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Tutorial

Quantum Entanglement Dynamics and Concurrence Preservation in a Noisy Two-Qubit System with External Control Field

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Abstract: Entanglement is an essential resource for quantum information processing; however, it is highly susceptible to decoherence induced by environmental interactions. In this work, we investigate the entanglement dynamics of a two-qubit system subjected to Markovian noise and explore the effectiveness of external control fields in preserving quantum correlations. Using the Lindblad master equation, we model the evolution of the system under two decoherence channels, namely, amplitude damping (spontaneous emission) and phase damping (dephasing). We introduce periodic driving and optimal control protocols to mitigate entanglement loss and analyze their impact using concurrence as a quantifier of two-qubit entanglement. Our numerical results reveal certain conditions under which sudden death of entanglement can be delayed or even prevented. Additionally, in the case of the initial state being a mixed state, we identify time regimes where entanglement revival occurs, providing insights into quantum error correction strategies. The results contribute to the broader understanding of quantum coherence preservation and have implications for quantum computing, secure quantum communication, and quantum memory design.

Keywords: quantum entanglement; two-qubit system; external control field; concurrence

1. Introduction

Quantum entanglement is a fundamental phenomenon in quantum mechanics [1–3]. Quantum entanglement occurs when the quantum state of two or more particles cannot be described independently, regardless of the distance between them. Mathematically, a bipartite quantum state $|\psi\rangle$ in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled if it cannot be expressed as a tensor product,

$$|\psi\rangle \neq |\phi_A\rangle \otimes |\phi_B\rangle. \tag{1}$$

First recognized by Einstein, Podolsky, and Rosen in 1935 as a paradox challenging the completeness of quantum mechanics, entanglement was later formalized by Schrödinger, who described it as the essential feature of quantum theory. The non-local nature of entanglement has been experimentally validated through Bell's theorem, confirming the violation of classical locality assumptions. Entanglement has profound implications for quantum information science [4], quantum computing [5], quantum cryptography [6], quantum communication, condensed matter systems [7], and many-body systems [8]. Among the simplest systems exhibiting entanglement is the two-qubit system, which serves as the building block for more complex quantum networks. The study of entanglement dynamics in such systems is crucial for understanding coherence properties [9], quantum gate performance [10], and error correction mechanisms [11]. The thermal and decoherence effects on the entanglement for the case of two-qubit system are studied in [12]. The entanglement dynamics for the pure state of a closed two-qubit system is investigated in [13]. Some analytical results are presented in [14] for the entanglement evolution of arbitrary two-qubit pure state under amplitude damping and particularly phase damping channel. There is a work on the classical-hidden-variable description for entanglement dynamics of two-qubit pure states in [15].

One of the central challenges in quantum systems is the interaction with the environment, leading to decoherence and entanglement degradation [16–19]. In this work, we investigate the entanglement evolution of a two-qubit system under an interacting Hamiltonian with dissipative effects modeled via the Lindblad master equation [20]. The specific

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Hamiltonian under consideration involves an XX-type interaction, which appears in various physical implementations such as trapped ions, superconducting qubits, and spin chains [21–23].

To quantify the entanglement dynamics, we employ the concurrence measure, which provides a direct metric for evaluating the degree of entanglement between two qubits [24,25]. By solving the Lindblad equation numerically, we obtain the time evolution of the system's density matrix and analyze how concurrence behaves in the presence of decoherence. This study provides insights into the resilience of entanglement under different parameter regimes, offering potential strategies for mitigating decoherence in practical quantum systems.

The applications of understanding entanglement dynamics in two-qubit systems extend across various domains of quantum technologies. In quantum computing, maintaining entanglement is critical for the fidelity of quantum algorithms, including Shor's factoring algorithm and Grover's search. In quantum cryptography, secure communication protocols such as quantum key distribution (QKD) rely on robust entanglement properties. Furthermore, in quantum sensing, entangled qubits can enhance precision measurements beyond classical limits, which is particularly valuable in metrology and gravitational wave detection.

The remainder of this paper is organized as follows. Section 2 introduces the theoretical framework, including the system Hamiltonian. The Lindblad formalism for open quantum systems with dissipative processes is described in Section 3. In Section 4 the entanglement measure applied in this work is introduced. Various initial states are represented in Section 5. Section 6 presents our numerical results, highlighting the concurrence evolution under various conditions. Finally, Section 7 provides conclusions and future research directions.

2. Hamiltonian

The Hamiltonian of a quantum system determines its energy levels and governs its evolution over time. In this study, we consider both the free Hamiltonian, which describes the intrinsic properties of the two-qubit system, and the control Hamiltonian, which introduces external interactions to manipulate the behavior of the system. We consider a two-qubit system interacting via an XX-coupling Hamiltonian with an external time-dependent control field. The total Hamiltonian is given by

$$H = H_0 + H_{\text{ctrl}}(t). \tag{2}$$

The free Hamiltonian consists of the energy terms associated with each qubit and their mutual interaction. This part of the Hamiltonian is responsible for the natural evolution of the qubits and is often derived from fundamental physical principles governing the specific quantum platform, such as superconducting qubits or trapped ions. We choose the free Hamiltonian as,

$$H_0 = \frac{\alpha_1}{2}\sigma_{1z} + \frac{\alpha_2}{2}\sigma_{2z} + J(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)\,. \tag{3}$$

Here, α_1 and α_2 are the qubit energy splittings, and J being the coupling strength, is responsible for entangling dynamics between the two qubits. In a two-qubit system, σ_{1z} and σ_{2z} refer to the Pauli-Z operators acting on qubit 1 and qubit 2, respectively. As well, σ_x is Pauli-X operator and σ_y is Pauli-Y operator. On the other hand, the control Hamiltonian introduces an external driving field that is used to influence the dynamics of the qubits. This control is basically essential for implementing quantum gates, mitigating decoherence effects, and optimizing entanglement preservation. The choice of control Hamiltonian depends on experimental feasibility and the desired quantum operations. By carefully tuning the strength and frequency of the control terms, we can enhance coherence and steer the system towards specific quantum states, making it a crucial tool in quantum information processing. The control Hamiltonian devised for this work is presented by

$$H_{\text{ctrl}}(t) = f(t)\big(\sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x + \sigma_z \otimes \sigma_z\big)\,, \tag{4}$$

where $f(t)$ represents a time-dependent external driving field, designed to dynamically manipulate the system. Here σ_z is Pauli-Z operator. The coherent interaction terms $\sigma_x \otimes \sigma_y$, $\sigma_y \otimes \sigma_x$, and $\sigma_z \otimes \sigma_z$ are chosen to effectively drive entanglement generation and counteract decoherence effects. Specifically, we consider a time-dependent external field acting on the qubits, represented by a prefactor in the Hamiltonian. This choice is motivated by its relevance in quantum computing, where control fields are used to implement logical operations and error correction schemes. By adjusting the strength and frequency of the control field, we can influence the coherence properties and enhance entanglement preservation in the system. The control field $f(t)$ is parameterized as

$$f(t) = A_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + A_3 \sin(\omega_3 t)^2\,, \tag{5}$$

where A_1, A_2, A_3 are control amplitudes and $\omega_1, \omega_2, \omega_3$ are control frequencies. The field combines sinusoidal and quadratic terms allowing for adaptive modulation of the system. The control parameters $A_1, A_2, A_3, \omega_1, \omega_2, \omega_3$ will be optimized using derivative-free optimization [26] to maximize concurrence. In our simulations we will apply this numerical optimization method to find the best parameters that maximize average entanglement over time. Such that we first start with an initial guess for the parameters. The optimizer evaluates the objective function with different parameter combinations. The optimization algorithm converges the parameters to a set of control parameters that best enhance the concurrence over time. Therefore the aim is to design control fields appropriately in order to stabilize entangled states and improve the robustness of quantum operations in noisy environments.

3. Lindblad Master Equation and Open System Evolution

Quantum systems are never completely isolated. They inevitably interact with their surrounding environment, thus we actually deal with an open quantum system [27]. This interaction leads to decoherence, which can significantly impact entanglement and quantum coherence. To model these effects, we employ the Lindblad master equation, a widely used formalism for describing the non-unitary evolution of open quantum systems.

The Lindblad equation describes the time evolution of the density matrix ρ under both unitary dynamics (governed by the Hamiltonian) and dissipative processes (due to interactions with the environment). The equation reads

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right), \tag{6}$$

where H is the total Hamiltonian of the system, and L_k is the Lindblad operators that describe the environmental interactions leading to dissipation and decoherence. For our two-qubit system, common sources of decoherence include *spontaneous emission* and *dephasing*. These are modeled using appropriate Lindblad operators.

The qubit lowering operator is utilized in L_1 and L_2 to induce energy relaxation

$$L_1 = \sqrt{\gamma}\sigma_- \otimes I, \quad L_2 = \sqrt{\gamma} I \otimes \sigma_-, \tag{7}$$

where γ is the decoherence rate and $\sigma_- = (\sigma_x - i\sigma_y)/2$ is the lowering Pauli operator, representing spontaneous relaxation to the ground state. These operators capture the loss of energy due to spontaneous emission, which is one of the primary causes of decoherence in qubit systems. Additionally, pure dephasing processes can be represented using the Lindblad operators L_3 and L_4,

$$L_3 = \sqrt{\gamma_\phi}\sigma_z \otimes I, \quad L_4 = \sqrt{\gamma_\phi} I \otimes \sigma_z, \tag{8}$$

where γ_ϕ represents the pure dephasing rate. These terms describe the loss of phase coherence without energy loss, which can be crucial in maintaining quantum information. Including these Lindblad operators in our model, we can analyze how different decoherence mechanisms affect entanglement and identify strategies to mitigate their impact through appropriate control techniques.

4. Entanglement Measures: Concurrence

Entanglement measures provide a quantitative way to assess the degree of quantum correlations between qubits. Among the most widely used measures for two-qubit systems is concurrence, which directly quantifies entanglement [24,25,28]. Concurrence is a measure specifically designed for two-qubit systems that quantifies entanglement. Its application has some advantages. It is computationally simple for two-qubit systems because its closed formula exists. This measure is intuitively meaningful and directly tells us how entangled the state is. Moreover, it is an entanglement monotone, such that it won't increase under local operations and classical communication. On the other side, this measure has limitations. For instance it is not generalizable in a straightforward way to higher-dimensional systems or multipartite entanglement. Finally, to directly apply this measure beyond two quibits it requires complex extensions.

For a two-qubit system with density matrix ρ (pure or mixed), concurrence C is defined as

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{9}$$

where λ_i are the square roots of the eigenvalues of the matrix $R = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, arranged in descending order. Here, σ_y is the Pauli matrix, and ρ^* represents the complex conjugate of ρ. We work in the standard basis $\{|11\rangle, |00\rangle, |10\rangle, |01\rangle\}$.

Concurrence takes values between 0 and 1, where 0 corresponds to a separable (unentangled) state, and 1 corresponds to a maximally entangled Bell state. By computing concurrence as a function of time, we can track the entanglement evolution of the system and understand its resilience against decoherence. This measure is particularly useful in practical quantum applications, where maintaining high concurrence is essential for robust quantum operations. Evaluating concurrence under different decoherence conditions allows us to develop strategies for preserving entanglement in quantum information processing and communication protocols.

5. Possible Entangled Initial States

In our study of entanglement dynamics, the choice of the initial state plays a crucial role in determining the evolution of quantum correlations. Various entangled states serve as useful starting points for analyzing decoherence effects and the efficacy of control mechanisms. In the following, we shortly discuss some commonly considered entangled initial states in a two-qubit system.

5.1. Bell States

Bell states represent maximally entangled states of two qubits and are foundational in quantum information science. These states are defined as

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned} \tag{10}$$

Bell states serve as ideal candidates for quantum teleportation, superdense coding, and quantum cryptography. Their maximally entangled nature makes them highly sensitive to decoherence, providing an excellent benchmark for studying entanglement preservation under open system dynamics.

5.2. Generalized Werner States

Werner states are a family of mixed states that interpolate between maximally entangled and completely mixed states. They are often used in quantum information studies due to their ability to model realistic noise conditions. The density matrix of a Werner state is typically written as

$$\rho_W = p|\text{Bell}\rangle\langle\text{Bell}| + \frac{1-p}{4}I, \tag{11}$$

where p determines the degree of entanglement, with $p = 1$ corresponding to a Bell state and $p = 0$ representing a completely mixed state. The identity operator is denoted by I.

5.3. GHZ-Like States

The Greenberger–Horne–Zeilinger (GHZ) state is usually defined for three or more qubits; however, a two-qubit GHZ-like state can be expressed as

$$|\text{GHZ}_2\rangle = \cos\theta|00\rangle + e^{i\phi}\sin\theta|11\rangle. \tag{12}$$

This state allows for a tunable degree of entanglement and is relevant in quantum control and error correction studies.

5.4. Arbitrary Superpositions of Product and Entangled States

More general initial conditions can be considered by parameterizing arbitrary superpositions of separable and entangled components as

$$|\psi(0)\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \tag{13}$$

where α, β, γ, and δ are complex coefficients satisfying normalization conditions. These states provide a broader test bed for evaluating how entanglement evolves under different initial conditions.

6. Numerical Results

To analyze entanglement dynamics, we solve numerically the Lindblad master equation for different initial states and compute concurrence as a function of time. To this end we have written a Python code wherein the derivative-free optimization is implemented to maximize concurrence. In our numerical computations we set $\alpha_1 = \alpha_2 = 0.5$ and $J = 1$.

6.1. Maximally Entangled State

First, we take as the initial state the maximally entangled state, $|\Phi^+\rangle$. The density matrix is $\rho = |\Phi^+\rangle\langle\Phi^+|$. This corresponds to the Werner state with $p = 1$. In Figure 1 the concurrence as a function of time is presented for two types of decoherence, namely, spontaneous emission and dephasing. In this figure the effect of control Hamiltonian is also shown. The initial state is not fully immune to the control Hamiltonian. It gets rotated or transitions into other entangled components since the system is not initially in an eigenstate of the control Hamiltonian. As it is evident from the figures, decoherence leads to entanglement loss over time. In fact, without the control Hamiltonian, the concurrence decays rapidly showing that decoherence dominates in the absence of control. It can be seen that the inclusion of control Hamiltonian mitigates the decoherence effects at late times but it accelerates the decoherence at early times.

We have also examined the case with the initial state being the Bell state, $|\Psi^+\rangle$, and it is found that the control Hamiltonian is ineffective for this initial state. The reason is that the control Hamiltonian acts diagonally on $|\Psi^+\rangle$ and so does not generate transitions.

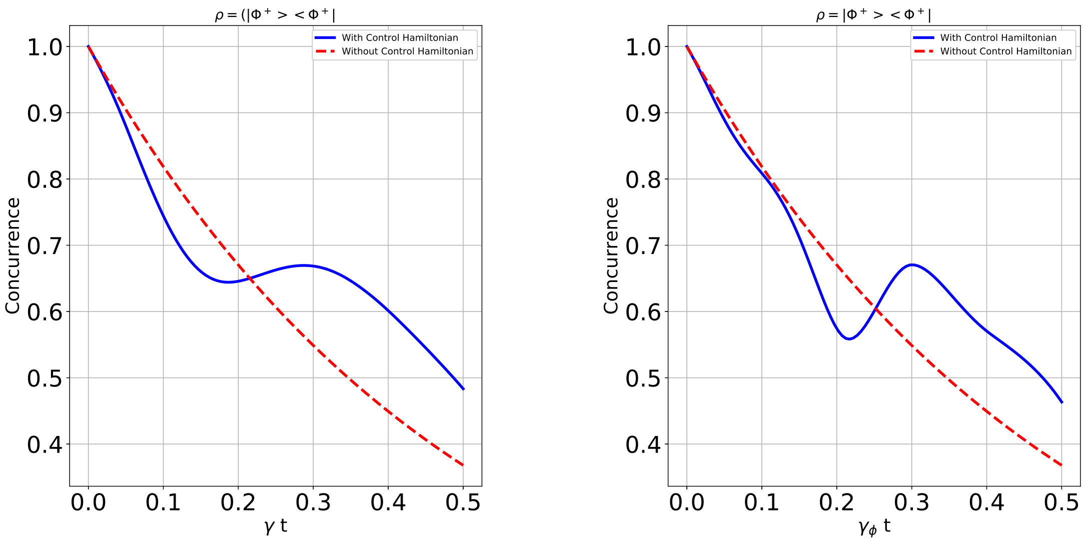

Figure 1. Time evolution of the concurrence for maximally entangled initial state $|\Phi^+\rangle$ when the included decoherence is *spontaneous emission* (left panel) and when the included decoherence is *dephasing* (right panel). The effect of control Hamiltonian is shown.

6.2. Superposition of Product and Entangled States

Here we take as our initial state $|\psi\rangle = 1/\sqrt{10}(2|00\rangle + |01\rangle + i|10\rangle + 2|11\rangle)$. This state corresponds to a general superpositions of separable and entangled components with $\alpha = \delta = 2/\sqrt{10}$, $\beta = 1/\sqrt{10}$ and $\gamma = i/\sqrt{10}$. We compute the concurrence for two types of decoherence and find the optimized control parameters for each case. The resulting parameters are given in Table 1 and the concurrence is shown as a function of time in Figure 2.

Table 1. The optimized control parameters for decoherence type being *spontaneous emission* or *dephasing*. The initial state is a superposition of product and entangled states.

	A_1	A_2	A_3	ω_1	ω_2	ω_3
$\gamma \neq 0$ (spontaneous emission)	0.362	0.714	-0.130	13.416	5.675	5.456
$\gamma_\phi \neq 0$ (dephasing)	-1.244	0.270	1.079	19.642	16.755	0.508

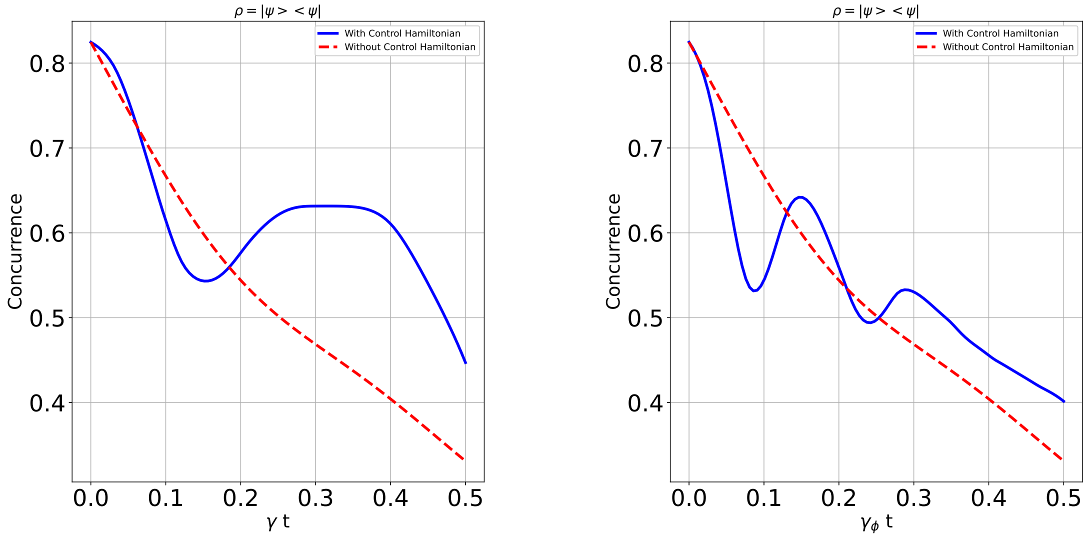

Figure 2. Time evolution of the concurrence for the initial state $|\psi\rangle$ when the included decoherence is *spontaneous emission* (left panel) and when the included decoherence is *dephasing* (right panel). The effect of the control Hamiltonian is shown and compared when there is no control Hamiltonian.

As seen in Figure 2 the control Hamiltonian is able to preserve the concurrence for a period of time in the case with decoherence being spontaneous emission.

6.3. Initially Mixed State

Now we take a state being an equal mixture of $|\Phi^+\rangle$ and a separable state $|ii\rangle$, where $|ii\rangle = |11\rangle, |00\rangle$. This state has partial entanglement and models a realistic noisy environment. This type of initial state is useful in order to study the control Hamiltonian or error correction so as to increase the entanglement dynamically. This initial state is already considered to study the thermal and phase decoherence effects on entanglement dynamics of the quantum spin systems [12]. The optimized parameters of the control Hamiltonian for the decoherence of type spontaneous emission are listed in Table 2 and that for dephasing decoherence are listed in Table 3. The numerical results for the time evolution of concurrence are depicted in Figure 3. The results indicate that entanglement decay varies significantly based on the chosen separable state $|11\rangle$ or $|00\rangle$. The inclusion of control Hamiltonian improves entanglement preservation for two distinct mixed initial states. In fact, if the control Hamiltonian is optimized correctly, it may counteract the decoherence effects and maintain or even increase concurrence. Therefore, the optimization ensures that the control field is tailored to sustain entanglement against decoherence effects. In other words, the optimized control field ensures that the system adapts to environmental noise, reducing the decay of concurrence over time. Moreover, comparing the results in Figures 1 and 3 we note that by incorporating the control Hamiltonian, while pure entangled states lose entanglement more rapidly the mixed states exhibit slower decoherence. It is worth mentioning that the strength of the Heisenberg-type coupling in our Hamiltonian (J) does not affect the concurrence dynamics. In fact, the interaction part of the free Hamiltonian includes terms that generate the transition, $|01\rangle \leftrightarrow |10\rangle$. Therefore, if our state has no support in $|01\rangle$ or $|10\rangle$ (which is the case in $|\Phi^+\rangle$ and $|00\rangle$), then the J-interaction does nothing. So unless decoherence or control Hamiltonian connects those components, changing J has no effect and it acts as a spectator term for this state.

Table 2. The optimized control parameters for decoherence type being *spontaneous emission*.

$\gamma \neq 0$	A_1	A_2	A_3	ω_1	ω_2	ω_3
$\|\Phi^+\rangle\langle\Phi^+\| + \|11\rangle\langle 11\|$	0.585	1.470	0.150	3.597	5.599	0
$\|\Phi^+\rangle\langle\Phi^+\| + \|00\rangle\langle 00\|$	0.922	1.243	-0.279	10.295	-1.90	6.258

Table 3. The optimized control parameters for decoherence type being *dephasing*.

$\gamma_\phi \neq 0$	A_1	A_2	A_3	ω_1	ω_2	ω_3
$\|\Phi^+\rangle\langle\Phi^+\| + \|11\rangle\langle 11\|$	0.223	0.276	0.348	3.191	8.545	4.945
$\|\Phi^+\rangle\langle\Phi^+\| + \|00\rangle\langle 00\|$	-2.628	-11.256	-2.188	51.388	75.113	17.965

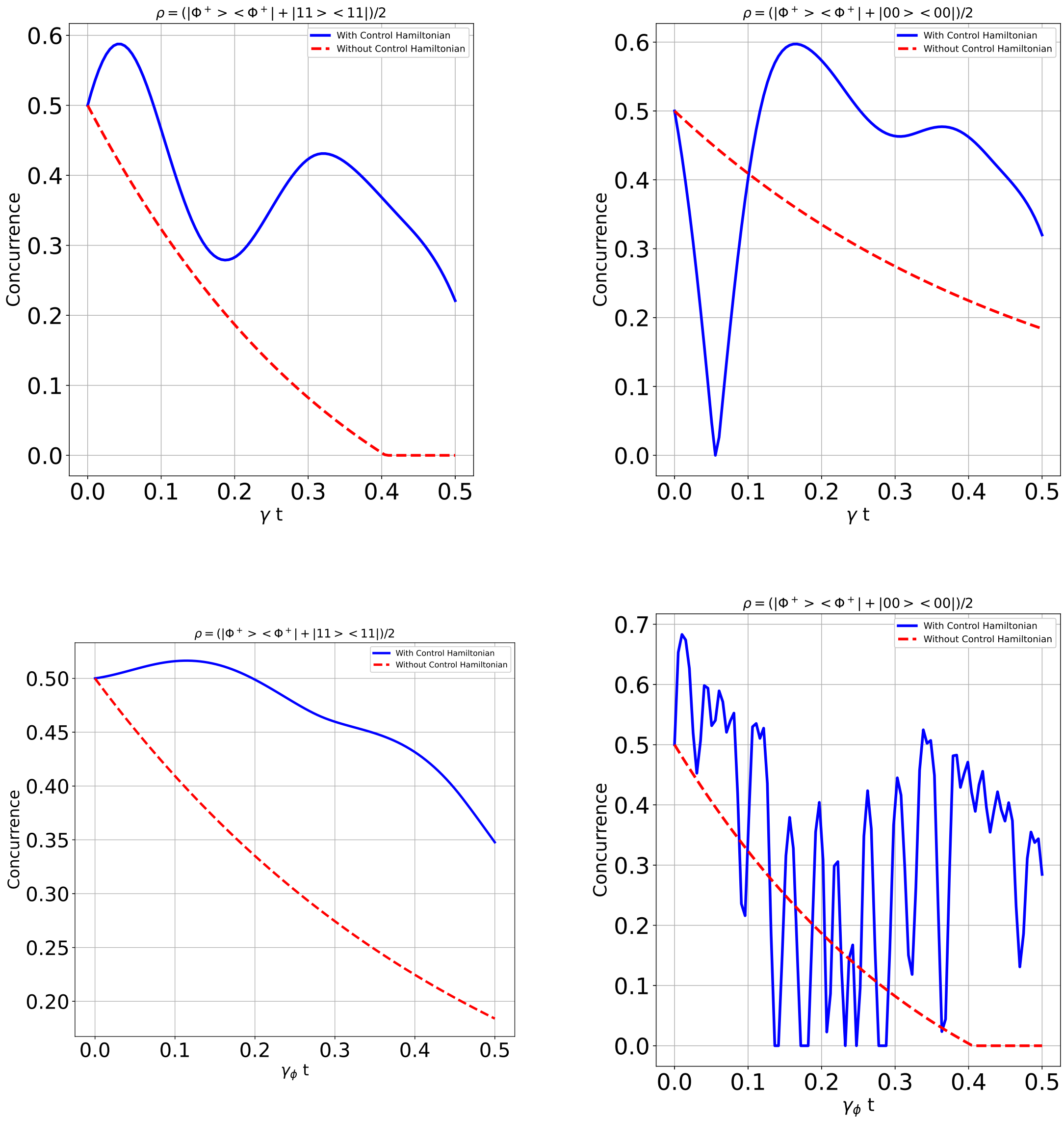

Figure 3. Time evolution of the concurrence for the initially prepared mixed state is defined as equal mixture of $|\Phi^+\rangle$ and $|ii\rangle$, where $|ii\rangle = |11\rangle, |00\rangle$. In the top panel the included decoherence is *spontaneous emission* and in the bottom panel the included decoherence is *dephasing*. The effect of control Hamiltonian is shown for comparison.

7. Conclusion

Our study highlights the impact of decoherence on entanglement evolution in two-qubit systems and demonstrates the effectiveness of a control Hamiltonian in mitigating these effects. By analyzing concurrence dynamics, we provide valuable insights into the robustness of quantum correlations under different environmental conditions. Our results share significant implications for quantum computing, cryptography, and quantum information processing, where preserving entanglement is essential for reliable operations. One important finding in this work indicates that the control Hamiltonian has a major impact when a mixed initial state is chosen with respect to the case that a pure entangled initial state or a superposition of product and entangled states is picked up.

Future research could explore optimized control strategies and extend our analysis to multi-qubit systems to further enhance quantum coherence preservation. Additionally, investigating different noise models, including non-Markovian effects, could provide a deeper understanding of realistic quantum environments. Another promising direction is the exploration of machine learning techniques for designing optimal control pulses that maximize entanglement resilience. We will broaden the scope of this study in the future, and wish to contribute to the ongoing advancements in quantum technologies and their practical implementations.

Author Contributions

Conceptualization and Supervision, K.G.; Project Execution and Writing, K.G. and R.R.; Methodology, K.G. and R.R.; Data Analysis, K.G. and R.R.; Project Management K.G. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

There is no data availability.

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