta^f \wedge {}^*\Omega^f) - a\,\ell^2\vartheta^f \wedge {}^*\left\{2L_{fg} - \frac{1}{2b\ell^{*2}}(\Omega^g \wedge \vartheta^g \wedge \vartheta^g + 2\Lambda\eta)\right\}\\ &= \ell^2\,{}^*\left\{\Sigma_{\mathrm{m}}(\text{hadrons}) + \frac{\mathrm{a}}{\mathrm{b}}\,{}^*(\vartheta^{\mathrm{g}} \wedge {}^*\Sigma_{\mathrm{m}}(\text{leptons})) \wedge \vartheta^{\mathrm{f}}\right\}\end{aligned} \tag{15.1.8}$$

even in vacuum contains a pressure-type contribution that is proportional to the mixing term. This form of the field equations for the f-metric permits the deduction of an important property of classical vacuum solutions of the f-g theory. If an Einstein space, e.g.,

$$\frac{1}{2}\,{}^*(\vartheta^g \wedge {}^*\Omega^g) + \Lambda\vartheta^g = 0, \tag{15.1.9}$$

is postulated for the metric of the macroscopic gravity—although this is not compulsory—then analogously, the Einstein-type field equations

$$\frac{1}{2}\,{}^*(\vartheta^f \wedge {}^*\Omega^f) + \Lambda^f_{\text{eff}}\vartheta^f = 0 \tag{15.1.10}$$

follow from (15.1.8) for the "strong" tensor fields f_{ij}. The thereby occurring "cosmological constant"

$$\Lambda^f_{\text{eff.}} = \frac{2a}{\kappa b}\Lambda - 2a\,\ell^2\,{}^*L_{fg} \tag{15.1.11}$$

is, however, partially of microscopic origin. This expresses itself within the structure of the solutions.

In the f-g theory, the combination

$$\vartheta^{m_f} := \vartheta^f - \vartheta^g \tag{15.1.12}$$

corresponds to an observable *massive* spin-2 field, while the metric structure, which serves as a substratum for a macroscopic spacetime, is defined by the square of the line element

$$\begin{aligned} ds^2 &= \frac{1}{1+\kappa}\, g^{\alpha\beta}\left(\vartheta^g_\alpha \otimes_s \vartheta^g_\beta + \kappa\vartheta^f_\alpha \otimes \vartheta^f_\beta\right) \\ &= \frac{1}{1+\kappa}\,(g_{ij} + \kappa\, f_{ij})\, dx^i \otimes_s dx^j . \end{aligned} \tag{15.1.13}$$

It is due to the assumption (15.0.1) that the ratio

$$\kappa = \ell^{*2}/\ell^2 = G_{\text{N}}/G_{\text{S}} = 0.38 \times 10^{-38} \tag{15.1.14}$$

of the (weak) Newtonian constant G_N of gravity to the coupling constant G_S of strong interactions is of such an elusive smallness that g_{ij} suffices to represent the metric spacetime structure beyond hadronic distances of the size of one fermi = 10^{-13} cm. In the quantized theory, these fields ought to be identical to the physically observable f^0-meson and the massless graviton in a first approximation.

Classical solutions of the f-g theory (AICHELBURG et al. 1971, 1972; AICHELBURG 1973; ARAGONE & CHELA-FLORES 1972; CHELA-FLORES 1974) as well as their coupling to Yang–Mills fields (DE ALFARO et al. 1979) have been studied from varying points of view. An important *exact* solution that meets the discussed standards was found by ISHAM & STOREY (1978). This solution refers to a two-tensor theory specified by the mixing term (15.1.5) with an arbitrary u. Here (15.1.9) is valid, and consequently, it is possible to describe macroscopic gravity by the Schwarzschild–de Sitter line element with

$$e^{-\lambda^{g,f}} := 1 - \frac{2\mu^{g,f}}{r} - \frac{1}{3}\Lambda^{g,f} r^2, \tag{15.1.15}$$

while

$$\begin{aligned} ds_f^2 &= f_{ij}\, dx^i \otimes_s dx^j \\ &= -\frac{3\gamma}{2} e^{-\lambda^f} dt^2 + 2\left[\gamma\left\{1 - \left(1 + \frac{9\gamma}{4}\right) e^{\lambda^f - \lambda^g} + \frac{9\gamma}{4}\, e^{2(\lambda^f - \lambda^g)}\right\}\right]^{1/2} dt\, dr \\ &\quad + e^{\lambda^g}\left\{\frac{2}{3} + \frac{3\gamma}{2}(1 - e^{\lambda^g - \lambda^f})\right\} dr^2 + \frac{2}{3} r^2 d\Omega^2 \end{aligned} \tag{15.1.16}$$

is the result for the f-metric. The integration constants are μ^g, μ^f, and γ. For u = 1/2, $\mu^g = \Lambda = 0$, this solution reduces to the configuration that was given by SALAM & STRATHDEE (1977).

It is of importance to note that both metrics are of Schwarzschild–de Sitter type, i.e., with the same constants of integration, they are *isometric*. For the representation of the f-metric, a particular choice of coordinates is required in order to make possible the interlocking of both geometric structures, compare GÜRSES (1981). Moreover, the condition (15.1.9) forces the stress–energy tensor $\sum^g$ of the gravitational field to be zero, a property that is reminiscent of one of the instanton solutions of the Yang–Mills fields or the solutions of SKY gravity (MIELKE 1981b). The fact, however, that the f-metric, due to the induced "cosmological" term (15.1.11), is not asymptotically flat makes these *microscopic* structures appropriate prototypes of hadronic "bags" for the confinement of quarks (SALAM & STRATHDEE 1978a).

Apart from these "microcosmological" configurations, there exists also a spherically symmetric solution in which, according to semilinear approximations (ARAGONE & CHELA-FLORES 1972; CHELA-FLORES 1974; CHELA-FLORES & SILVA-GALIZA 1980), the radial function exhibits a Yukawa-like exponential decrease

$$e^{-\lambda^f} \simeq 1 - \frac{2\mu^f}{r} \exp\left(-r\, m_f\, c/\hbar\right). \tag{15.1.17}$$

This is in harmony with the expectation that strong tensor forces within hadrons are only of rather short range that is limited by the Compton wavelength of the f-mesons of mass $m_f = 1 - 2$ GeV$/c^2$. This additional property of the tensor dominance model could be responsible for a possibly *partial* confinement of quarks (SALAM 1977) or even yield a *geon*-type structure of the hadrons (ISHAM et al. 1971c; MIELKE & SCHERZER 1981).

15.1.1 Particle Spectrum

Alternatively, the hypothesis of the tensor dominance of strong interactions can be related to independent translational and rotational gauge potentials within the Poincaré gauge theory. These are given by the tetrads E_j^{β} and Ricci's rotation coefficients $\Gamma_i^{\alpha\beta}$, respectively. With respect to anholonomic frame, these potentials transform under the Lorentz group according to the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ or $(1, 0) \oplus (0, 1)$, respectively, and are therefore have an holonomic spin $\overline{s}_a = 1$. After the transition to a holonomic frame, it is the additional spin-1 of the lower indices that induces gauge potentials of maximal (reducible) spin or helicity $\overline{s} = 2$ (Hehl et al. 1978b). More precisely, these gauge potentials comprise the following direct sum of representations:

$$E_{\alpha\beta} : (2^+) \oplus 2(1^-) \oplus 2(0^+) \oplus (1^+) \tag{15.1.18}$$

and

$$\Gamma_{\mu\alpha\beta} : (2^+) \oplus 2(1^-) \oplus (0^+) \oplus (2^-) \oplus 2(1^+) \oplus (0^-), \tag{15.1.19}$$

which are irreducible with respect to the Lorentz group (Sezgin & Van Nieuwenhuizen 1980). Since the qPG theory takes off from Yang–Mills-like Lagrangian 4-forms, this theory contains in a most natural way two spin-2 fields with independent dynamical degrees of freedom. And the term quadratic in torsion yields a dynamics that are closely related to that of Einstein's GR, and this due to its formal similarity with the teleparallelism theory. Thus it is highly suggestive, if not rather a surprise at a first glance, to relate the translational gauge degrees of freedom to the Einstein–Newton gravitational potentials of infinite range. These couple dynamically to matter via the canonical energy–momentum tensor roughly in the sense of Mach (cf. Wheeler 1964 concerning Mach's principle). Accordingly, the Planck length ℓ^* of "weak" gravity has been taken for granted as a coupling constant. On the other hand, the rotational spin-2 degrees of freedom couple similarly as in the nonabelian Yang–Mills theories, via the field equation to the canonical spin current of matter. This suggests that these "rotons" have a component of short range and therefore are to be characterized by a dimensionless and probably strong coupling constant κ. The latter assumption is evidently implied in the work of Sezgin & Van Nieuwenhuizen (1980; see also Sezgin (1981) concerning the quantum-theoretic particle content of the qPG theory being linearized on a Minkowski background space.[1] Henceforth, this theory, depending on ten parameters, has "ghost-free" and "tachyon-free" subsystems in which there are not only massless gravitons but also propagating massive spin-2 fields of short range.

[1]Facing the fact that solutions of the qPG theory with the duality property "live" on Einstein spaces with an induced effective cosmological constant, the Minkowski basis for the linearization procedure of Sezgin & Van Nieuwenhuizen (1980) is no longer justified for the complete nonlinear theory with $\Lambda_{\text{eff}} \neq 0$. Other quantization procedures are needed in order to isolate the "strong" tensor component; cf. Christensen & Duff (1980).

It is for this reason that we are proposing with HEHL et al. (1978b) that the qPG theory brings about naturally a tensor component of strong interactions. This then means that in contrast to the original f-g theory, there is no necessity to resort to ad hoc introduced mixing terms for the generation of mass. Common to both models, however, is the feature that the "strong" tensor gauge field within the Lagrangian 4-form is often accompanied by an induced constant-pressure term of microscopic origin. The provenance of this "cosmological term" has just been dealt with for the f-g theory. In the case of the qPG theory its occurrence for configurations with duality properties has to be understood as a consequence of the very general Ansatz. It follows from its contraction that a space of constant curvature having a radius proportional to $\ell = \ell^*/\sqrt{\kappa}$ is superposed in the metric background of the solutions. But under the hypothesis of tensor dominance of strong interactions, the Planck length ℓ^* scaled by $1/\sqrt{\kappa}$ is identical to the Compton wavelength of a proton according to (15.0.1). This result is reflected, e.g., in the spherically symmetric Baekler solution with nontrivial torsion and effective "cosmological" constant. Due to its physically small scale, the (anti-)de Sitter part of the solution is of utmost importance for the microphysical interpretation of the spherically symmetric configuration. Probably this is even in a far more important way true for the purely "microcosmological" torsion solution of BAEKLER et al. (1982). It is a consequence of these formulations that the f-g theory resides in a Riemann–Cartan space. This will certainly be of relevance for the combination of internal degrees of freedom, as isospin, with the external spin of particles. However, the hope that such theories prevent singularities in the development of the universe is unfortunately fallacious. Indeed, the collapse of the universe, filled with 10^{80} baryons, stops at a minimum radius of 1 cm, while this minimum radius would be increased to about 10^{13} cm as a result of the hypothesis of strong gravity and thus lead to final densities whose order of magnitude would only slightly surmount those of the nuclei of atoms (ISHAM et al. 1973b). This argument of TRAUTMAN (1973) is based on a classically treated spin fluid with a completely aligned spin; for Dirac fields, however, the collapse behavior that is predicted by GR would be even accelerated; cf. NESTER & ISENBERG (1977) in EC theory.

In order to clarify the hypothesis of tensor dominance as a model for the unification of (macroscopic) gravity and microphysics, it is not necessary to stop with the Poincaré gauge theory. For it is also possible to retrace characteristics in an appropriately scaled metric–affine theory (HEHL et al. 1978a, HEHL & ŠIJACKI 1980). This reminds us of the phenomenological properties of strong interaction, spin, scale invariance, and the bands of shear excitations of matter sources resembling those of the observed hadrons that follow the Regge trajectories.

Of an even more fundamental nature, however, is the question concerning the enormous relative difference between the macroscopic "weak" and the microscopic "strong" gravity. The relative strength of the coupling constants ℓ^2 and ℓ^{*2}, which has been taken as the basis for the model in (15.1.1), due to (15.0.1) is an enormously small number of order of magnitude $\kappa = 0,38 \times 10^{-38}$. It was again EINSTEIN in 1919 who, while anticipating the bag models of quantum chromodynamics, modified his gravitational field equations by an accordingly large constant term. Concerning a still classically regarded corpuscle, this compensating term should generate such an inwardly directed pressure as to balance repulsive electric forces. Later, Lanczos (1949) took up this idea in a modified way.

All these Ansätze point to a cosmological origin for the remarkable asymmetry of these coupling constants (De Sabbata & Rizzati 1977; Sivaram & Sinha 1979). Within Dirac's hypothesis (1938) concerning the origin of the big dimensionless numbers in physics, it is suggested that Newtons's gravitational constant G_N changes on cosmological time scales and likewise the Planck length ℓ^* derived from G_N with the result that in the original hadronic area of the universe there would be a fundamental length ℓ being enlarged by about 10^{20}. This hypothesis is supported indirectly by the empirical estimate of the relationship

$$\frac{M_U}{M_P} = \frac{\ell_H}{\ell^*} \cdot \frac{\ell}{\ell^*} = \frac{1}{\kappa} \frac{\ell_H}{\ell} \simeq 10^{80} \tag{15.1.20}$$

between the total mass M_U of the universe and the mass M_P of the proton while relating it to the product of the relative Hubble radius ℓ_H and the relative Compton wavelength ℓ of the nucleon. Here the standard of measurements is to be seen in Planck's length ℓ^*.

Another possible argument for the huge difference in the order of magnitude concerning the "weak" and "strong" gravity could be inferred from the analysis of instanton and meron configurations of a gravitationally coupled Yang–Mills theory, an idea that was presented by De Alfaro et al. (1979, 1980). With the occurrence of a microscopic contribution proportional to the inner curvature of the group manifold, the ideas explained here would come full circle within the corpus of Kaluza–Klein theory.

15.2 Einstein–Cartan Theory with Internal Degrees of Freedom

As has been pointed out by Heisenberg (1932), protons and neutrons are isospin doublets concerning the $SU(2)$ group of unitary transformations. This theoretical proposition is justified by the approximate charge independence of strong interactions. Such a group-theoretic approach has been extended in order to arrange the observed spectrum of the metastable hadronic particle-like states according to representations of the unitary groups $SU(f)$. Concerning this aspect, particular success has to be ascribed to the quark model (Gell-Mann & Ne'eman 1964; cf. also Kokkedee 1969).[2]

[2] It is impossible, however, to regard $SU(3)$ as an exact symmetry, since the hadrons that are combined in $SU(3)$ octets, decuplets, and so on have too distinct masses. Alternatively, there may be a ramification of a *dynamical* symmetry (Barut 1972) of the interacting system. This may hold to a far larger extent for the enlarged group $SU(4)$ that is postulated in the context of the charm hypothesis, and also for the grand unified theories (GUTs) that embarks upon $SU(5)$ as internal symmetry group.

Apart from the "internal" characteristics of particles such as isotopic spin, hypercharge, charm, and so on, they most certainly have also spin proper, i.e., the fields transform according to spinor representations of the group $SO(3) \approx SU(2)$ of the familiar rotations of the 3-dimensional space. In order to position internal and external spin within a common group-theoretic scheme, the group $SU(4) \approx SO(6) \supset SO(4) \approx SU(2) \times SU(2)$ was proposed by Wigner. Later on, the group $SU(6)$ was proposed by GÜRSEY et al. (1964) for the classification of excited particle states. Within the framework of this phenomenological classification scheme, the low-lying meson states are supposed to be contained within the 35-dimensional representation of $SU(6)$, while the energetically low-lying baryon states with quantum numbers $J = 1/2$ and $J = 3/2$ of total angular momentum are thought to fill out a 56-fold irreducible representation.

This classification is partly successful. One basic objection, however, cf. SALAM & STRATHDEE (1978b), is that the symmetry scheme of Wigner–Gürsey–Radicati–Sakita is essentially of a nonrelativistic nature. The simplest way of circumventing this problem is to extend $SU(6)$ to the linear group $SL(6, \mathbb{C}) \supset SL(2, \mathbb{C}) \times SU(3)$. Thereby a relativistic union of the internal symmetry group SU(3) with the (simply connected) covering group $SL(2, \mathbb{C})$ of the Lorentz group, acting "externally" on spacetime, is achieved. If the symmetry is regarded only as a local one, an $SL(6, \mathbb{C})$ gauge theory of strong interactions (ISHAM et al. 1972b; 1973a) results that is a large enough frame also for dealing with tensor forces in an apt way. Shortly afterwards, SALAM (1973) pointed out in his anticipatory study that these more involved models are nothing but a natural generalization of the $SL(2, \mathbb{C})$ gauge theory of WEYL (1929a, b, 1950), a generalization that leads to a theory with additional, "internal," degrees of freedom in an extended Riemann–Cartan space.

Here we are considering Salam's model for the generic case in which the unitary degrees of freedom of the building blocks (quarks, preons) of particles are characterized by both f flavors and c colors; cf. MIELKE (1981a). The structure of such an Einstein–Cartan theory with internal degrees of freedom is again based on a principal fiber bundle P(M, $\overline{\text{G}}$, π, δ) as a geometric arena. But this time, it is

$$\begin{aligned} \overline{G} &= SL(2f, \mathbb{C}) \otimes SL(2c, \mathbb{C}) \\ &\supset SU(f)_{\mathrm{L}} \otimes SU(f)_{\mathrm{R}} \otimes SU(c)_{\mathrm{L}} \otimes SU(c)_{\mathrm{R}} \end{aligned} \tag{15.2.1}$$

that has to serve as the structure group. Furthermore, it can be shown that the unitary subgroups, being marked by L and R, are related to the left-, or right-handed helicity of the fundamental fermions. Thus, there is a "noncompact" generalization of the chiral $[SU(4)\otimes]^4$ model of ELIAS et al. (1978) ready to hand for the case $f = c = 4$. For the sake of the following insertion of a $\overline{\text{G}}$-gauge-covariant multiplet of spinor fields into this theory, we therefore consider the generalized $\overline{\text{G}}$-bundle

$$\overline{L}(M) := M^4 \times \overline{G} \supset M^4 \times SL(2, \mathbb{C}) \tag{15.2.2}$$

instead of the bundle L(M) of linear frames (compare also with the formalism of ELIAS et al. (1978) in the pure Yang–Mills case). Following ISHAM et al. (1973b, 1974), our gauge-theoretic concept is developed with respect to a *Dirac basis*. We associate a 1-form

$$\overline{\gamma} = \overline{E}_j^{\alpha}(m)\gamma_\alpha \otimes dx^j = \overline{E}_j\, dx^j \tag{15.2.3}$$

with $\overline{\mathrm{L}}$(M). This 1-form is supposed to have holonomic components in the local expansion

$$\begin{aligned}\overline{E}_i = &\frac{1}{2}\left\{E_{i\alpha}^{(f)\,j}\,\gamma^\alpha + i\ {}^*E_{i\alpha}^{(f)\,j}\,\gamma^\alpha\gamma^5\right\}\lambda_j^{(f)}\\ &\oplus \frac{1}{2}\left\{E_{i\alpha}^{(c)\,j}\,\gamma^\alpha + i\ {}^*E_{i\alpha}^{(c)\,j}\,\gamma^\alpha\gamma^5\right\}\lambda_j^{(c)}\end{aligned} \tag{15.2.4}$$

with reference to elements of the ideal of $\overline{\mathrm{G}}$. Analogously, the $\overline{\mathrm{G}}$-invariant fiber metric

$$\frac{1}{4}\mathrm{Tr}(\overline{\gamma} \otimes_s \overline{\gamma}) = f_{ij}\, dx^i \otimes_s\, dx^j \tag{15.2.5}$$

can be introduced into the bundle $\overline{\mathrm{L}}$(M). It can be identified with the contribution of the strong tensor field in (15.1.13). If it is required, a complete multiplet of spin-2 fields—as is, for instance, realized in nature via the $SU(3)$-nonet f, f', A_2, and K* (1430)—can thus be achieved without the trace operation. On the other hand, the expansion (15.2.4) may be vaguely regarded as an equivalent of the generalization of the Kaluza–Klein ansatz for tensor gauge fields; cf. PERCACCI (1984).

As a further dynamical ingredient, a Lie-algebra-valued 1-form in the (associated) bundle is needed. In order to clarify the character of the relationship between $\overline{\mathrm{G}}$ and the subgroup $SL(2,\mathbb{C})$, we again refer to a Dirac basis as far as the local presentation

$$\overset{*}{\sigma}\overline{\omega} = \overline{\omega}_i^{\ j}\ \overline{L}_j\ \otimes\, dx^i = \overline{\omega}_i\, dx^i \tag{15.2.6}$$

of this connection 1-form is concerned. In terms of this basis, the infinitesimal generators of $SL(2N,\mathbb{C}) \supset SL(2,\mathbb{C}) \times U(N)$ can be realized by the matrices

$$\left\{\overline{L}_j = \sigma^{\alpha\beta} \otimes \lambda_i\,,\ \ \lambda_i\,,\ \ i\,\gamma^5\lambda_i \mid j = 1, \ldots 8N^2 - 1\right\}, \tag{15.2.7}$$

where the generalized Gell-Mann matrices λ_i are the N^2-1 vector operators of the $SU(N)$ subgroup. If the gauge potentials of the unitary groups—which are contained in $\overline{\mathrm{G}}$ according to the subgroup chain (15.2.1)—are defined by

$$\begin{aligned}A_i = &\frac{1}{2}\left\{A_i^{(f)j} + {}^*A_i^{(f)j}\,\gamma^5\right\}\lambda_j^{(f)}\\ &\oplus \frac{1}{2}\left\{A_i^{(c)j} + {}^*A_i^{(c)j}\,\gamma^5\right\}\lambda_j^{(c)},\end{aligned} \tag{15.2.8}$$

then the components of the total connection 1-form $\overline{\omega}$ will be obtained in holonomic notation as follows:

$$\overline{\omega}_i = \frac{1}{2}\Gamma_i^{(f)\alpha\beta j}\sigma_{\alpha\beta}\lambda_j^{(f)} \oplus \frac{1}{2}\Gamma_i^{(c)\alpha\beta j}\sigma_{\alpha\beta}\lambda_j^{(c)} \oplus A_i. \tag{15.2.9}$$

The Yang–Mills-like gauge potentials A_i aside, this total connection 1-form comprises also generalizations of the spacetime rotation Ricci coefficients. The gauge field strengths that belong to the 1-forms $\overline{\gamma}$ and $\overline{\omega}$ are the 2-form

$$\overline{\Theta} := \overline{D}\overline{\gamma} = d\overline{\gamma} - [\overline{\omega}, \overline{\gamma}] \tag{15.2.10}$$

of the internal "torsion" and the 2-form

$$\overline{\Omega} = d\overline{\omega} - \overline{\omega} \wedge \overline{\omega} \tag{15.2.11}$$

of the gauge curvature, respectively.

The completion of the gauge-theoretic scheme is achieved by the prescription of dynamics that are closely related to what has already been applied successfully in the tensor dominance model. Within the Einstein–Cartan theory with internal degrees of freedom, this is achieved by replacing the Weyl term $L_{\mathrm{W}}(\mathrm{f})$ in the total Lagrangian 4-form (15.1.1) by the generalized Lagrangian 4-form

$$\overline{L}_{\mathrm{w.}}(f) = \frac{i}{8}\,\mathrm{Tr}\,(\overline{\gamma} \wedge \overline{\gamma} \wedge {}^*\overline{\Omega}). \tag{15.2.12}$$

If the internal degrees of freedom are suppressed, i.e., for the case $f = c = 0$, the form (15.2.12), which is written in a Dirac basis, reduces to Weyl's version of the Lagrangian 4-form of the conventional Einstein–Cartan theory.

It was ISHAM et al. (1974) who first formulated a gauge theory with $SL(2, \mathbb{C})$ as a structure group. In the framework of an $SL(6, \mathbb{C})$ gauge theory, i.e., for $f = 3$ and $c = 0$, they also proposed a mixing term $\overline{L}_{\mathrm{fg}}$ that is responsible for the mediation between the realms of strong and weak gravity. The aim was to break the local $SL(6, \mathbb{C})$ invariance without violating the invariance of the subgroup $SU(3)^c$. In order to solve this problem, it seems to be much more promising to extend the "broken" Poincaré gauge theory by incorporating internal symmetries. An analysis of the quantum-theoretic particle content of such a tensor gauge theory with internal degrees of freedom, which should follow closely the procedure of SEZGIN & VAN NIEUWENHUIZEN (1980), is most likely to provide all the physically desirable multiplets of spin-1 and spin-2 particles. According to the proposal of DENNIS & HUANG (1977) and HUANG & DENNIS (1981a, b), such a mechanism of symmetry breaking can also be achieved by the explicit introduction of Higgs fields that carry not only hadronic but also leptonic quantum numbers.

The EC theory with internal degrees of freedom couples to fundamental spinor fields represented by the following multiplet:

$$\psi = \{\psi^{(q_f, q_c)} | q_f = 1, \ldots, f\,;\ q_c = 1, \ldots, c\}. \tag{15.2.13}$$

Remarkably enough, it shows that the generalized torsion (15.2.10) induces a nonlinearity in the Dirac equation that yields not only a self-interaction for the single Dirac fields, but also a "contact interaction" among the different fundamental fermion fields that occur in (15.2.13). Accordingly, this results in a $\overline{\text{G}}$-gauge-invariant nonlinear Heisenberg–Pauli–Weyl spinor equation

$$i\,\overline{\gamma} \wedge {}^*\overline{D}^{\{\}}\psi - 6\,\ell^2\,\overline{\gamma}^5\,\overline{\gamma} \wedge {}^*(\overline{\psi}\,\overline{\gamma}^5\,\overline{\gamma}\,\psi)\psi - m\psi\eta = 0 \tag{15.2.14}$$

with

$$\overline{D}^{\{\}} := d + i\,\overline{\omega}^{\{\}} \tag{15.2.15}$$

(MIELKE 1977, 1980, 1981a). This generalization, however, due to the proposition of *strong* tensor forces, contains the Compton wavelength (15.0.1) of the proton as a coupling constant. In his unified field theory, HEISENBERG (1967) originally made allowance only for the isospin group, i.e., $G = SU(2)$. If we followed that assumption, this would be equivalent to the introduction of an *isotorsion* into a generalized Riemann–Cartan space, similar to what can be found in FINKELSTEIN (1961a, b).

The problem, however, as to which of the gauge groups is to be valued as the optimal remains a subject of controversy within the scientific community. If one acknowledges not only the hypothesis of charmed quarks but also the necessity of additional color degrees of freedom concerning the fundamental building blocks, a theoretically elegant solution, for our purposes, is to be seen particularly in the assimilation of the unification model of strong, weak, and electromagnetic interactions, which was developed by PATI & SALAM (1974) on the basis of Yang–Mills gauge theories. Within the scheme A of this $SU(4)^f \otimes SU(4)^c$-gauge theory, the quarklike parts of the 4×4 fundamental spinor fields carry integer quanta of charge, while the lepton number is interpreted as a fourth color (lilac). Our generalization to an $SL(8, \mathbb{C})^f \otimes SU(8, \mathbb{C})^c$-gauge theory, however, is rather more related—following the evidence of (15.2.1)—to the $[SU(4)\otimes]^4$-model of ELIAS et al. (1978), in which quarks carry noninteger quanta of charge.

Nevertheless, in both schemes, the matrix of fundamental spinors can be identified with those hypothetical hadronic constituents and the known leptons as follows:

$$\psi = \left[\begin{array}{ccc|c} \mathcal{P}_a & \mathcal{P}_b & \mathcal{P}_c & \nu_e \\ \eta_a & \eta_b & \eta_c & e^- \\ \lambda_a & \lambda_b & \lambda_c & \mu^- \\ \hline c_a & c_b & c_c & \nu_\mu \end{array}\right] \begin{array}{c} \text{up} \\ \text{down} \\ \text{strange} \\ \text{charm} \end{array} \quad \begin{array}{c} \uparrow \\ \text{flavors} \\ \downarrow \end{array} \tag{15.2.16}$$

red yellow blue lilac
⟵ (body) – colors ⟶

Within this frame[3] of *color geometrodynamics* (CGMD, MIELKE 1980), as in quantum chromodynamics (QCD), there also emerge gluon fields, apart from the observable hadronic mesons, but here maximally with spin 2. The exact particle content of this model depends also on the mechanism that is applied to the physically necessary breaking of this high symmetry. If the exact local $SU(3)^c$-symmetry is preserved, as is usually taken for granted, the resulting vector gluons might be responsible for the *saturation* of the color degrees of freedom of the quarks—as is the case in chromodynamics—and the tensor gluons might possibly be responsible for their confinement (MIELKE 1977).

On account of the nonlinear coupling of the different spinor fields, being contained in the Eq. (15.2.14), the baryons should form semiclassically bound states of three quarks; cf. RAÑADA (1983, 1986). A dominant role, however, with regard to the problem of permanent confinement, is probably taken by the self-consistently generated attractive tensor forces.

This holds also for hadronic mesons, which are thought to be quark–antiquark pairs within QCD. Here it is the CGMD that provides for the alternative (MIELKE 1980), which is admittedly highly speculative, to capture the color gauge fields within the nontrivial topology of the space $M = \mathbb{R} \times S^1 \times S^2$. Its apparent apertures would give the impression of sinks for these color gauge fields—similarly to the wormhole model (WHEELER 1955, 1966), which has its origin in the theoretical insights of WEYL (1924, p. 611) and EINSTEIN & ROSEN (1935). Such a wormhole would function, so to speak, as a quark–antiquark system that is bound by a gluon string, cf. Fig. 15.1.

Moreover, it cannot be excluded that the Einstein–Cartan theory with internal degrees of freedom including the fermionic degrees of freedom leads to a "supergeometrodynamics" (BAAKLINI & SALAM 1979) in the sense of a supersymmetry.

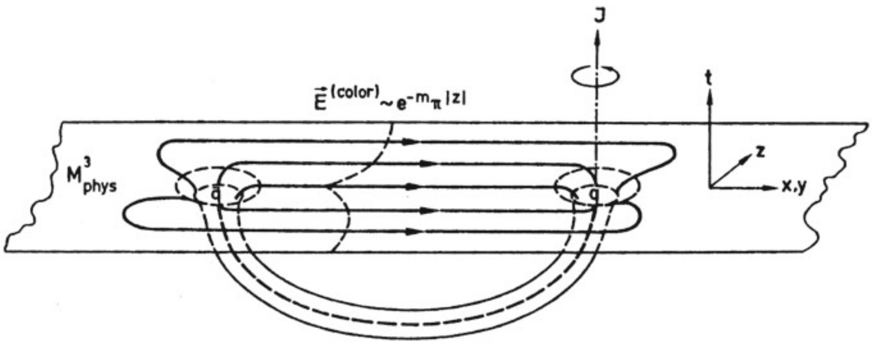

Fig. 15.1 Hadronic meson as "gluonic string" being trapped in a multiply connected Kerr–Newmann geometry

[3]This "color geometrodynamics" (CGMD) is approximately equivalent to NE'EMANs visionary notion (1965, p. 230): "This is a new type of 'geometrodynamics,' this time involving all interactions.".

15.3 Outlook: Gauge Unifications in Four Dimensions?

One of the first attempts to unify gravity with electromagnetism, the celebrated example of a U(1) gauge theory, was made by EINSTEIN & MAYER (1931), PERCACCI (1993), who explicitly advocated four dimensions (4D) in order to avoid generalizations to higher dimensions[4] à la Kaluza; cf. DENARDO & DOEBNER (1981). Indeed, from a modern perspective, dimension four is *special* not only for physics, where it represents macroscopic spacetime, but also for mathematics (DONALDSON 2006; ATIYAH et al. 2010) due to the invariants of Donaldson.

Eventually, Maxwell's abelian gauge group $U(1)$ was generalized to the global $SU(3)$ of GELL-MANN & NE'EMAN (1964), the $SU(4)$ of PATI & SALAM (1974), and even to $SU(5)$. Influenced by strong gravity ideas, the Pati–Salam model with leptons (or "leptoquarks") as fourth color, together with the local Lorentz group $SL(2, \mathbb{C})$ of gravity, has been embedded into *color geometrodynamics* (MIELKE 1980) as indicated previously; see also CASTRO (2012). Thereby, a $GL(8, \mathbb{C})^f \otimes GL(8, \mathbb{C})^c$ gauge unification of all basic particle forces emerged into a rather speculative scheme.

Since $SU(4)$ and $U(5)$ together with the local Lorentz group can also be embedded in higher orthogonal groups, gauge unifications based on local orthogonal groups (FADDEEV 2011) such as $SO(1, 9)$ and the embedding of the $SO(10)$ GUT into the group $SO(1, 13)$ have been proposed (PERCACCI 1991). Even "graviweak" groups such as $SO(4, \mathbb{C})$ have been considered by NESTI & PERCACCI (2010). Again, in order to be realistic, it is necessary to break (LECLERC 2006) such large symmetries à la Higgs.

More recently, exceptional groups (LISI & WEATHERALL 2010; LISI et al. 2010) such as $E(8)$ and hyperbolic Kac–Moody algebras (KLEINSCHMIDT & NICOLAI 2010) like $E(10)$ have been suggested to achieve this. However, these approaches may stumble into chirality issues (NESTI & PERCACCI 2010).

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[4] For similar reasons, generalizations of the MacDowell and Mansouri theory to six dimensions (BJORKEN 2013) as well as to supergravity (OBREGON et al. 2012) are not considered here.

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Chapter 16
Geometric Model of Quark Confinement?

In all current gauge-theoretic models of fundamental interactions, it is the material core, i.e., the emanating point of the gauge fields, that is taken to be constituted by spinor fields. Apart from the electron, neutrino, and those "excited" partners such as the muon, muonic neutrino, and the τ-lepton, which exist freely, hadronic, matter is based on quarks (GELL-MANN 1964). These hypothetic building blocks are thought to constitute all of the hadrons that are known at present. Up to now, high-energy experiments have not yielded any direct hints concerning their existence. It was only within the repetition of Millikan's experiment for the purpose of determining the quantum of charge with superconducting niobium spheres, LA RUE et al. (1981) found specimens with one-third the quantum of charge, but without confirmation.

However, independent of these considerations, it is incumbent on every realistic theory of strong interactions that is founded on the quark hypothesis to provide for a mechanism of permanent or partial (PATI & SALAM 1977) confinement of quarks.

An answer to the problem of saturation is in a sense already provided for by chromodynamics (GREENBERG & NELSON 1977): the basic assumption of the "colorlessness" of the total system against the structure group SU(3) allows just for bound states that consist of a quark–antiquark pair, or a combination of three colored quarks for which the total state represents a color singlet.

Additionally, the existence of free quanta of the associated spin-1 gauge fields, i.e., of gluons, is prohibited with reference to this postulate. What remains a problem in such gauge theories is consequently the interpretation of the strong interactions in the nucleus, which occur among the already compounded nucleons.

The permanent confinement of quarks cannot be proved by the usual treatment of quantum chromodynamics in terms of perturbation theory; cf. MARCIANO & PAGELS (1978). Rather, it is to be expected that a decisive role in this nonlinear

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field theory is taken over by global effects, as exemplified by instanton or monopole solutions. However, a convincing argument proving that quantum chromodynamics provides for a phase transition without necessitating additional preconditions, which could establish the confinement of quarks or at least that of the gluons, has not yet been found. The very reverse seems to be true in the classical treatment of such SU(3) color gauge theories: since adjacent vector gauge fields always point in the same direction, as far as the space of internal degrees of freedom is concerned, they repel each other in a manner similar to that in the case of like charges. Consequently, there are no geonlike structures (i.e., classical "glueballs") in a Yang-Mills gauge theory (Coleman & Smarr 1977, Weder 1977).

16.1 Bag Models

A promising and successful way out of this dilemma was provided by Chodos et al. (1974). Within the so-called MIT bag model, a color-sensitive "bag"constant B is introduced into the Lagrangian 4-form of the coupled Yang–Mills–Dirac system in order to compensate for the "quark gas." It is this supposition in addition to apt Cauchy initial conditions, regarding the fundamental spinor fields at the boundary of this extended and baglike hadronic structure, that leads to an astonishingly realistic description of the hadrons, see Hasenfratz & Kuti (1978). This is equally to be seen as a further development of the Dirac model (1962) of the extended electron which withstands the pressure of its own electromagnetic field by the Poincaré-type surface tension of its "bubble-type" structure. However, these and other theoretical constructions have to be introduced in a rather ad hoc way, due to the spin-1 character of the internal gauge fields.

In order to modify this discouraging situation, some are inclined to admit a tensor component within strong interactions, a theoretical modification that was considered by Salam (1977). Then all of the geometric theories with tensor gauge fields that have been presented so far in this study, i.e., the Poincaré gauge theory, the Kaluza–Klein theory and the f-g theory, have in common one feature: the occurrence of a "cosmological term" of microscopic origin. Within the qPG theory, a constant term is a consequence of the general duality ansatz by which the field equations are to be solved; within the Kaluza–Klein spaces, such a term is the result of integration over the internal curvature of the structure group taken as a manifold; and within the frame of the f-g theory, a related result is given by the symmetry-breaking mixing term and the interlocking of both metric structures. On the other hand, it has to be noted that the nontrivial vacuum expectation value of the Yang–Mills–Higgs theories, being spontaneously broken, is identical to a huge "microcosmological" constant, which possibly ought to be compensated for on a cosmological scale by the renormalization of its vacuum energy (Davies & Unwin 1981). In the microcosm, such terms are in principle unavoidable, so that the tensor component of the strong interaction is to be determined by Einstein's field equation

$$R^{\mathrm{f}}_{ij} - \frac{1}{2}\mathrm{f}_{ij}(R^{\mathrm{f}} - 2\Lambda_{\mathrm{eff}}) = \ell^2 \mathrm{T}_{ij} \tag{16.1.1}$$

for the strong gravity tensor.

And it was again Einstein who recognized their relevance for the construction of "baglike" models of (classical) particles: "The scalar curvature R takes over the part of a negative pressure that is of constant value $R_\circ$ outside of the electrically charged corpuscle. Within each of these corpuscles there is a negative pressure (positive $R - R_\circ$) whose gradient balances the electrodynamic force.[1]

In 1957, this idea was enlarged upon in a modifying way by LANCZOS. The possibility of providing an exemplary geometrodynamical mechanism for confinement via the Schwarzschild–de Sitter solution of the f-g vacuum field equations (16.1.1) was first pointed out by SALAM & STRATHDEE (1978a, b, c, cf. also CALDIROLA et al. 1978), and with reference to the Einstein–Cartan theory with internal degrees of freedom by MIELKE (1977a, b, 1980). The notion, however, that extended particles are to be represented by microuniverses of (anti-)de Sitter type, i.e., via a spacetime with *constant* curvature, is not completely new but has been already advocated by, for instance, PRASAD (1966, 1967) and ROMAN & KOH (1966). These notions have been revitalized by ROMAN & HAAVISTO (1977) regarding the already mentioned "spontaneous" generation of a microcosmological contribution on account of the breaking of the internal gauge symmetry by the Higgs fields.

Judged by today's advanced scientific standards, qPG theory may provide an effective theoretical framework, since it contains a strongly interacting tensor gauge field component (of the "rotons"), and this without the postulate of explicit terms inducing a symmetry breaking. Essentially, this tensor gauge field component is responsible for the negative (for $\kappa > 0$) pressure term

$$B = \frac{2\hbar c}{\ell^{*2}}\Lambda_{\mathrm{eff}} = -\frac{3\kappa}{2}\frac{\hbar c}{\ell^{*4}} \tag{16.1.2}$$

resulting from the effective cosmological constant within, e.g., the Baekler solution.

16.2 Confining Potentials from Strong Gravity

In order to clarify the confining properties of this configuration, we adopt the crude physical notion, following SALAM & STRATHDEE (1978c), that a hadron is represented by a Schwarzschild–de Sitter microuniverse, whereby the $2\mu/r$-singularity results from one of the quarks. The properties of this metric background are thought to be probed by the addition of a further quark—as it were a test particle—since its

[1]"Der Krümmungsskalar R spielt die Rolle eines negativen Druckes, der außerhalb der elektrischen Korpuskel einen konstanten Wert $R_\circ$ hat. Innerhalb jeder Korpuskel besteht ein negativer Druck (positives $R - R_\circ$), dessen Gefälle der elektrodynamischen Kraft das Gleichgewicht leistet" (EINSTEIN 1919, p. 352).

contribution as a source of an induced gravitational field is to be ignored. Within this model, we restrict ourselves to scalar quarks that have to serve not only as a source but also as test particles that solve the Klein–Gordon equation

$$[\Box - (mc/\hbar)^2]\varphi = 0 \tag{16.2.1}$$

within the given four-dimensional background space of constant curvature given by (3.9.1,2) for $\mu = 0$. Here

$$\Box = \frac{1}{\sqrt{|f|}}\partial_i(f^{ij}\sqrt{|f|}\partial_j) \tag{16.2.2}$$

denotes the local expression of the generally covariant Laplace–Beltrami operator (B.19). Stationary radial solutions of (16.2.1) are obtained by the application of the well-known separation ansatz

$$\varphi = m\frac{c}{\hbar}\frac{1}{\rho}F^{\iota}_{nm}(\rho^*)Y^m_{\iota}(\vartheta,\phi)e^{-it\omega mc^{2/\hbar}}. \tag{16.2.3}$$

Moreover,

$$\rho := \sqrt{\frac{1}{3}\Lambda_{\text{eff}}}\ r, \quad r = |\mathbf{x}| \tag{16.2.4}$$

is introduced as a radial coordinate and via a Pfaffian form, Wheeler's "tortoise" coordinate

$$\rho^* = \int_o^\rho \frac{dx}{1 \pm x^2} = \begin{cases} \text{artan}\ \rho, & \kappa > 0 \\ \text{Arth}\ \rho, & \kappa < 0). \end{cases} \tag{16.2.5}$$

Within the anti-de Sitter space, the above integration provides a globally defined coordinate. The (dimensionless) ratio of the Compton wavelength ℓ_m of the test particle to the Planck length ℓ^* is abbreviated by

$$\beta = \frac{\sqrt{2\pi\kappa}}{\ell^*}\Big/\frac{mc}{\hbar} = \frac{\ell_m}{\ell}. \tag{16.2.6}$$

Then it follows that the Eq. (16.2.1) is reducible to the radial Schrödinger equation

$$\left[\partial^2_{\rho^*} - V^{\iota}_{\text{eff}}(\rho^*) + \frac{\omega^2}{\beta^2}\right]F = 0 \tag{16.2.7}$$

with reference to the still to be determined functions $\mathrm{F} = \mathrm{F}^{\iota}_{\mathrm{nm}}(\rho^*)$.

After the transition to the "tortoise" coordinate ρ^*, the curved background space no longer enters the wave equation in a multiplicative manner but formally as an effective curvature potential (cf. MTW, p. 868). For the anti–de Sitter space, this

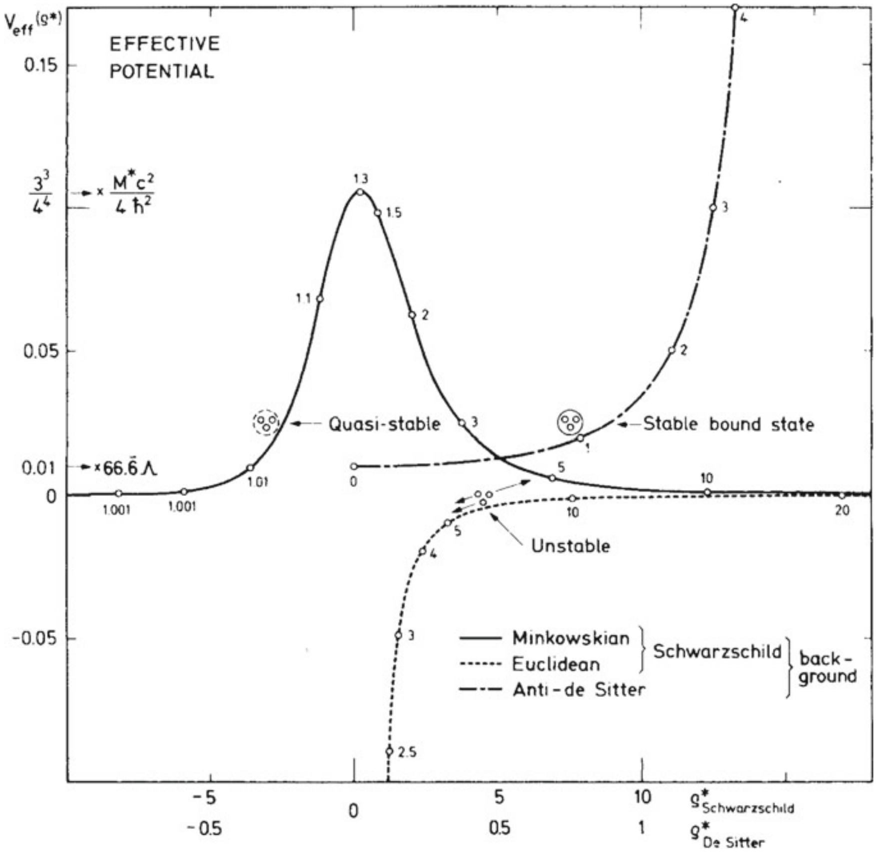

Fig. 16.1 Confining potentials in strong gravity

potential is implicitly given by

$$V_{\text{eff}}^{\iota}(\rho^*) = \frac{1 \pm \rho^2}{\rho^2}\left[\iota(\iota+1) - \left(2 - \frac{1}{\beta}\right)\rho^2\right], \begin{matrix}\kappa > 0 \\ (\kappa < 0).\end{matrix} \tag{16.2.8}$$

For the sake of illustration (Fig. 16.1), the simplest case $\iota = 0$ for $\kappa > 0$, i.e., the state without orbital angular momentum, is considered. Then the effective potential takes on the explicit form

$$V_{\text{eff}}^{o}(\rho^*) = \left(\frac{1}{\beta^2} - 2\right)[1 + \tan^2\rho^*] \tag{16.2.9}$$

and thus belongs to a family of potentials that in the context of oscillations of diatomic molecules was studied by PÖSCHL & TELLER (1933) (cf. also FLUGGE 1971, p. 89).

At $\rho^* = \frac{\pi}{2}$ there arises for $\beta^2 < 2$ a steeply rising potential wall that can never be penetrated by a test quark.

The radial Schrödinger equation (16.2.7) can be solved in terms of hypergeometric functions (PRASAD 1966, SALAM & STRATHDEE 1978c). The regular square-integrable solutions form a complete set of functions with quantized field energies

$$E_n = \beta mc^2 \left(2n + \iota + \frac{3}{2} + \sqrt{\frac{g}{4} + \frac{1}{\beta^2}} \right), n = 0, 1, 2 \ldots . \quad (16.2.10)$$

Exempt from the zero-point energy, these are exactly the energy levels that are known from the quantum-mechanical treatment of the three-dimensional *nonrelativistic harmonic oscillator* (FLUGGE 1971, p. 70). The result (16.2.10) may be considered the prototype of a mass spectrum for an excited hadron with integer spin, in particular if the invariance group SO(2,3) of the anti-de Sitter space is considered a dynamical symmetry of the interacting system; cf. BARUT (1972). Only discrete eigenvalues belong to the spectrum. There is no continuum to it and subsequently no dissociation of the quarks. This is equally shown by the investigation of coherent quantum-mechanical states in such potentials that was conducted by NIETO & SIMMONS (1979).

On account of the infinitely steep wall of the potential, which is fixed in this coordinate system at $\rho^* = \frac{\pi}{2}$, the geometrodynamical confinement mechanism incorporates the properties not only of the MIT bag but also of models with an external potential. In the latter, it is usually the potential $\mathrm{V(r)} = 2\mu/\mathrm{r} + \mathrm{cr}$ that is postulated for the *confinement* (MARCIANO & PAGELS 1978). The Coulomb-type source term $2\mu/\mathrm{r}$ is also necessary in quantum chromodynamics as well as in the Schwarzschild–de Sitter solution, in order to account for that "asymptotic freedom" (WILCZEK 2005) of the constituents that is observable in the scattering experiments at very high energy, i.e., at very small distances. This would in any case be expected in gauge field theories with nonabelian structure groups. Arguments for the conjecture that a *quantized* theory of gravity is also asymptotically free are to be found in SALAM (1977). Additionally, it has to be remarked that the Schwarzschild term, if taken alone, already induces such a partial confinement (DEPPERT & MIELKE 1979). Then spin-1 gluon fields, which saturate the quarks, would be unobservable from the outside, i.e., would become "transcendent" (WHEELER 1971).

It has to be admitted that our analysis of the confinement properties of an anti-de Sitter microcosm is only of model-like character. The quarks as hypothetical building blocks of matter are fermions. Consequently, it would thus be necessary to investigate a generally covariant and "minimally" coupled Dirac equation, within the frame of the EC theory with internal degrees of freedom even with a torsion-induced self-interaction of the different spinor fields. Moreover, due to the spin, a self-consistent axially symmetric solution of Kerr–Newman–de Sitter type would be an even more appropriate representation of the metric background. According to these notions, the hadrons would be identical to the *geons* or "black solitons," respectively (Wheeler 1955; Salam & Strathdee 1976), which are held together by their self-generated strong curvature (Vigier 1966; Mielke & Scherzer 1981).

The main problem for the de Sitter model of the *geometrodynamical mechanism* of *quark confinement* has to be seen in the embedding of the hadronic microcosm into spacetime (Salam & Strathdee 1978c). For here it has to be achieved via an appropriate mechanism of symmetry breaking. The effective "bag" constant (16.1.2) of the qPG theory with internal degrees of freedom becomes color sensitive, in such a way that the colored quarks are subject to the geometrodynamical mechanism of confinement, while the colorless singlet state, which represents the hadron, is not. Moreover, the issue of embedding—as is already assumed by Komar (1964), Ne'eman (1965) and Vigier (1966)–could lead to important insights concerning the physical origin of the structure group SU(f) of the "flavor" symmetries.

16.3 Black Soliton Mass Formula

Independent of its internal structure, the external region of a hadron of mass M, quantum number J of total angular momentum, and generalized charge $\bar{Q}$ is (almost) unequivocally described by the Kerr–Newman solution. This follows from the corresponding generalization of Wheeler's conjecture that "a black hole has no hair," i.e., that there are no further adjustable parameters to a black hole other than M, J, and Q (MTW, p. 876; cf. also Teitelboim 1972 and Bekenstein 1972a, b, c, 1975). In the Einstein–Cartan theory with internal degrees of freedom, a modification is obtained only insofar as the concept of charge had also to be extended with reference to the quantum numbers I, Y, and C of isotopic spin, hypercharge, and possibly also charm.

If the basis of the following considerations is seen in a model with additional charmed quarks, it will be observed that the quadratic Casimir operator of the broken SU(4) symmetry, with reference, for instance, to the octet representation within the 20-dimensional representation of the excited baryon states (Okubo 1975), has the following eigenvalues:

$$\mathrm{Tr}\overline{Q}^2 = -\frac{4b}{\alpha_e}Y + I(I+1) - \frac{1}{4}Y^2. \tag{16.3.1}$$

The amount of symmetry breaking enters via the phenomenological parameter b, while the electric charge Q_e is determined via the generalized Gell-Mann–Nishijima relation

$$Q_e = I_3 + \frac{1}{2}Y - \frac{2}{3}C. \tag{16.3.2}$$

The size of the horizon of the Kerr–Newman solution is characterized by the irreducible mass M_{ir}; concerning the tensor dominance model, it is thus suggestive to propose that M_{ir} as well as the modified Planck mass $M^* := \hbar\sqrt{8\pi}/\ell^* c$ are approximately identical to the mass of a proton, i.e., $M_{ir} = M^* = 1\mathrm{GeV}/c^2$. For

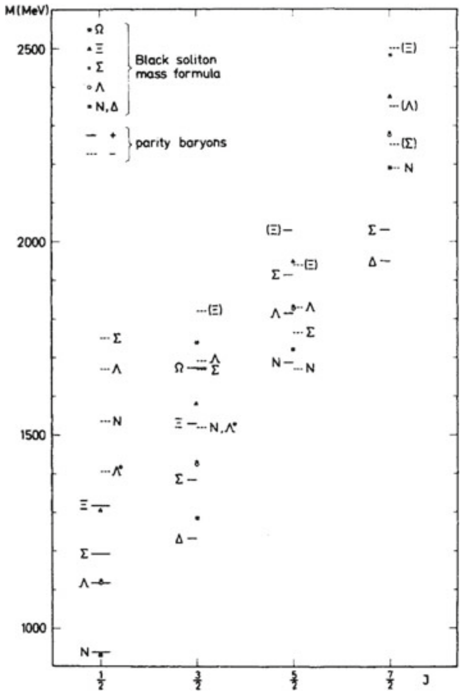

Fig. 16.2 Black soliton mass formula for baryons

such a rescaling, the CHRISTODOULOU & RUFFINI (1971) result leads to the mass formula

$$\frac{M^2}{M^{*2}} = \left\{1 - bY + \frac{\alpha_e}{4}[I(I+1) - \frac{1}{4}Y^2]\right\}^2 + \frac{1}{4}J(J+1) \tag{16.3.3}$$

for *extended* baryons (MIELKE 1977b, 1980). Thus in contrast to the comparable Gell-Mann–Okubo formula (GELL–MANN 1964) or to that of GÜRSEY & RADICATI (1964), M, J, and I are the characteristics of particles that are here defined *canonically* in the sense of field theory. In (16.3.3) these are coupled in a nonlinear way on account of the underlying (strong) geometric structure of the Einstein–Yang–Mills system. (However, if J = 0, this leads precisely back to the linear Gell-Mann–Okubo formula!)

As sort of a "metatheorem," it then is to be expected that an analogous result follows from every generally relativistic gauge field theory that couples to "gravitational charge," i.e., to the mass. This conjecture, which is also to be expected after quantization, is underscored by the fact that part of the mass spectrum of baryons is surprisingly well represented by (16.3.3). (In Fig. 16.2, b $= \frac{1}{5}$ has been chosen, while M* has been accommodated to the state $\Lambda(1116)$ of excited baryons.) A detailed comparison with the empirical masses of the nucleon and Δ resonances and a possible verification of the mass spectrum of mesons is to be found in MIELKE (1981).

The idea to generate a nonphenomenological relation for the mass increase of excited hadrons from the structure of the Kerr–Newman geometry can be traced back to SALAM (1973), and it was later considered in a series of highly speculative works (TENNAKONE 1974; KESKINEN & PERKO 1976; SIVARAM & SINHA 1977; MOFFAT 1978). Those latter works mention the Christodoulou–Ruffini formula without employing it in full scale for the construction of a fundamental mass formula for hadrons with nonvanishing total angular momentum (cf. SIVARAM & SINHA 1979).

The notion that a hadron corresponds to a microuniverse with regard to its interior, while its exterior characteristics are similar to a black hole (the black hole possibly being a quantized one; cf. MAL'TSEV & MARKOV 1980) in which the quarks are confined via either an enormously steep wall in the potential or the semipermeable "membrane" of the *black soliton's* horizon has even further physical consequences. As has already been mentioned, the Kerr–Newman geometry yields exactly Dirac's gyromagnetic ration for the electron, for an observer located at asymptotic infinity. Additionally, the Bekenstein–Hawking effect (HAWKING 1975) occurs as well with this quantum-theoretic treatment of the external fields in curved spacetime: a black hole with a given surface area of the event horizon emits radiation like an absolute blackbody of temperature

$$T_{\bullet} = \frac{2\hbar c}{S_A k_B}\left(r_+ - \frac{M}{M^{*2}}\frac{\hbar}{c}\right). \tag{16.3.4}$$

Here $k_B = 8.6 \times 10^{-5}\,\mathrm{eV/K}$ denotes the Boltzmann constant. Under the assumption of a strongly interacting tensor component, this fundamental process, which is established in the geometric frame, can be related quantitatively—similarly to the blackbody radiation of the MIT bags (CHODOS et al. 1974)—to Hagedorn's concept of a maximum temperature in hadronic fireballs (SALAM & STRATHDEE 1977). On the other hand, it clearly has to be seen that the probability of a quantum-mechanical breaking of a black hole into several smaller ones is nearly zero (HAWKING 1976). Measured by classical standards, such a process is prohibited in any case. Therefore, in our view, a black hole "rivals an elementary particle in its perfection …" (cf. WHEELER 1974, p. 279). Independently from the question whether it is a specified phase transition characterized by instanton solutions of the Yang–Mills theories or the ad hoc introduced boundary conditions of the MIT bag or the anti-de Sitter solutions of an extended geometrodynamics that solve the issue of quark confinement and hence the riddle of the structure of matter.

All of these different working hypotheses have their conceptual origin in Riemannian geometry. And this one fact—in agreement with an insight of VON–HELMHOLTZ (1868)—cannot be dismissed, i.e., that a Riemannian space indeed allows a pointwise rotation, but that free movability of *extended* (rigid) domains is possible only in spaces of *constant* curvature. Judged by today's scientific standards, this looks rather like an anticipation of the notion of extended hadrons as closed microuniverses that communicate with one another and with the macrocosm only via self-consistently generated gauge fields. This "prestabilized harmony" roughly in the sense[1] of the categories of LEIBNIZ's (1714) monadology may be the core answer of geometrodynamics to the fundamental issue of physics: "what is the structure of matter?"

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[1] See also WEYL (1924, p. 208), MTW, p. 1217, and SIEROKA & MIELKE (2014).

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Appendix A
Notation

(a) Mathematical symbols

$:=$	: Equal by definition
$\equiv$	: Identically equal
$\approx$	: Isomorphic
$\simeq$	: Approximately equal
$\sim$	: Asymptotic expansion
$\times$	: Topological product
$\otimes$	: Direct product
$\oplus$	: Direct sum
$\ltimes$	: Semidirect product
$\mathrm{G}_\circ$	: Connected component of the group
$\tilde{\mathrm{G}}$	: Simply connected covering group
$[\mathrm{A}, \mathrm{B}] := \mathrm{AB} - \mathrm{BA}$	: Commutator
$\{\mathrm{A}, \mathrm{B}\} := \mathrm{AB} + \mathrm{BA}$	: Anticommutator
$\mathrm{A}^* = \mathrm{C}\,\mathrm{A}$	: Complex-conjugate matrix
$\mathrm{A}^+ = \mathrm{A}^{*\mathrm{T}}$	: Hermitian adjoint matrix
$\overset{-1}{\pi}(\mathrm{N})$	: Original domain of N with respect to the mapping $\pi : \mathrm{M} \to \mathrm{N}$
$\mathrm{T}_{(\mu\nu)} := \frac{1}{2}(\mathrm{T}_{\mu\nu} + \mathrm{T}_{\nu\mu})$	: Symmetrized second-rank tensor
$\mathrm{T}_{[\mu\nu]} := \frac{1}{2}(\mathrm{T}_{\mu\nu} - \mathrm{T}_{\nu\mu})$	: Antisymmetrized second-rank tensor

(b) Metric and summation conventions
If not stated otherwise, Einstein's convention of summation over repeated indices is taken for granted:

$$x^A y_A := \sum_{A=0}^{n} \sum_{B=0}^{n} x^A y^B g_{AB}. \tag{A.1}$$

(c) Physical units
With the exceptions given, the following system of natural units is employed:

$$c = \hbar = 1. \tag{A.2}$$

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(d) Spin matrices
2 × 2 Pauli matrices

$$
\begin{aligned}
&(\sigma^a)^{\mathrm{T}} := (\sigma^0, \sigma^k) \quad k = 1, 2, 3. \\
&\sigma^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \mathbb{1}, \ \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{A.3}
$$

They obey the commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = \varepsilon^{ij}{}_k \, \sigma^k \tag{A.4}$$

and

$$\{\sigma^i, \sigma^j\} = g^{ij} \tag{A.5}$$

of the Clifford algebra C(0, 2) = $\mathbb{H}$ (field of quaternions).

4 × 4 Dirac matrices in the so-called Pauli realization:

$$(\gamma^\alpha)^{\mathrm{T}} := (\gamma^0, \gamma^k), \tag{A.6}$$

where

$$
\begin{aligned}
&\gamma^0 := \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \quad \gamma^k := \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}; \\
&\gamma^5 := i\gamma^\circ\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}.
\end{aligned} \tag{A.7}
$$

These matrices satisfy the defining relation

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta} \tag{A.8}$$

of the Clifford algebra C(1, 3).

Moreover, it follows from (A.3) and (A.5) that

$$\gamma^{0+} = \gamma^0, \quad \gamma^{k+} = -\gamma^k, \quad \gamma^{5+} = \gamma^5. \tag{A.9}$$

(e) Clifford algebras
According to ATIYAH et al. (1964), every Clifford algebra can be decomposed into a tensor product of the following elementary Clifford algebras:

$$\mathrm{C}(0,1) = \mathbb{C} \qquad \text{(field of complex numbers)}, \tag{A.10}$$

$$\mathrm{C}(1,0) = \mathbb{R} \oplus \mathbb{R}, \tag{A.11}$$

$$\mathrm{C}(1,1) = \mathrm{C}(2,0) = \mathrm{GL}(2,\mathbb{R}), \tag{A.12}$$

$$\mathrm{C}(0,2) = \mathbb{H} \qquad \text{(field of quaternions)}, \tag{A.13}$$

As an example, the Clifford algebra C(1, 3) of the familiar 4×4 Dirac matrices (A.6) can be decomposed as follows:

$$C(1,3) = C(0,2) \otimes C(1,1). \tag{A.14}$$

More generally, in n dimensions, the decomposition reads

$$C(1,n-1) = \left[\bigotimes_{i=1}^{[\frac{n-2}{2}]} C(0,2)\right] \otimes C(1,1). \tag{A.15}$$

This algebraic structure is reflected in the following realization of $\mathrm{N} \times \mathrm{N}$ Dirac matrices in an n-dimensional spacetime:

$$\begin{aligned}
(\widehat{\gamma}^{A})^{\mathrm{T}} &:= (\widehat{\gamma}^{\mu}, \widehat{\gamma}^{2p+1}, \widehat{\gamma}^{2p+2}), \\
\widehat{\gamma}^{\mu} &:= \gamma^{\mu} \otimes \left[\bigotimes_{i=1}^{[\frac{n-2}{2}]} \sigma^{1}\right], \\
\widehat{\gamma}^{2p+1} &:= i\mathbb{1} \otimes \left(\bigotimes_{i=1}^{p-2} \sigma^{0}\right) \otimes \sigma^{3} \otimes \left(\bigotimes_{j=1}^{l-p+1} \sigma^{1}\right), \\
\widehat{\gamma}^{2p+2} &:= i\mathbb{1} \otimes \left(\bigotimes_{i=1}^{p-2} \sigma^{0}\right) \otimes \sigma^{2} \otimes \left(\bigotimes_{j=1}^{l-p+2} \sigma^{1}\right),
\end{aligned} \tag{A.16}$$

where $4 < 2\mathrm{p} + 1 < 2\mathrm{p} + 2 \leq n = 2l + 2$; cf. KALINOWSKI (1984).

(f) Topological space

A topological space consists of a set X together with a collection of subsets of X, called open sets, that satisfy the following implicit axioms:

(i) The union of an arbitrary number of open sets is open.
(ii) The intersection of any finite number of open sets is open.
(iii) Both X itself and the empty set Ø are open.

(g) Hausdorff space

The open sets of a topological space are usually subject to the requirements of the following *separation axioms:*

T_1: For each two distinct points $\mathrm{m}_1, \mathrm{m}_2 \in \mathrm{X}$, there exist neighborhoods $\mathrm{U}_\mathrm{i} \ni \mathrm{m}_\mathrm{i}$ of each point that exclude the other point.

T_2: For any two distinct points $\mathrm{m}_1, \mathrm{m}_2 \in \mathrm{X}$ there exist neighborhoods $\mathrm{U}_\mathrm{i} \ni \mathrm{m}_\mathrm{i}$ that are disjoint, namely $\mathrm{U}_1 \bigcap \mathrm{U}_2 = \emptyset$.

A topological space X that satisfies the axiom T_2 is called a Hausdorff space.

(h) Diffeomorphism
A diffeomorphism $\sigma : M \to N$ between manifolds of the same dimension (e.g., $M \approx N \approx \mathbb{R}^n$) is an injective mapping for which σ and the corresponding inverse mapping $\overset{-1}{\sigma}$ are C^∞ functions, i.e., every partial derivative of σ exists and is continuous in some neighborhood.

(i) Inverse image ("pullback") of exterior forms
Let $\sigma : M \to N$ be a mapping from a manifold M to another manifold N and let α be a given exterior differential form on N. The differential of σ at the point m gives rise to a mapping $\sigma_* : T_m(M) \to T_{\sigma(m)}(N)$ within the space of tangent vectors along the curve s(t) or $\sigma(s(t))$ if constructed according to the directional derivative e^f. On the other hand, the transposed mapping of σ_* defines a linear mapping $\sigma_*^T : T^*_{\sigma(m)}(N) \to T^*_M(M)$ between the "dual" cotangent spaces. As such, it may be employed for the construction of a p-form on M

$$\overset{*}{\sigma}\alpha(e_{\alpha_1}, \ldots, e_{\alpha_p}) := \alpha(\sigma_* e_{\alpha_1}, \ldots, \sigma_* e_{\alpha_p}). \quad \text{(A.17)}$$

Consequently, $\overset{*}{\sigma}\alpha$ denotes the inverse image of the differential form α. For cross sections of fiber bundles over M or N, analogous mappings can be constructed (DRECHSLER & MAYER 1977, p. 67).

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Appendix B
Calculus of Exterior Differential Forms

Completely *antisymmetric* covariant tensor fields of degree $(p, 0)$ are of prime importance not only in differential geometry but also in the description of gauge fields. Following the book of FLANDERS (1963), a compendium of the calculus of forms will be given here.

The space of *exterior differential forms* of degree p is generated by the completely antisymmetric tensor field

$$\alpha^{(p)} = \frac{1}{p!} A_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \in C^{\infty}(\wedge^p T^*(M)). \tag{B.1}$$

With regard to the *exterior* (wedge or Grassmann) product

$$\begin{aligned}\alpha^{(p)} \wedge \beta^{(q)} := \frac{1}{(p!) \cdot (q!)} & A_{\alpha_1 \cdots \alpha_p} \cdot B_{\beta_1 \cdots \beta_q} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \\ & \wedge \vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_q}\end{aligned} \tag{B.2}$$

having the alternating property

$$\alpha^{(p)} \wedge \beta^{(q)} = (-1)^{pq} \beta^{(q)} \wedge \alpha^{(p)}, \tag{B.3}$$

these p-forms span the *exterior algebra*

$$D(M) := \sum_{p=0}^{n} \wedge^p T^*(M) \tag{B.4}$$

of differential forms over the field $\mathbb{R}$ of real numbers. The action of the exterior differential d within D(M) is defined as follows: applied to a 0-form or a real function

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f, it is nothing but the total differential[1] $\mathrm{df} = \vartheta^\alpha \partial_\alpha \mathrm{f}$. This action is extended to arbitrary forms by postulating the following additional properties:

$$d(\alpha^{(p)} \wedge \beta^{(q)}) = d\alpha^{(p)} \wedge \beta^{(q)} + (-1)^p \alpha^{(p)} \wedge d\beta^{(q)} \tag{B.5}$$

(generalized Leibniz rule) and

$$d\, d\alpha^{(p)} \equiv 0. \tag{B.6}$$

With these additional rules, d is defined for general p-forms.

These rules ensure that the exterior differential forms transform *covariantly* with respect to smooth transformations of the coordinates. Let

$$\sigma : M \to N, \quad \sigma \in \mathscr{D}(M) \tag{B.7}$$

be a diffeomorphism between two manifolds. Then there exists a "derived" mapping

$$\sigma_* : \wedge^p T^*(N) \to \wedge^p T^*(M) \tag{B.8}$$

that for a 0-form is obtained by composition the $\sigma_* \alpha^{(0)} := \alpha^{(0)} \circ \sigma$.

This construction extends to p-forms in a way (FLANDERS 1963, p. 56) that clearly exhibits their covariance:

$$\sigma_*(\alpha^{(p)} + \beta^{(p)}) = \sigma_*(\alpha^{(p)}) + \sigma_*(\beta^{(p)}), \tag{B.9}$$

$$\sigma_*(\alpha^{(p)} \wedge \beta^{(q)}) = \sigma_*(\alpha^{(p)}) \wedge \sigma_*(\beta^{(q)}), \tag{B.10}$$

$$d(\sigma_* \alpha^{(p)}) = \sigma_*(d\alpha^{(p)}). \tag{B.11}$$

Let $\varepsilon_{\alpha_1 \ldots \alpha_n}$ be the totally antisymmetric unit tensor (Levi-Civita tensor; see MTW, p. 87) with the conventional choice of signs $\varepsilon_{\hat{1} \ldots \tilde{n}} = +1$. In a pseudo-Riemannian manifold endowed with the metric tensor g_{ij}, this unit tensor allows one to define the *dual* of a p-form by

$$^*\alpha^{(p)} := \frac{1}{(n-p)! \cdot (p!)} \sqrt{|\det g_{\mathrm{ij}}|}\, \varepsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_{n-p}} \times A^{\alpha_1 \cdots \alpha_p} \vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_{n-p}}. \tag{B.12}$$

(Here the metric is used not only for the construction of the determinant but also for raising the indices of the local components of the p-form). Application of this so-called Hodge star operator * brings an (n-p)-form into existence.

Let us assume, as a special case, that a p-form is given as the exterior product of p canonical 1-forms ϑ. On the other hand, it is known that the ϑ^α generate a basis for a generic form. Then it follows from (B.12) that its *dual,* i.e.,

[1] This differential is given here in terms of an arbitrary "comoving" frame of reference; with respect to a coordinate basis, it would read $\mathrm{df} = (\partial \mathrm{f}/\partial x^i) dx^i$.

$$^*(\underbrace{\vartheta \wedge \cdots \wedge \vartheta}_{\text{p products}}) = \frac{p!}{(n-p!)} \underbrace{\vartheta \wedge \cdots \wedge \vartheta}_{\text{n}-\text{p products}} (\star), \tag{B.13}$$

consists of n-p exterior or alternating products of the 1-form ϑ. The *volume form* on the manifold M may consequently be expressed by

$$\begin{aligned} d_\mu(M) &:= {}^*1 = \sqrt{|\det g_{\alpha\beta}|}\, \vartheta^1 \wedge \cdots \wedge \vartheta^n = \eta \\ &= \sqrt{|\det\ g_{ij}|}\, dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n} \vartheta \wedge {}^*\vartheta. \end{aligned} \tag{B.14}$$

Except for the sign, the twofold application of the star operator leads back to the original form, i.e.,

$$^{**}\alpha^{(p)} = (-1)^{p(n+1)+s}\alpha^{(p)}. \tag{B.15}$$

Moreover, the Hodge star operator induces an inner product

$$\begin{aligned} \langle \alpha^{(p)}, \beta^{(p)} \rangle &:= \alpha^{(p)} \wedge {}^*\beta^{(p)} = \beta^{(p)} \wedge {}^*\alpha^{(p)} \\ &= (-1)^{s*}(\alpha^{(p)} \wedge {}^*\beta^{(p)})\eta \\ &= \frac{(-1)^s}{p!} A^{\alpha_1 \cdots \alpha_p} B_{\alpha_1 \cdots \alpha_p} d\mu(M) \in C^\infty(\wedge^n T^*(M)) \end{aligned} \tag{B.16}$$

on the space of p-forms that may be extended to the following globally defined symmetric *scalar product*:

$$(\alpha, \beta) := \int_M \alpha^{(p)} \wedge {}^*\beta^{(p)} = (\beta, \alpha) \tag{B.17}$$

(see, e.g., CHOQUET- BRUHAT et al. 1982), provided the integration over the pseudo-Riemannian manifold converges.

The adjoint of the exterior derivative d, i.e., the generalized *divergence* operator for p-forms, is defined by

$$\delta\alpha^{(p)} := (-1)^{pn+n+1+s}\ {}^*d^*\alpha^{(p)}. \tag{B.18}$$

The Laplace–Beltrami operator is an invariant operator of second order that does not change the degree of 0-forms if applied to scalars. For p-forms, the appropriate generalization is provided by the *de Rham–Lichnerowicz operator*

$$\Box\alpha^{(p)} := d\delta\alpha^{(p)} + \delta d\alpha^{(p)}. \tag{B.19}$$

In the case of a Riemannian manifold, its expansion in local components is intricate and may be found in the book of YANO (1970, p. 67).

For a closed manifold, i.e., one without boundary, the generalized Laplace–Beltrami operator (B.19) is positive definite with respect to the scalar product defined by (B.17). The proof follows immediately from the following identities:

$$\begin{aligned}(\alpha, \Box\alpha) &= (\alpha, d\delta\alpha) + (\alpha, \delta d\alpha)\\ &= (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha).\end{aligned} \tag{B.20}$$

In the de Rham theory of cohomology, an exterior differential form is called a *closed form* if it satisfies

$$d\alpha = 0. \tag{B.21}$$

A sufficiently smooth form α is *harmonic*, i.e., satisfies the relation

$$\Box\alpha = 0, \tag{B.22}$$

if and only if it is closed as well as *coclosed*, i.e., $\delta\alpha = 0$.

The corresponding cocycles Z(M) form a graded subalgebra of D(M). The subset B(M) := d D(M) of *exact* forms

$$\beta^{(p)} = d\alpha^{(p-1)} \in C^{\infty}(B(M)) \tag{B.23}$$

is a graded ideal in Z(M) due to the property (B.6) of the exterior derivative. The (graded) quotient space

$$H(M) := Z(M)/B(M) \tag{B.24}$$

is commonly called de Rham's *cohomology algebra* of M. The grading of H(M) is generated via

$$H(M) = \sum_{p=0}^{n} H^p(M) := \sum_{p=0}^{n} Z^p(M)/B^p(M). \tag{B.25}$$

Since the degree of a differential form cannot exceed the dimension n of the manifold, the following is a logical consequence:

$$H^p(M) = 0 \quad \text{for} \quad p > n, \tag{B.26}$$

and

$$H^n(M) = \wedge^n T^*(M)/B^n(M). \tag{B.27}$$

On the other hand, we have $B^0(M) = 0$. Consequently,

$$H^0(M) = Z^0(M) \tag{B.28}$$

is bound to consist solely of closed smooth functions over M. If M is connected, these functions have to be constant, a result that has

$$H^0(M) \approx \mathbb{R} \tag{B.29}$$

as its consequence.

In the case of finite-dimensional cohomology classes (as is the case for compact manifolds), the dimensions of the spaces $H^p(M)$ are called the *Betti numbers*

$$b_p := \dim H^p(M) \tag{B.30}$$

of M. The Poincaré polynomial

$$\mathrm{f_M}(t) = \sum_{p=0}^{n} b_p t^p, \tag{B.31}$$

with the Betti numbers as coefficients, gives for $t = -1$ the alternating sum

$$\chi(M) := \mathrm{f_M}(-1) = \sum_{p=0}^{n} (-1)^p b_p, \tag{B.32}$$

which yields the *Euler–Poincaré characteristic*, or Euler number for short.

If the base manifold is the n-dimensional sphere, S^n, then the cohomology classes are given by

$$\mathrm{H}^0(S^n) = H^n(S^n) \approx \mathbb{R} \tag{B.33}$$

and

$$H^p(S^n) = 0 \quad \text{for} \quad 1 \leq p \leq n-1. \tag{B.34}$$

Consequently, the corresponding Poincaré polynomial takes the form

$$\mathrm{f}_{S^n}(t) = 1 + t^n, \tag{B.35}$$

which yields

$$\chi(S^n) = \begin{cases} 0 \text{ for } n = 2k+1, \\ 2 \text{ for } n = 2k, \end{cases} \tag{B.36}$$

as the Euler number of the n-sphere.

In order to be able to integrate differential forms over the manifold, we proceed from a simplicial covering of M. In the case of the Euclidean vector space E^n endowed with a countable basis η_i, a p-dimensional standard simplex is defined by the set

$$\Delta_p := \left\{ x \in E^n | x = \sum_{i=0}^{p} \lambda^i \eta_i, \lambda^i \geq 0, \sum_{i=0}^{p} \lambda^i = 1 \right\}. \tag{B.37}$$

From this construction, a *smooth p-simplex* of a manifold M may be derived from the C^∞-mapping $c : \Delta_p \to M$. These simplices form a free $\mathbb{R}$-module $C_p(M, \mathbb{R})$, whose elements are called p-chains. The boundary ∂c of a chain will be generated by the mapping

$$\partial : C_p(M, \mathbb{R}) \to C_{p-1}(M, \mathbb{R}), \tag{B.38}$$

defined explicitly by

$$\partial c = \sum_{i=0}^{p} (-1)^i c \left(\sum_{j=0}^{i-1} \lambda^j \eta_j + \sum_{j=i}^{p-1} \lambda^j \eta_{j+1} \right). \tag{B.39}$$

As in the case of the exterior derivative, this operation is a linear derivation. As such, it has the property

$$\partial\, \partial\, c \equiv 0, \tag{B.40}$$

which means figuratively speaking that "the boundary of a boundary vanishes identically." The spaces $Z_p(M, \mathbb{R}) = (\mathrm{Ker}\partial)_p$, $B_p(M, \mathbb{R}) := (\mathrm{Im}\partial)_p$ as well as

$$H_\mathrm{p}(M, \mathbb{R}) := Z_\mathrm{p}(M, \mathbb{R})/B_\mathrm{p}(M, \mathbb{R}) \tag{B.41}$$

are called p-*cycle*, p-*boundary*, and p-dimensional *homology class*, respectively.

In general, p-forms may be regarded as p-dimensional volume elements of a p-dimensional (compact) submanifold $N_{(p)} \subset M$ of M. Consequently, they can be integrated, provided $N_{(p)}$ admits a simplicial covering by the chain $c_p \in C_p(M, \mathbb{R})$. Then *Stokes's theorem*[2] can be expressed in its most general form via

$$\int_{c_{p+1}} d\alpha^{(p)} = \int_{\partial c_{p+1}} \alpha^{(p)}. \tag{B.42}$$

Expressed differently, the integration

$$\langle\, , \rangle_M : \begin{cases} \bigwedge^p T^*(M) \times & C_p(M, \mathbb{R}) \to \mathbb{R} \\ \Psi & \Psi \\ \langle \alpha^{(p)}, & c_p \rangle_\mathrm{M} \to \int_{c_p} \alpha^{(p)} \end{cases} \tag{B.43}$$

provides us with an $\mathbb{R}$-bilinear mapping in terms of exterior forms and chains. Due to (B.42), the relation

$$\langle d\alpha, c \rangle_\mathrm{M} = \langle \alpha, \partial c \rangle_\mathrm{M} \tag{B.44}$$

holds in terms of this mapping.

---

[2] Special subcases of this general theorem have been considered or proved by Newton, Leibniz, Gauss, Green, Ostrogradsky, Stokes, and Poincaré.

Speaking in more precise terms, (B.44) should rather be considered as the mapping $H^P(M) \times H_p(M, \mathbb{R}) \rightarrow \mathbb{R}$ from the Cartesian product of the cohomology and homology group into the field of real numbers. The reason for this is that the divisors in the quotient spaces (B.24) and (B.41) do not contribute to (B.43), due to (B.44).

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Appendix C
Lie Groups

In most cases of relevance for physics, infinite-dimensional groups are used. Such a group is called a *Lie group* G if

(i) the elements $g \in G$ form an (analytic) manifold,
(ii) the mapping $G \times G \rightarrow G$ originating from the pointwise multiplication $(g_1, g_2) \rightarrow g_1 g_2^{-1}$ of (group) elements is analytic as well.

The most general example is given by $G = GL(N, \mathbb{C})$, i.e., by the general linear group of complex $N \times N$ matrices.

The tangent space $T_e(G)$ attached to the unit element $e \in G$ of the group possesses the structure of a vector space of dimension K = dim G, and as such is isomorphic to the *Lie algebra* of $\mathfrak{g}$ of G (KN I., p. 38). Let $\xi = \{\xi^j, j = 1, \ldots, K\}$ be an appropriate parametrization of G. Then the matrices

$$I_j := \frac{\partial g(\xi)}{\partial \xi^j}|_{g=e} \in T_e(G) \approx \mathfrak{g} \tag{C.1}$$

are the *infinitesimal generators* of G. Viewed oppositely, the transition from the Lie algebra $\mathfrak{g}$ to the group G is mediated by the *exponential mapping*

$$\exp : \mathfrak{g} \rightarrow G \ni g = \exp(\imath \xi^j I_j) \tag{C.2}$$

in terms of matrices. The vector space spanned by the infinitesimal generators is turned into the corresponding Lie algebra by prescribing the commutation relations

$$[I_i, I_j] = c_{ij}{}^k I_k. \tag{C.3}$$

Here the real *structure constants* c_{ij}^k are characteristic of the group G. They are always antisymmetric

$$c_{ij}{}^k = -c_{ji}{}^k \tag{C.4}$$

with respect to the first two indices. Moreover, they satisfy the cyclic Jacobi identity

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$$3c_{[i|s}{}^{p}c_{|jk]}{}^{s} := c_{is}{}^{p}c_{jk}{}^{s} + c_{js}{}^{p}c_{ki}{}^{s} + c_{ks}{}^{p}c_{ij}{}^{s} \equiv 0. \tag{C.5}$$

As a result, the *Lie product*

$$[\mathrm{AB}] := \mathrm{A}^{\mathrm{i}}\mathrm{B}^{\mathrm{j}}\mathrm{c}_{\mathrm{ij}}^{\mathrm{k}}\mathrm{I}_{\mathrm{k}}. \tag{C.6}$$

defined for generic Lie-algebra-valued vectors $\mathrm{A} = \mathrm{A}^{\mathrm{i}}\mathrm{I}_{\mathrm{i}}$ and $\mathrm{B} = \mathrm{B}^{\mathrm{j}}\mathrm{I}_{\mathrm{j}}$ is linear and antisymmetric. Moreover, it satisfies the Jacobi identity, as required.

For applications in physics it is not the group itself, but more often the group's *linear representations* that are considered. The latter are derived from a Lie homomorphism $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{N}, \mathbb{C})$, i.e., from an injective, continuous, and open mapping into the general group of complex matrices. This homomorphism inherits the matrix multiplication

$$\rho_A^C(g_1)\rho_C^B(g_2) = \rho_A^B(g_1 g_2) \tag{C.7}$$

from the multiplication rûle of the group in question. The "derived" homomorphism generates a representation of the Lie algebra via

$$[A, B] \longrightarrow \rho([A, B]) = [\rho(A), \rho(B)]. \tag{C.8}$$

An example of prime importance is the *adjoint representation*

$$\mathrm{Ad}A(B) := [A, B]; \quad A, B \in \mathfrak{g}, \tag{C.9}$$

which reads explicitly, with regard to the basis I_{j},

$$(\mathrm{Ad}\ A)_j{}^k = c_{ij}{}^k A^i. \tag{C.10}$$

With the aid of the trace operation Tr, a scalar product

$$(A, B) := \mathrm{Tr}(\mathrm{Ad}\ A\ \mathrm{Ad}\ B) = c^i_{\iota k} c^k_{si} A^\iota B^s =: \overset{\circ}{g}_{\iota s} A^\iota B^s \tag{C.11}$$

may be introduced that is determined by the *Cartan–Killing metric* $\overset{\circ}{g}_{ls}$ of the Lie algebra. This scalar product is invariant with respect to the group of automorphisms of $\mathfrak{g}$. In the case of the Lie algebra $\mathfrak{gl}(\mathrm{N}, \mathbb{C})$ of the general linear group, this product can be written as follows:

$$(A, B) = 2N\mathrm{Tr}(AB) - 2\mathrm{Tr}(A)\mathrm{Tr}(B) \tag{C.12}$$

BARUT & RĄCZKA 1980, p. 15). For the *special groups* that by definition satisfy $\det \mathrm{G} = 1$, the last term vanishes. This is a consequence of the matrix identity $\det \mathrm{G} = \exp^{\mathrm{Tr}\mathfrak{g}}$. The scalar product (C.12) is nondegenerate if and only if G is *semisimple*, i.e., if G does not contain an abelian invariant subgroup. A closer analysis (HELGASON 1962) of the Cartan–Killing metric

$$\overset{\circ}{g}_{\iota m} := c_{\iota k}{}^{i}\, c_{mi}{}^{k} \tag{C.13}$$

reveals that every *simple* Lie group, regarded as a manifold, corresponds either to a Riemannian manifold or a pseudo-Riemannian manifold. This may indicate why (C.13) can also be regarded as the metric of the group manifold.

The group that is frequently discussed in connection with the quark model (GELL-MANN & NE'EMAN 1964; KOKKEDEE 1969) is the simple group SU(N) or its Lie algebra. In terms of a real parametrization ξ^{j}, an arbitrary group element is generated by

$$g = \exp^{(-\frac{i}{2}\xi^{j}\lambda_{j})} \in\ SU(N),\ j = 1, \ldots, N^{2} - 1. \tag{C.14}$$

Here the λ_j denote the generalized Gell-Mann matrices, which are Hermitian and trace-free. Their commutation and anticommutation relations are specified by

$$[\lambda_i, \lambda_j] = 2i\mathrm{f}_{ij}{}^{k}\lambda_k \quad (c_{ij}{}^{k} = 2if_{ig}{}^{k}), \tag{C.15}$$

$$\{\lambda_i, \lambda_j\}_+ = \frac{4}{N}\delta_{ij} + 2d_{ij}{}^{k}\lambda_k. \tag{C.16}$$

It is a consequence of these relations that the product of two of such matrices can always be written in the following form:

$$\lambda_i\, \lambda_j = \frac{2}{N}\delta_{ij} + (d_{ij}{}^{k} + if_{ij}{}^{k})\lambda_k. \tag{C.17}$$

In these formulas, it is understood that the λ-matrices are normalized according to

$$\mathrm{Tr}(\lambda_i\, \lambda_j) = 2\delta_{ij}. \tag{C.18}$$

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