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Exact Results in Five-Dimensional Gauge Theories

On Supersymmetry, Localization and Matrix Models

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Abstract

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Gauge theories are one of the corner stones of modern theoretical physics. They describe the nature of all fundamental interactions and have been applied in multiple branches of physics. The most challenging problem of gauge theories, which has not been solved yet, is their strong coupling dynamics. A class of gauge theories that admits simplifications allowing to deal with the strong coupling regime are supersymmetric ones. For example, recently proposed method of supersymmetric localization allows to reduce expectation values of supersymmetric observables, expressed through the path integral, to finite-dimensional matrix integral. The last one is usually easier to deal with compared to the original infinite-dimensional integral.

This thesis deals with the matrix models obtained from the localization of different 5D gauge theories. The focus of our study is N=1 super Yang-Mills theory with different matter content as well as N=1 Chern-Simons-Matter theory with adjoint hypermultiplets. Both theories are considered on the five-spheres. We make use of the saddle-point approximation of the matrix integrals, obtained from localization, to evaluate expectation values of different observables in these theories. This approximation corresponds to the large-N limit of the localized gauge theory.

We derive behavior for the free energy of 5D N=1* super Yang-Mills theory at strong coupling. This result is important in light of the relation between 5D theory and the world-volume theories of M5-branes, playing a significant role in string theory. We have also explored rich phase structure of 5D SU(N) N=1 super Yang-Mills theory coupled to massive matter in different representations of the gauge group. We have shown that in the case of the massive adjoint hypermultiplet theory undergoes infinite chain of the third order phase transitions while interpolating between weak and strong coupling in the decompactification limit.

Finally, we obtain several interesting results for 5D Chern-Simons theory, suggesting existence of the holographic duals to this theory. In particular, we derive behavior of the free energy of this theory, which reproduces the behavior of the free energy for 5D theories with known holographic duals.

Keywords: Supersymmetric localization, Matrix Models, Chern-Simons theory, Supersymmetric Field Theories, (2, 0) theories

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To my family and my friends

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. Kallen, J. A. Minahan, A. Nedelin and M. Zabzine, *N^3-behavior from 5D Yang-Mills theory,* JHEP **1210** (2012) 184.

II J. A. Minahan, A. Nedelin and M. Zabzine, *5D super Yang-Mills theory and the correspondence to AdS_7/CFT_6,* J. Phys. A **46** (2013) 355401.

III J. A. Minahan and A. Nedelin, *Phases of planar 5-dimensional supersymmetric Chern-Simons theory,* JHEP **12** (2014) 048.

IV A. Nedelin, *Phase transitions in 5D super Yang-Mills theory,* Manuscript.

Papers not included in the thesis

V M. N. Chernodub and A. S. Nedelin, *Phase diagram of chirally imbalanced QCD matter*, Phys. Rev. D **83**, 105008 (2011)

VI M. N. Chernodub and A. S. Nedelin, *Pipelike current-carrying vortices in two-component condensates*, Phys. Rev. D **81**, 125022 (2010)

Contents

1. Introduction

This thesis is devoted to exact results in supersymmetric gauge theories. The principle of *gauge symmetry* is one of the cornerstones of modern theoretical physics. Theories that possess gauge symmetry have been known for a very long time. To be more precise, the first gauge theory to come into play was *Maxwell's theory of electromagnetism* in the middle of 19th century. However, the importance of gauge symmetry was not recognized until 50 years later when developing quantum mechanics for particles in an external electromagnetic field [44]. This work in the early 20th century led to the first formulation of a $U(1)$ gauge symmetry and can be considered the foundation for quantum electrodynamics (QED) formulated in the 1950's. This led to the rapid development of gauge theories in the form that we know today. Very soon after the birth of QED Yang and Mills generalized it to the case of non-abelian gauge groups in order to describe interactions of protons and neutrons. Later it was realized that all interactions in nature are nothing else but gauge theories. This was first worked out for the case of unified electromagnetic and weak interactions in the papers written by Glashow, Salam and Weinberg in 1960's [48, 113, 104]. It appeared that both interactions can be unified into one *electreoweak* interaction described as an $SU(2)\times U(1)$ gauge theory. Finally, the discovery of *asymptotic freedom* [52, 94] led to the triumph of Quantum Chromodynamics (QCD), i.e. the $SU(3)$ gauge theory of the strong interactions. All of these gauge theories unified together form the *Standard Model* of particle physics, which describes all fundamental forces present in nature, except gravity. We also should notice that though gauge theories are usually thought of as fundamental theories, they can also emerge in other branches of physics such as condensed matter theory, where they also can play an important role [56].

However, just because the theory is formulated does not mean it is solved. It is well known how to deal with weakly interacting theories. In this case well established tools of perturbation theory can be successfully applied. However, sometimes theory has a large coupling constant, which makes perturbation theory impossible. In particular, this situation takes place in QCD which has weak coupling at very high energies, but strong coupling at the scale of $\Lambda_{\text{QCD}} \sim 220\text{MeV}$. This is the typical scale of, for example, mesons and it is known to have rich physics such as confinement and dynamical chiral symmetry breaking. Most of the phenomena that occurs on this scale were observed both experimentally and numerically, but still lack a theoretical explanation. Similar issues with strongly interacting theories appear in condensed matter physics, such as strongly correlated electron systems. One class of these systems is the holy grail of modern condensed matter physics, high-T_c superconducting cuprates. Thus the development of tools appropriate for a description of non-perturbative effects in gauge theories is a very important problem in modern theoretical physics, both from fundamental and applied points of view.

In the past twenty years there has been quite substantial progress in understanding the non-perturbative dynamics of certain gauge theories. This progress is intimately related to *string theory* which is considered to be the main candidate for a the theory unifying all interactions in nature. There are two concepts incorporated in modern string theory that are particularly related to this progress, *duality* and *supersymmetry*.

By using the term "duality" physicists usually mean that different systems have the same physics. There are many different dualities in string theory. However, here we consider only one of them, the AdS/CFT or *gauge/gravity* duality conjectured by Maldacena in [79]. This duality makes the correspondence between certain gauge theories living in d dimensions and string theories living in a background containing $(d+1)$ dimensional *Anti-de Sitter space*. Thus, performing calculations on one side of the duality we can obtain results for the other side. Surprisingly, the most interesting case of a strongly coupled gauge theory corresponds to the supergravity limit of a string theory. Supergravity is much easier to deal with then the full string theory. Thus the gauge/gravity duality gives us hope to obtain results in strongly coupled theories. Unfortunately, this method is limited to the class of theories admitting a duality. However, the AdS/CFT duality can at least give some hints on general mechanisms involved in the strong coupling dynamics of gauge theories. In recent years there was some noticeable progress in applying this powerful tool to QCD [40, 69, 24], condensed matter physics [86, 56] and even hydrodynamics [96].

However, this thesis deals more with a second concept - supersymmetry. Supersymmetry is simply symmetry relating fermionic and bosonic degrees of freedom. It was first developed independently in pioneering works by Gervais and Sakita [46], Golfand and Likhtman [49], Akulov and Volkov [112] and Wess and Zumino [114, 115]. Later in the 1980's the full power of supersymmetry was realized in string theory where it has elegantly solved both the problem of the tachyon excitation in the closed bosonic string spectrum and the introduction of matter particles into the spectrum. Supersymmetry can also cure some problems of the Standard Model, such as the hierarchy problem for example. Because of all these reasons supersymmetry seems to be natural and could exist in nature . It has not yet been found, but the search at the LHC is still going on.

But even though supersymmetry will is not found it is useful to study supersymmetric theories. The reason is that presence of the supersymmetry, as well as any other symmetry, puts extra constraints on the dynamics of the theory and leads to simplifications. Thus, supersymmetric gauge theories can be considered toy-models, providing a qualitative picture of gauge theories in general.

One way to simplify supersymmetric gauge theory calculations, *supersymmetric localization*, was introduced by Nekrasov in 2002 [91] and later reformulated by Pestun in 2007 [93]. In particular Pestun considered $\mathcal{N}=2$ super Yang-Mills theory on the four-sphere and realized that localization can be used to simplify the full quantum field theory path integral to a finite dimensional matrix integral. The latter is easier to work with then the original infinite-dimensional integral. This reduction allowed Pestun to prove a conjecture for supersymmetric Wilson loops put forward by Erickson Semenoff and Zarembo in [41]. Different aspects of the matrix integral obtained by Pestun were later considered in a series of works by Russo and Zarembo [23, 98, 103, 99, 100, 101, 5].

At the same time the localization method was also used to derive results for three-dimensional ABJM theory [1]. Localization of this theory was performed by Kapustin, Willet and Yaakov in [64]. Interestingly, the corresponding matrix model is solvable and was solved following papers by Drukker, Marino and Putrov [85, 84, 37, 38].

Finally, in 2012 localization was also performed for $\mathcal{N}=1$ supersymmetric Yang-Mills on S^5 by Källen and Zabzine in [63] and later was generalized to the theory with additional $\mathcal{N}=1$ hypermultiplet by Källen, Qiu and Zabzine in [62]. In this thesis we discuss details and solutions of the matrix model obtained after localizing different five-dimensional gauge theories.

This thesis is organized as follows. We start by reviewing some background material on different topics related to five-dimensional super Yang-Mills and its localization. In particular, in chapter 2 we review details of supersymmetry transformations and five-dimensional super Yang-Mills and Chern-Simons actions on S^5. We also go through the main motivation for a large part of the research presented in this thesis. Namely, we describe how five-dimensional maximally supersymmetric Yang-Mills theory is important for the description of the dynamics of $M5$-branes. Finally, we give a brief overview of the properties of five-dimensional Chern-Simons theory, which has many features that differ from three-dimensional Chern-Simons theory. In chapter 3 we discuss the main idea of localization and review particular calculations for the case of five dimensions. Chapter 4 reviews standard technique used to the matrix model. In chapter 5 we describe general properties of matrix models obtained after localization of five-dimensional Yang-Mills-Chern-Simons theory. In chapter 6 we discuss results of paper I. In particular we show how N^3 behavior of the free energy of five-dimensional $\mathcal{N}=1$ super Yang-Mills with an adjoint hypermultiplet can be found using the matrix model obtained from localization. We also discuss how our results support the duality between five-dimensional super Yang-Mills and six-dimensional $(2,0)$ theories conjectured in [77, 35]. In chapter 7 we introduce Chern-Simons terms in the matrix model and discuss the possible existence of holographic duals of this theory. We also prove the existence of phase transitions between a theory dominated by Yang-Mills or Chern-Simons terms. Chapter 8 describes the rich phase structure of $\mathscr{N} = 1$ super Yang-Mills with massive matter multiplets in the decompactification limit, i.e. when the radius of the five-sphere is taken to be infinitely large. Finally, in chapter 9 we briefly summarize all results presented in this thesis as well as possible future directions.

Part I:
Background Material

2. $5D$ Super Yang-Mills theory on S^5

In this chapter we give a brief overview of supersymmetric gauge theories in five dimensions. The chapter will start with the construction of $5D$ supersymmetric Yang-Mills theory with arbitrary gauge group on the five-sphere. Then we will briefly discuss the $6D$ $(2,0)$ theories and their relation to $5D$ $SU(N)$ super Yang-Mills. The final section of this chapter shows how Chern-Simons terms can be included in the five-dimensional theory.

2.1 Vector Multiplets

In this section we introduce supersymmetry transformations and the $5D$ *vector multiplet* action on the five-sphere S^5. In our notations, derivations and reasoning we closely follow [58] and [62]. The review presented here is brief, but more details about supersymmetry on curved five-manifolds can be found in [58, 63, 62, 66].

The $\mathscr{N}=1$ vector multiplet of $5D$ super Yang-Mills contains a vector field A_μ, with $\mu = 1,..,5$ being Euclidian spatial indices, a real scalar σ and a spinor λ^I satisfying the $SU(2)_R$ *Majorana condition*

$$(\psi_I^\alpha)^* = \varepsilon^{IJ} C_{\alpha\beta} \psi_J^\beta \,, \tag{2.1}$$

where $I,J = 1,2$ are $SU(2)_R$ indices and $\alpha,\beta = 1,..,4$ are spinor indices of $Spin(5) \backsimeq Sp(2)$. ε^{IJ} is the $SU(2)_R$ invariant antisymmetric tensor raising and lowering R-symmetry indices. The choice of signs in this tensor is defined by $\varepsilon^{12} = 1$. Finally, $C_{\alpha\beta}$ is the charge conjugation matrix.

For the off-shell formulation we also need to introduce the triplet of auxiliary scalar fields, D_{IJ}, that satisfies the following relations

$$(D_{IJ})^\dagger = D^{IJ} \equiv \varepsilon^{II'} \varepsilon^{JJ'} D_{I'J'} \,, \quad D_{[IJ]} = 0 \,. \tag{2.2}$$

In our conventions all fields in the vector multiplet are Hermitian matrices for a non-abelian gauge symmetry.

The supersymmetry transformations and corresponding invariant action for $5D$ super Yang-Mills are well known for the case of flat space $\mathbb{R}^5$, however in this thesis we will concentrate on the case of super Yang-Mills theory on the five-sphere S^5 constructed in [58]. In this case the supersymmetry transformations take the following form

$$\begin{aligned}
\delta_\xi A_\mu &= i\varepsilon^{IJ}\xi_I\Gamma_\mu\lambda_J\,, \\
\delta_\xi\sigma &= i\varepsilon^{IJ}\xi_I\lambda_J\,, \\
\delta_\xi\lambda_I &= -\frac{1}{2}\Gamma^{\mu\nu}\xi_I F_{\mu\nu} + \Gamma^\mu\xi_I D_\mu\sigma + \xi_J D_{KI}\varepsilon^{JK} + 2t_I{}^J\xi_J\sigma\,, \\
\delta_\xi D_{IJ} &= -i\xi_I\Gamma^\mu D_\mu\lambda_J + [\sigma,\xi_I\lambda_J] + it_I{}^K\xi_K\lambda_J + (I\leftrightarrow J)\,,
\end{aligned} \tag{2.3}$$

where ξ_I is the supersymmetry transformation parameter. We choose it to be a Grassmann even spinor that satisfies the Killing spinor equation on S^5

$$D_\mu\xi_I \equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma^{ab}\right) = \Gamma_\mu t_I{}^J\xi_J \equiv \Gamma_\mu\tilde{\xi}_I\,, \tag{2.4}$$

In the equations above the following gauge field strength and covariant derivatives were used,

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]\,, \\
D_\mu\sigma &= \partial_\mu\sigma - i\left[A_\mu,\sigma\right]\,, \\
D_\mu\lambda_I &= \partial_\mu\lambda_I + \frac{1}{4}\omega_\mu^{ab}\Gamma^{ab}\lambda_I - i\left[A_\mu,\lambda_I\right]\,.
\end{aligned} \tag{2.5}$$

Here ω_μ^{ab} is the spin connection, with latin indices $a,b,\ldots$ being vielbein indices. Γ^a denotes coordinate-independent gamma-matrices, while $\Gamma_\mu = e_\mu^a\Gamma^a$ corresponds to coordinate-dependent ones. Finally we use the following definition for the antisymmetrized products

$$\Gamma^{\mu_1\mu_2\ldots\mu_n} \equiv \frac{1}{n!}\sum_\sigma(-1)^{P(\sigma)}\Gamma^{\mu_{\sigma(1)}}\Gamma^{\mu_{\sigma(2)}}\ldots\Gamma^{\mu_{\sigma(n)}}\,, \tag{2.6}$$

where we sum over all possible permutations σ.

The matrix $t_I{}^J$ can be defined from the $SU(2)_R$ Majorana conditions imposed on the spinors ξ_I and $\tilde{\xi}_I$. We can choose $t_I{}^J$ to be equal to any linear combination of the Pauli matrices with imaginary coefficients. For convenience we choose

$$t_I{}^J = \frac{i}{2r}\sigma_3\,. \tag{2.7}$$

In order to obtain the form (2.3) of supersymmetry transformations one requires the commutator of two supersymmetry transformations δ_ξ and δ_η to result in the combination of a gauge transformation δ_G, a translation $\mathscr{L}$, a dilation δ_D, a Lorentz rotation δ_L and an R-symmetry rotation δ_R,

$$\left[\delta_\xi\,,\,\delta_\eta\right] = \mathscr{L}(-iv) + \delta_G\left(\gamma + iv^\mu A_\mu\right) + \delta_D(\rho) + \delta_L(\Theta^{ab}) + \delta_R(R_{IJ})\,, \tag{2.8}$$

with the transformation parameters of given by

$$
\begin{aligned}
v^{\mu} &= 2\varepsilon^{IJ}\xi_I\Gamma^{\mu}\eta_J \ , \\
\gamma &= -2i\varepsilon^{IJ}\xi_I\eta_J\sigma \ , \\
\rho &= -2i\varepsilon^{IJ}(\xi_I\tilde{\eta}_K - \eta_I\tilde{\xi}_K) \ , \\
R_{IJ} &= -3i(\xi_I\tilde{\eta}_J + \xi_J\tilde{\eta}_I - \eta_I\tilde{\xi}_J - \eta_J\tilde{\xi}_I) \ , \\
\Theta^{ab} &= -2i\varepsilon^{IJ}(\tilde{\xi}_I\Gamma^{ab}\eta_J - \tilde{\eta}_I\Gamma^{ab}\xi_J) \ .
\end{aligned} \tag{2.9}
$$

One can also observe rigid supersymmetry transformations in curved spacetime from supergravity following the recipe of [42]. In particular one needs to start with the $5D$ off-shell supergravity coupled to Yang-Mills theory [74, 75]. Then in order to obtain *rigid supersymmetry* one should first give expectation values to the scalar triplet of the Weyl multiplet t^{IJ} and the metric. This fixes the background on which we put the super Yang-Mills theory. In order for the background to be supersymmetric we should set the gravitino ψ^I_μ and the fermion χ^I from the Weyl multiplet together with their supersymmetric variations to zero. This procedure was worked out for $5D$ manifolds in [58, 54, 92] and in the case of the S^5 background led to the same supersymmetry transformations as in (2.3). However notice that the algorithm described above, in principle, can work for more general manifolds. In particular in [54] the authors used it to describe supersymmetric Yang-Mills on squashed spheres.

The Lagrangian density of $5D$ super Yang-Mills that is invariant under supersymmetry transformations (2.3) is [58]

$$
\begin{aligned}
\mathscr{L}_{vect.} = \frac{1}{g_{YM}^2}\mathrm{tr}\Big[&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu\sigma D^\mu\sigma - \frac{1}{2}D_{IJ}D^{IJ} + 2\sigma t^{IJ}D_{IJ} - \\
&10t^{IJ}t_{IJ}\sigma^2 + i\lambda_I\Gamma^\mu D_\mu\lambda^I - \lambda_I\left[\sigma,\lambda^I\right] - it^{IJ}\lambda_I\lambda_J\Big] \ .
\end{aligned} \tag{2.10}
$$

All equations in this section were written for the theory on S^5. However one can always send the radius of the sphere to infinity $(r \to \infty)$ in order to recover the usual well-known relations for flat space super Yang-Mills theory.

2.2 Hypermultiplets

In this section we consider *hypermultiplets* for $5D$ supersymmetric theories. The $\mathscr{N}=1$ hypermultiplet contains a doublet of complex scalars q^A_I, a fermion ψ^A and *auxiliary scalars* F^A_I. Here $I = 1,2$ is the R-symmetry index and $A = 1,\ldots,2r$ is the index enumerating the r hypermultiplets. These fields obey the following reality conditions

$$
\left(q^A_I\right)^* = \Omega_{AB}\varepsilon^{IJ}q^B_J,\ \left(\psi^{A\alpha}\right)^* = \Omega_{AB}C_{\alpha\beta}\psi^{B\beta},\ \left(F^A_I\right)^* = \Omega_{AB}\varepsilon^{IJ}F^B_J, \tag{2.11}
$$

where Ω_{AB} is the antisymmetric invariant tensor of the $Sp(r)$ *flavor symmetry*.

Unfortunately off-shell supersymmetry cannot be realized for hypermultiplets with a finite number of auxiliary fields. This is because one cannot find a supersymmetry transformation that closes on the algebra.

However for localization it is only necessary to find a supersymmetry transformation that squares to set of transformations in (2.8). Such a transformation has been found in [58] and has the following form

$$\begin{aligned}
\delta q_I^A &= -2i\xi_I\psi^A\,, \\
\delta\psi^A &= \varepsilon^{IJ}\Gamma^\mu\xi_I D_\mu q_J + i\varepsilon^{IJ}\xi_I\sigma q_J^A - 3t^{IJ}\xi_I q_J + \varepsilon^{I'J'}\hat{\xi}_{I'}F_{J'}\,, \\
\delta F_{I'}^A &= 2\hat{\xi}_{I'}\left(i\Gamma^\mu D_\mu\psi^A + \sigma\psi^A + \varepsilon^{KL}\lambda_K q_L^A\right)\,, \qquad (2.12)
\end{aligned}$$

where $\hat{\xi}_{I'}$ is the constant spinor satisfying the following relations

$$\varepsilon^{IJ}\xi_I\xi_J = \varepsilon^{I'J'}\hat{\xi}_{I'}\hat{\xi}_{J'}, \quad \xi_I\hat{\xi}_{J'} = 0, \quad \varepsilon^{IJ}\xi_I\Gamma^\mu\xi_J + \varepsilon^{I'J'}\hat{\xi}_{I'}\Gamma^\mu\hat{\xi}_{J'} = 0\,. \qquad (2.13)$$

The supersymmetry transformation (2.12) squares to the sum of a gauge transformation δ_G, a translation $\mathscr{L}$, a dilation δ_D, a Lorentz rotation δ_L, an R-symmetry rotation δ_R and additionally an $SU(2)'$ rotation $\delta_{R'}$,

$$\delta_\xi^2 = \mathscr{L}(-iv) + \delta_G\left(\gamma + iv^\mu A_\mu\right) + \delta_D(\rho) + \delta_L(\Theta^{ab}) + \delta_R(R_{IJ}) + \delta_R'(R'_{I'J'})\,. \qquad (2.14)$$

with

$$\begin{aligned}
v^\mu &= \varepsilon^{IJ}\xi_I\Gamma^\mu\xi_J\,, \\
\gamma &= -i\varepsilon^{IJ}\xi_I\xi_J\sigma\,, \\
R_{IJ} &= 3i\left(\varepsilon^{KL}\xi_K\xi_L\right)t_{IJ}\,, \\
\Theta^{ab} &= -2i\varepsilon^{IJ}\tilde{\xi}_I\Gamma^{ab}\xi_J\,, \\
R'_{I'J'} &= -2i\hat{\xi}_{I'}\Gamma^\mu D_\mu\hat{\xi}_{J'}\,. \qquad (2.15)
\end{aligned}$$

The Lagrangian invariant under the supersymmetry transformations (2.12) can be written in the following form

$$\begin{aligned}
\mathscr{L}_{hyper.} = {} & \varepsilon^{IJ}D_\mu\bar{q}_I D^\mu q_J - \varepsilon^{IJ}\bar{q}_I\sigma^2 q_J - 2i\bar{\psi}\Gamma^\mu D_\mu\psi^B - 2\bar{\psi}\sigma\psi \\
& -i\bar{q}_I D^{IJ}q_J - 4\varepsilon^{IJ}\bar{\psi}\lambda_I q_J + \frac{15}{2}t^{KL}t_{KL}\varepsilon^{IJ}\bar{q}_I q_J - \varepsilon^{I'J'}\bar{F}_{I'}F_{J'}\,, \qquad (2.16)
\end{aligned}$$

where we have introduced

$$\bar{q}_A = \Omega_{AB}q^B, \quad \bar{\psi}_A = \Omega_{AB}\psi^B, \quad \bar{F}_A = \Omega_{AB}F^B\,. \qquad (2.17)$$

The final ingredients to be introduced are the *masses of the hypermultiplets*. The easiest way to to do this is to introduce an auxiliary vector multiplet. Then the mass terms in the Lagrangian can be obtained from the vacuum expectation values of the fields in the multiplet

$$\langle\sigma\rangle = M\,, \quad \langle D_{IJ}\rangle = -2t_{IJ}\langle\sigma\rangle = -2t_{IJ}M\,, \quad \langle A_\mu\rangle = \langle\lambda\rangle = 0\,, \qquad (2.18)$$

where M is the hypermultiplet mass matrix. Using (2.16) the mass terms for the hypermultiplet Lagrangian can be written down in the following form

$$\mathscr{L}_{mass} = \bar{q}_I \left(-\varepsilon^{IJ} M^2 + 2it^{IJ} M\right) q_J - 2\bar{\psi} M \psi \,. \tag{2.19}$$

Notice, that if we introduce the hypermultiplet masses, the supersymmetry transformations (2.16) should be changed to include these masses.

One important issue to be discussed here, as it will be useful for us later on , is *supersymmetry enhancement*. In flat space $\mathscr{N}=1$ supersymmetry can be enhanced to $\mathscr{N}=2$ if one introduces massless hypermultiplet in the adjoint representation of the gauge group. However, the generalization of this result to the five-sphere is not obvious, since $5D$ super Yang-Mills is not a conformal theory and thus the flat space theory can not be mapped canonically onto the sphere.

The easiest way to find the parameters for which the enhancement takes place is to consider the mass terms in (2.19) together with similar terms for the scalars σ and q_I in (2.10) and (2.16)

$$-\frac{4}{r^2}\sigma^2 + (\frac{15}{4r^2} - M^2)\varepsilon^{IJ}\bar{q}_I q_J + 2iMt^{IJ}\bar{q}_I q_J \,. \tag{2.20}$$

Looking at these terms we can try to find a special point at which enlargement of the global symmetries takes place. The value of parameters will then correspond to supersymmetry enhancement. In particular, one can notice that there is a special point $M=\frac{1}{2r}$ (or $M=-\frac{1}{2r}$) where the terms in (2.20) become

$$-\frac{4}{r^2}\sigma^2 + \frac{3}{r^2}\bar{q}_1 q^1 + \frac{4}{r^2}\bar{q}_2 q^2 \,. \tag{2.21}$$

The expression above suggests an enlargement of the global symmetry to $SO(1,2)\times SO(2)$. Thus we can conclude that in order to obtain super Yang-Mills theory with 16 supersymmetries on the five-sphere of radius r one needs to add an adjoint hypermultiplet with the mass $M=\frac{1}{2r}$. Notice that in flat space the scalar in the vector multiplet is massless, so for enhancement of the supersymmetry the hypermultiplet also should have massless scalars. However on the sphere the situation is different. Supersymmetry requires the vector multiplet scalar to have a mass term which is different from the conformal mass term. In order to get enlargement of the supersymmetry one should compensate for this difference by introducing a mass M for the hypermultiplets.

The same result has been obtained in [67, 53] in a more strict way. In particular, in [67] the authors explicitly constructed $5D$ super Yang-Mills theory with 16 supercharges on S^5. This led them to the Lagrangian of the same form as (2.10), (2.16), (2.19) with the mass parameter, M, being equal to $\frac{1}{2r}$.

2.3 Correspondence to $6D$ $(2,0)$ theories

In this section we will discuss $6D$ $(2,0)$ *theories*, which motivated a major part of the studies presented in this thesis.

$6D$ theories with $(2,0)$ supersymmetry is one of the most interesting and puzzling one in Nahm's classification of superconformal theories [89]. These theories were originally defined in terms of Type IIB string theory compactified on a $K3$ manifold [119] and later appeared in the context of the world-volume theory of $M5$-branes [108, 120]. We will discuss this relation with $M5$-branes in a little bit more detail below.

The matter multiplet of $(2,0)$ theory consists of four chiral fermions, five real scalars and, instead of a vector field, contains a two-form field $B_{\mu\nu}$, that has a self-dual field strength

$$H \equiv dB = \star dB\,. \tag{2.22}$$

In the case of an $SU(N)$ gauge group these theories play an important role in the dynamics of $M5$-branes. It is well known that the dynamics of the D-branes is determined by the open strings ending on them. A similar picture takes place for $M5$-branes. However, instead of strings there are $M2$-branes ending on $M5$-branes and thus defining their dynamics. The solution of this kind has been constructed in the case of a single $M5$-brane and is known to result in the *self-dual string* living on the world-volume of the brane [59]. In the case of parallel $M5$-branes there should exist similar string states corresponding to the $M2$-branes stretched between the $M5$-branes. Not very much is known about these string states, but the $(2,0)$ theories are believed to arise from the dynamics of these strings.

Another thing that makes these theories interesting is the wide variety of its compactifications. One of the simplest and most famous examples is a T^2 compactification leading to $4D$ $\mathcal{N}=4$ super Yang-Mills. From this point of view we can find a simple explanation for *S-duality* in $D=4$ as a consequence of the symmetry under the exchange of the two radii of the torus. Other important results arising from $(2,0)$ compactifications are, for example, the AGT correspondence [45] or the recently developed $3d/3d$ correspondence [25].

However, though $(2,0)$ theories are extremely important, they are not very well understood due to several reasons. First of all, the presence of the self-dual field $B_{\mu\nu}$ makes the construction of a Lorentz-invariant Lagrangian tricky. In fact, there is no known Lagrangian description in the case of the non-Abelian case.

What also makes it difficult to study the $(2,0)$ theories is the absence of any free parameter. These theories are believed to be fixed points of the six-dimensional renormalization group, which forbids them to have any dimensionful parameters. At the same time the fixed points are isolated which means that dimensionless parameters are excluded as well. The absence of free parameters makes it impossible to study the perturbative regime, assuming there was a proper Lagrangian description of the theory.

The only useful tool for their study that exists at the moment is the duality between them and M-theory on the $AdS_7 \times S^4$ background conjectured in [79]. In the planar limit this reduces to supergravity on the same background, which makes the calculations doable. One of the most important results obtained for $(2,0)$ theories from their supergravity duals is the N^3 behavior of their free energy and conformal anomaly [57, 70].

In this thesis we will examine the validity of that by considering $(2,0)$ theory compactifications. In particular we will discuss an S^1 compactification which leads to $5D$ maximally supersymmetric Yang-Mills theory. Recently it was proposed [77, 35] that $5D$ maximally supersymmetric Yang-Mills captures all degrees of freedom of the six-dimensional theories. In order to support this statement the authors of [77] considered a system of two slightly separated $M5$-branes, corresponding to an $SU(2)$ gauge group. The idea was to consider the BPS spectrum of $5D$ super Yang-Mills and to compare it with the Kaluza-Klein spectrum of charged self-dual strings. Comparing the spectra it was found that the spectrum of $5D$ dyonic instanton particles [78] is in perfect agreement with the spectrum of the self-dual strings wrapped along the compactification direction x_5. At the same time the spectrum of self-dual strings stretched along the non-compact directions matches the spectrum of the $5D$ 't Hooft-Polyakov monopoles. This match leads to the following identification between the radius of compactification and the $5D$ coupling, [1]

$$R_6 = \frac{g_{YM}^2}{8\pi^2}\,. \tag{2.23}$$

To understand how this equation arises, we should first recall that dyonic instantons in five dimensions are particles, which have mass $M_{inst} = \frac{8\pi^2|q|}{g_{YM}^2}$, where q is the winding number of the instanton. It also carries a conserved charge

$$J_5^0 = -\frac{1}{4g_{YM}^2}\mathrm{tr}\left(\varepsilon^{ijkl}F_{ij}F_{kl}\right)\,, \tag{2.24}$$

where $i,j,k,l = 0,1,2,3$. Now the relation (2.23) can be seen in several ways. For example, one can identify the masses of the Kaluza-Klein modes $M_6 = n/R_6$ with the masses of the instanton particles. This will lead to the following equation

$$M_6 \equiv \frac{n}{R_6} = \frac{8\pi^2|q|}{g_{YM}^2} \equiv M_{inst}\,. \tag{2.25}$$

As both n and q are quantized we arrive at the desired relation (2.23).

[1] Notice that our identification (2.23) differs from the one proposed in [77]. This happens because our coefficient in front of Yang-Mills action (2.10) differs from the standard one by factor of two.

Alternatively one can identify the conserved current J_5^0 with the momentum P_5 in the compactified direction. This identification can be obtained either from the corresponding component of the stress energy tensor

$$T_{05} \sim H_0^{ij} H_{5ij} = \varepsilon^{0ijkl} F_{kl} F_{ij} \tag{2.26}$$

or by comparison of the supersymmetry algebra in five dimensions with the dimensionally reduced general six-dimensional supersymmetry algebra [77]. This identification will lead to the same relation (2.23) between the $5D$ coupling and radius of the compactification.

The BPS spectrum matching led the authors of [77, 35] to conjecture that $5D$ maximally supersymmetric Yang-Mills theory captures all degrees of freedom of $6D$ $(2,0)$, suggesting that these two theories are the same. However, there is also an important renormalization issue, which makes the picture more complicated. As can be seen from the action (2.10) the Yang-Mills coupling squared has the dimension of length, i.e. $\left[g_{YM}^2\right] = [r]$. Thus, simple power coupling suggest that the theory is nonrenormalizable. At the same time the $(2,0)$ theory is believed to be UV finite , suggesting that the $5D$ theory is also UV finite in order for the conjecture in [77, 35] to hold. However, we now know that $5D$ maximally supersymmetric Yang-Mills is not finite. This was shown in [15], where the authors computed the six-loop four-point correlation function in the planar limit showing that it is divergent. This means that the relation between the five and six dimensional theories is not as straightforward and we can not claim that these two theories are equivalent.

Even the $(2,0)$ theory is only a UV completion of $5D$ maximally supersymmetric Yang-Mills, the study of the latter can still give us clues about the $(2,0)$ theories. Moreover, due to the matching analysis we can propose that the identification of the two theories works at least for the supersymmetric observables.

2.4 Chern-Simons terms

Now we can consider one of the most interesting ways of extending $SU(N)$ super Yang-Mills theory. Since we are considering an odd-dimensional space-time, we can include *Chern-Simons term* in our action.

Let's start with the description of the non-supersymmetric version of the $5D$ Chern-Simons action. In general, in a space-time with dimension $(2n-1)$, the action of the Chern-Simons theory can be constructed using the *Chern-Simons form* Ω_{2n-1}, defined as

$$d\Omega_{2n-1} = \mathrm{Tr}\left[F^n\right], \tag{2.27}$$

where $F = dA - iA \wedge A$ is a gauge field strength two-form and its powers F^n are defined by wedge products. In three space-time dimensions this form gives rise to the famous $3D$ Chern-Simons action

$$S_{CS_3} = \frac{k}{4\pi}\int d^3x\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right), \tag{2.28}$$

where k is the *Chern-Simons level* and plays the role of the coupling. It can be shown that, in order for this action to be *gauge invariant*, k should take only integer values.

Three dimensional Chern-Simons theory is *topological*, meaning that the observables of this theory are defined only by the global properties of the space which the theory lives on. This theory, as well as it's supersymmetric extensions, is well studied from many points of view. For a review the interested reader is referred to the book [82] and references therein.

In the $5D$ Chern-Simons theory the action is given by

$$\begin{aligned} S_{CS_5} = \frac{k}{24\pi^2}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\,\mathrm{tr}\big(A_\mu\partial_\nu A_\rho\partial_\alpha A_\beta - \\ \frac{3i}{2}A_\mu A_\nu A_\rho\partial_\alpha A_\beta - \frac{3}{5}A_\mu A_\nu A_\rho A_\alpha A_\beta\Big). \end{aligned} \tag{2.29}$$

Here A_μ is the gauge field, with the gauge group taken to be $U(N)$[2]. Field A_μ transforms under the gauge transformation $g \in U(N)$ as

$$A_\mu \to g^{-1}A_\mu g - ig^{-1}\partial_\mu g\,. \tag{2.30}$$

Under this transformation S_{CS_5} gets an additional term $S_{CS_5} \to S_{CS_5} + \delta S$ given by

$$\begin{aligned} \delta S = k\frac{i}{240\pi^2}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\,\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\alpha g g^{-1}\partial_\beta g\right) + \\ \frac{k}{48\pi^2}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\partial_\nu\mathrm{tr}\left(\partial_\mu g g^{-1}A_\rho A_\alpha A_\beta\right) \end{aligned} \tag{2.31}$$

The last term in the expression above a total derivative and we take it to be zero, assuming that $\partial M = \varnothing$, where M is the manifold which our theory lives on. However the first term in (2.31) is much more interesting and requires more care. Let's rewrite it in the form $\delta S = 2\pi ikQ(g)$, where

$$Q(g) \equiv \frac{i}{120\pi}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\,\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\alpha g g^{-1}\partial_\beta g\right), \tag{2.32}$$

Let's now consider two gauge transformations $g(x)$ and $g'(x)$ that differs infinitesimally, i.e.

$$g'(x) - g(x) = i\varepsilon(x)g(x)\,, \tag{2.33}$$

[2]The only groups admitting Chern-Simons term in five dimensions are $U(N)$ and $SU(N)$. For definiteness we assume $U(N)$, however all arguments presented in this section can of course be extended to $SU(N)$

with small $\varepsilon(x)$. For these two gauge transformations we obtain the following difference between $Q(g')$ and $Q(g)$

$$
\begin{aligned}
&Q(g') - Q(g) = \\
&-\frac{1}{32\pi^2}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\alpha g g^{-1}\partial_\beta \varepsilon\right) = \\
&-\frac{1}{32\pi^2}\int d^5x\varepsilon^{\mu\nu\rho\alpha\beta}\mathrm{tr}\partial_\beta\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\alpha g g^{-1} \varepsilon\right) ,
\end{aligned}
\tag{2.34}
$$

which results in a total derivative, thus giving us zero. Hence, we conclude that for any two gauge transformations g' and g that can be continuously deformed into each other, we obtain $Q(g) = Q(g')$. Thus $Q(g)$ depends only on the *homotopy class* of $g(x)$, which is

$$
g : S^5 \to SU(N)\,. \tag{2.35}
$$

These mappings can be characterized by the values of the homotopy group [90]

$$
\pi_5\left(SU(N)\right) \cong \pi_5\left(U(N)\right) \cong \mathbb{Z}\,, \quad \text{for } N \geq 3\,. \tag{2.36}
$$

Moreover $Q(g)$ explicitly gives the wrapping number $n \equiv \pi_5\left(U(N)\right)$ of the map $g(x)$ [19]. Therefore $Q(g)$ takes only integer values. Returning to the Chern-Simons action this leads to

$$
\delta S_{CS_5} = 2\pi k n\,, \quad n \in \mathbb{Z}\,. \tag{2.37}
$$

Hence, we see that the Chern-Simons action is not invariant under the gauge transformations itself. In fact, for gauge transformations that can not be connected to the identity continuously, the action receives a *constant shift*. However, observables in the field theory are related to the *partition function*

$$
Z_{CS_5} \equiv \int \mathscr{D}A\; e^{-iS_{CS_5}}\,, \tag{2.38}
$$

rather than the action itself. Just as in the case of three-dimensional Chern-Simons theory [30, 29] gauge invariance of the partition function

$$
k \in \mathbb{Z}\,. \tag{2.39}
$$

This condition of level quantization plays an important role in Chern-Simons physics. In particular it protects the level k from renormalization as it cannot be varied continuously. On the other hand it is does not prevent the level from getting constant integer shifts from loop corrections.

Though $5D$ and $3D$ Chern-Simons theories share some interesting properties, such as quantization of the level for example, there are also some differences. The most striking is the presence of local degrees of freedom in the higher dimensional case. As we have mentioned above, in three dimensions

Chern-Simons theory is a topological theory, meaning that observables in this theory depend only on the global properties of the space the theory lives in. As a result, the theory should not have any propogating degrees of freedom. A more precise way to understand this is to consider the equations of motion for our theory. In the case of $3D$ Chern-Simons theory the variation of the action (2.28) leads to the equations of motion

$$F_{\mu\nu} = 0\,. \tag{2.40}$$

So in three dimensions the space of solutions is simply given by all flat connections modulo gauge transformations, leading to the absence of local degrees of freedom.

However, in the case of five (and, generally any higher odd number) dimensions the equations of motion become more complicated

$$\varepsilon^{\rho\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 0\,, \tag{2.41}$$

and their solutions admit local degrees of freedom. In particular, Hamiltonian analysis leads to the following number of degrees of freedom (d.o.f.) [9, 10]

$$\#\text{d.o.f.} = 2N - 2 - N\,, \tag{2.42}$$

where N is the rank of the $U(N)$ gauge group. As we see for any $N \geq 3$ there is a nonzero number. This is quite interesting result. Indeed action of the theory does not depend on the metrics and thus theory is topological. Usually, but not necessarily, topological theories don't have local degrees of freedom. This peculiar fact makes more detailed study of $5D$ Chern-Simons attractive.

The supersymmetric version of the 5D Chern-Simons was derived in [63], where the action was found to be

$$\begin{aligned} S_{SCS_5} = {} & S_{CS_5}(A - \sigma\kappa) - \\ & \frac{k}{8\pi^2} \int \operatorname{tr}\left(\Psi \wedge \Psi \wedge \kappa \wedge F(A - \sigma\kappa)\right)\,, \end{aligned} \tag{2.43}$$

where A is the gauge field one-form, F is the field strength two-form and Ψ is the fermionic one-form, $\Psi_\mu = -2\xi\Gamma_\mu\lambda$. Finally, κ is a *contact form*. This is the form that is defined[3] in such a way that $\kappa \wedge d\kappa \wedge d\kappa$ is non-vanishing everywhere. Hence, $\kappa \wedge d\kappa \wedge d\kappa$ can serve as the volume element on the five-manifold our theory is defined on.

There are several reasons why the inclusion of a Chern-Simons term can be interesting to study. First of all, pure $U(N)$ and $SU(N)$ Chern-Simons theories are superconformal fixed points in five dimensions [105, 60], which in general can be seen as infinite coupling limits of certain super Yang-Mills theories.

[3]Notice that contact form is not necessarily definable on arbitrary manifold. Differentiable manifolds admitting such forms are called *contact manifolds*.

In general, these fixed points can be divided into three classes: super Yang-Mills theories with the exceptional gauge groups, super Yang-Mills theory with $USp(N)$ gauge group and, finally, $SU(N)$ or $U(N)$ Chern-Simons theory.

Another reason to include Chern-Simons terms is that they can be generated by integrating out massive hypermultiplets in the super Yang-Mills theory [105, 60], provided that the hypermultiplets transform in complex representations of the gauge group. If we consider the masses of the hypermultiplets as the UV cut-off, this leads to the generation of the Chern-Simons term in the one-loop correction to the classical theory. Hence inclusion of this term can be important in some cases if we wish to have a complete description of the theory.

Finally Chern-Simons theory can be important in the relationship between $5D$ super Yang-Mills and $6D$ (2,0) theories. In particular, one can argue [18, 16, 17] that the $5D$ Chern-Simons term can be generated by the anomaly terms in the six-dimensional theory.

3. Supersymmetric Localization

In this chapter we discuss *supersymmetric localization*, which is the main tool used in this thesis for obtaining exact results in $5D$ super Yang-Mills and Chern-Simons theories. We start with a brief description of the main idea behind localization. After this we consider particular results of localization for the vector- and hypermultiplets of $5D$ super Yang-Mills theory. At the end of this chapter we summarize results of localiztion in five dimensions, which we use extensively in the following chapters of the thesis.

3.1 The basic idea of localization

A primary goal of any quantum field theory is to calculate expectation values for different observables. These observables can be local, such as correlation functions, or can be non-local operators, such as Wilson or 't Hooft loops. In order to calculate the expectation value of some observable in a quantum field theory one should evaluate the path integral

$$\langle O\rangle = \int \mathscr{D}\phi \, O e^{-S}\,, \tag{3.1}$$

where ϕ are the fields in the theory, O is an observable we wish to study and S is the action for the theory. Thus the observable expectation value is given by an infinite-dimensional path integral. These objects are quite hard to work with and are usually computed perturbatively using an expansion in the coupling constant. However, this approach is limited only to weak coupling and more general techniques are required to study field theories in different regimes. One such technique is supersymmetric localization.

The main idea behind localization is that in some situations integrals can equal their semiclassical approximations exactly. In the case of quantum field theory this idea is usually formulated as follows [83]. Let's assume there is a quantum field theory with action $S(\phi)$, where ϕ represents the fields of this theory. The action of the theory is also assumed to be invariant under some Grassmann-odd symmetry δ, i.e. $\delta S = 0$. Finally, we suppose that the symmetry transformation δ squares to some Grassmann-even symmetry $\mathscr{L}_B$ of the theory, i.e.

$$\delta^2 = \mathscr{L}_B\,, \quad \mathscr{L}_B S = 0\,. \tag{3.2}$$

In physical theories the natural choice for the Grassmann-odd symmetry is *supersymmetry* while the Grassmann-even symmetry is taken to be a combination of the Lorentz and gauge symmetries. Thus the localization method can work for supersymmetric gauge theories.

Now let's consider the following deformation of the expectation value (3.1) of an observable $O(\phi)$

$$O(t) = \int \mathscr{D}\phi\, O e^{-S-t\,\delta V}\,, \tag{3.3}$$

where V is an Grassmann-odd operator invariant under the $\mathscr{L}_B$ symmetry, i.e. $\delta^2 V = 0$. Now computing the derivative we can easily see that $O(t)$ doesn't depend on t provided that the observable O is also invariant under the symmetry transformation δ

$$\frac{d}{dt}\langle O\rangle = -\int \mathscr{D}\phi\, O\,\delta V\, e^{-S-t\,\delta V} = -\int \mathscr{D}\phi\,\delta\left(O V\, e^{-S-t\,\delta V}\right) = 0\,, \tag{3.4}$$

where in the last step we treat the integral as a total derivative. In order for this integral to be zero we need two conditions to be satisfied. First, the measure of the path integral should be invariant under the symmetry δ, or equivalently, δ should not be an anomalous symmetry. Second, this equation can be spoiled by the boundary terms, but in our case such terms do not appear.

Assuming these conditions are satisfied, then $\langle O\rangle = 0$ is constant in t. So instead of computing (3.1) we can compute (3.3) for any value of t. Now we notice that in the limit $t \to \infty$ we have a dramatic simplification. Provided that δV has a positive definite bosonic part δV_B, in this limit the integral (3.3) localizes to the submanifold of field space satisfying

$$\delta I|_{bos.} = 0\,, \tag{3.5}$$

which is called localization locus. In this case the expectation value of the observable can be rewritten as

$$\langle O\rangle = \int_{\text{Locus}} D\phi\, O(\phi) \exp\left(-t\delta V - S - \log Z_{1-\text{loop}} - t^{-1}\log Z_{2-\text{loop}} - \dots\right), \tag{3.6}$$

where integral is evaluated over the localization locus and $Z_{\text{n}-\text{loop}}$ stands for n-loop contribution of fields fluctuating around the locus into the partition function. As seen from this expression in the limit $t \to \infty$ higher loop terms are infinitely suppressed and one-loop determinant becomes exact. Notice that the full infinite dimensional path integral (3.3) is reduced to the integral over the localization locus. In many interesting cases the last one appears to be a finite dimensional integral, leading to the reformulation of the theory in terms of the matrix model.

The idea of reduction of the integration space came from mathematics [6, 14] and was first applied to physics in the context of supersymmteric quantum mechanics in the 1980's [116]. The original quantum field theory application of localization appeared in work on topological field theory (see for example [118, 117]). Later in the 2000's the method was extended to $\mathcal{N}=2$ four-dimensional super Yang-Mills theory in order to derive Seiberg-Witten solution [91]. Finally, in the late 2000's the method was formulated in the form we have presented it here for Lorentz invariant gauge theories possessing enough supersymmetry. This was done first in [93] for the case of $\mathcal{N}=2$ super Yang-Mills theory with adjoint hypermultiplet on S^4. Later these results were generalized to Chern-Simons-matter theories on S^3 in [64] and $\mathcal{N}=(2,2)$ gauge theories on S^2 [34]. Finally in [63, 62, 58] the method was extended to five-dimensional gauge theories on S^5 and later [95, 111] on more complicated five-manifolds.

In the following sections we briefly review results of the localization of $5D$ $\mathcal{N}=1$ super Yang-Mills theory on S^5 with arbitrary matter content and show which matrix model corresponds to this gauge theory. In our considerations we closely follow [58] and [62].

3.2 Vector Multiplet

Before describing details of localization for the action derived in Section 2.1 let's discuss the issue of positive definiteness of this action. As it can be seen from the kinetic terms for the scalar and auxiliary fields σ and D in the Lagrangian (2.10), in order for the Yang-Mills action to be positive definite the integration contour for both of these fields in the path integral should be rotated by i

$$\sigma \to -i\sigma\,, \quad D_{IJ} \to -iD_{IJ}\,. \tag{3.7}$$

After this short remark, we can localize the vector multiplet with the following choice of the function V [58]

$$\begin{aligned} V_{vec.} &= \int \mathrm{tr}\left[(\delta\lambda)^\dagger \lambda\right] \\ &= \int \mathrm{tr}\left[\frac{1}{2}\varepsilon^{IJ}\xi_I\Gamma^{\mu\nu}\lambda_J F_{\mu\nu} - i\varepsilon^{IJ}\xi_I\Gamma^\mu\lambda_J D_\mu\sigma - i\varepsilon^{IJ}\xi_K\lambda_J\left(D_{LI}+2\sigma t_{LI}\right)\varepsilon^{KL}\right]. \end{aligned} \tag{3.8}$$

It can be shown (for details see [58]) that this function indeed satisfies $\delta^2 V_{vec.} = 0$ and thus can be used for localization. $\delta V_{vec.}$ contains both bosonic terms and terms bilinear in fermion fields. Due to the condition (3.5) defining the localization locus we are interested only in the bosonic part which equals to

$$\begin{aligned} \delta V|_{bos.} = \int \mathrm{tr}\Bigg[&\frac{1}{4}\left(F_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu\rho\alpha\beta}v^\rho F^{\alpha\beta}\right)\left(F^{\mu\nu} - \frac{1}{2}\varepsilon^{\mu\nu\rho\alpha\beta}v_\rho F_{\alpha\beta}\right) \\ &+\frac{1}{2}\left(v^\mu F_{\mu\nu}\right)\left(v_\alpha F^{\alpha\nu}\right) + D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D_{IJ}+2\sigma t_{IJ}\right)\left(D^{IJ}+2\sigma t^{IJ}\right)\Bigg], \end{aligned} \tag{3.9}$$

where $v^\mu = \varepsilon^{IJ}\xi_I\Gamma^\mu\xi_J$ is the Reeb vector. Provided that σ and D are purely imaginary fields, the expression above is positive definite and we can write down the equations for the localization locus

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\alpha\beta}v^\rho F^{\alpha\beta}\,, \quad v^\mu F_{\mu\nu} = 0\,, \quad D_\mu\sigma = 0\,, \quad D_{IJ} + 2\sigma t_{IJ} = 0\,. \tag{3.10}$$

The first equation implies the second. The first equation describes so called contact instantons [55, 63]. These instantons live on the $\mathbb{CP}^2$ base of the Hopf fibration[1], while the second equation implies that the vector field A_μ is zero component along the S^1 fiber. In the following we concentrate on the sector with zero instanton number and will not discuss other sectors in detail. If we deal only with zero topological number we must consider only flat connections, so that fields A_μ vanish up to gauge transformations. This reduces the localization locus to

$$A_\mu = 0\,, \quad \sigma = \sigma_0 = const\,, \quad D_{IJ} = -2t_{IJ}\sigma_0\,. \tag{3.11}$$

The classical action of the vector multiplet on this localization locus then equals

$$S^0_{vec.} = \int d^5x\sqrt{g}t_{IJ}t^{IJ}\mathrm{tr}\left[\sigma_0^2\right] = \frac{8\pi^3 r}{g^2_{YM}}\mathrm{tr}\left(\phi^2\right) \tag{3.12}$$

where we have introduced a dimensionless scalar field $\phi \equiv \sigma r$.

The calculation of the one-loop determinant is less straightforward and can be performed in two ways. One way is to write down terms quadratic in the fields and then diagonalize the accompanying differential operator using a basis of spherical harmonics. Then the one-loop determinants can be expressed as products its eigenvalues. However, this method is uncomplicated only for the case of three-dimensional gauge theories [64, 82]. For higher dimensions this method becomes quite tedious.

Another way of calculating determinants was introduced for 4D theory in [93] and generalized to 5D in [63]. It uses an analogy with localization of finite-dimensional integrals in mathematics. This technique reduces the calculation of the one-loop determinant to an evaluation of the superdeterminant of the operator $i\mathscr{L}_v - i[\sigma,]$ with $\mathscr{L}_v$ being the Lie derivative along the Reeb vector. The superdeterminant can then be calculated using an index theorem. Though this method employs more sophisticated mathematical concepts it makes calculations much simpler.

One-loop determinants for five-dimensional theories on S^5 were evaluated in both ways. In particular it was first evaluated in [63, 62] using index theorems. Later the same calculations were performed using spherical harmonics in [66]. Both led to the same expression for the one-loop determinant

$$Z^{\text{vector}}_{1-\text{loop}}(\phi) = \prod_\mu \prod_{t\neq 0}\left(t - \langle i\phi,\beta\rangle\right)^{(1+\frac{3}{2}t+\frac{1}{2}t^2)}\,, \tag{3.13}$$

[1]Here we mean that five-sphere S^5 is considered as Hopf fibration over $\mathbb{CP}^2$.

where β are the roots of the Lie algebra of the corresponding gauge group.

The last result we can extract from the localization of the vector multiplet is the classical action of the supersymmetric Chern-Simons term (2.43). Notice that this term does not contain entries quadratic in the fields and thus will not contribute to the one-loop determinants. The only contribution from this term is the following classical action on the localization locus (3.11)

$$
\begin{aligned}
S^0_{SCS_5} = S_{CS_5}(-i\sigma_0\kappa) = \frac{k}{24\pi^2}\mathrm{tr}\left(\sigma_0^3\right)\int d^5x\kappa\wedge d\kappa\wedge d\kappa \\
= \frac{\pi k}{3}\mathrm{tr}\left(\phi^3\right),
\end{aligned}
\tag{3.14}
$$

where in the last step we have used the fact that $\kappa\wedge d\kappa\wedge d\kappa$ is a measure of the volume.

3.3 Hypermultiplet

Finally, we can consider localization of the hypermultiplet action corresponding to the Lagraingain (2.16). Here we again follow [58].

As can be seen from the hypermultiplet Lagrangian (2.16) and its massive terms (2.19), in order to have a positive definite action we should rotate the integration contour for the auxiliary fields $F_{I'}$ as we did in (3.7) for the fields of vector multiplet

$$
F_{I'} \to iF_{I'}\,, \quad M \to iM\,, \tag{3.15}
$$

where we have also rotated the masses of the hypers to imaginary values. This rotation can be understood directly from the way we have introduced these masses. Indeed, the mass terms (2.19) in the Lagrangian have been derived by introducing an extra vector multiplet and assigning it expectation values (2.18). Hence we should consider masses M on the same footing with the scalar σ of vector multiplet, which was rotated to imaginary axis in (3.7).

For localization of the hypermultiplet we choose the following regulator V

$$
V_{hyper} = \int (\delta\psi)^\dagger\,\psi\,. \tag{3.16}
$$

Then using the explicit expression (2.12) for the supersymmetry transformation of the ψ field we find that the bosonic part of δV_{hyper} is given by

$$
\begin{aligned}
\delta V_{hyper}\big|_{bos} = \int \frac{1}{2}\varepsilon^{IJ}\left(v^\mu D_\mu\bar{q}_I - 3t_I{}^K\bar{q}_K\right)\left(v^\nu D_\nu q_J - 3t_J{}^K q_K\right) \\
+\frac{1}{8}\varepsilon^{KL}\left(D^\mu\bar{q}_K - v^\mu\left(v^\nu D_\nu\bar{q}_K\right) + 2w^{\mu\nu}{}_K{}^I D_\nu\bar{q}_I\right) \\
\times\left(D_\mu q_L - v_\mu\left(v^\nu D_\nu q_L\right) + 2w_{\mu\nu I}{}^J D^\nu q_J\right) \\
+\frac{1}{2}\varepsilon^{IJ}\bar{q}_I(\sigma+m)^2 q_J + \frac{1}{2}\varepsilon^{I'J'}\bar{F}_{I'}F_{J'}\,,
\end{aligned}
\tag{3.17}
$$

where the Wick rotations (3.7) and (3.15) have been taken into account. We have also introduced the following notation $w^{\mu\nu}{}_{IJ} \equiv \xi_I \Gamma^{\mu\nu} \xi_J$. We then see that each term in this sum is positive definite as desired. Hence we arrive at the following equations for the localization locus

$$\begin{aligned} v^\mu D_\mu q_J - 3t_K{}^L q_L &= 0\,, \\ D_\mu q_K - v_\mu\,(v^\nu D_\nu q_K) + 2w_{\mu\nu K}{}^J D^\nu q_J &= 0\,, \\ (\sigma + M) q_J = 0\,,\ \ F_{J'} &= 0\,. \end{aligned} \tag{3.18}$$

These equations have simple solutions on the Coulomb branch, which is our main interest,

$$q_J = 0\,, \qquad F_{I'} = 0\,. \tag{3.19}$$

Since the hypermultiplet fields are trivial on the localization locus, they do not contribute to the classical action. The one-loop determinant was evaluated for this locus using index theorems in [62] and spherical harmonics in [66]. Both calculations resulted in the following expression

$$Z^{\text{hyper}}_{1-\text{loop}}(\phi) = \prod_\mu \prod_t \left(t - \langle i\phi, \mu \rangle - im + \frac{3}{2} \right)^{-(1+\frac{3}{2}t+\frac{1}{2}t^2)}\,, \tag{3.20}$$

where again $\phi = \sigma r$. We have also introduced a similar notation for the dimensionless mass parameter $m \equiv Mr$. Notice that in order for localization locus to work, only real masses $m \in \mathbb{R}$ should be considered. Finally, the μ in above expression stands for the weights of the representation R the hypermultiplet transforms under.

3.4 Summary of Localization

Let us very briefly summarize the localization of $5D$ super Yang-Mills theory on S^5. We started by considering the partition function with a Lagrangian given by the sum of (2.10), (2.16) and (2.19). Supersymmetric localization resulted in the locus

$$\sigma = \sigma_0 = const\,, \quad D_{IJ} = -2t_{IJ}\sigma_0\,, \quad A_\mu = 0\,, \quad q_J = 0\,, \quad F_{I'} = 0\,, \tag{3.21}$$

so that the only nonzero fields in the locus are the constant scalar and auxiliary fields. Hence, our path integral will be reduced to a finite-dimensional matrix integral over the locus σ_0,

$$Z = \int [d\phi]\, e^{-\frac{8\pi^3 r}{g^2_{YM}}\text{Tr}(\phi^2) - \frac{\pi k}{3}\text{Tr}(\phi^3)} Z^{\text{vect}}_{1-\text{loop}}(\phi) Z^{\text{hyper}}_{1-\text{loop}}(\phi) + \mathscr{O}\left(e^{-\frac{16\pi^3 r}{g^2_{YM}}} \right)\,, \tag{3.22}$$

where $\phi = r\sigma_0$ with r being the radius of the five-sphere. The one-loop contributions of the vector- and hypermultiplets are given by the infinite products (3.13) and (3.20). We have also introduced a dimensionless mass $m = rM$. Finally the last term which we do not write down explicitly, corresponds to the contributions from the nonzero instanton sectors. These terms are not considered in detail in this thesis due to reasons that will be discussed in the following chapters. The interested reader is referred to [66, 68] for further information. Notice that the partition function in (3.22) can be applied to a theory with arbitrary matter content, i.e. we can always consider different number of hypermultiplets in different representations. However, in this thesis we mainly focus on the case of one adjoint hypermultiplet with either a $U(N)$ or $SU(N)$ gauge group and only briefly discuss other possible cases.

4. Matrix Models Technique

In this chapter we briefly review basic methods commonly used to construct solutions of matrix models. The interested reader can find more advanced tools, details of calculations presented in this chapter, and numerous applications of matrix models in reviews [31, 32, 33, 81] and references therein.

4.1 The partition function of matrix model

The technique presented in this chapter was first introduced in [22] for the matrix models described by the partition function

$$Z = \int DMe^{-S[M]}\,, \tag{4.1}$$

where $S[M]$ is the action of the matrix model. Throughout this chapter we consider the action to have the following polynomial form

$$S[M] \equiv \frac{1}{g^2}V(M) = \frac{1}{g^2}\sum_{k=2} t_k \mathrm{Tr}\left(M^k\right)\,, \tag{4.2}$$

where M is $N\times N$ Hermitian matrix, $DM = \prod_{i=1}^{N} dM_{ii}\prod_{i<j} d\left(\mathrm{Re}M_{ij}\right) d\left(\mathrm{Im}M_{ij}\right)$ is a measure of the matrix integral and g^2 is the coupling constant.

The matrix model action (4.2) contains only traces of powers of M and thus possesses a "gauge" symmetry

$$M \to M' = U\,M\,U^{-1}\,, \tag{4.3}$$

where U is an $N\times N$ unitary matrix. Thus, the partition function (4.1) has redundant degrees of freedom (d.o.f.) which can be read off. In particular, by applying the unitary transformation (4.3) one can always diagonalize the matrix M by finding an appropriate unitary matrix U_0 such that

$$\Lambda \equiv \mathrm{diag}\left(\phi_1\,,\ldots,\,\phi_N\right) = U_0\,M\,U_0^{-1}\,. \tag{4.4}$$

The matrix Λ contains only N d.o.f. instead of the N^2 found in M. We can then "gauge fix" the matrix integral (4.1) in a manner similar to the Faddeev-Popov procedure in quantum field theory. To do so we should insert the identity written in the form

$$1 = \int D\Lambda DU\,\delta^{(N^2)}\left(U\,M\,U^{-1} - \Lambda\right)\Delta^2\left(\Lambda\right)\,, \tag{4.5}$$

into the partition function (4.1). Here $\Delta^2(\Lambda)$ is the Jacobian of the "gauge" transformation (4.3) and will be defined later. Then (4.1) becomes

$$Z = \mathrm{Vol}\,U(N) \int \prod_{i=1}^{N} d\phi_i \, \Delta^2(\{\phi\})\, e^{-\sum_i S(\phi_i)}\,, \tag{4.6}$$

where $\mathrm{Vol}\,U(N)$ is the volume of the group. Since the integral is $U(N)$ independent, $\mathrm{Vol}\,U(N)$ is an overall constant and can be omitted.

The Jacobian $\Delta^2(\phi)$ is found as follows. As the dominant contribution to the integral in (4.5) comes from matrices U close to the matrix U_0 diagonalizing M, one can change integration variables to $U = (1+T)U_0$, where T is an antihermitian matrix with small entries. After some manipulations it can be shown that the δ-function takes the form

$$\begin{aligned}\delta^{(N^2)}\left(UMU^{-1} - \Lambda\right) &= \delta^{(N^2)}\left(\Lambda' - \Lambda + \left[T, \Lambda'\right]\right) \\ &= \delta^{(N)}\left(\Lambda - \Lambda'\right)\delta^{(N^2-N)}\left(T_{ij}\left(\phi_i' - \phi_j'\right)\right)\,,\end{aligned} \tag{4.7}$$

where in the last step we used that $\Lambda - \Lambda'$ is diagonal and $[T, \Lambda']$ has only non-diagonal terms. We then define the Jacobian Δ^2 using the expressions (4.5) and (4.7)

$$\begin{aligned}1 &= \int D\Lambda DT\, \Delta^2(\Lambda)\, \delta^{(N)}\left(\Lambda - \Lambda'\right)\delta^{(N^2-N)}\left(T_{ij}\left(\phi_i' - \phi_j'\right)\right) \\ &= \Delta^2(\phi') \int \prod_{i<j} d\left(\mathrm{Re}T_{ij}\right) d\left(\mathrm{Im}T_{ij}\right) \delta^{(N^2-N)}\left(T_{ij}(\phi_j' - \phi_i')\right)\,,\end{aligned} \tag{4.8}$$

which finally leads to

$$\Delta(\phi) = \prod_{i<j}(\phi_j - \phi_i) = \det \phi_i^{j-1}\,. \tag{4.9}$$

This expression is often referred to as the Vandermonde determinant. Finally, we end up with the following simple form for the partition function

$$\begin{aligned}Z &= \int \prod_i d\phi_i e^{-S_{eff}(\phi)}\,, \\ S_{eff.}(\phi) &= \frac{1}{g^2}\sum_i V(\phi_i) - 2\sum_{i<j}\log\left(\phi_i - \phi_j\right)\,, \\ V(\phi_i) &= \sum_i t_k \phi_i^k\,.\end{aligned} \tag{4.10}$$

In general, it is hard to work out this matrix integral. Instead of doing it exactly we will restrict our attention to the limit

$$N \to \infty\,, \qquad \lambda = g^2 N \to \text{fixed}\,. \tag{4.11}$$

Notice that the contribution from each sum in the effective action is of order $O(N)$. Hence the whole effective action is of order $O(N^2)$ in the limit (4.11). Thus the partition function can be approximated by the value of the integrand at the saddle point, which extremizes the effective action

$$\frac{\partial S_{eff}}{\partial \phi_k} = \frac{1}{g^2} V'(\phi_k) - 2\sum_{j\neq i} \frac{1}{\phi_k - \phi_j} = 0\,, \tag{4.12}$$

The limit (4.11) is called the 't Hooft or planar limit. It was first introduced by 't Hooft in the context of QCD with a large number of colors [109]. An important property of this limit is that all physical observables (for example, partition functions) can be expanded in series in $1/N$. Each term in this expansion corresponds to a sum of diagrams with the same topology. There are also many simplifications happening in this limit so that some rough predictions about complicated theories like QCD can be made [80]. In the end of the 1990's, planar limit has become extremely important again due to the AdS/CFT duality [79, 2]. This duality makes a correspondence between a certain class of gauge theories in Minkowski flat space-time on one side and string theory on Anti-de Sitter (AdS) space on the other. The full quantum string theory on the AdS background is not fully solved. However, in the 't Hooft limit the picture simplifies a lot because then only classical string theory (no string loops), which is much easier to work with, is needed to describe the gauge theory.

Turning back to the matrix model saddle-point equation (4.12), we can say that it describes the genus-zero contribution to the free energy. In order to obtain higher-genus corrections a complicated technique such as, for example, orthogonal polynomials should be applied [33]. However here we restrict ourselves to the saddle-point technique.

As we deal with large matrices, it is natural to introduce a continuous eigenvalue density function

$$\rho(\phi) \equiv \frac{1}{N}\sum_i \delta(\phi - \phi_i)\,, \tag{4.13}$$

where the ϕ_i's solve (4.12). We assume that the eigenvalue density has finite support consisting of one or more intervals. This density $\rho(\phi)$ is subject to the normalization condition, that can be easily obtained from its definition

$$\int d\phi \rho(\phi) = 1\,, \tag{4.14}$$

where the integral is taken over all intervals $\mathscr{C}_i$ over the density support. The same is presumed for all follow up derivations unless a different integration domain is specified. Using the eigenvalue density we can rewrite the saddle-point equation (4.12) as a singular integral equation with Cauchy kernel

$$\frac{1}{2g^2} V'(\phi) = \fint d\phi' \frac{\rho(\phi')}{\phi - \phi'}\,, \tag{4.15}$$

Before we start solving this equation let us make one remark. Notice that the discrete equations of motion (4.12) can be considered as an equilibrium condition for a system of N particles on a line repelling each other with a logarithmic potential and placed in an external central potential $V(\phi)$. Then we should expect these eigenvalues to condense around the extremal points of this potential V. In particular, if the potential has k extremal points we can expect the eigenvalues to be situated on disjoint intervals $\mathscr{C}_i\,, \quad i=1,..,k$. In the following two sections we will consider cases of single- and many-cut solutions for a generic potential together with some simple examples of particular potentials.

4.2 One-cut solution

To start, let's assume that the eigenvalues are all located on a single interval $\mathscr{C}=[a,b]$. In the following discussion we will refer to this configuration as a *one-cut solution*. The standard way to solve it is to introduce the resolvent defined in the following way

$$\omega(\phi) \equiv \int_{\mathbb{R}} d\psi\, \rho(\psi) \frac{1}{\phi-\psi}\,. \tag{4.16}$$

There are several properties of the resolvent that can help us define it. First, notice that the resolvent is an analytic function in the entire ϕ complex plane except along the interval $\mathscr{C}$. Across this interval the resolvent jumps by an amount that is easily calculated using the Sokhotskyi-Plemelj formula

$$\omega(\phi \pm i0) = \int d\psi \frac{\rho(\psi)}{\phi-\psi \pm i0} = \fint d\psi \frac{\rho(\psi)}{\phi-\psi} \mp i\pi\rho(\psi)\,, \tag{4.17}$$

so that we obtain the following relations for the resolvent between the two sides of the cut along $\mathscr{C}$

$$\omega(\phi+i0)+\omega(\phi-i0) = \frac{1}{\lambda}V'(\phi)\,, \tag{4.18}$$

$$\omega(\phi+i0)-\omega(\phi-i0) = -2\pi i\rho(\phi)\,, \tag{4.19}$$

Another condition that can help us fix the resolvent comes from its asymptotic behavior which follows from the normalization condition (4.14)

$$\omega(\phi) \to \frac{1}{\phi} \quad \text{when} \quad \phi \to \infty\,. \tag{4.20}$$

Due to (4.18) and (4.19) it is natural to assume the following form for the resolvent

$$\omega(\phi) = \frac{1}{2\lambda}\left(V'(\phi) - P(\phi)\sqrt{(\phi-a)(\phi-b)}\right)\,, \tag{4.21}$$

resulting in a very simple expression for the eigenvalue density

$$\rho(\phi) = \frac{1}{2\pi\lambda} P(\phi) \sqrt{(b-\phi)(\phi-a)} \,. \tag{4.22}$$

In this expressions we have introduced the polynomial $P(\phi)$. If the central potential $V(\phi)$ is a polynomial of degree k, then $P(\phi)$ has degree $k-2$, i.e. there are $k-1$ coefficients that should be determined in this polynomial. Together with the two positions a and b of the cut endpoints we have $k+1$ coefficients to determine. At this point the asymptotic behavior (4.20) of the resolvent should be applied. In particular, we want all terms of the expansion at $\phi \to \infty$ up to the $1/\phi$ term to be canceled. This will give us all $k+1$ relations to determine all unknown coefficients.

Example: Quartic Matrix Model. Now we can consider a particular example of the *quartic matrix model* with potential

$$V(\phi) = \frac{1}{2} m\phi^2 + \frac{1}{4}\lambda\phi^4 \,, \tag{4.23}$$

where we assume that m can be both positive and negative while coupling the λ is assumed to be positive $\lambda > 0$. The corresponding resolvent, according to (4.21), then takes the form

$$\omega(\phi) = \frac{1}{2} m\phi + \frac{1}{2}\lambda\,\phi^3 - (c_2\phi^2 + c_1\phi + c_0)\sqrt{\phi^2 - a^2} \,. \tag{4.24}$$

In order to have the appropriate asymptotic behavior we fix the coefficients and the endpoints of the cut to

$$\begin{aligned} c_0 &= \frac{1}{4}\lambda a^2 + \frac{1}{2} m \,,\; c_1 = 0 \,,\; c_2 = \frac{\lambda}{2} \,, \\ a^2 &= \frac{2}{3\lambda}\left(\sqrt{m^2 + 12\lambda} - m\right) . \end{aligned} \tag{4.25}$$

Finally, eigenvalue density for the one-cut solution can be obtained from the resolvent (4.24) using (4.19)

$$\rho(\phi) = \frac{\lambda}{2\pi}\left(\phi^2 + \frac{1}{2}a^2 + \frac{m}{\lambda}\right)\sqrt{a^2 - \phi^2} \,. \tag{4.26}$$

Notice that this solution has a local minimum around the origin, which can be seen in Fig.4.1(a). If we continue m down to negative values then at some point when $m = -2\sqrt{\lambda}$, the minimum touches the origin (see Fig.4.1(c)). Decreasing m further it becomes impossible to maintain a single-cut solution.

4.3 Many-cuts solution

Now we can consider a more general solution with n cuts, where n is less than or equal to the number of critical points of $V(\phi)$. The eigenvalue density support then consists of a union of n disjoint intervals $\mathscr{C}_i = [a_i, b_i]\,,\; i = 1,..,n$.

To find this n-cut solution one introduces a resolvent the same way as for the one-cut solution in (4.16). This resolvent will have discontinuities along every cut in the eigenvalue density support, which results in the same boundary conditions (4.18) and (4.19) as before. The asymptotic behavior (4.20) of the resolvent is preserved as well. Then the natural way to generalize the ansatz (4.21) is

$$\omega(\phi) = \frac{1}{2\lambda}\left(V'(\phi) - P(\phi)\sqrt{\prod_{i=1}^{n}(\phi - a_i)(\phi - b_i)}\right), \tag{4.27}$$

where $P(\phi)$ is a polynomial of degree $k-n-1$, where n is the number of cuts. So in order to completely determine the coefficients of this polynomial together with the $2n$ positions of the branch points we need $k+n$ conditions. However, the asymptotic behavior (4.20) of the resolvent $\omega(\phi)$ fixes only $k+1$ coefficients, leaving $n-1$ more coefficients that should be determined from some other conditions. These conditions come from fixing the filling fractions f_i defined as follows

$$f_i \equiv \frac{N_i}{N} = \int_{\mathscr{C}_i} d\phi\,\rho(\phi), \quad i = 1,..,n. \tag{4.28}$$

The filling fractions show how many eigenvalues belong to each interval in the support. We note that filling fractions are often difficult to calculate as the integrals in (4.28) typically result in elliptic functions. However, symmetries can help find a solution as we show next for the example of a quartic potential.

Example: A Quartic Matrix Model. Referring back to the quartic matrix model in (4.23), we note that for $m<0$ there are two critical points for the potential at $\phi = \pm\sqrt{-m/\lambda}$. Thus we can look for a two-cut solution. In particular, according to (4.27) we expect a resolvent of the form

$$\begin{aligned}\omega(\phi) = \frac{1}{2}m\phi + \frac{1}{2}\lambda\,\phi^3 - (c_1|\phi| + c_0)& \\ \times\sqrt{(\phi^2 - a_1)(\phi - a_2)(\phi - b_1)(\phi - b_2)}&.\end{aligned} \tag{4.29}$$

In order to avoid problems determining the filling fractions we assume the solution is symmetric w.r.t. $\phi \to -\phi$, which is natural due to the symmetry properties of the potential. Then the two cuts are symmetric with respect to the origin, so that $a_2 = -a_1 = -a$ and $b_2 = -b_1 = -b$ and the asymptotic behavior of the resolvent will completely determine coefficients c_0 and c_1 as well as the branch point positions

$$c_0 = 0,\ c_1 = \frac{\lambda}{2},\ a^2 = \frac{1}{\sqrt{\lambda}}\left(2 - \frac{m}{\sqrt{\lambda}}\right),\ b^2 = \frac{1}{\sqrt{\lambda}}\left(-2 - \frac{m}{\sqrt{\lambda}}\right). \tag{4.30}$$

Finally, the eigenvalue density is

$$\rho(\phi) = \frac{\lambda}{2\pi}|\phi|\sqrt{(a^2 - \phi^2)(\phi^2 - b^2)}. \tag{4.31}$$

This solution is valid only when $b^2 > 0$ or equivalently $m < -2\sqrt{\lambda}$. At the point $m = -2\sqrt{\lambda}$ the endpoints b and $-b$ of the two cuts coalesce and we obtain the one-cut solution in (4.26).

4.4 Phase Transitions

Another subject we consider in this thesis is phase transitions that take place in the planar limit of matrix models. These phase transitions have been obtained in different contexts: in the triangulation analysis of $2D$ gravity [27, 3, 65, 20], in lattice gauge theory [51] and in $2D$ Yang-Mills theory [97]. Recently similar phase transitions were found in matrix models obtained by localizing different supersymmetric gauge theories. We will return to a detailed analysis of these transitions later.

These phase transitions have many common properties. First, they appear when the theory crosses between solutions with a different number of cuts, or equivalently when new branch points emerge or coalesce with each other. Second, the phase transitions are generally third order. This is quite unusual, since ordinary phase transitions in physical systems are usually first order, second order or crossover transitions. The softening observed in matrix model transitions is related to the effects of the large-N limit.

The best way to understand these properties is to consider a particular example of such a phase transition. One of the simplest models where a third order phase transition takes place is matrix model with a quartic potential considered above.

Indeed we have seen that one can start with positive parameter m where one-cut solution (4.26) is valid. If m is decreased down to negative values, the central potential of the matrix model becomes a double-well potential which becomes deeper and deeper as m becomes more and more negative. At some point the potential will become deep enough to repel eigenvalues from the origin and separate the one cut into two resulting in the two-cut solution in (4.31). The evolution of this solution is shown on the Fig.4.1, with the critical point at $m = m_c = -2\sqrt{\lambda}$.

In order to show that the transition between the one- and two-cut solutions is indeed a third order phase transition we evaluate the free energy of the model. By definition the free energy is given by

$$F \equiv -\log Z = \frac{N^2}{\lambda} \int d\phi \rho(\phi) V(\phi) - \frac{N^2}{2} \int \int d\phi d\phi' \log\left(\phi - \phi'\right)^2 , \quad (4.32)$$

where we have used the continuous limit of the matrix model partition function (4.10). If the eigenvalue distribution $\rho(\phi)$ is known it is straightforward to evaluate the integrals in (4.32) using standard tools of complex analysis.

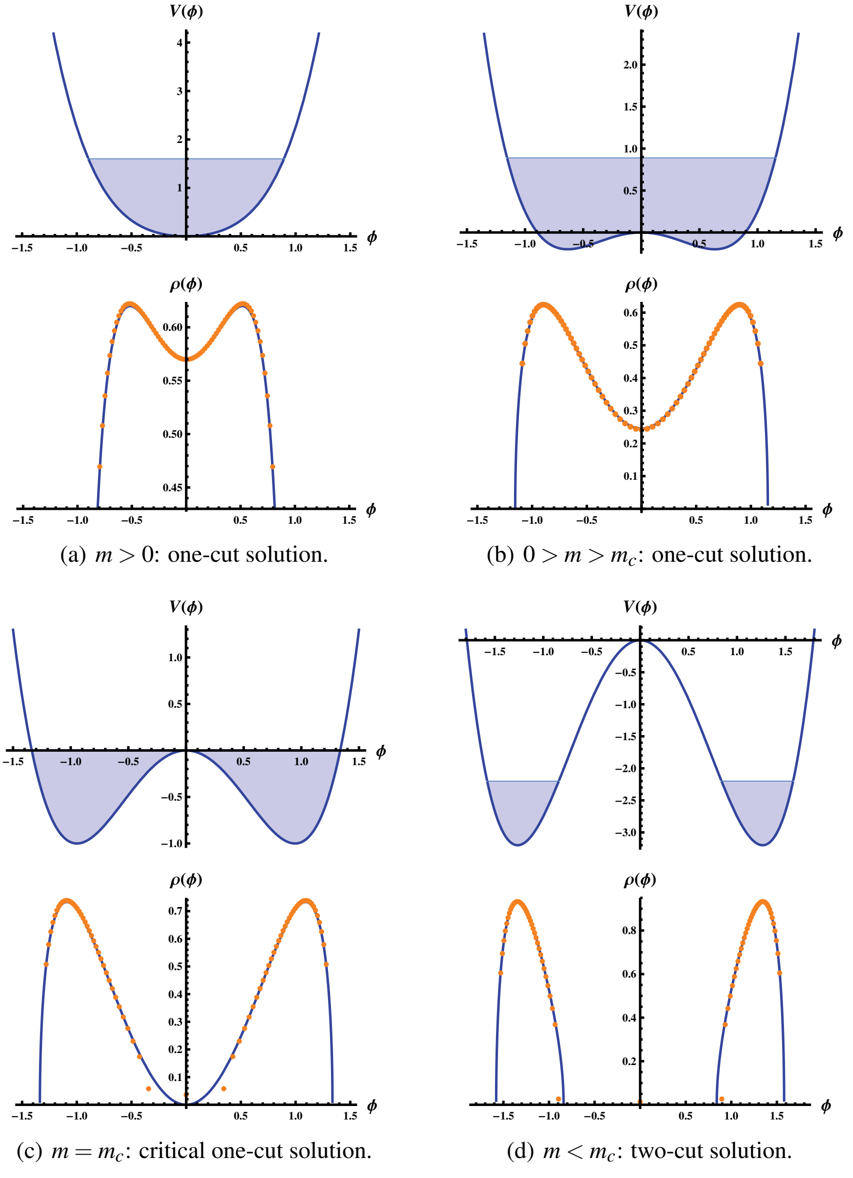


Figure 4.1. Solutions for the matrix model with quartic potential for $\lambda = 5$ and various values of m, corresponding to different phases. Critical point is at $m_c = -2\sqrt{\lambda} \approx -4.47$

In the case of the quartic potential (4.23) this calculation leads to the following free energy for the one-cut solution (4.26)

$$F^{(1)} = \frac{3}{8} - \frac{1}{384} a^2 m \left(a^2 m - 40\right) - \log \frac{a^2}{4}, \tag{4.33}$$

where a is the branch point position in (4.25). The same calculation for the two-cut solution (4.31) results in

$$F^{(2)} = \frac{3}{8} - \frac{m^2}{4\lambda} + \frac{1}{4}\log\lambda\,. \tag{4.34}$$

Now in order to determine the order of the phase transition we should evaluate the derivatives of the free energies with respect to the parameters of theory above and below the transition point. If the n^{th} derivative of the free energy has a discontinuity at the critical point, then the phase transition is of order n. In particular for the case of the matrix model with quartic potential one finds

$$\begin{aligned}
&\partial_m\left(F^{(2)} - F^{(1)}\right)\Big|_{m=m_c} = \partial_m^2\left(F^{(2)} - F^{(1)}\right)\Big|_{m=m_c} = 0\,,\\
&\partial_m^3\left(F^{(2)} - F^{(1)}\right)\Big|_{m=m_c} = -\frac{1}{4\lambda^{3/2}}\,,
\end{aligned} \tag{4.35}$$

implying that the phase transition is *third order*.

At this point we have finished our review of matrix models methods and move on to a particular model (3.22) obtained localizing of 5D super Yang-Mills theory.

Part II: Developments

5. General properties of the matrix model

In this chapter we determine general properties of the matrix model (3.22) which results from localizing of $5D$ super Yang-Mills. In particular before we start solving the matrix model it is instructive to discuss such issues as convergence of the partition function, renormalization of couplings and the decompactification limit of the matrix model. This chapter summarizes the results of paper II.

Suppose we consider $5D$ super Yang-Mills with gauge group G and hypermultiplet of mass M in representation R of this group. Then, as we have seen in chapter 3, localization reduces the full path integral to the finite-dimensional matrix integral, which we also refer to as the partition function of the matrix model

$$Z = \int [d\phi]\, e^{-\frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2) - \frac{\pi k}{3}\mathrm{Tr}(\phi^3)} Z^{\mathrm{vect}}_{1-\mathrm{loop}}(\phi) Z^{\mathrm{hyper}}_{1-\mathrm{loop}}(\phi) + \mathscr{O}\left(e^{-\frac{16\pi^3 r}{g_{YM}^2}}\right) , \quad (5.1)$$

where the one-loop contributions are given by the infinite products

$$Z^{\mathrm{vect}}_{1-\mathrm{loop}}(\phi) = \prod_{\beta}\prod_{t\neq 0}(t - \langle \beta, i\phi\rangle)^{(1+\frac{3}{2}t+\frac{1}{2}t^2)} , \quad (5.2)$$

and

$$Z^{\mathrm{hyper}}_{1-\mathrm{loop}}(\phi) = \prod_{\mu}\prod_{t}\left(t - \langle i\phi, \mu\rangle - im + \frac{3}{2}\right)^{-(1+\frac{3}{2}t+\frac{1}{2}t^2)} . \quad (5.3)$$

Here $m = Mr$ is the dimensionless mass of the hypermultiplet, β are the roots of G and μ are the weights of representation R. The partition function (5.1) also depends on the Yang-Mills coupling g_{YM}, the Chern-Simons level k and radius r of the five-sphere.

As previously mentioned, we are interested in the planar limit of the matrix model (5.1) due to both technical reasons and its physical significance. In this limit the last term, which corresponds to the contribution of the instantons of the gauge theory, is suppressed. Indeed, the argument of the exponent equals $-N/\lambda$ up to a constant of order one, which is approaches infinity in the limit of infinite N and fixed λ. Hence, from now on we omit all instanton contributions.

5.1 Renormalization of coupling constants

Let us start with the regularization of the partition function (5.1). It contains one-loop contributions (5.2) and (5.3) which are both equivalent to the divergent infinite product

$$P \sim \prod_{t=1}^{\infty} (t+x)^{(1+\frac{3}{2}t+\frac{1}{2}t^2)} \, (t-x)^{(1-\frac{3}{2}t+\frac{1}{2}t^2)} \ . \tag{5.4}$$

where the divergent part is given by the expression

$$\log P = \sum_{t=1}^{\infty} \left(-3x - \frac{x^2}{2} \right) + \text{convergent part} . \tag{5.5}$$

There is also an x-independent divergent part which we omitted. The simplest way to regularize P is to cut off the sum at $t_0 = \pi \Lambda_0 r$, where Λ_0 plays the role of the UV cut-off in our theory. Then the divergent part of each product contributes $\log \mathscr{P} \sim -\pi \Lambda_0 r \left(\frac{x^2}{2} + 3x \right)$, resulting in the regularized partition function

$$\begin{aligned} \log(Z^{\text{vect}}_{1-\text{loop}}(\phi) Z^{\text{hyper}}_{1-\text{loop}}(\phi)) &= -\frac{\pi \Lambda r}{2} \sum_{\beta} (\langle \beta, i\phi \rangle)^2 + \frac{\pi \Lambda r}{2} \sum_{\mu} (\langle i\phi, \mu \rangle)^2 + \dots \\ &= \pi \Lambda r \, (C_2(\text{adj}) - C_2(R)) \, \text{Tr}(\phi^2) + \dots \ , \end{aligned} \tag{5.6}$$

where the ellipsis stands for the convergent parts and $C_2(R)$ is the quadratic Casimir operator for representation R, defined by $\text{Tr}(T_A T_B) = C_2(R) \delta_{AB}$. We have also omitted the linear terms $\langle \beta, i\phi \rangle$ and $\langle i\phi, \mu \rangle$ [1]. One can see from the expressions above that the divergent contributions are proportional to $\text{Tr}(\phi^2)$, suggesting that it can be absorbed into the redefinition of the Yang-Mills coupling constant

$$\frac{1}{g_{eff}^2} = \frac{1}{g_{YM}^2} - \frac{\Lambda}{8\pi^2} \left(C_2(\text{adj}) - \sum_I C_2(R) \right) \ , \tag{5.7}$$

where the sum in the second term is over all hypermultiplets of the theory. This same expression can be also obtained in flat space using perturbation theory [43]. Note also that in the case of one adjoint hypermultiplet the divergent parts from the vector and matter multiplets cancel. Hence the effective coupling equals the bare one, i.e. $g_{eff} = g_{YM}$. It is worth mentioning that in the absence of hypermultiplets (pure $\mathscr{N} = 1$ super Yang-Mills), renormalization of the coupling (5.7) can be derived using the decoupling limit of the massive adjoint hypermultiplet. The decoupling limit means that one should include the mass of the hypermultiplet and consider the limit in which this mass is

[1] The linear terms are often zero. For example $\langle \beta, i\phi \rangle$ can always be omitted due to the symmetries of the root system and the terms linear in $\langle i\phi, \mu \rangle$ vanish for semi-simple groups.

considerably larger than the typical scale of the model. In this case the mass of the hypermultiplet plays the role of the UV cut-off Λ. The decoupling limit analysis is carried out in detail in paper II.

After renormalizing the coupling constant one can rewrite the regularized partition function as

$$Z = \int\limits_{Cartan} d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2) - \frac{\pi k}{3}\mathrm{Tr}(\phi^3)} \det_{Ad}\Big(S_3(i\phi)\Big)\, \det_R^{-1}\left(S_3\left(i\phi + im + \frac{3}{2}\right)\right), \tag{5.8}$$

where $S_3(x)$ is the triple sine function that can be written in the form of infinite product

$$S_3(x) = 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4} - \frac{3}{2}x} \prod_{t=1}^{\infty} \Big[\left(1 + \frac{x}{t}\right)^{(1+\frac{3}{2}t+\frac{1}{2}t^2)} \left(1 - \frac{x}{t}\right)^{(1-\frac{3}{2}t+\frac{1}{2}t^2)} e^{\frac{x^2}{2} - 3x}\Big], \tag{5.9}$$

This is not the definition of the triple sine function but rather one of its representations. For the definition, properties and relations for this function see [73].

Also note that the integration in (5.8) is over the Cartan subalgebra rather than the complete algebra. The reason for this is ϕ can always be diagonalized with a unitary gauge transformation, so one performs a procedure similar to the diagonalization described in chapter 4[2]. For a general group this procedure reduces the full integral to the integral over the Cartan subalgebra and results in a Vandermonde term of the form $\prod_{+}\langle\beta, i\phi\rangle^2$. In (5.8) the Vandermonde factor is canceled by a similar factor coming from the vector multiplet's one-loop contribution.

It is also useful to write (5.8) in the form

$$\begin{aligned} Z &= \int\limits_{Cartan} d\phi\, e^{-\mathscr{F}}, \\ \mathscr{F} &= \frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}(\phi^2) + \frac{\pi k}{3}\mathrm{Tr}(\phi^3) - \sum_{\beta} \log S_3(\langle i\phi, \beta\rangle) \\ &\quad + \sum_{\mu} \log S_3\left(\langle i\phi, \mu\rangle + im + \frac{3}{2}\right), \end{aligned} \tag{5.10}$$

where we have introduced the prepotential $\mathscr{F}$. This form is especially convenient for the saddle-point method.

[2]Here this unitary transformation is actually the gauge transformation of the localized field theory and thus should be fixed

5.2 Convergence of the Matrix Integral

We want our matrix integral to be convergent. To find the conditions for convergence we can check the limit of large ϕ and demand the prepotential $\mathscr{F}$ to be positive-definite.

In order to define this limit, we use the following asymptotic formula for the triple sine function with $|\mathrm{Im} z| \to \infty$

$$\log S_3(z) \sim -\mathrm{sgn}(\mathrm{Im} z)\pi i \left(\frac{1}{6} z^3 - \frac{3}{4} z^2 + z + O(1) \right), \tag{5.11}$$

which directly leads to

$$\begin{aligned} \mathscr{F} \;=\; & \frac{8\pi^3 r}{g_{eff}^2} \mathrm{Tr}(\phi^2) + \frac{\pi k}{3} \mathrm{Tr}(\phi^3) + \frac{\pi}{6} \Big(\sum_\beta |\langle \phi, \beta \rangle|^3 - \sum_\mu |\langle \phi, \mu \rangle|^3 \Big) \\ & - \frac{\pi}{2} m \sum_\mu \mathrm{sgn}(\langle \phi, \mu \rangle)(\langle \phi, \mu \rangle)^2 - \pi \sum_\beta |\langle \phi, \beta \rangle| \\ & - \frac{\pi}{2} \left(m^2 + \frac{1}{4} \right) \sum_\mu |\langle \phi, \mu \rangle| + \dots , \end{aligned} \tag{5.12}$$

where the ellipsis stands for the terms subleading in large ϕ. This expression is dominated by the cubic terms in the limit of large ϕ and thus they determine the convexity of the prepotential $\mathscr{F}$. However, we should notice that there are some exclusions, where the cubic terms are canceled. An example is the case of a single adjoint hypermultiplet or the case of a ive-dimensional superconformal fixed point with a $USp(N)$ gauge group [61].

Let's for instance study the case of the $SU(N)$ theory with N_f fundamental hypermultiplets and zero Chern-Simons level ($k = 0$). The leading terms in the prepotential for this theory are given by

$$\mathscr{F} = \frac{\pi}{6} \left(\sum_{j \neq i} \sum_i |\phi_i - \phi_j|^3 - N_f \sum_i |\phi_i|^3 \right) + \dots . \tag{5.13}$$

In order for $\mathscr{F}$ to be a convex function the inequality $N_f < 2N$ should be satisfied. In turn, the effective coupling g_{eff}^2 is allowed to be negative as it doesn't spoil the convergence of the matrix integral (5.10).

In general, for different gauge groups and hypermultiplet content the expression (5.12) should be analyzed separately. However we can see that the asymptotic expression (5.12) exactly matches the flat space prepotential observed in [60]. The same paper also provides a detailed convex analysis for a large variety of theories.

5.3 Decompactification limit

Here we study the behavior of the prepotential $\mathscr{F}$ in the decompactification limit, i.e. in the limit where the radius of the five-sphere is taken to infinity.

In order to consider the decompactification limit one should first restore the radius dependence in ϕ and m before taking the limit of infinite radius r. Hence, we replace these with

$$\phi \to r\phi\,, \quad m \to rm\,, \quad r \to \infty\,. \tag{5.14}$$

Substituting this into the prepotential (5.10) one finds

$$\begin{aligned}\frac{1}{2\pi r^3}\mathscr{F} = \frac{4\pi^2}{g_{eff}^2}\mathrm{tr}\left(\phi^2\right) + \frac{k}{6}\mathrm{tr}\left(\phi^3\right) + \mathrm{tr}_{Ad}\left(\frac{1}{12}|\phi|^3 - \frac{1}{2r^2}|\phi|\right) \\ -\sum_I \mathrm{tr}_{R_I}\left(\frac{1}{12}|\phi+m|^3 + \frac{1}{16r^2}|\phi+m|\right) + \ldots\,,\end{aligned} \tag{5.15}$$

where the ellipsis indicates terms subleading in $1/r$. Up to constants that can be absorbed into the redefinition of the couplings (5.15) reproduces the quantum prepotential in the flat-space limit [60] provided the $1/r^2$ terms are omitted.

Formally, the $1/r^2$ terms vanish in the decompactification limit. However we will keep them in the prepotential as these terms will be useful in the later discussions of finite r effects in the large radius limit.

In this chapter we described general properties of the matrix model (5.1) with arbitrary gauge group and multiplet content. In the following chapters the focus will be on particular examples of theories with unitary, i.e. $SU(N)$ or $U(N)$, gauge groups and mainly with a single massive hypermultiplet in the adjoint representation. However, cases with other hypermultiplet content will be briefly discussed as well.

6. Planar limit of $\mathcal{N} = 1^*$ super Yang-Mills

The main goal of this chapter is to derive the N^3 behavior of the free energy in $5D$ super Yang-Mills theory. To derive this behavior we will discuss $\mathcal{N} = 1$ super Yang-Mills theory with a single massive hypermultiplet in the adjoint representation. For brevity we will sometimes refer to this theory as $\mathcal{N} = 1^*$ super Yang-Mills. The gauge group is taken to be $SU(N)$ or $U(N)$. In the planar limit it doesn't matter which one we take. We also drop the Chern-Simons term assuming that the level is $k = 0$. We will also evaluate the expectation value of the circular Wilson loop in $5D$ super Yang-Mills and discuss how it is related to the corresponding observable in the $(2,0)$ theory.

6.1 Saddle point

Our matrix model (5.10) is very complicated due to the form of the one-loop contributions (5.2) and (5.3). Unfortunately it is not possible to use methods such as, for example, orthogonal polynomials, to obtain an exact result. However the planar limit (4.11) provides a good approximation. This allows us to neglect instanton contributions, as previously discussed. But more important for us, large-N limit, because this is the limit where AdS/CFT calculations are valid for $(2,0)$ theories.

As discussed in chapter 4, in the large N-limit the matrix integral can be approximated by the value of the integrand at the saddle point. This saddle point can be derived directly minimizing the prepotential (5.10) and using the following expression for the logarithmic derivative of the triple sine function

$$\frac{S_3'(x)}{S_3(x)} = \frac{\pi}{2}(x-1)(x-2)\cot(\pi x)\,, \tag{6.1}$$

which can be found in [76, 72] together with many other useful relations. This equation can also be obtained directly from (5.9). Using (6.1) together with the explicit expressions for the $SU(N)$ roots and the symmetry properties of the root system, one arrives at the following saddle point equation

$$\begin{aligned}\frac{8\pi^3 N}{\lambda}\phi_i \quad = \quad & \pi\sum_{j\neq i}\Bigg[\left(2-(\phi_i-\phi_j)^2\right)\coth(\pi(\phi_i-\phi_j)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi_i-\phi_j-m)^2\right)\tanh(\pi(\phi_i-\phi_j-m)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi_i-\phi_j+m)^2\right)\tanh(\pi(\phi_i-\phi_j+m))\Bigg]\,, \end{aligned} \tag{6.2}$$

where

$$\lambda \equiv \frac{g_{YM}^2 N}{r}, \tag{6.3}$$

is the dimensionless 't Hooft coupling. Introducing the eigenvalue density as in (4.13) we can write down the continuous limit of (6.2)

$$\begin{aligned} \frac{8\pi^3}{\lambda}\phi \quad = \quad & ⨍ d\psi\,\rho(\psi)\Bigg[\left(2-(\phi-\psi)^2\right)\coth(\pi(\phi-\psi)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi-\psi-m)^2\right)\tanh(\pi(\phi-\psi-m)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi-\psi+m)^2\right)\tanh(\pi(\phi-\psi+m))\Bigg]. \end{aligned} \tag{6.4}$$

Thus the saddle point for the matrix model (5.1) is given by the solution to this singular integral equation. However this equation differs significantly from (4.15). Unfortunately, the standard resolvent technique used to solve (4.15) does not apply here. There is no good method to solve this equation explicitly as only the solutions to singular integral equations with Cauchy, i.e $(x-y)^{-1}$ and Hilbert, i.e. $\coth(x-y)$, kernels are known.

Instead in papers I and II we were able to find reasonable approximations to equations of motion (6.4) in the different limits of λ. Then we were able to solve these approximate equations. To check the validity of these approximations we compared their solutions with numerical solutions of the actual equations (6.4). In the following sections we briefly describe these solutions and compare the strong coupling results with results using the AdS/CFT correspondence applied to $(2,0)$ theory.

6.2 Solutions

To get the right intuition about the interaction of the eigenvalues let's consider the kernel (6.4). This kernel has a singularity at $\phi=\psi$ coming from $2\pi\coth(\pi(\phi-\psi))$ term and leads to a repulsion between the eigenvalues at short distances. As we increase the separation between eigenvalues the seemingly dominant attractive $(\phi-\psi)^2$ coming from the vector multiplet cancels with a similar term coming from the hypermultiplet, leaving a repulsive potential. As a consequence one ends up with a repulsive force between the eigenvalues at large distances as well.

6.2.1 Weak Coupling

We start with the weak coupling limit $\lambda \ll 1$. In this case the central potential is very steep around the origin. Thus all eigenvalues tend to condense near

$\phi = 0$. Hence it is reasonable to assume that the separation between any two eigenvalues satisfies $|\phi - \psi| \ll 1$ in this regime. With this assumptions the full saddle point equation (6.4) is approximately

$$\frac{16\pi^3}{\lambda}\phi \approx 2\fint d\psi \frac{\rho(\psi)}{\phi - \psi} . \tag{6.5}$$

This is equation (4.12) with central potential $V(\phi) = 8\pi^3 \phi^2$. Following the standard technique described in chapter 4 results in the Wigner semicircle distribution for the eigenvalues

$$\rho(\phi) = \frac{2}{\pi\phi_0^2}\sqrt{\phi_0^2 - \phi^2} \qquad \phi_0 = \sqrt{\frac{\lambda}{2\pi^3}} , \tag{6.6}$$

The corresponding free energy can then be calculated by substituting the eigenvalue density (6.6) and the quadratic central potential $V(\phi)$ into (4.32)[1]. Computing all integrals involved in this expression results in the usual free energy for the Gaussian matrix model

$$F = -\log Z \approx -N^2 \log\sqrt{\lambda} . \tag{6.7}$$

6.2.2 Strong Coupling

Net we examine what happens at strong 't Hooft coupling $\lambda \gg 1$. This regime is much more interesting. As is typical for a weakly coupled gauge theory, the free energy scales as N^2. In order to have the N^3 behavior found for the $(2,0)$ theories one must go to strong coupling.

In the case of large 't Hooft coupling $\lambda \gg 1$ the central potential of the matrix model becomes very shallow. Thus one expects the repulsion of the eigenvalues to push them away from each other to large separations. Hence we can assume $|\phi - \psi| \gg 1$ in the kernel of (6.4). Then equation (6.4) can be approximated by

$$\frac{16\pi^3}{\lambda}\phi = \pi\left(\frac{9}{4} + m^2\right)\int d\psi\, \rho(\psi)\,\mathrm{sign}(\phi - \psi) . \tag{6.8}$$

Notice that this integral equation is not singular and can be solved by differentiating both sides w.r.t. ϕ leading to a constant distribution[2]

$$\begin{aligned} \rho(\phi) &= \frac{32\pi^2}{(9+4m^2)\lambda}, \qquad |\phi| \le \phi_m , \qquad \phi_m = \frac{(9+4m^2)\lambda}{64\pi^2} \\ &= 0, \qquad |\phi| > \phi_m , \end{aligned} \tag{6.9}$$

[1]We omit an extra constant in the expansion of the one-loop contributions, which can be found in paper III

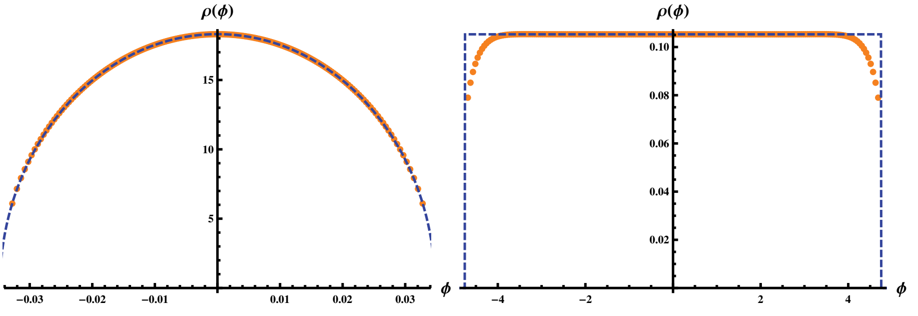
[2]Notice that in papers I and II the same eigenvalue density was obtained in slightly different ways.

where the endpoint position ϕ_m is determined from the normalization condition (4.14).

In order to find the free energy one can use the asymptotic expression (5.11) for the triple sine function in the prepotential (5.10). The resulting expression for the free energy then looks like

$$
\begin{aligned}
F \equiv -\log Z = & -\frac{8\pi^3 N^2}{\lambda}\int d\phi\,\rho(\phi)\,\phi^2 \\
& + \frac{\pi N^2}{2}\left(\frac{9}{4}+m^2\right)\int\int \rho(\phi)\rho(\psi)\,|\phi-\psi|\,,
\end{aligned}
\tag{6.10}
$$

After substituting for the eigenvalue density (6.9) one can finally obtain the



(a) $\lambda = 0.15$. (b) $\lambda = 300$.

Figure 6.1. Density of eigenvalues $\rho(\phi)$ for the solution of saddle point equations (6.2) with $m = \frac{1}{2}$ and $N = 200$. Orange dots shows results of numerical simulation while dashed lines correspond to solutions (6.6) *(left)* and (6.9) *(right)*.

free energy for $\mathcal{N} = 1^*$ super Yang-Mills

$$
F = -\frac{g_{YM}^2 N^3}{96\pi r}\left(\frac{9}{4}+m^2\right)^2\,, \tag{6.11}
$$

demonstrating N^3 behavior for the free energy at strong coupling. In paper I we have shown that if one or more hypermultiplets are chosen to be in the fundamental representation then the free energy scales as N^2 at strong coupling.

We have also compared our analytical results in (6.6) and (6.9) with the numerical solutions for the full equations of motion (6.2). Results of this comparison are shown for particular values of the parameters in Fig. 6.1. In these graphs the orange dots stand for results of the numerical solution while the dashed lines shows our analytical results obtained after the approximations described previously. As one can see the numerical and analytical results match very well, suggesting that our approximations are reasonable.

6.3 Wilson Loops

As discussed in chapter 3, localization works only for supersymmetric observables. Hence there are strict limitations on what can be evaluated using this method. However some interesting possibilities are still left. The most important is the Wilson loop. The main motivation for the original localization paper [93] was the study of circular Wilson loops in four-dimensional $\mathcal{N}=4$ super Yang-Mills. In particular localization was used to prove the conjecture proposed by Erickson, Semenoff and Zarembo in [41] for large-N and by Drukker and Gross in [36] for any N. This conjecture claims that the expectation value of the circular Wilson loop can be evaluated from the expectation value in the Gaussian matrix model

$$\langle W\rangle_{\mathcal{N}=4} \sim \langle \mathrm{Tr} e^{M}\rangle_{gaussian}\,, \tag{6.12}$$

In order to prove this conjecture Pestun shown that localization of the partition function results in the Gaussian matrix model. Then the Wilson loop reduces to the operator e^M and thus the conjecture (6.12) is proven.

Let us now show how e^M arises in the matrix model. Here we derive this for the case of the $5D$ theory, but the story is the same in all other dimensions as well. We want to consider the expectation value of the supersymmetric Wilson loop given by

$$\langle W(C)\rangle_R = \frac{1}{\mathrm{dim}R}\langle \mathrm{Tr}_R P\exp\oint_C d\tau\left(iA_\mu(x)\dot{x}^\mu + \sigma\dot{y}\right)\rangle\,, \tag{6.13}$$

where R is the representation of the gauge group in which Wilson loop transforms and σ is the scalar of the vector multiplet. The path of the Wilson loop $x^\mu(\tau)$ is parametrized by the parameter τ and $y(\tau)$ can be thought of as the parametrization of the Wilson loop in the internal space. In order for the Wilson loop to preserve half of the supersymmetries the contour C should wrap the equator of S^5 and also the relation $\dot{y}^2 = |\dot{x}|^2$ should be satisfied.

As the Wilson loop preserves half of the supersymmetry its expectation value can be localized following the prescriptions in chapter 3. Furthermore, the localization locus (3.21) and the one-loop determinants remain the same are not effected by Wilson loop linear in fields as well. Thus the only contribution of the Wilson loop to the matrix integral comes from its classical contribution at the locus. We concentrate on the case of a Wilson loop in the fundamental representation, which then, has the following expectation value

$$\langle W\rangle \sim \frac{1}{N}\int\prod_i d\phi_i\,\sum_i e^{2\pi\phi_i}\,e^{-\mathscr{F}}\,, \tag{6.14}$$

where $\mathscr{F}$ is the prepotential given in (5.10). In the planar limit term $\sum_i e^{2\pi\phi_i}$ has negligible back-reaction on the saddle point position. Hence, one can use the (6.6) and (6.9) for weak and strong coupling respectively. The Wilson loop expectation value can then be approximated by

$$\langle W \rangle = \int d\phi\, \rho(\phi)\, e^{2\pi\phi}\,, \tag{6.15}$$

where $\rho(\phi)$ is the eigenvalue density corresponding to the saddle point of the localized partition function.

In particular, in the case of weak coupling $\lambda \ll 1$, using (6.6), one obtains

$$\langle W \rangle \approx \sqrt{\frac{4\pi}{\lambda}}\, I_1\left(\sqrt{\frac{\lambda}{\pi}}\right) \approx \exp\left(\frac{\lambda}{8\pi}\right)\,, \tag{6.16}$$

where I_1 is the modified Bessel function of the first kind.

At strong coupling $\lambda \gg 1$, using the distribution (6.9), one obtains

$$\langle W \rangle \approx \frac{32\pi^2}{(9+4m^2)\,\lambda} \int\limits_{-\phi_m}^{\phi_m} e^{2\pi\phi} d\phi \sim \exp\left[\frac{\lambda}{8\pi}\left(\frac{9}{4}+m^2\right)\right]. \tag{6.17}$$

In these expressions we omit the prefactors in front of the exponent, because only the argument of the exponent is relevant for us. Interestingly, the only difference between the expressions for weak and strong coupling is the factor of $\left(\frac{9}{4}+m^2\right)$. This differs from the $4D$ case where the strong and weak coupling results have different λ-behavior. For example, see the calculations for the circular Wilson loop expectation values in four-dimensional $\mathcal{N}=2$ theories on S^4 [98, 23].

6.4 Comparison with supergravity calculations

As mentioned in chapter 2 we want to compare our results for the matrix model calculations with the values of corresponding observables in $6D$ $(2,0)$ theories. As discussed in chapter 2 the only well established way to find expectation values of observables in the $(2,0)$ theories is through the AdS/CFT correspondence.

In genral the AdS/CFT correspondence conjectures a duality between a conformal field theory in d-dimensional flat space and a certain string theory in $d+1$-dimensional anti-de Sitter space. This correspondence was first conjectured [79] in the context of four-dimensional $\mathcal{N}=4$ super Yang-Mills theory, which is dual to type IIB superstring theory on $AdS_5\times S^5$ background. Since then similar dualities were conjectured for many other supersymmetric gauge theory. For example, in three dimensions this duality takes place between ABJM theory with Chern-Simons level k and M-theory on $AdS_4\times S^7/Z_k$ background [1].

We are especially interested in the duality between $6D$ $(2,0)$ theories and M-theory on $AdS_7 \times S^4$ background. This correspondence was also proposed in original paper by Maldacena [79]. According to the prescription of this duality the radii of the AdS-space and the sphere are given by [2]

$$R_{AdS} = 2R_{S^4} \equiv l = 2l_{pl}\,(\pi N)^{1/3} \,, \tag{6.18}$$

where l_{pl} is the Planck length.

Calculations in frame of M-theory are not possible. Hence one would like to take the limit in which M-theory reduces to supergravity theory. The supergravity regime corresponds to the case where the radius of AdS space is much larger then Planck length. Then from (6.18) it is clear that this regime is valid when the dual $(2,0)$ theory has a large number of colors N. In this case we can compute the free energy and Wilson surface and then compare it to the matrix model results in the previous section.

In particular, we want to compare results for the $(2,0)$ theory compactified on $S^5 \times S^1$ with the analogous results for $\mathcal{N} = 1^*$ super Yang-Mills theory on S^5. The same space should be the boundary of the AdS-space in the setting of holography. This leads to the following Euclidian AdS_7 metric

$$ds^2 = \ell^2(\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_5^2) \,, \tag{6.19}$$

where $d\Omega_5^2$ is the metric of the unit five-sphere and ρ corresponds to the bulk direction with the boundary at $\rho \to \infty$. The Euclidian time direction is compactified such that

$$\tau \equiv \tau + \frac{2\pi R_6}{r} \,, \tag{6.20}$$

where R_6 and r are the radii of S^1 and S^5 respectively.

According to the AdS/CFT correspondence dictionary the free energy of the boundary theory corresponds to *supergravity action* in the bulk of AdS space. Full action of AdS supergravity should include three terms

$$I_{AdS} = I_{bulk} + I_{surf.} + I_{c.t.} \,, \tag{6.21}$$

where the bulk action is

$$I_{bulk} = -\frac{1}{16\pi G_N}\mathrm{Vol}\left(S^4\right)\int d^7x\,\sqrt{g}\,(R - 2\Lambda) \,, \tag{6.22}$$

with Newton constant $G_N = 16\pi^7 l_{pl}^9$ [79]. I_{surf} is the surface contribution

$$I_{surf} = -\frac{1}{8\pi G_N}\int\limits_{\partial AdS} d^6x\,\sqrt{\gamma}K \,, \tag{6.23}$$

where γ is the metric induced by g on the AdS boundary and K is the intrinsic curvature of this boundary.

The bulk and surface contributions have divergent parts and need to be regulated. The regularization is done by introducing *counterterms*, which depend only on the induced metric on the boundary (γ) and cancel all divergent parts in I_{bulk} and I_{surf} [8, 39, 28, 7]. In (6.21) these counterterms are contained in $I_{c.t.}$.

In order to calculate (6.22) one should introduce a cut-off ρ_0 for large ρ which corresponds to the UV cut-off in the dual gauge theory. If one considers the ε-expansion in the boundary theory it is reasonable to make the identification $\varepsilon = e^{-\rho_0}$. Evaluation of the integral in (6.22) results in the bulk action of the following form

$$I_{bulk} = \frac{4\pi R_6}{3r}\left(\frac{1}{64}\varepsilon^{-6} - \frac{3}{32}\varepsilon^{-4} + \frac{15}{64}\varepsilon^{-2} - \frac{5}{16} + O(\varepsilon^2)\right), \tag{6.24}$$

As expected, the bulk action contains divergent terms with negative powers of ε. The surface term I_{surf} will contribute to the divergent terms but not to the finite part of the action so we ignore it. The counterterms should cancel all divergences coming from the bulk and surface terms. In general the counterterms can contribute to the finite part of bulk action, but here we use the minimal subtraction regularization. In this scheme counterterms only cancel the divergent pieces in the action. Thus, one obtains [39]

$$I_{AdS} = -\frac{5\pi R_6}{12r}N^3, \tag{6.25}$$

which should be compared with the strong coupling free energy of the $5D$ theory (6.11).

Another observable we can evaluate is the expectation value of a circular Wilson loop. In the $(2,0)$ theories the Wilson loop operator becomes a surface operator. Indeed, the analog of expression (6.13) in $6D$ theory should contain the two-form field $B_{\mu\nu}$ in the exponential. To contract the indices of this field we need to integrate it over some surface. In order to have a Wilson loop wrapping the equator after compactifying to five dimensions, this surface wraps the compactification circle S^1 and the S^5 equator.

The expectation value of Wilson surfaces in $(2,0)$ theories was evaluated using the AdS/CFT picture in [12]. According to the AdS_7/CFT_6 dictionary, a Wilson surface is related to the extremized world-volume of an $M2$-brane with tension $T^{(2)} = \frac{1}{(2\pi)^2 l_{pl}^3}$ by exponentiating its negative action,

$$\langle W \rangle \sim e^{-T^{(2)}\int dV}. \tag{6.26}$$

One direction of the $M2$-brane wraps the S^5 equator, another direction wraps the compactification circle S^1 and the third direction falls into the bulk of the AdS_7 space. The calculation of the volume for this $M2$-brane configuration using minimal subtraction results in the following expression for the Wilson surface expectation value

$$\langle W \rangle \sim \exp\left(\frac{2\pi N R_6}{r}\right). \tag{6.27}$$

Now we are ready to compare results obtained from the matrix model and supergravity calculations. Lets remember that the original conjecture proposed in [77, 35] states the equivalence of $5D$ maximally supersymmetric Yang-Mills and compactified $(2,0)$ theories. Though this conjecture is not true due to the existence of the UV divergences in $5D$ super Yang-Mills, one can still expect supersymmetric observables of these two theories to match. As the original conjecture was stated for the maximally supersymmetric Yang-Mills it would be natural first to consider the supersymmetry enhancement point $m = i/2$, obtained in chapter 2[3]. Then the comparison of the expectation values of the Wilson loop on S^5 (6.17) and the Wilson surface in the $(2,0)$ theory (6.27) leads to the identification between the radius of compactification and the Yang-Mills coupling

$$R_6 = \frac{g_{YM}^2}{8\pi^2}, \tag{6.28}$$

which reproduces the usual identification (2.23) proposed in [77, 35]. However in this case there is the mismatch of $4/5$ between the free energy (6.11) and the supergravity action (6.25). The same mismatch was observed in [67].

Interestingly we can fix this mismatch by relaxing the identification in (6.28) and finding the value of the mass at which both comparisons of the free energy and Wilson loops can work. The comparison of the free energy with the supergravity action leads to the relation

$$R_6 = \frac{g_{YM}^2}{40\pi^2}\left(\frac{9}{4}+m^2\right)^2. \tag{6.29}$$

At the same time the comparison of the Wilson loop on S^5 with the Wilson surface of $(2,0)$ theory leads to different relation

$$R_6 = \frac{g_{YM}^2}{16\pi^2}\left(\frac{9}{4}+m^2\right). \tag{6.30}$$

Now we can see that in order for these two relations to be consistent one needs to fix mass of the hypermultiplet at $m = 1/2$ so that

$$R_6 = \frac{5}{2}\frac{g_{YM}^2}{16\pi^2}, \qquad m = \frac{1}{2}. \tag{6.31}$$

[3]Notice that extra factor of i appears due to the Wick rotation (3.15) performed in chapter 3. We should also notice that it is not completely clear if localization can be performed at this point as m should be real according to prescriptions in chapter 3.

Hence we can conclude that in order to match the prefactors of the large-N behavior of observables in $5D$ and $6D$ theories one needs to move away from the supersymmetry enhancement point to the point with the mass of the hypermultiplet $m = 1/2$ and modify the usual identification between the radius of compactifiction and $5D$ Yang-Mills coupling.

Regarding the last one we should notice that it is not clear if our result for the identification between compactification radius and coupling constant should be exactly the same as in the original conjecture. The latter one was formulated for the theories on the flat space with Minkowski metric, while the theories we consider are defined on spheres and have massive hypermultiplets. Hence it is also possible that in our setting the relation between the two theories can be slightly reformulated.

To summarize, we have shown that $5D$ supersymmetric Yang-Mills theory with an adjoint hypermultiplet at strong coupling has the same N^3 behavior as the free energy of the $6D$ $(2,0)$ theories. We also calculated the expectation values of the circular Wilson loops in both theories and found that they match exactly at the supersymmetry enhancement point provided the usual identification (2.23) between the compactification radius and the $5D$ Yang-Mills coupling is satisfied. However in this case there is mismatch of $4/5$ between the free energy of the $5D$ and $6D$ theories. We have also found that both free energies and Wilson loops can be exactly matched if the mass of the hypermultiplets is $m = 1/2$ and the identification between compactification radius and $5D$ Yang-Mills coupling is modified according to (6.31).

We should also notice that there are problems with mapping the $(2,0)$ theory onto the $S^1 \times S^5$ background, which we blindly performed in this section. To be more precise the $(2,0)$ theory on flat space can be conformally mapped onto $\mathbb{R} \times S^5$. Here $\mathbb{R}$ is time-like and requires a Euclidian rotation in order to be compactified. However this rotation results in the Euclidian version of $(1,1)$ theory rather than $(2,0)$ (see paper II). This issue and its effect on the relation between $5D$ super Yang-Mills theory on S^5 and $(2,0)$ theories should be better understood in the future.

7. Supersymmetric Chern-Simons theory

In the previous chapter we considered $5D$ $\mathcal{N}=1$ super Yang-Mills with an adjoint hypermultiplet with Chern-Simons level $k=0$. The aim of this chapter is to generalize this theory by including a nontrivial Chern-Simons term. We will also consider the case of infinite Yang-Mills coupling $g_{YM}^2 \to \infty$, which will be referred to as pure Chern-Simons theory, since the Yang-Mills term does not contribute in this limit. This case is of special interest as the pure supersymmetric Chern-Simons theory is a conformal fixed point. Hence, one could look for holographic duals. Here we explore the possibility of their existence.

We also describe the theory with both Chern-Simons and Yang-Mills terms included. We examine the interplay between these terms and show that this gives rise to a phase transition at certain values of the couplings.

Finally we show that when a Chern-Simons term is included there is a distinction between $SU(N)$ and $U(N)$ gauge groups, even in the large-N limit. In the following we argue why the choice between the $SU(N)$ or $U(N)$ gauge group matters.

7.1 Saddle point equation

We again assume that the matrix integral (5.1) is dominated by a saddle point. The saddle-point can then be found by minimizing the prepotential $\mathcal{F}$ in (5.10) and making use of the expression for the triple sine function's derivative (6.1). The saddle point equation is similar to the equations of motion in (6.2), with only the l.h.s. being different

$$\begin{aligned}\frac{\pi N}{\tilde{\lambda}}\left(\phi_i^2+2\kappa\phi_i-\mu\right) \quad = \quad & \pi\sum_{j\neq i}\Bigg[\left(2-(\phi_i-\phi_j)^2\right)\coth(\pi(\phi_i-\phi_j)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi_i-\phi_j-m)^2\right)\tanh(\pi(\phi_i-\phi_j-m)) \\ & +\frac{1}{2}\left(\frac{1}{4}+(\phi_i-\phi_j+m)^2\right)\tanh(\pi(\phi_i-\phi_j+m))\Bigg],\end{aligned} \tag{7.1}$$

where

$$\tilde{\lambda}\equiv\frac{N}{k}, \qquad \kappa\equiv 8\pi^2\frac{\tilde{\lambda}}{\lambda}. \tag{7.2}$$

Here we have introduced the analog of 't Hooft coupling $\tilde{\lambda}$ for the case of the Chern-Simons theory analogously to ABJM theory [1]. The parameter κ measures the relevance of the Yang-Mills term. Finally, μ is a *Lagrange multiplier*, which is included in order to enforce the tracelessness condition for the $SU(N)$ gauge group. For a $U(N)$ gauge group we can just set $\mu = 0$.

For technical reasons it is easier to keep the equations of motion in the discrete form (7.1), introducing the continuous limit only when it necessary. Equation (7.1) has the same kernel as pure Yang-Mills theory since the one-loop determinants are not altered by the Chern-Simons term. As a result, all properties of the eigenvalue interactions remain unchanged from those described in chapter 6. Hence, we use the same strategy for solving (7.1) where we first make an approximation based on the general properties of the kernels and then solve the approximate equations. Afterwards we will compare these results with numerical solutions of the full equations of motion (7.1).

7.2 Weak Coupling

In this section by *weak coupling* we mean that at least one of the inequalities $\lambda \ll 1$ or $\tilde{\lambda} \ll 1$ is satisfied. At weak coupling the cubic central potential of the matrix model (5.1) is very steep and thus, forces the eigenvalues to condense at one of the critical points balanced by the singularity of the kernel which leads to eigenvalue repulsion. These balanced forces lead to an eigenvalue distribution that forms a short cut around one of the critical points. It is then reasonable to assume that at weak coupling the separation between eigenvalues is small, i.e. $|\phi_i - \phi_j| \ll 1$. This leads to the equations of motion

$$\frac{\pi N}{\tilde{\lambda}} \left(\phi_i^2 + 2\kappa\phi_i - \mu \right) \quad \approx \quad 2\sum_{j\neq i} \frac{1}{\phi_i - \phi_j} \,. \tag{7.3}$$

It is worth noting that the weak coupling matrix model also arises in the context of $2D$ quantum gravity whose many aspects have been well-studied. [33, 32]. We will repeat some of the results for $2D$ gravity.

The equations (7.3) are of the same form as equation (4.12). Thus, in order to solve these equations we can use the resolvent technique which leads to the density of eigenvalues

$$\rho(\phi) = \frac{1}{2\tilde{\lambda}} \sqrt{-B + \frac{4\tilde{\lambda}}{\pi}\tilde{\phi} - (\tilde{\phi}^2 - \kappa^2 - \mu)^2} \,, \tag{7.4}$$

where we have introduced $\tilde{\phi} \equiv \phi + \kappa$. In general, this solution has two cuts, or, equivalently, four branch points. The parameter B controls the filling fractions on these two cuts.

If the Yang-Mills coupling is much smaller than the Chern-Simons then the effect of the latter one is small and there is only a single cut along the real axis in the $\tilde{\phi}$ plane. The one-cut solution has

$$B = -(\kappa^2 + \mu - b^2)(\kappa^2 + \mu + 3b^2)\,, \tag{7.5}$$

where b satisfies the equation

$$b(\kappa^2 + \mu - b^2) = \frac{\tilde{\lambda}}{\pi}\,, \tag{7.6}$$

and the eigenvalue density becomes

$$\rho(\phi) = \frac{1}{2\tilde{\lambda}}(\tilde{\phi} + b)\sqrt{\frac{2\tilde{\lambda}}{b\pi} - (\tilde{\phi} - b)^2}\,. \tag{7.7}$$

This solution should be considered separately for an $SU(N)$ or $U(N)$ gauge groups. For the $U(N)$ case one sets $\mu = 0$ in all the above expressions. For $SU(N)$ the situation is more complicated due to the tracelessness constraint forcing the eigenvalue sum to zero. For the density (7.7) this constraint leads to the relations

$$4b^2(\kappa - b) = \frac{\tilde{\lambda}}{\pi}\,, \qquad \mu = (\kappa - 3b)(b - \kappa)\,, \tag{7.8}$$

while the eigenvalue density is still described by (7.7) This solution describes a phase of the theory dominated by the Yang-Mills term.

Now we examine the opposite situation of the pure Chern-Simons theory where the Yang-Mills term is ignored. In this case we set the Yang-Mills coupling to infinity, so that $\kappa = 0$. The general solution (7.4) is still valid. However, turning off the Yang-Mills term leads to several interesting effects. For $U(N)$ the eigenvalue density still takes the form (7.7) for the one-cut solution, but now, according to (7.6), b should satisfy

$$b^3 = -\frac{\tilde{\lambda}}{\pi}\,. \tag{7.9}$$

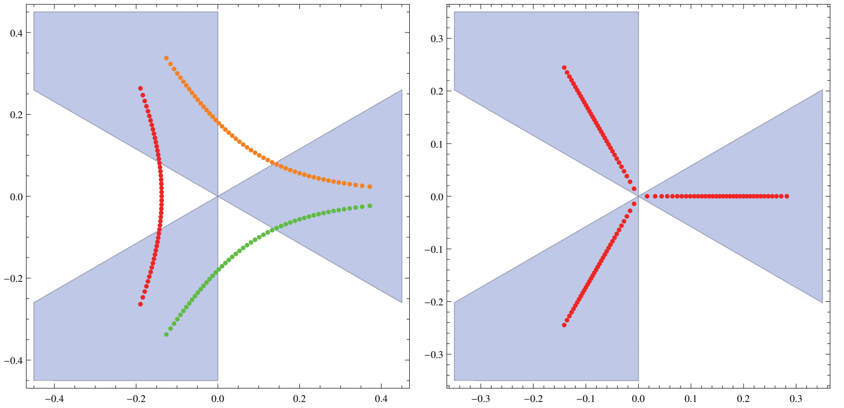
This equation has three roots corresponding to the three different one-cut solutions shown in Fig.7.1(a).

For $SU(N)$ it is not possible to find one-cut solution. However there exists an interesting analog to the one-cut solution. Let's consider the general solution (7.4) with $B = 0$ and $\mu = 0$. In this case the branch point of one cut hits the side of another so that the eigenvalue density equals to

$$\rho(\phi) = \frac{\phi^2}{2\tilde{\lambda}}\sqrt{\frac{4\tilde{\lambda}}{\pi\phi^3} - 1}\,. \tag{7.10}$$

This eigenvalue density describes a solution composed of three cuts. Each cut emerges from the origin and goes to the branch point located at $\phi^* =$

$(4\tilde{\lambda}/\pi)^{1/3}\omega^n$ where $n = 0,1,2$ and $\omega \equiv e^{2\pi i/3}$. Thus, this solution has a Z_3 symmetry with respect to the origin (see Fig.7.1(b)). This Z_3 symmetry emerges form the Z_3 symmetry of the saddle point equation (7.3), which can be directly checked by the discreet rotations of eigenvalues $\phi_i \to e^{2\pi n i/3}\phi_i$, $n = 1,2,3$, preserving the Z_3 symmetry of the pure Chern-Simons theory Since the eigenvalues sum up to zero for the Z_3 symmetric solution , i.e. $\langle\phi\rangle = 0$ this solution is valid for both $U(N)$ and $SU(N)$.



(a) Eigenvalues of the three single-cut solutions for $N = 51$, $\tilde{\lambda} = 0.1$

(b) Eigenvalues of the Z_3 solution for $N = 123$, $\tilde{\lambda} = 0.02$

Figure 7.1. Eigenvalues for the pure CS model at weak coupling. The blue regions are the integration regions in the complex plane where $\mathrm{Re}(\phi^3) > 0$ so that the path integral converges.

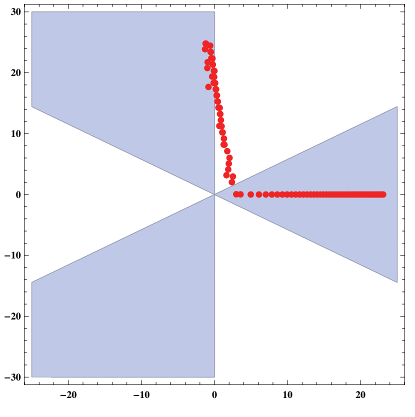


Figure 7.2. Example of strong coupling solution $\tilde{\lambda} = 750$, $N = 87$.

To determine which solution is favored, one should evaluate the corresponding free energies. For either solution one can just use (4.32) and substitute in the cubic potential $V(\phi) = \pi k \phi^3/3$. Details of these calculations can be found in paper III[1]. The result is

$$F = N^2\left(\frac{1}{2} - \frac{1}{3}\log\frac{\tilde{\lambda}}{\pi}\right) \tag{7.11}$$

for the Z_3 solution and

$$F = N^2\left(\frac{1}{2} - \frac{1}{3}\log\frac{\tilde{\lambda}}{\pi} + \frac{1}{2}\log 2\right), . \tag{7.12}$$

for the one-cut solution. The free energy of theory scales as N^2 for both solutions. However, the Z_3 solution has lower free energy, and thus it is energetically favored.

Later in this chapter we will investigate the crossover between the Yang-Mills and the Chern-Simons regimes.

7.3 Strong Coupling

Now let's consider the strong coupling regime where $\lambda, \tilde{\lambda} \gg 1$. For the equations of motion (7.1) we make approximations similar to those made in section 6.2.2. With the Chern-Simons term the eigenvalues can lie on the complex plane. But the kernels in (7.1) have poles along the imaginary axis which can affect the solutions once the eigenvalue separation is of order one. We will largely ignore this effect and make approximations based on the separations of the real parts of the eigenvalues. Later we show numerical evidence that this approximation is valid at leading order.

We know that the kernels in (7.1) are repulsive for all separations and the central potential is shallow when the Yang-Mills and Chern-Simons couplings are both large. Hence the eigenvalues tend to repel each other to large distances. Assuming the separation is not along the imaginary axis we then have separations between the eigenvalues along the real axis, i.e. $|\mathrm{Re}(\phi_i - \phi_j)| \gg 1$. Under this assumption (7.1) becomes

$$N\frac{\pi}{\tilde{\lambda}}(\phi_i^2 + 2\kappa\phi_i + \mu) = \left(\frac{9}{4} + m^2\right)\pi\sum_{j\neq i}\mathrm{sign}(\mathrm{Re}(\phi_i - \phi_j)). \tag{7.13}$$

If we assume that $\mathrm{Re}(\phi_i)$ are all ordered, we get the following equation for ϕ_i

$$\phi_i^2 + 2\kappa\phi_i + \mu = \chi\,\frac{2i - N}{N}, \tag{7.14}$$

[1] In calculations of paper III we also add constant term $C = N^2\left(\frac{1}{8}\log 2 + \log\pi + \frac{7\zeta(3)}{16\pi^2}\right)$ to the free energy. This term comes from the subleading contribution in small separations expansion of one-loop determinants in our matrix model (5.1).

where χ is defined as

$$\chi \equiv \left(\frac{9}{4}+m^2\right)\tilde{\lambda}\,. \tag{7.15}$$

In order to obtain eigenvalue density $\rho(\phi)$, we should consider the continuous limit of (7.14) in the following way

$$x \equiv \frac{i}{N}\,, \qquad \frac{1}{N}\sum_i \to \int dx\,, \qquad \rho(\phi) = \frac{dx}{d\phi}\,, \tag{7.16}$$

which is consistent with the usual definition (4.13) of the eigenvalue density. Using this continuous limit it is straightforward to find the following expression from (7.14)

$$\rho(\phi) = \frac{1}{\chi}(\phi+\kappa)\,. \tag{7.17}$$

Up to here all considerations were independent of the choice between $U(N)$ and $SU(N)$. Hence the eigenvalue density itself does not depend on this choice. However position of the cut does, as we will see below.

For $U(N)$ we set $\mu = 0$ and find that the cut goes between the endpoints ϕ_- and ϕ_+, where

$$\phi_\pm = -\kappa + \sqrt{\kappa^2 \pm \chi}\,. \tag{7.18}$$

For $SU(N)$ we keep the Lagrange multiplier μ and impose a tracelessness condition $\langle\phi\rangle = 0$. Then finding the positions of the support endpoints can be reduced to solving the following system of equations

$$\begin{aligned} \phi_+^2 + 2\kappa\phi_+ - \mu &= \chi \\ \phi_-^2 + 2\kappa\phi_- - \mu &= -\chi \\ \tfrac{1}{3}(\phi_+^3 - \phi_-^3) + \tfrac{1}{2}\kappa(\phi_+^2 - \phi_-^2) &= 0\,, \end{aligned} \tag{7.19}$$

Finding $\phi_\pm$ and μ from these equations one can determine the position of the cut. It is interesting to note that for both $SU(N)$ and $U(N)$ equations (7.18) and (7.19), which determine the positions of the endpoints have subtle behavior at certain points. For example, it is easy to see from (7.18) that in the $U(N)$ case for $\kappa^2 > \chi$ all eigenvalues lie along the real axis. However, if we decrease κ and go to the phase dominated by the Chern-Simons term, one endpoint of the cut will move into the complex plane. In the last section of this chapter we will return to this transition and discuss its details, as well as details of a similar transition that takes place in the $SU(N)$ case.

Now let's consider what happens if the Yang-Mills term is turned off ($\kappa = 0$), corresponding to pure Chern-Simons theory. In this case we still assume our approximations are applicable, so that equations (7.14) and (7.17) hold as well. At strong coupling it is natural to expect of solutions to evolve from the weak coupling solutions (7.4) and (7.10), described in the previous section. However, we note that the kernels in (7.1) admit a Z_3 symmetry only in the weak coupling approximation (7.3). This symmetry breaks down when the coupling is increased.

Let us consider the $U(N)$ case, corresponding to $\mu = 0$. In this case the eigenvalues can be separated into two parts. Indeed, from equation (7.14) it is clear that for $i \geq N/2$ the eigenvalues ϕ_i are real, so that the cut goes between the origin and the endpoint $\phi_+ = \sqrt{\chi}$. The rest of the eigenvalues, i.e. $i < N/2$, become purely imaginary. Hence, half of the cut goes along imaginary axis. In fact, there are three possible ways that the cut can go along the imaginary axis. For the first two possibilities all imaginary eigenvalues have the same sign, corresponding to the cut either going up from the origin to $\phi_- = i\sqrt{\chi}$ or the cut going down from the origin to $\phi_- = -i\sqrt{\chi}$. Both situations give rise to one-cut solutions in the weak coupling that have complex free energy. The third possible configuration has one-fourth of the eigenvalues lying along the positive imaginary axis and one-fourth along the negative imaginary axis. Hence, we end up with three cuts emerging from the origin. This solution connects to the Z_3-symmetric solution and so we call it Z_3-solution, even though the symmetry is no longer there at strong coupling. As we will see further, the contributions from the two conjugated cuts along the imaginary axis cancel each other in the free energy, so that the resulting free energy is real.

All of these solutions have part of their eigenvalues lying on the imaginary axis. However, this is inconsistent with approximations we made to obtain equations (7.13). In order to make the solutions consistent with these approximations we should assume that the positions of the eigenvalues satisfy $|\mathrm{Im}(\phi_i - \phi_j)| \gg |\mathrm{Re}(\phi_i - \phi_j)| \gg 1$, i.e. the cuts do not go *exactly* along the imaginary axis but instead diverge slightly from it. In the limit of infinite coupling constant $\tilde{\lambda} \to \infty$ the ratio of the real and imaginary parts of the eigenvalue separations should go to zero. In Fig.7.2 we show the eigenvalue positions for the numerical solution of (7.1) with the coupling constant at $\tilde{\lambda} = 750$. As we see, in general, this picture supports all statements made so far about the solution. In particular, approximately half of the eigenvalues lies on the positive real half-axis, while another half goes up into the imaginary plane at an angle with the real axis that is close to $\pi/2$. Moreover, it was checked for many different parameters m and $\tilde{\lambda}$, that both endpoints of the cut fit the values $\phi_+ = \sqrt{\chi}$ and $\phi_- = i\sqrt{\chi}$ very well.

In Fig. 7.2 the distribution of the eigenvalues along the imaginary axis is not very smooth, contrary to the distribution along real axis which forms an almost perfect line. This is due to the following reasons. First, the kernels in (7.1) contain coth and tanh functions, both having poles along the imaginary axis. These poles significantly decrease the precision of numerical methods.

However there is a more fundamental reason for this chaotic distribution. Let's consider the one-cut solution for small coupling constant $\tilde{\lambda}$. When $\tilde{\lambda}$ is increased the cut starts growing according to its $\tilde{\lambda}^{1/3}$ dependence. However, at a particular value of the coupling we expect this cut to break into two cuts. In order to support this statement we have considered the special case when the mass of the adjoint hypermultiplet is set to $m = \pm i/2$ (see appendix C of paper III). This is the supersymmetry enhancement point. At this point equations of motion reduce to

$$\frac{\pi N}{\tilde{\lambda}} \phi_i^2 \quad = \quad 2\pi \sum_{j \neq i} \coth(\pi(\phi_i - \phi_j)), \tag{7.20}$$

which in the continuous limit corresponds to a singular integral equation with a Hilbert action and can be solved analytically using the resolvent technique. Analyzing this solution we have found that at the critical point $\tilde{\lambda}_c \approx 0.976$ the eigenvalue density goes to zero in the middle of the cut (see Fig. 7.3) signaling a phase transition at this point, in analogy with the phase transition described in chapter 4. We also argue that new breaks appear in the cut along the imaginary axis every time we increase $\sqrt{2\tilde{\lambda}}$ by 2. For a general value of mass we expect the same thing to happen when $\sqrt{\chi}$ is increased by 2. Each break of the cut signals a phase transition. Thus, to go from weak to strong coupling a theory should undergo a chain of equidistant phase transitions. Hence, at strong coupling, the chaotic behavior in Fig.7.2 is the aftermath of crossing many phase transition points.

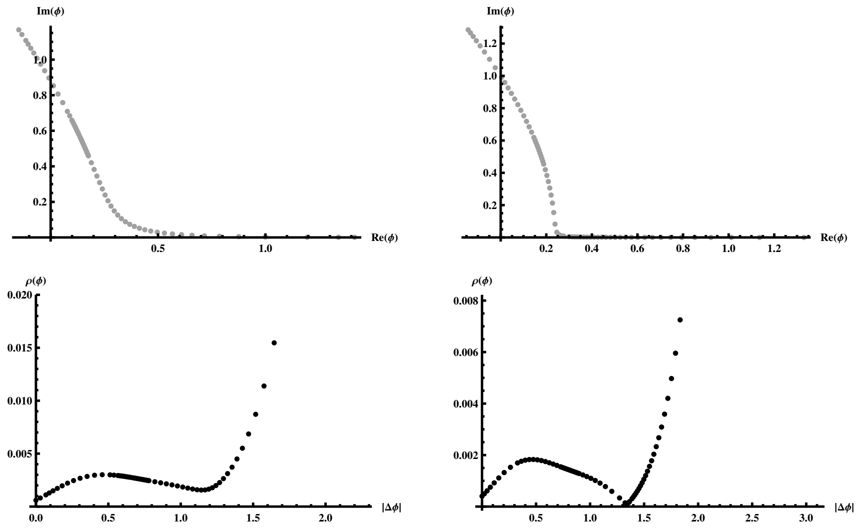
Now let's discuss the $SU(N)$ case. If we set $\kappa = 0$, the equations (7.19) lead straightforwardly to the following values for the cut endpoints and Lagrange multiplier

$$\mu = \pm \frac{i}{\sqrt{3}} \chi, \;\; \phi_+ = \frac{\sqrt{2}}{3^{1/4}} \chi^{1/2} e^{\pm i\pi/12}, \;\; \phi_- = -\frac{\sqrt{2}}{3^{1/4}} \chi^{1/2} e^{\pm 7i\pi/12}. \tag{7.21}$$

However, these endpoints can not be connected by any cut as they belong to different branches. Hence, it is natural to conclude, that there is no analog of the one-cut solutions considered above for the $SU(N)$ case. At the same time numerical analysis shows the presence of a one-cut solution that goes along the imaginary axis and gives $\langle \phi \rangle = 0$.

Finally, we compute the free energy of the solutions we have found for the $U(N)$ case. Under the approximation used to obtain (7.13), the free energy can be rewritten in the following form

$$F \equiv -\log Z \quad \approx \quad \sum_i \frac{\pi k}{3} \phi_i^3 - \frac{(9+4m^2)\pi}{4} \sum_{i<j} |\mathrm{Re}(\phi_i - \phi_j)|. \tag{7.22}$$



(a) $\tilde{\lambda} = 0.8$ (b) $\tilde{\lambda} = 0.976$

Figure 7.3. Cut behavior for different $\tilde{\lambda}$

For the Z_3 solution this becomes

$$
\begin{aligned}
F \approx \sum_{i>N/2} \frac{\pi k}{3}\phi_i^3 - \frac{(9+4m^2)\pi}{4}\frac{N}{2}\sum_{i>N/2}\phi_i \\
- \frac{(9+4m^2)\pi}{4}\sum_{N/2<i<j}(\phi_j - \phi_i)\,,
\end{aligned}
\tag{7.23}
$$

with the contributions from two complex branches canceling each other. Performing the summation in the expression we find

$$
F \approx -\frac{(9+4m^2)^{3/2}\pi}{60} N^2 \tilde{\lambda}^{1/2}\,, \tag{7.24}
$$

If we consider the free energy for solutions that have eigenvalues only along the positive or negative imaginary axis, the free energy gets an imaginary part in addition to the real one. This real part still equals (7.24). These analytical results reproduce the corresponding numerical simulations done in paper III.

Notice that if we substitute $\tilde{\lambda} = N/k$ into the expression (7.24) we obtain $F \sim N^{5/2}$. This reproduces the strong coupling free energy behavior obtained for five-dimensional superconformal theories in [61]. These theories are of a special interest due to the existence of holographic dual in AdS_6 space. Thus, our result gives a hint at the possible existence of a holographic dual to 5D supersymmetric Chern-Simons theory, which is also a conformal theory. However, we should notice that the chain of phase transitions on the way from weak to strong coupling can make the search for a gravity dual more complicated or even impossible.

7.4 Yang-Mills – Chern-Simons phase transitions

Both in strong and weak coupling we considered two phases of the theory in the presence of Yang-Mills and Chern-Simons terms. One phase is dominated by the Yang-Mills term when the κ parameter is large enough. The other phase appears when κ is small enough and the theory is dominated by the Chern-Simons term. In this section we discuss how this transition develops in weak and strong coupling of the $U(N)$ and $SU(N)$ theories.

7.4.1 Weak coupling

We have found that in the weak coupling regime the eigenvalue distribution takes the form (7.7) and the parameter b is found from (7.6). When λ is significantly smaller then $\tilde{\lambda}$ this solution has its cut along the real axis. However, if we start increasing λ, at some value of κ the left branch point will coincide with the zero of the density function in (7.7) at $\tilde{\phi} = -b$. If we increase λ further this branch point moves off the real axis into the complex plane (either up or down), or, alternatively, one cut splits into three. What happens at the critical point depends on which configuration is energetically more favorable and on the choice of the gauge group. For example, in the case of $SU(N)$ the cut is forced to split into three in order to satisfy tracelessness condition $\langle\phi\rangle = 0$.

For both the $U(N)$ and $SU(N)$ cases these transitions take place at $b = \left(\tilde{\lambda}/2\pi\right)^{1/3}$ according to the eigenvalue distribution (7.7). Making use of equation (7.6) and assuming $\mu = 0$ we get the following critical point for the $U(N)$ case

$$\kappa_c \equiv \sqrt{3}\left(\frac{\tilde{\lambda}}{2\pi}\right)^{1/3} \quad \Rightarrow \quad \lambda = \frac{2}{\sqrt{3}}(2\pi)^{7/3}\tilde{\lambda}^{2/3}. \tag{7.25}$$

Similarly, using relation (7.8) one finds a critical point for the $SU(N)$ theory at

$$\kappa_c = \frac{3}{2}(\tilde{\lambda}/2\pi)^{1/3} \quad \Rightarrow \quad \lambda = \frac{4}{3}(2\pi)^{7/3}\tilde{\lambda}^{2/3}. \tag{7.26}$$

Now let's describe the details of the transition between the phases, corresponding to $\kappa > \kappa_c$ and $\kappa < \kappa_c$. In particular, we wish to determine if the transition between these two phases is a phase transition, and, if so, what is the order of this phase transition. To answer these questions one should examine the free energy in the vicinity of the transition point. The general expression for the free energy of matrix model (5.1) in the large-N and weak coupling limits can be evaluated directly from the prepotential (5.10) using the approximations described above [2]

$$F = N^2 \left(\frac{1}{2L} \int d\phi \;\; \rho(\phi)(\tfrac{1}{3}\phi^3 + \kappa\phi^2 - \tfrac{2}{3}\kappa^3) \right. \\ \left. - \frac{1}{2} \int d\phi d\phi' \rho(\phi)\rho(\phi') \log(\phi - \phi')^2 \right) , \tag{7.27}$$

where we have introduced the parameter $L \equiv (\tilde{\lambda}/2\pi)$ and the integrals are taken along the cuts.

Let's start with $U(N)$ case. Near the critical point (7.25) we can parametrize the eigenvalue density (7.4) with the small parameters ε and δ

$$\kappa^2 = 3L^{2/3} - \varepsilon \,, \qquad B = -12L^{4/3} + 4L^{2/3}\varepsilon + L^{1/3}\delta \,. \tag{7.28}$$

Using these expressions one can expand the free energy (7.27) a series in ε and δ as

$$F = N^2 \left(-\frac{3}{4} - \frac{1}{3}\log L + \frac{3}{4} L^{-2/3}\varepsilon - \frac{1}{8} L^{-4/3}\varepsilon^2 - \frac{1}{40} L^{-5/3}\varepsilon\,\delta + \ldots \right), \tag{7.29}$$

where the ellipsis stands for the subleading terms in ε and δ. This expression is valid on both sides of the transition. In order to distinguish between the one-cut and the split cut solutions, one should determine the relation between δ and ε in both cases.

For one-cut solution (7.7) below the critical point ($\varepsilon < 0$), this relation can be found directly from (7.5) and (7.6). In particular, in paper III we obtained $\delta \approx 8(-\varepsilon/3)^{3/2}$, and so the free-energy becomes

$$F = N^2 \left(\text{regular terms} + \frac{1}{15\sqrt{3}} L^{-5/3} (-\varepsilon)^{5/2} + \ldots \right) . \tag{7.30}$$

This shows that the third derivative of F diverges at the critical point $\varepsilon = 0$ and thus the phase transition is third order.

Now let's continue into the region above the phase transition ($\varepsilon > 0$). As we discussed before there are two possible solutions in this phase. Either there is a one-cut solution with the left endpoint moving into the complex plane or the cut splits into three cuts. For the first case all derivatives remain the same and the free energy can be obtained from (7.30) by analytical continuation of ε to the positive real axis.

[2] Here we shift the free energy in two ways. First we subtract a constant term $C = N^2 \left(\frac{1}{8}\log 2 + \log\pi + \frac{7\zeta(3)}{16\pi^2} \right)$ in the same way as in section 7.2 (see the corresponding footnote there). Second we add the term $-\frac{2}{3}\kappa^3$, which depends on the coupling constants but is regular and thus can not change the order of the transition.

For the second solution, which connects to the Z_3-symmetric solution at infinite Yang-Mills coupling, the derivation is more complicated. In total there are four branch points for this particular solution. At the critical point three of these points coincide at $\tilde{\phi} = -L^{1/3}$. As we increase ε and move away from the critical point, the branch points spread apart and the eigenvalue density near the point $\tilde{\phi} = -L^{1/3}$ can be written as

$$\rho(\delta\phi) \approx \frac{1}{4\pi L^{5/6}} \sqrt{4(\delta\phi)^3 + 4\varepsilon(\delta\phi) - \delta}\,, \tag{7.31}$$

where $\delta\phi = \tilde{\phi} + L^{1/3}$ is assumed to be small. In order to find the relation between δ and ε one should impose the condition that $\int \rho(\delta\phi)$ is positive definite. The integral in this expression is taken between the branch points. The positions of these points are determined by δ and ε, hence, by imposing positive definiteness on the integral over the distribution one can find a relation between them. This calculation was performed in paper III and led to

$$\delta \approx (1.40907)\, \varepsilon^{3/2}\,. \tag{7.32}$$

This result was obtained numerically, as the integral $\int \rho(\delta\phi)$ contains complete elliptic integrals of the first and second kind. Substituting this into (7.29) we obtain for the split-cut free energy

$$F = N^2 \left(\text{regular terms} - (0.035223)\, L^{-5/3}\, \varepsilon^{5/2} + \ldots\right)\,, \tag{7.33}$$

where the regular terms are the same as in (7.30). Notice that this free energy is lower than the one for the single cut solution in (7.30). Indeed while the regular terms are the same, the coefficient of the $\varepsilon^{5/2}$ term is imaginary for the one-cut solution, and negative real for the split-cut solution. Hence the last one is energetically favorable. This result is not surprising, as the Z_3-symmetric solution has lower free energy compared to the one-cut solution at the pure Chern-Simons point. As the coefficients are different in (7.30) and (7.33), there is a discontinuity in the third derivatives for this solution as well.

We continue with the $SU(N)$ case. The analysis is similar, but also includes the Lagrange multiplier μ. To get the one-cut solution below the phase transition we set $\kappa = \kappa_c - \varepsilon'$, where κ_c is given by (7.26) and ε' is a small positive real number. Then from (7.8) one obtains the ε'-expansion of b and the free energy

$$F = N^2 \left(\text{regular terms} + \frac{4\sqrt{2}}{15\sqrt{3}} L^{-5/6} (-\varepsilon')^{5/2}\right)\,. \tag{7.34}$$

Hence, this also has a third order phase transition at the critical point, given by κ_c in (7.26).

7.4.2 Strong coupling

In the strong coupling limit, when both $\lambda \gg 1$ and $\tilde{\lambda} \gg 1$, we have found the solutions (7.17). In the *case of* $U(N)$ the endpoints of the cut are given by (7.18). As discussed in section 7.3 this solution has a special point at

$$\kappa = \kappa_c = \sqrt{\chi}\,, \tag{7.35}$$

where χ is given by (7.15). For $\kappa > \kappa_c$ all the eigenvalues are real, while for $\kappa < \kappa_c$ some of them are shifted from the real axis onto the complex plane. The free energy at strong coupling using the assumption $|\mathrm{Re}(\phi_i - \phi_j)| \gg 1$ can be well approximated by

$$\begin{aligned} F \approx N^2 \Bigg(& \frac{\pi}{\tilde{\lambda}} \int_{\phi_-}^{\phi_+} d\phi \left(\tfrac{1}{3}\phi^3 + \kappa\phi^2 - \tfrac{2}{3}\kappa^3\right) \rho(\phi) \\ & - \frac{(9+4m^2)\pi}{8} \int_{\phi_-}^{\phi_+} d\phi d\phi' |\phi - \phi'| \rho(\phi)\rho(\phi') \Bigg)\,. \end{aligned} \tag{7.36}$$

Using the eigenvalue density (7.17) together with the endpoint positions $\phi_\pm$ from (7.18) one finds

$$F \approx N^2 \left(\text{regular terms} + \frac{8\pi}{15(9+4m^2)\,\tilde{\lambda}^2} (-\varepsilon)^{5/2} \right)\,, \tag{7.37}$$

where $\varepsilon = \kappa_c^2 - \kappa^2$ is a parameter defining how far the theory is from the critical point and κ_c is given by (7.35). Hence, once again we obtain a third order phase transition at the critical point.

The last situation we discuss is *the* $SU(N)$ *case at strong coupling*. The solution described in section 7.3 has an eigenvalue density given by (7.17) and cut endpoints given by (7.19). After some manipulations it can be shown that the system of equations defining the endpoints together with the Lagrange multiplier μ has a critical point at

$$\kappa = \kappa_c = \frac{2\sqrt{2}}{3} \chi^{1/2}\,. \tag{7.38}$$

To explore the behavior of the theory around the critical point one should again introduce a small parameter ε' defined as $\varepsilon' = \kappa_c - \kappa$. Then one should first find the ε'-expansion of μ from the relations (7.19) and then evaluate the free energy from (7.36). This calculation, as shown in paper III, results in

$$F \approx N^2 \left(\text{regular terms} + \frac{4\pi}{5\sqrt{6}} \kappa_c^{1/2} \tilde{\lambda}^{-1} (-\varepsilon')^{5/2} \right)\,. \tag{7.39}$$

Hence, the transition stays third order at strong coupling for the $SU(N)$ theory as well.

To summarize, we have found that the matrix model (5.1) experiences a phase transition when both the Yang-Mills and Chern-Simons terms are present.

Table 7.1. *Critical values of κ for different regimes of the matrix model (5.1).*

	$U(N)$	$SU(N)$
strong	$\chi^{1/2}$	$\frac{2\sqrt{2}}{3}\chi^{1/2}$
weak	$\sqrt{3}\left(\frac{\tilde{\lambda}}{2\pi}\right)^{1/3}$	$\frac{3}{2}\left(\frac{\tilde{\lambda}}{2\pi}\right)^{1/3}$

The transition happens when the cut along the real line, when the theory is dominated by the Yang-Mills term, starts either moving onto the complex plane or, alternatively splits into three branches at one of the endpoints. This critical point is characterized by the parameter κ defined in (7.2). The value of κ_c depends on whether the gauge group is $SU(N)$ or $U(N)$ if the theory is in the strong or weak coupling regime. The critical values for the different cases are summarized in table 7.1. All transitions are found to be third order phase transitions. As we have discussed in chapter 4, third order phase transitions are typical for matrix models. However, phase transitions in matrix models are usually accompanied by a break or a merger of the cuts, while in our case this does not necessarily take place.

7.5 Wilson loops

Finally, we discuss the behavior of Wilson loops in pure Chern-Simons theory. It is especially interesting, since the Chern-Simons term in the action is odd under charge conjugation and, thus in principle, Wilson loops in the fundamental and anti-fundamental representations can differ.

As discussed in chapter 6, after localizing the expectation value of the $\frac{1}{2}$-BPS Wilson loop wrapping the equator of the five-sphere reduces to the matrix model expectation value

$$\langle W\rangle^{\pm} = \int d\phi\rho(\phi)e^{\pm 2\pi\phi}\,, \tag{7.40}$$

where $\rho(\phi)$ is eigenvalue distribution at the saddle point of the matrix integral (5.1), and the $+(-)$ sign refers to the fundamental (anti-fundamental) representation.

One can now evaluate (7.40) for the density distributions discussed previously in this chapter. In particular, if we consider the Z_3-symmetric weak coupling solution (7.10) , (7.40) results in

$$\langle W\rangle^{\pm}_{Z_3} = \ {}_1F_3\left(\frac{1}{2};\frac{1}{3},\frac{2}{3},2;\pm\frac{32\pi^2\tilde{\lambda}}{27}\right)\,. \tag{7.41}$$

This expression is real and in the limit $\tilde{\lambda} \ll 1$ its log is approximately

$$\log\langle W\rangle^{\pm}_{Z_3} \approx \pm\frac{4\pi^2\tilde{\lambda}}{3} \tag{7.42}$$

Already we see that the log of the Wilson loop has a different sign for fundamental and anti-fundamental representations.

We have also seen that in the $U(N)$ case the matrix model has one-cut solutions (7.7) with b given by one of the roots of (7.9). For example, the real root b corresponds to the one-cut solution crossing the real axis in Fig.7.1(a). In this case expectation value of the Wilson loop is

$$\langle W\rangle^{\pm}_{1-\mathrm{cut}} = \frac{e^{-a}}{a}\left(a\,{}_0F_1\left(2,-\frac{a^2}{2}\right)+J_2\left(\sqrt{2}a\right)\right) \tag{7.43}$$

where ${}_0F_1$ is the confluent hypergeometric function and $a=\pm(8\pi^2\tilde{\lambda})^{1/3}$. This expression is real and for $\tilde{\lambda} \ll 1$ its log behaves as

$$\log\langle W\rangle^{\pm}_{1-\mathrm{cut}} \approx \mp\frac{3}{2}(\pi^2\tilde{\lambda})^{1/3}. \tag{7.44}$$

In order to obtain the Wilson loop expectation values for the two other one-cut solutions one rotates $\phi \to \phi e^{\pm 2\pi i/3}$, or equivalently the same rotation on a in (7.43), which for $\tilde{\lambda} \ll 1$ leads to

$$\log\langle W\rangle^{\pm}_{n} \approx \mp\frac{3}{2}(\pi^2\tilde{\lambda})^{1/3}e^{2\pi i n/3}, \tag{7.45}$$

where $n=1,2$ corresponds to the two rotated solutions. For the rotated solutions the log of the Wilson loop expectation value is complex. For the solution crossing the real axis the log has a sign opposite to the Z_3-symmetric solution (7.42). Also note that the $\tilde{\lambda}$-dependence of the logs in (7.44) and (7.45) differs from the behavior in (7.42). The last one is linear in $\tilde{\lambda}$ which is more natural for a Wilson loop at weak coupling where perturbation theory is valid, while the one-cut solutions have the stranger $\tilde{\lambda}^{1/3}$ behavior.

Finally, we test the Wilson loop expectation value in the strong coupling limit $\tilde{\lambda} \gg 1$ of pure Chern-Simons theory. As discussed in section 7.3 the solution in this regime exists only for the $U(N)$ case. The corresponding eigenvalue distribution is given by (7.17) with $\kappa = 0$. The cut goes along the real axis from the origin to $\phi_+ = \chi^{1/2}$ and along the imaginary axis from the origin to either $\phi_- = i\chi^{1/2}$ or $\phi_- = -i\chi^{1/2}$. Alternatively, the Z_3 solution has two conjugate cuts going along the imaginary axis from the origin to both $\phi_- = \pm i\chi^{1/2}$ points.

For all three solutions the real part of the Wilson loop expectation value in the fundamental representation is found to be

$$\mathrm{Re}\left(\langle W\rangle^{+}_{strong}\right) \approx \frac{1}{2\pi}\chi^{-1/2}.e^{2\pi\chi^{1/2}}, \tag{7.46}$$

In the cases when the imaginary components of the eigenvalues are all of the same sign the Wilson loop has an imaginary part which is of order χ and can be neglected. For the Z_3-solution the imaginary contributions from the conjugate branches cancel each other and the Wilson loop ends up being real and equal to (7.46). Then the log of the Wilson loop is approximately

$$\log\left(\langle W\rangle^{+}_{strong}\right) \approx 2\pi\chi^{1/2} = 2\pi . \sqrt{\tilde{\lambda}\left(\frac{9}{4}+m^2\right)} \tag{7.47}$$

However, the situation is different for the loop in the anti-fundamental representation where there is no exponentially large factor and the leading order behavior of the Wilson loop's real part is given instead by $\sin 2\pi\chi$ which does not have a well defined large-$\tilde{\lambda}$ limit.

The behavior of the Wilson loop expectation value at the critical points was also examined in paper III. Here it was shown that Wilson loops in both the fundamental and anti-fundamental representations have a discontinuity in their second derivatives, which is in good correspondence with a third order phase transition.

7.6 A comment on the choice of contour and solution

The last remark we make concerns the contour of integration in the matrix model (5.1) and the choice between the different competing solutions found in the different regimes of this matrix model.

In chapter 5 we discussed the convergence of the matrix integral and, in particular, wrote down the asymptotic behavior of the prepotential $\mathscr{F}$ for large vales of ϕ. As noticed there, in the case of an adjoint hypermultiplet the leading cubic terms coming from the one-loop determinants cancel. When the Chern-Simons term is relevant, it then determines the behavior of the matrix model integrand at large ϕ. Hence, in order for the matrix integral to converge an integration contour should go through any pair of the three shaded wedges in Fig.7.1, where the cubic term has positive definite contribution. Thus we have a choice between three possible contours of integration for each eigenvalue that determine the cut configuration. If we want our solution to be continuously connected to the pure Yang-Mills solution described in chapter 6, the contour should go at least through the wedge that covers the positive real half-axis. This already excludes the one-cut solution that crosses the real axis and leaves us with two remaining contours. Now if the integrals for all eigenvalues go through the same two wedges then we should choose the one-cut solution whose end points are in these wedges. However, another possibility is to choose integration contours where half the eigenvalues are integrated over a contour that goes from the positive real wedge to the upper allowed wedge, while the other half are on contours that go to the lower allowed wedge. In this case one choses the Z_3-symmetric solution. The latter choice is preferable since , as we have seen, Z_3-solutions always result in lower energies then one-cut solutions. Finally, for one-cut solutions there are problems with the free energy as well as the Wilson loop expectation values, both of which have imaginary parts. This means the theory has some instabilities for these solutions. At the same time observables for Z_3-solutions are always real as it should be in the ordinary field theory. This suggests that physically Z_3-solutions are the ones that really describe $5D$ supersymmetric Chern-Simons theory.

8. Phase transitions in massive theories

In this chapter we show how phase transitions arise in the decompactification limit of theories with massive hypermultiplets. These phase transitions happen for different matter content and different theories. They were obtained in three-dimensional supersymmetric $U(N)$ Chern-Simons theory with N_f massive fundamental hypermultiplets [11, 102] and in four-dimensional $SU(N)$ $\mathscr{N}=2$ super Yang-Mills theory with $2N$ massive fundamental hypermultiplets [100, 99]. In this chapter we show how phase transitions similar to the ones mentioned above appear in $5D$ $\mathscr{N}=1$ super Yang-Mills with N_f fundamental hypermultiplets.

Different suppersymmetric theories with one massive adjoint hypermultiplet have an even more interesting phase structure. In this case the theory experiences an infinite chain of third-order phase transitions while going from weak to strong coupling in the decompactification limit. Such a phase structure was observed for mass-deformed ABJM theory [4] as well as for $4D$ $\mathscr{N}=2^*$ super Yang-Mills theory [121, 103, 99]. As we discuss in this chapter, a similar story takes place in the decompactification limit of the matrix model (5.1) with one massive adjoint hypermultiplet.

In the following we use the results of section 5.3, where we considered the decompactification limit of the matrix model (5.1). In particular, to derive the equations of motion and the free energy for the different theories we will use the asymptotic form (5.15) of the prepotential $\mathscr{F}$ with the particular hypermultiplet content for each case.

8.1 Fundamental hypermultiplets

we start with $\mathscr{N}=1$ super Yang-Mills with N_f hypermultiplets transforming under the fundamental representation of an $SU(N)$ gauge group. Everywhere in this chapter we turn off the Chern-Simons term by setting $k=0$. As discussed in chapter 5, convex analysis shows that this theory makes sense only when $N_f < 2N$, which we assume to be satisfied in this section.

Before discussing the solutions of the matrix model, we note that the renormalization of the Yang-Mills coupling (5.7) takes the following form

$$\frac{1}{g_{eff}^2} = \frac{1}{g_0^2} - \frac{\Lambda_0 N}{8\pi^2}\left(1 - \frac{1}{2}\zeta\right), \tag{8.1}$$

where $\zeta \equiv N_f/N$ is the Veneziano parameter. According to this formula the coupling constant g_{eff}^2 can be negative, leading to a repulsive central potential for the matrix model (5.1). For convenience we redefine the UV cut-off Λ_0 so that the effective coupling constant can be written in the following form

$$\frac{1}{g_{eff}^2} = -\frac{\Lambda N}{8\pi^2}\,. \tag{8.2}$$

As usual, we take the large-N limit, where the matrix integral (5.10) is dominated by its saddle point. The decompactification limit of the saddle point equation can be obtained by minimizing the asymptotic expression (5.15) for the prepotential $\mathscr{F}$. This leads to the following result

$$\begin{aligned} -2\Lambda\phi = \frac{1}{4}\zeta(\phi+m)^2\mathrm{sign}(\phi+m) + \frac{1}{4}\zeta(\phi-m)^2\mathrm{sign}(\phi-m) - \\ \int d\psi\rho(\psi)(\phi-\psi)^2\mathrm{sign}(\phi-\psi)\,, \end{aligned} \tag{8.3}$$

where the eigenvalue density $\rho(\phi)$ is introduced in the usual way according to (4.13). We assume that the eigenvalue distribution has finite support $[-a, a]$. In analogy with the three- and four-dimensional results described in [11, 102] and [100, 99] respectively, we expect the following picture: The matrix model should have two phases. One phase corresponds to the regime for which the inequality $a < m$ is satisfied and corresponds to small Λ (equivalently we can say that this is the strongly coupled phase of the theory). If we increase Λ, at some point we have $a > m$, corresponding to the second phase of the theory. This phase will contain a pair of sharp peaks at $\phi = \pm m$, created by terms corresponding to the massive hypermultiplets in (8.3).

The best way to solve the integral equation (8.3) is to take three derivatives of this equation, thus reducing it to an algebraic equation

$$\begin{aligned} -2\Lambda = \frac{1}{2}\zeta(\phi+m)\mathrm{sign}(\phi+m) + \frac{1}{2}\zeta(\phi-m)\mathrm{sign}(\phi-m) \\ -2\int d\psi\rho(\psi)(\phi-\psi)\mathrm{sign}(\phi-\psi)\,, \end{aligned} \tag{8.4}$$

$$0 = \frac{1}{2}\zeta\,\mathrm{sign}(\phi+m) + \frac{1}{2}\zeta\,\mathrm{sign}(\phi-m) - 2\int d\psi\rho(\psi)\mathrm{sign}(\phi-\psi)\,, \tag{8.5}$$

$$0 = \zeta\left(\delta(\phi+m) + \delta(\phi-m)\right) - 4\rho(\phi)\,. \tag{8.6}$$

We start with $m > a$ (phase I). In this case the terms with masses in (8.5) cancel each other and we are left with the simple relation

$$0 = \int_{-a}^{a} d\psi\rho(\psi)\mathrm{sign}(\phi-\psi) = \int_{-a}^{\phi} d\psi\rho(\psi) - \int_{\phi}^{a} d\psi\rho(\psi)\,, \tag{8.7}$$

which can be satisfied only if the eigenvalue density is highly peaked near the endpoints of the distribution at $\phi = \pm a$. The natural ansatz is then[1]

$$\rho^{(I)}(\phi) = \frac{1}{2}\left(\delta(\phi - a) + \delta(\phi + a)\right), \tag{8.8}$$

where the overall coefficient is fixed by the normalization condition (4.14). In order to determine the endpoint a of the support, one should substitute (8.8) into the original equation (8.3). This leads to

$$a^{(I)} = \Lambda + \frac{1}{2}\zeta m. \tag{8.9}$$

By construction, this solution works when $a < m$ or, equivalently when $\Lambda < m\left(1 - \frac{1}{2}\zeta\right)$. Thus the position of the *critical point* is at

$$\Lambda_c = m\left(1 - \frac{1}{2}\zeta\right). \tag{8.10}$$

As we see from (8.9) the length of the cut increases with increasing Λ. Therefore, if one increases Λ, at some point the theory enters phase II, corresponding to $m < a$. To find the ansatz appropriate for this regime let's use the third derivative equation (8.6). In phase I the δ-functions do not contribute to this equation and it is equivalent to $\rho(\phi) = 0$, which is satisfied by (8.8) up to the isolated points $\phi = \pm a$. However, in phase II these delta functions contribute to the eigenvalue distribution and a reasonable ansatz should include them on top of the solution (8.8), giving

$$\begin{aligned}\rho^{(II)}(\phi) = \frac{1}{4}\left[(2-\zeta)\delta(\phi + a)\right. &+ \zeta\delta(\phi + m) \\ &\left.+ \zeta\delta(\phi - m) + (2-\zeta)\delta(\phi - a)\right],\end{aligned} \tag{8.11}$$

where the coefficients in front of $\delta(\phi \pm m)$ are chosen to satisfy equation (8.6) and the coefficients in front of $\delta(\phi \pm a)$ are obtained from the normalization condition (4.14).

Substituting the ansatz (8.11) into (8.3) one can determine the position of the support endpoint a

$$a^{(II)} = \left(1 - \frac{1}{2}\zeta\right)^{-1}\Lambda. \tag{8.12}$$

The solution (8.11) is valid, by construction, when $m < a$, which is equivalent to $\Lambda > \Lambda_c$. Thus all values of Λ cover two phases, and unlike $3D$ Chern-Simons with N_f fundamental hypermultiplets, without any intermediate phase [11].

[1] Note that (8.8) does not satisfy (8.6) at the isolated points on boundaries of the eigenvalue distribution. Further in this section, where the effects of finite r are discussed, we give an explanation of the validity of this solution.

The validity of the solutions (8.8) and (8.11) can be checked numerically. In Fig. 8.1 we show the numerical solution to the full equations of motion with finite r using orange dots. On the same plots the vertical dashed lines represent the position of δ-functions in the analytical solutions described in this section. As one can see from these graphs, the analytical solution reproduces the numerical one with very good precision.

The derivation of the solutions (8.8) and (8.11) in the form presented here is not very rigorous. However there is a good way to justify these solutions. To do this we consider a large but finite radius for the five-sphere and include terms subleading in the large-r expansion in (8.3). This leads to a repulsion of the eigenvalues at small separations, which washes out δ-functions and turns them into the peaks of order $1/r$ width. The analytical form of these peaks can be found explicitly. Then the δ-functions with their coefficients can be obtained in the limit $r \to \infty$. Calculation described above was done in appendix C of paper IV and, in the decompactification limit, leads to solutions (8.8) and (8.11).

We now determine if the transition between phases I and II is a phase transition and, if so, what is its order. To answer this we evaluate the free energy of the matrix model (5.10) in the decompactification limit, which according to (5.15) takes the form

$$\frac{1}{2\pi r^3 N^2}F = -\frac{1}{2}\Lambda\int d\phi\rho(\phi)\phi^2 + \frac{1}{12}\int\int d\phi d\psi\rho(\phi)\rho(\psi)|\phi-\psi|^3 - \frac{\zeta}{24}\int d\phi\rho(\phi)\left(|\phi+m|^3+|\phi-m|^3\right) \tag{8.13}$$

Substituting the eigenvalue distributions (8.8) and (8.11) and evaluating integrals one can get

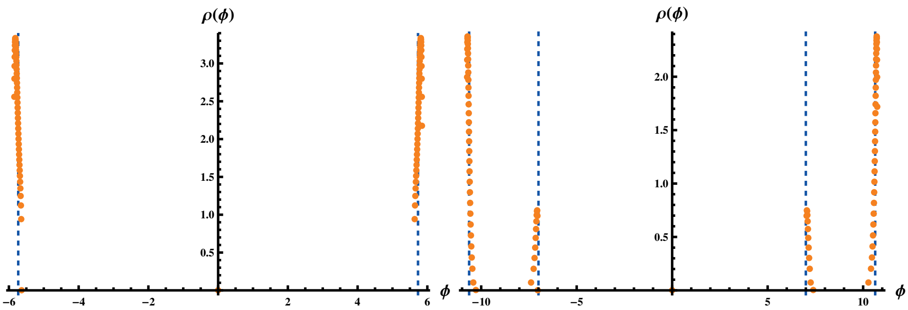
$$\begin{aligned} F^{(I)} &= -\frac{\pi r^3 N^2}{24}\left(8\Lambda^3+12m\zeta\Lambda^2+6m^2\zeta^2\Lambda+m^3\zeta(4+\zeta^2)\right), \\ F^{(II)} &= -\frac{\pi r^3 N^2}{6}\left(\frac{4}{2-\zeta}\Lambda^3+3m^2\zeta\Lambda+m^3\zeta^2\right), \end{aligned} \tag{8.14}$$

where $F^{(I)}$ and $F^{(II)}$ are the free energies in phases I and II. It is then easy to check that the free energy has a discontinuity in its third derivative w.r.t. Λ

$$\partial_\Lambda\left(F^{(II)}-F^{(I)}\right)\Big|_{\Lambda=\Lambda_c} = \partial_\Lambda^2\left(F^{(II)}-F^{(I)}\right)\Big|_{\Lambda=\Lambda_c} = 0, \tag{8.15}$$

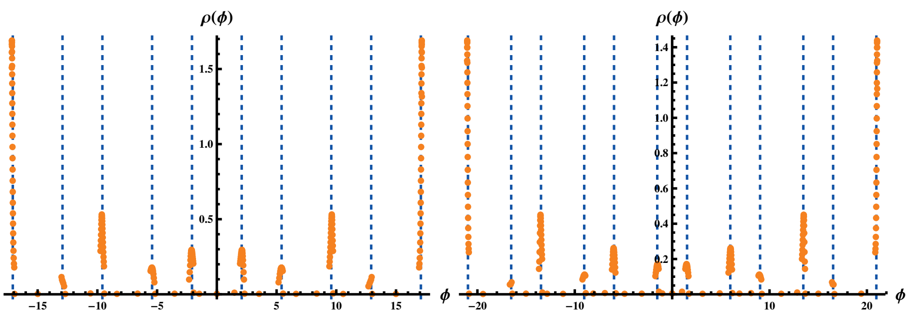
$$\partial_\Lambda^3\left(F^{(II)}-F^{(I)}\right)\Big|_{\Lambda=\Lambda_c} = 2\pi r^3 N^2\frac{\zeta}{\zeta-2}. \tag{8.16}$$

Hence at the critical point $\Lambda_c = \left(1-\frac{1}{2}\zeta\right)m$, the theory experiences a third order phase transition when the decompactification and large-N limits are taken simultaneously. As discussed in chapter 4 third-order phase transitions are typical in matrix models. However the picture here is slightly different from the one presented in chapter 4. There the phase transitions were created by merging or breaking the cuts. Here there is no strict concept of a cut as the support of eigenvalue density is a set of isolated points. Similar third-order phase transitions were observed in $3D$ Chern-Simons-Matter theory [11, 102] and $4D$ $\mathcal{N}=2$ SYM with $2N$ massive fundamental hypermultiplets [99].



(a) $\Lambda = 4.\,(\Lambda < \Lambda_c)$ (b) $\Lambda = 8.\,(\Lambda > \Lambda_c)$

Figure 8.1. Eigenvalue density $\rho(\phi)$ calculated with the following values of parameters: $r=10, m=7, \zeta=\frac{1}{2}, N=100$ corresponding to $\Lambda_c \approx 5.27$. The orange dots show the results for the numerical solution, while the dashed lines show the positions of δ-functions in the analytical solutions (8.8) and (8.11).



(a) $t=58.\left(t_c^{(4)} < t < t_c^{(5)}\right)$ (b) $t=69.\left(t_c^{(5)} < t < t_c^{(6)}\right)$

Figure 8.2. Free energy of $\mathcal{N}=1$ SYM with one adjoint hypermultiplet. The orange dots show the results for the numerical solution, while the dashed lines on (a) and (b) represent the positions of δ-functions in the analytical solutions (8.32). The parameters of the theory are taken to be $r=20, m=7.5, N=200$.

The numerical and analytical analysis in paper IV shows that for finite radius r the phase transition washes out and becomes a smooth crossover. The phase transition takes place only in the decompactification limit.

One can also evaluate Wilson loop expectation values using (6.15) and the eigenvalue distributions (8.8) and (8.11). The calculation shows a discontinuous second derivative of the Wilson loop expectation value w.r.t. Λ at the critical point $\Lambda_c = \left(1 - \frac{1}{2}\zeta\right)m$,

$$\begin{aligned}
&\partial_\Lambda \left(\langle W \rangle^{(II)} - \langle W \rangle^{(I)} \right)\Big|_{\Lambda=\Lambda_c} = 0\,, \\
&\partial_\Lambda^2 \left(\langle W \rangle^{(II)} - \langle W \rangle^{(I)} \right)\Big|_{\Lambda=\Lambda_c} = \frac{\zeta}{2-\zeta} 4\pi^2 r^2 \cosh(2\pi r m)\,,
\end{aligned} \tag{8.17}$$

which is consistent with a third-order phase transition.

8.2 One adjoint hypermultiplet

In this section we discuss the more interesting case of phase transitions taking place in $5D$ $\mathcal{N} = 1$ super Yang-Mills with a massive adjoint hypermultiplet. Similar investigations of the $3D$ mass-deformed ABJM model [4] and $4D$ $\mathcal{N} = 2^*$ super Yang-Mills [103, 99] revealed an interesting phase structure in these theories. In particular, it was found that in the decompactification limit both theories undergo an infinite chain of phase transitions on the way from weak to strong coupling. These phase transitions are third-order for ABJM and fourth-order for $4D$ $\mathcal{N} = 2^*$ super Yang-Mills theory.

In the case of one adjoint hypermultiplet (5.7) shows that the coupling constant g^2_{eff} does not get any renormalized. This happens because the divergent parts of the one-loop contributions from the vector- and hypermultiplets cancel each other. Therefore, from here on we will just use the bare coupling constant, denoting it as g^2_{YM}.

As usual, in the in large-N limit the matrix integral (5.10) is dominated by the solution of the saddle point equation (6.2). In the decompactification limit this equation takes the form

$$\begin{aligned}
&\frac{16\pi^2}{t}\phi = \int d\psi \rho(\psi) \left[-(\phi-\psi)^2 \mathrm{sign}(\phi-\psi) \right. \\
&\left. + \frac{1}{2}(\phi-\psi+m)^2 \mathrm{sign}(\phi-\psi+m) + \frac{1}{2}(\phi-\psi-m)^2 \mathrm{sign}(\phi-\psi-m) \right],
\end{aligned} \tag{8.18}$$

which can also be obtained by minimizing the prepotential (5.10). Here we have also introduced 't Hooft coupling $t = g^2_{YM}N$. Notice that this 't Hooft coupling is dimensionful and differs from the 't Hooft coupling (6.3) used in previous chapters. Equations (8.18) can be solved with an approach similar to the one used in previous section for fundamental hypermultiplets. Namely, we take three derivatives of the equation of motion (8.18) resulting in

$$\begin{aligned} \frac{16\pi^2}{t} &= \int d\psi \rho(\psi)\left[-2|\phi-\psi| + |\phi-\psi+m| + |\phi-\psi-m|\right], && (8.19)\\ 0 &= \int d\psi \rho(\psi)\left[-2\,\mathrm{sign}(\phi-\psi) + \mathrm{sign}(\phi-\psi+m)\right. \\ &+ \left.\mathrm{sign}(\phi-\psi-m)\right], && (8.20)\\ 0 &= -4\rho(\phi) + 2\rho(\phi+m) + 2\rho(\phi-m)\,. && (8.21) \end{aligned}$$

Let's sketch how one solves these equations for different values of the parameters. We assume that the eigenvalue distribution has finite support $[-a,a]$ and first assume that $2a < m$ is satisfied. Then (8.20) becomes (8.7) and can be solved with (8.8). Substituting (8.8) into (8.18) results in the δ-functions positioned at

$$a^{(0)} = m - \frac{8\pi^2}{t}\,. \tag{8.22}$$

By construction this is valid only if $2a^{(0)} < m$. Using (8.22) this inequality can be reformulated into the condition for the 't Hooft coupling $t < \frac{16\pi^2}{m}$.

Now assume that the coupling is increased and the system passes the point $t = \frac{16\pi^2}{m}$, so that $m < 2a < 2m$. Then we expect a situation analogous to 4D $\mathcal{N} = 2^*$ super Yang-Mills, and even more so with the *ABJM* theory. The solution below the transition contains two sharp peaks at the endpoints of the distribution. As the system passes the transition point and the length of the distribution support becomes larger than the mass of the hypermuliplet ($2a > m$), the kernel in (8.18) develops special points where sign functions in the second line change sign assuming there are peaks at the ends of the distribution. These special points are found at $\phi = \pm(m-a)$. To proceed we split the support $[-a,a]$ into three intervals

$$\rho^{(1)}(\phi) = \begin{array}{ll} \rho_A(\phi), & m-a < \phi < a\,, \\ \rho_B(\phi), & -m+a < \phi < m-a\,, \\ \rho_C(\phi), & -a < \phi < -m+a\,, \end{array} \tag{8.23}$$

with the symmetry conditions $\rho_B(\phi) = \rho_B(-\phi)\,, \rho_A(\phi) = \rho_C(-\phi)$. Now the equations of motion can be solved in general form in each of these three regions. In particular in region A (8.21) reads

$$-4\rho_A(\phi) + 2\rho_C(\phi - m) = 0\,, \tag{8.24}$$

where we used that $\phi + m > a$ and $(\phi - m) \in C$. Using the symmetries of the eigenvalue distribution one then obtains

$$2\rho_A(\phi) = \rho_A(m-\phi)\,, \tag{8.25}$$

which can be satisfied only if $\rho_A(\phi) = 0$ everywhere on the interval A, except its endpoints. It is then natural to assume that the eigenvalue distribution consists of two sharp peaks around $\phi = a$ and $\phi = m - a$, which can be approximated by the following density

$$\rho_A(\phi) = c_0 \delta(\phi - a) + c_1 \delta(\phi - m + a)\,. \tag{8.26}$$

In region B (8.20) can be written as

$$\int_{-a}^{a} d\psi \rho(\psi) \mathrm{sign}(\phi - \psi) = \int_{a-m}^{\phi} d\psi \rho_B(\psi) - \int_{\phi}^{m-a} d\psi \rho_B(\psi) = 0\,, \tag{8.27}$$

which leads to the eigenvalue distribution sharply peaked at the endpoints $\phi = \pm(m-a)$ of the interval B.

Hence, using the symmetry properties we arrive at the ansatz

$$\begin{aligned} \rho^{(1)}(\phi) = c_0 \delta(\phi - a) + c_1 \delta(\phi - m + a) \\ + c_1 \delta(\phi + m - a) + c_0 \delta(\phi + a)\,. \end{aligned} \tag{8.28}$$

The coefficients c_1 and c_2 by substituting the ansatz back into (8.18), which results in

$$\begin{aligned} \frac{16\pi^2}{t} = 2a(-2c_0 + c_1) + 2c_0(a+m) + 2c_1(2m-a) \\ + (-2c_1 + c_0)\left(|\phi - a + m| + |\phi + a - m|\right)\,. \end{aligned} \tag{8.29}$$

This equation is satisfied if $c_0 = 2c_1$. Combined with the normalization condition (4.14) this gives

$$c_0 = \frac{1}{3}, \quad c_1 = \frac{1}{6}\,. \tag{8.30}$$

Substituting these values of c_0 and c_1 back into (8.29) we find that the endpoint lies at

$$a^{(1)} = 2m - \frac{24\pi^2}{t}\,. \tag{8.31}$$

By construction this solution is valid when $m < 2a^{(1)} < 2m$. Using (8.31) one can rewrite this inequality as the condition for the ’t Hooft coupling, $\frac{16\pi^2}{m} < t < \frac{24\pi^2}{m}$.

We can continue with this logic for $t > \frac{24\pi^2}{m}$. Consider the parameters satisfying $2m < 2a < 3m$. The previous solution contains two pairs of peaks - one pair of primary peaks on the boundaries of the distribution and one pair of resonances at $\phi = \pm(m-a)$. Now it is reasonable to expect that this resonance peaks will, in turn, create secondary resonances at $\phi = \pm(2m-a)$

because of the cusps in the kernel in (8.18). Hence we expect to find a solution containing six δ-functions located at $\phi = \pm a\,, \pm(m-a)\,, \pm(2m-a)$. If the 't Hooft coupling is increased further so that $2a > 3m$, these secondary resonances will create another pair of resonances and so on. Each time the length of the eigenvalue support is increased by m, a new pair of resonances will emerge. Thus if we want to consider the most general solution we should assume that the parameters of the theory are chosen so that $mn < 2a < m(n+1)$ where $n \in \mathbb{Z}$. Then the peaks at the endpoints $\phi = \pm a$ will create resonances at $\phi = \pm(a-m)$. These resonances will in turn create secondary resonances at $\phi = \pm(a-2m)$ and so on up to the last resonance pair that can fit inside the distribution support, located at $\phi = \pm(a-nm)$. Then the ansatz for the eigenvalue density would be

$$\rho(\phi) = \sum_{k=0}^{n} c_k \left(\delta(\phi - a + mk) + \delta(\phi + a - mk)\right) , \tag{8.32}$$

In order to fix the coefficients c_k we break the support $[-a,a]$ into $(2n+1)$ intervals. Then substituting (8.32) into (8.18) one gets n relations for the $(n+1)$ coefficients c_k. The remaining relation is obtained from the normalization condition (4.14). Details of these derivations can be found in paper IV. The final result is given by

$$c_k = \frac{n+1-k}{(n+1)(n+2)}\,, \tag{8.33}$$

while the position of the support endpoint is

$$a^{(n)} = (n+1)m - \frac{4\pi^2}{t}(n+1)(n+2)\,. \tag{8.34}$$

The 't Hooft coupling is in the range

$$\frac{8\pi^2}{m}(n+1) < t < \frac{8\pi^2}{m}(n+2)\,, \tag{8.35}$$

so that the n^{th} transition point is located at

$$t_c^{(n)} = \frac{8\pi^2}{m}(n+1)\,. \tag{8.36}$$

These derivations lack rigor, because the solutions contain delta functions located at isolated points at which the kernels in (8.18) suffer discontinuities. In paper IV we make this more precise by following the same method used for massive fundamental matter. Namely, we consider the equations of motion (6.2) for the case of finite but large r and also include the first terms subleading in r. These subleading terms will turn on a repulsive interaction between the eigenvalues at small separations and will wash out the δ-functions, turning them into peaks of $\sim 1/r$ width. Then we can split the support into the same intervals as before and solve the corresponding equations of motion, allowing us to find the analytical form of these peaks. In the end we take the decompactification limit $r \to \infty$ and see that these peaks shrink to δ-functions while their coefficients coincide with (8.33). In appendix D of paper IV we did this calculation for the cases of $n = 0$ (no resonances) and $n = 1$, reproducing the solutions (8.8) and (8.28) correspondingly.

In Fig. 8.2 the numerical solutions for large but finite r are shown with the dots, while the positions of the δ-functions in (8.32) are shown with the dashed lines. As we see from this figure the analytical results reproduce the numerical ones very well.

Finally, let's address the question about the order of the phase transitions taking place at the critical points $t_c^{(n)}$. To answer this question we evaluate the free energy and its derivatives at these critical points. The free energy in the decompactification limit can be read directly from (5.10) by choosing the appropriate representation of the hypermultiplet

$$\begin{aligned}\tilde{F} \equiv \frac{1}{2\pi r^3 N^2} F = \frac{4\pi^2}{t} \int d\phi \rho(\phi)\phi^2 + \frac{1}{12} \int \int d\phi d\psi \rho(\phi)\rho(\psi) \Big[|\phi - \psi|^3 - \\ \frac{1}{2}|\phi - \psi + m|^3 - \frac{1}{2}|\phi - \psi - m|^3 \Big] ,\end{aligned} \tag{8.37}$$

After substituting eigenvalue density (8.32) with coefficients from (8.33) and evaluating all integrals one obtains

$$\begin{aligned}\tilde{F}^{(n)} = \frac{1}{12t^3} \Big[256\pi^6(1+n)^2(2+n)^2 - 64\pi^4 mt(1+n)(2+n)(3+2n) + \\ 24\pi^2 t^2 m^2 (1+n)(2+n) - m^3 t^3 (3+2n) \Big] .\end{aligned} \tag{8.38}$$

Taking derivatives shows that the third derivative of the free energy is discontinuous at the critical points $t_c^{(n)}$ in (8.36), with the discontinuity given by

$$\partial_t \left(\tilde{F}^{(n+1)} - \tilde{F}^{(n)} \right) \Big|_{t=t_c^{(n+1)}} = \partial_t^2 \left(\tilde{F}^{(n+1)} - \tilde{F}^{(n)} \right) \Big|_{t=t_c^{(n+1)}} = 0 , \tag{8.39}$$

$$\partial_t^3 \left(\tilde{F}^{(n+1)} - \tilde{F}^{(n)} \right) \Big|_{t=t_c^{(n+1)}} = -\frac{m^6}{512\pi^6 (2+n)^3} . \tag{8.40}$$

Hence at every critical point $t_c^{(n)}$ the theory goes through a third order phase transition. From the (8.40) above we can also see that as the 't Hooft coupling t increases, or equivalently, the phase number n increases, the phase transitions become weaker and turn into crossover transitions in the limit of infinite coupling constant.

Another observable that can test the order of the phase transition is the Wilson loop expectation value, which in the planar limit localizes to the simple integral (6.15). Doing this integration with the eigenvalue distribution (8.32) we obtain discontinuities in the second derivatives at the critical points $t_c^{(n)}$ for these observables

$$\begin{aligned}
&\partial_t \left(\langle W \rangle^{(n+1)} - \langle W \rangle^{(n)} \right)\Big|_{t=t_c^{(n+1)}} = 0\,, \\
&\partial_t^2 \left(\langle W \rangle^{(n+1)} - \langle W \rangle^{(n)} \right)\Big|_{t=t_c^{(n+1)}} = \frac{m^4 \sinh\left(\pi m(2+n)\right)}{16\pi^2(2+n)^2\sinh(\pi m)}\,,
\end{aligned} \tag{8.41}$$

which is consistent with third-order phase transitions at these critical points.

In the decompactification limit of mass-deformed ABJM theory, where there is also an infinite number of third order phase transitions. These transitions also become weaker with increasing coupling. A chain of phase transitions was also found in $4D$ $\mathcal{N} = 2^*$ super Yang-Mills theory. However these phase transitions appear to be fourth order.

It is interesting to understand what happens in the limit of large 't Hooft coupling, where $2a \gg m$, or, equivalently, when the solution contains a large number of resonance pairs. With n pairs t is in the range $\frac{8\pi^2}{m}(n+1) < t < \frac{8\pi^2}{m}(n+2)$, so that for large n we have the approximation $n \approx \frac{mt}{8\pi^2}$. The end-point of the support is then given by

$$a \approx mn - \frac{4\pi^2}{t}n^2 = \frac{m^2 t}{16\pi^2}\,, \tag{8.42}$$

and the δ-functions in the eigenvalue distribution (8.32) average to the constant distribution,

$$\begin{aligned}
\rho(\phi) &= \frac{8\pi^2}{m^2 t}\,, \quad |\phi| \leq a\,, \\
&= 0\,, \qquad\quad |\phi| > a\,,
\end{aligned} \tag{8.43}$$

Now let's return to the strong coupling solution (6.9) presented in chapter 6. After using the following relations to restore the r-dependence

$$m \to mr\,, \quad \lambda \to \frac{t}{r}\,, \quad \phi \to r\phi\,, \quad \rho(\phi) \to \frac{1}{r}\rho(\phi)\,, \tag{8.44}$$

and taking the decompactification limit $r \to \infty$, one can reproduce the eigenvalue density (8.43). Hence, we conclude that at very strong coupling the solutions obtained in this chapter average to the usual strong coupling solution discussed in chapter 6. Thus these two solutions are consistent with each other. Similar results were obtained for the mass-deformed ABJM theory in [4] and for four-dimensional $\mathcal{N} = 2^*$ theory in [103].

9. Summary

Recently there has been much progress in obtaining exact results for supersymmetric gauge theories and a substantial part of this progress is due to supersymmetric localization [93]. Localization allows one to reduce the full path integral, corresponding to the expectation value of some supersymmetric observable, to a matrix integral. Although it is not guaranteed that the resulting matrix integral will be easily solvable, it is still much easier to work with than the original path integral.

In particular, in this thesis we concentrated on results for five-dimensional gauge theories. One of the motivations to study these theories is their interesting relation with $6D$ $(2,0)$ theories [77, 35]. The latter are assumed to be low energy limits of world-volume theories of $M5$-branes. A better understanding of these theories can play an important role in string theory. Another motivation for the study of $5D$ gauge theories comes from the existence of a corresponding Chern-Simons theory. This theory is particularly interesting as it is a conformal fixed point in five dimensions and can, in principle, admit a holographic dual. We discussed some properties of the $5D$ theories, as well as details of their relation with $6D$ theories, in chapter 2.

To work with these supersymmetric gauge theories we used results of localization performed in [63, 62, 58, 66]. Localization reduces the partition function of $5D$ super Yang-Mills theory with a Chern-Simons term to the matrix integral in (3.22)

In this summary we will briefly review the main results presented in this thesis and discuss possible future directions.

9.1 Correspondence between $5D$ and $6D$ theories

One of the most important results obtained in this thesis is related to the relation between $5D$ maximally supersymmetric Yang-Mills theory and $6D$ $(2,0)$ theories. The latter does not have a Lagrangian description, making it difficult to work with. The most useful tool for their study is the AdS_7/CFT_6 duality. Using this duality the N^3 behavior of the free energy and anomalies was derived in the large-N limit [70, 57].

Our main goal was to reproduce this N^3 behavior in $5D$ super Yang-Mills theory. In **papers I** and **II** we solved the matrix model obtained by localizing $\mathscr{N}=1$ super Yang-Mills with one adjoint multiplet on S^5. Some details of this solution were given in chapter 6. We showed that the free energy has the N^3 behavior in the planar limit.

We also calculated the expectation value of the circular Wilson loop in $5D$ super Yang-Mills and the Wilson surface in $(2,0)$ theory. The second was evaluated using the AdS/CFT correspondence. A comparison of these two results at the supersymmetry enhancement point $m = i/2$ led to the identification between the radius of compactification of the $(2,0)$ theory and the $5D$ Yang-Mills coupling proposed in [77],

$$R_6 = \frac{g_{YM}^2}{8\pi^2} \,. \tag{9.1}$$

However in this case there is $4/5$ mismatch between the free energies of $5D$ super Yang-Mills and the $(2,0)$ theory. This mismatch is puzzling and remains unsolved. It would be interesting to study this problem in the future.

We have also found that it is possible to match both free energy and Wilson loop if we move away from the supersymmetry enhancement point. In particular we should fix mass of the hypermultiplet at $m = 1/2$ and modify identification between the radius of compactification of $(2,0)$ theory and $5D$ Yang-Mills coupling, so that

$$R_6 = \frac{5}{2}\frac{g_{YM}^2}{16\pi^2} \,. \tag{9.2}$$

In the future it would be interesting to find other supersymmetric observables for which a comparison could be made. One possibility is the 't Hooft operator which is the magnetic dual of the Wilson loop operator. In five dimensions the 't Hooft operators are some surface operators corresponding to the world-surface of monopole strings. Presumably they can be supersymmetrized and localized, similar to the 't Hooft loops in four dimensions [47, 50]. In the $(2,0)$ theory there is no difference between 't Hooft and Wilson operators as the theory is self-dual and both are surface operators. Hence the $5D$ 't Hooft loop should correspond to a $6D$ Wilson surface that does not wrap the compactification direction. If we can do the calculations for both $5D$ and $6D$ theories we we would have an additional check on the identification of parameters.

9.2 Supersymmetric Chern-Simons-Matter theories

Another theory considered in this thesis was $5D$ supersymmetric Chern-Simons with an adjoint hypermultiplet. This theory is particularly interesting because it is a conformal fixed point in five dimensions suggesting the possibility of an AdS_6 dual.

We studied the large-N behavior of the Chern-Simons matrix model , which includes a cubic term in its potential. The only groups that admits this term are

$SU(N)$ and $U(N)$. We also introduced the analog of the 't Hooft coupling $\tilde{\lambda} = N/k$, where k is the Chern-Simons level. The main results are the following

- There is a sharp difference between the behavior of the $U(N)$ and $SU(N)$ theories due to the cubic term in the action. In particular at strong coupling the free energy of the $U(N)$ theory exhibits $N^{5/2}$ behavior, while $SU(N)$ does not.
- There is a chain of phase transitions going from weak to strong coupling in the pure Chern-Simons theory, with a new phase every time $\sqrt{\tilde{\lambda}\left(\frac{9}{4}+m^2\right)}$ increases by 2. The existence of these phase transitions complicates the search for a supergravity dual and perhaps prevents its existence.
- There is a distinction between Wilson loops in the fundamental and anti-fundamental representations. From one point of view this result is expected due to the odd parity of Chern-Simons action under charge conjugation. It would be interesting to understand the nature of this result from field theory point of view in the future.
- There is a third-order phase transition between the Yang-Mills phase and the Chern-Simons phase. The critical valuesat weak and strong coupling are given in 7.1.

9.3 Phase transitions in massive theories

In **paper IV**, inspired by the series of recent works [121, 103, 99, 4, 11, 102] we investigated the behavior in the decompactification limit $\mathscr{N} = 1$ super Yang-Mills with massive hypermultiplets in different representations of the gauge group. Taking the large-N and decompactification limits simultaneously we obtained the following results.

- For the case of fundamental hypermultiplets there is a phase transition when the points $\phi = \pm m$ move inside eigenvalue density support $[-a,a]$. Two resonances arise at these points signifying the emergence of a new phase. The crossover to the new phase is third-order.
- For the case of adjoint hypermultiplet there is a more interesting phase structure. Here there are a series of third-order phase transitions at critical values of the 't Hooft coupling. At each of these critical values a pair of resonances moves on to the eigenvalue distribution. In the strong coupling limit these distribution spikes become dense and reproduce the smooth strong coupling distribution.

The results of paper IV presented in chapter 8 are similar to decompactification results for three- and four-dimensional theories. Hence these phase transitions are common to all supersymmetric theories with massive matter multiplets. In the future it would be interesting to understand these phase transitions from the field theory point of view. We would also like to test the mass-deformed $5D$ $USp(N)$ super Yang-Mills for the existence of similar phase transitions. This study can be especially interesting because of the existing holographic dual of the undeformed theory.

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Summary in Swedish

Gaugeteorier spelar en viktig roll i den moderna teoretiska fysiken. Alla interaktioner i naturen kan beskrivas i termer av olika gaugeteorier. Till exempel Standardmodellen för partikelfysik inkluderar gaugeteorier av starka kraftens interaktioner, motsvarande $SU(3)$ gauge grupp, och elektrosvag växelverkan, motsvarande $SU(2) \times U(1)$ gauge grupp. De spelar en viktig roll inte bara i partikelfysik utan även i många andra grenar av fysiken.

Ett av de mest utmanande problemen med gaugeteorier är beskrivningen av deras dynamik när interaktionen mellan partiklar i teorin är stark. Svagt samverkande teorier kan behandlas med metoder från störningsteori, medan det inte finns några väl utvecklade metoder för starkt kopplade teorier. För att lösa detta problem betraktar man teorier som medger sådana metoder. Till exempel kan man undersöka gaugeteorier som besitter supersymmetri.

Supersymmetri är en symmetri som relaterar två slags partiklar som finns i naturen: bosoner och fermioner. Det uppstår naturligt i strängteorin där det eliminerar vissa grundläggande problem av denna teori och tillåter en att beskriva materiepartiklarna. Supersymmetri kan också bota flera problem i standardmodellen. Alla dessa fina egenskaper tyder på att supersymmetri kan existera i naturen. Ett antal experiment pågår vid LHC som bland annat syftar till att få bevis för supersymmetri i naturen.

Supersymmetri som alla andra symmetrier sätter extra begränsningar för teorin. Därför kan det i princip förenkla teorier tillräckligt för att göra dem lösbara i ett eller annat sätt. En av de metoder som hjälper till att få exakta resultat i supersymmetriska gaugeteorier är lokalisering.

Den huvudsakliga idén med lokalisering är att man kan reducera en oändligt-dimensionell integreral i kvantfältteori till ändligt-dimensionell matris-integral, vilket kallas en matrismodell. I denna avhandling undersöker vi några resultat i matrismodellen som erhålls från lokalisering av $5D$ supersymmetrisk Yang-Mills teori. Examensarbetet innehåller resultat från fyra artiklar.

I artikel I och II finner vi lösningar på matrismodeller som erhållits från lokalisering av $SU(N)$ $5D$ $calN = 1$ super Yang-Mills teori med hypermultiplet i adjoint representation av gaugegruppen. Med hjälp av dessa lösningar får vi uttryck för den fria energin och väntevärdet för en Wilsonloop. Detta resultat ger oss möjlighet att kontrollera de senaste hypoteserna om dualitet mellan $5D$ super Yang-Mills och $6D$ $(2,0)$ superconformal teori. Denna $6D$ teori är viktigt på grund av dess relation med fysiken av $M5$-branes.

I artikel III utför vi en detaljerad studie av egenskaperna hos matrismodellen som erhållits från lokalisering av $U(N)$ $5D$ supersymmetriska Chern-

Simons-materia teori. Denna teori är viktigt eftersom det är det enklaste exemplet på högre dimensionell Chern-Simons. Det är dessutom ett av de få $5D$ konformala teorier. Resultaten som presenteras i denna avhandling tyder på förekomsten av holografiska duala teorier för $5D$ supersymmetriska Chern-Simons teori. Men man bör lägga märke till att vi också fann en serie fasövergångar som denna teori genomgår på väg från svag till stark koppling. Förekomst av dessa fasövergångar kan göra sökandet efter holografiskt duala teorier komplicerad eller till och med omöjligt.

Slutligen i artikel IV använder vi samma matrismodell som i artikel I och II. Vi beskriver möjliga fas strukturer i $5D$ $calN = 1$ super Yang-Mills med olika hypermultiplar. Som vi visar, en teori med hypermultiplet i fundamentala representationen upplever en tredjedel ordnings fasövergång, om vi varierar kopplingskonstanten mellan stark och svag koppling. Slutligen, för fallet med adjoint hypermultiplet går teorin igenom en oändlig kedja av tredje ordningens fasövergångar på väg från svag till stark koppling.

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