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The underlying geometry of the CAM gauge model of the Standard Model of particle physics

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The Composition Algebra-based Methodology (CAM) [B. Wolk, *Pap. Phys.* **9**, 090002 (2017); *Phys. Scr.* **94**, 025301 (2019); *Adv. Appl. Clifford Algebras* **27**, 3225 (2017); *J. Appl. Math. Phys.* **6**, 1537 (2018); *Phys. Scr.* **94**, 105301 (2019), *Adv. Appl. Clifford Algebras* **30**, 4 (2020)], which provides a new model for generating the interactions of the Standard Model, is geometrically modeled for the electromagnetic and weak interactions on the parallelizable *sphere operator fiber bundle* $B_n = (T\mathbb{M}, S^n \to \mathscr{S}^n, SO(n+1), \pi)$ consisting of base space, the tangent bundle $T\mathbb{M}$ of space–time $\mathbb{M}$, projection operator π, the parallelizable spheres $S^n = \{S^1, S^3\}$ conceived as operator fibers $S^n \to \mathscr{S}^n$ attaching to and operating on $T_p\mathbb{M}$ $\forall p \in \mathbb{M}$ as p varies over $\mathbb{M}$, and as structure group, the norm-preserving symmetry group $SO(n+1)$ for each of the division algebras which is simultaneously the isometry group of the associated unit sphere. The massless electroweak $SU(2)_L \otimes U(1)_Y$ Lagrangian is shown to arise from $B_{3\otimes 1}$'s generation of a local coupling operation on sections of Dirac spinor and Clifford algebra bundles over $\mathbb{M}$. Importantly, CAM is shown to be a new genre of gauge theory which subsumes Yang–Mills Standard Model gauge theory. Local gauge symmetry is shown to be at its core a geometric phenomenon inherent to CAM gauge theory. Lastly, the higher-dimensional, topological architecture which generates CAM from within a unified eleven $(1, 10)$-dimensional geometro-topological structure is introduced.

Keywords: Parallelizable spheres; division algebras; differential geometry; differential topology; fiber bundles; complexified Clifford algebras; Standard Model; Yang–Mills theory; string theory; compactification.

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> "Physics is an attempt conceptually to grasp reality as it is thought independently of its being observed."
>
> — Albert Einstein[a]

[a] *Autobiographical notes*, reprinted in S. Hawking, *A Stubbornly Persistent Illusion, The Essential Scientific Works of Albert Einstein* (Running Press Book Publishers, Philadelphia, 2007), p. 376.

Table 1. CAM coupled operators.

$U(1)$	$\eta\partial_{\mathbb{C}}$	$\bar{\eta}\eta_{\mathbb{C}}$	—
$SU(2)$	$\eta\partial_{\mathbb{H}}$	$\bar{\eta}\eta_{\mathbb{H}}$	—
$SU(3)$	$\eta\partial_{\mathbb{O}_{\mathcal{Z}}}$	$\bar{\eta}\eta_{\mathbb{O}_{\mathcal{Z}}}$	$\bar{\eta}\eta_{\mathbb{O}}$

1. The CAM Framework

The CAM framework[b] uses the unique operator structure of the equally unique composition algebras to generate the Standard Model (SM) Lagrangian of elementary-particle physics as well as the Lagrangian for general relativity. The theory has proven to be a successful alternative to Yang–Mills (YM) gauge theory for generating the SM pre-Higgs left-chiral electroweak and quantum chromodynamic Lagrangians in intrinsic local gauge symmetric form.[1–5] CAM's current additional successes include intrinsic accommodation of chiral asymmetry for $SU(2)$,[2] natural provision of a noninteractive right-chiral neutrino,[2] imposition of experimentally verified phenomenological constraints on gauge mediated proton decay,[5] and introduction of a Higgs-like field intrinsically coupled to the space–time continuum.[95]

What is interesting about these theoretic successes is that each evidences qualitative and quantitative differences between the CAM and YM models. And each of these theoretic successes points to CAM as the more fundamental framework for generating the elementary-particle forces. For example, the CAM phenomenology of proton decay is confirmed by experiment, whereas YM proton decay predictions have consistently been experimentally invalidated.[5]

2. Primary Motivation

This paper postulates the parallelizable spheres as the core geometric entities which induce the CAM formalism.[c] The motivation for this postulate is immediately evident upon inspection of the CAM operators in Table 1, used for generating the $U(1)$, $SU(2)$, $SU(2) \otimes U(1)$ and $SU(3)$ Lagrangians.[1–5]

[b]The CAM gauge model is an alternative to Yang–Mills theory for generating the Standard Model Lagrangian of elementary-particle physics (Ref. 1–5 and Sec. 1 herein), while also generating the Lagrangian of general relativity theory.[95] Composition algebras are algebras $\mathbb{A}$ such that for any two elements the algebraic norm of their product equals the product of their norms:[13,14] $\|xy\| = \|x\|\|y\|\ \forall x, y \in \mathbb{A}$. These composition algebras exist only in $1, 2, 4$ and 8 dimensions, corresponding to $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (Refs. 11–14) and their split versions $\mathbb{K}' = \{\mathbb{C}', \mathbb{H}', \mathbb{O}'\}$.[12,13] Only the $\mathbb{K}$ algebras are division algebras (composition algebras without zero divisors).[12,13] However, since the nondivision, composition algebra $\mathbb{O}'$ was used along with the $\mathbb{K}$ algebras in generating both QCD's $SU(3)_c$ and GR's Lagrangians, the term CAM is the appropriate designation for the formalism in general.

[c]The unique, closed set of parallelizable unit spheres $S^P = \{S^0, S^1, S^3, S^7\}$.[6–11,14–17] $S^0 = \{-1, 1\}$ is trivially isomorphic to the unit reals.[14] S^7 will be considered in subsequent work, and will be shown to generate the $SU(3)_c$ Lagrangian for the strong force — the fiber bundle geometry of which will prove to be more involved than that of $SU(2) \otimes U(1)$'s herein.

But though the CAM formalism has been revealed using the algebraic structure of the division algebras, the essential beauty of the division algebras resides elsewhere — namely, within the fundamental *geometric* structure[11–14]

$$\mathbb{R}_u \cong S^0\,, \quad \mathbb{C}_u \cong S^1\,, \quad \mathbb{H}_u \cong S^3\,, \quad \mathbb{O}_u \cong S^7\,. \tag{1}$$

That is to say, the unit division algebras $\mathbb{K}_u$, which carry with them the division algebraic operator structure, are isomorphic to the unique, closed set of smooth parallelizable unit spheres S^P. This intimate relation between CAM's algebraic operator structure and what is considered its more fundamental manifold structure is the pivotal topic of this paper. For it is held to be self-evident that, with regard to the fundamental laws of physics, to be theoretically and aesthetically viable detailed algebraic form must of necessity be intrinsically contained within and naturally emanate from unified geometric structure.

In keeping with this modern theoretical approach of striving for a "unified geometric description of the fundamental physical interactions,"[11,16,17,21,22,25,36,40,d] we are thus logically compelled to a contemplation of the parallelizable spheres as the natural underlying structure for a CAM geometric program.

3. Synopsis

But for the existence of the parallelizable spheres, the elementary-particle forces would not exist, if at all, in their known form. The CAM formalism is generated by these spheres via their coaction with space–time. Demonstrating and elaborating on these two assertions is the goal of this and the subsequent papers which will consider CAM's underlying geometric structure and which proffers the CAM model's structure as emanating from a $(1,10)$-dimensional space–time.

Just as YM is a gauge theory, so CAM will also be seen herein to be a gauge theory — of a sort more fundamental than YM gauge theory. Just as YM began as a formalism of theoretical physics which only thereafter realized a geometric, fiber bundle underpinning,[18,19] so CAM will herein also be given a source fiber bundle geometry. Just as YM is modeled on a certain fiber bundle structure,[11,16,17,20,21] so CAM is also modeled on a certain fiber bundle structure which differs in a nontrivial way from that of the YM structure. Furthermore, just as string/M-theory takes YM as contained within a compactified higher-dimensional space–time,[20,23,60,61] so the

[d]Reference 40, Preface. As the author further writes, "The most remarkable characteristic of this theoretical approach is the firm and fundamental conviction that all hypotheses concerning the real physical world can be given — before quantization — in purely geometric terms" (p. 2). The program set forth herein for the CAM formalism follows this approach, as an underlying geometric structure is shown to first generate the CAM formalism which in turn generates a relativistic, local gauge symmetric Lagrangian, which pursuant to the fundamental postulates underlying the canonical quantization (or the path-integral approach) remains the Lagrangian for constructing a quantum field theory.[37] CAM's possession of these two attributes of (1) having a geometric base, and (2) generating pre-quantization Lagrangians, distinguishes it from all other solely algebraic formalisms (see Fn. (ii), herein), which have neither.

CAM fiber bundle structure will be seen to arise from the compactification of a $(1, 10)$-dimensional space–time.

Of the various differences between the CAM and YM fiber bundle structures, one particular distinction rates immediate articulation. At its most fundamental level theoretical physics must not simply describe the universe but must also explain why it exists as it does.[22] As Weinberg emphasized[e]

> The aim of physics at its most fundamental level is not just to describe the world but to explain why it is the way it is.

This notion forms a qualitatively dispositive distinction as between the two theoretic structures — both YM and CAM describe the SM interactions, but whereas YM fiber bundle theory is but a descriptive gauge theory[23] CAM's fiber bundle theory is explanatory as well.

This paper develops the basic theoretical and conceptual structure of CAM's fiber bundle geometry for $SU(2) \otimes U(1)$. This fiber bundle structure is based on the existence of two of the three parallelizable spheres $S^n = \{S^1, S^3\}$ of the unique, closed set of three nontrivial parallelizable spheres $S^X = \{S^1, S^3, S^7\}$.[11,14–17]

The $S^n = \{S^1, S^3\}$ will be contained within the sphere operator fiber bundle

$$B_n = \big(T\mathbb{M}, S^n \to \mathscr{S}^n, SO(n+1), \pi\big) \tag{2}$$

with projection operator π and fiber operators $\mathscr{S}^n \vee T_p\mathbb{M}$ operating point-wise over $\mathbb{M}$ via the wedge sum action $\vee$ which projects the division algebraic coupling operator $\circ$ in the tangent space $T_p\mathbb{M}$ over the tangent bundle $T\mathbb{M}$. CAM's unique coupling operator $\circ$[1,2,5] is inherently encoded within each fiber operator $S^n \vee T_p\mathbb{M} \to \mathscr{S}^n \vee T_p\mathbb{M} = \{\mathscr{S}^1 \vee T_p\mathbb{M}, \mathscr{S}^3 \vee T_p\mathbb{M}\}$ and is operationally attached to $T_p\mathbb{M}\ \forall p \in \mathbb{M}$ via S^n's unique attribute (among the spheres) of parallelizability. The structure group $SO(n+1)$ acts simultaneously as the isometry (automorphism) group of the subject unit sphere as well as the norm-preserving symmetry group of its associated division algebra.[11,13,16,17] The B_n structure is proffered as the fiber bundle framework which induces creation of the pre-Higgs $SU(2)_L \otimes U(1)_Y$ Lagrangian.

Within this fiber bundle arena, the gauge fields and currents are sections of the complexified Clifford algebra bundle $\mathbb{C}l_{1,3}\mathbb{M}$ while matter-wave fields are sections of the Dirac spinor bundle $S\mathbb{M}$. These sections are locally coupled by the CAM operator ϕ_X which is generated via $\mathscr{S}^n$'s local $\circ$-coupling action on the CAM operator set $\Omega \equiv \{\eta, \partial\}$ intrinsically embedded in $T_p\mathbb{M}$. The $SU(2)_L$, $U(1)_{\mathrm{EM}}$ and $SU(2)_L \otimes U(1)_Y$ Lagrangians thereby manifest.

The CAM fiber bundle structure will also be shown to offer a straightforward geometric explanation for the existence of local gauge symmetry within the fundamental laws of particle physics. Lastly, the CAM architecture will be shown to arise from a $(1, 10)$-dimensional space–time.

[e]Reference 22, p. 219.

4. Geometro-Algebraic Structures

Numerous well-established geometric and algebraic structures over Minkowski space–time $\mathbb{M}$ play a fundamental part in the CAM fiber bundle geometry. There is a tremendous topical depth to all of these structures and objects, but the fairly minimal depth as given herein is what is needed in order to set forth the essential structure of CAM fiber bundle theory.

4.1. *Minkowski space–time*

There exists the single, absolute, unique Lorentzian space–time manifold $\mathbb{M}$ with metric $\boldsymbol{\eta}$ having standard basis components $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.[21,24,25] The remainder of CAM's geometry revolves around structures inherent to $(\mathbb{M}, \boldsymbol{\eta})$ and their associations with S^n.

Relativistic physics demands locality in physical theories,[24–26] and thus physics initiates in or about the tangent spaces of $\mathbb{M}$: $T_p\mathbb{M}$,[24,27] which designates the collection of all vectors tangent to $\mathbb{M}$ at $p \in \mathbb{M}$.[11,16,17] The union of all $T_p\mathbb{M}$ over $\mathbb{M}$ is the tangent bundle $T\mathbb{M}$:[46]

$$T\mathbb{M} = \bigcup_{p\in\mathbb{M}} T_p\mathbb{M}\,, \tag{3}$$

of which the fiber at p is $T_p\mathbb{M}$. The bundle $T\mathbb{M}$ is a smooth manifold in its own right.[11,16,17,64]

$(\mathbb{M}, \boldsymbol{\eta})$ induces three structures in $T_p\mathbb{M}$ which play a pivotal part in CAM's $SU(2) \otimes U(1)$ fiber bundle makeup, to wit

(1) the Levi-Civita connection $\nabla^{\boldsymbol{\eta}}$,[21,24,31]
(2) the Clifford algebra $\mathrm{Cl}_{1,3}$,[27–30] and
(3) CAM's operator set $\Omega \equiv \{\eta, \partial\}$.[1,2]

CAM fiber bundle theory exploits this tripartite internal space–time structure for its theoretical construction.

4.1.1. *The connection* $\nabla^{\boldsymbol{\eta}}$

A metric $\boldsymbol{g}$ defined on a Riemannian manifold M induces a unique metric-preserving and torsion-free connection — the Levi-Civita connection ∇ which in the frame field $\{e_\mu\}$ acts on T_pM as the covariant operator $\nabla \mapsto \nabla_\mu$.[16,21,31] The covariant operator's action on a vector field $X = X^\mu e_\mu$ is given by $\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\xi_{\mu\nu} X_\xi$,[24,32] where $\Gamma^\xi_{\mu\nu}$ are the connection coefficients defined with respect to $\{e_\mu\}$ as[31,32]

$$\nabla_\mu e_\nu = \Gamma^\xi_{\mu\nu} e_\xi\,. \tag{4}$$

As applied to CAM fiber bundle theory, we have $(\mathbb{M}, \boldsymbol{\eta})$ inducing the flat Levi-Civita connection $\nabla^{\boldsymbol{\eta}}$: $(\mathbb{M}, \boldsymbol{\eta}) \mapsto (\mathbb{M}, \nabla^{\boldsymbol{\eta}})$.[11,21,24,33] In the coordinate basis $\{\partial_\mu\}$ of $\mathbb{M}$ associated to the local coordinates $\{x^\mu\}$ the metric $\boldsymbol{\eta} \to \eta_{\mu\nu}$ induces the covariant operator $\nabla^{\boldsymbol{\eta}} \mapsto \nabla_{\partial_\mu}$ on $T_p\mathbb{M}$ for which the $\Gamma^\xi_{\mu\nu}$ are the Christoffel symbols (of the second kind) defined by $\nabla_{\partial_\mu}\partial_\nu = \Gamma^\xi_{\mu\nu}\partial_\xi$.[11,21,31,32]

In relativity theory, the Christoffel symbols $\Gamma^\xi_{\mu\nu}$ are the components of the gravitational field,[34] which vanish everywhere on $\mathbb{M}$.[24,35] Thus, using $\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\xi_{\mu\nu}X_\xi$, we see that ∇_{∂_μ} forms the partial differential operators ∂_μ on $T_p\mathbb{M}$. Because the ∂_μ globally satisfy the holonomic commutation relations $[\partial_\mu, \partial_\nu] = 0$,[31] they therefore simultaneously form a covariantly constant coordinate frame field over $\mathbb{M}$: $\{e_\mu\} \to \{\partial_\mu\}$,[11,21,31] for which we have the flat connection equation[31]

$$\nabla^{\boldsymbol{\eta}}\partial_\mu = 0\,, \tag{5}$$

and for which both the curvature tensor $R(e_\mu, e_\nu)$ and torsion tensor $T(e_\mu, e_\nu)$ vanish on $\mathbb{M}$.[16,31]

4.1.2. *The Clifford algebra* $\mathrm{Cl}_{1,3}$

$(\mathbb{M}, \eta_{\mu\nu})$ induces the Clifford algebra $\mathrm{Cl}_{1,3}$ via generation of the basis elements $\{\gamma_\mu\}$ through the anticommutation relation:[2,14,17,28]

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,. \tag{6}$$

Also known as the space–time algebra, $\mathrm{Cl}_{1,3}$ is induced by $\mathbb{M}$ in $T_p\mathbb{M}$ at all points $p \in \mathbb{M}$[28,32,43] and is the natural, minimal algebraic construct belonging to the geometry of $\mathbb{M}$.[28] $\mathrm{Cl}_{1,3}$ consists of 16 linearly independent elements generated by the $\{\gamma_\mu\}$:[17,30]

$$\Gamma_z = [\{1\}, \{\gamma_\mu\}, \{\gamma_0\gamma_k\}, \{\gamma_j\gamma_k : j \neq k\}, \{i \equiv \gamma_0\gamma_1\gamma_2\gamma_3\}, \{i\gamma_\mu\}]\,, \tag{7}$$

where z runs from 1 to 16 with each Γ_z representing one of the elements. $\mathrm{Cl}_{1,3}$ is spanned by Γ_z and constitutes a 16-dimensional vector space.[17,32,36,37]

The Γ_z are abstract entities unto themselves, and their common depiction as matrices are but representations of these elements.[12,30,38] For instance, complexifying $\mathrm{Cl}_{1,3} \xrightarrow{\otimes\mathbb{C}} \mathbb{C}l_{1,3}$ with $\mathbb{C}$-unit i, we can then write the 4×4 chiral representation of the Clifford generators $\{\gamma_\mu\}$ and the chirality operator $\boldsymbol{\gamma}_5$ using the Pauli matrices σ_k satisfying $\sigma_k = \frac{1}{\mathrm{i}}\gamma_k\gamma_0$ and the identity element σ_0:[2,32,39,40]

$$\gamma_\mu = \begin{pmatrix} & \sigma_\mu \\ \bar{\sigma}_\mu & \end{pmatrix}, \quad \boldsymbol{\gamma}_5 = \mathrm{i}i = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}. \tag{8}$$

A generic field $\mathcal{F}(x_\mu)$ of $\mathbb{C}l_{1,3}$ can be written as[36]

$$\mathcal{F}(x_\mu) = \sum_{z=1}^{16} \alpha_z \Gamma_z\,, \tag{9}$$

in which we take each α_z to represent complexified scalar, vector, tensor or spinor field components. For instance, $1\Psi + \gamma_\mu B^\mu + \lambda\gamma_j\gamma_k$ and $\gamma_k A^k_{\mu\nu} + \mathrm{i}\gamma_0\gamma_k B^k_\mu + \boldsymbol{\gamma}_5 C_\mu$ are examples of $\mathbb{C}l_{1,3}$ fields.

4.1.3. *The operator set* $\Omega \equiv \{\eta, \partial\}$

Using ∇_{∂_μ} and $\mathrm{Cl}_{1,3}$, the CAM operators[1,2]

$$\partial = \partial/\partial t + \boldsymbol{i}\partial/\partial x + \boldsymbol{j}\partial/\partial y + \boldsymbol{k}\partial/\partial z \tag{10}$$

and

$$\eta = \gamma_0 + \boldsymbol{i}\gamma_1 + \boldsymbol{j}\gamma_2 + \boldsymbol{k}\gamma_3 \tag{11}$$

can be constructed, where the $\{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are the basis elements of the quaternions $\mathbb{H}$ with $\mathbf{1}$ the unit identity element (being notationally suppressed) and $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ Hamilton's imaginary elements identified with the even subalgebra Cl_3^+ of $\mathrm{Cl}_3 \subset \mathrm{Cl}_{1,3}$.[2,28,30] The quaternionic basis elements are a subset of Γ_z, induced by the $\mathrm{Cl}_{1,3}$ generators $\{\gamma_\mu\}$ via the relations:[27,28,30]

$$\mathbf{1} = \gamma_0\gamma_0\,, \quad \boldsymbol{i} = \gamma_2\gamma_3\,, \quad \boldsymbol{j} = \gamma_3\gamma_1\,, \quad \boldsymbol{k} = \gamma_1\gamma_2\,. \tag{12}$$

The operator ∂ is nothing other than the generalization of the Cauchy–Riemann operator $\partial_x + \mathrm{i}\partial_y$ of complex analysis to $(1, 3)$ dimensions which is used throughout Clifford algebraic analysis[17,28,f] and which is isomorphic to the special-relativistic operator ∂_μ.[2,27,28]

As to the operator η, it can be utilized to generate the Lie algebra of the Lorentz group,[2] and is a simple and aesthetically symmetric composition of the four unit Clifford–Dirac elements $\{\gamma_\mu\}$:[1,2,14,17,27,28]

$$\begin{aligned}\eta &\equiv \gamma_0\gamma_0\gamma_0 + \gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_3 \\ &= \gamma_0 + \boldsymbol{i}\gamma_1 + \boldsymbol{j}\gamma_2 + \boldsymbol{k}\gamma_3\,,\end{aligned} \tag{13}$$

where we have used the relations $\gamma_i\gamma_j = -\gamma_j\gamma_i$, and (12).[27,28]

Since both ∇_{∂_μ} and $\mathrm{Cl}_{1,3}$ are manifestly embedded within $T_p\mathbb{M}$, so is Ω. The relation between $\{\Omega, \bar{\Omega}\}$ and its generation of the algebra of the Poincaré group was previously shown.[2] The Poincaré group represents the inherent symmetry of $\mathbb{M}$ — its true group of automorphisms,[41] and it is thus not unreasonable to suppose that Ω will play a vital role in construction of the CAM fiber bundle geometry.

4.2. *The algebra bundle* $\boldsymbol{A}\mathbb{M}$

Noted above was that $\mathbb{M}$ intrinsically generates and contains the space–time algebra $\mathrm{Cl}_{1,3}$ which is induced by $\mathbb{M}$ in $T_p\mathbb{M}$ at all points $p \in \mathbb{M}$. Nature requires complexification of $\mathrm{Cl}_{1,3} \xrightarrow{\otimes\mathbb{C}} \mathbb{C}l_{1,3}$ within $T\mathbb{M}$, for without the $\mathbb{C}$-unit i, the chirality operator $\boldsymbol{\gamma}_5$ would not be constructable and there would be no chiral matter in nature.[14] Furthermore, $\mathbb{C}$-unit complexification was required in order to establish Ω's part in generating the Poincaré algebra of transformations of $\mathbb{M}$.[2] Importantly, this mandated i-plexification generates the Pauli algebra $\mathbb{P}$ in $T\mathbb{M}$ since

[f] See Ref. 28, Chap. 3 and Subsec. 3.3.

$\mathbb{H} \cong \text{Cl}_3^+ \subset \text{Cl}_3 \subset \text{Cl}_{1,3}$,[2,28,30] and $\mathbb{P} \cong \mathbb{C} \otimes \mathbb{H}$,[14] which in turn trivially implies the existence of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H}$ within $T\mathbb{M}$.[14]

The complexified space–time algebra $\mathbb{C}l_{1,3}$ is taken to form the fibers of the space–time algebra bundle $\mathbb{C}l_{1,3}\mathbb{M} \equiv A\mathbb{M}$,[11,16,32] with $\mathbb{C}l_{1,3}(T_p\mathbb{M})$, the fibers $\forall p \in \mathbb{M}$:[29,32,42]

$$A\mathbb{M} = \bigcup_{p \in \mathbb{M}} \mathbb{C}l_{1,3}(T_p\mathbb{M})\,. \tag{14}$$

Thus, $A\mathbb{M}$ is a bundle of complexified Clifford algebras over $\mathbb{M}$. It is the natural complexified algebra bundle associated with $T\mathbb{M}$ and $\mathbb{M}$.[28,30,36] The connection $\nabla^{\boldsymbol{\eta}}$ induces a connection on $A\mathbb{M}$,[29,33,42] and all Clifford algebra operations carry over into the $A\mathbb{M}$ bundle.[32,42]

4.3. *The Lorentz frame bundle*

Associated with $T\mathbb{M}$ is the Lorentz frame bundle[16]

$$L\mathbb{M} = \bigcup_{p \in \mathbb{M}} L_p\mathbb{M}\,, \tag{15}$$

of which the fiber $L_p\mathbb{M}$ represents the collection of all tangent Lorentzian frames of reference at p. Since all Lorentz frames can be reached by operation of the Lorentz group $O(1,3)$ on some fixed Lorentz frame, the fiber becomes $O(1,3)$ itself[11] and thus, $L\mathbb{M}$ is a principal fiber bundle.

4.4. *The spinor bundle* $S\mathbb{M}$

We have[11,16,17,32]

$$w_k(T\mathbb{M}) = 0 \tag{16}$$

for the Stiefel–Whitney classes $k = 1, 2$. Therefore $\mathbb{M}$ is a spin manifold,[40,42] and but for such result the Dirac equation could not be considered.[17] The essential relation between spinors and the space–time algebra $\mathbb{C}l_{1,3}$ reveals the fundamental geometro-algebraic nature of CAM's makeup,[17,44,45,g] and we therefore briefly review this relationship before defining the spinor bundle.

Since $\mathbb{C}l_{1,3}$ is an associative algebra it has a matrix representation.[30,36] An $n \times n$ matrix representation C_n of an algebra can be conceived as a set of linear operators on an n-dimensional vector space V whose vectors v are represented column-wise[36,39,44,45]

$$\nu = (v_1, v_2, \ldots, v_n)^T\,. \tag{17}$$

The vectors ν of the vector representation space V acted upon by the matrix representation C_n of a Clifford algebra are spinors.[32,36,44] In particular, quantum

[g] As Zee notes, "spinor representations exist because Clifford algebras exist," (Ref. 38, p. 561).

mechanical matter-wave fields ψ such as the electron are Hilbert space $\mathcal{H} = \mathbb{C}^4$ vectors (Dirac spinors) ψ on $\mathbb{M}$[17,36,45]

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \tag{18}$$

of the four-dimensional spinor representation space acted upon by a matrix representation of $\mathbb{C}l_{1,3}$,[11,32,36] with basis elements $\{\gamma_\mu\}$ given a matrix representation such as in (8)'s chiral representation.

The Dirac spinor bundle $S\mathbb{M}$ along with subsets the chiral Weyl bundles $(W\mathbb{M}, \overline{W\mathbb{M}})$ are given by[16]

$$\begin{aligned} S\mathbb{M} &= (\mathbb{M}, \mathbb{C}^4, SL(2,\mathbb{C}) \oplus \overline{SL(2,\mathbb{C})}, \pi)\,, \\ W\mathbb{M} &= (\mathbb{M}, \mathbb{C}^2, SL(2,\mathbb{C}), \pi)\,, \\ \overline{W\mathbb{M}} &= \left(\mathbb{M}, \mathbb{C}^2, \overline{SL(2,\mathbb{C})}, \pi\right). \end{aligned} \tag{19}$$

To construct these spinorial fiber bundles a structure group over $\mathbb{M}$ with spinor representation which lifts to a spin group is requisite.[11,16] In particular, we consider the structure group of the Lorentz frame bundle $L\mathbb{M}$ with a definite space and time orientation — the proper, orthochronous group with Lorentz transformation Λ:[16,40]

$$SO_0(1,3) \equiv \{\Lambda \in O(1,3) |\det \Lambda = +1, \Lambda^0_0 > 0\}\,. \tag{20}$$

Because of (16), $SO_0(1,3)$ lifts without obstruction to the spin group $\mathrm{Spin}_{1,3} \cong SL(2,\mathbb{C})$ over $\mathbb{M}$,[12,16,46] and thus $\mathbb{M}$ admits a spin structure.[40] The connection on $S\mathbb{M}$ is induced by $\nabla^{\boldsymbol{\eta}}$.[11,16,24,32] The space–time bundle $A\mathbb{M}$ is trivial on $\mathbb{M}$,[32] and therefore the standard Dirac operator $\partial\!\!\!/$ of relativistic quantum mechanics is generated thereon.[32,42,43]

$S\mathbb{M}$ corresponds to the $(1/2, 0) \oplus (0, 1/2)$ representation of $SL(2,\mathbb{C})$,[15,39] while $W\mathbb{M}$ and $\overline{W\mathbb{M}}$, respectively correspond to the $(1/2, 0)$ and $(0, 1/2)$ representations. Thus, we see that $\mathcal{H}$-space matter-wave fields ψ are sections of $S\mathbb{M}$.[11]

4.5. *The parallelizable spheres $S^n = \{S^1, S^3\}$*

The largest number q of independent vector fields that can exist on some manifold M is $n = \dim(M)$.[11] When $q = n$ the manifold is parallelizable.[15,17] One can continuously assign a globally smooth covariantly constant frame field $\{e_\mu\}$ on TM at all points of a parallelizable manifold M with connection ∇,[11,31,32] for which the covariant derivative in any direction vanishes: $\nabla_V e_\mu = 0$ $\forall V \in T_pM$ $\forall p \in (M, \nabla)$.[11,31,h] This yields the teleparallel connection equation over (M, ∇):[31]

$$\nabla e_\mu = 0\,. \tag{21}$$

[h]In this paper, differentio-geometric structures such as a connection and a metric are used in discussing the critical trait of parallelizability of S^n. However, it is important to emphasize here that the parallelizability of $S^X : X = 1, 3, 7$ is more fundamentally a differentio-topological topic and structure,[8,10] and further falls under algebraic K-theory.[6,7,10,11] Specifically, no notion of a metric structure on S^X need be defined in order for S^X to be parallelizable or proven such[8,9,65] (see, e.g. Ref. 65, Subsec. 21.3).

If a manifold M is not parallelizable, then one cannot continuously assign a globally smooth covariantly constant frame field $\{e_\mu\}$ on TM for all points of M such that (21) is satisfied, for example for the 2-sphere S^2.[11,31] If in addition $[e_\mu, e_\nu] = 0$, then $\{e_\mu\}$ is a coordinate frame field $\{e_\mu\} \to \{\partial_\mu\}$ for which (5) is also satisfied, for example for $\mathbb{M}$.[11,31,32]

The spheres $S^n = \{S^1, S^3\}$ are respectively defined as follows:[14,16,21,32]

$$S^1 = \left\{ (x_1, x_2); x_j \in \mathbb{R} \middle| \sum_{j=1}^{2} (x_j)^2 = 1 \right\},$$
$$S^3 = \left\{ (x_1, x_2, x_3, x_4); x_j \in \mathbb{R} \middle| \sum_{j=1}^{4} (x_j)^2 = 1 \right\}, \tag{22}$$

where S^1 and S^3 are respectively isomorphic to the unit complex (quaternion) elements $\mathbb{C}_u$ and $\mathbb{H}_u$,[11,14] which in turn form the simply-connected symmetry groups $U(1)$ and $SU(2)$.[16,21,38] For example, $U(1)$ is the circle generated by the unit complex elements:[21,27,38]

$$S^1 = \left\{ z = x_1 + \mathrm{i}x_2 \in \mathbb{C} \middle| \|z\|^2 = \sum_{j=1}^{2} (x_j)^2 = 1 \right\}. \tag{23}$$

The norm symmetry groups of the unit division algebras $\mathbb{C}_u$ and $\mathbb{H}_u$ are the special orthogonal groups $SO(2)$ and $SO(4)$, for which the norms of S^1 and S^3 are preserved.[13,i]

As for $\mathbb{M}$ so the S^n are also parallelizable,[8,11,14,16] and thus, the teleparallel connection equation $\nabla e_\mu = 0$ is satisfied over S^n as well. However, the torsion tensor $T(e_\mu, e_\nu)$ need not vanish on a manifold M though the curvature tensor $R(e_\mu, e_\nu)$ does,[16,31] and in such a case the flat connection equation as in (5) is not satisfied. Such is the case for instance on S^3 and S^7.[11,16] Thus for CAM's fiber bundle makeup the more general teleparallel connection equation is considered the pivotal operative equation permitting S^n to attach to and operate on $\mathbb{M}$.

5. $\mathscr{S}$ Structure Theory

5.1. *The invariant algebraic operation* $\circ$

Since $S^1 \cong U(1)$ and $S^3 \cong SU(2)$,[11,14,16,17,37] group theory requires that $S^1(S^3)$ possess at least one elemental binary operation $*_1(*_3)$, respectively on them.[38,39] Furthermore, group closure requires that the binary combination $\alpha * \beta$ of any two elements $\{\alpha, \beta\} \in S^1(S^3)$ generates another element $\delta \in S^1(S^3)$.[38,39,47] Thus the operations $*_1(*_3)$ are to be invariant closed binary operations on $S^1(S^3)$.

[i] Reference 13, Subsec. 12.1.

The closed, binary operation $\circ$ which is unique to the division algebras takes the form[1–5,28,j]

$$\alpha \circ \beta = \left(\alpha_0\beta_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \alpha_0\boldsymbol{\beta} + \boldsymbol{\alpha}\beta_0 + \frac{1}{2}\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]\right), \tag{24}$$

when acting within a space with local Euclidean metric for the spatial components permitting the standard $\mathbb{R}^3(\mathbb{R}^7)$ dot and cross products, where $\alpha = (\alpha_0, \boldsymbol{\alpha}); \beta = (\beta_0, \boldsymbol{\beta})$ are division algebraic operator fields[16,36] separated into their real and imaginary components, and the right-hand side $(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})$ and $(\boldsymbol{\alpha}\times\boldsymbol{\beta} = \frac{1}{2}[\boldsymbol{\alpha},\boldsymbol{\beta}])$ operations are the standard metric-equipped $\mathbb{R}^3(\mathbb{R}^7)$ dot and cross products. Since a metric g on a manifold M is the structure that smoothly assigns to each point $p \in M$ a metric g_p on T_pM[21] by which dot and cross products manifest,[21,39,48] the form of (24) requires operation within a manifold equipped with a metric. In point of fact, $\alpha_0\beta_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ implies the Minkowski metric: $\alpha_\mu\beta^\mu = \eta_{\mu\nu}\alpha^\nu\beta^\mu \equiv \alpha_0\beta_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\beta}$.

We have seen that S^1 and S^3 are isomorphic to the unit complex (quaternion) elements, respectively.[11,16,38] It follows that $\circ = *$ for both S^1 and S^3. Since a S^n is parallelizable its tangent spaces T_mS^n $\forall m \in S^n$ are isomorphic, with the isomorphism independent of the curve γ joining any two points of S^n that a vector is parallel-transported over.[11,16,17] Thus the division algebraic structure of S^n remains invariant as one parallel-transports over S^n. Since the binary $\circ$-coupling is part of this division algebraic structure of S^n it follows that the $\circ$-operation is an invariant, smooth algebraic operation on S^n.[15,47] Thus the $\circ$-operator is an invariant, smooth and closed elemental binary operation on both S^1 and S^3.

5.2. *The $S^n \to \mathscr{S}^n$ structural attachment*

As noted above, from a differentio-geometric viewpoint we have the teleparallel connection equation $\nabla e_\mu = 0$ necessarily globally vanishing on $S^n, \mathbb{M}$ (and therefore also in $T\mathbb{M}$);[31] all three manifolds $\{S^n, \mathbb{M}\}$ are parallelizable. For a covariantly constant frame field $\{e_\mu\}$ established over $\mathbb{M}$ in $T\mathbb{M}$ we can thus always associate a covariantly constant frame field $\{e_\nu\}$ over S^n in TS^n which smoothly varies with $\{e_\mu\}$ as p moves over $\mathbb{M}$ $\forall p \in \mathbb{M}$.[11,16,17,31,k]

The tangent space T_qS^n for $q \in S^n$ with S^n the fiber of a bundle over $\mathbb{M}$ will thus be said to attach to $T_p\mathbb{M}$ at all points $p \in \mathbb{M}$ since an associated, common set of basis frame fields can be established between the tangent spaces of S^n and $\mathbb{M}$ at all space–time points. In this way we view $S^n \to \mathscr{S}^n$ as structurally attached to

[j] See in particular, Refs. 2 and 5, and references therein for a more detailed discussion of this operation.
[k] See Ref. 16, pp. 258–260 (Subsec. 7.3.2).

$\mathbb{M}$ at all space–time points $p \in \mathbb{M}$ via mutual association of the global frame fields within the respective tangent bundles.[l]

Importantly, since $\nabla e_n = 0$ is true only for S^n (and S^7), no other unit spheres besides these can possess this smooth structural frame attachment to $T\mathbb{M}$ over $\mathbb{M}$. Thus, for example, since there does not exist a single, global nowhere-vanishing parallel frame field $\{e_\alpha\}$ on the 2-sphere S^2,[11,31] it follows that the 2-sphere cannot smoothly attach to $T\mathbb{M}$ throughout $\mathbb{M}$ as the S^n do.

5.3. *The* $\circ$*-operational attachment and wedge action* $\vee$

Accompanying this structural frame attachment we can also conceive of an operational fiber attachment between $\mathscr{S}_p^n$ and $\mathbb{M}$ at p via a morphism φ which preserves the $\circ$-operator division algebraic structure of S^n in double projection $\varphi \equiv \kappa \cdot \pi : S^n \mapsto \mathbb{M}$, where κ is the tangent bundle projection of $T_p\mathbb{M}$ into $\mathbb{M}$.

For any $s,\, r \in S^n$ with $s \circ r = t \in S^n$ we have $\{r, s, t\} \in \mathbb{C}_u$ or $\mathbb{H}_u$. Recalling that $\mathrm{Cl}_{1,3}$ is intrinsic to and that $\mathbb{C}l_{1,3}$ is a physically necessary structure of $\mathbb{M}$ both which are induced in $T_p\mathbb{M}$, and noting $\mathbb{C} \subseteq \mathbb{C}l_{1,3}$ and $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \cong Cl^+ \subseteq \mathrm{Cl}_{1,3}$ we have $\{r, s, t\} \in T_p\mathbb{M}$. We can thus define a map $f : S^n \to T_p\mathbb{M}$ at $p \in \mathbb{M}$ $\forall x \in \mathbb{C}_u$ or $\mathbb{H}_u$ given by $f : x \to x \in T_p\mathbb{M}$ which is invariant with respect to the composition rule of elements: $f(x \circ y) = f(x)\big(\bullet_{T_p\mathbb{M}}\big)_p f(y)$, in which $\big(\bullet_{T_p\mathbb{M}}\big)_p$ is the applicable binary operation on $T_p\mathbb{M}$ at p. We have $f(s \circ r)_p = f(t)_p = t = (s \circ r)_p = \big(f(s) \bullet_{T_p\mathbb{M}} f(r)\big)_p = (s \bullet_{T_p\mathbb{M}} r)_p$, from which we see that $(\bullet_{T_p\mathbb{M}})_p \cong \circ$.

We can thus conceive of the projection $\kappa : \big(\bullet_{T_p\mathbb{M}}\big) \mapsto \circ(\mathbb{M})_p$ within the morphism φ into $\mathbb{M}$ as inducing an S^n bundle action on $\mathbb{M}$, where we consider the induced action as occurring locally at the space–time event p via the attachment of the mutually related covariantly constant frame fields of TS^n and $T\mathbb{M}$.[m]

To mathematically model the structural attachment of the $\mathscr{S}^n$ to all points of $\mathbb{M}$ together with projection of the division algebraic operation into $T\mathbb{M}$ the wedge sum operation $\vee$ between topological spaces is utilized.[97] The wedge operation $\vee$ identifies distinct points on two or more manifolds to a single point. Thus for

[l]As Robinson has stated, "The idea of various geometries attached to space–time and the physically observable effects they have on what we can see is in many ways the dominant theme of particle physics." (Ref. 39, p. 161). Further, since "the tangent space $T_p\mathbb{M}$ is attached to a point $p \in \mathbb{M}$" (Ref. 28, p. 16), "we may visualize velocity (a tangent vector $v \in T_p\mathbb{M}$) as an arrow *touching* the surface at a given point ($p \in \mathbb{M}$) . . . which is strictly *confined* to a point" (Ref. 31, pp. 21–22).

Mathematically speaking a space (e.g. the tangent space) is not conceived of as attached to a manifold in "a direct geometrical way" (see Ref. 48, p. 43), the attachment sometimes being considered but a "mere pictorial resource" (Ref. 65, Subsec. 6.1.1). In this paper, a fundamentally different tack is taken. From a physical and operational standpoint in generating the fundamental particle interactions in space–time the CAM model conceives (and postulates) the parallelizable spheres (comprehended as *operator structures*) as so attached via $T_p\mathbb{M}$.

[m]Since the underlying division algebraic composition of elements does not require a metric[12,14] and since the composition is smooth it is sufficient that S^n be a differentio-topological entity with no equipped metric (vice a metric-equipped differentio-geometric entity). In this light the f-mapping of $\circ$ into the metric-equipped $\big(T_p\mathbb{M}, \eta_{\mu\nu}, \bullet_{T_p\mathbb{M}}\big)$ is seen to induce $\mathscr{S}^n$'s $\circ$-operation to manifest in the metricized form of (24).

example $S^1 \vee S^1$ is equivalent to a figure-8 — being two circles touching at a point.[97] Within the current theory, it is irrelevant as to which point of S^n is used, since by parllelizabilty any and every point induces the division algebraic operation. We thus take the wedge action $\mathbb{M} \vee S^n$ as occurring over all points of $\mathbb{M}$ and as attaching the parallelized operator-spheres $\mathscr{S}^n$ to each point therein, where we emphasize that as always we are dealing with maps and operations occurring between and in the tangent spaces of these manifolds.[64]

5.4. *Bundle candidates*

We wish to condense these various notions into a fiber bundle structure which will represent the generating geometric structure of CAM's geometro-algebraic formalism. A few alternatives we might consider are:

(1) The principal operator fiber bundle $P(\mathbb{M}, S^n, \pi)$;
(2) The operator fiber bundle $E_n = (\mathbb{M}, T_p\mathbb{M}, S^n, \pi)$;
(3) The sphere operator fiber bundle $B_n = \big(T\mathbb{M}, S^n, SO(n+1), \pi\big)$;[n]
(4) The automorphism operator fiber bundle $A_n = (\mathbb{M}, S^n, A(n), \pi)$, with $A(n)$ the automorphism group of the associated division algebra.

Options (1) and (2) suffer from a slight potential flaw. Namely, both have as structure group the spheres themselves.

Use of $P(\mathbb{M}, S^n, \pi)$ implies that the fiber S^n is isomorphic to a group,[38,65] since principal fiber bundles have as structure group the fiber itself.[11,46,65] A principal fiber bundle is nothing else than a local group on a manifold.[46] Thus this structure might work for geometrically representing S^1 and S^3 as operator fibers, since these spheres are isomorphic to $U(1)$ and $SU(2)$. But, unlike S^1 and S^3, the sphere S^7 is not isomorphic to a group.[11,16] As such, when developing the theory to generate QCD's $SU(3)_c$ Lagrangian via $S^7 \to \mathscr{S}^7$ the use of $P(\mathbb{M}, S^7, \pi)$ would prove problematic. The same issue arises with the operator fiber bundle of option (2): $E_n = (\mathbb{M}, T_p\mathbb{M}, S^n \to \mathscr{S}^n, \pi)$. So, unless one is willing to generalize the type of structure which can exist in the place of a structure group for a fiber bundle, other options should be considered.

We might instead postulate the parallelizable *sphere operator fiber bundle* $B_n = \big(T\mathbb{M}, S^n \to \mathscr{S}^n, SO(n+1), \pi\big)$, in which we now have the base space $T\mathbb{M} = \bigcup_{p\in\mathbb{M}} T_p\mathbb{M}$ with operator fibers the spheres S^n_p attaching to and operating on $T_p\mathbb{M}$ $\forall p \in \mathbb{M}$ in the specific manner $(S^n \to \mathscr{S}^n)$ via the $\vee$-action as set forth herein as p varies over $\mathbb{M}$. Use of $T\mathbb{M}$ as the base space emphasizes that the S^n are attaching to and operating through $T_p\mathbb{M}$ via mutual compatible covariantly constant frame fields. In addition we now take the structure group as the norm symmetry group $SO(n+1)$ for each of the division algebras, namely $SO(2)$, $SO(4)$ and $SO(8)$.[13]

[n] We leave open imposition of the more refined double cover structure carrying a temporal dimension: $SO(n+1) \to \mathrm{Spin}(n, 1)$.

It should be clear that use of B_n now permits incorporation of S^7 into the geometric framework.[o] This option is also quite aesthetic, for it is naturally a sphere fiber bundle[62,63,66] with structure group the inherent norm symmetry group and with the main postulated theoretic construct being that the S^n bundle is to be an operator fiber bundle on and over $T\mathbb{M}$.[p]

Furthermore, the structure groups of B_n may provide for further interesting and explanatory physics. One need only consider the family generation issue and its oft-conjectured relation to $SO(8)$'s unique triality symmetry,[13,38,57,85] or the fact that the spin covering groups Spin(n) of $SO(n)$ and their complexifications are used to describe neutral and electrically charged fermions and are used as the structure groups of spinor bundles.[31,32,43–45]

For instance, concerning $SO(8)$'s triality symmetry and its relation to S^7, we can write three representations of $SO(8)$ related to one another via triality symmetry as:[87]

$$\begin{aligned} SO(8) &= \{x \mapsto pxp : x \in \mathbb{O}, p \in \mathbb{O}_u\}\,, \\ SO(8) &= \{x \mapsto px : x \in \mathbb{O}, p \in \mathbb{O}_u\}\,, \\ SO(8) &= \{x \mapsto xp : x \in \mathbb{O}, p \in \mathbb{O}_u\}\,. \end{aligned} \tag{25}$$

Notice that the triality generating operator is $p \in \mathbb{O}_u \cong S^7 \to \mathscr{S}^7$ operating via octonionic multiplication,[87] i.e. via the CAM operator $\circ$. The mapping pxp corresponds to the vector representation 8^v, the mapping px to the spinor representation 8^+, and xp to the dual spinor representation 8^-.[13,85,87,88] Triality is the implicit mapping between these representations determined through use of some $p \in S^7$ cyclically acting on an element x of $\mathbb{O}$.[85,87,88] Thus, the three copies of $SO(8)$ are equivalent under triality symmetry,[87] and it is the sphere S^7 in the role of an operator which generates this symmetry equivalence.

Additionally, the set

$$[B_n] = \{B_0, B_1, B_3, B_7, B_{i\otimes j}, B_{i\otimes j\otimes k}, \ldots\}\,, \quad i, j, k, \ldots = 0, 1, 3, 7\,, \tag{26}$$

where $B_{i\otimes j\cdots} \Leftrightarrow S^i \otimes S^j \cdots$, is uniquely closed under the product composition $\otimes$ since $\otimes$-compositions of parallelizable manifolds are themselves parallelizable.[59] These $\otimes$-composition product manifolds will provide for the unified interaction Lagrangians of $SU(2) \otimes U(1)$ and $SU(3) \otimes SU(2) \otimes U(1)$.

[o] The S^7 structure will prove a bit more involved, but nevertheless still consistent with this fiber bundle framework. For instance, the base space for $\mathscr{S}^7$ theory will make use of $T\mathbb{M} \otimes T\mathbb{M}$ vice simply $T\mathbb{M}$.

[p] This notion of a parallelizable manifold (S^n) smoothly acting point-wise on another manifold ($\mathbb{M}$) is analogous to the mathematical notion of Lie group elements, which are points on differentiable, parallelizable manifolds, smoothly acting on various kinds of objects — such as vectors or other manifolds[32,39,71,86] (see, e.g. Ref. 71, pp. 411–412; Ref. 86, Subsec. 3.2). The essential difference is the mechanism of action; for Lie groups action on other objects is achieved via a representation of the group, whereas with the parallelizable spheres herein action is achieved via an algebraic structure intrinsic to the sphere.

Lastly, note that the $SO(n+1)$ contain the division algebraic automorphism groups as subgroups:[13] $U(1) \subseteq SO(2)$; $SU(2) \subset SO(4)$; $G_2 \subset SO(8)$. These automorphism groups are associated with the fourth bundle option above. Though we will presuppose B_n for this introductory paper the fiber bundle $A_n = (\mathbb{M}, S^n, A(n), \pi)$, where $A(n)$ is the automorphism group of the associated division algebra, deserves future in-depth consideration along with B_n.

The pivotal attribute of $A(n)$ of A_n is that it maintains invariance of the algebraic binary operation $\circ$ as one moves over the associated sphere S^n. Since this algebraic operation is the defining, core operation of the CAM formalism which simultaneously permits association of teleparallel frame fields between S^n and $\mathbb{M}$,[16] $A(n)$ might be called the *CAM symmetry group* and A_n the *CAM operator bundle*.

This invariance property of the CAM operation gives weight to use of A_n. Interestingly, A_1 and A_3 reduce to the principal bundles $P(\mathbb{M}, S^1, \pi)$ and $P(\mathbb{M}, S^3, \pi)$. Also, in using A_7 the $SU(3)$ symmetry group naturally arises as the subgroup of G_2 which maintains invariance of the complex octonion element ℓ.[87] However, with A_n we lose $SO(8)$ triality. Furthermore, $SO(n+1)$ is the isometry (automorphism) group of the associated unit sphere and is thus the symmetry group which maintains the structure of the manifold itself which in turn intrinsically contains CAM's binary operation $\circ$ — thus $SO(n+1)$ preserves the CAM algebraic operator structure while simultaneously preserving the underlying geometric structure. Lastly, we note that since we have $G_2 \subset SO(8)$ we still may arrive at $SU(3)$ as the symmetry group of the subgroup of G_2 which maintains invariance of the complex octonion element ℓ.

Thus, despite the intriguing properties of A_n, the operator bundle B_n better provides the attributes for geometrically representing and generating the CAM formalism and for inducing the elementary-particle force Lagrangians, and is chosen for such. Deeper future geometric analysis and theoretic or phenomenological considerations may further compare and contrast A_n with B_n and highlight their respective parts within CAM. At this stage of theoretic development, the more important objective is to have a viable and robust fiber bundle structure to house the sphere operator fibers within, and B_n — which contains A_n — meets this criteria.

5.4.1. *The sphere operator fiber $\mathscr{S}^n$*

Because of the aforementioned structural and operational attachments we conceive of B_n's operator S^n as attached to $\mathbb{M}$ via $T_p\mathbb{M}$ at all $p \in \mathbb{M}$ and acting therein. These notions of structural and operational attachment of S^n to $\mathbb{M}$ via their respective tangent bundles form the foundation for the postulate that the parallelizable spheres are operator fibers operating directly on $\mathbb{M}$ via $T_p\mathbb{M}$.

The bundles B_1 and B_3 are thus postulated as operator bundles with parallelizable spheres S^n postulated as operator fibers $\mathscr{S}^n$ operating point-wise via the coupling operator $\circ$ on $\mathbb{M}$ in the tangent space $T_p\mathbb{M}$ over the tangent bundle $T\mathbb{M}$

$$S^n \vee T_p\mathbb{M} \to \mathscr{S}^n \vee T_p\mathbb{M} = \left\{\mathscr{S}^1 \vee T_p\mathbb{M}, \mathscr{S}^3 \vee T_p\mathbb{M}; \forall p \in \mathbb{M}\right\}, \tag{27}$$

where for brevity we will henceforth write $\mathscr{S}^n T_p\mathbb{M}$ as the operation. We reiterate that this global operator structure over $\mathbb{M}$ is only possible because of the mutually compatible global teleparallel structure of the manifolds $\{T\mathbb{M}, \mathbb{M}, S^n\}$ revealed by (21), and that this teleparallel structure exists inherently within these manifolds.[16,31,q] We further reiterate the critical point that since $\nabla e_n = 0$ is globally true only for the unique parallelizable spheres of one, three (and seven) dimensions, no other unit spheres S^N besides these can possess this postulated B_n operator structure. Thus, for example, S^2 cannot generate a smooth operator structure $B_2 = (T\mathbb{M}, S^2 \to \mathscr{S}^2, SO(3), \pi)$ over $\mathbb{M}$ for which $S^2 T_p\mathbb{M} \to \mathscr{S}^2 T_p\mathbb{M}$ for all $p \in \mathbb{M}$, as in (27).

5.5. *Conceiving B_n's operation*

Armed with the operator bundle structure B_n we can now abstractly conceive of how a Lagrangian manifests through B_n's coaction with space–time.

Given that the $\circ$-coupling operator is the invariant operator embedded in $S^n \to \mathscr{S}^n$ at all points $m \in S^n$ and that $\mathscr{S}^n$ attaches to $\mathbb{M}$ via $\mathscr{S}^n T_p\mathbb{M}$'s attachment to $T_p\mathbb{M}$ at all $p \in \mathbb{M}$, the $\circ$-operator is the natural candidate for the operating mechanism which effects the postulated B_n-action on $\mathbb{M}$.

Given that the CAM operator η generates the Lie algebra of $SL(2,\mathbb{C})$ — the double cover of the Lorentz group — and that $\Omega \equiv \{\eta, \partial\}$ generates the Poincaré algebra,[2] along with Ω's intrinsic inducement in $T_p\mathbb{M}\ \forall p \in \mathbb{M}$ taken with the mandated complexification $\mathrm{Cl}_{1,3} \to \mathbb{C}l_{1,3}$, the set $\{\Omega, \bar{\Omega}\} \equiv \{\eta, \bar{\eta}, \partial, \bar{\partial}\}$ is a reasonable candidate for the space–time entities which $\mathscr{S}^n_p$ will operate on in B_n's $\circ$-action in $T_p\mathbb{M}$, and are premised as such.

B_n's $\circ$-action of the sphere operators $S^n_p \to \mathscr{S}^n_p$ on an operator subset $\omega \subset \{\Omega, \bar{\Omega}\}$ in $T_p\mathbb{M}$ generates a coupled operator ϕ at p whose function it is to operate

[q]This theoretical notion of structural attachment permitted by a concurrent *global* parallelizability of the manifolds (per (21)) will come further into consideration when gravity is addressed within the geometry of the CAM model. This will require, *inter alia*, introduction of a general Levi-Civita connection ∇^g for some Riemannian space–time manifold $(\mathbb{B}, g)$ on which $R(e_\mu, e_\nu)$ does not globally vanish (thus calling into question the global applicability of (21) and thus also the global structural attachment between S^n and $\mathbb{B}$), but for which via the principle of equivalence (21) may be recovered in a limited domain of $\mathbb{B}$ locally approximating $\mathbb{M}$ for a sufficiently small neighborhood of a space–time point $p \in \mathbb{B}$,[35,49,50] and thus S^n and $\mathbb{M}$ can be considered locally structurally attachable.

Thus, the fiber bundle makeup of CAM is consistent with the general relativistic framework for gravity. General relativity deals with the phenomenon that a global free-float frame is not possible within a gravitationally endowed space–time region $\mathbb{X} \subset \mathbb{B}$.[49] If such a global free-float frame were possible throughout $\mathbb{X}$, then the S^n would still globally attach to space–time therein, which would mean that use of CAM's differential operator ∂ (i.e. special relativity) would be sufficient for the description of physics within $\mathbb{X}$ — which is in contradiction with general relativity theory. But the impossibility of a global free-float frame in a region of space–time is equivalent to the impossibility of a global covariantly constant frame field in this region such that parallel transport between two events therein is path independent[31] — thus the S^n cannot globally attach to space–time within $\mathbb{X}$. Thus, the need for a general relativistic formulation of CAM when considering the gravitational interaction.

on and couple any interacting array of j sections $\mathfrak{s}_{j(p)}$ existing at the space–time event p and belonging to either $S_p\mathbb{M}$ or $A_p\mathbb{M}$:

$$\mathscr{S}_p^n : \omega \overset{\circ}{\longmapsto} (\phi_\omega)_p \mathfrak{s}_{j(p)}\,, \quad \omega \subset \{\Omega, \bar{\Omega}\}\,, \quad \mathfrak{s}_{j(p)} \in S_p\mathbb{M} \cup A_p\mathbb{M}\,. \tag{28}$$

Since ϕ couples sections of bundles at p it "stitches" the fiber bundles together at the coupling event p. As B_n's $\circ$-coupling operation is invariant on $\mathscr{S}^n$ and exists at all $p \in \mathbb{M}$ we may write the bundle section coupler as: $(\phi_\omega)_p \equiv \phi_\omega$. This bundle-stitching process induced by ϕ_ω thus occurs throughout space–time at all p where the subject interactions occur between sections $\mathfrak{s}$ of the fiber bundles $\{S\mathbb{M}, A\mathbb{M}\}$.

The coupled operator's ϕ_ω action on and coupling of the sections $\mathfrak{s}_{j(p)}$ generates a set of k ϕ_ω-associated fields $\mathcal{F}^k_{\phi_\omega}$ at p from which a local gauge invariant Lagrangian $\mathcal{L}_p$ manifests and governs the local dynamics and interactions induced at p

$$\phi_\omega : \mathfrak{s}_{j(p)} \mapsto \mathcal{L}_p\left(\mathcal{F}^k_{\phi_\omega}\right). \tag{29}$$

Equations (28) and (29) combine to

$$\mathscr{S}_p^n : \omega \overset{\circ}{\longmapsto} \phi_\omega\left(\mathfrak{s}_{j(p)}\right) \mapsto \mathcal{L}_p\left(\mathcal{F}^k_{\phi_\omega}\right), \tag{30}$$

which abstractly exhibits how the CAM fiber bundle geometry encompassed in B_n generates the CAM geometro-algebraic operator formalism along with its attendant results.[1–5,r]

Thus the CAM formalism is theorized to be locally executable space–time operator machinery intrinsically encoded within an underlying geometric source — in particular within the smooth manifold structure of the parallelizable spheres co-acting locally on the space–time manifold via the operator fiber bundle B_n.

5.6. $\mathscr{S}^3 \otimes \mathscr{S}^1$

The sections $\mathfrak{s}_i$ of the $S\mathbb{M}$ and $A\mathbb{M}$ bundles applicable to electromagnetic $\mathscr{S}^1 \rightarrow U(1)$ theory are[1,2]

Spinor: ψ
Space–time Algebra:
$\not{A}_\mu(x) = \gamma_0 A^0(x) + \gamma_1 A^1(x) + \gamma_2 A^2(x) + \gamma_3 A^3(x)$
$\not{\mathbb{K}}^\mu = \boldsymbol{\gamma}_5 g_1 \bar{\psi}\gamma^\mu\psi$

where we use the notation $\not{\mathbb{K}}$ to signify that we have a section of $A\mathbb{M}$ with components attached to Clifford elements other than the $\{\gamma_\mu\}$.

[r]See Ref. 2, Subsec. 1.1 and Eq. (1), which matches (30) with the CAM formalism.

A satisfying and natural outcome of the geometry is that though the matter-wave field ψ itself is a section of the spinor bundle the current $\not{K}^\mu$ which ψ induces is a section of the space–time bundle, thus the couplings are naturally between sections (the gauge and current fields) of the same bundle $A\mathbb{M}$.

Setting $\omega_1 = (\eta, \partial)$; $\omega_2 = (\bar{\eta}, \eta)$ and applying (28) gives:[1,2,s]

$$\begin{aligned} \mathscr{S}^1_p : \omega_1 \overset{\circ}{\longmapsto} (\phi_{\omega_1})_p \equiv (\eta \circ \partial)_p \equiv \eta\partial_{\mathbb{C}} \\ = (\gamma_0 \partial/\partial t - \boldsymbol{\gamma} \cdot \nabla, \gamma_0 \nabla + \boldsymbol{\gamma} \partial/\partial t + \boldsymbol{\gamma} \times \nabla)\,, \\ \mathscr{S}^1_p : \omega_2 \overset{\circ}{\longmapsto} (\phi_{\omega_2})_p \equiv (\bar{\eta} \circ \eta)_p \equiv \bar{\eta}\eta_{\mathbb{C}} = (C, 2\gamma_0 \boldsymbol{\gamma})\,, \end{aligned} \tag{31}$$

where C is a constant which has no effect on the Euler–Lagrange field equations. Applying (29), the operator ϕ_{ω_1} acts on the sections $\mathfrak{s}_{j(p)} = (\psi, \not{A}_\mu(x)) \in (S\mathbb{M}, A\mathbb{M})$,[1] while ϕ_{ω_2} operates on the section $\not{K}^\mu$,[1,2] with the resulting chirally symmetric local gauge invariant Lagrangian $\mathcal{L}_{\bigoplus_i \phi_{\omega_i}}$ for $U(1)$ following as a matter of course.[1,2]

The sections of the $S\mathbb{M}$ and $A\mathbb{M}$ bundles applicable to weak $\mathscr{S}^3 \to SU(2)$ theory are[2]

Spinor: Ψ doublet
Space–time Algebra:
$\not{\boldsymbol{W}}^i_\mu(x) = \gamma_1 W^1_\mu(x) + \gamma_2 W^2_\mu(x) + \gamma_3 W^3_\mu(x)$
$\not{\boldsymbol{K}}^\mu = (1 + \boldsymbol{\gamma}_5) g_2 \bar{\Psi} \gamma^\mu \boldsymbol{\tau} \Psi$

Following the regime of (28) and (29) the same operators, $\bigoplus_i \phi_{\omega_i}$, are generated for $\mathscr{S}^3$; however, in this instance because the Lie algebra $\mathfrak{su}(2)$ structure constants are not identically zero we get for $SU(2)$'s $(\phi_{\omega_2})_p$:[2]

$$(\phi_{\omega_2})_p \equiv (\bar{\eta} \circ \eta)_p \equiv \bar{\eta}\eta_{\mathbb{H}} = \left(C, 2\gamma_0 \boldsymbol{\gamma} + \frac{1}{2}[\boldsymbol{\gamma}, \boldsymbol{\gamma}] \right). \tag{32}$$

The extra operator term $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ generates the nonlinear $\not{\boldsymbol{W}}^i_\mu(x)$ gauge field self-interaction terms of the $SU(2)$ field strength tensor.[2] Furthermore due to the structure of $\not{\boldsymbol{K}}^\mu$ for $\mathscr{S}^3$ theory, chiral asymmetry may spontaneously arise in the local gauge symmetric $SU(2)_L$ Lagrangian.[2]

As the product manifold $S^3 \otimes S^1$ is parallelizable,[59] the product operator geometry $\mathscr{S}^3 \otimes \mathscr{S}^1$ of the operator bundle $B_{3\otimes 1}$ can be used to generate the pre-Higgs locally gauge symmetric, left-chiral electroweak Lagrangian $\mathcal{L}_{\bigoplus_i \phi_{\omega_i}}$ locally on $\mathbb{M}$ in $T\mathbb{M}$[2]

$$(\mathscr{S}^3 \otimes \mathscr{S}^1) T_p\mathbb{M} \mapsto (SU(2)_L \otimes U(1)_Y)_p\,. \tag{33}$$

[s] See Ref. 1, Eq. (8); Ref. 2, Eq. (9), Fn. (7), and Subsec. 2.2.2 and (3)'s generation of the $U(1)$ current.

6. The Geometry of Local Gauge Symmetry

In YM bundle theory local gauge transformations are changes in fiber coordinates of an associated principal fiber bundle,[11,46] for example for the gauge group $U(1)$. Consider the parameter θ for electromagnetism's principal fiber bundle $P(\mathbb{M}, U(1))$.[11] All global gauge transformations are generated by the gauge group transformation element $z = e^{\mathrm{i}\,\theta} \in U(1)$[28,31,37] and identically operate instantaneously throughout space–time. All local gauge transformations are generated by $z(p) = e^{\mathrm{i}\,\theta(\boldsymbol{x},t)}$,[31,37] where $e^{\mathrm{i}\,\theta(\boldsymbol{x},t)} \in U(1)$ with $p = (\boldsymbol{x}, t) \in \mathbb{M}$ and $\theta(\boldsymbol{x}, t)$ varying locally throughout space–time as one moves from point p_n to point p_m therein.[11,16]

Within CAM, the geometry of local gauge symmetry is based on the parallelizability of the spheres and is clearly visualized.

In the CAM model, we utilize the fundamental geometric fact that $U(1)$ is contained in the circle since isomorphic thereto:[11,21] $U(1) \cong S^1 \mapsto \mathscr{S}^1$. Thus any choice of gauge group element $z = e^{\mathrm{i}\,\theta(\boldsymbol{x},t)} \in U(1)$ at $p \in \mathbb{M}$ is more fundamentally a geometric choice of a point z on the 1-sphere operator fiber $\mathscr{S}^1$ of the circle operator bundle B_1 acting in $T_p\mathbb{M}$ at $p = (\boldsymbol{x}, t) \in \mathbb{M}$.

If θ is varied from point p_n to p_m of $\mathbb{M}$ the element $z_{p_n} = e^{\mathrm{i}\,\theta_{p_n}} \in S^1_{p_n} \mapsto \mathscr{S}^1_{p_n}$ concomitantly varies to $z_{p_m} = e^{\mathrm{i}\,\theta_{p_m}} \in S^1_{p_m} \mapsto \mathscr{S}^1_{p_m}$. As discussed above, since S^1 is parallelizable $\mathscr{S}^1$ attaches to and operates in $T_p\mathbb{M}$ with the algebraically invariant $\circ$-coupling operator at all space–time points p. Thus with the variation $\theta_{p_n} \to \theta_{p_m} \Rightarrow z_{p_n} \to z_{p_m}$ the $\mathscr{S}^1$ coupling mechanism exists and is operatively invariant on $T_p\mathbb{M}$ $\forall p \in \mathbb{M}$, thereby producing the correct coupled, local gauge invariant Lagrangian $\forall p \in \mathbb{M}$.

This presents an example of a symmetry principle enunciated by Weyl:[41] if a condition which determines an effect possesses a certain symmetry, then the effect will exhibit the same symmetry. $\mathscr{S}^1$ is the condition which determines the local gauge invariant laws of physics for $U(1)$ as the effect — such as for electromagnetism.[11] Since the $\circ$-coupling is an invariant on $\mathscr{S}^1$, its generation of the laws of physics in a local gauge invariant form will remain invariant as $z \in \mathscr{S}^1$ varies with variation of the point $p \in \mathbb{M}$ and the operator fiber $\mathscr{S}^1$ operates via the attachment and action $\mathscr{S}^1_z T_p\mathbb{M}$, where this reads: "at the point $z \in S^1$ the sphere operator fiber $S^1 \to \mathscr{S}^1$ attaches to and acts on $\mathbb{M}$ at the point $p \in \mathbb{M}$ via TS^1 and $T\mathbb{M}$'s mutually compatible covariantly constant frame fields."

Consider now the alternative scenario — if S^1 were not parallelizable. Then its Euler–Poincaré characteristic $\chi \neq 0$,[11,65] and S^1 would possess at least one singular point z at which all vector fields on S^1 vanish.[65,84] At any point $p \in \mathbb{M}$ a simple phase (gauge) transformation $e^{i\alpha}$ of $S^1_n T_p\mathbb{M} \to \mathscr{S}^1_n T_p\mathbb{M}$ would then bring this singular point z in contact with p, i.e. $\mathscr{S}^1_n T_p\mathbb{M} \mapsto \mathscr{S}^1_z T_p\mathbb{M}$, and there would then be no attachment of any covariantly constant frame fields between TS^1 and $T\mathbb{M}$ at p — and thus there could be no B_n action at p and therefore no Lagrangian. Thus at any point p where a Lagrangian is initially considered locally gauge symmetric a simple gauge transformation would make it not so, thereby violating the fundamental principle that the laws of physics are to be invariant under gauge transformations. Thus since $\mathscr{S}^1$ could not be said to attach to and $\circ$-operate in $T_p\mathbb{M}\ \forall p \in \mathbb{M}$ it could not then generate a locally gauge symmetric Lagrangian throughout $\mathbb{M}$. Local gauge symmetry in elementary-particle physics for $SU(2)$ and $U(1)$ is thus seen at its core as an attribute of the parallelizability of the 1 and 3-spheres.

7. Contrasting the YM and CAM Gauge Theories

7.1. *CAM is a gauge theory*

Often stated as "the principle of gauge theory"[t] for SM physics is that matter-wave fields are to be described by sections of gauge group vector bundles in locally gauge invariant field equations.[11,16,17,21,26,32,46] Stated in this way the principle presupposes that YM theory must be used in constructing SM, for it is within the fiber bundle structure of YM that matter-wave fields are sections of gauge group bundles for which local gauge invariant field equations result.

This form of the gauge principle thus assumes that YM is the only formalism that can generate the SM laws of physics within a locally gauge invariant environment, and thus YM theory is presumptively considered the SM local gauge theory. In short, the term "Yang–Mills theory" has often been held synonymous with the term "gauge theory" or "non-Abelian gauge theory" for SM physics.[11,21,27,32,39,47,51–53,u]

But it is now known that CAM also generates the local gauge invariant SM gauge group Lagrangians with inherently accompanying gauge fields.[1–5] To make amends of this, gauge invariance must not be conceived of simply as a set of mathematical tools, but instead as, "the *physical principle* governing the fundamental forces between the elementary particles."[v]

Given this understanding, a simple way to state the gauge principle is taken from Weyl, Yang and Mills[26,40,54,55,w]

> The gauge field is to be a necessary accompaniment of the matter-wave field within a locally gauge symmetric setting.

Stated in this way the gauge principle has two requirements: (1) Weylian accompaniment, and (2) Yang–Millian locality.

The Weylian accompaniment requirement stems from Weyl's dynamical view of matter as both the "inducing agent" of a gauge field as well as the interacting agent involved in a gauge field's "transferring interactions from matter to matter,"[x] whereas the YM locality condition of local gauge symmetry is a requirement for relativistic field theories.[18,26]

[t]Reference 21, p. 222.
[u]Thus Kane writes, "The Standard Model is called a Yang–Mills gauge theory" (Ref. 23, p. 1–2).
[v]Reference 91, p. 5 (emp. supp.).
[w]Reference 54, p. 331; Ref. 26, p. 192.
[x]Reference 55, p. 609.

Using this form of the gauge principle as the guiding principle we see that the YM and CAM systems are but two differing methodologies for effecting this principle and thereby producing local gauge symmetric SM gauge group Lagrangians with coupled matter and gauge fields. They are thus to be considered as two separate gauge theoretic models for generating the local gauge symmetric Standard Model of particle physics.

Therefore it is more accurate to state that in YM-based SM gauge theory matter-wave fields are described by sections of simply-connected gauge group vector bundles with gauge fields being the connections of the associated principal fiber bundle of the structure group through which the required accompaniment-coupling is generated; whereas in CAM-based SM gauge theory matter-wave fields are sections of the spinor bundle while gauge fields and four-currents are sections of the space–time algebra bundle which interact via CAM's bundle coupling technology, thereby intrinsically generating a local gauge invariant theory.

7.2. *Distinctions*

Though both gauge theories intrinsically produce coupled, locally gauge invariant Lagrangians for the SM simply-connected gauge groups nevertheless there are important conceptual, theoretical and phenomenological differences between them, some of which follows:

(1) YM theory is a gauge theory based on the groups $SU(N) : N \geq 1$. The CAM model is a gauge theory based on the parallelizable spheres $S^N : N = 1, 3, 7$. In YM-based gauge theory a different choice of gauge group corresponds to a different elementary-particle force.[39] Thus in YM forces are initially and primarily related to gauge groups. In CAM-based gauge theory a different choice of parallelizable sphere (or composition thereof) corresponds to a different elementary-particle force. Thus in CAM the forces are initially and primarily related to spheres. No longer do we generate a force by specifying a Lie gauge group; we now generate a force by specifying a parallelizable sphere.

(2) In CAM the sphere operators $\mathscr{S}_p$ are conceived as operationally attaching to space–time. In YM the gauge group fibers G_p are conceived only as "standing, in some sense, above"[27] or "sitting right over"[21] space–time and operating not on space–time but on matter-wave sections of another fiber bundle over space–time.

(3) YM deals with connections of various fiber bundles. CAM deals only with the connection ∇ on the tangent bundle $T\mathbb{M}$, and is therefore more fundamentally geometric than YM. As Baez noted regarding general relativity's relation to YM theory, "general relativity is even *more* (emp. Baez) geometrical [than Yang–Mills theory], since it concerns, not just any old bundle, but the tangent bundle!"[y]

[y]Reference 21, pp. 365–366.

(4) A fundamental question remains as to the origin of the SM gauge group symmetries.[27] The YM-based SM remains silent on why the symmetries are $U(1)$, $SU(2)$ and $SU(3)$. No theoretical basis for the symmetries exists within YM. As Kane noted, "The Standard Model is a descriptive theory. It does not explain why its particular electroweak and color forces are what they are, and whether they are inevitable . . . ."[z] Comparatively, CAM specifically establishes where and why the $U(1)$, $SU(2)$ and $SU(3)$ symmetries arise through generation of the characteristic structure constants of these simply-connected symmetry groups, and these alone.[aa] According to the CAM model these forces, and these alone, are inevitable. In this way CAM theory mandates the symmetry groups $U(1)$, $SU(2)$ and $SU(3)$ for the elementary-particle forces of nature.

(5) CAM finalizes the singular elementary-particle forces at $SU(3)$, since S^7 is the last parallelizable sphere. YM contains no such constraint, which has proven to be an issue for YM theory in regards to proton decay phenomenology.[5] This CAM constraint thus answers the question echoed throughout the theoretical physics community as stated by Schwichtenberg:[bb] "[T]he three fundamental forces described by the standard model correspond to the symmetry groups $U(1)$, $SU(2)$ and $SU(3)$. Why is there no fundamental force following from $SU(4)$? Nobody knows!" Now we know.

(6) In YM numerous curvatures over the various gauge group bundles necessarily arise. In CAM the only required connection is $\nabla^{\boldsymbol{\eta}}$ of $\mathbb{M}$, and thus no group bundle curvatures exist in CAM.

(7) In YM the gauge fields are connections. In CAM the gauge fields are not connections, but Clifford fields. Now Clifford elements operate on spinor fields,[16,27,36,38] which are the matter-wave fields in both CAM and YM. Furthermore gauge fields also operate on the spinor matter-wave fields in transferring interactions.[36,39,40,53] Therefore it is reasonable to postulate that the gauge fields should have a direct nexus with the Clifford elements and thus take their values in the Clifford space–time bundle, i.e. be Clifford fields. This is argued to be a more natural procedure than the YM gauge theory's procedure of the gauge fields being the connection coefficients of a specified principal gauge group bundle thereafter effecting operation on the spinor matter-wave fields via a representation ρ on an associated vector bundle of the matter-wave field.

(8) In YM gauge theory, there is "no necessary correlation whatsoever between the geometry of the base manifold and the geometry of the fiber."[cc] Thus although on one hand the tangent bundle is "intimately associated" with the

[z]Reference 23, pp. 1–3.
[aa]See Ref. 5, Fn. (15) and related content for further discussion of the structure constant topic.
[bb]Reference 86, p. 3.
[cc]Reference 48, p. 379.

base space manifold $\mathbb{M}$, on the other hand there "is no pre-existing reason to add a spinor space" as a fiber bundle for associating material fields.[48] In CAM, the relation between the base space $\mathbb{M}$ and the spinor and Clifford bundles is more subtle and intimate. For instance in CAM bundle theory there *is* a pre-existing reason to add a spinor space to $\mathbb{M}$, since CAM generates the Dirac operator $\not{\partial}$ which operates on spinors. Although it remains to be seen whether the spinor and Clifford fields can "arise as a result of any natural structure on the base manifold,"[48] nevertheless the structure of these fields is intimately and necessarily correlated with the form of the operators generated by $\mathscr{S}$-action on $\mathbb{M}$. Thus (30)'s generated operators determine to various extent the form of the fields that can be operated on and thus the nature of the fiber bundles of which these fields are sections.

(9) CAM naturally accommodates intrinsic spontaneous generation of chiral asymmetry for $SU(2)$. YM does not, and only imposes parity violation *ad hoc*[36,39,52] or derives parity violation from the renormalization effects within the causal gauge invariance of second order tree graphs for leptonic couplings.[56]

(10) CAM intrinsically accommodates a right-chiral neutrino; YM does not.

(11) CAM places constraints on proton decay which are consistent with experiment; YM does not.

(12) CAM generates Einstein's equations of general relativity and provides a place for the Higgs field to enter the physics arena and inherently couple to the space–time metric.

7.3. *Connections*

Despite these differences, the principal connection between the YM and CAM gauge theories is readily shown. This theoretic crossroads between the two approaches reveals the Yang–Mills methodology as derivative of the CAM methodology, a subject which was previously alluded to.[dd]

7.3.1. *YM gauge theory*

The core of Yang–Mills gauge theory rests upon the building blocks of a chosen non-Abelian gauge symmetry group and a subsequently derived connection associated with the gauge potential field of the symmetry group.[18,26,91,94] The applicable covariant derivative arises thereafter from this derivation, and a Yang–Mills Lagrangian and the equations of motion are developed under the guidance of a mandated local gauge symmetry.[17,39,46,91] Reduction to the Abelian symmetry group $U(1)$ results in Maxwell's electromagnetism.[36,53,91,94]

Construction of modern non-Abelian gauge theory begins with identification of the Lie algebra for an arbitrarily chosen non-Abelian group for which the local

[dd] Reference 5, Fn. (14).

symmetry transformation of a wave function is:[12,91,94,ee]

$$U\psi = \exp\left(-ig\sum_k \theta^k(x)G_k\right)\psi\,, \tag{34}$$

where the internal space parameters $\theta^k(x)$ are continuous functions of the space–time coordinates (x) and G_k are the generators of the chosen symmetry group's Lie algebra abiding by the ubiquitous Lie bracket relations:

$$\left[G_i, G_j\right] = \mathrm{i}c_{ijk}G_k\,. \tag{35}$$

Here we make the pivotal observation that within YM gauge theory the initial choice of a simply-connected local gauge symmetry group for (34)–(35) is arbitrary — we could with equal theoretical justification choose the symmetry group $SU(6)$ or $SU(2)$, $SU(4)$ rather than $SU(5)$ or $SU(3)$, Spin(13) as for Spin(10); in short, no internal guidance is given within YM gauge theory for guiding the choice of gauge group so as to be in line with the SM gauge groups. No inherent directive or self-constraining mechanism exists within YM gauge theory which would mandate the SM gauge groups or restrict use of symmetry groups for generating a Yang–Mills gauge interaction. It is here where CAM gauge theory essentially differs with YM theory, and ironically it is the core interconnection between the YM and CAM gauge constructs which reveals this distinction.

Continuing with the Yang–Mills approach, we write a particle's wave function in the internal basis of the chosen symmetry group's internal space:[91]

$$\psi(x) = \sum_\alpha \psi_\alpha(x)u_\alpha\,, \tag{36}$$

where u_α are the basis vectors for the internal symmetry space. Calculating the differential change in the wave function as the particle moves through an external potential field gives

$$d\psi = \sum_\alpha \left[\partial_\mu\psi_\alpha\, dx^\mu u_\alpha + \psi_\alpha\, du_\alpha\right]. \tag{37}$$

Focusing on the change in the internal space bases du_α will yield the connection.[91,94] Consider the infinitesimal internal precession associated with the external space–time displacement dx and caused by the external potential field:

$$U(dx) = \exp\left(-ig\sum_k d\theta^k G_k\right), \tag{38}$$

where $d\theta^k = \partial_\mu\theta^k\, dx^\mu$. The associated internal displacement of the u_α is of course $U(dx)u = u + du$, which may be written as[91]

$$U(dx)u_\alpha = \exp\left(-ig\sum_k \partial_\mu\theta^k\, dx^\mu (G_k)_{\alpha\beta}\right)u_\beta\,, \tag{39}$$

[ee]See, e.g. Ref. 12, Subsec. 9.1; Ref. 91, Subsec. 3.1.

and which yields after expansion to first order the change in internal space bases:

$$du_\alpha = -ig \sum_k \partial_\mu \theta^k \, dx^\mu (G_k)_{\alpha\beta} u_\beta \,. \tag{40}$$

The connection is defined from (40) as[53,91,94]

$$(A_\mu)_{\alpha\beta} = \sum_k \partial_\mu \theta^k (G_k)_{\alpha\beta} \,. \tag{41}$$

Looking back to (37), we thus have for the total change in the wave function

$$d\psi = \sum_{\alpha\beta} \left[\partial_\mu \psi_\alpha \delta_{\alpha\beta} - ig(A_\mu)_{\alpha\beta}\psi_\alpha\right] dx^\mu \, u_\beta \,, \tag{42}$$

from which we can resect the general covariant derivative operator D_μ:[91,94]

$$D_\mu \psi_\beta = \sum_\alpha \left[\delta_{\alpha\beta}\partial_\mu - ig(A_\mu)_{\alpha\beta}\right]\psi_\alpha \,. \tag{43}$$

By way of example, using (43) for the one-dimensional Abelian symmetry group $U(1)$ gives the well-known covariant derivative or minimal coupling for the electromagnetic interaction:[39,47,52,91]

$$D_\mu = \partial_u - iqA_\mu \,, \tag{44}$$

use of which permits the applicable local gauge symmetric Lagrangian and equations of motion to follow.[17,36,53,94]

7.3.2. *CAM as primordial YM gauge theory*

Within the CAM model, we begin further back in the process, before emergence of any applicable Lie symmetry group. CAM theory in fact reveals and generates the applicable local symmetry groups for the elementary-particle interactions — the SM local symmetry groups — by generating the Lie algebra structure constants necessarily attached to and associated with these symmetry groups.

The fundamental theorems of Lie group theory reveal the central role that the structure constants have therein,[92] and therefore also within particle physics gauge theory. The structure constants possess a certain symmetry which the generators lack — namely, they are invariant with respect to changes in the representation of a group.[38,39,92] In contrast, the generators lack this symmetry.[39]

Furthermore, the structure constants are characteristic of a Lie group, and thus all Lie groups (and for our purposes the simply-connected ones) are classifiable from them.[92,93] Any particular set of structure constants are necessarily and uniquely related to (and so isolate) a distinguished, simply-connected Lie group and therefore its associated geometric manifold.[36,38,92,93] Finally from the structure constants the adjoint representation immediately follows,[38] from which all theoretic SM results are derivable since the group structure is independent of the representation.[38,39,92]

It is from (35) that the structure constants indicate a specific Lie algebra and thus mandate its associated simply-connected Lie symmetry group to be used in (34). From this indication the flow of YM gauge theory begins and its theoretic structure is established — from (34) through (44) and beyond. The structure constants thus indicate and generate the building blocks of YM gauge theory, namely the local gauge symmetry groups and connections along with associated covariant derivatives.

It has been previously shown that CAM indicates and mandates the SM gauge groups through generation of the applicable structure constants for the Lie algebras of the $U(1)$, $SU(2)$ and $SU(3)$ symmetry groups, and these alone.[2,4,5] For the $SU(2)$ weak interaction the operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ (based in the quaternions $\mathbb{H}$) generated the structure constants for the Lie algebra $\mathfrak{su}(2)$,[2] just as the same operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ — but in this instance based in the octonions $\mathbb{O}$ — generated the structure constants for the $\mathfrak{su}(3)$ Lie algebra of the $SU(3)$ interaction.[4] There are of course no structure constants for the $U(1)$ interaction based in $\mathbb{C}$ and so the operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ vanishes and does not come into play.[1,2] It is in this way that CAM gauge theory begins further back in the process than YM gauge theory, indicating the applicable simply-connected symmetry groups through generation of their respective Lie algebras that YM gauge theory must introduce *ad hoc*.

We thus arrive at the result that Yang–Mills gauge theory is derivable from, and therefore less fundamental than, CAM gauge theory. This also means that the gauge theoretic structure of YM is utilizable within CAM gauge theory.

For instance, noted above was that whereas YM deals with connections of various fiber bundles, CAM deals only with the connection ∇ on the tangent bundle $T\mathbb{M}$. This means that CAM gauge theory needs only the connection on $T\mathbb{M}$ to generate the SM interactions. But this is not to say that YM gauge theoretic connections and all their related fiber bundle structures (such as bundle curvature) cannot be utilized within CAM's theoretical analysis. On the contrary, they can be; it simply means that neither the covariant derivative nor connection (nor the symmetry group for that matter) is the nascent, determinative structure to be used for deriving or postulating an elementary-particle force. Thus, if CAM gauge theory places an *ab initio* restriction or prohibition on use of a certain covariant derivative or symmetry group, then it and its associated connection cannot be used in violation of this restriction in contemplating the fundamental laws of elementary-particle physics. It is this YM theoretic assumption of the viability of extension of the covariant derivative and connection in postulating an $SU(5)$ (or other symmetry gauge group) interaction which has created internally unresolvable issues in YM theory for such things as proton-decay phenomenology.[5]

We conclude that CAM is a more primordial gauge theoretic structure than YM. Furthermore, since CAM gauge theory generates the building blocks for the gauge theoretic mechanisms which are integral to YM gauge theory, and thus generates and subsumes YM gauge theory itself, CAM might be more easily grasped if considered as a refinement of YM gauge theory or as primordial YM gauge theory.

8. Discussion

For YM fiber bundle gauge theory to work it is essential that the applicable principal and associated fiber bundles have certain exact symmetries. As Penrose contemplated, this contingency "raises fundamental questions as to the origin of such symmetries, and what these symmetries actually are,"[ff] for which YM theory does not provide an answer.

In CAM fiber bundle gauge theory, however, the answers to these questions are firmly rooted in the geometric structure of the formalism itself. No longer is the SM symmetry group "put into the theory by hand."[gg] The origin and operation of the symmetries of the SM forces inherently exist within the dynamic symbiosis of the parallelizable spheres and space–time, which generates CAM's geometro-algebraic formalism.

This notion is fundamentally important, and so warrants emphasis. Symmetry does indeed dictate design,[18,57] but what dictates the symmetry? What distinguishes this from that symmetry as operative in the structure and function of the fundamental laws of physics and so in the real world? It would seem logical to postulate that some mechanism or structure functioning behind the scenes generates and picks out a symmetry as that symmetry which is to apply to the design of reality in distinction to the plethora of other mathematical symmetries which do not. Is there something more fundamental lying behind the operative symmetries of reality which generates those symmetries in the first instance? This paper theorizes in the affirmative for the elementary-particle forces — the parallelizable spheres dictate these particular symmetries of reality which in turn dictate reality's design.

Thus though we can conceive YM theory as a formalism for mathematically representing the SM interactions, the CAM model goes further and can be viewed as also generating the structure of these interactions. The gauge theory of YM is but descriptive, while the gauge theory of CAM is both descriptive and generative. This theoretic and conceptual contrast between the theories justifies the introductory statement that but for the particular structure of the three unique parallelizable spheres the fundamental elementary-particle forces would not exist in the form in which they do, if at all.

That the elementary-particle forces have a core structure contingent upon the most symmetric of immutable topological structures — the spheres — and what is more only those spheres endowed with the uniqueness and universality of parallelizability — is an aesthetically compelling paradigm for comprising a corpus of fundamental physical law which once grasped cannot easily be castaway.[22,41,58]

The paradigm strongly resonates with Yau's aesthetic conviction that[hh]

> The deepest ideas of math, if shown to be true, would almost invariably have consequences for physics and manifest themselves in nature in general,

[ff]Reference 27, p. 354.
[gg]Reference 14, p. 29.
[hh]Reference 60, p. 78.

in providing (in part) a resolution to the theoretical questions which have resonated in the physics community for some time — as echoed by Peskin[96] and Furey[78,79,ii]

What makes $SU(3) \otimes SU(2) \otimes U(1)$ so special[78]?

> Of the infinite number of imaginable gauge groups, why *this* gauge group[79]?
>
> Of all the ways that nature could be built, how do we know that the Standard Model is the correct one[96] (Sec. 1)?

through theoretical authentication of Dixon's aesthetic intuition that[jj]

> The parallelizable spheres . . . are the only ones of their kind . . . and, I believe, an inevitable part of the design of reality.

9. Higher-Dimensional Unification

Given the above aesthetic paradigm one is justified in casting still further out and postulating that the elementary-particle forces are "the result of the hidden (topological structure of the spheres) that conducts its business behind the scenes,"[kk] where here "the scenes" is taken as the metric-equipped four-dimensional space–time continuum as part of the $(1, 10)$-dimensional geometro-topological structure[22,23,27,51,61]

$$\mathcal{M}^{1,10} \sim \overbrace{\mathcal{M}^{1,3}}^{\text{noncompact}} \times \overbrace{\mathscr{X}^{7}}^{\text{compactified}}, \tag{45}$$

where $\mathcal{M}^{1,3} = (\mathbb{M}, \boldsymbol{\eta})$ is the simplest choice.[61]

[ii] "In part" for two reasons: (1) the role of $\mathscr{S}^7$ has not yet been mathematically set forth — though from this paper's structure $\mathscr{S}^7$'s role can be fairly intuited, and (2) as Furey notes,[79] "Even if we were to understand why nature's local symmetries should be given by $SU(3) \otimes SU(2) \otimes U(1)$, we would still be at a loss to explain SM's particle content." In this regard we must emphasize that we hold it to be self-evident that *if* the detailed structure of some algebraic framework (such as the division algebraic systems proffered by Dixon[14,81] or Furey,[78,79] Gogberashvili[80] or Günaydin and Gürsey,[82] Dray and Manogue[87] or Pushpa and Bisht,[83] or otherwise) is in fact nature's choice for accommodating the particulars of SM particle and quantum number content, *then* — in keeping with this paper's core motivation and the modern ideal of perfecting an explanation of the fundamental physical interactions via a unifying geometry — assimilation of the proffered algebraic system into an underlying, unifying and inducing geometric structure (such as with the CAM formalism herein) is mandated. *In short, with regard to the fundamental laws of physics, to be theoretically and aesthetically viable detailed algebraic form must of necessity be intrinsically contained within and naturally emanate from unified geometric structure.*

It is by the light of this unity-through-geometry mandate that Eq. (1) addresses Furey's additional inquiry: "If SM's group representation structure is indeed a result of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, then what is it exactly that is so special about these algebras?"[79]

[jj] Reference 14, p. 160.

[kk] Reference 60, p. xvii. The term "topological structure of the spheres" has been substituted for Yau's original term "geometrical structure" since Yau considers geometrical structure as coinciding with the existence of a metric on a space (see, e.g. Ref. 60, pp. 18, 25, 79) (and in particular Calabi–Yau geometric structures), but as has been emphasized in this paper the qualities and operations contemplated of the spheres (such as parallelizability and algebraic composition) do not require equipping the spheres with this extra metric structure.

9.1. *A nonmetric source space*

As the first critical theoretic and philosophic departure from standard conceptions in string/M-theory for conceiving a higher-dimensional structure for generating the CAM model, we take $\mathbb{R}^7 \subset \mathcal{M}^{1,10}$ (from which $\mathscr{X}^7$ will arise) to be a nonmetric topological space $(\mathbb{R}^7, \tau)$ endowed with the *usual topology* τ.[11,16,24,62,84,ll] The space $(\mathbb{R}^7, \tau)$ is not Euclidean space $\mathbb{E}^7$ endowed with the standard Euclidean metric $d(p, q) = \sqrt{\sum_{i=1}^{7}(p_i - q_i)^2}$.[62,63,65,84,mm] Since in standard Kaluza–Klein theory and its string/M-theory successors higher-dimensional space–time is presumed to have a metric structure,[50] this forms a fundamental distinction between the approaches.

Despite being devoid of all metric structure $(\mathbb{R}^7, \tau)$ is a well-defined, deep-structured topological space which is locally homeomorphic to $\mathbb{E}^7$.[62,65,70] This homeomorphism exists because $(\mathbb{R}^7, \tau)$ and $\mathbb{E}^7 \equiv (\mathbb{R}^7, d)$, though having different topological bases, correspond to the same topology.[11,16,24,62–65,70,84,nn] This correspondence of topologies is an interesting mathematical result which warrants application in fundamental theoretical physics.

For if we can generate the self-same deep topological structure as a metrizable space does without recourse to the additional metric structure, then the metric structure has become superfluous for the purposes at hand and *novacula Occami*, simplicity and aesthetics mandate that we forgo it. Such a theoretic program of abstraction and nonmetric generalization is also physically justified since our intuition can easily be led astray when considering particular representations of spaces in metricized Euclidean space $\mathbb{E}^n$.[66,oo]

The operative philosophic notion here is that there is no logical reason theoretical physics cannot nor should not (with sundry reasons why it should) consider a more general yet well-defined, structured higher-dimensional space in which we "free ourselves from the straitjacket of having to work inside some Euclidean space."[pp] As Armstrong emphasized, "In defining neighborhoods in a Euclidean space we used very strongly the Euclidean distance between points. In constructing an abstract space we would like to retain the concept of neighborhood but *rid ourselves of any dependence on a distance function*."[qq]

[ll] Reference 11, pp. 9 and 13; Ref. 16, p. 81; Ref. 62, p. 43.
[mm] Reference 11, p. 28; Ref. 62, Subsec. 2.2; Ref. 65, Subsec. 1.2.6.
[nn] Reference 24, App. A; Ref. 62, pp. 18, 24 and 43; Ref. 65, Subsec. 1.2.8; Ref. 70, p. 47; Ref. 84, pp. 1–2.
[oo] See Ref. 66, pp. 10–11, Figs. 1.14–1.15, for an excellent example of this intuition-misleading phenomenon using the Möbius strip $\mathscr{M}$ generated by $i = 1/2$ and $k = 3/2$ twists. We have $\mathscr{M}_i \simeq \mathscr{M}_k$. Yet despite $\mathscr{M}_i \simeq \mathscr{M}_k$ when using representations of $\mathscr{M}_i$ and $\mathscr{M}_k$ in $\mathbb{E}^n$ no amount of stretching, bending or twisting can deform $\mathscr{M}_i$ into $\mathscr{M}_k$ (Ref. 66) — which would lead one's intuition to conclude, incorrectly, that the spaces are not homeomorphic.
[pp] Reference 66, p. 14.
[qq] Reference 66, pp. 12–13, emphasis author.

Considering only the three-dimensional spatial component of four-dimensional space–time, for two-and-a-half millennia from Aristotle through Newton to Descartes its structure was presumed to be $\mathbb{E}^3$ endowed with the standard Euclidean metric $d(p, q)$.[65,66] But as Aldrovandi and Pereira have noted:[rr]

> We are by now sure that the standard Euclidean one which has been taken for granted for millennia is not valid at distances of interest at the quantum level. The last few decades have brought to light doubts about its validity also at very large, cosmological distances.

This experimentally backed, slow-developing realization regarding our directly measurable 3D space further emphasizes the need for theoretical physics not to incipiently constrain itself to a metric space when considering higher-dimensional theory. One is in fact logically led in the alternative direction of a more general space. Furthermore, there is no experimental mandate requiring there be a metric associated with this higher-dimensional space. Such a mandate is a theoretical straitjacket. More general, all-encompassing spaces are valid as fair-game which may serve the qualitative purposes of theoretical physics for modeling our reality in a much simpler and more aesthetic way than demanding a superfluous higher-dimensional structure of Euclidean or general metric geometry.

In short, the fundamental laws of physics are not to be arbitrarily constrained by a physically and mathematically unfounded proclamation that the necessary, "*a priori*, eternal truths" of Euclidean geometry govern either the construction or perception of our universe.[35,67,73,ss] The basic philosophy of higher-dimensional CAM thus follows the modern "heretical movement"[tt] in theoretical physics, from general relativity to quantum theory, of liberating ourselves from the "undue and superfluous restrictions" engendered within the classical comforts of a metricized "ambient Euclidean space."[uu]

Since topology can be thought of as a kind of generalization of Euclidean geometry,[11] it is natural that our theoretical considerations begin within its domain. In point of fact, "topology has taken on the role of providing the foundations for just about every branch of mathematics that has any use for a concept of 'space',"[vv] and all the relevant, deep structural properties of a space such as continuity, open sets, boundedness, connectedness, compactness, contractibility, convergence, dimension, point-distinguishability, qualitative proximity and even differentiability are "quite independent of any notion of distance" or metric.[64,65,70,ww]

[rr] Reference 65, Preface to 2nd edition.
[ss] Reference 35, pp. 15–20; Ref. 67, pp. 8–13; Ref. 73, p. 329 and Subsec. VII.6 ("Mathematics and Reality").
[tt] As the mathematician and theoretical physicist Carl Friedrich Gauss feared his ideas on non-Euclidean geometry would be viewed (Ref. 35, p. 5), and for which Gauss was attacked by philosophers (Ref. 73, p. 329).
[uu] Reference 64, p. 1.
[vv] Reference 70, p. 4.
[ww] Reference 65, p. 8.

As the geometric notion of distance is not invariant under homeomorphism and therefore not a topological symmetry,[11,62,64,66,70] a metric d — being defined so as to correspond to our intuitive notion of the geometric distance between two points[11,65,70] — "contains extraneous (nontopological) information" and thus its metric structure is irrelevant to our foundational topological purposes.[65,70] This being the case we require that *any and all metric structure of CAM's theorized higher-dimensional compactified topological space $\mathscr{X}^7$ be extracted*.

To adequately liberate the topology of $\mathscr{X}^7$ from all metric structure, we are then compelled to initiate its compactification from a nonmetric source space. We are thus and in this way naturally led to postulate the aforesaid nonmetric space $(\mathbb{R}^7, \tau)$ as our pre-compactification source space: $(\mathbb{R}^7, \tau) \subset \mathcal{M}^{1,10}$.

9.2. *Source space compactification*

As a second theoretic departure we apply the specific topological procedure of one-point compactification Θ to the source space $(\mathbb{R}^7, \tau)$:[11,16,17,62,65,66,xx]

$$\Theta : \mathbb{R}^7 \overset{c}{\dashrightarrow} \mathscr{X}^7 , \tag{46}$$

which yields $\mathscr{X}^7$ as a space homeomorphic to the topological 7-sphere:[11,16,62,yy]

$$\mathscr{X}^7 \simeq S^7_\tau . \tag{47}$$

We know that the 7-sphere is isomorphic to the unit octonions.[11,14,17] Yet no division algebraic composition rule exists on (47)'s S^7_τ. This seems a contradiction at first blush, but it is not. For the division algebraic binary operation is a smooth algebraic structure belonging to S^7 as a smooth manifold and not to S^7 as a topological space — there is no notion of smooth algebraic composition defined in the topology of S^7. Smoothness and smooth structures are not purely topological properties,[64,65] and so such structure will have to be appended.[zz]

In this regard it is imperative here to recognize that there are different layers of structure which might exist in S^7. In fact, there are generally four layers of increasingly refined structure:[64,69,70,aaa]

[xx]Reference 11, Chap. 6 and Subsec. 10.2; Ref. 16, Subsec. 2.3.5; Ref. 17, p. 15: Ex. (v); Ref. 62, Sec. 19.

[yy]Per Moira, the one-point compactification of $(\mathbb{R}^7, \tau)$ happens to be homeomorphic to S^7_τ. Otherwise, (45)'s higher-dimensional structure would not be a viable CAM unification candidate.

[zz]For example, the geometric objects that the Lie groups $U(1)$ and $SU(2)$ are associated with must be differentiable (smooth) manifolds with a group $\rightarrow$ manifold operation which induces a differentiable map of the manifold into itself;[16,17,38,86] this group $\rightarrow$ manifold operation for the unit 1 and 3-spheres is of course division algebraic multiplicative binary composition. Thus the purely topological manifolds S^1_τ and S^3_τ lacking smooth structure are insufficient as the manifolds to associate with $U(1)$ and $SU(2)$, and the additional smooth structure will have to be appended. Though S^7 is not a group, the same reasoning applies as to the associated unit octonions.

[aaa]See Ref. 64, Chap. 1. As the author Lee[64,70,72] states in giving a nice explanation of these various structures on a unit-sphere:[72] "One reason it's easy to get confused about these layers of structure is that when there are obvious "natural" choices such as the ones I described above, we often don't even mention that a choice is being made. For example, if an author writes

footnote aaa (*Continued on next page*)

(1) S^7_s as a set;
(2) $S^7_\tau \equiv (S^7, \tau)$ as a topological space with topology τ structure;[bbb]
(3) $S^7_{\mathscr{A}} \equiv (S^7, \tau, \mathscr{A})$ as a differentio-topological[ccc] manifold with smooth $\mathscr{A}$ structure;
(4) $S^7_g \equiv (S^7, \tau, \mathscr{A}, g)$ as a differentio-geometric manifold with Riemannian metric g structure.

9.3. *Transport of algebraic structure into* S^7_τ

Equation (36)'s homeomorphism $\mathscr{X}^7 \simeq S^7_\tau$ admits seven-dimensional manifolds other than S^7_τ, such as $\mathscr{X}^7 = \mathcal{S}_i; i = 1 - 28$, where $\mathcal{S}_i$ for some i is one of the 28 orientation-preserving exotic spheres homeomorphic to S^7_τ.[8,16,64,68,69] Although the differentiable $\mathcal{S}_i$ structures may contain division algebraic structure,[89,90,ddd] for the purposes of this introductory paper we will maintain use of the 7-sphere with standard smooth $\mathscr{A}$ structure (a maximal smooth atlas equipped with the standard smooth structure[8,16,64]).[eee]

CAM's compactification scenario will thus require a subsequent topological deformation[16,68] event

$$\alpha : \mathscr{X}^7 \overset{d}{\dashrightarrow} S^7_\tau \tag{48}$$

necessarily occurring before the transport of $\mathscr{A}$ structure induced by (49) below. This topological deformation is required because the $\mathcal{S}_i$ are not diffeomorphic to $S^7_{\mathscr{A}}$,[9,63,68] while the smooth $\mathscr{A}$ structure being transported via (49) morphs $\mathscr{X}^7$ into $S^7_{\mathscr{A}}$. Thus without (48) a mapping such as (49), which would make $\mathscr{X}^7 = \mathcal{S}_i$ an $\mathscr{A}$-carrier diffeomorphic to $S^7_{\mathscr{A}}$, could not exist.

Following topological deformation, development of (47) to the operator sphere $\mathscr{X}^7 \mapsto \mathscr{S}^7$ occurs as the result of a transport of $\mathscr{A}$ structure[64] event induced by

footnote aaa (*Continued*)
"Let S^2 be the unit sphere in $\mathbb{R}^3$," you have to decide from the context whether she's thinking of it as a set, or a topological space with the subspace topology, or a smooth manifold with the induced smooth structure, or as a Riemannian manifold with the induced Riemannian structure ... Technically speaking, S^2 is only a set. The topological space is an ordered pair (S^2, τ) where τ is a topology; the smooth manifold is an ordered triple $(S^2, \tau, \mathscr{A})$ where $\mathscr{A}$ is a smooth structure; and the Riemannian manifold is an ordered quadruple $(S^2, \tau, \mathscr{A}, g)$ where g is a Riemannian metric. To keep the notation from getting too cumbersome, we usually omit all those additional structures when they're understood from context."

[bbb]There is a further delineation here, in which a topological manifold is defined as a topological space which also has the properties of being Hausdorff, second-countable and everywhere locally homeomorphic to an open subset of $\mathbb{E}^n$.[64,70] Topological manifolds are the simplest type of manifold.[63,64,70] Given this definition it can be shown that both $(\mathbb{R}^7, \tau)$ and S^7_τ are in fact topological manifolds.[21,62,64]

[ccc]We follow Milnor's use of the term differential topology,[8] vice the use of the phrase "smooth manifold theory."[64]

[ddd]For instance, in constructing the first exotic 7-sphere, Milnor used boundaries $S^3 \times S^3$, for which each S^3 is identified with the unit quaternions with standard multiplicative structure.[90] This raises an intriguing possibility for future study.

[eee]Reference 8, p. 2; Ref. 16, p. 174; Ref. 64, pp. 5 and 20.

the canonical isomorphism

$$\varphi : \mathscr{X}^7 \equiv S^7_\tau \to S^7_{\mathscr{A}} , \quad \varphi(s_n) = s_n \, \forall s_n \in S^7 , \tag{49}$$

where $S^7_{\mathscr{A}}$ is the differentio-topological 7-sphere equipped with the smooth division algebraic structure β which is taken as a subset of the $\mathscr{A}$ structure: $\beta \subseteq \mathscr{A}$. We know that the β structure is smooth structure since, as previously discussed, the division algebraic composition of elements yields a smooth global frame field on the division algebra's associated isomorphic unit parallelizable sphere;[11,16,64] for instance, octonion multiplication yields a smooth global frame on S^7.[fff]

The transport of β structure $S^7_{\mathscr{A}} \overset{\beta}{\hookrightarrow} S^7_\tau$ induces the division algebraic binary multiplicative operation $\cdot$ on S^7_τ via

$$s_1 \cdot s_2 \equiv \varphi(s_1) * \varphi(s_2) , \tag{50}$$

for all $s_n \in S^7_\tau$, where $*$ is the binary operation of multiplication on $S^7_{\mathscr{A}}$ existing as part of the β structure of $S^7_{\mathscr{A}}$.

9.3.1. *The β structure*

We briefly consider the essential β structure of $S^7_{\mathscr{A}}$ in order to emphasize its purely algebraic nature by first considering the complex numbers which can then in turn be generalized to the quaternion and octonion algebraic structure.[12,14,28] We have the fundamental algebraic identity[12]

$$\left(x_0^2 + x_1^2\right)\left(y_0^2 + y_1^2\right) = \left(x_0 y_0 - x_1 y_1\right)^2 + \left(x_0 y_1 + x_1 y_0\right)^2 , \tag{51}$$

which by introducing the imaginary unit $\mathrm{i}^2 = -1$ and defining the complex number $q = q_0 + \mathrm{i}\, q_1$ and the algebraic norm $\|q\| = \sqrt{q_0^2 + q_1^2}$ can be expressed as[12,13]

$$\|xy\| = \|x\|\|y\| , \tag{52}$$

where we note the multiplicative composition operation

$$x \cdot y = (x_0 y_0 - x_1 y_1) + \mathrm{i}\,(x_0 y_1 + x_1 y_0) . \tag{53}$$

The same applies to the algebra of the real numbers.[13]

Consider now the algebraic identity[12]

$$\left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right)\left(y_0^2 + y_1^2 + y_2^2 + y_3^2\right) = \left(z_0^2 + z_1^2 + z_2^2 + z_3^2\right) , \tag{54}$$

with

$$\begin{aligned} z_0 &= x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 , \\ z_1 &= x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2 , \\ z_2 &= x_0 y_2 + x_2 y_0 + x_3 y_1 - x_1 y_3 , \\ z_3 &= x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1 . \end{aligned} \tag{55}$$

[fff] See, e.g. Ref. 64, pp. 20, 179 and 200. The spheres $S^X = \{S^1, S^3, S^7\}$ are the only spheres which admit globally smooth parallel frame fields via their associated division algebraic structure.[8,11,16,64]

Equation (54) can also be expressed in the form of (52) by introducing the unit quaternions $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, defining a quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ and the extended algebraic norm,[12,13,74] and where the algebraic multiplicative composition $\cdot$ of two quaternions is given by[12,14,74]

$$\begin{aligned} x \cdot y &= \big(x_0 + \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{k}x_3\big)\big(y_0 + \boldsymbol{i}y_1 + \boldsymbol{j}y_2 + \boldsymbol{k}y_3\big) \\ &= z_0 + \boldsymbol{i}z_1 + \boldsymbol{j}z_2 + \boldsymbol{k}z_3\,. \end{aligned} \tag{56}$$

This algebraic structure can then be generalized to the octonions $\mathbb{O}$, which is the last algebra with this structure.[12–14]

The above β structure constitutes purely algebraic structure. However, in the literature, a Euclidean metric is often attached to this algebraic structure. For instance, quaternion composition is often put in the form of (24) from which the quaternions $\mathbb{H}$ are designated as arriving within a metric space.[ggg] In this way the quaternions are appended to a metric space $\mathbb{R}^4$.[hhh]

However, the quaternions $\mathbb{H}$ (and the other division algebras) are fundamentally not a metric space, but an algebraic structure — a noncommutative unital division ring possessing two binary operations, that of multiplication $\cdot$ and addition $+$,[65,74] and the unit quaternions are a purely algebraic structure $(\mathbb{H}_u, \cdot)$ under the binary operation of multiplication.

We thus emphasize that the transport of smooth $\mathscr{A}$ structure contemplated herein neither contains nor postulates any metric structure, and CAM's $\beta \subseteq \mathscr{A}$ structure is a smooth, nonmetric algebraic structure. Furthermore, only when acting within a space with the Minkowski metric structure $\boldsymbol{\eta}$ can the algebraic composition of two quaternions (octonions) be put in the form of (24). As previously stated in Subsec. 5.1, the form of (24) only comes about in CAM via the $\mathscr{S}^n$ bundle operator's local action in $T_p\mathbb{M}$ (which is endowed with $\eta_{\mu\nu}$ thereby permitting this metric-based form).

9.4. *CAM compactification sequence*

The entire CAM compactification sequence in which transport of algebraic structure $S^7_{\mathscr{A}} \overset{\beta}{\hookrightarrow} S^7_\tau$ follows one-point compactification Θ and topological deformation α is

$$\mathbb{R}^7 \overset{c}{\dashrightarrow} \mathscr{X}^7 \overset{d}{\dashrightarrow} S^7_\tau \overset{t}{\dashrightarrow} S^7_{\mathscr{A}} \mapsto \mathscr{S}^7\,, \tag{57}$$

where $\overset{c}{\dashrightarrow}$ denotes the one-point compactification Θ procedure, $\overset{d}{\dashrightarrow}$ the topological deformation event α, $\overset{t}{\dashrightarrow}$ the transport of β structure $S^7_{\mathscr{A}} \overset{\beta}{\hookrightarrow} S^7_\tau$, and $\mapsto$ the operator bundle B_n postulate of Subsec. 5.4.

[ggg]See Ref. 28, Subsecs. 1.3 and 1.3.2, in which the author writes, "The 4D linear space $\mathbb{R}^4$, endowed with the quaternion product, is denoted $\mathbb{H}$."
[hhh]Defined in Ref. 28, Subsec. 1.2.1 as a metric space.

As we have the submanifold chain $S^0 \subset S^1 \subset S^3 \subset S^7$,[12,14,64] we thus have

$$S^1 \subset \mathscr{S}^7\,, \quad \mathscr{S}^3 \subset \mathscr{S}^7\,, \tag{58}$$

thereby permitting the $\mathscr{S}^1 \vee T\mathbb{M}$, $\mathscr{S}^3 \vee T\mathbb{M}$ and $(\mathscr{S}^3 \otimes \mathscr{S}^1) \vee T\mathbb{M}$ operator technologies as set forth herein as well as $\mathscr{S}^7$ technologies to be considered in subsequent work. Using (45) we can thus write

$$\mathcal{M}^{1,3} \times \mathscr{X}^7 \Rightarrow \mathbb{M} \vee_{\mathbb{K}} S^7\,, \tag{59}$$

where $\vee_{\mathbb{K}}$ indicates that it is only the parallelized spheres $S^n \to \mathscr{S}^n$ associated with the division algebras $\mathbb{K}$ which can and do take part in the wedge action between the manifolds.

9.5. *Conclusion*

We conclude that the CAM compactification postulate encompassed within (45)–(48) evidences various fundamental differences (both mathematical and physical) from the Klein compactification postulate which is generalized and utilized in string/M-theory.[23,50,51,60,61,76,77,iii] The two contrasting postulates can in their essence be simply put:

Klein compactification postulate

Higher metric dimensions compactify into a Planck-size space with intricate geometry.

CAM compactification postulate

Higher nonmetric dimensions compactify into a space homeomorphic to the topological 7-sphere.

Some basic contrasts follow from the two compactification scenarios:

(1) With the Klein postulate, the extra metric dimensions $\mathscr{X}^7$ of $\mathcal{M}^{1,10}$ cannot be measured because they are curled up into an experimentally essentially nonobservable Planck-size space.[23,36,50,51,60,61,76,77] With the CAM postulate, the extra dimensions $\mathscr{X}^7$ of $\mathcal{M}^{1,10}$ cannot be measured because they contain no metric by which to be measured. They form but a nonmetricized operator space — however its higher-dimensional existence is indicated by the very existence of the elementary-particle forces which it generates;

[iii] See, e.g. Ref. 76, Ch. 1, Fn. (99).

(2) Given the CAM postulate, every point p of space–time is occupied not by a Planck-size Calabi–Yau manifold $\mathscr{C}$ or a G_2 manifold derived from the compactification of a higher-dimensional metricized geometrical space,[23,60,61,77] but instead by an operator sphere fiber $\mathscr{S}^7$ generated from the one-point compactification of a higher-dimensional nonmetric topological manifold;

(3) Within CAM 11-dimensional unification there is no chirality issue such as occurred within the Klein regime,[36,51] since there is, again, no higher-dimensional metric tensor and no use of a higher-dimensional Dirac equation which would require a seven-dimensional chirality operator;

(4) CAM's metric-free extra dimensions do not inherit potential instability from the classical instability arguments on extra dimensions;[jjj]

(5) Within CAM unification the SM symmetry group is not obtained via its existence as the isometry group of a certain manifold as is done in the Kaluza–Klein regime,[51] since there is no higher-dimensional metric whose invariance need be maintained. Instead the (Lagrangian of the) SM symmetry group is generated by the differentio-topologically induced bundle operator space B_n itself acting locally on the differentio-geometric base space $(T\mathbb{M}, \eta)$ of $\mathbb{M}$, which is the local manifold (arrived at via the principle of equivalence[31,35,49,50,76]) of some global Riemannian manifold $(\mathbb{B}, g)$;[kkk]

(6) Although the CAM model's higher-dimensional topological manifold $\mathscr{X}^7$ does not have the refined geometric structure of string/M-theory's $\mathscr{X}^7$ manifold it nevertheless is far from being the devoid nonentity of an outdated absolute space.[73,75,lll] As Aldrovandi and Pereira noted, "The study of this primitive structure [i.e. a topological space] makes use of very simple concepts ... but the structure itself may be very involved and may leave an important (eventually dominate) imprint on the physical objects present in the space under consideration."[mmm] CAM's $\mathscr{X}^7$ plays such a dynamic and dominant part in the form and function of our relativistic reality which invites extensive investigation into its causes, its effects and any capabilities we might be able to discern and develop in order to potentially detect and affect it.

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[jjj]Reference 27, Subsec. 31.12.

[kkk]See Fn. (q), herein.

[lll]Recall from Fn. (ddd) that future study keeps open the possibility of other structures $\mathscr{X}^7 \simeq S^7_\tau$ which are not diffeomorphic to $S^7_{\mathscr{A}}$ yet may contain the division algebraic structure necessary for the purposes of operating on $T\mathbb{M}$ as given through (27).

[mmm]Reference 65, pp. 8–9.

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