Journal of Physics A: Mathematical and Theoretical

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OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **51** (2018) 455203 (15pp)
https://doi.org/10.1088/1751-8121/aae4cc

Extended Calogero models: a construction for exactly solvable *kN*-body systems

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Received 19 July 2018, revised 17 September 2018
Accepted for publication 27 September 2018
Published 16 October 2018

Abstract

We propose a systematic procedure for the construction of exactly solvable *kN*-body systems which are natural generalisations of Calogero models. As examples, we present two new $3N$-body models and determine explicit expressions for their eigenvalues and eigenfunctions.

Keywords: exactly solvable systems, Calogero models, pre-superpotential

1. Introduction

Exactly solvable (ES) quantum many-body problems attract considerable research activity due to their connections with many branches of physics, e.g. [1–8]. In 1969, Calogero obtained the exact solution for a three-particle system with pairwise interactions via square and inverse square potentials [9], and later generalized this result to the N-body case [10]. In 1974, Wolfes extended Calogero's three-body problem by adding terms which are inverse squares of certain linear combinations of the three-particle coordinates [11]. In [12], Sutherland proposed ES models with trigonometric potentials [12]. In the 1980s, Olshanetsky and Perelomov carried out a survey and gave a classification of ES models according to the root systems of simple Lie algebras [13].

Over recent decades, models of the Calogero type (i.e. those where the potential is of the form 'oscillator/inverse square') have received considerable attention, and many interesting properties have been discovered [14–26]. There are also many works which have attempted to obtain new ES models by extending existing ones through separation of variables [27–29].

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More complicated extensions, which have connections with orthogonal polynomials, can be obtained by $\mathcal{PT}$ (parity and time reversal) symmetric quantum mechanics [30–32]. In this work, we propose a systematic method for constructing ES kN-body systems in one dimension. Such models consist of N interacting blocks, each of which contains k particles. The blocks interact through their centres of mass, while particles in each block interact via A or G_2 type potential. As examples, we provide two new ES $3N$-body models and obtain their corresponding eigenvalues and eigenfunctions.

The paper is organized as follows. In section 2, we describe a general procedure for constructing ES many-body quantum Hamiltonians in terms of pre-superpotentials. By choosing an appropriate form of pre-superpotential, we derive a rational ansatz whose solutions give rise to ES models. All Calogero type models associated with the root systems of simple Lie algebras satisfy this ansatz. We list the Hamiltonians of such Calogero systems, and their corresponding ground state energies and wave functions. In section 3, we combine distinct A or G_2 type models together to form a new family of ES models through a coupling function. Various types of coupling functions will be studied. We show that every member in this family satisfies the rational ansatz, thus proving these new models remain ES. As examples, in section 4 we present two new $3N$-body systems. Applying appropriate coordinate transformations, we separate the $3N$-body eigenvalue problem into equations for radial, angular, and center-of-mass parts coordinates. We solve these equations to give the eigenvalues and eigenfunctions of the $3N$-body models. We summarize our work in the final section.

2. General discussion and results

Throughout this paper we set $\hbar = 2m = 1$. We start with the basic relation [26],

$$\hat{p}^2 e^W = -\sum_{i=1}^{N}\left[\left(\frac{\partial W}{\partial x_i}\right)^2 + \frac{\partial^2 W}{\partial x_i^2}\right]e^W, \quad \hat{p}^2 = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}.$$

This relation guarantees that e^W is an eigenfunction of the following Hamiltonian

$$\hat{H} = \hat{p}^2 + \sum_{i=1}^{N}\left[\left(\frac{\partial W}{\partial x_i}\right)^2 + \frac{\partial^2 W}{\partial x_i^2}\right],$$

with zero eigenvalue, i.e.

$$\hat{H}e^W = 0, \tag{1}$$

provided that e^W is square-integrable. Such a function W is called a *pre-superpotential*. Now we set W to be of the form

$$W = \sum_{i=1}^{M}\alpha_i \log|\vec{v}_i\cdot\vec{x}| - \frac{\omega}{2}\sum_{i=1}^{N}x_i^2, \tag{2}$$

where $\vec{x} = (x_1, x_2, \cdots, x_N)$, and $\vec{v}_i$'s are some distinct vectors. Then (1) becomes

$$\left\{\hat{p}^2 + \omega^2\sum_{i=1}^{N}x_i^2 + \sum_{i=1}^{M}\frac{\alpha_i(\alpha_i-1)|\vec{v}_i|^2}{(\vec{v}_i\cdot\vec{x})^2} + 2\sum_{i<j}^{M}\frac{\alpha_i\alpha_j}{(\vec{v}_i\cdot\vec{x})(\vec{v}_j\cdot\vec{x})}(\vec{v}_i\cdot\vec{v}_j) - E_0\right\}e^W = 0,$$

where $E_0 = 2\omega\sum_{i=1}^{M}\alpha_i + N\omega$. Thus if we choose $\vec{v}_j$ and α_j such that the following so-called *rational ansatz* is satisfied

$$\sum_{i<j}^{M} \frac{\alpha_i \alpha_j}{(\vec{v}_i \cdot \vec{x})(\vec{v}_j \cdot \vec{x})} (\vec{v}_i \cdot \vec{v}_j) = 0, \tag{3}$$

and denote the corresponding Hamiltonian as $\hat{H}_{\rm Cal}$, we then have

$$\hat{H}_{\rm Cal} = \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{M} \frac{\alpha_i(\alpha_i - 1)|\vec{v}_i|^2}{(\vec{v}_i \cdot \vec{x})^2},$$

$$\hat{H}_{\rm Cal}\ e^W = \left(2\omega \sum_{i=1}^{M} \alpha_i + N\omega\right)\ e^W. \tag{4}$$

In other words, if (3) is satisfied, the Hamiltonian (4) admits a ground state e^W with corresponding energy E_0.

It is straightforward to show that $e^{-\widehat{W H_{\rm Cal}}}e^W$ gives

$$e^{-\widehat{W H_{\rm Cal}}}e^W = \hat{p}^2 - 2\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\alpha_j v_{ji}}{\vec{v}_j \cdot \vec{x}} \frac{\partial}{\partial x_i} + 2\omega \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + E_0 \tag{5}$$

where v_{ji} denotes the ith component of the vector $\vec{v}_j$. We find $e^{-\widehat{W H_{\rm Cal}}}e^W \mathcal{P}_n(t) \subset \mathcal{P}_n(t)$ for any positiver integer n, where $\mathcal{P}_n(t)$ is defined by

$$\mathcal{P}_n(t) = {\rm span}\{1, t, t^2, \cdots, t^n\}, \quad t = \sum_{i=1}^{N} x_i^2.$$

That is, $e^{-\widehat{W H_{\rm Cal}}}e^W$ preserves the infinite flag of spaces

$$\mathcal{P}_0(t) \subset \mathcal{P}_1(t) \cdots \subset \mathcal{P}_n(t) \subset \cdots.$$

It is not difficult to obtain

$$e^{-\widehat{W H_{\rm Cal}}}e^W\ L_n^{(\alpha)}(4\omega t) = (4\omega n + E_0)\ L_n^{(\alpha)}(4\omega t), \quad \alpha = 4\sum_{j=1}^{M} \alpha_j + 2N + 1,$$

where $L_n^{(\alpha)}(4\omega t)$ is the Laguerre polynomial of degree n. Moreover, the results from [23–25] can be generalized to show the exact solvability of the Hamiltonian $\hat{H}_{\rm Cal}$. Indeed, from (5), we find

$$\hat{g}^{-1} e^{-\widehat{W H_{\rm Cal}}}e^W \hat{g} = \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + E_0 - N\omega,$$

$$\hat{g} = \exp\left\{\frac{1}{4\omega}\left[\hat{p}^2 - 2\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\alpha_j v_{ji}}{\vec{v}_j \cdot \vec{x}} \frac{\partial}{\partial x_i}\right]\right\} \cdot \exp\left\{-\frac{1}{4\omega}\hat{p}^2\right\} \cdot \exp\left\{\frac{\omega}{2}\sum_{i=1}^{N} x_i^2\right\}.$$

In terms of the ladder operators

$$\hat{a}_j = \frac{\partial}{\partial x_j} + \omega x_j, \quad \hat{a}_j^\dagger = -\frac{\partial}{\partial x_j} + \omega x_j,$$

we have

$$\hat{g}^{-1}\widehat{e^{-W}H_{\mathrm{Cal}}e^{W}}\hat{g}=\frac{1}{2}\sum_{i=1}^{N}\{\hat{a}_i,\hat{a}_i^{\dagger}\}+E_0-N\omega.$$

We see that Hamiltonian $\hat{H}_{\mathrm{Cal}}$ can be mapped to independent harmonic oscillators and thus is ES. It is straightforward to show that the transformed 'number operators' $\hat{J}_{ii}=e^{W}\hat{g}\hat{a}_i^{\dagger}\hat{a}_i\hat{g}^{-1}e^{-W}$, $i=1,2,\cdots,N$ are conserved

$$[\hat{J}_{ii},\hat{H}_{\mathrm{Cal}}]=0,$$

as expected.

It can be shown that the rational ansatz (3) has non-trivial solutions. With appropriate α_j's, root vectors of simple Lie algebras satisfy (3) and the corresponding $\hat{H}_{\mathrm{Cal}}$ in (4) give the Hamiltonians of the Calogero type models. On the other hand, it is worth noting that [27] provides an example of an ES Hamiltonian corresponding to $\vec{v}_j$'s in (3) which are not related to a root system of a Lie algebra.

We now list some known results, taken from [15, 16, 20, 26], for later use. While the general discussion above is valid for parameters M and N are independent, the fact that the results below are expressed in terms of root systems imposes a relation between M and N.

2.1. A type Calogero model

For the Calogero model associated with A type root system, the positive root vectors are

$$\hat{\mathbf{e}}_i-\hat{\mathbf{e}}_j=(\cdots,1,\cdots,-1,\cdots)\ \ 1\leqslant i<j\leqslant N$$

where $\hat{\mathbf{e}}_i$ denotes a standard basis element in N-dimensional Euclidean space $\mathbb{R}^N$, and the dots represent zeros. We set $\vec{v}_1=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2$, $\vec{v}_2=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_3,\ldots,\vec{v}_M=\hat{\mathbf{e}}_{N-1}-\hat{\mathbf{e}}_N$, where $M=N(N-1)/2$. We also set $\alpha_1=\cdots=\alpha_M=\alpha$ in (2), such that

$$\begin{aligned}W&=\alpha\sum_{i<j}^{N}\log|x_i-x_j|-\frac{\omega}{2}\sum_{i=1}^{N}x_i^2,\\ \hat{H}_A&=\hat{p}^2+\sum_{i<j}^{N}\frac{2\alpha(\alpha-1)}{(x_i-x_j)^2}+\omega^2\sum_{i=1}^{N}x_i^2,\\ \hat{H}_Ae^{W}&=[N\omega+N(N-1)\omega\alpha]e^{W}.\end{aligned}\tag{6}$$

2.2. BC type Calogero model

For the Calogero model associated with BC type root system, the positive root vectors are

$$\begin{aligned}\hat{\mathbf{e}}_i\pm\hat{\mathbf{e}}_j&=(\cdots,1,\cdots,\pm1,\cdots),\quad 1\leqslant i<j\leqslant N,\\ \hat{\mathbf{e}}_i&=(\cdots,1,\cdots),\quad i=1,2,\ldots,N.\end{aligned}$$

We set $\vec{v}_1=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2$, $\vec{v}_2=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_3,\ldots,\vec{v}_{M'/2}=\hat{\mathbf{e}}_{N-1}-\hat{\mathbf{e}}_N$, $\vec{v}_{M'/2+1}=\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2,\ldots,\vec{v}_{M'}=\hat{\mathbf{e}}_{N-1}+\hat{\mathbf{e}}_N$ where $M'=N(N-1)$, and $\vec{v}_{M'+i}=\hat{\mathbf{e}}_i$, $i=1,\ldots,N$. We also set $\alpha_1=\cdots=\alpha_{M'}=\beta_1$, $\alpha_{M'+1}=\cdots=\alpha_M=\beta_2$ in (2), where $M=M'+N$. Then

$$W = \beta_1 \sum_{i<j}^{N} \{\log|x_i - x_j| + \log|x_i + x_j|\} + \beta_2 \sum_{i=1}^{N} \log|x_i| - \frac{\omega}{2} \sum_{i=1}^{N} x_i^2,$$
$$\hat{H}_{BC} = \hat{p}^2 + \sum_{i<j}^{N} \left\{ \frac{2\beta_1(\beta_1 - 1)}{(x_i - x_j)^2} + \frac{2\beta_1(\beta_1 - 1)}{(x_i + x_j)^2} \right\} + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{N} \frac{\beta_2(\beta_2 - 1)}{x_i^2},$$
$$\hat{H}_{BC} e^W = [2N(N-1)\omega\beta_1 + 2N\omega\beta_2 + N\omega]e^W. \tag{7}$$

If $\beta_2 = 0$, then BC type reduces to D type.

2.3. E_8 type Calogero model

Before defining this model, we need to introduce some notation. For $j = 1, \cdots, 7$ let $a_j \in \mathbb{Z}_2$ and let $\mathcal{I}$ denote the set of septuples $\mathfrak{a} = (a_1, \cdots a_7)$ such that

$$\sum_{i=1}^{7} a_i = 0.$$

Then for the Calogero model associated with E_8 type root system, the positive root vectors are

$$\hat{\mathbf{e}}_i \pm \hat{\mathbf{e}}_j = (\cdots, 1, \cdots, \pm 1, \cdots) \qquad 1 \leqslant i < j \leqslant 8,$$
$$\hat{\mathbf{e}}_8 + \sum_{i=1}^{7} (-1)^{a_i} \hat{\mathbf{e}}_i = ((-1)^{a_1}, \cdots, (-1)^{a_7}, 1), \qquad (a_1, \cdots, a_7) \in \mathcal{I}. \tag{8}$$

We set $\vec{v}_1 = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$, $\vec{v}_{56} = \hat{\mathbf{e}}_7 + \hat{\mathbf{e}}_8$, while the remaining $\vec{v}_{57}, ..., \vec{v}_{120}$ have the form (8). We also set $\alpha_1 = \cdots = \alpha_{120} = \beta$ in (2), so

$$W = \beta \sum_{i<j}^{8} \{\log|x_i - x_j| + \log|x_i + x_j|\} + \beta \sum_{\mathfrak{a} \in \mathcal{I}} \log\left|x_8 + \sum_{i=1}^{7} (-1)^{a_i} x_i\right| - \frac{\omega}{2} \sum_{i=1}^{8} x_i^2,$$
$$\hat{H}_{E_8} = \hat{p}^2 + \sum_{i<j}^{8} \left\{ \frac{2\beta(\beta - 1)}{(x_i - x_j)^2} + \frac{2\beta(\beta - 1)}{(x_i + x_j)^2} \right\} + \sum_{\mathfrak{a} \in \mathcal{I}} \frac{8\beta(\beta - 1)}{(x_8 + \sum_{i=1}^{7} (-1)^{a_i} x_i)^2} + \omega^2 \sum_{i=1}^{8} x_i^2, \tag{9}$$
$$\hat{H}_{E_8} e^W = (240\omega\beta + 8\omega)e^W.$$

2.4. F_4 type Calogero model

For the Calogero model associated with F_4 type root system, the vectors are

$$\hat{\mathbf{e}}_i \pm \hat{\mathbf{e}}_j = (\cdots, 1, \cdots, \pm 1, \cdots) \qquad 1 \leqslant i < j \leqslant 4,$$
$$\hat{\mathbf{e}}_1 + \sum_{i=2}^{4} (-1)^{a_i} \hat{\mathbf{e}}_i = (1, \pm 1, \pm 1, \pm 1),$$
$$\hat{\mathbf{e}}_i = (\cdots, 1, \cdots), \qquad i = 1, 2, 3, 4.$$

We set $\vec{v}_1 = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2, ..., \vec{v}_{12} = \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4$, $\vec{v}_{13} = (1, 1, 1, 1), ..., \vec{v}_{20} = (1, -1, -1, -1)$ and $\vec{v}_{20+i} = \hat{\mathbf{e}}_i$, $i = 1, ..., 4$. We also set $\alpha_1 = \cdots = \alpha_{12} = \nu$, $\alpha_{13} = \cdots = \alpha_{24} = \mu$ in (2), so

$$
\begin{aligned}
W &= \nu \sum_{i<j}^{4} \{\log|x_i - x_j| + \log|x_i + x_j|\} + \mu \sum_{i=1}^{4} \log|x_i| \\
&\quad + \mu \sum_{a_i \in \mathbb{Z}_2} \log\left|x_1 + \sum_{i=2}^{4} (-1)^{a_i} x_i\right| - \frac{\omega}{2} \sum_{i=1}^{4} x_i^2, \\
\hat{H}_{F_4} &= \hat{p}^2 + \sum_{i<j}^{4} \left\{ \frac{2\nu(\nu-1)}{(x_i+x_j)^2} + \frac{2\nu(\nu-1)}{(x_i-x_j)^2} \right\} + \sum_{i=1}^{4} \frac{\mu(\mu-1)}{x_i^2} + \omega^2 \sum_{i=1}^{4} x_i^2 \\
&\quad + \sum_{a_i \in \mathbb{Z}_2} \frac{4\mu(\mu-1)}{(x_1 + \sum_{i=2}^{4} (-1)^{a_i} x_i)^2}, \\
\hat{H}_{F_4} e^W &= (4\omega + 24\omega\mu + 24\omega\nu) e^W.
\end{aligned}
$$

2.5. G_2 type Calogero model

For the Calogero model associated with G_2 type root system, the vectors $\vec{v}_i$, $i = 1, ..., 6$ are given by

$$
\begin{aligned}
&\vec{v}_1 = (1, -1, 0), \qquad \vec{v}_2 = (1, 0, -1), \qquad \vec{v}_3 = (0, 1, -1), \\
&\vec{v}_4 = (1, 1, -2), \qquad \vec{v}_5 = (1, -2, 1), \qquad \vec{v}_6 = (-2, 1, 1).
\end{aligned}
$$

We also set $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1$, $\alpha_4 = \alpha_5 = \alpha_6 = \beta_2$ in (2), so

$$
\begin{aligned}
W &= \beta_1 \sum_{i<j}^{3} \log|x_i - x_j| + \beta_2 \sum_{l \neq i < j \neq l}^{3} \log|x_i + x_j - 2x_l| - \frac{\omega}{2} \sum_{i=1}^{3} x_i^2, \\
\hat{H}_{G_2} &= \hat{p}^2 + \sum_{i<j}^{3} \frac{2\beta_1(\beta_1 - 1)}{(x_i - x_j)^2} + \sum_{l \neq i < j \neq l}^{3} \frac{6\beta_2(\beta_2 - 1)}{(x_i + x_j - 2x_l)^2} + \omega^2 \sum_{i=1}^{3} x_i^2, \\
\hat{H}_{G_2} e^W &= (3\omega + 6\beta_1\omega + 6\beta_2\omega) e^W.
\end{aligned}
\tag{10}
$$

The E_6 and E_7 cases have constraints on the coordinates [20]. These do not lend to a convenient physical interpretation, so we omit them.

3. Construction of new models

In this section, we present a systematic approach for constructing ES kN-body systems in one dimension. Such models describe systems of N interacting blocks, each of which has k particles interacting via an A type or G_2 type potential.

The kN-body system is proposed to have a Hamiltonian $\hat{H}$ given by

$$
\hat{H} = \sum_{i=1}^{N} \hat{H}_i + \mathcal{C}(X_1, X_2, \cdots, X_N). \tag{11}
$$

Here $X_i = \frac{1}{k} \sum_{j=1}^{k} x_{ij}$, $i = 1, 2, \cdots, N$, is the center-of-mass of the ith block, $\mathcal{C}(X_1, \cdots, X_N)$ is called the *coupling function*, and $\hat{H}_i$ is the Hamiltonian for the ith block,

$$\hat{H}_i = \hat{p}_i^2 + \omega^2 \sum_{j=1}^{k} x_{ij}^2 + V_i, \;\; \hat{p}_i^2 = -\sum_{j=1}^{k} \frac{\partial^2}{\partial x_{ij}^2}, \;\; i = 1, 2, \cdots, N. \tag{12}$$

The potential V_i above is of the inverse square form:

$$V_i = \sum_{l=1}^{m_i} \frac{g_{il}}{(\vec{v}_{il} \cdot \vec{x})^2}.$$

In this model, $\hat{H}_i$ is assigned m_i vectors, $\vec{v}_{il}$ is the lth vector associated with $\hat{H}_i$,

$$\vec{v}_{il} = (0, 0, \cdots, 0, \underbrace{v_{il1}, v_{il2}, \cdots, v_{ilk}}_{\text{in the } i\text{th block}}, 0, \cdots, 0),$$

and $\vec{x}$ is the collection of coordinates of the form,

$$\vec{x} = (x_{11}, \cdots, x_{1k}, \cdots, \underbrace{x_{i1}, \cdots, x_{ik}}_{i\text{th block}}, \cdots, x_{N1}, x_{N2}, \cdots, x_{Nk}). \tag{13}$$

We take the coupling function to be of the inverse square form, i.e.

$$\mathcal{C}(X_1, \cdots, X_N) = \frac{1}{k} \sum_{i=1}^{r} \frac{\beta_i(\beta_i - 1)|\vec{\mu}_i|^2}{(\vec{\mu}_i \cdot \vec{X})^2},$$

where $\vec{X} = (X_1, \cdots, X_N)$ and $\vec{\mu}_i = (\mu_{i1}, \cdots, \mu_{iN}), \;\; i = 1, 2, \cdots, r$. In fact, we have

$$k\vec{\mu}_i \cdot \vec{X} = \vec{\mu}_i' \cdot \vec{x},$$

where $\vec{x}$ is given by (13), and $\vec{\mu}_i'$ is some 'expansion' of $\vec{\mu}_i$,

$$\vec{\mu}_i' = (\underbrace{\mu_{i1}, \cdots, \mu_{i1}}_{k \text{ copies of } \mu_{i1}}, \underbrace{\mu_{i2}, \cdots, \mu_{i2}}_{k \text{ copies of } \mu_{i2}}, \cdots, \cdots, \underbrace{\mu_{iN}, \cdots, \mu_{iN}}_{k \text{ copies of } \mu_{iN}}).$$

It is then clear that the inner product $\vec{\mu}_i' \cdot \vec{v}_{il}$ is well defined. Putting $\vec{x}$, $\vec{v}_{il}$ and $\vec{\mu}_i'$ into (3), and using the relations

$$\vec{v}_{il} \cdot \vec{v}_{i'l'} = \delta_{ii'} \vec{v}_{il} \cdot \vec{v}_{il'},$$
$$\vec{v}_{il} \cdot \vec{\mu}_j' = \mu_{ji} \sum_{s=1}^{k} v_{ils},$$
$$\vec{\mu}_i \cdot \vec{\mu}_j = k(\vec{\mu}_i' \cdot \vec{\mu}_j'),$$

we arrive at $S_1 + S_2 + S_3 = 0$ where

$$S_1 = \sum_{i=1}^{N} \sum_{l<l'}^{m_i} \frac{\alpha_{il}}{(\vec{v}_{il} \cdot \vec{x})} \frac{\alpha_{il'}}{(\vec{v}_{il'} \cdot \vec{x})} (\vec{v}_{il} \cdot \vec{v}_{il'}),$$

$$S_2 = \sum_{i,l,j} \frac{\alpha_{il}}{(\vec{v}_{il} \cdot \vec{x})} \frac{\beta_j}{(\vec{\mu}_j' \cdot \vec{x})} (\mu_{ji} \sum_{s=1}^{k} v_{ils}),$$

$$S_3 = \frac{1}{k} \sum_{i<j}^{r} \frac{\beta_i}{(\vec{\mu}_i \cdot \vec{X})} \frac{\beta_j}{(\vec{\mu}_j \cdot \vec{X})} (\vec{\mu}_i \cdot \vec{\mu}_j). \tag{14}$$

We want S_1, S_2 and S_3 in (14) to vanish individually. First, let us examine S_1, it is nothing but a sum of ansatzes (3) of each $\hat{H}_i$. For the ith Hamiltonian $\hat{H}_i$, we can choose $\vec{v}_{il}$'s to be the root system of a simple Lie algebra, with appropriate α_{il}'s such that S_1 vanishes. For S_2, we can make it vanish by choosing $\vec{v}_{il}$'s to be root vectors of Lie algebra A or G_2, i.e. by choosing the potential V_i to have the form

$$V_i = \sum_{j<l}^{k} \frac{2\lambda_i(\lambda_i - 1)}{(x_{ij} - x_{il})^2}, \tag{15}$$

or

$$V_i = \sum_{j<l}^{3} \frac{2\lambda_{i1}(\lambda_{i1} - 1)}{(x_{ij} - x_{il})^2} + \sum_{s\neq j<l\neq s}^{3} \frac{6\lambda_{i2}(\lambda_{i2} - 1)}{(x_{ij} + x_{il} - 2x_{il})^2}. \tag{16}$$

To make S_3 vanish, we can just choose $\vec{\mu}_i$'s to be the root vectors of some Lie algebra. That is we choose the coupling function $\mathcal{C}$ to be one of the A, BC, E_8, F_4 or G_2 types. It is seen from (6) and (10) that each $\hat{H}_i$ in (11) admits a ground state e^{W_i} with ground energy $E_0^{(i)}$. So we can readily give the ground state wavefunction and energy of the Hamiltonian (11), for each choice of $\mathcal{C}$, as follows.

3.1. A type coupling

If $\mathcal{C}$ is A type, i.e.

$$\mathcal{C} = \frac{1}{k}\sum_{j<s}^{N} \frac{2\alpha(\alpha - 1)}{(X_j - X_s)^2}, \ \alpha > 0, \tag{17}$$

we have the pre-superpotential

$$W_A = \sum_{i=1}^{N} W_i + \alpha \sum_{j<s}^{N} \log|X_j - X_s|.$$

So e^{W_A} is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{N} E_0^{(i)} + N(N-1)\omega\alpha.$$

3.2. BC type coupling

If $\mathcal{C}$ is BC type, i.e.

$$\mathcal{C} = \frac{1}{k}\sum_{j<s}^{N} \left\{ \frac{2\beta_1(\beta_1 - 1)}{(X_j - X_s)^2} + \frac{2\beta_1(\beta_1 - 1)}{(X_j + X_s)^2} \right\} + \frac{1}{k}\sum_{j=1}^{N} \frac{\beta_2(\beta_2 - 1)}{X_j^2}, \ \ \beta_1 > 0, \ \beta_2 > 0,$$

we have the pre-superpotential

$$W_{BC} = \sum_{i=1}^{N} W_i + \beta_1 \sum_{j<s}^{N} \{\log|X_j - X_s| + \log|X_j + X_s|\} + \beta_2 \sum_{i=1}^{N} \log|X_i|.$$

Then $e^{W_{BC}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{N} E_0^{(i)} + 2N(N-1)\omega\beta_1 + 2N\omega\beta_2.$$

We remind that when $\beta_2 = 0$, BC type reduces to D type.

3.3. E_8 type coupling

If $\mathcal{C}$ is E_8 type (so $N = 8$), i.e.

$$\mathcal{C} = \frac{1}{k}\sum_{i<j}^{8}\left\{\frac{2\beta(\beta-1)}{(X_i-X_j)^2} + \frac{1}{k}\frac{2\beta(\beta-1)}{(X_i+X_j)^2}\right\} + \frac{1}{k}\sum_{\mathfrak{a}\in\mathcal{I}}\frac{8\beta(\beta-1)}{(X_8+\sum_{i=1}^{7}(-1)^{a_i}X_i)^2},\ \beta>0,$$

we have the pre-superpotential

$$W_{E_8} = \sum_{i=1}^{8} W_i + \beta\sum_{i<j}^{8}\{\log|X_i-X_j| + \log|X_i+X_j|\} + \beta\sum_{\mathfrak{a}\in\mathcal{I}}\log|X_8 + \sum_{i=1}^{7}(-1)^{a_i}X_i|.$$

Then $e^{W_{E_8}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{8} E_0^{(i)} + 240\omega\beta.$$

3.4. F_4 type coupling

If $\mathcal{C}$ is F_4 type (so $N = 4$), i.e.

$$\mathcal{C} = \frac{1}{k}\sum_{i<j}^{4}\left\{\frac{2\nu(\nu-1)}{(X_i+X_j)^2} + \frac{1}{k}\frac{2\nu(\nu-1)}{(X_i-X_j)^2}\right\} + \frac{1}{k}\sum_{i=1}^{4}\frac{\mu(\mu-1)}{X_i^2} + \frac{1}{k}\sum_{a_i\in\mathbb{Z}_2}\frac{4\mu(\mu-1)}{(X_1+\sum_{i=2}^{4}(-1)^{a_i}X_i)^2},$$

we have the pre-superpotential

$$W_{F_4} = \sum_{i=1}^{4} W_i + \nu\sum_{i<j}^{4}\{\log|X_i-X_j| + \log|X_i+X_j|\} + \mu\sum_{i=1}^{4}\log|X_i| + \mu\sum_{a_i\in\mathbb{Z}_2}\log|X_1+\sum_{i=1}^{4}(-1)^{a_i}X_i|.$$

Then $e^{W_{F_4}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{4} E_0^{(i)} + 24\omega\nu + 24\omega\mu.$$

3.5. G_2 type coupling

If $\mathcal{C}$ is G_2 type (so $N = 3$), i.e.

$$\mathcal{C} = \frac{1}{k}\sum_{j<s}^{3}\frac{2\beta_1(\beta_1-1)}{(X_j-X_s)^2} + \frac{1}{k}\sum_{l\neq j<s\neq l}^{3}\frac{6\beta_2(\beta_2-1)}{(X_j+X_s-2X_l)^2},\ \beta_1>0,\ \beta_2>0,$$

we have the pre-superpotential

$$W_{G_2} = \sum_{i=1}^{3} W_i + \beta_1 \sum_{j<s}^{3} \log|X_j - X_s| + \beta_2 \sum_{l\neq j<s\neq l}^{3} \log|X_j + X_s - 2X_l|.$$

Then $e^{W_{G_2}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{N} E_0^{(i)} + 6\omega(\beta_1 + \beta_2).$$

4. Exactly solvable 3*N*-body problems

As examples of the general results above, we consider the $k = 3$ case and construct two ES 3N-body systems.

Model 1: We choose all V_i's in (12) to be G_2 type and set $g_{il} = g_i$ for all $l = 1, 2, 3$, i.e. each $\hat{H}_i$ is of the form

$$\hat{H}_i = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} + \omega^2 \sum_{j=1}^{3} x_i^2 + \frac{2}{9} \sum_{j<l}^{3} \frac{g_i}{(x_{ij} - x_{il})^2} + \frac{2}{3} \sum_{s\neq j<l\neq s}^{3} \frac{\lambda_i}{(x_{ij} + x_{il} - 2x_{il})^2}. \tag{18}$$

We choose $\mathcal{C}$ in (11) to be A type, i.e. $\mathcal{C}$ is given by (17). Putting (18) and (17) into (11) gives the Hamiltonian

$$\hat{H} = -\sum_{i=1}^{N}\sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} + \omega^2 \sum_{i=1}^{N}\sum_{j=1}^{3} x_{ij}^2 + \frac{2}{9}\sum_{i=1}^{N}\sum_{j<s}^{3} \frac{g_i}{(x_{ij} - x_{is})^2}$$
$$+ \frac{2}{3}\sum_{i=1}^{N}\sum_{l\neq j<s\neq l}^{3} \frac{\lambda_i}{(x_{ij} + x_{is} - 2x_{il})^2} + \sum_{i<j}^{N} \frac{6\alpha(\alpha-1)}{(x_{i1} + x_{i2} + x_{i3} - x_{j1} - x_{j2} - x_{j3})^2}.$$

In order to solve the Schrödinger equation $\hat{H}\Psi = E\Psi$, we first make a transformation for each triplet $\{x_{i1}, x_{i2}, x_{i3}\}$

$$x_{i1} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}}r_i\left(-\frac{1}{2}\cos\theta_i + \frac{\sqrt{3}}{2}\sin\theta_i\right),$$
$$x_{i2} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}}r_i\cos\theta_i,$$
$$x_{i3} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}}r_i\left(-\frac{1}{2}\cos\theta_i - \frac{\sqrt{3}}{2}\sin\theta_i\right) \tag{19}$$

such that $X_i = Y_i/\sqrt{3}$. Then $\hat{H}$ becomes

$$\hat{H} = -\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial Y_i^2} + \frac{\partial^2}{\partial r_i^2} + \frac{1}{r_i}\frac{\partial}{\partial r_i} + \frac{1}{r_i^2}\frac{\partial^2}{\partial\theta_i^2}\right) + \omega^2\sum_{i=1}^{N}(Y_i^2 + r_i^2)$$
$$+ \sum_{i=1}^{N}\left(\frac{g_i}{r_i^2\sin^2 3\theta_i} + \frac{\lambda_i}{r_i^2\cos^2 3\theta_i}\right) + \sum_{i<j}^{N}\frac{2\alpha(\alpha-1)}{(Y_i - Y_j)^2}.$$

This means we can partially factorize the eigenfunction Ψ:

$$\Psi = \psi_{(n_1,\cdots,n_N)}(Y_1,\cdots,Y_N)\prod_{i=1}^{N} R_i(r_i)\prod_{i=1}^{N}\Theta_{n_i}^{(i)}(\theta_i),$$

which leads to $2N+1$ independent equations:

$$\left[-\frac{\partial^2}{\partial\theta_i^2}+\frac{g_i}{\sin^2 3\theta_i}+\frac{\lambda_i}{\cos^2 3\theta_i}\right]\Theta_{n_i}^{(i)}(\theta_i) = (B_{n_i}^{(i)})^2\Theta_{n_i}^{(i)}(\theta_i), \qquad i=1,2,\cdots,N, \tag{20}$$

$$\left[-\frac{\partial^2}{\partial r_i^2}-\frac{1}{r_i}\frac{\partial}{\partial r_i}+\frac{(B_{n_i}^{(i)})^2}{r_i^2}+\omega^2 r_i^2\right]R_i(r_i) = E_i R_i(r_i), \qquad i=1,2,\cdots,N, \tag{21}$$

and

$$\hat{H}_Y\psi_{(n_1,\cdots,n_N)} = E_Y\psi_{(n_1,\cdots,n_N)},$$
$$\hat{H}_Y = \sum_{i=1}^{N}\left[-\frac{\partial^2}{\partial Y_i^2}+\omega^2 Y_i^2\right]+\sum_{i<j}^{N}\frac{2\alpha(\alpha-1)}{(Y_i-Y_j)^2}.$$

The total energy E is given by

$$E = E_Y + \sum_{i=1}^{N} E_i.$$

For each i, equation (20) has a known solution, with eigenvalue and eigenfunction given by

$$\Theta_{n_i}^{(i)}(\theta_i) = \sin^{2\nu_i}(3\theta_i)\cos^{2\eta_i}(3\theta_i)P_{n_i}^{(2\nu_i-1/2,2\eta_i-1/2)}(\cos 6\theta_i),$$
$$B_{n_i}^{(i)} = 6(n_i+\nu_i+\eta_i),\ \ n_i=0,1,2,\cdots,\ \ i=1,2,\cdots,N,$$
$$\nu_i = \frac{3+\sqrt{9+4g_i}}{12},\ \ \eta_i = \frac{3+\sqrt{9+4\lambda_i}}{12}, \tag{22}$$

where $P_{n_i}^{(2\nu_i-1/2,2\eta_i-1/2)}(\cos 6\theta_i)$ is the Jacobi polynomial of degree n_i.
Now we look at (21): for each i, (21) is recognized from Calogero's work [9], with solution given by

$$R_i(r_i) = r_i^{B_{n_i}^{(i)}} L_{k_i}^{(B_{n_i}^{(i)})}(\omega r_i^2)\times\exp\left\{-\frac{\omega}{2}r_i^2\right\},$$
$$E_i = 2\omega(2k_i + B_{n_i}^{(i)}+1),\ \ k_i=0,1,2,\cdots\ \ i=1,2,\cdots,N, \tag{23}$$

where $L_{k_i}^{(B_{n_i}^{(i)})}$ is Laguerre polynomial of degree k_i with parameter $B_{n_i}^{(i)}$.
To solve the last equation $\hat{H}_Y\psi_n = E_Y\psi_n$, we adopt the approach of [17] involving Dunkl operators. Define

$$\hat{D}_j = -\mathrm{i}\frac{\partial}{\partial Y_j}+\mathrm{i}\alpha\sum_{l\neq j}^{N}\frac{1}{Y_j-Y_l}\sigma_{jl},\ \ \hat{a}_j^{\pm} = D_j \pm \mathrm{i}\omega Y_j,\ j=1,2,\cdots N,$$

$$\hat{A}_m^{\pm} = \sum_{j=1}^{N}(\hat{a}_j^{\pm})^m,\ \ m=1,2,\cdots,N,$$

$$[H_Y,\hat{A}_m^{\pm}] = \pm 2m\omega\hat{A}_m^{\pm},$$

where the σ_{ij} interchange coordinates, i.e. $\sigma_{ij}f(\cdots x_i,\cdots,x_j,\cdots) = f(\cdots x_j,\cdots,x_i,\cdots)$. The solutions are given by

$$\psi_{(n_1,\cdots,n_N)} = \prod_{i=1}^{N}(\hat{A}_i^+)^{n_i}\psi_0,\quad \psi_0 = \prod_{i<j}^{N}|Y_i - Y_j|^{\alpha}\exp\left\{-\frac{\omega}{2}\sum_{i=1}^{N}Y_i^2\right\},$$
$$E_Y = 2n\omega + N\omega + N(N-1)\alpha\omega$$

where

$$n = \sum_{i=1}^{N} i n_i,\quad n_i = 0,1,2,\cdots.$$

The total energy E for $\hat{H}$ is then

$$E = 2n\omega + N\omega + N(N-1)\alpha\omega + 2\omega\sum_{i=1}^{N}[2k_i + 6(n_i + \nu_i + \eta_i) + 1].$$

Model 2: We again choose V_i to be G_2 type but choose $\mathcal{C}$ to be D type. This gives rise to the following Hamiltonian:

$$\hat{H} = -\sum_{i=1}^{N}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_{ij}^2} + \omega^2\sum_{i=1}^{N}\sum_{j=1}^{3}x_{ij}^2 + \frac{2}{9}\sum_{i=1}^{N}\sum_{j<s}^{3}\frac{g_i}{(x_{ij}-x_{is})^2}$$
$$+\frac{2}{3}\sum_{i=1}^{N}\sum_{l\neq j<s\neq l}^{3}\frac{\lambda_i}{(x_{ij}+x_{is}-2x_{il})^2} + \sum_{i<j}^{N}\frac{6\beta(\beta-1)}{(x_{i1}+x_{i2}+x_{i3}-x_{j1}-x_{j2}-x_{j3})^2}$$
$$+\sum_{i<j}^{N}\frac{6\beta(\beta-1)}{(x_{i1}+x_{i2}+x_{i3}+x_{j1}+x_{j2}+x_{j3})^2}. \tag{24}$$

It can be seen that when transformation (19) is applied again, the equations for r_i's and θ_i's are the same as (20) and (21), as well as the solutions (22) and (23). The equation for Y_i is

$$\hat{H}_Y\psi_n = E_Y\psi_n,$$
$$\hat{H}_Y = \sum_{i=1}^{N}\left[-\frac{\partial^2}{\partial Y_j^2} + \omega^2 Y_i^2\right] + \sum_{i<j}^{N}\frac{2\beta(\beta-1)}{(Y_i-Y_j)^2} + \sum_{i<j}^{N}\frac{2\beta(\beta-1)}{(Y_i+Y_j)^2}.$$

In order to solve this equation, we again use results from [17]:

$$\hat{D}_j = -\mathrm{i}\frac{\partial}{\partial Y_i} + \mathrm{i}\beta\sum_{s\neq j}^{N}\left\{\frac{1}{Y_j - Y_s}\sigma_{js} + \frac{1}{Y_j + Y_s}t_j t_s\sigma_{js}\right\},$$
$$\hat{a}_j^{\pm} = \hat{D}_j \pm \mathrm{i}\omega Y_j,\quad j = 1,2,\cdots,N,$$
$$\hat{A}^{\pm} = \sum_{i=1}^{N}(\hat{a}_i^{\pm})^2,$$

where the t_i change coordinate signs, i.e. $t_i f(\cdots Y_i\cdots) = f(\cdots - Y_i\cdots)$. Eigenfunctions and eigenvalues of (24) are given by

$$\psi_n = (\hat{A}^+)^n \psi_0, \quad \psi_0 = \prod_{i<j}^{N} |Y_i - Y_j|^\beta |Y_i + Y_j|^\beta \exp\left\{ -\frac{\omega}{2} \sum_{i=1}^{N} Y_i^2 \right\},$$

$$E_Y = 4n\omega + 2\beta N(N-1)\omega + N\omega, \quad n = 0, 1, 2, \cdots .$$

In this case, the total energy E is then

$$E = 4n\omega + 2\beta N(N-1)\omega + N\omega + 2\omega \sum_{i=1}^{N} [2k_i + 6(n_i + \nu_i + \eta_i) + 1].$$

5. Conclusion

In this work, we have presented a general approach for constructing ES kN-body systems in one dimension. In our construction the coupling function $\mathcal{C}$ plays a crucial role. We give examples which demonstrate that, in some instances, these can be chosen in relation to the root system of a simple Lie algebra. For each listed choice of $\mathcal{C}$, we give the ground state and ground-state energy of the corresponding ES model. As non-trivial examples, we have presented two $3N$-body systems. We have solved the two models by separating their Schrödinger equations into the centers-of-mass, radial, and angular parts. The equations for radial and angular parts are familiar ones, and can be solved analytically. The equation for the centers-of-mass is not generally separable, but can be solved by using Dunkl operators [17].

For more general kN-body systems with $k > 3$, we have found that the procedure for separating variables does not generalise in an obvious manner. The solution to this problem will be the subject of future investigations.

Acknowledgments

This work was partially supported by the Australian Research Council through Discovery Project DP150101294. Zhe Chen is supported by the Australian Government Research Training Program Scholarship.

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