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Primordial Gravitational Wave in de Sitter Space and the Ermakov-Lewis Invariant

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The governing equation of primordial gravitational waves in de Sitter space takes the form of a harmonic oscillator with a time-dependent frequency. The Ermakov-Lewis invariant of this time-dependent harmonic oscillator is obtained using the mode solutions of the primordial gravitational wave and an auxiliary equation, which is dual to the mode equations, in de Sitter space. Additionally, the dual symmetry of the mode functions of the primordial gravitational wave is briefly mentioned by employing the invariance of the Schwarzian derivative.

Keywords: Time-dependent harmonic oscillator, Ermakov-Lewis invariant, Schwarzian derivative, Primordial gravitational wave

I. INTRODUCTION

It is widely accepted that an accelerating phase in the early Universe is crucial for understanding the history of the Universe and explaining cosmic background radiation (CMB) anisotropy, large-scale structure (LSS) and primordial gravitational waves. The CMB anisotropy, LSS formation and the primordial gravitational wave are known to have been seeded by the quantum fluctuations in an inflationary period.

The governing equations, commonly referred to as the Sasaki-Mukhanov equations [1, 2], which are responsible for the CMB, LSS, or primordial gravitational waves are presented in the form of a time-dependent harmonic oscillator (TDHO). The TDHO appears quite often in the branch of physics *for example* in optics, black hole physics and cosmology. The dynamics of a scalar field in an expanding background can also be described by a damped harmonic oscillator or a Caldirola-Kanai harmonic oscillator [3,4].

We use the Ermakov-Lewis invariant approach to study a TDHO problems in this work. The time-dependent equations of motion can be transformed into the time-independent equations by introducing an auxiliary field satisfying the Penny equation [5]. The Ermakov-Lewis invariant method has been used to define an invariant vacuum for the quantum fluctuations in inflation [6].

In Ermakov systems for the TDHO, there exists an invariant quantity whose physical meaning is proposed to be a conserved energy in a time-independent harmonic oscillator systems [7] or a conservation of an angular momentum [8]. This invariant quantity can provide useful tools for constructing the theoretical models of inflation. We can trace the underlying symmetries of the time-dependent harmonic oscillator problems from the Schwarz equation, which can be obtained from the Ermakov systems using the nonlocal transformations. The Schwarzian derivative is known to be invariant under the $SL(2, R)$ transformation.

In this study, we present a solution to the auxiliary equations with the mode solutions of the primordial gravitational wave in de Sitter space. We then show that these solutions yield the constancy of the Ermakov-Lewis invariant. Finally, we provide a brief overview of the Schwarz equation through the nonlocal transformation from the Ermakov systems. The invariance of the Schwarz derivative under the $SL(2, R)$ transformation

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provides the dual symmetry of the mode functions of the primordial gravitational wave [9].

II. PRIMORDIAL GRAVITATIONAL WAVES IN DE SITTER SPACE

With a linearised Friedmann-Lemaitre-Robertson-Walker metric for the tensor modes

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j, \tag{1}$$

where h_{ij} is a traceless and transverse tensor perturbation, the quadratic action for h_{ij} for the Einstein-Hilbert action in de Sitter space is given by

$$\delta_2 S = \frac{M_p^2}{8}\int d\eta d^3x a^2\left[h'_{ij}h^{ij\prime} - (\partial_k h_{ij})(\partial^k h^{ij})\right], \tag{2}$$

where the prime denotes the derivative with respect to the conformal time η and we have used the mode decomposition of h_{ij} as

$$h_{ij}(x,\eta) = \sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^{3/2}}\epsilon^\lambda_{ij}(k)h^\lambda_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3}$$

where ϵ^λ_{ij} is a polarization tensor and $M_p^2 = 1/8\pi G$ is the reduced Planck mass. Varying (2) with respect to h_{ij} leads to the equation of motion for each polarization mode of the gravitational wave

$$h''_k + 2\frac{a'}{a}h'_k + k^2 h_k = 0, \tag{4}$$

where we have ignored the polarization superscript λ. This equation reminds us the equation of motion of a massless scalar field. If we introduce $u_k = \frac{h_k}{a}$, the equation of u_k becomes

$$u''_k + \left(k^2 - \frac{a''}{a}\right)u_k = 0. \tag{5}$$

This takes the form of the time-dependent harmonic oscillator with a time-dependent frequency $\omega(\eta) = k^2 - \frac{a''}{a}$. This equation consists of Ermakov systems with the auxiliary equation

$$\rho''_k + \omega^2(\eta)\rho_k = \frac{\Omega^2}{\rho_k^3}, \tag{6}$$

where Ω^2 is an arbitrary constant. Multiplying ρ_k to (5) and u_k to (6) and then subtracting (6) from (5) yields

$$\rho_k u''_k - u_k\rho''_k = \frac{d}{d\eta}(\rho_k u'_k - \rho'_k u_k) = -\frac{\Omega^2 u_k}{\rho_k^3}. \tag{7}$$

Multiplying the integration factor $(\rho_k u'_k - \rho'_k u_k)$ on the both sides, we have

$$(\rho_k u'_k - u_k\rho'_k)\frac{d}{d\eta}(\rho_k u'_k - u_k\rho'_k) = -\frac{\Omega^2 u_k}{\rho_k^3}(\rho_k u'_k - u_k\rho'_k), \tag{8}$$

or

$$\frac{1}{2}\frac{d}{d\eta}(\rho_k u'_k - u_k\rho'_k)^2 = -\frac{1}{2}\Omega^2\frac{d}{d\eta}(u_k/\rho_k)^2 \tag{9}$$

then we obtain the invariant quantities (Ermakov-Lewis invariant) by integrating

$$I = \frac{1}{2}\left[(\rho_k u'_k - u_k\rho'_k)^2 + \Omega^2\left(\frac{u_k}{\rho_k}\right)^2\right]. \tag{10}$$

The physical meaning of this invariant is suggested in several literature; conservation of an angular momentum [8] , conservation of energy [7] and so on.

In the next section we will briefly introduce how to solve the time-dependent harmonic oscillator using the Ermakov-Lewis invariant method.

III. ERMAKOV-LEWIS INVARIANT

Equation (5) describes the time-dependent harmonic oscillator with a time-dependent frequency $\omega^2(\eta) = k^2 - a''/a$ and with a unit mass. Using the Ermakov-Lewis invariant method that will be described here shortly, we can obtain the solution to the time dependent harmonic oscillator.

Transforming to a new variable by introducing an auxiliary function $\rho(\eta)$

$$v_k = u_k/\rho_k(\eta), \tag{11}$$

the Eq. (5) is transformed as

$$\frac{d^2 v_k}{d\tau^2} + \Omega^2 v_k = 0, \tag{12}$$

with ρ satisfying Eq. (6), where we have used

$$\tau(\eta) = \int^\eta \frac{1}{\rho_k^2(\eta')}d\eta'. \tag{13}$$

Because the solution to (12) is given by

$$v_k = e^{\pm i\Omega\tau}, \tag{14}$$

the solution to the time-dependent harmonic oscillator becomes

$$u_k(\eta) = \rho_k(\eta) e^{\pm i\Omega \int \frac{1}{\rho_k^2} d\eta}. \tag{15}$$

Once we get the solution to (6), then we can obtain the solution to (5) from (15).

The Hamiltonian for v_k is given by

$$H_k(v,\pi;\tau) = \frac{1}{2}\pi_k^2 + \frac{1}{2}\Omega^2 v_k^2 \equiv const., \tag{16}$$

where π_k is a conjugate momentum to v

$$\pi_k = \frac{\partial v_k}{\partial \tau} = (\rho_k u_k' - \rho_k' u_k). \tag{17}$$

In the original variable, the Hamiltonian (16) becomes

$$I_k = H_k(v,\pi;\tau) = \frac{1}{2}\left[(\rho_k u_k' - \rho_k' u_k)^2 + \Omega^2 \left(\frac{u_k}{\rho_k}\right)^2\right], \tag{18}$$

and this is the Ermakov-Lewis invariant (10) which is conserved

$$\frac{dI_k}{dt} = \frac{\partial I_k}{\partial t} + \{I_k, H_k\} = 0 \tag{19}$$

where $\{,\}$ is a Poisson bracket for the classical systems and can be thought of as a Dirac bracket for the quantum systems. This implies the conservation of energy in the transformed coordinate space.

The solutions to (6) is given by [5,10]

$$\rho = \sqrt{a\psi^2 + b\varphi^2 + 2c\psi\varphi}, \tag{20}$$

where $ab - c^2 = \frac{\Omega^2}{W^2}$ and ψ and φ are two linearly independent solutions of (5) and W is a Wronskian of ψ and φ. If we choose $c = 0$ and $a = 1$, then we can set to $b = \Omega^2/W^2$, and then (20) can be written as

$$\rho = \left[\psi^2 + \frac{\Omega^2}{W^2}\varphi^2\right]^{1/2}. \tag{21}$$

As an example, we consider the de Sitter space with $a = e^{Ht}/H$ with a constant H, where t is a physical time related to the conformal time as $dt = a(\eta)d\eta$, and we have

$$\frac{a''}{a} = \frac{2}{\eta^2}, \quad \eta = -\frac{1}{H}e^{-Ht} = -\frac{1}{aH}, \tag{22}$$

then (5) can be solved exactly [11]

$$u_k(\eta) = \sqrt{|\eta|}\left[C_k J_{3/2}(k|\eta|) + D_k Y_{3/2}(k|\eta|)\right], \tag{23}$$

where $J_{3/2}(x)$ and $Y_{3/2}(x)$ are the Bessel and Neumann functoin with the order 3/2, respectively. The Wronskian of the two independent solutions of (23) are

$$W = u_1 u_2' - u_1' u_2 = \frac{2}{\pi}. \tag{24}$$

With the solution (23), the solution of the auxiliary equation (6) from (20) leads to

$$\rho_k = \left[\frac{2}{k\pi}\left(\frac{\sin(x)}{x} - \cos(x)\right)^2 + \frac{\pi\Omega^2}{2k}\left(\sin(x) + \frac{\cos(x)}{x}\right)^2\right]^{1/2}, \tag{25}$$

where $x = k|\eta|$ and we have used

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\cos(x) + \frac{\sin(x)}{x}\right),$$
$$Y_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos(x)}{x} + \sin(x)\right). \tag{26}$$

If we choose Ω as (Ω is an arbitrary constant)

$$\Omega^2 = W^2 = \frac{4}{\pi^2}, \tag{27}$$

then we get

$$\rho_k = \sqrt{\frac{2}{\pi k}}\left(1 + \frac{1}{k^2|\eta|^2}\right)^{1/2}. \tag{28}$$

With (23) and (25), we can check the Ermakov-Lewis invariant I_k (10) becomes constant,

$$I_k = \frac{1}{2}\alpha^2\Omega^2 + \frac{2\beta^2}{\pi^2}. \tag{29}$$

IV. SYMMETRY IN TIME-DEPENDENT SYSTEMS

In this section we will briefly mention the underlying symmetries of the time-dependent harmonic oscillator. The Ermakov systems are transformed by the nonlocal transformation into the Schwarz equation. When we perform the nonlocal transformation

$$z' = u^{-2}, \tag{30}$$

where we have omitted the subscript k of u_k in this section, then (5) is transformed into the third-order differential equations and it turns out that the Schwarz equation

$$S[z] = 2\omega^2(\eta), \tag{31}$$

where $S[z]$ is the Schwarzian derivative

$$S[z] = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 = \left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2. \quad (32)$$

With the similar transformation $\xi' = \rho^{-2}$, (6) becomes

$$S[\xi] = -2\Omega^2\xi'^2 + 2\omega^2, \quad (33)$$

which is a variant of the Schwarz equation. The Schwarzian derivative $S[z]$ in (32) is known to be invariant under the $SL(2,R)$ transformations

$$\tilde{z} = \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \quad (34)$$

i.e. $S[\tilde{z}] = S[z]$. The Schwarz equation (31) has a solution of the quotient form

$$z = \frac{\varphi(t)}{\psi(t)}, \quad (35)$$

where $\varphi(t)$ and $\psi(t)$ are the two linearly independent solution of (5).

Since $S[z]$ is invariant under the transformation (34), it can be proved that u''/u is also invariant under the transformation (30) [12]

$$\frac{\tilde{u}''}{\tilde{u}} = -\frac{1}{2}S[\tilde{z}] = -\frac{1}{2}S[z] = \frac{u''}{u}, \quad (36)$$

which leads to [12]

$$\tilde{u}(\eta) = C_1 u(\eta) + C_2 u(\eta)\int^{\eta}\frac{d\eta'}{u^2(\eta)}. \quad (37)$$

These properties of dual symmetry have been employed in the investigation of the scale invariance [9] and the enhancement of the power spectrum [13].

V. SUMMARY AND DISCUSSIONS

We have considered the primordial gravitational wave in de Sitter space and found that the governing equation, the Sasaki-Mukhanov equation, of the primordial gravitational waves takes the form of the harmonic oscillator with a time-dependent frequency. The Ermakov-Lewis invariant method is used to study the TDHO problems in de Sitter space. With the solutions to the gravitational wave modes, u_k and the auxiliary field, ρ_k, we have shown that the Ermakov-Lewis invariant becomes constant and then is conserved.

The nonlocal transformation of the variables transforms the gravitational wave mode equations and the auxiliary field equation into third-order differential equations, known as the Schwarz equation. The Schwarz derivative is known to have a symmetry under the $SL(2,R)$ transformation. This dual symmetry allows to study the scale invariance as well as the enhancement of the power spectrum in an early phase of the universe.

Although we have focused on the primordial gravitational wave for this work, we can extend it to the scalar mode equations responsible for the large-scale structures and the cosmic background radiation anisotropy. It would be interesting to study the role of the Ermakov-Lewis invariant and the symmetric properties of the Schwarzian derivative to constrain the inflationary model or to provide additional observables to help understand the evolution of our universe.

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