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Black holes, instability and scalar-tensor gravity

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To my little sister, Maria.
The world is big, young lady.

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"Eu quase que nada não sei. Mas desconfio de muita coisa."
Guimarães Rosa, Grande Sertão: Veredas

ABSTRACT

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In this work, we review three topics which are relevant on its own but which are also interconnected through the AdS/CFT correspondence: (i) black holes in AdS and its thermodynamics, (ii) nonlinear instability of AdS and (iii) scalar- tensor theory of gravity. Each one of these topics find applications in holography using the above mentioned correspondence. We review the various coordinate systems used to write the AdS metric and discuss the main black holes with AdS asymptotics as well as their thermodynamical properties. We also review current results on linear and nonlinear stability for various spacetimes, presenting a heuristic explanation for the nonlinear instability of AdS. The discussion about alternative theories of gravity is restricted to the case of scalar-tensor theories (Horndeski theories, specially). We study the multipole expansion of the electromagnetic field in the solitonic background of a shift-symmetric scalar-tensor model (up to second order in the scalar field coupling constant with the Gauss-Bonnet term). We find that the multipoles are everywhere regular and finite except for the monopole $l = 0$, which diverges at the origin of the spacetime coordinates.

keywords: Scalar-Tensor gravity. Black holes. Anti-de Sitter spacetime. Stability.

RESUMO

CÔNSOLE, F. C. C. **Buracos negros, instabilidade e gravidade escalar-tensorial**. 2019. 83p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2019.

Neste trabalho, temos como objetivo fazer uma revisão sobre três temas de grande relevância por si só mas, que também se interligam através da correspondencia AdS/CFT: (i) buracos negros em AdS e sua termodinâmica, (ii) a instabilidade não-linear de AdS e (iii) teorias escalar-tensoriais da gravidade. Cada um destes temas encontram aplicações em holografia usando a correspondencia citada acida. Revisamos as diversas formas de escrever a métrica de AdS e discutimos os principais buracos negros assintóticamentes AdS assim como suas propriedades termodinâmicas. Nós também revisamos os resultados atuais sobre a estabilidade linear e não-linear para diversos espaços-tempos, reproduzindo uma explicação heurísitca sobre a instabilidade não-linear do espaço-tempo AdS. A discussão das teorias alternativas à Relatividade Geral é restrita ao caso das teorias escalar-tensorias da gravidade (a teoria de Horndeski, especialmente). Nós estudamos a expansão multipolar do campo electromagnético em um espaço-tempo que é solução do modelo "*shift-symmetric scalar tensor gravity*" (até segunda ordem na constante de acoplamento do campo escalar com o termo de Gauss-Bonnet) com características solitônicas. Encontramos que os multipolos são regulares e finitos em todo espaço-tempo com exceção do monopolo $l = 0$, que diverge na origem do sistema de coordenadas.

Palavras-chave: Gravidade escalar-tensorial. Buracos negros. Espaço-tempo Anti-de Sitter. Estabilidade.

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

AdS	Anti-de Sitter
BH	Black Hole
GB	Gauss-Bonnet
GR	General Relativity
GW	Gravitational Waves
QNM	Quasinormal Modes
RN	Reissner-Nordström
ST	Scalar-Tensor

CONTENTS

1 INTRODUCTION

General Relativity (GR) is the best theory of gravity that physicists have created up to today, in the sense that its predictions are in accordance with all observations that have ever been made with an incredible accuracy. Nonetheless, the majority of GR tests that have been carried out is within the weak field regime such as solar system tests. An example of an indirect test of GR in strong regime is through the emission of gravitational waves in a binary system. For the first time in 1974, gravitational waves were detected indirectly through a binary system of neutron stars[5] (a neutron star next to a pulsar - a radiating neutron star) in orbit around the center of mass of the system. This system presented variation of the emission of radio waves exactly as predicted in the theory of GR through the emission of gravitational waves. Recently, GR was tested in the strong field regime using two intriguing and fascinating predictions of GR: (i) black holes (BHs) which are predict to be the end state of the gravitational collapse of very massive stars with a gravitational field so strong that not even light can escape and (ii) gravitational waves (GWs), ripples of spacetime that propagates at the speed of light. On 14th of September 2015 the same GW signal was observed at two earth based light interferometers and its analysis fits extremely well with the templates of BHs merger. When two BHs merge, the final result is a BH with a smaller mass than the sum of the masses of the two initial BHs. The difference in mass is irradiated through GWs. The analyze of the data showed that the signal comes from the coalescence of two black holes with the emission of 3 solar masses of energy through GWs. [6] This observation was followed by others BH-BH mergers and one neutron star-neutron star merger, where the latter allow us to look at both gravitational waves and electromagnetic waves coming from the same event and possibly discover some hints of how matter behaves in regimes of high pressure and strong gravity as is the case in neutron stars. As precision detection is advancing it will be possible to test GR in these strong gravity regimes and look, for example, for additional fundamental fields which may be relevant in the strong gravity regime.

In GR, the laws of Physics are governed by two principles: the *equivalence principle* and the principle of *general covariance*. The former means that locally, the motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame. The latter states that the metric of the spacetime is the only quantity "pertaining to spacetime" that can appear in the laws of physics. [7] Einstein postulated the equation that governs GR, partly motivated by the the relation between the tidal forces in Newtonian gravity and the relative acceleration between nearby geodesics in curved spacetime. Nowadays, however, it's common to obtain the equations of motion of the system using the principle of least action, which is omnipresent in modern Physics. So in this context, one can derive the equation that Einstein postulated by varying the Einstein-Hilbert action*

$$S_{E-H} = \frac{1}{16\pi} \int d^4x \sqrt{-g} R. \tag{1.1}$$

R is a scalar quantity constructed from the *metric field* known as Ricci scalar. In this setup, the geometry of spacetime is described by the metric field, $g_{\mu\nu}$, which is the principal object in GR. The equation that governs the dynamics of the metric field is given by the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.2}$$

where $R_{\mu\nu}$ and $T_{\mu\nu}$ are the Ricci tensor and the energy-momentum tensor of energy and matter, respectively. The energy momentum tensor is defined as the variation of the matter action with respect to the metric

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{1.3}$$

The phrase by John Wheeler illustrate the idea behind GR: "*Spacetime tells matter how to move; matter tells spacetime how to curve*". Einstein's field equation (1.2) is a set of coupled nonlinear partial differential equations which are very hard to solve (analytically and numerically) without the use of symmetry. Notably, just after the final formulation of GR by Einstein, Karl Schwarzschild discovered a static solution to Einstein's equation in vacuum, $T_{\mu\nu} = 0$, assuming spherical symmetry. This solution is known nowadays as the Schwarzschild solution and describes a non-rotating black hole of mass m. In spherical coordinates (t, r, θ, ϕ), the metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2 d\phi^2, \qquad f(r) = 1 - \frac{2m}{r}. \tag{1.4}$$

At $r = 2m$ and $r = 0$, the metric function g_{rr} diverges indicating that the spacetime at these coordinate values is not well behaved. Nonetheless, a more careful analysis based on physical invariant quantities tells us that the apparent singularity at $r = 2m$ is due to a bad choice of coordinates and in fact, all physical quantities computed at this coordinate value remain finite. On the other hand, the coordinate value $r = 0$ is really a physical singularity with scalar quantities diverging there: GR predicts its own demise. In fact, it was proven that singularities must form within GR, under general assumptions on the nature of the matter such as the positivity of the energy density but it's conjectured that these singularities must be hidden inside an event horizon and do not affect the spacetime outside. This is the so called *cosmic censorship conjecture*†. If a naked singularity is visible to an outside observer, an end is placed to the predictive power of GR. It's believed,

* We use geometric units where the Newton constant and the speed of light are set to unity, $G_N = c = 1$.

† A rigorous statement of the cosmic censorship and its proof is one of the biggest open problems in GR.

however, that these singularities will be resolved by a quantum description of spacetime. This is a difficult topic that physicists have been trying to understand better for quite a long time and develop a consistent quantum theory of gravity.

One approach to such a quantum description of gravity is through the *AdS/CFT correspondence*. Here *AdS* stands for Anti-de Sitter spacetime (see Section 2.1) and CFT for conformal field theory (see Section B). This correspondence, in a specific limit, relates physical observables in a classical theory of gravity in AdS spacetime to a physical observable in a strongly coupled conformal field theory that lives in at least one lower dimension[‡]. This duality is an example of the *holography principle*. [8] The holographic principle states that for any consistent quantum theory of gravity the number of microscopic degrees of freedom necessary to describe a region of spacetime must be proportional to the area of that region instead of the volume of the region. The principle has its origins in the Bekenstein-Hawking formula for the entropy of a black hole, which is proportional to the area of the BH rather than the volume as it is usual for thermodynamical systems (see Section 2.4). Since the duality is of the weak/strong type, one can study classical GR and use the correspondence to learn about strongly coupled quantum field theories (QFTs), such as the ones believed to describe superconductors at high critical temperature. [9] In this context, BHs plays an essential role since it's a thermodynamic system and its temperature is associated with the QFT temperature through the correspondence. Also, BH formation in the bulk is interpreted as thermalization of the dual theory, so the study of stability of spacetimes (see Chapter 3), besides being important on its own, is related to thermalization properties of strongly coupled QFTs.

Alternatives theories of gravity (that is, theories of gravity different from GR) have been extensively studied in the last few years. Motivation mainly comes from cosmology: when trying to explain the accelerated expansion of the universe through a model that has GR as a limit, since in the solar system scale GR is very well tested. Recently, however, motivation to study alternative models of gravity has also come from holography. These models can be used as toy models when trying to understand strongly coupled QFTs. This works as follows. Introduce the necessary ingredients (fields) in the gravity theory that is believed to approximately describe the properties of the QFT of interest even though there's no exact duality between the two theories. This approach is called bottom-up in contrast to the top-down approach which starts with a formulation of string theory and derives the precise relation between the two theories.

Outline of this dissertation

This dissertation is organized as follows: In chapter 2 we review Anti-de Sitter

‡ The duality is sometimes referred as bulk-boundary duality since the CFT can be seen as living on the conformal boundary of AdS.

(AdS) spacetime and show its metric in different coordinate systems, each one of them being more suitable to discuss specific problems. We briefly discuss a scalar field in the fixed AdS background and show the origin of the Breitenlohner-Freedman (BF) bound. Then, we show the most common BHs solutions within AdS spacetime and mention the no hair conjecture giving a sketch of the no-scalar-hair theorem. This is followed by a discussion of BH thermodynamics and the chemistry of BHs. Chapter 3 is devoted to the study of *stability*. We briefly discuss the stability of the most popular spacetimes in theoretical physics. The linear stability problem is not as difficult to solve as the nonlinear stability, so we begin that chapter with the discussion of quasinormal modes (QNMs) and mention one application in astrophysics and holography. The nonlinear instability of AdS is analyzed based on reference [10] and at the end of the chapter we discuss the generalizations that have been considered in the literature. We deal with scalar-tensor theories of gravity in Chapter 4. We briefly mention how to go beyond GR and introduce new degrees of freedom while preserving properties of GR. An Einstein-scalar-Gauss-Bonnet model is discussed both for the asymptotically flat and asymptotically AdS case. Chapter 5 contains the final remarks and conclusions. Let us mention that this dissertation does not intend to be a comprehensive review on any of the subjects discussed in here and we refer the reader to the original articles whenever we think it appropriate.

2 ANTI-DE SITTER, BLACK HOLES AND THERMODYNAMICS

This chapter discusses Anti-de Sitter spacetime (AdS), we show some coordinate systems used to describe it and discuss some of its distinguishing properties. We briefly mention the origin of the Breitenlohner-Freedman (BF) bound for a scalar field in fixed AdS background and then we discuss some black hole solutions in AdS spacetime as well as their thermodynamics.

2.1 Anti-de Sitter and its avatars

This section is devoted to a brief introduction to Anti-de Sitter (AdS) spacetime and the coordinate systems used to write down the AdS metric. We follow the references [1,11]. To begin, AdS is a maximally symmetric spacetime. This means that it has the maximum number of symmetries a D-dimensional spacetime can have. In General Relativity, symmetries are described by Killing vectors, this is, vectors ξ^μ satisfying the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.1}$$

It can be show that the second derivative of a Killing tensor is proportional to the Riemann tensor

$$\nabla_\mu \nabla_\nu \xi^\rho = R^\rho_{\mu\nu\delta}\xi^\delta. \tag{2.2}$$

Indeed, acting with the commutator $[\nabla_\mu, \nabla_\nu]$ on the Killing vector ξ^λ and summing cyclic permutations of the indices

$$\nabla_{[\mu}\nabla_\nu \xi_{\lambda]} = R^\rho_{[\lambda\mu\nu]}\xi_\rho = 0 \tag{2.3}$$

where the right hand side of the equation vanishes by the first Bianchi identity. The left hand side of (2.3) can be written as

$$\nabla_\mu \nabla_\nu \xi_\lambda + \nabla_\lambda \nabla_\mu \xi_\nu + \nabla_\nu \nabla_\lambda \xi_\mu = 0. \tag{2.4}$$

Using the Killing equation and rearranging the above equation we have

$$\nabla_\lambda \nabla_\mu \xi_\nu = -[\nabla_\mu, \nabla_\nu]\xi_\lambda = R^\rho_{\lambda\mu\nu}\xi_\rho \tag{2.5}$$

which shows that the second covariant derivative of a Killing vector at a given point is proportional to the Killing vector at the same point. Therefore, given the Killing vector $\xi^\mu(x_0)$ and its derivative $\nabla_\nu \xi^\mu(x_0)$ in the spacetime point x_0, it's sufficient to completely characterize the Killing vector everywhere. Using the fact that in a D-dimensional spacetime we can only have, at most, D independent vectors and the anti-symmetry property of the Killing equation, the number of independent Killing vectors a D-dimensional spacetime can have is $D + D(D-1)/2 = D(D+1)/2$. Any D-dimensional spacetime with this maximal number of independent Killing vectors is said to be maximally symmetric. A simple example of a maximally symmetric spacetime is 4-dimensional Minkowski spacetime. It has $4(4+1)/2 = 10$ independent Killing vectors; 4 translations in space and time coordinates, and 3 Lorentz boosts and 3 spatial rotations. It also can be shown that in a maximally symmetric spacetime, the Riemann tensor is proportional to the metric in the following way

$$R_{\mu\nu\rho\sigma} = k\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{2.6}$$

for some constant k. The idea to proof this result goes as follows: in *Riemann normal coordinates*[*] at a point x_0, use isotropy of space to write the metric at x_0 as $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$. By assumption, the metric is isotropic at x_0, then the curvature tensor must be invariant under Lorentz rotations. We use the fact that the only invariant tensors under Lorentz rotations are the Minkowski metric $\eta_{\mu\nu}$ and the Levi-Civita tensor ϵ_{ijkl}. This implies that the Riemann curvature tensor must have the following form

$$R_{\mu\nu\rho\sigma}(x_0) = a\eta_{\mu\nu}\eta_{\rho\sigma} + b\eta_{\mu\rho}\eta_{\nu\sigma} + c\eta_{\mu\sigma}\eta_{\nu\rho} + d\epsilon_{\mu\nu\rho\sigma}. \tag{2.7}$$

a,b, c and d are constants.The symmetries of the Riemann tensor[†] implies that $a = d = b + c = 0$ and the Riemann tensor simplifies to

$$R_{\mu\nu\rho\sigma}(x_0) = b(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{2.8}$$

Therefore, in an arbitrary coordinate system, the Riemann tensor takes the form

$$R_{\mu\nu\rho\sigma}(x_0) = b(g_{\mu\rho}(x_0)g_{\nu\sigma}(x_0) - g_{\mu\sigma}(x_0)g_{\nu\rho}(x_0)). \tag{2.9}$$

Using the homogeneity property of a maximally symmetric spacetime, we can write the Riemann tensor in an arbitrary point x as

$$R_{\mu\nu\rho\sigma}(x) = b(x)(g_{\mu\rho}(x)g_{\nu\sigma}(x) - g_{\mu\sigma}(x)g_{\nu\rho}(x)). \tag{2.10}$$

Contracting the above equation to obtain the Ricci tensor and the Ricci scalar

$$R_{ij}(x) = (D-1)b(x)g_{ij} \tag{2.11}$$
$$R(x) = D(D-1)b(x) \tag{2.12}$$

substituting $b(x)$ in Eq. (2.10), the Riemann tensor can be finally written as

$$R_{ijkl}(x) = \frac{R}{D(D-1)}(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{2.13}$$

[*] Riemann normal coordinate always can be constructed around a given point, p, such that the metric at this point is the Minkowski metric, $g_{\mu\nu}(p) = \eta_{\mu\nu}$, and the Christoffel symbols vanish at this point, $\Gamma^{\mu}_{\nu\rho}(p) = 0$.

[†] The symmetries of the Riemann tensor used: $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$, $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

Furthermore, D-dimensional AdS spacetime is a solution to Einstein's equation with a negative cosmological constant in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0. \tag{2.14}$$

Substituting the expressions for the Ricci tensor and the Ricci scalar in Eq. (2.14), we find that the cosmological constant is related to the Ricci scalar as

$$\Lambda = \frac{D-2}{2D} R. \tag{2.15}$$

from now on we write $D = d+1$, where d is the number of spatial dimensions. The AdS$_{d+1}$ metric can be cast in many different ways. In what follows we will review some of the main coordinate systems that are used in the literature. As usual for constant curvature spacetimes (spheres and hyperboloids), the (d+1)-dimensional AdS metric, AdS$_{d+1}$, can be obtained by embedding the hypersurface

$$-(X^0)^2 + \sum_{i=1}^{d} (X^i)^2 - (X^{d+1})^2 = -L^2 \tag{2.16}$$

into a $(d+2)$ dimensional Minkowski spacetime with two timelike coordinates, $\mathbb{R}^{2,d}$ with coordinates (X^0, X^i, X^{d+1}), $i = 1, ..., d$. L has length dimension and is called the AdS radius. The metric is

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d} (dX^i)^2 - (dX^{d+1})^2. \tag{2.17}$$

The advantage of this construction of AdS$_{d+1}$ spacetime is that it makes explicit that it has $SO(2,d)$ as the isometry group. Using the following coordinates transformations

$$X^0 = L \cosh\rho \cos\tau, \qquad X^i = L \sinh\rho\, \Omega_i, \qquad X^{d+1} = L \cosh\rho \sin\tau \tag{2.18}$$

the constraint (2.16) is automatically satisfied and we obtain AdS spacetime in so-called *global coordinates* (τ, ρ, Ω_i)

$$ds^2 = -L^2 \left(\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2 \right). \tag{2.19}$$

The τ coordinate is periodic with period 2π and introduces the problem of closed timelike curves. One can evade this complication by going to the *universal covering group* of the hyperboloid (2.16) which essentially consists of taking the τ coordinate to be in the range $-\infty < \tau < \infty$ without identifying τ with $\tau + 2\pi$. What is usually called AdS space in the literature is the universal covering group of AdS. With the change of coordinates $\tan\theta = \sinh\rho$, $(0 \le \theta < \pi/2))$ we arrive at the *conformal* metric for AdS$_{d+1}$:

$$ds^2 = \frac{L^2}{\cos^2\theta} \left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2 \right) = \frac{L^2}{\cos^2\theta} \left(-d\tau^2 + d\Omega_d^2 \right). \tag{2.20}$$

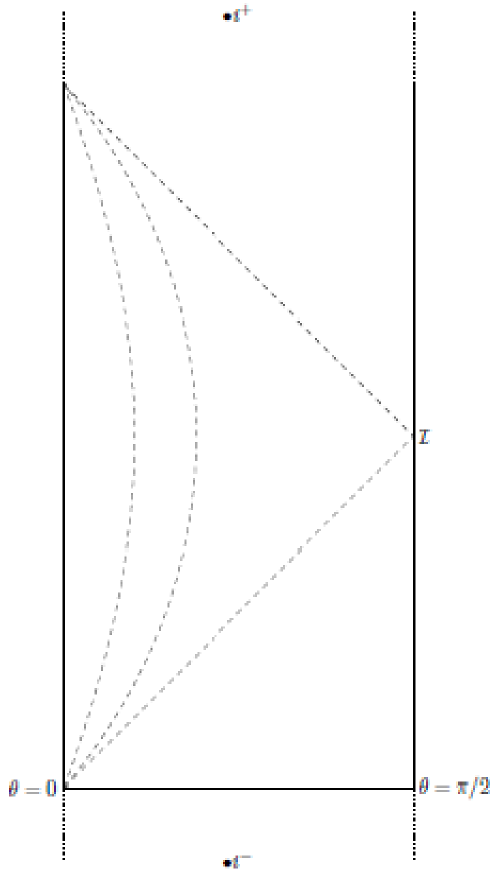

Figure 1 – Penrose diagram for AdS.
Source: BLAU. [1]

This coordinate system for AdS_{d+1} tell us that it's conformal to the upper half of the Einstein static universe $\mathcal{R}$ x $\mathcal{S}^d$. With this metric we can draw a Penrose diagram for AdS_{d+1} spacetime (Fig. 1). Penrose diagrams are useful because they enable us to encapsulate the causal structure of the entire spacetime in a compact picture. This is done by a coordinate transformation that preserves the timelike, null or spacelike separation between points. As usually we suppress $d-1$ angular coordinates. Note that the θ coordinate is compact while τ is not. We cannot make both of these coordinates compact without changing the fact that light rays travel at 45º in the diagram. Fig. 1 tell us another important feature of AdS_{d+1} space: it has a timelike conformal boundary. In other words, spatial infinity $\rho \to \infty$ corresponds to $\theta = \pi/2$ in the conformal metric (2.20) and the induced metric on the conformal boundary $\theta = \pi/2$ has Lorentzian signature

$$ds^2_{|\theta=\frac{\pi}{2}} = -d\tau^2 + d\Omega^2_{d-1} \tag{2.21}$$

with topology $\mathcal{I} = \mathcal{R}$ x $\mathcal{S}^{d-1}$. One can see that a point p in the future of a spacelike hypersurface Σ as the one drawn at the bottom of Fig. 1 has past directed timelike or null casual geodesics that do not intercept Σ. That is, AdS has no Cauchy hypersurface. For this reason, AdS is not *globally hyperbolic* and the initial data on a hypersurface is not enough to describe the evolution of a system; we also need to prescribe conditions on the boundary.

Yet another coordinate system which is used in the AdS/CFT correspondence (CFT stands for conformal field theory. See Appendix B) is the Poincaré coordinate (t, z, x^i). [1] It's obtained by the following coordinate transformation:

$$\begin{aligned} X^0 &= \frac{1}{2z}(z^2 + L^2 - t^2 + x_1^2 + ... + x_{d-1}^2) \\ X^{d+1} &= \frac{1}{2z}(z^2 - L^2 - t^2 + x_1^2 + ... + x_{d-1}^2) \\ X^i &= \frac{L}{z}x^i, \qquad (i = 1, ..., d) \end{aligned}$$

where $z > 0$ and the AdS_{d+1} metric becomes

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx_1^2 + ... + dx_{d-1}^2 + dz^2\right). \tag{2.22}$$

The motivation to consider this coordinate system for AdS_{d+1} is because it makes the Poincaré symmetry of AdS explicit. Nonetheless, the Poincaré coordinates do not cover the whole AdS spacetime, since the relation between the coordinates are

$$z = \frac{L^2}{X^0 - X^{d+1}} \tag{2.23}$$

and we must have $z > 0$, the Poincaré coordinates only cover the region $X^0 > X^{d+1}$ of the entire AdS spacetime.
In gravitational physics, a useful coordinate system for the AdS spacetime is the system of *static coordinates* (t, r, Ω_i), which are obtained from the global coordinates (2.19) by the following coordinate transformation:

$$t = L\tau, \qquad r = L\sinh\rho. \tag{2.24}$$

Then the AdS_{d+1} metric becomes

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2. \tag{2.25}$$

One intriguing property of AdS is that light rays reach the conformal boundary in finite proper time. To see this, let us consider the trajectory of a radial outgoing light ray. We use the conformal metric (2.20) and set $d\Omega^2 = 0$ since we want to analyze purely radial null geodesics. Null geodesics satisfies the equation $ds^2 = 0$, which implies

$$-d\tau^2 + d\theta^2 = 0, \qquad \frac{d\theta}{d\tau} = \pm 1, \tag{2.26}$$

we choose the plus signal for outgoing light rays and the solution is given by

$$\tau(\theta) - \tau(\theta_0) = \theta - \theta_0 \tag{2.27}$$

remembering that $\theta = \pi/2$ (the conformal boundary) corresponds to $\rho = \infty$, we conclude that the (coordinate) time a light ray takes to go from the deep inside ($\theta = 0$) of AdS to the conformal boundary is $\delta\tau = \pi/2$. The facts presented in this section about AdS_{d+1} i.e., the presence of a timelike boundary which is conformal to Minkowski spacetime, the need to prescribe boundary conditions and the fact that AdS_{d+1} has $SO(2,d)$ as the symmetry group, which is the same symmetry group of *conformal transformations* in d dimensions (see Appendix B), make it possible to construct the dictionary between the bulk and the boundary theory.

2.2 Scalar fields in Anti-de Sitter

There's a non trivial property of scalar fields in AdS_{d+1} background: they can be *stable* even with negative mass as long as their mass isn't too negative. The reason for this is that AdS spacetime gives a contribution for the mass term of the scalar field, generating an effective mass $m_{ef}^2 = m^2 + d^2/4L^2$. To check this, we consider a minimally coupled scalar field, ϕ, in fixed AdS spacetime, whose action is given by[‡]

$$S = \int d^{d+1}x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu - \frac{1}{2}m^2\phi^2\right) \tag{2.28}$$

where ∇_μ is the covariant gravitational derivative with respect to the AdS metric and, $\mu = 0, ..., d$. The equation of motion for the scalar field is obtained by varying the above action with respect to the scalar field and reads

$$\left(\nabla^\mu\nabla_\mu - m^2\right)\phi = 0, . \tag{2.29}$$

The above equation can be written explicitly using the Poincaré form (2.22) of the AdS metric and reads

$$\frac{1}{L^2}\left(z^2\partial_z^2 - (d-1)\partial_z + z^2\eta^{ij}\partial_i\partial_j - m^2L^2\right)\phi(z, x^i) = 0. \tag{2.30}$$

Where $i = 0, 1, ..., d-1$ and η_{ij} is the Minkowski metric. Assuming translation invariance and taking the Fourier transform with respect to x^i coordinates

$$\phi(z, x^i) = \int \frac{d^dx}{(2\pi)^d}e^{ik_ix^i}\phi_k(z), \tag{2.31}$$

Eq. (2.30) becomes

$$\left(z^2\partial_z - (d-1)z\partial_z - (m^2L^2 + z^2k^2)\right)\phi_k(z) = 0 \tag{2.32}$$

There is an exact solution for this equation in terms of the Bessel's functions but generally, the case of interest is the behaviour of this scalar field in the asymptotic regime which in the Poincaré coordinates means $z \to 0$. In this case, the term z^2k^2 can be neglect in

‡ We set $c = \hbar = 1$.

relation to m^2L^2 and substituting the ansatz $\phi_k(z) \sim c_k z^\Delta$, we obtain an algebraic second order equation for Δ

$$m^2L^2 = \Delta(\Delta - d). \tag{2.33}$$

The solutions are $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}$. Reality of $\Delta_\pm$ implies that

$$m^2L^2 \geq -\frac{d^2}{4}. \tag{2.34}$$

This is the Breitenlohner-Freedman (BF) bound. It was shown [12] that for theories satisfying the BF bound (2.34) the energy of the spacetime is positive and the theory is stable. When $\Delta_+ \neq \Delta_-$, the asymptotic solution in positions space is given by

$$\phi(z \to 0, x^i) = \phi_0(x^i)z^{\Delta_-} + \phi_1(x^i)z^{\Delta_+} + ... \tag{2.35}$$

where $\cdots$ denote corrections to the expansion. We can define an inner product on AdS [13] given by

$$(\phi_1, \phi_2) == -i \int_{\Sigma_\tau} dz d^d x \sqrt{-g} g^{tt} \left(\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*\right) \tag{2.36}$$

then, the solution (2.35) will have different properties according to the values of $\Delta_\pm$. The modes are split in two sets

- $-\frac{d^2}{4} \leq m^2L^2 < -\frac{d^2}{4} + 1$;
- $-\frac{d^2}{4} + 1 \leq m^2L^2$.

In the first set, both modes, Δ_- and Δ_+, are normalizable[§] with respect to the inner product defined in Eq. (2.36). While in the second set, the modes z^{Δ_-} are non-normalizable and the modes z^{Δ_+} are normalizable. We can think of the non-normalizable mode as living on the boundary of AdS, since

$$\phi_0(x) = \lim_{z\to 0} \frac{1}{z^{\Delta_-}} \phi(z, x) \tag{2.37}$$

This function has *scaling dimension* Δ_- (see Appendix B), since under a scale transformation $x^i \to x'^i = \lambda x^i$, the field transform as

$$\phi_0(x) \Rightarrow \phi_0'(x') = \lim_{z\to 0} \frac{1}{z^{\Delta_-}} \phi'(z, \lambda x) \tag{2.38}$$

$$= \lim_{z'\to 0} \frac{\lambda^{-\Delta_-}}{z'^{\Delta_-}} \phi'(\lambda z', \lambda x) \tag{2.39}$$

$$= \lambda^{-\Delta_-} \phi_0(x) \tag{2.40}$$

where in the first step we have defined $z' = z/\lambda$ and in the second step we used the transformation property of the scalar field. This fact indicates that we can think of $\phi_0(x)$

§ Normalizable in the sense that we can define a convergent norm for ϕ. Non-normalizable means that the norm of ϕ with respect to the inner product do not converge to a finite value.

as being a source for an operator in the CFT with scaling dimension Δ_+, since this would make the action

$$S = \int d^d x \phi_0(x) \mathcal{O}(x). \tag{2.41}$$

invariant under scale transformation; the sum of the scaling dimension of each term in the action vanish, using the fact that $\Delta_- + \Delta_+ = d$.
For the normalizable modes, it's possible to quantize the action (2.28) and construct the Hilbert space of the theory in the bulk which is identified with the Hilbert space in the dual theory. In resume, normalizable modes corresponds to sates of the dual theory and non-normalizable modes corresponds to a source for an operator with scaling dimension Δ_+.

2.3 Black holes in Anti-de Sitter

In common language, a black hole (BH) is a region of the spacetime from which nothing can escape, not even light. In technical terms, given a spacetime provided with a metric $(\mathcal{M}, g_{ab})$, a black hole is the region $\mathcal{B} = \mathcal{M} - \mathcal{I}^-(\mathcal{I}^+)$. That is, the entire spacetime excluded the *chronological past* of the *future null infinity.* The *event horizon*, $\mathcal{H}$, is the boundary of $\mathcal{B}$. This definition encapsulates the idea that what is outside the event horizon can reach out spatial (or null) infinity while what has passed through the event horizon cannot escape to spatial or null infinity.

There are black holes within the AdS spacetime analogous to the asymptotically flat Schwarzschild, Reissner-Nördstron and Kerr BH usually called Schwarzschild-AdS, Reissner-Nördstron-AdS and Kerr-AdS black holes, respectively[¶]. We don't give the derivation of these solutions here and simply write the relevant metric for our purposes. One can check that they're solutions to the Einstein field equation in vacuum with a negative cosmological constant by explicitly calculating the Ricci tensor and substituting into the equation.

2.3.1 Schwarzschild-AdS

The spacetime for a spherically symmetric BH with AdS asymptotics is described in the static coordinates by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{d-1}, \qquad f(r) = 1 + \frac{r^2}{L^2} - \frac{2m}{r^{d-2}}. \tag{2.42}$$

Where d is the number of spatial dimensions. The resemblance with the Schwarzschild metric is immediate; there's an additional term r^2/L^2 in the warp function f. The constant

¶ There's a famous example of BH in 2+1 dimensions called BTZ (the letters coming from Bañados, Teitelboim, and Zanelli) black hole. It has properties similar to the Kerr BH; an event horizon, an inner horizon and an ergosphere however it is asymptotically AdS. This BH has been studied as a toy model to gain insight into quantum gravity. [14]

m is related to the mass M of the black hole by

$$m = \frac{8\pi M}{(d-1)Vol(\mathcal{S}^{d-1})} \tag{2.43}$$

the locus of the event horizon can be found to be at $r = r_h$, largest value of r that fulfills the equation $f(r_h) = 0$. Sometimes it's useful to write the metric in terms of r_h instead of the parameter m. Then the metric function becomes

$$f(r) = 1 + \frac{r^2}{L^2} - \frac{r_h^{d-2}}{r^{d-2}}\left(1 + \frac{r_h^2}{L^2}\right). \tag{2.44}$$

If we send $L \to \infty$, (which corresponds to setting $\Lambda = 0$) we recover the Schwarzschild spacetime. In section 2.4 we review some interesting properties of the Schwarzschild-AdS black hole concerning its thermodynamics. Surprisingly, one can generalize this black hole solution by changing the unity in the metric function f by k, where $k = 0, -1$ and then replacing the (d-1)-sphere Ω^2_{d-1} by the euclidean plane $\mathbb{R}^{d-1}$ or the (d-1)-dimensional torus $\mathbb{T}^{d-1}$ for $k = 0$ and hyperboloid $\mathbb{H}^{d-1}$ for $k = -1$, respectively. These are known as *topological black holes* [15] more or less motivated by the fact that the event horizon no longer has a spherical symmetry.

2.3.2 Reissner-Nordströn-AdS

A spherically symmetric charged BH in AdS spacetime is a gravitational solution to the Einstein equation with negative cosmological constant coupled to a $U(1)$ gauge filed A_μ with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Assuming that the BH carries only electric charge, the only non-vanishing component of this field tensor which gives the electric field is

$$F_{rt} = \frac{Q}{r^{d-1}}, \tag{2.45}$$

in the static coordinates (t, r, Ω_i) the corresponding spacetime is described by the metric [16]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{d-1}, \qquad f(r) = 1 - \frac{2m}{r^{d-2}} + \frac{Q^2}{r^{2(d-2)}} + \frac{r^2}{L^2}, \tag{2.46}$$

where the parameter m is related to the mass of the BH and the parameter Q to the total electric charge. Inspection on the curvature invariants indicates that this solution possesses a single physical singularity located at $r = 0$. The metric function f can be written as

$$f(r) \equiv \frac{\Delta(r)}{L^2 r^{2d-4}}, \qquad \Delta(r) = L^2 r^{2d-4} - 2mL^2 r^{d-2} + Q^2 L^2 + r^{2d-2}. \tag{2.47}$$

$\Delta(r)$ is a polynomial of degree $2d-2$ for $d \geq 3$, so in $d = 3$, $\Delta(r)$ has 4 roots that we will not give here because they're large expressions. [17] The event horizon is located at the largest value of r that is a solution of $f(r_h) = 0$ which implies $\Delta(r_r) = 0$. In $d = 3$ and for $\Lambda = 0$, there are two horizons (an event and a Cauchy horizon) which reads $r_\pm = m \pm \sqrt{m^2 - q^2}$.

The condition $m^2 \geq q^2$ prevents the appearance of a naked singularity. When Λ is small compared to m^{-2} we expect the same structure to be present in the RN-AdS as in the flat RN black hole: for values of q^2 less than some critical value, q^2_{cr},there will be an outer horizon (the event horizon) and an inner event horizon (a Cauchy Horizon). These two horizon coincide when $q^2 = q^2_{cr}$ and we obtain what is called an extremal RN black hole. For $q^2 > q^2_{cr}$ we have two complex roots, i.e. the event horizon does not shield the physical singularity from external observation, which is referred to as a naked singularity.
It's convenient to write the mass m in terms of r_h:

$$2m = r_h + \frac{Q^2}{r_h} + \frac{r_h^3}{L^2}. \tag{2.48}$$

The area of the event horizon at given time $t = t_0$ is obtained by fixing $r = r_h$ and integrating the angular variable θ and φ

$$A = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \;\, r_+^2 \sin\theta = 4\pi r_+^2 \tag{2.49}$$

In section 2.4 we will study the thermodynamical properties of some asymptotically AdS black holes (the RN-AdS BH displays a rich thermodynamic phase structure and possesses some similarities with a van der Waals liquid gas system [18]) and the area of the event horizon divided by 4 is associated with the entropy of the black hole. Taking the limit $L \to \infty$, one recovers the asymptotically flat RN black hole. It is believed that the flat counterpart of the RN-AdS solution is not relevant to astrophysics, since any net charge would tend to neutralize within an astrophysical environment. But it has interesting and curious properties that make it instructive to study, and in particular it has similar properties than the spinning black holes that are astrophysically relevant.

2.3.3 Kerr-AdS

Before discussing the asymptotically AdS spinning black hole, we briefly review some properties of the asymptotically flat spinning black hole. It took nearly 50 years after the publication of the final version of GR to find the solution of a spinning black hole. [19] In 1963 through a *tour de force*, Roy Kerr found the most astrophysically important solution of a black hole: the Kerr black hole. In *Boyer-Lindquist coordinates*, (t, r, θ, ϕ), its metric reads

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{2arsin^2\theta}{\rho^2} (dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &+ \frac{sin^2\theta}{\rho^2} \left(\left(r^2 + a^2\right)^2 - a^2 \Delta sin^2\theta\right) d\phi^2 \end{aligned} \tag{2.50}$$

where

$$\Delta(r) = r^2 - 2mr + a^2, \qquad \rho^2(r,\theta) = r^2 + a^2 cos^2\theta. \tag{2.51}$$

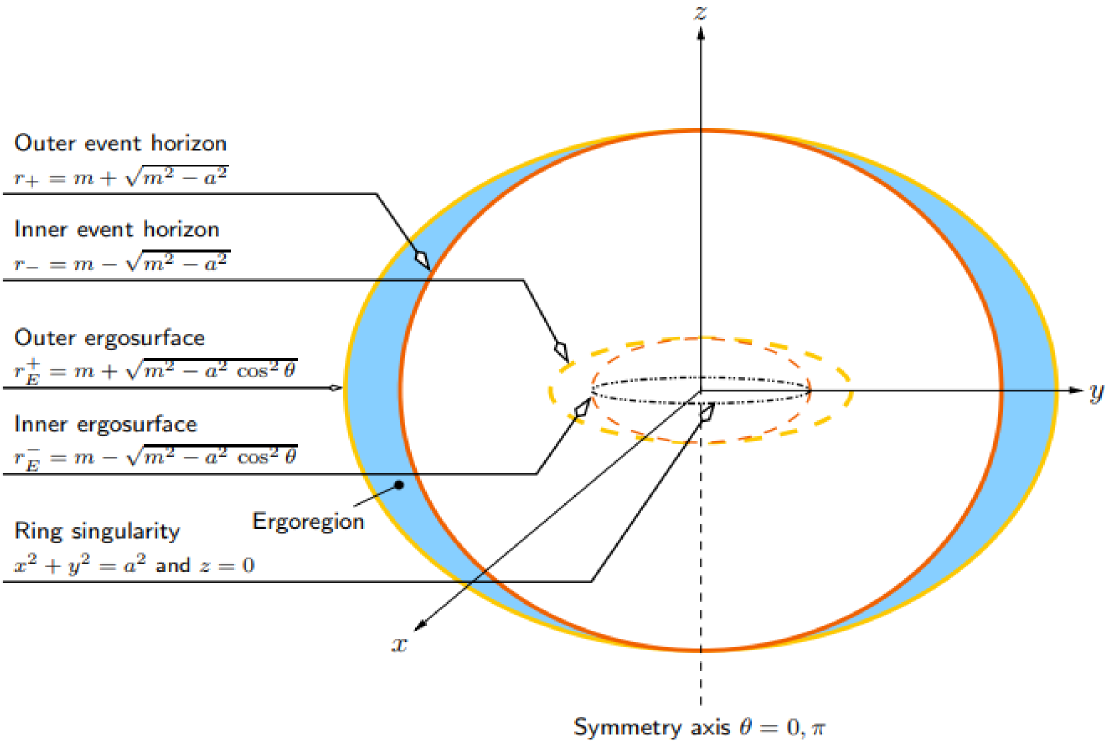

Figure 2 – Horizons, singularities and ergosphere of asymptotically flat Kerr black hole.
Source: VISSER. [2]

m is the BH mass and $J = ma$ is the BH angular momentum. The event horizon is located at $r_+ = m + \sqrt{m^2 - a^2}$, the large root of $\Delta(r) = 0$. The induce metric on the event horizon $\mathcal{H}$ is

$$ds^2 = \left(r_+^2 + a^2 \cos^2\theta\right) d\theta^2 + \left(\frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\right) \sin^2\theta d\phi^2 \tag{2.52}$$

Let $h = (r_+^2 + a^2)^2 \sin^2\theta$ be the determinant of the induced metric on the event horizon, then the area of the event horizon is given by

$$A = \int_{\mathcal{H}} \sqrt{h} d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta (r_+^2 + a^2) \sin\theta = 4\pi(r_+^2 + a^2). \tag{2.53}$$

This black hole possesses novel characteristics that are not found in the Schwarzschild and Reissner-Nordströn black holes. It's *stationary*, different from the Schwarzschild and RN black holes which are *static*. Both terms encompasses the idea of time independent metric but static is a stronger condition. One way to characterize them is to say that stationary metrics are invariant under time translations while static metrics are invariant under time translation and invariant under time reflections $t \to -t$. The Kerr geometry also possesses an infinite-redshift surface** known as (outer) ergosurface outside the horizon defined by $K^\mu K_\mu = 0$, where $K = \partial_t$ is the Killing vector timelike at infinity. In fact, if a black hole

** That is, any light ray emitted from this surface will be infinitely redshifted when observed at infinity.

event horizon is present in a stationary and axisymmetric spacetime, there will be an ergoregion. For the metric (2.50), the ergosurface is located at

$$r_{ergo} = m + \sqrt{m^2 - a^2 \cos^2\theta} \tag{2.54}$$

the region between the ergorsurface and the event horizon is the *ergoregion*. Due to the presence of an ergosphere[††], it's possible to extract energy from the Kerr black hole by a process know as *Penrose process*. Another difference from Schwarzschild BH is the nature of the singularity: for $\rho = 0$ the curvature invariants diverges. But $\rho = 0$ defines a ring in space given by $r = 0$ and $\theta = \pi/2$. We should remember the reader at this point that in the Boyer-Lindquist coordinates r is not an euclidean radial coordinate but rather related to the euclidean coordinates (x, y, z) by

$$x = (r^2 + a^2)^{1/2} \sin\theta \cos\phi, \quad y = (r^2 + a^2)^{1/2} \sin\theta \sin\phi, \quad z = r\cos\theta. \tag{2.55}$$

The generalization of the Kerr BH to include a negative cosmological constant was found in 1968 for $d + 1 = 4$ spacetime dimensions [20] and to all $d + 1 > 5$ spacetime dimensions in. [21] The $d + 1 = 4$ Kerr-AdS black hole metric is given by

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Sigma}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{\rho^2}\sin^2\theta\left(adt - \frac{r^2 + a^2}{\Sigma}d\phi\right)^2 \tag{2.56}$$

with

$$\Delta_r = \left(r^2 + a^2\right)\left(1 + \frac{r^2}{L^2}\right) - 2Mr, \qquad \Sigma = 1 - \frac{a^2}{L^2} \tag{2.57}$$

$$\Delta_\theta = 1 - \frac{a^2}{L^2}\cos^2\theta, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2.58}$$

The solution is valid for $a^2 < L^2$ and becomes singular for $a^2 = L^2$. The event horizon, $\mathcal{H}$, is located at $r = r_+$, the largest root of $\Delta_r(r) = 0$. The induced metric on the event horizon is formally obtained from Eq. (2.56) by setting $r = r_+$, $dt = dr = 0$ and the fact that r_+ is solution to $\Delta_r(r) = 0$ and is given by

$$ds^2_{\mathcal{H}} = \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(\frac{(r_+^2 + a^2)}{\Sigma}\right)^2 d\phi^2. \tag{2.59}$$

The determinant, h, of the induced metric is $h = \sin^2\theta(r_+^2 + a^2)^2/\Sigma^2$. Then the area, A, of the event horizon is given by

$$A = \int_{\mathcal{H}} \sqrt{h} d\theta d\phi = 4\pi \frac{\left(r_+^2 + a^2\right)}{\Sigma} \tag{2.60}$$

†† In this region is impossible to "stand still" since the time-translation Killing vector becomes spacelike [19] and the observer is forced to rotate with the BH. However, the observer can still move toward or away the event horizon.

comparison with Eq. (2.53), the area of an asymptotically flat Kerr BH, we see that it increases with L decreasing (or alternatively, with Λ increasing in absolute value) and that in the limit $L \to \infty$ reduces to the area of the asymptotically flat Kerr BH given in (2.53).

2.3.4 Gauss-Bonnet-AdS

The Einstein-Hilbert action (Eq. 1.1) contain only one curvature term, the Ricci scalar. Theories with higher order curvature terms, (such as $R^2, R^{\mu\nu}R_{\mu\nu}, R_{\mu\nu}R^{\mu\sigma}R^{\nu}_{\sigma}, ...$) appears as low energy limit of string theories and in the context of AdS/CFT correspondence the additional higher order curvature terms are seeing in the dual theory as corrections to the large N expansion. Considering only second order curvature terms, BH solutions were found in. [22] The second order curvature term is the Gauss-Bonnet (GB) term, $\mathcal{G}$, a specific combination of curvature quantities which is a topological invariant[‡‡] in (3+1) dimensions. It is given by

$$\mathcal{G} = R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{2.61}$$

The action of Einstein-Gauss-Bonnet gravity is given by

$$S = \frac{1}{16\pi}\int d^{d+1}x\sqrt{-g}\left(R - 2\Lambda + \alpha\mathcal{G}\right). \tag{2.62}$$

where α is the GB coefficient with dimension $(lenght)^2$. In $d = 3$ Einstein-GB gravity reduces to GR and from now on we consider $d \geq 4$. With an static and spherically symmetric ansatz of the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 h_{ij}dx^i dx^j, \tag{2.63}$$

BH solution was found. [23] h_{ij} is the metric on a $d-1$ dimensional hypersurface with constant curvature, which can be positive, negative of null. The metric functions ν and λ are assumed to depend only on the radial coordinate r. The solutions are given by

$$e^{2\nu} = e^{-2\lambda} = k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{64\pi\tilde{\alpha}M}{(d-1)\Sigma_k r^d} - \frac{4\tilde{\alpha}}{L^2}}\right). \tag{2.64}$$

where $\tilde{\alpha} = \alpha(d-2)(d-3)$ and $k = 0, 1, -1$. Σ_k is the volume of the $d-1$ dimensional hypersurface and M the gravitational mass of the solution. The solution given by Eq. (2.64) is asymptotically AdS and has two branches: "-" and "+" sign. When $M = 0$, we see from Eq. (2.64) that the coupling constant must satisfies $4\tilde{\alpha}/L^2 \leq 1$ in order for the solution to be real. Thermodynamics properties of this solutions is analyzed in Section 2.4.

‡‡ The integral over the entire four dimensional spacetime (*manifold*) of the GB term is a characteristic of the spacetime (manifold) and do not contribute to the equations of motion.

2.3.5 No hair conjecture

The black holes spacetime discussed so far are fully characterized by the parameters $\{M, J, Q\}$, related to the mass, angular momentum and charge, respectively, and the AdS radius L which can always be set to unity by a coordinate redefinition. It is because of the few parameters needed to describe the *classical* spacetime of a black hole that leads to the statement that black holes are simple objects. This is certainly not true for other astrophysically relevant objects. For example, to describe the exterior region of a neutron star with sufficient accuracy, one would need to know all the distribution of matter inside the star and therefore, all the multipole moments of the configuration of matter inside the star, while two black holes with the same parameters and asymptotics are exactly equal. The simplicity of black holes is described by the famous phrase *black holes have no hair.* [24] "Hair" is used as a metaphor for complicated things and with the idea that if everyone were bald, we would be more alike. For 4-dimensional electro-vacuum GR with flat asymptotics, there are theorems known as *uniqueness theorems* [25,26] assuring us that the only possible parameters that can be observed at infinity are $\{M, J, Q\}$. However, when one of the theorem's assumptions is violated, there can be black hole solutions with hair, that is, with an additional parameter[§§] beyond the standard ones describing the exterior region of the black hole spacetime. And indeed, a variety of hairy black hole have been found within theories with non-linear matter sources. [27]

Scalar fields are the simplest type of field one can conceive and in 2012 scientists at CERN have found a fundamental scalar field in nature: the Higgs field. So, to consider if a scalar field in a black hole spacetime contributes with an additional type of hair for this black hole is a very natural question to ask, and indeed, there's a set of no-scalar-hair theorems. [28] In general, these theorems make assumptions about the coupling between the scalar field and gravity, which may be minimal or non-minimal, the inherited spacetime symmetries for the scalar field and the types of potential for the scalar field. Violation of one of these assumptions can make it possible to find a hairy black hole. For example, Kerr black holes with scalar hair have been found [29] violating the assumption about the symmetries of the scalar field and the spacetime being the same.

Following [30] we sketch the proof a no-scalar-hair theorem for a minimally coupled scalar field in a generic BH spacetime with flat asymptotic. The idea is to consider the scalar field equation with a potential $U(\phi)$ on a fixed curved background

$$\Box\phi = U'(\phi). \tag{2.65}$$

Where $'$ represents derivative with respect to the scalar field and $\Box = \nabla^\mu\nabla_\mu$ is the covariant d'Alembertian. We assume the background to be stationary, asymptotically flat

[§§] In the literature, the hair is usually classified as *primary* or *secondary*. Primary hair being a new global charge and secondary hair being not independent from the standard ones.

and at spatial infinity we should have $\phi = \phi_0$, $U'(\phi_0) = 0$. We also assume that the scalar field to respect the symmetries of the background. The stationary assumption implies the existence of a timelike Killing vector ξ at infinity and if we're considering vacuum solutions to Einstein's equation, it will have to be axysymmetric as well by Hawking's theorem [31], implying the existence of second Killing vector ζ that has closed orbits. Then, multiplying Eq. (2.65) by $U'(\phi)$ and integrating over the volume $\mathcal{V}$

$$\int_{\mathcal{V}} d^4x\sqrt{-g}\, U^{'}(\phi)\Box\phi = \int_{\mathcal{V}} d^4x\sqrt{-g}\, U^{'2}(\phi). \tag{2.66}$$

We construct the volume $\mathcal{V}$ in such way that it's spatially bounded by a timelike 3-surface at infinity, $\mathcal{S}_0$, and by the BH event horizon, $\mathcal{H}$, and temporally bounded by two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. $\mathcal{S}_1$ is a partial Cauchy hypersurface and $\mathcal{S}_2$ is obtained from $\mathcal{S}_1$ by a shifting each point of $\mathcal{S}_1$ by a unit parameter distance along integral curves of ξ^μ. Using Integration by parts on the l.h.s of Eq. (2.66) and rearranging the equation gives

$$\int_{\mathcal{V}} d^4x\sqrt{-g}\,[U^{''}(\phi)\nabla^\mu\phi\nabla_\mu\phi + U^{'2}(\phi)] = \int_{\partial\mathcal{V}} d^3x\sqrt{|h|}\, U^{'}(\phi)n^\mu\nabla_\mu\phi. \tag{2.67}$$

On the r.h.s, the boundary integration is over the surfaces $\mathcal{S}_0$, $\mathcal{H}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. The integration over the timelike 3-surface, $\mathcal{S}_0$, vanish because we assume $\phi \to \phi_0$ at spatial infinity. The integration over $\mathcal{H}$ vanish because the normal vector to the horizon, η^μ, is a linear combination of ξ and ζ which are orthogonal to the horizon. The last part of the boundary integral is over the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ which do not vanish but exactly cancel each other by the stationarity condition. On the l.h.s. of Eq. (2.67) we have the sum of a non-negative term, $U^{'2}(\phi)$, with $U^{''}(\phi)$ multiplying $\nabla_\mu\phi\nabla^\mu\phi$. However, the gradient of the scalar cannot be timelike anywhere or null everywhere as it is orthogonal to both of the Killing vectors. So, as long as $U^{''}(\phi) > 0$, the only possible way for the integration of two non-negative terms to vanish is if the scalar field is trivial $\phi = \phi_0$ (which implies $U^{'}(\phi) = 0$) outside the event horizon. The $U'' > 0$ condition can be seen as a local stability condition for the scalar field. It must be said that there is not only one no-scalar-hair theorem but in fact, there's a few of them, each one with different assumptions suitable for the theory of interest.

It turns out that the possibility of forming non-trivial hair is a necessary ingredient in the case of AdS/CFT applications to condensed matter and the different boundary conditions of AdS invalidates the class of no-scalar-hair theorems proved for asymptotically flat spacetimes. As an example, the holographic models used to describe high T_c superconductors need that the scalar field becomes unstable to formation of non trivial hair near the black hole horizon at low temperature. [32] This instability to form scalar hair at low temperature mimics the second order phase transition that happens in the superconductor.

2.4 Black hole thermodynamics

It was a milestone in the history of physics when Hawking discovered that due to quantum effects, black holes can emit radiation with a perfect thermal spectrum. [33] The temperature associated with the black hole is given by

$$T = \frac{\kappa}{2\pi} \tag{2.68}$$

in units where the Newton constant, the speed of light, the reduced Planck constant and the Boltzmann constant are equal to one. κ is the surface gravity and it is defined by the equation

$$\nabla^a(\xi^b \xi_b) = -2\kappa \xi^a \tag{2.69}$$

where ξ^a is a Killing vector normal to the event horizon. The surface gravity can be though of as the acceleration of a *static* ¶¶ observer near the BH event horizon as measured by an observer near infinity. To avoid confusion, we explicitly state that T given by (2.77) is the temperature as measured by an observer near infinity.

Bekenstein did not accept that throwing stuff inside a black hole would diminish the entropy of the universe since the no-hair theorem states that the only charges characterizing the black hole were their mass, charge and angular momentum. He would rather "save" thermodynamics than accept that thermodynamics does not work when considering black holes. Motivated by a geometric theorem stating that the horizon area of black holes never decreases and associating this result with the second law of thermodynamics, he proposed to consider a black hole having an entropy proportional to its event horizon area. [34] At the time this idea was not very well accepted, but everything changed with the discovery of Hawking that when considering quantum effects, a black hole is not a perfect sponge anymore; it emits radiation.

Hawking's results originated from the analysis of particle creation in curved space-time. He considered the gravitational collapse to form a Schwarzschild BH and then considering a free scalar field a with vacuum initial state ***, he calculated the expected number of particles of the scalar field at spatial infinity at late times. The conclusion is that the expected number of particles is equal to the emission of particles by a black body at temperature given by (2.77). This important discovery allows one to treat black holes as thermodynamical objects. But this discovery also brought a new paradox to theoretical physics: *the information loss paradox*.

Let us state the 4 laws of black hole thermodynamics based on. [35,36]

• 0. The 0-th law states that the surface gravity κ of a stationary black hole is constant all along the event horizon.

¶¶ A static observer keeps its spatial coordinates fixed.

*** State containing no incoming particle fromm the past null infinity $\mathcal{I}^-$.

• 1. The first law of BH thermodynamics states that the increase in the BH mass is mediated by the changes in the BH area, A, angular momentum, J, and charge, Q

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi dQ. \tag{2.70}$$

where κ is the surface gravity, Ω_H is the angular velocity of the event horizon, and Φ is the electric potential at the horizon. Equation 2.70 is the gravitational version of the first law of thermodynamics

$$dU = TdS - PdV \mu dN + \Phi dQ. \tag{2.71}$$

where μ is the chemical potential and N the particle number.
• 2. *The generalized 2nd law* states that in a system containing black hole and matter, the total amount of entropy never decreases

$$\delta S_{BH} + \delta S_{ext} \geq 0. \tag{2.72}$$

• 3. Also known as Nernst's law, the 3rd law states that it is impossible to reduce the surface gravity to zero by a finite sequence of operations and it is not really a fundamental law in thermodynamics.

We give a basic idea of how to calculate the temperature of a stationary black hole by the *euclidean path integral* method. It's not as rigorous as Hawking's derivation but gives the same results saving us a lot of work. First, we consider the euclidean version of the stationary metric, which we obtain by performing a Wick rotation in the time coordinate: $t \to i\tau$

$$ds^2 = f(r) d\tau^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{d-1}. \tag{2.73}$$

we assume the function $f(r)$ has a simple zero on the horizon r_h (this is, $f(r_h) = 0$ and $f'(r_h) \neq 0$) and expand this function near the horizon: $f(r) \sim f'(r_h)(r - r_h)$. Then (2.73) becomes

$$ds^2 = f'(r_h)(r - r_h) d\tau + \frac{1}{f'(r_h)(r - r_h)} dr^2 + r^2 d\Omega^2_{d-1} \tag{2.74}$$

$$= d\rho^2 + \rho^2 d\varphi^2 + r_h^2 d\Omega^2_{d-1} \tag{2.75}$$

where in the last equality we have introduced the following quantities

$$\rho = \sqrt{\frac{4(r - r_h)}{f'(r_h)}}, \qquad \varphi = \frac{1}{2} |f'(r_h)| \tau \tag{2.76}$$

the first two terms in the metric (2.74) look like polar coordinates for $\mathbb{R}^2$ *if* φ has period 2π. The crucial argument is: the Lorentzian metric (2.42) is not singular at the horizon and we have no reason to believe that the Euclidean version is going to be singular at the horizon. But the metric (2.74) has a *conical singularity*[†††] at $\rho = 0$ which corresponds to

[†††] There are geometric quantities that diverges when computed at $\rho = 0$ unless φ is 2π periodic.

$r = r_h$. To avoid this conical singularity the coordinate φ must be periodic with period given by $\beta = \frac{4\pi}{|f'(r_h)|}$. Then, using the fact that the partition function of a statistical system at temperature $T = 1/\beta$ is given by the closed periodic (in euclidean time) path integral of period β, we arrive at the Hawking temperature associated with the black hole

$$T = \frac{|f'(r_h)|}{4\pi}. \tag{2.77}$$

From this expression we see that the Schwarzschild BH is unstable (since its temperature is $T = 1/8\pi m$): the radiation emitted by the black hole will make it loose mass and so increase its temperature making the loss of mass increase. The situation changes when we put the black hole in a box, that is, a Schwarzschild-AdS BH can be stable or unstable depending on it's mass.

• SAdS

The temperaure of a d-dimensional SAdS black hole can be obtained by applying equation 2.77 to the metric function (2.44)

$$T = \frac{dr_h^2 + (d-2)L^2}{4\pi L^2 r_h}, \tag{2.78}$$

for $r_h \ll L$ the temperature behaves as $T \approx 1/r_h$ while for $r_h \gg L$ we have $T \approx r_h/L^2$. These two limits are known as small ($r_h \ll L$) and large ($r_h \gg L$) SAdS black holes. For small SAdS black holes the temperature decreases when we increase r_h while for large SAdS black holes the temperature increases when we increase r_h so, there must be a point of inflection. Indeed, there's a minimum temperature at $r_{h_{min}} = \sqrt{(d-2)/d}L$ given by $T_{min} = \sqrt{d(d-2)}/2\pi L$. The heat capacity is given by

$$C \propto \frac{\partial M}{\partial T} = \frac{\partial M}{\partial r_h}\left(\frac{\partial T}{\partial r_h}\right)^{-1} \tag{2.79}$$

the factor $\partial M/\partial r_h$ is always non-negative for $d \geq 2$ and then the sign of the heat capacity C is determined by the sign of the derivative $\partial T/\partial r_h$. From the analysis above we see that small black holes have negative heat capacity signaling that they are unstable. While large SAdS black holes have positive heat capacity signaling that they may be stable thermodynamically, that is, they can be in thermal equilibrium with their Hawking radiation. A more rigorous way to check thermodynamical stability of these large black holes is to calculate the free energy, F, and to show that it is minimized by the black hole solution. [7] In fact, there is a critical temperature above which large black holes minimize the free energy, that is, they're the preferred state. Small black hole are unstable and not the preferred state. This tells us that there is a phase transition know as *Hawking-Page transition* from small to large black holes at $r_h = L$. This phase transitions can be interpreted as a confinement/deconfinement phase transition in the dual quark gluon plasma through the AdS/CFT correspondence. [37]

• RN-AdS

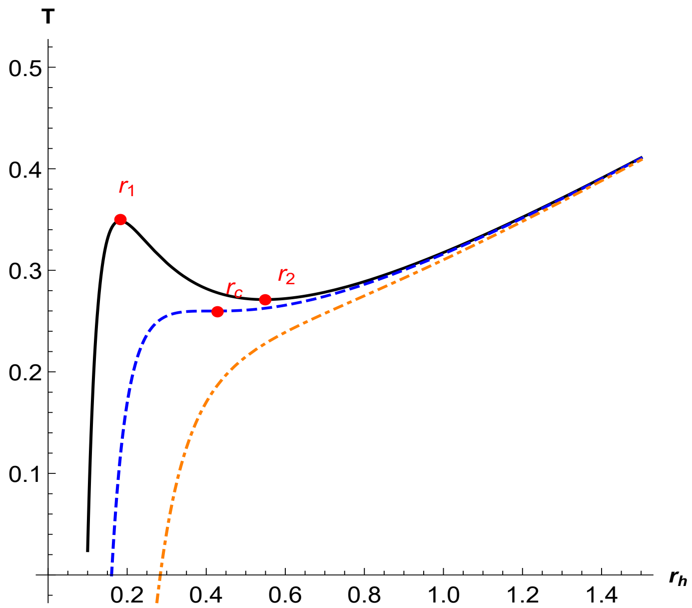

Figure 3 – Temperature T of $d+1=4$ RN-AdS black hole as a function of horizon radius r_h for different values of the charge Q: we plotted the solid black line with $0 < 36Q^2 < 1$, the dashed blue line with $36Q^2 = 1$ and the dot-dashed orange line with $36Q^2 > 1$. For $36Q^2 < 1$, the points of inflection that causes the heat capacity to diverges are r_1 and r_2 and between the range $r_1 < r < r_2$ the RN-AdS is unstable. For $36Q^2 = 1$, r_c also causes the divergence of the heat capacity.
Source: By the author.

Applying Eq. (2.77) to a $d = 3$ RN AdS black hole 2.46 and setting the AdS radius to unity, we obtain

$$T = \frac{1}{4\pi}\left(3r_h - \frac{Q^2}{r_h^3} + \frac{1}{r_h}\right) \tag{2.80}$$

using the expression for the mass in terms of r_h (see (2.48)) we can find the heat capacity at constant charge

$$C_Q = \frac{\partial M}{\partial T} = \frac{\partial M}{\partial r_h}\frac{\partial r_h}{\partial T} = 2\pi r_h^2\left(\frac{r_h^2 - Q^2 + 3r_h^4}{3r_h^4 + 3Q^2 - r_h^2}\right) = \frac{8\pi^2 r_h^5 T}{3r_h^4 + 3Q^2 - r_h^2}. \tag{2.81}$$

We can see from the denominator that there are configurations in the parameter space (M, Q) that cause the heat capacity to diverge. We can obtain the values of r_h for which the heat capacity diverges by solving the quadratic equation in r_h^2

$$3r_h^4 - r_h^2 + 3Q^2 = 0, \quad \Rightarrow \quad r_\pm = \frac{1}{6} \pm \frac{1}{6}\sqrt{1 - 36Q^2}. \tag{2.82}$$

The critical value for the radial coordinate is $r_c = 1/6$, when $36Q^2 = 1$. When the condition $36Q^2 \leq 1$ is satisfied and the charge, Q, is fixed, the heat capacity is positive for $r < r_-$, becomes negative for $r_- < r < r_+$ and then becomes positive again for $r > r_+$. Hence there are two phases of stable black hole, one with horizon radius less than the other separated by a thermodynamically unstable black hole with intermediate horizon radius (Fig. 3).

• Gauss-Bonnet-AdS

The event horizon radius, r_h, is given by the largest value of r that is a solution of $e^{2v} = 0$, where $e^{2\nu}$ is given by Eq. (2.64), and can be inverted to express the mass of the BH as a function of the horizon radius

$$M = \frac{(d-1)\Sigma_k r_h^{d-2}}{16\pi}\left(k + \frac{\tilde{\alpha}k^2}{r_h^2} + \frac{r_h^2}{L^2}\right). \tag{2.83}$$

Then, the temperature of the GB-AdS black hole is obtained using Eq. (2.77) with $(e^{2\nu})'_{|r_h}$ instead of $f'(r_h)$

$$T = \frac{1}{4\pi}\left(e^{2\nu}\right)' = \frac{d\, r_h^4 + (d-1)kL^2 r_h 2 + (d-4)\tilde{\alpha}k^2L^2}{4\pi r_h(r_h^2 + 2\tilde{\alpha})k}. \tag{2.84}$$

When higher order curvature terms are introduced, the entropy is no longer given by one quarter of the horizon area, but the 1st law of BH thermodynamics is still valid and one can obtain the entropy using it

$$S = \int T^{-1}dM = \int_0^{r_h} T^{-1}\left(\frac{\partial M}{\partial r_h}\right) dr_h. \tag{2.85}$$

We assume that the entropy of the BH goes to zero when the horizon radius goes to zero. The explicit form of the entropy can be find by substituting the expressions for M and T into the integral and evaluating it. The result is

$$S = \frac{\Sigma_k r_h^{d-1}}{4}\left(1 + \frac{(d-1)2\tilde{\alpha}k}{(d-3)r_h^2}\right). \tag{2.86}$$

With the quantities M, T, S given above, we can construct the thermodynamic potential and study the globally preferred solutions. In Fig. 4 we show the BH temperature for $d+1 = 5$ without the GB term for different horizon topology, $k = -1, 0, 1$. In the case of hyperbolic horizon (solid black line), black holes only exists with horizon radius greater than a minimun value and are always thermodynamically stable. Thermodynamics properties of planar BHs (dashed blue line) are independent of the GB coupling constant although both solutions are very different they present the same thermodynamics and are always thermodynamically stable. Spherical horizons (orange dot-dashed line) are not always thermodynamically stable; they exhibit a first order phase transitions between small (unstable) and large black holes (stable).

GB-AdS spherical black holes with $d+1 = 5$ and $d+1 > 5$ have very different thermodynamic properties as can be seen from the expression of the temperature Eq. (2.84). For $d+1 = 5$ the BH temperature has $T = 0$ for $r \to 0$ while for $d \geq 6$, the BH temperature behave as $T \to \infty$ for $r_h \to 0$ and shows a behavior similar to BH without GB term. The temperature T of a $d+1 = 5$ GB-AdS BH is show in Fig. 5 as a function of the horizon radius r_h for different values of the coupling constant. The introduction of the GB term brings out a new region where small BHs are stable: considering the dashed

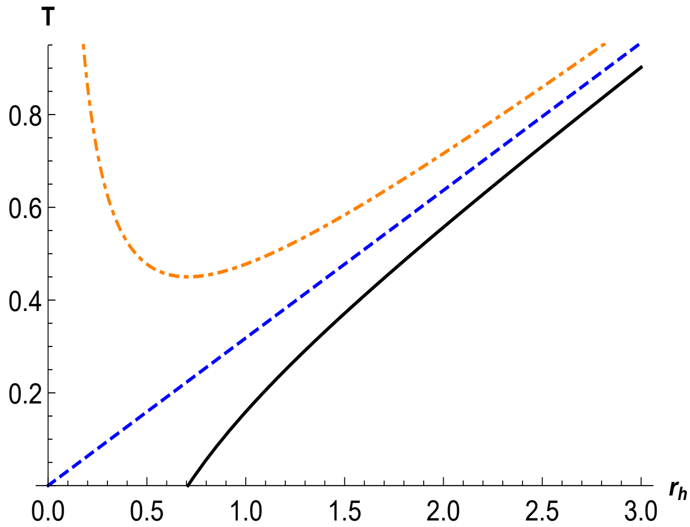

Figure 4 – Temperature T of $d+1=5$ topological BHs without the GB term ($\alpha=0$) as a function of the horizon radius r_h. The solid black line is the BH with hyperbolic horizon $k=-1$. The blued dashed line is the BH with planar horizon $k=0$ and the orange dot-dashed line is the BH with spherical horizon $k=1$.
Source: By the author.

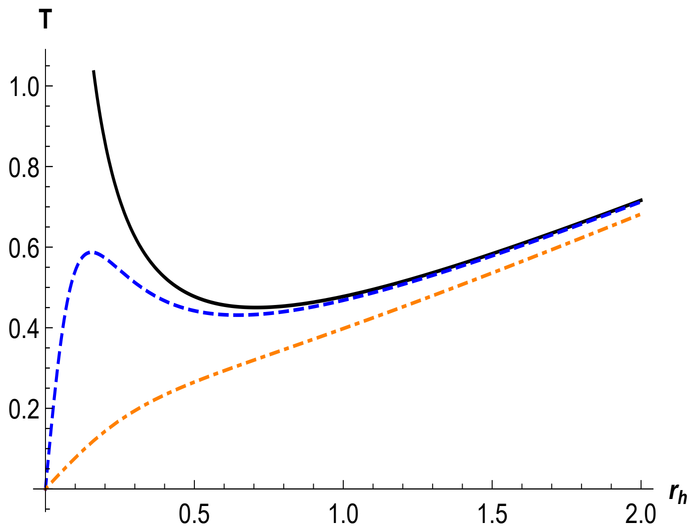

Figure 5 – Temperature T of $d+1=5$, $k=1$ GB-AdS black hole as a function of horizon radius r_h for different values of the coupling constant α: we plotted the solid black line with $\tilde{\alpha}=0$, the dashed blue line with $\tilde{\alpha}=0.01$ and the dot-dashed orange line with $\tilde{\alpha}=0.1$.
Source: By the author.

blue line in Fig. 5, the heat capacity starts positivity from $r_h = 0$, becomes negative and then positive again (analogous to the RN-AdS BH discussed above). BHs with spherical horizon (without the GB term) in AdS do not posses the first stable region. We should mention that the preferred thermodynamics configuration will be the one which minimizes the free energy $F = M - TS$. BHs in the GB-AdS theory with spherical horizon and fixed parameters (r_h, α) is not the solution which minimizes the free energy (thermal AdS does).

2.4.1 Black hole chemistry

Black hole chemistry is essentially incorporating the PdV term of the 1st law of thermodynamics in the thermodynamics of black holes and studying the phase structure of the black holes. There's no obvious way to introduce the concept of pressure for a black hole, which historically justifies the omission of the PdV term in the first law of BH thermodynamics. But from the cosmological perspective, it's natural to consider the cosmological constant as the pressure when treating the vacuum as a perfect fluid. To see this, consider Einstein's equation with cosmological constant and treat the vacuum as a perfect fluid, $G_{ab} = -\Lambda g_{ab} = 8\pi T_{ab}$. The stress-energy tensor of a perfect fluid is $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where p and ρ are the pressure and energy density, respectively and u^a is the velocity field of the fluid. Comparing both sides of the equation leads to the identification

$$p = -\frac{\Lambda}{8\pi}, \tag{2.87}$$

and when $\Lambda < 0$ we obtain a positive pressure. The generalized 1st law of thermodynamics (treating Λ as a dynamical variable) was obtained in [38] and is given by

$$dM = TdS + VdP + \Omega dJ + \Phi dQ \tag{2.88}$$

where V, the *thermodynamic volume*, is defined to be the conjugate variable to the pressure

$$V \equiv \left(\frac{\partial M}{\partial P}\right)_{S,Q,J} \tag{2.89}$$

The thermodynamic volume is independent of the geometric volume for most black holes [39] and in general, cannot be given a geometric interpretation. Comparison between the first law of thermodynamics (2.71) and the generalized first law of BH thermodynamics (2.88) leads us to the conclusion that M is no longer associated with the internal energy but rather with the chemical enthalpy[‡‡‡]: the internal energy E plus the energy PV required to displace the vacuum energy of its environment. [40] Once the identifications are properly made, we can start studying the phase structure of some black holes in AdS space. Black hole chemistry is a novel area of research with interesting phenomena appearing: black holes with phase structure similar to van der Waals fluids, solid liquid transition, triple

‡‡‡ One is related to the other by a Legendre transformation.

points and reentrant phase transitions have beeb found. See [40,41] for a review. In particular, one can understand stability of BH by looking at the BH compressibility k, defined to be

$$k = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,Q,J} . \tag{2.90}$$

3 STABILITY

This chapter is devoted to the study of stability of solutions to Einstein equation. We introduce the concept of quasinormal modes which can indicate linear stability using as reference [42] and discuss the role that these modes play in the holographic duality. Then we study the nonlinear stability of AdS spacetime based on reference. [10]

3.1 Introduction

The question about *stability* of solutions to theories of gravity is important to understand better the physical significance of these solutions. This is, if the solution is unstable (a sufficiently small perturbation leads to an unbounded growth), we probably will not find the object described by the solution in nature, unless the timescale of the instability is much larger than another timescale (e.g. the age of the universe) present in the system.

Regarding the stability of spacetimes, it was proven that Minkowski spacetime and de Sitter spacetime are nonlinearly stable. [43,44] The mechanism responsible for the stability in Minkowski and de Sitter is related to the fact that linear fields decay sufficiently fast at infinity. Stability of black hole spacetimes is an active topic of research nowadays and recently it was proven that the asymptotically flat Schwarzschild [45] and non-extreme Kerr [46] spacetimes are linearly stable. The nonlinear stability problem of these spacetimes is much harder and has not yet been proven.

Concerning asymptotically AdS spacetimes, the linear stability of pure AdS has been proven [47] but no proof has been given with respect to the nonlinear stability. In fact, it's conjectured that AdS spacetime is nonlinearly unstable [Dafermos, 2006], which means that arbitrarily small perturbations will lead to the formation of black holes. What makes difficult to proof the nonlinearly (in)stability of AdS is the fact that AdS has a timelike boundary and therefore, energy cannot disperse to infinity. It should be said that the stability question of AdS is dependent on the boundary conditions chosen for AdS of which the most common is the reflecting boundary condition on conformal infinity $\mathcal{I}$, which does not permit any flux of energy through the boundary. Recently, the nonlinear instability of a Einstein null dust system has been proven for this boundary condition. [48] A remarkable attempt to resolve the problem of AdS stability was given in. [10] Their numerical results support the conjecture of instability of AdS and they also give some heuristic arguments explaining the reason why AdS is nonlinearly unstable under arbitrarily small perturbations against the formation of black holes. After this work, a large number of papers on the subject was published examining different models and assumptions.

Before discussing the results of [10] about the nonlinear instability of AdS, we review the topic of quasinormal modes (QNMs) which can indicate stability or instability of black holes at the linear level and also has applications in astrophysics and holography.

3.2 Quasinormal modes

Linear perturbations around solutions to Einstein's field equation can indicate stability or instability of such geometry and they can be used to learn more about black holes and stars. When the spacetime is perturbed, it will oscillate. These oscillations, that go to spatial infinity or fall into a black hole, are known as quasinormal modes. The term "quasi" here indicates that we're dealing with a dissipative (the black hole event horizon behaves as a one-way membrane) and open system such that energy is dispersed through gravitational waves and/or electromagnetic and scalar waves if matter is present. Formally, in a vacuum spacetime the linearized Einstein's equation around a fixed black hole spacetime whose metric is known takes the form [42]

$$\delta R_{\mu\nu} = 0, \tag{3.1}$$

where the metric deviations from the fixed background is assumed to be small. In general, Eq. (3.1) can be rewritten in the form of a wave equation for the metric perturbations. Once the equations are obtained what remains to be done is solve them with the *appropriate* boundary conditions: purely outgoing waves at infinity and purely incoming waves at the event horizon. The boundary conditions cause the spectrum to become discrete. The radial part of the wave equation behaves as

$$\psi \sim e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t} = e^{\omega_I t}\cos(\omega_R t) \tag{3.2}$$

where ω_R, the real part of the complex frequency, is related to the period of oscillation

$$T = \frac{2\pi}{\omega_R} \tag{3.3}$$

and the imaginary part ω_I gives a characteristic time scale

$$\tau = \frac{1}{\omega_I}. \tag{3.4}$$

A first and sometimes naive* analysis may give us indications about the stability of the spacetime under these perturbations, since

$$\begin{aligned} \omega_I &< 0, \qquad \text{exponential damping (stable);} \\ \omega_I &> 0, \qquad \text{exponential growth (unstable).} \end{aligned}$$

* Naive since higher order corrections can render a linearly unstable spacetime to become nonlinearly stable in the full nonlinear regime.

The wave equation obtained solely perturbing the spacetime metric is similar to the one obtained when analyzing classical fields in fixed black hole spacetimes. To illustrate this, we consider the simplest case of a scalar field in an asymptotically flat Schwarzschild spacetime whose dynamics are governed by the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right)=0. \tag{3.5}$$

The line element of the Schwarzschild black hole reads:

$$ds^2=-f(r)dt^2+f(r)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2, \qquad f(r)=1-\frac{2M}{r}. \tag{3.6}$$

To solve Eq. (3.5), we make the following separation of variables

$$\psi(t,r,\theta,\phi)=\sum_{l,m}e^{-i\omega t}Y_{lm}(\theta,\phi)\frac{\varphi(r)}{r} \tag{3.7}$$

where Y_{lm} are the spherical harmonics. Plugging this ansatz into Eq. (3.5) we obtain

$$\frac{1}{f}\partial_t^2\psi+\frac{1}{r^2}\left(\partial_r\left(r^2f\partial_r\psi\right)\right)+\frac{1}{r^2\sin\theta}\left(\partial_\theta\left(\sin\theta\partial_\theta\psi\right)\right)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi=0 \tag{3.8}$$

and the equation for $\varphi(r)$ is

$$\omega^2\varphi+\frac{f}{r}\left[\partial_r\left(fr^2\partial_r\left(\frac{\varphi}{r}\right)\right)\right]-\frac{f\left(l(l+1)\right)}{r^2}\varphi=0. \tag{3.9}$$

In order to put Eq. (3.9) into a Schrödinger-like form, we need to get rid of the first order derivative. We accomplish this by using the following coordinate transformation

$$\frac{dr_*}{dr}=\frac{1}{f} \qquad \Longrightarrow \qquad r_*=r+2M\ln(r-2M) \tag{3.10}$$

the coordinate r_* is called the tortoise coordinate and the black hole event horizon (which in the coordinates (t,r,θ,ϕ) is located at $r=2M$), is located at $r_*=-\infty$. Finally, the radial equation for $\phi(r)$ can be put in a Schrödinger-like form

$$\left(\omega^2+\partial_{r_*}^2\right)\phi=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]\phi. \tag{3.11}$$

In the above equation, r is to be understood as a function of r_*. The effective potential on the r.h.s of Eq. (3.11) can be generalized for different types of perturbation [49] as

$$V_l(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M(1-s^2)}{r^3}\right]\phi. \tag{3.12}$$

where $s=0,1,2$ for scalar, electromagnetic and *axial*[†] gravitational perturbations, respectively. The axial perturbations have parity $(-1)^{l+1}$ while polar perturbations have parity

[†] Axial perturbations are the ones that induce frame dragging and rotation to the black hole. On the other hand, *polar* perturbations are perturbations on the already non-vanishing metric components.

$(-1)^l$. Symmetry consideration constrain the possible values of l for the different types of perturbations. For scalar, electromagnetic and gravitational perturbations, we have $l \geq 0$, $l \geq 1$ and $l \geq 2$, respectively.

The QNMs are intimately related with gravitational wave astronomy and can be used to test GR. They have the extraordinary property to depend only on the BH parameters and are independent of the initial perturbation. [3,49] As discussed in Section (2.3.5), astrophysically relevant black holes in GR depend only on their parameters (M, J). Considering the coalescence of two black holes, just after the merger the final state consists of a single black hole emitting gravitational waves. Then, the precise measurement of the 1st ringdown QNM in earth based interferometers (such as LIGO and VIRGO) enables us to invert the relation between the measured quantities (ω_R, ω_I) to obtain the final BH parameters (M, J). If the precision is accurate enough to measure the 1st and 2nd ringdown modes, it's possible to test GR in the strong gravity regimes of the merging of two black holes, two neutron stars or the merger between a black hole and a neutron star. In the case involving neutron stars, the measurement and analysis of the signal will give us information about the equation of state of the matter inside the neutron star, which is not well known currently.

3.2.1 Holography and quasinormal modes

In addition to the applications of QNMs in astrophysics, they can also be used to study hydrodynamic properties of quantum systems through the AdS/CFT correspondence. This requires the study of QNMs of asymptotically AdS BH which corresponds to perturbations of the dual CFT on the boundary. [50,51]

Considering asymptotically AdS BHs, conservation of energy implies we should adopt reflective boundary conditions at infinity (the analogous of the potential (3.12) for the AdS case diverge at infinity so the wave function must vanishes there) and at the horizon we should require only incoming waves. These boundary conditions will only be satisfied by some discrete set of values of complex ω and in general, it will not be possible to solve the perturbation equations exactly determining the spectrum of the QNMs and one should use numerics. [52]

The imaginary part of ω, ω_I, governs the damping (or growth) of the BH perturbations. From the AdS/CFT correspondence, a large and static BH in AdS approximately corresponds to a thermal state in the conformal field theory and then perturbations of the BH are interpreted as perturbations on the CFT. Therefore, the thermalization timescale of the CFT will be dictated by the $\tau = 1/\omega_I$, which for strongly coupled CFTs are usually difficult to calculate. Scalar QNMs for Schwarzschild-AdS were computed in [50] and for $l = 0$ and it was verified that for large black holes ($r_h \gg L$) the thermalization timescale scales as $\tau \sim 1/T$, where T is the BH temperature. While electromagnetic and gravitational

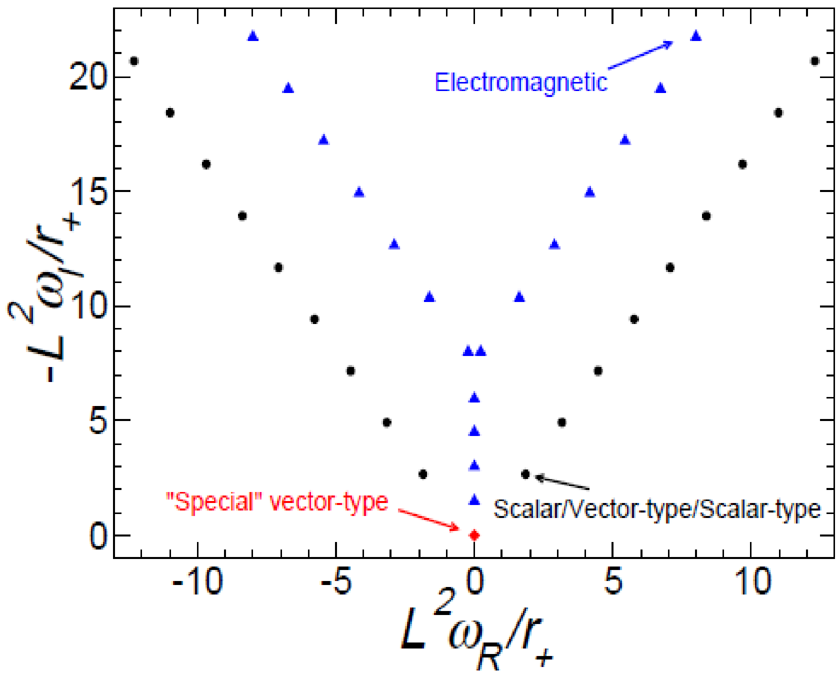

Figure 6 – Scalar, electromagnetic and gravitational quasinormal modes for (3+1) dimensional and large Schwarzschild-AdS black hole computed for $l = s$.
Source: BERTI; CARDOSO; STARINETS. [3]

QNMs for Schwarzschild-AdS were computed in [52] and it was found that $\omega_I < 0$ indicating linear stability of this black hole against such perturbations. Also, QNM frequencies are practically independent of l for $l \ll r_+/L$ (Fig. 6).

3.3 Nonlinear instability

The influential paper [10] (from now on referred as BR) gives numerical evidence that AdS is nonlinearly unstable against generic initial perturbations that lead to the formation of black holes and gives a heuristic argument to explain this result based on a nonlinear perturbation analysis. The time scale for black hole formations is of the order $1/\epsilon^2$, where ϵ is the amplitude of the initial perturbation.
In this section, we show the heuristic arguments in favor of the conjecture as presented by BR. They work within a 3+1 dimensional spacetime and assume spherical symmetry of the spacetime. Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be static. In order to evade the absence of dynamical degrees of freedom due to Birkhoff's theorem, it's necessary to add matter. For simplicity the matter chosen is a minimally coupled massless scalar field, ϕ, with spherical symmetry.

Their model is then given by the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R - 2\Lambda + \partial_\mu\phi\partial^\mu\phi\right). \tag{3.13}$$

The equations of motion are given by

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 8\pi G\left(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(\partial\phi)^2\right) \tag{3.14}$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0. \tag{3.15}$$

The ansatz for an asymptotically AdS_4 spacetime is chosen to be (compare with the conformal metric (2.20))

$$ds^2 = \frac{L^2}{\cos^2 x}\left(-Ae^{2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x d\Omega^2\right). \tag{3.16}$$

The functions A and δ depend only on (t, x) and L is the AdS_4 radius. Using the ansatz (3.16) and the auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1}e^\delta\dot{\phi}$, the field equations become

$$\begin{aligned}
&\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)', \qquad \dot{\Pi} = \frac{1}{\tan^2 x}(\tan^2 x Ae^{-\delta}\Phi)' \\
&A' = \frac{1 + 2\sin^2 x}{\sin x\cos x}(1 - A) - \sin x\cos x A(\Phi^2 + \Pi^2) \\
&\delta' = -\sin x\cos x(\Phi^2 + \Pi^2)
\end{aligned} \tag{3.17}$$

where a prime $'$ denotes derivative with respect to the radial coordinate x and a dot $\dot{}$ denotes derivative with respect to the time coordinate t. The first two equations in (3.17) are a wave equation written as two first order equations and are coupled to the remaining two elliptic equations in (3.17) usually referred to as constrains. The task is to solve the *initial-boundary value problem* for the system (3.17) under small perturbations of the AdS spacetime $A = 1, \delta = 0, \phi = 0$.

Heuristics - weakly nonlinear perturbation analysis

The initial data is assumed to be smooth and small

$$(\phi, \dot{\phi})_{|t=0} = (\epsilon f(x), \epsilon g(x)), \tag{3.18}$$

where f and g are fixed functions satisfying the boundary conditions and ϵ is a parameter assumed to be small. We expand the fields in power series near the pure AdS solution

$$\phi = \epsilon\phi_1 + \epsilon^3\phi_3 + ... \tag{3.19}$$

$$\delta = \epsilon^2\delta_2 + \epsilon^4\delta_4 + ... \tag{3.20}$$

$$A = 1 + \epsilon^2 A_2 + \epsilon^4 A_4 + ... \tag{3.21}$$

where $(\phi_1, \dot{\phi}_1)_{|t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)_{|t=0} = (0, 0)$ for $j \geq 2$. Inserting the above expansion in the field Eq. (3.17) and collecting the terms with the same power in ϵ results in a hierarchy of linear equations which can be solved order by order.

• First order. The first order in ϵ equation for the scalar field ϕ is

$$\ddot{\phi}_1 + L\phi_1 = 0, \qquad L = -\frac{1}{\tan^2 x}\partial_x(\tan^2 x\partial_x). \tag{3.22}$$

The operator L is essentially self-adjoint on $L^2([0,\pi/2],\tan^2 x dx)$. The eigenvalues, ω_j, and eigenvectors, $e_j(x)$, of this linear operator are known and have the following values [47]

$$\omega_j^2 = (3+2j)^2, \qquad e_j(x) = d_j \cos^3 x \;\; {}_2F_1(-j, 3+j, \frac{3}{2}; \sin^2 x). \tag{3.23}$$

The d_j's are normalization constants guaranteeing that $(e_i, e_j) = \delta_{ij}$[‡] We see that all the eigenvalues of the operator L are strictly positive which means that the solution is linearly stable. So any solution of Eq. (3.22) can be written as

$$\phi_1 = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j) e_j(x) \tag{3.24}$$

where the amplitudes a_j and the phases β_j are determined by the initial data.

• Second order. Looking at the system (3.17), the equations for the metric functions A and δ are quadratic in the scalar field, which means that the first perturbation on the metric appears at second order. So, once the first-order solution is known, the back-reaction to the metric can be found. The second order in ϵ equations for the metric functions A and δ are

$$A_2' + \frac{1+2\sin^2 x}{\sin x\cos x}A_2 = \sin x\cos x(\dot{\phi}_1^2 + \phi_1'^2) \tag{3.25}$$

$$\delta_2' = -\sin x\cos x(\dot{\phi}_1^2 + \phi_1'^2). \tag{3.26}$$

The equation for δ_2 can be trivially integrated once the solution ϕ_1 is known while the equation for A_2 can be solved by the integrating factor method to give

$$A_2 = \frac{\cos^3 x}{\sin x}\int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right)\tan^2 y dy \tag{3.27}$$

$$\delta_2 = -\int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right)\sin y\cos y dy. \tag{3.28}$$

• Third order. The first non trivial property of the system appears at third order. The equation for the scalar field is

$$\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2) \tag{3.29}$$

where $S := 2(A_2+\delta_2)\ddot{\phi}_1 + (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 + (A_2' + \delta_2')\phi_1'$. In order to analyze solutions of Eq. (3.29), we project onto the basis $\{e_j\}$ of the linear solution obtaining an infinite system of

‡ The inner product of f and g on the Hilbert space $L^2([0,\pi/2],\tan^2 x dx)$ is denoted and given by $(f,g) := \int_0^{\pi/2} f(x)g(x)\tan^2 x dx$.

decoupled forced harmonic oscillators for the generalized Fourier coefficients $c_j(t) = (\phi_3, e_j)$

$$\ddot{c}_j + \omega_j^2 c_j = S_j := (S, e_j). \tag{3.30}$$

The principal point in this heuristic discussion is that there are resonant terms in S_j, that is, terms proportional to $\cos \omega_j t$ or $\sin \omega_j t$ which make the solution grow linearly with t and invalidates perturbation theory. More exactly, let $\mathcal{J} = \{j \in \mathbb{N}_0 : a_j \neq 0\}$ be a set of indices of non zero modes in the linearized solution (3.24). Then it can be shown that for each triad $(j_1, j_2, j_3) \in \mathcal{J}^3$ such that $\omega_j = \omega_{j_1} + \omega_{j_2} - \omega_{j_3}$ gives rise to a resonant term in S_j. [53] Some of these resonant modes can be eliminated by the *Poincaré–Lindstedt method* (see Appendix A). But, since there is an infinite number of resonant terms, we cannot eliminate all of them. This resonant mode, it's believed, will trigger an (turbulent) instability by transferring energy from low to high energy modes. Eventually, energy will be concentrated is such small scale that a black hole will form. Note that BR do not claim that there will be BH formation *for all* small perturbations. If there's only one initial mode in the perturbation, instead of a linear combination of them, it's possible to add nonlinear corrections systematically in order to avoid the resonant terms.

A number of comments on papers that followed BR are necessary. The generalization to consider a complex scalar field was done in [54] and similar results were found. The purely gravitational problem of the nonlinear stability of (3+1) dimensional AdS was studied in [55] and similar results were also found. In this study, they had to give up the spherical symmetry which implies more coupled differential equations to solve and numerical calculations become considerably difficult. The analysis of BR was made for (3+1) dimensional spacetime. The authors of [56] looked at the same model but for (4+1) dimensional spacetime and arrived at the conclusion that there's no BH formation, which was subsequently contested in [57] where they argue that the same mechanism presented in BR is valid for higher dimensional AdS spacetimes and so, AdS_{d+1} is unstable to black hole formation under arbitrarily small scalar perturbations for all $d \geq 3$. They say the erroneous conclusion of [56] was caused by the insufficient spatial resolution in their numerical code and the lesson to be learned is to be careful about the conclusions from numerical simulations of asymptotically AdS spacetimes. Gravitational perturbations were considered in [58] for AdS_5 and there is also numerical evidence of black hole formation on time scale of the order $\mathcal{O}(\epsilon^{-2})$, where ϵ is the amplitude of the perturbation. The addition of high order curvature terms was analyzed in [59] for spherically symmetric (4+1) dimensional spacetime minimally coupled to a massless scalar field. The high order curvature considered is the Gauss-Bonnet (GB) term and evidence is given that the inclusion of the GB term renders the spacetime to be stable under such arbitrarily small perturbations. From the AdS/CFT viewpoint, higher order curvature corrections translate to finite N and 'tHooft coupling corrections in the dual theory and the absence of BH formation leads to the conclusion that the perturbations on the dual theory do not thermalize.

4 SCALAR-TENSOR GRAVITY

This chapter is devoted to discuss how to modify GR as a theory of gravity while preserving some of its properties. We restrict the discussion to scalar-tensor theories of gravity with higher order curvature terms. At the end of the chapter, we discuss the multipole expansion for the electromagnetic field in a fixed solitonic background.

4.1 Introduction

In order to modify GR, one can adopt the following line of reasoning. The Einstein equation is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4.1}$$

relating the geometry of the spacetime with its matter content. The l.h.s. is a symmetric and divergence-free rank-2 tensor depending only on the metric and its first and second derivatives. A natural question to ask is if the l.h.s is the only possible choice while preserving these characteristics. A theorem proved by Lovelock [60] guarantees that, in four dimensions, the unique symmetric rank-2 divergence-free tensor depending only on the metric and its first and second derivatives are the Einstein tensor and the metric itself and therefore, GR is the only theory constructed from the metric with such properties. So, to modify GR while requiring it to have second-order field equations, be diffeomorphism covariant and arise from an action principle one should relax some assumptions. The possibilities are

- consider higher dimensions;
- add degrees of freedom.

In the first case, the theories are known as *Lovelock theories* and are the most general diffeomorphic invariant gravity theory in higher dimensions with second order equation of motion for the metric field. Lovelock theories have the property that gravitational waves (GWs) can propagate faster or slower than the speed of light, so GWs detection can rule out some of these theories as candidates to describe gravity. The second alternative allows for a plethora of modifications depending on what additional degree of freedom you consider (scalar, vector, tensor...) and how to couple it with the metric field. The simplest modification one can think in this context is to add a scalar field giving rise to what is know as scalar-tensor (ST) theories of gravity. Another useful theorem classifying all ST theories of gravity in four dimensions was proved by Horndeski [61] and states that the only class of four dimensional ST theories giving rise to 2nd order equations of motion for both the metric and the scalar field are the Horndeski Theories (also known as generalized galileons)*:

$$S = \int d^4x\sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right) \tag{4.2}$$

where

$$\begin{aligned}
&\mathcal{L}_2 = K(\phi, X), \qquad \mathcal{L}_3 = -G_3(\phi, X)\nabla^\alpha\nabla_\alpha\phi, \\
&\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}[(\nabla^\alpha\nabla_\alpha\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \\
&\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}[(\nabla^\alpha\nabla_\alpha\phi)^3 - 3\nabla^\alpha\nabla_\alpha\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3],
\end{aligned} \tag{4.3}$$

and K, G_3, G_4, G_5 are arbitrary functions of the scalar field ϕ and $X = -\partial_\mu\phi\partial^\mu\phi/2$. $G_{\mu\nu}$ is the Einstein tensor and f_X denotes the derivative of the function f with respect to X. If one considers the subclass of Horndeski theories in which the scalar field has shift symmetry $\phi \to \phi + c$, then we have what is called the shift-symmetric Horndeski theory. The most general shift-symmetric Horndeski theory was proven [63] to be given by Horndeski action (Eq. 4.2) with the substitutions

$$\begin{aligned}
K(\phi, X) &\to K(X), \qquad G_3(\phi, X) \to G_3(X), \\
G_4(\phi, X) &\to G_4(X), \qquad G_5(\phi, X) \to G_5(X).
\end{aligned} \tag{4.4}$$

We should comment on the requirement of the equations of motion to be second order. Higher order equations, in general, introduce additional degrees of freedom known as ghosts which cause instability of the theory (Ostrogradsky instability). Although the action for Horndeski theory contain second order derivatives of the scalar field, it can be show [61] that the equations of motion remains second order and are free from Ostrogradsky instability. However, second order equations of motion are not necessary conditions for the absence of the Ostrogradsky instability in theories with multiple fields. [64] Exist theories with higher order equations of motion free from Ostrogradsky instability, these are the so-called degenerate higher-order scalar-tensor (DHOST) theories. DHOST theories were first found by performing a *disformal transformation*

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\nabla_\mu\phi\nabla_\nu\phi \tag{4.5}$$

in a Horndeski theory. The resulting equations contain higher order derivatives. The condition for the existence of an inverse disformal transformation is given by

$$C\left(C - XC_X + 2x^2D_X\right) \neq 0. \tag{4.6}$$

With the above condition satisfied, both theories (the Horndeski and the theory with higher order equation of motion) are related by a field redefinition and contain the same

* The Horndeski theories was rediscovered a few years ago [62] motivated by cosmology.

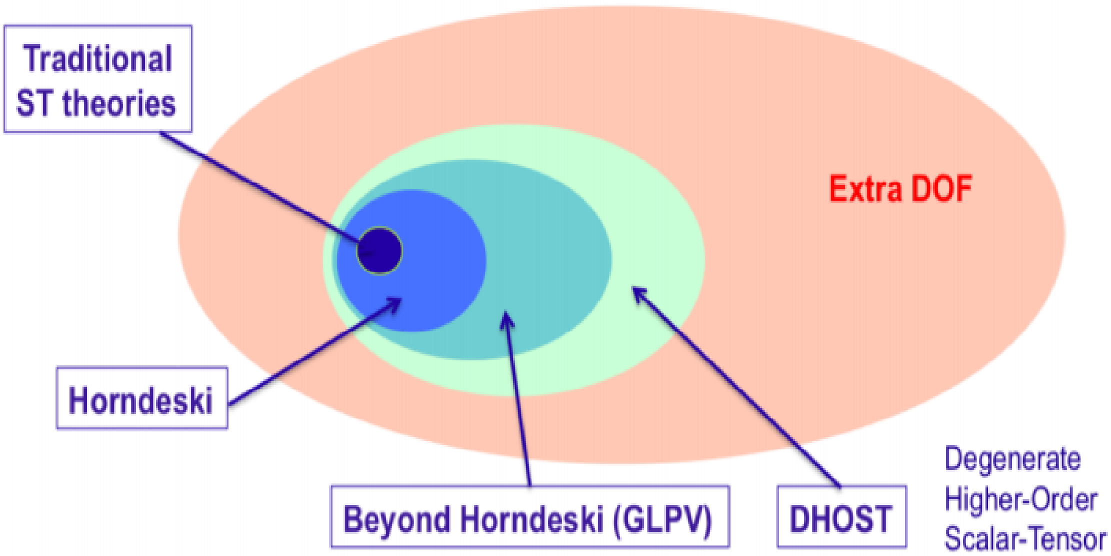

Figure 7 – Map of scalar-tensor theories of gravity.
Source: LANGLOIS.[4]

quantity of degrees of freedom. [4] A map of the space of the ST theories is shown in the Fig. 7.

In what follows we consider only the subclass of Horndeski theories which are shift invariant. Reference [65] proved a no-hair theorem for the generalized galileons: static, spherically symmetric black hole do not sustain non trivial galileon field outside the horizon. To prove the theorem, the symmetry of the galileon $\phi \to \phi + c$, which gives rise to a conserved Noether current, the symmetries of the background and regularity of physical quantities on the BH event horizon was used. Evading the assumption about regularity of the norm of the Noether current on the event horizon was a step towards constructing hairy BHs in the model. [63]

4.2 Gauss-Bonnet scalar-tensor gravity

In the low energy limit, string theory reduces to GR, essentially. First order corrections in α'[†] introduce second order curvature terms and a scalar field known as the *dilaton field*. A special case of interest in which there is an asymptotically flat and static hairy black hole solution [66] is given by the action[‡]

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha' e^{\phi}\mathcal{G}\right). \tag{4.7}$$

† The string length scale squared.
‡ In this section we set $16\pi G = c = 1$.

where the second order curvature term is the Gauss-Bonnet (GB) term, $\mathcal{G}$, (see Section 2.3.4) which is given by

$$\mathcal{G} = R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{4.8}$$

The scalar field is nontrivial outside the horizon characterizing what we call hair but it is a secondary hair (see Section 2.3.5) in the sense that it depends on the BH mass and does not have an independent parameter. The black hole solutions exist up to a certain value for the coupling constant α above which there's a naked singularity. Recently, it was proven in [67] that when choosing the arbitrary functions of Horndeski theory (Eq. 4.3) to be

$$K = G_3 = G_4 = 0, \qquad G_5 = -2\gamma\, ln|X|. \tag{4.9}$$

where γ is a constant, one obtains a model with a linear coupling between the scalar field and the GB term

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\gamma}{2}\phi\mathcal{G}\right) \tag{4.10}$$

and hairy black hole solutions in this shift-symmetric Horndeski theory are quite general unless one excludes by hand the coupling between the scalar field and the Gauss-Bonnet term. This can be seen by the scalar field equation

$$\nabla^\alpha\nabla_\alpha\phi + \frac{\gamma}{2}\mathcal{G} = 0. \tag{4.11}$$

The GB term only vanishes identically in flat spacetime, requiring ϕ to have a nontrivial configuration for the above equation to be satisfied. The model given by Eq. (4.10) is invariant under $\phi \to \phi + a_\mu x^\mu + c$ (a_μ, c being constant) because the GB term is a topological invariant in 4 spacetime dimensions. This leads to a conserved Noether current which can be used to understand better the model. Although not stated in [63], the norm of the Noether current diverges at the BH horizon not violating, therefore, the theorem on no galileon hair proved in [65], which assume the norm of the Noether current to be finite at the horizon.

The study of the addition of a cosmological constant to the model given by Eq. (4.10) was first analyzed in [68] and then generalized for arbitrary coupling of the scalar field to the GB term in. [69] The action for the linear coupling is then given by

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda + \frac{\gamma}{2}\phi\mathcal{G} - \partial_\mu\phi\partial^\mu\phi\right) \tag{4.12}$$

in order to look for solutions to the above model one should make an ansatz about the metric. For simplicity, the ansatz chosen is a static spherically symmetric one and reads

$$ds^2 = N(r)\sigma^2(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\phi^2), \qquad \phi = \phi(r) \tag{4.13}$$

The equations of motion are given by varying the action (Eq. (4.12)) with respect to the metric and the scalar field and its explicit form are found in. [68] They're a set of four coupled nonlinear differential equations with unknown analytical solution and have to be solved numerically with the appropriate boundary conditions. When the coupling constant γ vanishes, there's an exact static and spherically symmetric BH solution given by the Schwarzschild-AdS BH (Eq. (2.42)). So we can think of finding perturbative solutions in the coupling constant γ. The asymptotic behaviour of the metric functions is found to be AdS only up to linear order in γ because the curvature sources the scalar field equation even at $r \to \infty$. [68] It was found that with the addition of a cosmological constant BHs with regular horizon and non-trivial scalar field only exist for some intervals of the coupling constant, there exists a gap in the coupling constant parameter where there's no black hole solution with regular event horizon. Also, for specific values of the parameters (Λ, γ, r_h), the scalar field derivative can diverge at the horizon.

Solitonic solutions to the model given by the action Eq. (4.12) were also found, differently of what was shown in the flat counterpart case where solitonic solution was proven to not exist. [70] The addition of a U(1) gauge field to the model was discussed in. [71] The action is given by

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda + \frac{\gamma}{2}\phi\mathcal{G} - \partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{4.14}$$

the solitonic background spacetime is described by the metric (4.13) with the metric functions up to second order in the coupling constant given by

$$N = 1 + x^2 + \zeta^2\left(-2 - \frac{10}{3}x^2 + \frac{2}{x}\arctan(x)\right), \quad \sigma = 1 - \frac{\zeta^2}{9}\frac{1}{1+x^2}, \tag{4.15}$$

where we have defined $\zeta^2 = \gamma^2\Lambda^2/9$ and $x = \sqrt{-\Lambda/3}\,r$. Motivated by [72,73] where have been found static and everywhere regular multipole moments with finite energy of the electromagnetic field, we study the multipole expansion for the electromagnetic field in the solitonic background given by Eq. (4.13) with the metric functions N and σ given by Eq. (4.15).

4.2.1 Electromagnetic multipoles

We choose the ansatz for the vector potential to be

$$A_\mu dx^\mu = A_t(r,\theta)dt + A_\varphi(r,\theta)d\varphi. \tag{4.16}$$

To study purely an electric field on the solitonic background given by the metric functions (4.15), we choose $A_\varphi = 0$ and then use separation of variables

$$A_t(r,\theta) = \sum_{l=0}^{\infty} R_l(r)\mathcal{P}_l(\cos\theta) \tag{4.17}$$

where $\mathcal{P}_l(\cos\theta)$ is the Legendre polynomials with $l \geq 0$. Plugging Eq. (4.17) into the Maxwell's equation in the background given by Eq. (4.15) we find the equation satisfied

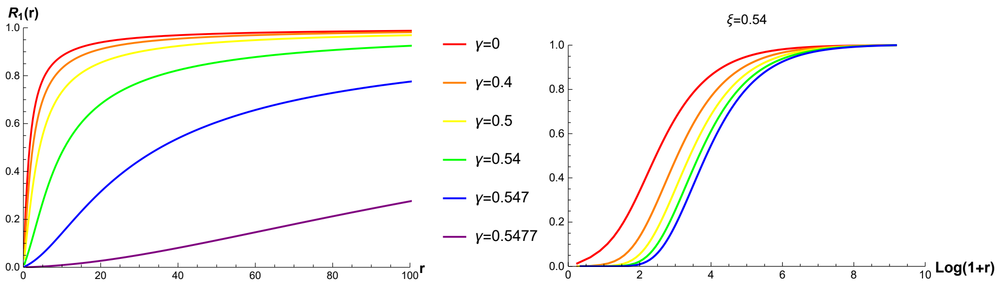

Figure 8 – The radial function R_l for the electric multipoles. On the left, we plot the dipole $l = 1$ radial function for increasing values of the coupling constant γ. On the right, we plot the first five radial functions for fixed value of $\gamma = 0.54$.
Source: By the author.

by the radial function for a given l

$$R_l'' = l\,(l+1)\,\frac{R_l}{x^2 N} + R_l'\left(\frac{\sigma'}{\sigma} - \frac{2}{x}\right), \tag{4.18}$$

with $'$ denoting derivative with respect to the radial coordinate x. For $\zeta = 0$, there is analytical solution given in terms of hypergeometric functions for any $l \geq 1$. The case $l = 0$, the monopole, has trivial solution $R_0 = 1$. While in the case $\zeta \neq 0$, we have to solve Eq. (4.18) numerically except for the monopole where the radial equation reduces to

$$R_0''(x) - R_0'(x)\left(\frac{\sigma'(x)}{\sigma(x)} - \frac{2}{x}\right) = 0 \tag{4.19}$$

which has the solution

$$R_0(x) = c_1 + c_2\left(\frac{\zeta^2 - 9}{x} + \zeta^2 \tan^{-1}(x)\right) \tag{4.20}$$

with c_1, c_2 arbitrary constants. So we see that the monopole $l = 0$ is not regular everywhere, diverging at the origin unless one sets $c_2 = 0$ or a specific value for the coupling constant $\zeta^2 = 9$, but this case is excluded because ζ is assumed to be small in the derivation of the metric functions N and σ. Choosing $c_2 = 0$ leads to the trivial solution $R_0 = c_1$, but in this case, the boundary conditions, $R_l(0) = 0$ and $R_l(r \to \infty) = 1$, is not satisfied. Therefore, "spherically symmetric, static and charged generalisations of the scalar-tensor solitons with approximate AdS asymptotics are not possible." [71]

We show the numerical solution for the the multipoles $l = 1, ..., 5$ in Fig. 8. The equations were solved numerically up to second order in ζ and a critical value for the coupling constant was found above which the solutions vanish. Solving the equations numerically for all orders in γ, it was shown in [71] that the solutions exist for all values of the coupling constant ζ. The boundary conditions imposed are $R_l(0) = 0$ and $R_l(r \to \infty) = 1$.

The energy-momentum tensor of the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2, \tag{4.21}$$

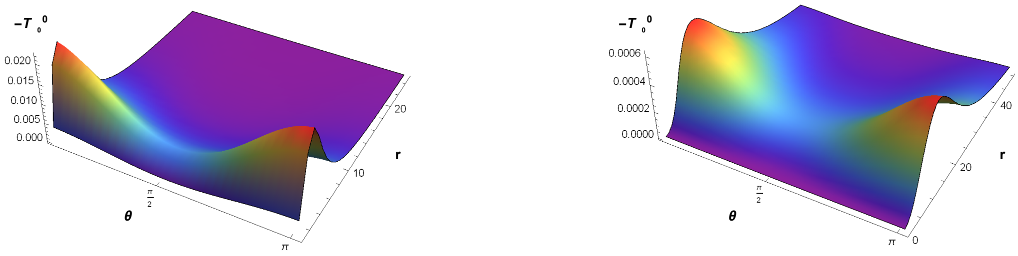

Figure 9 – The energy density, $-T_0^0$, for the electric quadrupole $l = 2$ for $\gamma = 0$ (left) and $\gamma = 0.54$ (right) as functions of the radial coordinate r and the polar coordinate θ. Source: By the author.

we have computed the energy density, $-T_t^t$, of the electric and magnetic multipoles as seen by a static observer with four velocity $u^\mu \propto (1, 0, 0, 0)$. The energy density for the quadrupole $l = 2$, is shown in Fig. 9 for the electric field and in Fig. 10 for the magnetic field as functions of the radial coordinate r and the polar coordinate θ and specific values of the coupling constant: $\gamma = 0$ on the left and $\gamma = 0.54$ on the right. The distribution of the energy density is symmetric about $\theta = \pi/2$, concentrates near the origin and decreases when γ increases. To study the magnetic multipoles, we choose $A_t = 0$ in the ansatz (4.16) and assume the separation of variables

$$A_\varphi(t, \theta) = S_l(r) U_l(\theta). \tag{4.22}$$

We use the electromagnetic duality to relate the functions R_l and $\mathcal{P}_l$ of the electric case to S_l and U_l of the magnetic case. The duality relates them by

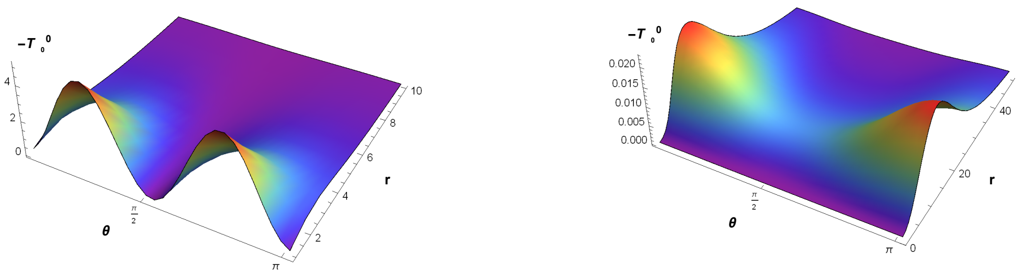

Figure 10 – The energy density, $-T_0^0$, for the magnetic quadrupole $l = 2$ for $\gamma = 0$ (left) and $\gamma = 0.54$ (right) as functions of the radial coordinate r and the polar coordinate θ. Source: By the author.

$$S_l(r) = \frac{r^2}{\sigma(r)} \frac{dR_l}{dr}, \qquad U_l(\theta) = \sin\theta \frac{d\mathcal{P}_l}{d\theta}. \tag{4.23}$$

In conclusion, we have found that the multipole expansion of the electromagnetic field in the solitonic background with metric given by (4.13) and the metric functions by (4.15) is everywhere finite and regular except for the monopole $l = 0$. Our result is valid up to second order in the coupling constant γ and it has been generalized to arbitrarly orders in reference. [71] The energy density of the multipoles $l \geq 1$ is concentrated near the origin of the coordinates system $r = 0$, and indicates that the solutions possesses finite energy.

5 CONCLUSIONS

This dissertation deals with extensions of General Relativity in the form of scalar-tensor gravity. We have been primarily interested in asymptotically Anti-de Sitter spacetime which is quite popular in theoretical physics nowadays because of its role in describing strongly coupled QFTs through the AdS/CFT correspondence. We have two main reasons to have studied the topics in this dissertation: (i) astrophysical/cosmological in the sense of a better understanding of gravity theories in these spacetimes by compare its prediction with what is observed through gravitational wave detections and others observations. (ii) use these theories as toy models to understand strongly coupled quantum field theories better through the above mentioned correspondence. We have mentioned the existing applications of the topics discussed in here in holography such as the description of holographic superconductors through the coupling of a charged scalar field to an AdS black hole and the role that the quasinormal modes of a black hole plays in the process of thermalization in the dual theory. We also have discussed new current research topics such as BH chemistry and the nonlinear instability of Anti-de Sitter spacetime reviewing the main results.

Chapter 2 is devoted to study anti-de Sitter spacetime and black hole solutions inside AdS. We only have discussed a few among the most popular BH solutions with AdS asymptotics (Schwarzschild-AdS, Reissner-Nordström-AdS, Kerr-AdS and Gauss-Bonnet-AdS) which we think is enough to illustrate the main points and features of these BHs. BH thermodynamics in AdS is a very active topic of research and we have mentioned some thermodynamic properties of AdS BHs.

We dealt with the instability issues of solutions to Einstein's equation with AdS asymptotic in Chapter 3. QNMs can indicate linear (in)stability of BHs and this is also a very active area of research. Its relation with GWs was clarified and the interpretation within the dual field theory was discussed. Currently, there's enough evidence (both numerical and analytical) in favor of the nonlinear instability of AdS. An analytical proof of the nonlinear instability, however, is a work in progress and a great deal of new results on the subject are expected in the coming years.

Some specific scalar-tensor theories of gravity have been discussed in Chapter 4. They're mainly motivated in order to explain the accelerated expansion of the universe. The scalar-tensor gravity theories discussed in this dissertation are not favored by the analyzes of the GW observations but this does not mean that they are useless: they also find applications in holography. From the bottom-up approach it is not always straightforward to find the dual theory, since the phase structure is, in general, more complicated. This is a topic that will be addressed in the future. We also have discussed the multipole expansion of the electromagnetic field in a shift-symmetric scalar-tensor theory in which the background has solitonic-like properties and our numerical results show the existence of everywhere regular multipoles except for monopole $l = 0$.

To summarize, all the topics discussed in here are central in order to better understand theoretical physics, specially BHs. These object plays such a major role in theoretical physics nowadays that they can be called the "harmonic oscillator of the 21st century"[74] and a proper understanding of Nature certainly requires a deep understanding of these objects.

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Appendix

APPENDIX A – POINCARÉ-LINDSTEDT METHOD

In this appendix, we deal with the Poincaré-Lindstedt method, which eliminates resonant terms appearing in nonlinear perturbation theory. Some resonant terms appearing in Eq. (3.30) can be eliminated by a specific time coordinate transformation and frequency shift (but not all of them). To illustrate this procedure, we study the following example of a 1-dimensional anharmonic oscillator [75]

$$\ddot{x} + x + \epsilon x^3 = 0, \tag{A.1}$$

for initial conditions $(x(0), \dot{x}(0)) = (1, 0)$ and ϵ assumed to be a small parameter. Dot $\dot{}$ represents derivative with time coordinate t. We assume the existence of a perturbative solution

$$x(t) = x_0(t) + \epsilon x_1(t) + ... \tag{A.2}$$

The solution at 0th order in ϵ is $x_0 = \cos t$. The 1st order in ϵ equation is

$$\ddot{x}_1 + x_1 + x_0^3 = 0. \tag{A.3}$$

Using some trigonometric identities, we rewrite $x_0^3 = \cos^3 t$ as $x_0^3 = (3\cos t + \cos 3t)/4$. The 1st order in ϵ equation is a inhomogeneous second order differential equation. It's solution is given by

$$x_1 = \frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8} t \sin t. \tag{A.4}$$

The last term on the r.h.s of Eq. (A.4) is responsible for invalidating the perturbation theory since it grows linearly with t and is called secular term. The Poincaré-Lindstedt method enable us to eliminate the secular term $t \sin t$. Considering the transformation

$$\tau = \omega t, \qquad \omega = \omega_0 + \epsilon\omega_1 + ... \tag{A.5}$$

Eq. (A.1) becomes

$$\omega^2 x''(\tau) + x(\tau) + \epsilon x(\tau)^3 = 0, \tag{A.6}$$

with the same initial conditions as the original problem. Where a prime $'$ denote derivative with respect to τ. We chose $\omega_0 = 1$ since it's the 0th order frequency. The solutions to the 0th and 1st order equations are

$$x_0(\tau) = \cos \tau, \tag{A.7}$$

$$x_1(\tau) = \frac{1}{32}(\cos 3\tau - \cos \tau) + \left(\omega_1 - \frac{3}{8}\right) \tau \sin \tau. \tag{A.8}$$

Choosing $\omega_1 = 3/8$ we eliminate the secular term. This procedure can be applied for higher order secular terms if necessary and the final perturbative solution up to 1st order is

$$x(t) = \cos\left(\left(1 + \frac{3}{8}\epsilon\right) t\right) + \frac{1}{32}\epsilon \left[\cos\left(3\left(1 + \frac{3}{8}\epsilon\right) t\right) - \cos\left(\left(1 + \frac{3}{8}\epsilon\right) t\right)\right]. \tag{A.9}$$

APPENDIX B – CONFORMAL FIELD THEORY

A conformal field theory (CFT) is a particular example of Quantum Field Theory (QFT) in which besides Poincaré invariance, it also has *conformal invariance*. For our purposes this will always be equivalent to the statement that the theory is invariant under scale transformations $x^\mu \to \lambda x^\mu$. To put in a different way, this means that the theory has no length or mass scale and we only care about angles between curves at a given point being preserved under such transformations. When studying CFTs one realize the great difference between two dimensional and higher dimensional CFTs. The conformal group in two dimensions has as elements of the group, all the analytical functions which makes it infinite dimensional. This property of CFTs in two dimensions is responsible for a variety of exact results obtained for 2-dimensional theories. [76] All the conformal groups with dimension higher than three are finite dimensional.

It's not straightforward to find a system in which conformal transformations are symmetries of it. Our Universe is by no means conformal invariant since there is universal length scales intrinsic to it such as the radius of the hydrogen atom. However, one example of a conformal invariant system of great importance to Physics is the critical phenomena such as phase transitions in statistical mechanics. The correlation length diverges at the critical values of the parameters and the system is conformal invariant. It's worth mentioning that CFTs also play an important role in *string theory*, due to the reparametrization invariance of the world-sheet, which is a 2-dimensional hypersurface embedded in spacetime. In the rest of this appendix, we discuss in more details the conformal transformations, the conformal group, and its algebra using as references. [76,77]

B.1 Conformal transformations

A *conformal transformation* is a transformation of the spacetime coordinates, $x^\mu \to x'^\mu$, which leaves the metric invariant up to a scale factor $\Omega(x)$

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{B.1}$$

Where the primed quantities are the ones obtained after the transformation. Under a generic coordinate transformation the metric, being a rank-2 tensor, transform as

$$g_{\rho\sigma}(x) \to g'_{\rho\sigma}(x') = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} g_{\mu\nu}(x), \tag{B.2}$$

Restricting the analysis for the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$, from Eqs. (B.1) and (B.2), a conformal transformation must obeys

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Omega(x)\eta_{\mu\nu}. \tag{B.3}$$

As usual in the study of symmetries, we study infinitesimal transformations satisfying Eq. (B.3). Finite transformations is expected to be obtained from the infinitesimal ones by acting with them an appropriate number of times. The infinitesimal transformation is given by

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu, \tag{B.4}$$

we assume that $\epsilon^\mu \ll 1$. since Eq. (B.3) has partial derivatives of the transformed coordinates with respect to the old ones, let us compute these quantities

$$\frac{\partial x'^\rho}{\partial x^\mu} = \delta^\rho_\mu + \frac{\partial \epsilon^\rho}{\partial x^\mu}, \tag{B.5}$$

substituting these partial derivatives in Eq. (B.3) and only taking into account first-order terms in ϵ

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \eta_{\rho\sigma}\left(\delta^\rho_\mu + \frac{\partial \epsilon^\sigma}{\partial x^\nu}\right)\left(\delta^\sigma_\nu + \frac{\partial \epsilon^\sigma}{\partial x^\nu}\right) = \eta_{\mu\nu} + \left(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}\right), \tag{B.6}$$

inspection on the right-most term of Eq. (B.6) tell us that it should be proportional to the metric in order for Eq. (B.3) to hold. So the last term in Eq. (B.6) should be

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu}. \tag{B.7}$$

Tracing both sides of Eq. (B.7) with $\eta^{\mu\nu}$, and using $\eta^{\mu\nu}\eta_{\mu\nu} = d$, we obtain $f(x) = (\partial_\mu \epsilon^\mu)2/d$. Then, substituting the form of the function f in Eq. (B.7),

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\left(\partial_\rho \epsilon^\rho\right)\eta_{\mu\nu}. \tag{B.8}$$

Acting on Eq. (B.8) with ∂^ν, commuting the derivatives and relabeling dummy indices,

$$\partial_\mu\left(\partial_\nu \epsilon^\nu\right) + \partial^\nu\partial_\nu \epsilon_\mu = \frac{2}{d}\partial_\mu\left(\partial_\nu \epsilon^\nu\right). \tag{B.9}$$

Acting on this last expression again with ∂_ν gives

$$\partial_\mu\partial_\nu\left(\partial_\alpha \epsilon^\alpha\right) + (\partial^\alpha\partial_\alpha)\partial_\nu \epsilon_\mu = \frac{2}{d}\partial_\mu\partial_\nu\left(\partial_\alpha \epsilon^\alpha\right), \tag{B.10}$$

exchanging $\mu \leftrightarrow \nu$, in Eq. (B.10),

$$\partial_\nu\partial_\mu\left(\partial_\alpha \epsilon^\alpha\right) + (\partial^\alpha\partial_\alpha)\partial_\mu \epsilon_\nu = \frac{2}{d}\partial_\nu\partial_\mu\left(\partial_\alpha \epsilon^\alpha\right), \tag{B.11}$$

adding Eq. (B.10) with Eq. (B.11), and using the commutativity property of partial derivatives,

$$2\partial_\mu\partial_\nu\left(\partial_\beta \epsilon^\beta\right) + \partial^\alpha\partial_\alpha\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = \frac{4}{d}\partial_\nu\partial_\mu\left(\partial_\beta \epsilon^\beta\right), \tag{B.12}$$

Using Eq. (B.8) in the second term on the l.h.s of Eq. (B.12) and rearranging,

$$\left[\,\eta_{\mu\nu}\partial^\alpha\partial_\alpha + (d-2)\partial_\mu\partial_\nu\,\right]\left(\partial_\beta \epsilon^\beta\right) = 0. \tag{B.13}$$

Finally, contracting the above equation with $\eta^{\mu\nu}$ gives

$$[d\,\partial^\alpha\partial_\alpha + (d-2)\partial^\alpha\partial_\alpha]\left(\partial_\beta\epsilon^\beta\right) = (d-1)\,\partial^\alpha\partial_\alpha\left(\partial_\beta\epsilon^\beta\right) = 0. \tag{B.14}$$

This equation implies that, for $d \geq 3$, $(\partial_\alpha\epsilon^\alpha)$ can be at most linear in x^μ, therefore, ϵ^μ can be at most quadratic in x^μ*.

B.2 Infinitesimal conformal transformations

In order to find the commutation relations satisfied by the conformal algebra, we look at the infinitesimal transformations. From Eq. (B.14) we wee that a general infinitesimal conformal transformation has the following form

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{B.15}$$

where a_μ, $b_{\mu\nu}$, $c_{\mu\nu\rho}$ are constants and assumed to be small. To obtain the generators corresponding to each of the three terms in (B.15), we consider that under a transformation of the coordinates and the fields, collectively denoted as Φ, behave as

$$x^\mu \to x'^\mu, \tag{B.16}$$

$$\Phi(x) \to \Phi'(x'), \tag{B.17}$$

where x' is understood to be a function of the old coordinates x and the new fields Φ' at the new position x', is assumed to be a function of the old function at the old position, that is, $\Phi'(x') = \mathcal{F}(\Phi(x))$. The changes in the fields Φ comes in two ways: (i) the functional change, which depends on the function $\mathcal{F}$ and (ii) the change in the argument $x \to x'$. Generic infinitesimal transformations can be written as

$$\begin{aligned} x'^\mu &= x^\mu + \epsilon_\alpha \frac{\delta x^\mu}{\delta\epsilon_\alpha}, \\ \Phi'(x') &= \Phi(x) + \epsilon_\alpha \frac{\delta\mathcal{F}}{\delta\epsilon_\alpha(x)}(x). \end{aligned} \tag{B.18}$$

Where the parameters $\{\epsilon_\alpha\}$ are assumed to be small and kept to first order. We define the generators of the transformation, G^α, through the functional difference

$$\Phi'(x) - \Phi(x) \equiv -i\epsilon_\alpha G^\alpha \Phi(x). \tag{B.19}$$

On the other hand, we can Taylor expand $\Phi'(x')$ up to first order in ϵ

$$\Phi'(x') = \Phi'(x) + \epsilon_\alpha \frac{\delta x^\mu}{\delta\epsilon_\alpha}\partial_\mu\Phi'(x). \tag{B.20}$$

* The case $d = 2$ requires special attention, as said in the introduction of this appendix (Eq. (B.14) do not folllows from Eq. (B.13)). In the case $d = 1$, Eq. (B.14) do not impose restrictions on the functional dependence of ϵ and any smooth function is a conformal transformation which is a trivial result since in this case there is no notion of angle.

Isolating $\Phi'(x)$ in Eq. (B.20) and substituting it into the Eq. (B.19), we have

$$\Phi'(x') - \epsilon_\alpha \frac{\delta x^\mu}{\delta \epsilon_\alpha} \partial_\mu \Phi'(x) - \Phi(x) = -i\epsilon_\alpha G^\alpha \Phi(x). \tag{B.21}$$

Using Eq. (B.18) we can rewrite Eq. (B.21) as

$$i\epsilon_\alpha G^\alpha \Phi(x) = \epsilon_\alpha \frac{\delta x^\mu}{\delta \epsilon_\alpha} \partial_\mu \Phi'(x) - \epsilon_\alpha \frac{\delta \mathcal{F}}{\delta \epsilon_\alpha(x)}(x). \tag{B.22}$$

From Eq. (B.18) we have $\partial_\mu \Phi' = \partial_\mu \Phi$ up to 0th order in ϵ. Then, Eq. (B.22) can be rewritten as

$$iG^\alpha \Phi = \frac{\delta x^\mu}{\delta \epsilon_\alpha} \partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \epsilon_\alpha}. \tag{B.23}$$

If we impose that the fields are unaffected by the transformation, such that $\mathcal{F}(\Phi) = \Phi$, the last term on the r.h.s of Eq. (B.23) vanish and we obtain the following expression for the generators:

$$G_a = -i \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu. \tag{B.24}$$

To discover the physical meaning of each term in (B.15), we study them separately. Setting $b_{\mu\nu} = 0$ and $c_{\mu\nu\rho} = 0$, the transformation is

$$x^\mu \to x'^\mu = x^\mu + a^\mu, \tag{B.25}$$

an infinitesimal translation in spacetime ($a^\mu \ll 1$). Therefore,

$$\delta x^\mu \equiv x'^\mu - x^\mu = a^\mu \quad \Rightarrow \quad \frac{\delta x^\mu}{\delta a^\nu} = \delta^\mu_\nu. \tag{B.26}$$

Using Eq. (B.24) and Eq. (B.26), the generator of translations is given by

$$P_\mu = -i\partial_\mu \, . \tag{B.27}$$

Let us look at the linear term. If we substitute $\epsilon_\mu = b_{\mu\nu} x^\nu$ into Eq. (B.8), we find that the symmetric part of $b_{\mu\nu}$ is proportional to the metric

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\gamma{}_\gamma \eta_{\mu\nu}, \tag{B.28}$$

so we decompose $b_{\mu\nu}$ as a symmetric part and an antisymmetric part

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu} \tag{B.29}$$

with α being a small constant and $m_{\mu\nu} = -m_{\nu\mu}$. If we set $m_{\mu\nu} = 0$ and consider only the transformation proportional to the metric, we have

$$x^\mu \to x'^\mu = (1 + \alpha)\, x^\mu, \tag{B.30}$$

an infinitesimal dilation (or infinitesimal scale transformation, since $\alpha \ll 1$). Therefore,

$$\delta x^\mu \equiv x'^\mu - x^\mu = \alpha x^\mu \quad \Rightarrow \quad \frac{\delta x^\mu}{\delta \alpha} = x^\mu. \tag{B.31}$$

Using Eq. (B.24) and Eq. (B.30), the generator of dilation is given by

$$D = -ix^{\mu}\partial_{\mu}. \tag{B.32}$$

Now we consider only the antisymmetric part of $b_{\mu\nu}$. The transformation is given by

$$x^{\mu} \to x'^{\mu} = \left(\delta^{\mu}_{\nu} + m^{\mu}{}_{\nu}\right) x^{\nu} \tag{B.33}$$

with $m^{\mu}{}_{\nu} \ll 1$ and $m_{\mu\nu} = -m_{\nu\mu}$. Therefore, using the antisymmetry

$$\delta x^{\mu} \equiv x'^{\mu} - x^{\mu} = m^{\mu\nu}x_{\nu} \quad \Rightarrow \quad \frac{\delta x^{\mu}}{\delta m^{\alpha\beta}} = \left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right) x_{\nu} \tag{B.34}$$

Substituting Eq. (B.34) into the expression for the generator of the transformation, Eq. (B.24), we obtain

$$L_{\alpha\beta} = -i\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right) x_{\nu}\partial_{\mu} = i\left(x_{\alpha}\partial_{\beta} - x_{\beta}\partial_{\alpha}\right) \tag{B.35}$$

which we recognize as the Lorentz generators; they generates rotations and boost in Minkowski spacetime.

The only term that has left to be scrutinized is the quadratic one, $c_{\mu\nu\rho}$. To obtain more information about the $c_{\mu\nu\rho}$, we act with ∂_{ρ} on Eq. (B.7), permute the indices and take a linear combination to obtain

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \eta_{\mu\nu}\partial_{\nu}f + \eta_{\nu\rho}\partial_{\mu}f - \eta_{\mu\nu}\partial_{\rho}f. \tag{B.36}$$

Substituting $\epsilon_{\mu} = c_{\mu\nu\rho}x^{\nu}x^{\rho}$ and $f = 2(\partial_{\alpha}\epsilon^{\alpha})/d$ into Eq. (B.36), we obtain an expression for the parameter $c_{\mu\nu\rho}$ given by

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu}, \qquad b_{\mu} = \frac{1}{d}c^{\nu}_{\nu\mu}. \tag{B.37}$$

The infinitesimal transformation is given by

$$\begin{aligned} x^{\mu} \to x'^{\mu} &= x^{\mu} + \eta^{\mu\alpha}c_{\alpha\nu\rho}x^{\nu}x^{\rho} = x^{\mu} + \eta^{\mu\alpha}\left(\eta_{\alpha\rho}b_{\nu} + \eta_{\alpha\nu}b_{\rho} - \eta_{\nu\rho}b_{\alpha}\right)x^{\nu}x^{\rho} \\ &= x^{\mu} + 2x^{\mu}(b_{\alpha}x^{\alpha}) - b^{\mu}x^{\alpha}x_{\alpha}. \end{aligned} \tag{B.38}$$

Therefore,

$$\delta x^{\mu} \equiv x'^{\mu} - x^{\mu} = 2x^{\mu}(b^{\alpha}x_{\alpha}) - b^{\mu}x^{\alpha}x_{\alpha} \quad \Rightarrow \quad \frac{\delta x^{\mu}}{\delta b^{\alpha}} = 2x_{\alpha}x^{\mu} - x^{\alpha}x_{\alpha}\delta^{\mu}_{\alpha}. \tag{B.39}$$

This transformation is called *special conformal transformation*. Substituting Eq. (B.39) into the expression for the generator of this transformation, Eq. (B.24), we obtain

$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu}\right). \tag{B.40}$$

It's not straightforward to see the physical meaning of this generator, but it can be shown that a special conformal transformation is equivalent to a inversion, $x^{\mu} \to x^{\mu}/x^{2}$, followed by a translation of b^{μ}, followed by another inversion.

B.3 Conformal algebra

Having the explicit expressions for all the generators of the conformal transformations, we calculate the commutation relations between them

$$D = -ix^{\mu}\partial_{\mu} \qquad \text{dilation} \tag{B.41}$$

$$P_{\mu} = -i\partial_{\mu} \qquad \text{translation} \tag{B.42}$$

$$L_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \qquad \text{rotation} \tag{B.43}$$

$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}\right) \qquad \text{special conformal transformation} \tag{B.44}$$

The commutation relation between dilation and translation is

$$\begin{aligned}
[D, P_{\mu}] &= DP_{\mu} - P_{\mu}D = \left(-ix^{\alpha}\partial_{\alpha}\left(-i\partial_{\mu}\right)\right) - \left(\left(-i\partial_{\mu}\right)\left(-ix^{\alpha}\partial_{\alpha}\right)\right) \\
&= -x^{\alpha}\partial_{\alpha}\partial_{\mu} + \partial_{\mu}\left(x^{\alpha}\partial_{\alpha}\right) = \partial_{\mu} = iP_{\mu}.
\end{aligned}$$

The commutation relation between dilation and special conformal transformation is

$$\begin{aligned}
[D, K_{\mu}] &= DK_{\mu} - K_{\mu}D = -ix^{\alpha}\partial_{\alpha}\left(-i\left(x_{\mu}x^{\alpha}\partial_{\alpha} - x^2\partial_{\mu}\right)\right) - \left(-i\left(x_{\mu}x^{\alpha}\partial_{\alpha} - x^2\partial_{\mu}\right)\right)\left(-ix^{\alpha}\partial_{\alpha}\right) \\
&= -x^{\alpha}\partial_{\alpha}\left(x_{\mu}x^{\alpha}\partial_{\alpha} - x^2\partial_{\mu}\right) + \left(x_{\mu}x^{\alpha}\partial_{\alpha} - x^2\partial_{\mu}\right)\left(x^{\alpha}\partial_{\alpha}\right) \\
&= -x^{\alpha}\partial_{\alpha}(x_{\mu}x^{\nu}\partial_{\nu}) + x^{\alpha}\partial_{\alpha}(x^2\partial_{\mu}) + x_{\mu}x^{\nu}\partial_{\nu}(x^{\alpha}\partial_{\alpha}) - x^2\partial_{\mu}(x^{\alpha}\partial_{\alpha}) \\
&= -x^{\alpha}(\partial_{\alpha}x_{\mu})x^{\nu}\partial_{\nu} - x^{\alpha}x_{\mu}(\partial_{\alpha}x^{anu})\partial_{\nu} - x^{\alpha}x_{\mu}x^{\nu}\partial_{\alpha}\partial_{\nu} + 2x^{\alpha}x_{\alpha}\partial_{\mu} \\
&+ x^{\alpha}x^2\partial_{\mu}\partial_{\alpha} + x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu} - x^2x^{\alpha}\partial_{\alpha}\partial_{\mu} \\
&= x^2\partial_{\mu} - 2x_{\mu}x^{\nu}\partial_{\nu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}\right) = -iK_{\mu}.
\end{aligned}$$

The commutation relation between translation and rotation is

$$\begin{aligned}
[P_{\rho}, L_{\mu\nu}] &= P_{\rho}L_{\mu\nu} - L_{\mu\nu}P_{\rho} = \partial_{\rho}\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) - \left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\partial_{\rho} \\
&= \eta_{\rho\mu}\partial_{\nu} + x_{\mu}\partial_{\rho}\partial_{\nu} - \eta_{\rho\nu}\partial_{\mu} - x_{\nu}\partial_{\rho\mu} - x_{\mu}\partial_{\nu\rho} + x_{\nu}\partial_{\mu}\partial_{\rho} \\
&= \eta_{\rho\mu}\partial_{\nu} - \eta_{\rho\nu}\partial_{\mu} = i\left(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu}\right)
\end{aligned}$$

The commutation relation between rotations is is

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] &= L_{\mu\nu}L_{\rho\sigma} - L_{\rho\sigma}L_{\mu\nu} \\
&= -\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right) + \left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right)\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \\
&= -x_{\mu}\partial_{\nu}\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right) + x_{\nu}\partial_{\mu}\left(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right) + x_{\rho}\partial_{\sigma}\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) - x_{\sigma}\partial_{\rho}\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \\
&= -x_{\mu}\eta_{\nu\rho}\partial_{\sigma} - x_{\mu}x_{\rho}\partial_{\nu}\partial_{\sigma} + x_{\mu}\eta_{\nu\sigma}\partial_{\rho} + x_{\mu}x_{\sigma}\partial_{\nu}\partial_{\rho} + x_{\nu}\eta_{\mu\rho}\partial_{\sigma} + x_{\nu}x_{\rho}\partial_{\mu}\partial_{\sigma} - x_{\nu}\eta_{\mu\sigma}\partial_{\rho} - x_{\nu}x_{\sigma}\partial_{\mu}\partial_{\rho} \\
&+ x_{\rho}\eta_{\sigma\mu}\partial_{\nu} + x_{\rho}x_{\mu}\partial_{\sigma}\partial_{\nu} - x_{\rho}\eta_{\sigma\nu}\partial_{\mu} - x_{\rho}x_{\nu}\partial_{\sigma}\partial_{\mu} - x_{\sigma}\eta_{\rho\mu}\partial_{\nu} - x_{\sigma}x_{\mu}\partial_{\rho}\partial_{\nu} + x_{\sigma}\eta_{\rho\nu}\partial_{\mu} + x_{\sigma}x_{\nu}\partial_{\rho}\partial_{\mu} \\
&= \eta_{\rho\nu}\left(x_{\sigma}\partial_{\mu} - x_{\mu}\partial_{\sigma}\right) + \eta_{\nu\sigma}\left(x_{\mu}\partial_{\rho} - x_{\rho}\partial_{\mu}\right) + \eta_{\mu\rho}\left(x_{\nu}\partial_{\sigma} - x_{\sigma}\partial_{\nu}\right) + \eta_{\sigma\mu}\left(x_{\rho}\partial_{\nu} - x_{\nu}\partial_{\rho}\right) \\
&= i\left(\eta_{\rho\nu}L_{\mu\sigma} + \eta_{\nu\sigma}L_{\rho\mu} + \eta_{\rho\mu}L_{\sigma\nu} + \eta_{\sigma\mu}L_{\nu\rho}\right).
\end{aligned} \tag{B.45}$$

In a similar way, we obtain the remaining non trivial commutation relation between special conformal transformations and translations,

$$[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right),$$

and between the special conformal transformations and rotation,

$$[K_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right).$$

These commutation relations define the *conformal algebra*. The number of generators of an algebra is an important quantity that characterize the associated group (the dimension of the group). We can count them in this case: 1 generator for dilation, d for the translations, d for the special conformal transformations and $d(d-1)/2$ for the Lorentz transformations. The total number of generators of the conformal group in d dimensions is, therefore, $(d+2)(d+1)/2$.

The AdS_{d+1} metric as given in Eq. (2.17) has an explicit $SO(2,d)$ invariance. The group $SO(2,d)$ is the group of orthogonal real (d+2) dimensional matrices with unit determinant and 2 timelike components, generated by antisymmetric (d+2) dimensional matrices. The number of generator of $SO(2,d)$ is, therefore, $(d+2)(d+1)/2$. Which is exactly the same number of generators of the conformal transformations. This apparent coincidence raise the question if the two groups, the group of conformal transformations in d dimensions and the group $SO(2,d)$ are isomorphic. Turns out that, in fact, they're isomorphic. This can be seen as follows. We consider a linear combination of the generators as

$$\begin{aligned} J_{\mu\nu} &\equiv L_{\mu\nu}, \qquad & J_{-1\mu} &\equiv \frac{1}{2}(P_\mu - K_\mu) \\ J_{-10} &\equiv D \qquad & J_{0\mu} &\equiv \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{B.46}$$

and $J_{mn} = -J_{nm}$ with $m, n \in \{-1, 0, 1, ..., d\}$. We can write a unique expression comprising all the commutation relations of the conformal algebra using the definition (B.46) as

$$[J_{mn}, J_{pq}] = i\left(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp}\right) \tag{B.47}$$

where $\eta_{mn} = \mathrm{diag}(-1,-1,1,...1)$. This is the commutation relations for the Lie algebra $so(2,d)$ (compare with (B.45), the commutation relation of the generators of the Lorentz group $SO(1,d-1)$).

The results above show that the conformal group in flat d-dimensional spacetime is the same as the symmetry group $SO(2,d)$ of AdS_{d+1} spacetime. This is one indication that theories living on both spacetimes can be seen as dual to one another, but solely the matching of the symmetries is not enough. Additionally, one should associate to each physical observable in one theory, another physical observable in the other. The duality AdS/CFT was proved for specific examples of gravity theories and conformal field theories and is undeniable the profound impact it caused in theoretical physics, providing framework to attack some of the most intricate problems.