Imperial College London

QUANTUM CORRELATIONS: FOUNDATIONAL AND PRACTICAL ASPECTS

Alexander Paige

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Centre for Doctoral Training in Controlled Quantum Dynamics
Department of Physics
Imperial College London

Declarations

The research presented in this thesis was carried out over the last three and a half years by me under the supervision of Myungshik Kim. I hereby declare that the material presented is a result of my work and that of my acknowledged collaborators, except where otherwise suitably referenced. Throughout the PhD I have been involved in the publication of the following papers, which contain most of the work presented here:

- A. J. Paige, Benjamin Yadin, and M. S. Kim. "Quantum correlations for anonymous metrology" *Quantum*, **3**, 178 (2019)

- A. J. Paige, Hyukjoon Kwon, Selwyn Simsek, Chris N. Self, Johnnie Gray, and M. S. Kim. "Quantum Delocalized Interactions" *Physical Review Letters* **125**, 240406 (2020)

- A. J. Paige, A. D. K. Plato, and M. S. Kim. Classical and nonclassical time dilation for quantum clocks. *Physical Review Letters*, **124**, 160602 (2020)

- Hadrien Chevalier, A. J. Paige, and M. S. Kim. "Witnessing the nonclassical nature of gravity in the presence of unknown interactions" *Phys. Rev. A* **102**, 022428 (2020)

- Hyukjoon Kwon, A. J. Paige, and M. S. Kim. "Condition on the Rényi Entanglement Entropy under Stochastic Local Manipulation" *Physical Review Letters*, **125** 100502 (2020)

Abstract

Quantum correlations have been an integral if sometimes discomforting aspect of quantum theory for over 85 years. From the early thought-experiments where they were used to argue that quantum mechanics was incomplete, through to their central position as resources in the modern theory of quantum information theory, quantum correlations have proved a hugely rich topic of study. This thesis explores a range of practical and foundational aspects within this ever developing field.

Having reviewed fundamental material, we turn to consider the quantum correlations for a task termed anonymous metrology. This is used to demonstrate an operational distinction within the hierarchy of quantum correlations. Building from this we formulate the concept of quantum delocalised interactions, making use of quantum games to study the necessary correlations. With these we establish strong connections to the concurrence monotone and additionally a link with quantum teleportation. We then move to considering entanglement under motion when relativity is accounted for. We demonstrate how the entanglement between internal energy and motional states can be affected by boosts in an analogous way to the known behaviour for spins, and in a more straightforward manner. We then show how this understanding sheds light on the topic of proper time for quantum clocks. Finally we consider a protocol where entanglement can be used to witness non-classicality of gravity. We present improvements which help to bring the scheme closer to experimental feasibility, together with pointing out a potential loophole and how to close it.

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Contents

Chapter 1

Introduction

Chapters should always start with quotes, even if they are made up and misattributed.

– Winston Churchill

1.1 Historical context and motivation

Quantum mechanics is arguably the most philosophically consequential physical theory that has thus far been discovered. It forces us to shift our view of reality, and despite being roughly a century old, we are still working through its consequences and marvelling at its multifarious phenomena. Central to many of these is the curious manner in which quantum systems can be correlated.

The laws of nature at the start of the 20th century seemed relatively ordered, with matter diligently obeying Classical Mechanics, and allowing itself to be predictably pushed and pulled around by Maxwellian Electromagnetism and Newtonian Gravity. From time to time things got a bit heated but we had Thermodynamics to take care of that. To willfully misquote Lord Kelvin's infamous speech [1], physics was looking good with only a couple of clouds left to deal with. Lord Kelvin is often lightly mocked for his comments in 1900, with some form of indication that he would have been better off obeying the old adage si tacuisses, philosophus mansisses. However, I believe most scientists would be inordinately proud of making a speech highlighting two specific problems for which the solutions were soon to completely revolutionize physics.

The first of the clouds was the failure of the Michelson-Morley experiment [2] to detect evidence for the luminiferous ether. The solution was famously provided by Einstein's theory of Special Relativity [3], and for extra credit he decided to put in a further ten years of work to produce General Relativity [4], perhaps the greatest intellectual achievement of any single human mind. These theories demanded a re-

think of our understanding of space and time and the implications were certainly profound. However, relativity still allows for a deterministic, comfortably local understanding of nature, niceties which would not go unchallenged by the other great breakthrough of the early 20th century.

The second of Lord Kelvin's clouds was the ultra-violet catastrophe. Quantum mechanics was the answer, and unlike Relativity it was truly a child of many fathers, with Planck [5], Einstein [6] Heisenberg [7], and Schrödinger [8], providing foundational insights, alongside vital contributions from many others. It was quickly appreciated that quantum mechanics was non-deterministic, with the Born rule providing probabilities from wave functions [9], and there was soon an awareness of something even stranger lurking in the new formalism. In 1935 Einstein, Podolsky and Rosen published their famous challenge to the completeness of quantum mechanics [10], on the grounds that with a particular two particle setup, they could assign "elements of reality" to non-commuting momentum and position variables. Though as Einstein later put it in a letter to Max Born, what he truly objected to was the way quantum theory seemed to permit spukhafte Fernwirkung, spooky action at a distance. Quantum correlations had been noticed, and the particular effect was deemed so counter-intuitive as to lead the authors to conclude that "we have thus shown that the wave function does not provide a complete description of the physical reality," and to this day, despite our subsequent understanding, we still refer to their work as the EPR paradox.

The EPR paradox added fuel to the ongoing debate between Einstein and Bohr [11] and prompted Schrödinger to coin the term Verschränkung in a letter to Einstein, a word which he translated as Entanglement. Schrödinger added to the list of charges [12], arguing that it was "discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy," and he was unequivocal on the significance of quantum correlations, opening his paper by stating: "When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."

In the intervening years the state of affairs with regards to locality in quantum theory was complex. David Bohm managed to formulate his eponymous hidden variables theory [13], in spite of an earlier ostensible proof by von Neumann that this was not possible [14], although Grete Hermann had apparently pointed out a flaw in the argument and been ignored [15]. Bohm and Aharanov presented a

useful reformulation of the EPR paradox using spins [16], but it was not until the seminal work of John Bell [17], nearly three decades after the paradox was put forward, that a decent degree of clarity was established. It is interesting that such a fundamental question took so long to be properly addressed, but quantum theory had opened up a whole plethora of interesting problems to tackle, and there is comfort in the instrumentalist view somewhat bluntly put by David Mermin as "shut up and calculate."

Fortunately, John Bell was, to borrow a term from Hardy and Spekkens [18], more of the "shut up and contemplate" school of thought. He proved that no local hidden variable theory could be consistent with the predictions of quantum mechanics [17], a result which is now known as Bell's Theorem, and he also rediscovered the flaw in von Neumann's argument against all hidden variable theories [19]. The proof that local hidden variable theories could not reproduce the predictions of quantum mechanics rested on their inability to violate what we now call Bell inequalities. These inequalities were later generalised [20] with an eye to experimental realisation, and in 1972 the first empirical evidence against local hidden variables was reported [21]. Subsequent experiments gave improved demonstrations of Bell inequality violations [22, 23, 24, 25], culminating in 2015 with proclaimed loophole free realisations [26, 27, 28]. Although it could be said that the setting-independence loophole was not truly closed in these experiments, for which the use of cosmic photons has been proposed [29] and demonstrated [30], and although the unscientific notion of superdeterminism can never be disproved, the verdict does seem clear. Bell's inequalities are violated. Regardless of the wishes of even one so intellectually exalted as Einstein, nature simply is that weird.

For all its significance, Bell non-locality turns out to be only one piece in the complex edifice of quantum correlations. This was not well understood at first, and indeed it was not until the close of the 20th century that modern entanglement theory started to take shape, with the importance of entanglement being increasingly realised in the growing field of quantum information theory. It is an interesting quirk of history that we knew the universe was quantum mechanical decades before Shannon laid the foundations of classical information theory [31], emphasising the power of technology in shaping the questions which scientists choose to ask. With quantum theory standing ready, once there was an information theory it was only a matter of time before people tried prepending quantum to it, and the field started to establish foundational results throughout the 60s and 70s [32, 33, 34, 35, 36]. Initially the main focus of the research was on the capacity of quantum systems as a means for straightforward communication, but quantum cryptography was added by Bennett and Brassard in 1984 with their aptly named BB84 protocol [37], and

quantum computation started to gain traction[1] in the 90s with the work of Deutsch and Jozsa [43], followed by the groundbreaking algorithm of Shor [44].

It was also in the 90s that a series of key results helped spur the formation of a new perspective on entanglement. 1991 brought a proposal for a form of quantum cryptography based on Bell's theorem [45], 1992 gave quantum dense coding [46], and 1993 introduced the dramatically named quantum teleportation [47]. Entanglement was increasingly being viewed as a resource, a quantity which in certain settings enables tasks that would be otherwise impossible.

This shift in perspective brought rapid advancements to our understanding of quantum correlations. Entanglement began to be formulated in the context of local operations and classical communication [48, 49], whereby it can be manipulated and distilled but not created. Measures of entanglement were formalised and developed [50, 51, 52, 53, 54], alongside entanglement witnesses [55, 56, 57, 58, 59, 60]. The emerging topic became truly vast(see [61] for an excellent review) and provided the archetype for the concept of quantum resource theories [62], which has been applied to multiple other areas including purity [63], coherence [64], asymmetry [65], and thermodynamics [66].

With growing interest in the field it also became clearer that one could distinguish various types of quantum correlations, with the relationships between entanglement, Bell non-locality and Schrodinger's steering [12] further elucidated through pioneering work from Wiseman et al. [67] building from earlier work by Werner [68]. In addition the early 2000s brought the introduction of quantum discord [69, 70], a correlation weaker than entanglement, which spurred a whole new set of investigations to quantify and interpret this new quantity [71, 72, 73, 74, 75]. And for completeness, though it falls outside the focus of this work, it is worth mentioning that researchers have even studied correlations beyond those allowed by quantum theory [76, 77].

The physical possibilities and consequences of the various quantum correlations have naturally been studied in a multitude of contexts. These have ranged from their relevance to the power of quantum computation [78, 79, 80, 81, 82] and their uses in metrology [83, 84, 85, 86, 87] right through to many body physics [88, 89, 90, 91, 92] and investigations regarding black-holes [93, 94, 95, 96, 97]. The breadth and significance of this single topic is both remarkable and beautiful, a worthy example for that most eloquent of Feynman quotes: "Nature uses only the longest threads to weave her patterns, so that each small piece of her fabric reveals the organization of the entire tapestry."

[1] This was a significant period but not the start of interest in quantum computation, with early discussions on the difficulties in classically simulating quantum systems from Poplavskii [38] and Feynman [39], together with foundational work from Bennett [40], Benioff [41], and Toffoli [42].

We have come a long way since the original proposal of the EPR paradox over 85 years ago. The path has not always been smooth, with progress at times slow and occasionally the odd misstep, however, one must always remember that the concepts at play are far from intuitive. I can illustrate with personal experience, as I still recall my own first exposure to these ideas was anything but perspicacious acceptance. It was during my undergraduate, and a friend of mine in the year above had just been learning about Bell's theorem in his lectures. He explained to me what I now know to be the CHSH game [20], laying out how classically one could achieve a win probability of no more than 0.75 and yet armed with the right entangled state one could surpass this limit. Following his claim of this quantum advantage I paused for thought and then, unable to keep the incredulity from my voice, asked him "Do you actually believe that?" He patiently explained that it wasn't really a matter of belief, you can work through the maths and furthermore the effect had been experimentally demonstrated. I found all of this fascinating, and yet rather disconcerting. At a later date I spent some pen and paper time playing around with entangled states, and managed for several hours to be completely convinced that I'd worked out how they could be used to achieve instantaneous signalling. My point is that quantum correlations and their consequences do not seem to come naturally to the classically evolved human mind. Either that or I am an idiot, I'll leave that judgement as an exercise for the reader.

Over 85 years of study, and the topic of quantum correlations is still breaking new ground, exposing new concepts, and advancing our understanding of nature. This thesis presents the results of my own studies in this ever expanding field, the topics and results of which will now be summarised.

1.2 Thesis outline

Here we give brief outlines of the material presented in each of the chapters.

In Chapter 2 we shall review the physical and mathematical fundamentals necessary for our study of quantum correlations. This will be divided into three main sections: Mathematical preliminaries, where we review vectors in Hilbert spaces and operators on Hilbert spaces; Fundamentals of quantum mechanics, where we briefly go over the essential elements of quantum theory together with details of some key example systems and relevant quantum information theoretic quantities; Fundamentals of quantum correlations, where we describe entanglement, Bell non-locality, steering and finally quantum discord. This chapter is used to establish results that will be subsequently called upon in the following chapters.

In Chapter 3 we study quantum correlations in an operational context. We introduce and study a task termed anonymous metrology, whereby some unknown variable is encoded in a quantum state in a manner which grants access to information on the parameter whilst hiding where (and therefore by whom, hence the anonymity) the encoding was performed. We perform a study of the quantum correlations required for this task, distinguishing two strengths of anonymity which we refer to as weak and strong. We show that the weak anonymity requires only discord, whereas the strong anonymity requires entanglement. We also establish that the resourceful states do not sit neatly into the known correlation sets, with discord not sufficient for weak anonymity and entanglement not sufficient for strong.

Chapter 4 builds off the work in Chapter 3, formalising the notion of delocalised interactions and studying this concept in depth. This is achieved by formulating quantum games which capture this notion and then studying two particular instances. A connection with the concurrence entanglement measure is established, in that it can be used to bound the win probability gain made available by quantum states. We show that there are various cases where the bounds can be tight, and a connection with quantum teleportation is found. In addition a trace distance inequality is considered as an alternate approach to capturing the concept and this is compared with the quantum games. Finally, the results of demonstrations of the quantum games on an IBM superconducting qubit device are presented, including a successful demonstration.

Chapter 5 shifts to a different topic in quantum correlations, namely understanding how quantum entanglement can be affected by motion in a relativistic setting. We first present the known effect that boosts can alter the entanglement between the internal spin and motional state of a particle. This is a consequence of the somewhat involved special relativistic Wigner rotations. We then show how similar behaviour can be established for internal energy states much more straightforwardly,

by simply properly accounting for mass-energy equivalence. We then demonstrate understanding the correct form for the boost in this case enables us to better understand proper time for quantum clocks. This in turn leads us to appreciate that without entanglement between the internal and motional states, we should expect additional effects beyond those of classical time-dilation. This is due to the fact that without entanglement between internal and motional states we do not have a unique Lorentz factor. We use a theoretical clock model together with the experimental example of atomic clocks to demonstrate this.

In Chapter 6 we consider quantum entanglement in a fascinating new context whereby it could be the means by which gravity is shown to be non-classical. We present a previously proposed experiment and perform a careful re-derivation of the main result. The rest of the chapter is then focused on improving the protocol. To this end an improved witness is found, and a procedure for hypothesis testing proposed and tested. The net result is to dramatically reduce the interaction times required and increase the noise tolerance, therefore bringing the proposal closer to experimental feasibility. The statistical analysis allows one to distinguish the effect of gravity even in situations where other interactions like the Casimir-Polder effect is dominant, and we show it is robust to uncertainty in the strength of the non-gravitational interaction terms. Then we present a final loophole that remains for the skeptic, and demonstrate how to close it via state tomography.

We finish with Chapter 7 in which we present conclusions and outlook. We summarise the work and present perspectives, together with putting forward suggestions for future research directions.

Chapter 2

Fundamentals

Do you know Hilbert? No? Then what are you doing in his space?

– Bad joke of unknown origin

2.1 Overview

In this chapter we review the fundamentals necessary for later chapters. We start with mathematical preliminaries, laying out the two concepts which essentially encapsulate all of quantum theory, namely vectors in Hilbert spaces and operators on Hilbert spaces. We also take this opportunity to present mathematical results which shall prove useful in the proceeding material. Armed with these concepts and tools we then turn to lay out the required fundamentals of quantum mechanics, defining states followed by their measurements and transformations. We then present two important example systems, that of qubits and continuous variable systems, before finishing by defining some useful information theoretic quantities, namely the quantum trace distance, the fidelity, and the quantum Fisher information. The third and final section deals with the requisite fundamentals of quantum correlations. We start by discussing entanglement, first for pure states where we highlight the importance of the Schmidt decomposition, the partial ordering from Nielsen's theorem and the von-Neumann entropy of entanglement. We then consider mixed state entanglement and introduce entanglement witnesses, followed by entanglement measures and monotones, giving a few relevant examples. After this we move to Bell non-locality and quantum steering, for which we emphasise the operational meaning and contrast the mathematical formulations with that of entanglement. Finally we present quantum discord, defining it via mutual information and laying out the non-discordant states.

2.2 Mathematical preliminaries

2.2.1 Vectors in Hilbert spaces

Quantum mechanics revolves around vectors in Hilbert spaces so we begin by clearly defining what these are. Even some practicing physicists allow themselves to be a bit fuzzy on Hilbert spaces, but they are simply vector spaces with added structure, namely an inner product and a completeness condition under the corresponding norm and distance measure. We shall proceed to build up these pieces accordingly together with presenting a few useful results and definitions along the way. The exposition here will be minimal, for a thorough grounding in vector spaces and more general linear algebra see [98], for a specific focus on Hilbert spaces see [99].

A vector space $\mathcal{V}$ over a field $\mathbb{F}$ (quantum theory uses complex vector spaces so $\mathbb{F} = \mathbb{C}$) is a set of elements (vectors) that is closed under vector addition and scalar multiplication, so for every $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ we have $\mathbf{v}+\mathbf{w} \in \mathcal{V}$, and given any scalar $a \in F$ we have $a\mathbf{v} \in \mathcal{V}$. Additionally these vector operations must satisfy the following eight axioms

1. Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

2. Commutativity of addition: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

3. Existence of additive identity: $\mathbf{v} + \mathbf{0} = \mathbf{v}$, $\forall \mathbf{v} \in \mathcal{V}$.

4. Existence of additive inverse: $\forall \mathbf{v} \in \mathcal{V}$, $\exists(-\mathbf{v}) \in \mathcal{V}$, s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

5. Associativity of scalar multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$.

6. Existence of scalar multiplicative identity: $1\mathbf{v} = \mathbf{v}$, $\forall \mathbf{v} \in \mathcal{V}$.

7. Distributivity of vector sums: $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

8. Distributivity of scalar sums: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

A key concept for vector spaces is that of a basis. A basis $\mathcal{B}$ of a vector space $\mathcal{V}$ is a linearly independent subset of $\mathcal{V}$ that spans $\mathcal{V}$, which means that every vector in $\mathcal{V}$ can be expressed as some linear combination of the vectors in $\mathcal{B}$. This allows us to define the dimension of a space $\dim \mathcal{V}$ as the number of elements in a valid basis for it.

An inner product space is a vector space with an inner product defined on it. An inner product $\langle \mathbf{v}, \mathbf{w} \rangle$, maps any two vectors $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ to a scalar value, and must satisfy the following three conditions

1. Second argument linearity[1]: $\langle \mathbf{v}, a\mathbf{w}\rangle = a\langle \mathbf{v}, \mathbf{w}\rangle$, $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$.

2. Conjugate symmetry: $\langle \mathbf{v}, \mathbf{w}\rangle^* = \langle \mathbf{w}, \mathbf{v}\rangle$.

3. Positive definiteness: $\langle \mathbf{v}, \mathbf{v}\rangle = \|\mathbf{v}\|^2 > 0$, for $\mathbf{v} \neq \mathbf{0}$.

Now that we have inner products we can define orthogonal and orthonormal bases. An orthogonal basis $\mathcal{B}$ is a basis where every vector is orthogonal to every other vector, i.e. $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for $\mathbf{v}_i, \mathbf{v}_j \in \mathcal{B}$, with $i \neq j$. An orthonormal basis is an orthogonal basis where the vectors are normalised to unity so

$$\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{2.1}$$

for $\mathbf{v}_i, \mathbf{v}_j \in \mathcal{B}$.

Vectors in an inner product space satisfy the Cauchy-Schwarz Inequality

$$|\langle \mathbf{v}, \mathbf{w}\rangle|^2 \leq \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle. \tag{2.2}$$

This can be seen as follows. If $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, then the statement is trivially true, so from now on we take them both to be non-zero. We have that $\|\mathbf{v} - \lambda\mathbf{w}\|^2 \geq 0$, the left-hand side can be expanded to $\|\mathbf{v}\|^2 - \lambda^*\langle \mathbf{v}, \mathbf{w}\rangle^* - \lambda\langle \mathbf{v}, \mathbf{w}\rangle + \lambda^*\lambda\|\mathbf{w}\|^2$. Setting $\lambda = \frac{\langle \mathbf{v},\mathbf{w}\rangle^*}{\|\mathbf{w}\|^2}$, then gives the result as claimed. Note that $|\langle \mathbf{v}, \mathbf{w}\rangle|^2 = \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle$ requires $\mathbf{v} = \lambda\mathbf{w}$, i.e. the vectors must be parallel.

A Hilbert space is an inner product space where the norm $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$ and corresponding distance measure $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$, turn it into a complete metric space. A metric space is simply a set together with a metric or distance measure defined on the set. A metric is a function that must satisfy the three axioms:

1. Identity of discernibles: $d(\mathbf{v}, \mathbf{w}) = 0 \iff \mathbf{v} = \mathbf{w}$.

2. Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$.

3. The triangle inequality: $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{u}) + d(\mathbf{u}, \mathbf{w})$.

It is clear that $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$, satisfies the symmetry condition and the identity of discernibles follows from positive definiteness of the inner product. The proof of the triangle inequality proceeds as follows. We write $d(\mathbf{v}, \mathbf{w})^2 = \|\mathbf{v} - \mathbf{u} + \mathbf{u} - \mathbf{w}\|^2$, and expand this out to write it as $d(\mathbf{v}, \mathbf{u})^2 + d(\mathbf{u}, \mathbf{w})^2 + \langle \mathbf{v}+\mathbf{u}, \mathbf{u}-\mathbf{w}\rangle + \langle \mathbf{u}+\mathbf{w}, \mathbf{v}-\mathbf{u}\rangle$. Taking the maximum of the latter two terms and then applying the Cauchy-Schwarz

[1]It is common amongst mathematicians to use first argument linearity but we use second for consistency with the physicists' preferred Dirac notation convention which we shall be adopting. Note that together with conjugate symmetry this means our inner product is anti-linear in the first argument.

inequality we rearrange to find that this expression is $\leq (d(\mathbf{v},\mathbf{u}) + d(\mathbf{u},\mathbf{w}))^2$, hence the triangle inequality holds.

The completeness of a metric space can intuitively be understood as the idea that there are no points missing from the space. More formally we say that if a series of vectors $\sum_{k=0}^{\infty} \mathbf{v}_k$ converges absolutely in the sense that $\sum_{k=0}^{\infty} \|\mathbf{v}_k\| < \infty$, then the series must converge to an element of the Hilbert space.

The final mathematical definition that we need is that of dual spaces. Given a vector space $\mathcal{V}$ over field $\mathbb{F}$, the dual space $\mathcal{V}^*$ is the set of all linear maps $\phi : \mathcal{V} \to \mathbb{F}$. For Hilbert spaces, the Riesz representation theorem gives us that for every $\phi \in \mathcal{H}^*$ there exists a unique $\mathbf{f}_\phi \in \mathcal{H}$ such that for any $\mathbf{v} \in \mathcal{H}$ we have $\phi(\mathbf{v}) = \langle \mathbf{f}_\phi, \mathbf{v} \rangle$. This final result enables the use of bra vectors which are a staple in quantum theory, and our mention of linear maps leads nicely into the next topic we must cover.

2.2.2 Operators on Hilbert spaces

Besides vectors in Hilbert spaces, quantum mechanics makes constant use of operators on Hilbert spaces, so we briefly review these, together with some key definitions and results. Again we shall tread lightly on this topic, so for a more detailed discussion we recommend the first chapter of John Watrous textbook [100], and for a mathematical introduction to operator theory see [101].

Quantum operators, which we shall simply refer to as operators, are linear mappings between Hilbert spaces $A : \mathcal{H}_1 \to \mathcal{H}_2$, with $A(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha A(\mathbf{v}) + \beta A(\mathbf{w})$. Note that they can, and often are, mapping from one Hilbert space to itself $\mathcal{H}_1 \to \mathcal{H}_1$. With this general notion of an operator, a vector in a complex Hilbert space $\mathcal{H}$ can be viewed as an operator mapping $\mathbb{C} \to \mathcal{H}$, and a dual vector can be viewed as a mapping $\mathcal{H} \to \mathbb{C}$.

Finite dimensional operators have corresponding matrices defined elementwise as $M_{ij} = \langle \mathbf{u}_i, A\mathbf{v}_j \rangle$ for orthornormal basis vectors with $\mathbf{v}_j \in \mathcal{H}_1$ and $\mathbf{u}_i \in \mathcal{H}_2$. As such the definitions and results presented have corresponding matrix interpretations. Accordingly, the kernel $\ker(A)$ of an operator A is the subspace of vectors $\mathbf{u}$ for which $A\mathbf{u} = \mathbf{0}$. In accordance with the rank-nullity theorem we may define the rank of operator A as $\text{rank}(A) = \dim(\mathcal{H}_1) - \dim(\ker(A))$. For the case of an operator A mapping to and from the same space, i.e. $\mathcal{H}_1 \to \mathcal{H}_1$, we often make use of eigenvectors $\mathbf{u}$ and eigenvalues λ satisfying $A\mathbf{u} = \lambda\mathbf{u}$. In addition, with basis vectors $\{\mathbf{v}_j\}_{j=1}^n$ we define the trace and determinant respectively as

$$\text{Tr}\{A\} = \sum_{j=1}^{n} \langle \mathbf{v}_j, A\mathbf{v}_j \rangle, \tag{2.3}$$

$$\det\{A\} = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{j=1}^{n} \langle \mathbf{v}_j, A\mathbf{v}_{\pi(j)} \rangle, \tag{2.4}$$

where S_n is the set of n element permutations and $\mathrm{sgn}(\pi)$ is $+1$ for even permutations and -1 for odd. Note that these definitions do not depend on the choice of basis, and in fact the trace equals the sum of the eigenvalues and the determinant equals their product.

There are a number of useful classes of operators which have specific names and definitions which we shall now lay out. For operator $A : \mathcal{H}_1 \to \mathcal{H}_2$. the *Adjoint Operator* $A^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$, is defined such that $\langle \mathbf{w}, A\mathbf{v} \rangle = \langle A^\dagger \mathbf{w}, \mathbf{v} \rangle$, for $\mathbf{v} \in \mathcal{H}_1$ and $\mathbf{w} \in \mathcal{H}_2$. For complex matrices the adjoint operation is equivalent to transposition together with complex conjugation $M^\dagger = (M^T)^*$. If A commutes with its adjoint $[A, A^\dagger] \equiv AA^\dagger - A^\dagger A = 0$, then A is said to be a *Normal Operator*. If $A = A^\dagger$ then A is said to be a *Hermitian Operator*[2] and such operators have real eigenvalues. A Hermitian operator A is *Positive Semi-definite* if all of its eigenvalues are greater than or equal to zero. A positive semi-definite operator is a *Projection Operator* if it satisfies $P^2 = P$. An operator U which when followed by its adjoint produces the identity, i.e. $U^\dagger U = \mathbb{1}_{\mathcal{H}_1}$, is a *Linear Isometry*, and a Linear Isometry that maps to and from the same space, i.e. $\mathcal{H}_1 \to \mathcal{H}_1$, is called a *Unitary Operator*.

One of the most useful results when dealing with operators is the spectral theorem. This states that any normal operator A can be expressed in terms of its orthonormal eigenvectors $\mathbf{v}_i$ and eigenvalues λ_i as

$$A = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\dagger. \tag{2.5}$$

The right-hand expression is referred to as the spectral decomposition. As a matrix equation this corresponds to the diagonalization $M = UDU^\dagger$ where D is diagonal and U is unitary.

Another useful result that builds off the spectral theorem is the singular value decomposition. This states that any operator A can be expressed as

$$A = \sum_{i=1}^{\mathrm{rank}(A)} s_i \mathbf{v}_i \mathbf{w}_i^\dagger, \tag{2.6}$$

for positive real numbers s_i and orthonormal sets of vectors $\mathbf{w}_i \in \mathcal{H}_1$ and $\mathbf{v}_i \in \mathcal{H}_2$.

For matrices we tend to express the singular value decomposition with a slight

[2]There are mathematical considerations about the correct usage of the terms Hermitian and self-adjoint, but this shall not be important for our purposes.

difference. For an $m \times n$ complex matrix M the factorization is of the form

$$M = U\Sigma V^\dagger, \tag{2.7}$$

where U is an $m \times m$ unitary matrix, Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and V is an $n \times n$ unitary matrix. This can be derived constructively as follows. The matrix $M^\dagger M$ is seen to be Hermitian which means we can diagonalize it $M^\dagger M = V\Lambda V^\dagger$ in accordance with the spectral theorem. Since it is also positive semi-definite we can write this out as $M^\dagger M = \sum_i \sigma_i^2 \mathbf{v}_i \mathbf{v}_i^\dagger$, where $\mathbf{v}_i$ are the eigenvectors of $M^\dagger M$. We now define $\mathbf{u}_i = \frac{M\mathbf{v}_i}{\sigma_i}$, and this can be straightforwardly shown to be a normalised eigenvector of $MM^\dagger$. Stacking these together to form a matrix equation we have $U = MV\Sigma^{-1}$, where U, V have ith columns $\mathbf{u}_i, \mathbf{v}_i$ and Σ is a diagonal matrix with entries σ_i^{-1}. Note that in cases where we lack full rank we simply add vectors to complete the unitary matrices, together with adding zeros to Σ. The equation can now be rearranged to the stated form $M = U\Sigma V^\dagger$.

Finally we need to consider functions of operators, which relies on the spectral theorem. Every function $f : \mathbb{C} \to \mathbb{C}$ may be extended to act on normal operators A with eigenvectors $\mathbf{v}_i$ and eigenvalues λ_i, via $f(A) = \sum_i f(\lambda_i)\mathbf{v}_i\mathbf{v}_i^\dagger$. It is also often useful to make use of their Taylor expansions.

With an understanding of functions of operators, we now conclude the mathematical preliminaries by deriving a pair of results which are often useful for operator manipulation. The first result, commonly referred to as Hadamard's lemma, states that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ... \tag{2.8}$$

This can be seen by Taylor expanding the function $f(s) = e^{As}Be^{-As}$ about $s = 0$, and then setting $s = 1$. Taking derivatives we find

$$\frac{df}{ds} = e^{As}ABe^{-As} - e^{As}BAe^{-As} = e^{As}[A, B]e^{-As}, \tag{2.9}$$

$$\frac{d^2 f}{ds^2} = e^{As}A[A, B]e^{-As} - e^{As}[A, B]Ae^{-As} = e^{As}[A, [A, B]]e^{-As}, \tag{2.10}$$

where this pattern continues for higher order terms. Thus the Taylor expansion about $s = 0$ gives

$$f(s) = B + [A, B]s + \frac{1}{2!}[A, [A, B]]s^2 + ..., \tag{2.11}$$

and setting $s = 1$ (which is within the radius of convergence assuming that the operator terms are well-behaved) gives the result as originally stated.

The second result is a special case of the Baker-Campbell-Hausdorff (BCH) theorem. If $[A,[A,B]] = [B,[A,B]] = 0$, then

$$e^{A+B} = e^A e^B e^{-[A,B]/2}. \tag{2.12}$$

This can be seen by considering the Taylor expansions of the two functions $g(s) = e^{s(A+B)}$ and $h(s) = e^{sA}e^{sB}e^{-s^2[A,B]/2}$. Taking derivatives of $g(s)$ we find

$$\frac{d^n g}{ds^n} = (A+B)^n g(s). \tag{2.13}$$

Differentiating $h(s)$ we find

$$\begin{aligned} \frac{dh}{ds} &= Ae^{sA}e^{sB}e^{-s^2[A,B]/2} + e^{sA}Be^{sB}e^{-s^2[A,B]/2} + e^{As}e^{Bs}(-s[A,B]e^{-s^2[A,B]/2}), \\ &= (A + e^{sA}Be^{-sA} - s[A,B])h(s), \\ &= (A+B)h(s). \end{aligned} \tag{2.14}$$

Here we have used the fact that $[A,[A,B]] = [B,[A,B]] = 0$ and Hadamard's lemma. From this we see

$$\frac{d^n h}{ds^n} = (A+B)^n h(s). \tag{2.15}$$

We have that $g(0) = h(0)$ and thereby all the derivatives at $s = 0$ are equal, therefore by Taylor's theorem the two functions are equivalent $g(s) = h(s)$. Setting $s = 1$ we recover the result as originally stated.

2.3 Fundamentals of quantum mechanics

There are numerous excellent introductions to quantum mechanics. For a general introduction see [102], for an introduction with a focus on quantum computation and quantum information theory see [103], and for even more focus on quantum information theory and particularly the mathematical aspects see [100].

2.3.1 Quantum states

In quantum theory, systems are identified with some finite or infinite dimensional complex Hilbert space $\mathcal{H}$. The complete physical description of a closed system is captured by its quantum state, which we denote using Dirac notation [104] as a ket $|\psi\rangle \in \mathcal{H}$. Each ket has a corresponding dual vector which we denote as a bra $\langle\psi| \in \mathcal{H}^*$, and inner products between two states $|\phi\rangle, |\psi\rangle$ are accordingly denoted as $\langle\phi|\psi\rangle$. States are normalised $\langle\psi|\psi\rangle = 1$, which allows for the Born rule, whereby the probability of observing the system in state $|\phi\rangle$, when it was prepared in state $|\psi\rangle$ is given by $|\langle\phi|\psi\rangle|^2$. A given physical state $|\psi\rangle$ has a global phase freedom such that $e^{i\theta}|\psi\rangle$ is physically equivalent for all $\theta \in \mathbb{R}$. Strictly speaking this means that quantum mechanical states are in fact rays, not vectors.

A ket such as $|\psi\rangle$ is commonly referred to as a pure state. Pure states denote complete knowledge of the system, but often this is not the case. Classical uncertainty is endemic to all experimental endeavours, so we may be 90% sure we prepared $|\psi\rangle$, but there is a 10% chance that we actually prepared $|\phi\rangle$. For practical purposes we need a way to include these mixed states in the formalism. To accommodate these situations we represent states with density operators ρ. These are linear operators on $\mathcal{H}$, which must be Hermitian $\rho = \rho^\dagger$, positive semi-definite $\rho \geq 0$, and normalised $\mathrm{Tr}(\rho) = 1$. A pure state $|\psi\rangle$ is represented as the outer product $\rho = |\psi\rangle\langle\psi|$, so has $\mathrm{Tr}(\rho^2) = 1$, whereas a general mixed state with probabilities p_i for states $|\psi_i\rangle$ is written as $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, and has $\mathrm{Tr}(\rho^2) < 1$. Note that the decomposition of mixed states into these convex combinations is not unique, in fact there are an infinite number of decompositions for every mixed state. The probability of observing state ρ in the pure state $|\psi\rangle$ is given by $\langle\psi|\rho|\psi\rangle$.

We now turn to the treatment of composite systems, which naturally are vital to the study of correlations. Given two quantum subsystems A and B, with individual Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, the state space of the joint system is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. Note that the dimension of the new space is the product $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$, which is the root cause of the exponential scaling problem for quantum states of many systems. This is in contrast to classical states of maximal certainty which when combined can be described in a direct sum vector space $\dim(\mathcal{V}_A) \oplus \dim(\mathcal{V}_B)$, which can intuitively be thought of as stacking the vectors

on top of each other, where the dimensions only add.

We shall denote composite bipartite pure states interchangeably as $|\psi\rangle_A \otimes |\phi\rangle_B = |\psi\rangle_A|\phi\rangle_B = |\psi\phi\rangle_{AB}$, and mixed states as ρ_{AB} or when appropriate $\rho_A \otimes \rho_B$, with the obvious generalizations to more than two subsystems. When we consider operators that only act non-trivially on part of such composite systems we shall often abbreviate full expressions like $O_A \otimes \mathbb{1}_B$ by omitting identity operators thus O_A. Given a composite system ρ_{AB}, the state of a subsystem can be obtained by tracing out the other subsystems, so $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i \langle i|_B \rho_{AB} |i\rangle_B$, where the sum is over any orthonormal basis in $\mathcal{H}_B$. With this we define the concept of purifications. The pure state $|\psi\rangle_{AB}$ is a purification of the mixed state ρ_A if $\mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|) = \rho_A$. All purifications of $\rho_A = \sum_j q_j |\phi_j\rangle_A\langle\phi_j|$, can be written as $|\psi\rangle_{AB} = \mathcal{U}_{B'\to B} \sum_i \sqrt{q_j}|\phi_i\rangle_A |i\rangle_{B'}$, for some isometry $\mathcal{U}_{B'\to B}$.

2.3.2 Measurements

Quantum theory is probabilistic, as such it makes predictions for the statistics of the results from measurements of observables such as energy, position and momentum. In the formalism, we represent physical observables with Hermitian operators $O = O^\dagger$ on $\mathcal{H}$. The expectation value of observable O for a system in the pure state $|\psi\rangle$ is given by $\langle O\rangle = \langle\psi|O|\psi\rangle$. For a density operator ρ, we have $\langle O\rangle = \mathrm{Tr}(\rho O)$. This is in direct accordance with the Born rule and the definition of expectation values, as can be seen by using a spectral decomposition $O = \sum_i o_i |i\rangle\langle i|$, which gives expectation value $\langle O\rangle = \sum_i o_i \langle i|\rho|i\rangle$. The variance of O is likewise calculated as $(\Delta O)^2 = \langle O^2\rangle - \langle O\rangle^2$.

We often deal with observables which do not commute $[A, B] \neq 0$. This results in a limit to the precision with which the two observables can be simultaneously predicted. This is the essence of the uncertainty principle, which is commonly stated via the formula

$$(\Delta A)(\Delta B) \geq \frac{1}{2}|\langle [A, B]\rangle|. \tag{2.16}$$

In words, there is a trade-off between our uncertainty on A and B. For instance, if we can predict A almost exactly so that $(\Delta A) \to 0$, then we have maximal uncertainty about B as $(\Delta B) \to \infty$.

This covers the statistics of our measurements but we need to understand what happens in a particular instance as well. When we measure O and obtain the value o_i, we have projected the state onto the subspace spanned by the eigenvectors with this eigenvalue. Writing $O = \sum_j o_j P_j$, we obtain the result o_i with probability $p_i = \mathrm{Tr}(P_i\rho)$ and our system is then in the new state $P_i\rho P_i/p_i$. The process of the state being instantly changed by measurement is often called "collapsing" the wavefunction. This type of measurement is referred to as a projective measurement.

We can actually consider more general types of measurement, where we obtain outcomes corresponding to elements in a Positive Operator Valued Measure (POVM). A POVM is a set of positive semi-definite operators $\{\Pi_i\}$ that sum to the identity $\sum_i \Pi_i = \mathbb{1}$. The probability of obtaining outcome i is $\mathrm{Tr}\{\Pi_i \rho\}$, and writing $\Pi_i = M_i^\dagger M_i$, which is always possible for positive semi-definite operators, we have the new state after this measurement outcome as $M_i \rho M_i^\dagger / p_i$. These are the effect on a subsystem of a projective measurement performed on the larger system of which it is a part.

2.3.3 State transformations

We have considered states and observables, but besides the "collapse" induced by measurements, we have not described how states change. We shall now lay out the important aspects of state transformations.

The standard transformations to a pure quantum state $|\psi\rangle$, such as time evolution, spatial translations and rotations, are all enacted by unitary operators U resulting in some new state $U|\psi\rangle$. Time reversal is anti-unitary but we shall not be making use of it in this work.

The most important example is time evolution which is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi\rangle, \tag{2.17}$$

where H is the Hamiltonian (energy) operator. This equation gives us that a system starting in state $|\psi(0)\rangle$ and evolving under Hamiltonian operator H, will at time t be in the state

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle. \tag{2.18}$$

So in other words we time evolve the state by acting on it with the unitary operator $U(t) = e^{-iHt/\hbar}$.

The most general evolution for a quantum state ρ is a quantum channel. This is some map $\mathcal{E} : \rho \to \mathcal{E}(\rho)$ which must be completely positive and trace preserving (CPTP). Trace preserving means that $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho)$. Completely positive means that taking $\mathcal{I}$ as the identity channel, we must have that $\mathcal{E}_A \otimes \mathcal{I}_B(\rho_{AB}))$ is positive semi-definite for all choices of B as a valid Hilbert space and all positive semi-definite ρ_{AB} within the full space AB. This last condition is to ensure that if $\mathcal{E}$ acts on some subsystem then the whole state is still a valid quantum state.

These quantum channels all correspond with some unitary transformation on a

larger composite system. As such we can express the action of a channel as

$$\mathcal{E}(\rho_A) = \mathrm{Tr}_B(U_{AB}\rho_A \otimes \sigma_B U_{AB}^\dagger). \tag{2.19}$$

Here σ_B is some initial ancilla system, and the unitary U_{AB} in general interacts the two subsystems A and B. Note that this view of general maps as the result of unitary evolution on larger composite systems is directly analogous to how general POVMs are the result of projective measurements on a larger composite system.

Taking a pure ancilla state $\sigma = |0\rangle\langle 0|$ and some basis $\{|\psi_i\rangle\}$, for the B subsystem, we can write the map in terms of Kraus operators $K_i = \langle\psi_i|U_{AB}|0\rangle$, as

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger. \tag{2.20}$$

where $\sum_i K_i^\dagger K_i = \mathbb{1}$. Quantum channels can all be represented by Kraus operators, but the decompositions are not unique.

2.3.4 Qubits

With the fundamentals of quantum mechanics defined, now is an appropriate moment to lay out two very important example systems. In principle quantum mechanics preferences no Hilbert space over any other. In practice the solutions to a huge number of physical problems rely on two particular examples. The first is the two dimensional qubit system.

Qubits are two level systems, the quantum mechanical equivalent of a bit, hence the portmanteau name from quantum bit. Physical examples include two electronic energy levels in atoms or ions, the polarization state of light, and current directions in superconducting loops. For whatever physical system we work with, we tend to assign some physically motivated computational basis with states $|0\rangle$ and $|1\rangle$, e.g. the ground $|g\rangle$ and excited $|e\rangle$ states of some electronic transition for an ion. All qubit pure state can be simply written as $\alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$, or as the equivalent column vector $(\alpha, \beta)^T$.

The set of Hermitian operators on the qubit Hilbert space is spanned by the Identity and the three Pauli operators, which in the computational basis we write as

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.21}$$

Our convention is such that the computational basis vectors $|0\rangle, |1\rangle$, are eigenvectors of σ^z and we shall use $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ to denote the eigenstates of σ^x.

All density operators for qubits can be expressed as $\rho = \frac{1}{2}(I + \sum_{i=x,y,z} r_i \sigma^i)$,

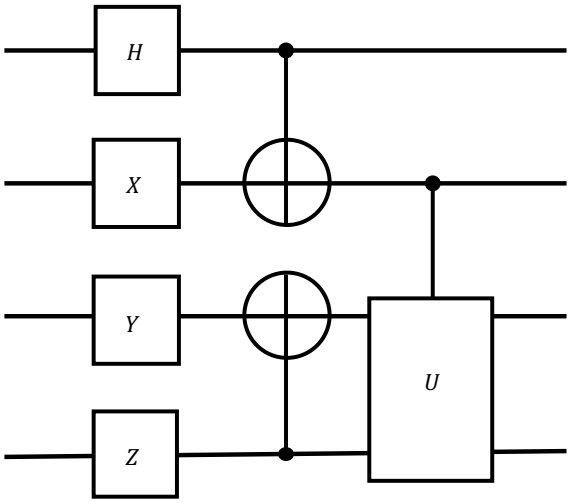

Figure 2.1: An example section of a quantum circuit. The lines represent qubits with the gates applied to them from left to right. The gate H represents the Hadamard operator $|+\rangle\langle 0| + |-\rangle\langle 1|$. The gates X, Y, Z represent the corresponding Pauli operators. The dot-line-circle elements are controlled not (CNOT) gates, which correspond to $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$, where the first subsystem is the qubit with the dot on it. The final gate is an example of a general controlled unitary where in this instance some known unitary U acts on two qubits.

where the vector $\mathbf{r} = (r_x, r_y, r_z)$ is real and has $\|\mathbf{r}\| \leq 1$. This defines a ball, where the bounding sphere represents pure states and the inside are mixed states. In addition, all observables for qubits can be expressed as linear combinations of the same four operators.

Qubits are the basic unit in what is termed circuit or gate based quantum computing. The standard approach is directly analogous to bit-wise classical logic circuits with bits replaced by qubits and logic gates replaced by unitary operators. These quantum circuits are compactly expressed using circuit diagrams as demonstrated in Fig 2.1.

An important type of unitary often used in such circuits is the controlled unitary. This is analogous to controlled gates in classical computation. A unitary operator is enacted on some subsystem conditioned on the state of a different subsystem. For instance the unitary U conditioned on a single qubit's computational basis state would be $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U$. The controlled not (CNOT) gate as illustrated in Fig. 2.1 is an example of this.

Finally, in the case where we have two qubits the computational basis is naturally written as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, but we shall also make frequent use of the Bell basis which consists of the four Bell states $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, and $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. The significance of these states will become more apparent when we turn to consider quantum correlations.

2.3.5 Continuous variable system

The second important example is the continuous variable system. These are more complicated than simple qubits, sitting at the other extreme with respect to dimensionality. They are infinite dimensional systems, with physical examples including the motion of quantum particles and the electromagnetic field of light.

We shall consider a 1D continuous variable system, for which we have position x and momentum p observables which satisfy the canonical commutation relation $[x, p] = i\hbar$, or $[X, P] = i$ depending on convention. The generalization to higher dimensions is straightforward and note that in the case of a light field these operators do not correspond to an actual position and momentum and are called field quadratures. Physicists often like to talk in terms of eigenstates for these operators such that $x|x_0\rangle = x_0|x_0\rangle$ and $p|p_0\rangle = p_0|p_0\rangle$, with $\langle x_1|x_2\rangle = \delta(x_1 - x_2)$, and $\langle p_1|p_2\rangle = \delta(p_1 - p_2)$, where we are using the Dirac delta function. Some mathematicians would rightly wring their hands and point out that operators with purely continuous spectra do not have eigenvalues or eigenstates, and that delta functions are functionals not functions so they do not count as proper eigenstates. We are sympathetic to this view but shall press on with this terminology regardless. Certainly physicists do not regard these as physical states since they do not correspond to square-integrable functions, but they are a convenient fiction.

We shall present two useful approaches to continuous variable systems. The first is most commonly used for the motion of particles. It is termed wave mechanics and is based around wave functions. The second is useful for light fields and trapped particles and centers around ladder operators.

For the wave mechanics approach to continuous variables, we represent states with complex-valued probability amplitudes. These are termed wave functions, for instance $\psi(x)$, where the choice of argument indicates we are using the position representation, which is the most commonly used. The connection with the ket representation is seen by writing $\mathbb{1} = \int_{-\infty}^{\infty} |x\rangle\langle x|$, and then for a state $|\psi\rangle$, we expand it in the position basis as

$$|\psi\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|\psi\rangle dx = \int_{-\infty}^{\infty} \psi(x)|x\rangle dx. \tag{2.22}$$

The position operator in this representation is straightforward $\langle x|x|\psi\rangle = x\psi(x)$. The momentum operator can be found for this representation as follows. By evaluation of $\langle x|[x, p]|y\rangle$ using both the canonical commutator relation and by expanding the commutator, we find $(x-y)\langle x|p|y\rangle = i\hbar\delta(x-y)$. Then taking $(x-y)\delta(x-y) = 0$, and differentiating we get $(x - y)\frac{\partial}{\partial x}\delta(x - y) = -\delta(x - y)$. With these two equations we see that $\langle x|p|y\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x - y) + \alpha\delta(x - y)$, where from now on we set

$\alpha = 0$ as this term can be eliminated by a global phase shift. Using this we find $\langle x|p|\psi\rangle = -i\hbar\frac{\partial\psi(x)}{\partial x}$.

The momentum representation is defined analogously. To switch between the two we write $\langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\langle x|p\rangle\langle p|\psi\rangle$. We therefore just need $\langle x|p\rangle$ which can be found by writing out $\langle x|p|p\rangle$ to get the equation $-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle$, which gives $\langle x|p\rangle = ce^{ipx/\hbar}$. The constant $c = \frac{1}{\sqrt{2\pi\hbar}}$ can be found using $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikx}$.

Two useful transformations to understand within these representations are spatial translations and momentum boosts. A spatial translation $T(x_0) = e^{-ipx_0/\hbar}$ acts as $T(x_0)|x\rangle = |x + x_0\rangle$. This can be derived by considering the action of the position operator on $T(x_0)|x\rangle$. As such we write $xe^{-ipx_0/\hbar}|x\rangle = e^{ipx_0/\hbar}e^{-ipx_0/\hbar}xe^{-ipx_0/\hbar}|x\rangle$, then by using Hadamard's lemma this can be evaluated as $(x + x_0)e^{-ipx_0/\hbar}|x\rangle$, as required. In terms of the wave function we have $\langle x|T(x_0)|\psi\rangle = \psi(x - x_0)$. The state has been shifted in the positive x direction, as can be seen by noting the probability amplitude $\psi(0)$ that was at $x = 0$ is now at $x = x_0$. The momentum boost is directly analogous with $B(p_0) = e^{ip_0x/\hbar}$ and $B(p_0)|p\rangle = |p + p_0\rangle$.

For the second approach to continuous variable systems we do not change our state representation from kets, but instead form new convenient operators. We define the lowering/annihilation operator as

$$a = \sqrt{\frac{A}{2\hbar}}(x + \frac{i}{A}p), \tag{2.23}$$

where A is some physically motivated constant, and the $\hbar$ is absent if we are using the $[X, P] = i$ convention. We also define the raising/creation operator as the adjoint $a^\dagger$. These operators can then be seen to satisfy the Bosonic commutation relation $[a, a^\dagger] = 1$.

These operators are not Hermitian so do not correspond to observables. However, we can define the number operator $n = a^\dagger a$ which is Hermitian, and its eigenvectors $n|n_0\rangle = n_0|n\rangle$ form a discrete basis for the Hilbert space, termed the number basis or Fock basis. The raising and lowering operators do as their names imply with $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. There is a vacuum state $|0\rangle$ for which $a|0\rangle = 0$.

The Hamiltonian (energy operator) for a particle of mass m subject to a Harmonic trap of frequency ω, can be rewritten with these operators (setting $A = m\omega$) as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}). \tag{2.24}$$

This is also the Hamiltonian for a single mode Bosonic field such as light, therefore the same mathematics is used to describe the two situations.

We close this section with a couple of example states that frequently crop up

in this continuous variable setting. The first is the coherent state which is the eigenstate of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$. The coherent state can be written out in the number basis as $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$. Coherent states form an overcomplete basis for the space such that $\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \mathbb{1}$. This enables the Glauber-Sudarshan P representation, whereby we can write any state as $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$, though it should be noted that $P(\alpha)$ is not always positive. The coherent state is most commonly used to represent the state of light from a laser but it is also used in the particle motion setting.

The second common state is the thermal state. This is a mixed state with energy probabilities related by the Boltzmann distribution $P(E) \propto e^{-\beta E}$, with $\beta = (k_B T)^{-1}$, where T is temperature and k_B the Boltzmann constant. It can be written out in the number basis as $\rho_{th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$, with $\bar{n} = (e^{\beta\hbar\omega} - 1)^{-1}$. It can also be expressed with a Glauber-Sudarshan P representation as $P(\alpha) = \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\bar{n}}$. This state is used to represent thermal light and thermalised states of motion in a harmonic trap.

2.3.6 Information theoretic quantities

The final topic that we need to cover here is that of useful information theoretic quantities. We shall consider distance measures between states and a central measure in quantum metrology. In terms of measuring the distance between states we shall mostly focus on the trace distance, but we shall also define the quantum state fidelity and describe how these two quantities are related. The key metrological quantity is the quantum Fisher information.

First we define the classical trace distance, which is sometimes referred to as the L_1 distance, or Kolmogorov distance. For two probability distributions p, and q, made up of probabilities $p(x)$ and $q(x)$ defined over the same index set $\{x\}$, the classical trace distance is defined as

$$T_c(p, q) \equiv \frac{1}{2} \sum_x |p(x) - q(x)|. \tag{2.25}$$

This defines a metric on probability distributions and is in clear correspondence with the quantum definition of trace distance that now follows.

The (quantum) trace distance, is defined as

$$T(\rho, \sigma) \equiv \frac{1}{2}\|\rho - \sigma\|_1, \tag{2.26}$$

where $\|A\|_1 \equiv \mathrm{Tr}(\sqrt{A^\dagger A})$ is the trace norm. We now review a series of standard results that are useful when working with the trace distance.

The first and most important result is the variational form for the trace distance, which is

$$T(\rho,\sigma) = \sup_{\Pi} \mathrm{Tr}[\Pi(\rho-\sigma)], \tag{2.27}$$

where Π is a POVM element $0 \leq \Pi \leq 1$. An important addition is that the maximal value is obtained by choosing Π as the projector P onto the positive subspace of $\rho - \sigma$.

To prove this we start by noting that $\rho - \sigma$ is a Hermitian operator with some positive and some negative eigenvalues. Since it is Hermitian we can write it as $\rho - \sigma = UDU^\dagger$. Separating the positive and negative eigenvalues such that $D = D_+ + D_-$, we have $\rho - \sigma = Q - S$, where $Q = UD_+U^\dagger$ and $S = -UD_-U^\dagger$. These positive operators are manifestly orthogonal $QS = -UD_+D_-U^\dagger = 0$. Using the notation $|A| = \sqrt{A^\dagger A}$, it follows that $|\rho - \sigma| = Q + S$, and therefore $T(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}|\rho-\sigma| = \frac{1}{2}(\mathrm{Tr}Q + \mathrm{Tr}S)$. We now use $0 = \mathrm{Tr}(\rho-\sigma) = \mathrm{Tr}Q - \mathrm{Tr}S$, to arrive at $T(\rho,\sigma) = \mathrm{Tr}Q$. Letting P be the projector onto the support of Q, we have $\mathrm{Tr}(P(\rho-\sigma)) = \mathrm{Tr}(P(Q-S)) = \mathrm{Tr}(Q) = T(\rho,\sigma)$. If we let Π be any other POVM element we have $\mathrm{Tr}(\Pi(\rho-\sigma)) = \mathrm{Tr}(\Pi(Q-S)) \leq \mathrm{Tr}(\Pi Q) \leq \mathrm{Tr}(Q) = T(\rho,\sigma)$. This concludes the proof.

This result is especially useful in proving other properties of the trace distance. Perhaps most importantly it leads to the operational interpretation for the trace distance as follows. Consider the task of single copy state discrimination where we are given the state ρ or σ, with equal probability, and we are tasked with answering which state it actually is. Without any measurement our maximum probability of guessing the actual state correctly is $\frac{1}{2}$, since we will just guess a state. To do better one applies some POVM measurement Π_ρ, Π_σ, where if we get the first outcome we guess ρ and σ for the second. The probability of guessing correctly is now $p_C = \frac{1}{2}\mathrm{Tr}(\Pi_\rho\rho) + \frac{1}{2}\mathrm{Tr}(\Pi_\sigma\sigma)$. Now using $\Pi_\rho + \Pi_\sigma = I$ we have $p_C = \frac{1}{2}[1 + \frac{1}{2}\mathrm{Tr}(\Pi_\sigma(\sigma-\rho))]$. Now by the variational form theorem, the maximum value is given by $p_C = \frac{1}{2}[1 + T(\rho,\sigma)]$, therefore the trace distance quantifies our ability to distinguish between the two states.

It turns out that the trace distance defines a metric on the space of quantum states. Thus it must satisfy the triangle inequality which can be seen by using Eq. (2.27) to write $T(\rho,\sigma) = \mathrm{Tr}(P(\rho-\sigma)) = \mathrm{Tr}(P(\rho-\tau)) + \mathrm{Tr}(P(\tau-\rho)) \leq T(\rho,\tau) + T(\tau,\sigma)$.

Another important property is that the trace distance is contractive under trace-preserving quantum operations.

$$T(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \leq T(\rho,\sigma). \tag{2.28}$$

To prove this we start by noting that since we have shown that $T(\rho,\sigma) = \mathrm{Tr}(Q)$,

and $\mathcal{E}$ is trace preserving, we have $\mathrm{Tr}(Q) = \mathrm{Tr}(\mathcal{E}(Q))$. Now consider the projector P chosen such that $T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \mathrm{Tr}[P(\mathcal{E}(\rho) - \mathcal{E}(\sigma))]$. We now have $\mathrm{Tr}(\mathcal{E}(Q)) \geq \mathrm{Tr}(P\mathcal{E}(Q)) \geq \mathrm{Tr}[P(\mathcal{E}(Q) - \mathcal{E}(S))] = T(\mathcal{E}(\rho), \mathcal{E}(\sigma))$, which concludes the proof.

It is useful to know how the trace distance behaves with regards to mixtures of states. For this we use the strong convexity of the trace distance, which gives us that

$$T(\sum_i p_i\rho_i, \sum_i q_i\sigma_i) \leq \sum_i p_i T(\rho_i, \sigma_i) + T_c(p, q). \tag{2.29}$$

To prove this we consider the projector P that is optimal in discriminating $\sum_i p_i\rho_i$, and $\sum_i q_i\sigma_i$. We can then write

$$\begin{aligned} T(\sum_i p_i\rho_i, \sum_i q_i\sigma_i) &= \mathrm{Tr}[P(\sum_i p_i\rho_i - \sum_i q_i\sigma_i)], \\ &= \sum_i p_i \mathrm{Tr}(P\rho_i) - \sum_i q_i \mathrm{Tr}(P\sigma_i), \\ &= \sum_i p_i \mathrm{Tr}[P(\rho_i - \sigma_i)] + \sum_i (p_i - q_i)\mathrm{Tr}(P\sigma_i), \\ &\leq \sum_i p_i T(\rho_i, \sigma_i) + T_c(p, q). \end{aligned}$$

The first term in the last line follows from $T(\rho, \sigma) = \max_P \mathrm{Tr}[P(\rho - \sigma)]$, and the second term from writing $\sum_i (p_i - q_i)\mathrm{Tr}(P\sigma_i) = \sum_{i:p_i>q_i}(p_i - q_i)\mathrm{Tr}(P\sigma_i) - \sum_{i:q_i>p_i}(q_i - p_i)\mathrm{Tr}(P\sigma_i)$, and noting that $0 \leq \mathrm{Tr}(P\sigma_i) \leq 1$ implies this is less than or equal to $\sum_{i:p_i>q_i}(p_i - q_i), = T_c(p, q)$, the classical trace distance.

A useful corollary of this, termed convexity of the trace distance, is that

$$T(\sum_i p_i\rho_i, \sigma) \leq \sum_i p_i T(\rho_i, \sigma). \tag{2.30}$$

This follows from the strong convexity result by setting $q_i = p_i$ and $\sigma_i = \sigma$.

The other main quantity of interest when comparing quantum states is the quantum state fidelity. It is defined as

$$F(\rho, \sigma) = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)\right]. \tag{2.31}$$

Note that some authors prefer to define the Fidelity as this quantity squared. For pure states this definition corresponds to $|\langle\psi_\rho|\psi_\sigma\rangle|$.

We shall not go into the same level of detail regarding the properties of the fidelity as we did for the trace distance. Instead we just note two important results that will prove useful to us.

The first result is that the fidelity satisfies

$$F(\rho_1^{\otimes n}, \rho_2^{\otimes n}) = F(\rho_1, \rho_2)^n. \tag{2.32}$$

This follows from the definition and the fact that $\sqrt{A \otimes A} = \sqrt{A} \otimes \sqrt{A}$, together with $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$. This is in contrast to the trace distance which does not scale in this multiplicative fashion for multiple copies of a system.

The second result is the relationship between fidelity and trace distance, as captured by the Fuchs–van de Graaf inequalities [105], which states that

$$1 - F(\rho, \sigma) \leq T(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{2.33}$$

For pure states the right-hand inequality becomes tight, so

$$T(|\psi\rangle, |\phi\rangle) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)^2}, \tag{2.34}$$

where here for the sake of brevity we are writing functions of pure states as $f(|\psi\rangle)$ rather than $f(|\psi\rangle\langle\psi|)$, and shall make use of this notational short-hand throughout.

The quantum Fisher information is the quantum analogue of the classical Fisher information, so first we briefly discuss this quantity.

Consider a parameter θ which we want to estimate, and some random variable X for which the probability depends on θ. We write the probability density function of X conditioned on θ as $f(X;\theta)$. The Fisher information is then defined as

$$\mathcal{I}(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta)\right)^2 \middle| \theta\right], \tag{2.35}$$

where $|\theta$ denotes that we are taking the expectation value at a particular θ. Subject to regularity conditions, this can be rewritten as

$$\mathcal{I}(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\middle|\theta\right]. \tag{2.36}$$

This form is more easily interpretable as it is now the curvature of the log-likelihood curve evaluated at the point θ. The maximum likelihood value sits at a maximum on this curve, and if the nearby curvature is small then there are lots of nearby values with similar likelihood. Conversely a large curvature means we have a nice clean peak in the likelihood, and thus the Fisher information helps capture the error in our estimate.

This is formally captured by the Cramer-Rao bound, which states that the vari-

ance for some unbiased estimate of θ given n samples, is bounded by

$$(\Delta\theta)^2 \geq \frac{1}{n\mathcal{I}(\theta)}. \tag{2.37}$$

Note that a factor of n is sometimes absorbed into the definition of the Fisher information.

The quantum setting is somewhat more involved [106]. In this work we shall only consider the situation where we want to estimate some parameter θ which has been unitarily encoded in a state via observable H, so we write

$$\rho(\theta) = e^{-iH\theta}\rho_0 e^{iH\theta}. \tag{2.38}$$

The statistical precision with which we can measure θ via measurements applied to $\rho(\theta)$, satisfies the quantum Cramer-Rao bound

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}(\rho, H)}, \tag{2.39}$$

where the quantum Fisher information is

$$\mathcal{F}(\rho, H) = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle\psi_i|H|\psi_j\rangle|^2, \tag{2.40}$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ is a (θ independent due to the unitary evolution) spectral decomposition, and we do not include terms in the sum for which $\lambda_i = \lambda_j = 0$.

This concludes the discussion on information theoretic quantities that we shall use in the rest of this work, along with bringing to an end our exposition of the necessary quantum fundamentals.

2.4 Fundamentals of quantum correlations

Quantum correlations come in several strengths and they are all rich subjects in themselves, with entanglement [107], Bell non-locality [108], quantum steering [109], and quantum discord [110] each with a developed literature of their own. We shall consider all of these to varying degrees, and we shall focus on bipartite correlations throughout.

2.4.1 Pure state entanglement

As discussed previously, in quantum mechanics the pure state of an isolated system is described by some ket $|\psi\rangle_A$, drawn from some Hilbert space $\mathcal{H}_A$. Likewise for some distinct isolated system with $|\phi\rangle_B$, and $\mathcal{H}_B$. However, a pure state for both systems taken together must be drawn from a total Hilbert space that is the tensor product of the two separate Hilbert spaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. This Hilbert space contains simple states like $|\psi\rangle_A \otimes |\phi\rangle_B$, which are termed product states, and indeed if we took the direct sum $\mathcal{H}_A \oplus \mathcal{H}_B$ to form the total Hilbert space then all states would be of this type, where we can always assign a single pure state to each individual system. However, the tensor product structure allows for composite states like $|\psi_1\rangle_A \otimes |\phi_1\rangle_B + |\psi_2\rangle_A \otimes |\phi_2\rangle_B$, which cannot be written in product state form. For states like this we can no longer assign a single pure state to either of the individual systems. For the most complete description we must describe the systems together. Such pure states are termed entangled.

We say that a pure state is entangled if it cannot be written as a product state. However, consider the example state $\frac{1}{2}(|0\rangle|1\rangle + |1\rangle|0\rangle + |0\rangle|0\rangle + |1\rangle|1\rangle)$. Written in this form it is not manifestly clear that this is in fact the product state $|+\rangle|+\rangle$. This example is fairly simple but in higher dimensions one does not really want to exhaustively work through the factoring. Any arbitrary state can be written as $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j\rangle$, for separate orthonormal bases $\{|i\rangle\rangle\}$ and $\{|j\rangle\rangle\}$. Fortunately any such state can be rewritten as $|\psi\rangle = \sum_k \sqrt{\lambda_k}|\psi_k\rangle|\phi_k\rangle$, with the sets $\{|\psi_k\rangle\}$ and $\{|\phi_k\rangle\}$ are both orthonormal. This can be proved as follows. The singular value decomposition tells us that we can take the matrix a made up from components a_{ij} and decompose it as $a = U\Lambda V^\dagger$, where U and V are unitary and Λ is diagonal with components we denote as $\sqrt{\lambda_k}$. We therefore write

$$\begin{aligned} |\psi\rangle &= \sum_{i,j}\sum_k U_{ik}\Lambda_{kk}V^\dagger_{kj}|i\rangle|j\rangle, \\ &= \sum_k \Lambda_{kk}(\sum_i U_{ik}|i\rangle)(\sum_j V^\dagger_{kj}|j\rangle), \\ &= \sum_k \sqrt{\lambda_k}|\psi_k\rangle|\phi_k\rangle. \end{aligned} \tag{2.41}$$

Here we have defined $|\psi_k\rangle = \sum_i U_{ik}|i\rangle$ and $|\phi_k\rangle = V^\dagger_{kj}|j\rangle$. The fact that these form orthonormal sets follows from the unitarity. The values $\sqrt{\lambda_k}$ are called Schmidt coefficients, and the number of non-zero Schmidt coefficients is called the Schmidt number or Schmidt rank.

We have defined which pure states are entangled and with Schmidt decompositions it is straightforward to identify them, but this does not get us very far. Indeed, looking from the point of view of Schmidt decompositions, all we have done is separate out those states which have a Schmidt number of 1 and declared all the others to be entangled. The problem is that we have defined entangled states but we have not fully defined entanglement. For instance, consider the states $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\sqrt{0.9}|00\rangle + \sqrt{0.1}|11\rangle$. With our current understanding we can say that both are entangled, but we cannot say much more, such as whether one is more entangled than the other.

The direction for our next step can be seen with the following thought experiment. Consider Alice and Bob are in different labs, separated by a large distance. They can only communicate via talking on the telephone. Alice has quantum system A and Bob has quantum system B. The systems are initially in the joint state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$, which is entangled. Now Alice decides that she actually wants them to share the state $|11\rangle_{AB}$. To achieve this she measures her system in the $\{|0\rangle_A, |1\rangle_A\}$ basis, and communicates the result to Bob. Applying the rules for measurement in quantum mechanics, we see that they will have the state $|00\rangle_{AB}$ or $|11\rangle_{AB}$, each with a 50% probability. In the case where they get $|00\rangle_{AB}$ then they simply both apply σ^x to transform the state into $|11\rangle_{AB}$. They have therefore managed to deterministically (with unit probability) transform their entangled state $|\phi^+\rangle_{AB}$, to the unentangled state $|11\rangle_{AB}$. But now suppose Alice regrets her decision and wants to transform back to the entangled state. Unfortunately with their current setup it turns out that this is impossible. We therefore identify a new feature to add to our conception of entanglement. We say that entanglement cannot be deterministically increased by local operations and classical communication (LOCC).

This simple step actually brings a large amount of novel structure to entanglement. To illustrate, we return to the question of comparing the entangled states $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\sqrt{0.9}|00\rangle + \sqrt{0.1}|11\rangle$. We can transform from the former to the latter state by using a local POVM with elements $\Pi_1 = 0.9|0\rangle\langle 0| + 0.1|1\rangle\langle 1|$ and $\Pi_2 = 0.1|0\rangle\langle 0| + 0.9|1\rangle\langle 1|$, followed by applying $\sigma^x \otimes \sigma^x$ when the Π_2 case is obtained. But there is no such protocol to deterministically go the other way. So we say $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is more entangled than $\sqrt{0.9}|00\rangle + \sqrt{0.1}|11\rangle$.

With this idea we can extend a partial ordering for entanglement over all pure states, via a result known as Nielsen's theorem [111]. This states that $|\psi\rangle_{AB}$ may be

deterministically transformed via LOCC to $|\phi\rangle_{AB}$ if and only if the eigenvalues of $\rho_A = \mathrm{Tr}_A(|\psi\rangle_{AB}\langle\psi|)$ are majorized by the eigenvalues of $\sigma_A = \mathrm{Tr}_A(|\phi\rangle_{AB}\langle\phi|)$. The d dimensional vector x is said to majorize the d dimensional vector y (written $x \succ y$) if $\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow}$ for all k in the range $1, ..., d$ with equality holding when $k = d$. Here the downward arrow denotes the vectors are taken in descending order. Note that the eigenvalues of the reduced density matrices used in Nielsen's theorem are simply the λ_i from the Schmidt decompositions (note the lack of square root), and we include any repeated zero values.

Now that we have a conception of which pure states are more entangled than others, it is natural to seek some measure that captures this. We require some function E which when $|\psi\rangle$ can be transformed to $|\phi\rangle$ under deterministic LOCC, must satisfy $E(|\psi\rangle) \geq E(|\phi\rangle)$. We refer to this as monotonicity, however on its own it is not enough to specify a unique function. However, if we add the condition that $E(|\phi^+\rangle) = 1$ and an extensivity condition $E(|\phi\rangle^{\otimes k}) = kE(|\psi\rangle)$, then this does specify a unique entanglement measure. This is the von-Neumann entropy of the reduced state

$$E(|\psi\rangle_{AB}) = S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A), \tag{2.42}$$

where $\rho_A = \mathrm{Tr}(|\psi\rangle\langle\psi|_{AB})$, and we shall use base 2 for our logarithms unless otherwise stated.

To see why this is the case we need a key result regarding distillation of Bell states in the asymptotic limit of many copies [112]. The result states that given N copies of state $|\psi\rangle$, in the limit as $N \to \infty$ there exist deterministic LOCC protocols to reversibly convert between $|\psi\rangle_{AB}^{\otimes N}$ and $|\phi^+\rangle_{AB}^{\otimes NS(\rho_A)}$, where $S(\rho_A)$ is the von-Neumann entropy for the reduced state of $|\psi\rangle_{AB}$.

Armed with this result we now prove the assertion that the von-Neumann entropy is uniquely specified by the above conditions. Monotonicity applied to the asymptotic interconversions enforces $E(|\psi\rangle_{AB}^{\otimes N}) \geq E(|\phi^+\rangle_{AB}^{\otimes NS(\rho_A)})$ and $E(|\psi\rangle_{AB}^{\otimes N}) \leq E(|\phi^+\rangle_{AB}^{\otimes NS(\rho_A)})$, therefore $E(|\psi\rangle_{AB}^{\otimes N}) = E(|\phi^+\rangle_{AB}^{\otimes NS(\rho_A)})$ in the limit of large N. Using extensivity and $E(|\phi^+\rangle_{AB}) = 1$, we arrive at $E(|\psi\rangle_{AB}) = S(\rho_A)$.

At this point entanglement theory seems well behaved. This is mainly because we have been restricting our attention to pure states. It is high time to throw a spanner in the works, and we therefore turn to consider entanglement for mixed states.

2.4.2 Mixed state entanglement

We first present two arguments which demonstrate the need to extend the conception of entanglement to mixed states. The first is that previously we considered only deterministic LOCC. In this context we said that we could not convert $\sqrt{0.9}|00\rangle +$

$\sqrt{0.1}|11\rangle$ to $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. However, consider using the POVM $\Pi_1 = \frac{1}{9}|0\rangle\langle 0| + |1\rangle\langle 1|$, $\Pi_2 = \frac{8}{9}|0\rangle\langle 0|$. With probability 0.2 we obtain the result corresponding to Π_1 and our state is then $|\phi^+\rangle$. So this looks like we have been able to increase the entanglement, it is just that we have only been able to do it stochastically rather than deterministically. This is termed Stochastic LOCC (SLOCC) and is a subject of study in its own right [113, 114, 115, 116], but the key point is that the possibility of such manipulations necessitates the inclusion of classical probabilities into our understanding of entanglement. As discussed previously, the way to include classical probabilities is with density operators and mixed states.

The second argument is more prosaic, and this is that we essentially never work with pure states experimentally. Uncertainty is an inherent part of empirical science, so if we want to be able to interpret the results from actual experiments in terms of entanglement, then we simply must extend the concepts to the mixed state regime.

As in the case of pure states, we begin by defining the states which are not entangled. A state is separable if it can be written in the form

$$\rho_{AB} = \sum_{ij} p_{ij} \rho_A^{(i)} \otimes \rho_B^{(j)}. \tag{2.43}$$

A state is entangled if it is not separable.

Unfortunately, unlike the case of pure states, for mixed states there is in general no easy way to check if a state is separable or entangled. In fact the quantum separability problem has been shown to be NP Hard [117]. As a results of this, there are multiple approaches when considering whether certain mixed states are entangled.

We shall start by introducing perhaps the bluntest but most practically effective tool, which is entanglement witnesses. An entanglement witness is a Hermitian observable W which satisfies $\langle W \rangle = \mathrm{Tr}(W\sigma_{AB}) \geq 0$ for all separable states σ_{AB}, but which has $\langle W \rangle = \mathrm{Tr}(W\rho_{AB}) < 0$ for certain entangled states ρ_{AB}. The entangled states for which the expectation value $\langle W \rangle$ are negative, are said to be witnessed by W. It is important to appreciate that no witness can witness all entangled states, but that a witness can always be found for any entangled state. This follows nicely from a geometric interpretation illustrated schematically in Figure 2.2, wherein we can think of the witness as defining a hyperplane. This is possible because we can view the expectation value $\mathrm{Tr}(W\rho)$ as an inner product $\langle W, \rho \rangle$ for the vector space of Hermitian operators. This is termed the Hilbert-Schmidt inner product. With this we can view W as defining the normal to a hyperplane and then all states with $\mathrm{Tr}(W\rho_{AB}) \geq 0$ fall on the hyperplane or to one side of it, and all states with $\mathrm{Tr}(W\rho_{AB}) < 0$ fall on the other side. It is now apparent how a witness can work for some entangled states but not others. Additionally, since the set of separable states

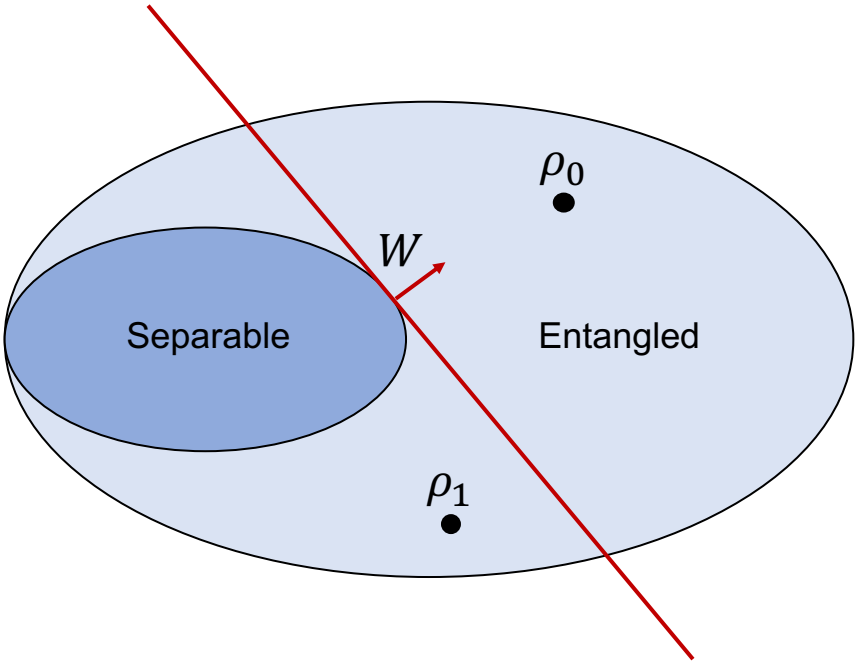

Figure 2.2: Schematic illustration of an entanglement witness. The witness defines a hyperplane with all the separable states on one side of it. All the entangled states on the other side (such as ρ_0) are witnessed by this choice of witness, but those on the same side as the separable states (such as ρ_1) are not.

is convex, we can imagine always being able to find a hyperplane which witnesses a certain entangled state. Formalising this intuitive view is non-trivial and proceeds via the Hahn-Banach theorem [118, 119].

The construction of witnesses is not always straightforward, however there is one particularly useful result which often proves helpful in such matters and will also help lead us beyond witnesses. This result is known as the Peres-Horodecki criterion [120, 121].

The Peres-Horodecki criterion, sometimes referred to as the positive partial transpose (PPT) condition, states that the partial transpose of a separable state must be positive, and thus if it is not then the state must be entangled.

This can be proved straightforwardly by first considering taking the partial transpose of the form for separable states presented in Eq. (2.43), to obtain

$$\rho_{AB}^{T_B} = \sum_{ij} p_{ij} \rho_A^{(i)} \otimes \rho_B^{(j)T}. \tag{2.44}$$

In accordance with the properties of density matrices, we have that $\rho_B^{(j)T}$ is a valid state and therefore the whole state $\rho_{AB}^{T_B}$ is a valid state, and as such must have only positive eigenvalues. Therefore if the partial transpose of some state has a negative eigenvalue then it cannot be written in the form of Eq. 2.43 and so we know the state must be entangled. In the case of two qubits or a qubit and a qutrit

(a three dimensional system) the PPT criterion has been shown to be necessary and sufficient, however for higher dimensions it is not guaranteed that an entangled state will give a negative eigenvalue.

In practise we cannot perform the partial transpose operation on a state, since the transpose is a positive but not completely positive map and therefore is not physical as per our previous discussion on quantum channels. However, we can use the Peres-Horodecki criterion to construct witnesses. If we consider some entangled state ρ_{AB}, take the partial transpose $\rho_{AB}^{T_B}$, and find that this state has a negative eigenvalue with eigenvector $|\chi_-\rangle$, then we can write $\mathrm{Tr}\big(|\chi_-\rangle\langle\chi_-|\rho_{AB}^{T_B}\big) < 0$. But the trace is invariant under partial transposition and therefore we also have that $\mathrm{Tr}\big(|\chi_-\rangle\langle\chi_-|^{T_B}\rho_{AB}\big) < 0$. This is precisely what we want for a witness, it only remains to note that the reverse step together with the Peres-Horodecki criterion guarantees us that $\mathrm{Tr}\big(|\chi_-\rangle\langle\chi_-|^{T_B}\sigma_{AB}\big) \geq 0$, for all separable states σ_{AB}. Therefore $|\chi_-\rangle\langle\chi_-|^{T_B}$ is an entanglement witness.

Entanglement witnesses are certainly useful but they have some undesirable behaviour. Namely, that they cannot be consistently used to compare entangled states. Consider Figure 2.2, and the comparison of ρ_0 and ρ_1. The first is declared entangled by our witness but it is inconclusive on the second, however we cannot draw conclusions from this regarding whether we can potentially use LOCC to transform from one state to the other, or in any way argue that one state is more entangled than the other.

A more powerful tool is that of entanglement monotones and measures [51, 52, 107]. These are names given to certain families of useful functions of states, and it should be noted that these terms have not always been consistently defined. Here we shall present the definitions of Plenio and Virmani, which we do via reference to the following desirable properties.

1. Positivity: $M(\rho) \in \mathbb{R}^+$.

2. Vanishing for separable states: $M(\sigma) = 0$ for σ separable.

3. Monotonically decreasing on average under LOCC:

$$M(\rho) \geq \sum_i p_i M\left(\frac{K_i \rho K_i^\dagger}{\mathrm{Tr}\left(K_i \rho K_i^\dagger\right)}\right), \tag{2.45}$$

   where $p_i = \mathrm{Tr}\left(K_i \rho K_i^\dagger\right)$ and the K_i are Kraus operators for some LOCC protocol.

4. Reducing to the entropy of entanglement for pure states: $M(|\psi\rangle) = E(|\psi\rangle)$ with E as defined in Eq. (2.42).

We can now define an entanglement monotone to be a function which satisfies 1,2, and 3. An entanglement measure satisfies 1,2, and 4, and instead of 3 it must satisfy the weaker condition of being montonically decreasing under deterministic LOCC. It should be mentioned that though these are the most agreed upon properties, sometimes other desirable properties are also sought, such as convexity $M(\sum_i p_i\rho_i) \leq \sum_i p_i M(\rho_i)$, and additivity $M(\rho^{\otimes n}) = nM(\rho)$.

An immediate example can be constructed from the Peres-Horodecki criterion. This is termed the negativity and is defined as

$$\mathcal{N}(\rho) = \frac{\left\|\rho^{T_B}\right\|_1 - 1}{2}. \tag{2.46}$$

This is simply the absolute sum of the negative eigenvalues of ρ^{T_B}. This can be seen by first writing $\rho^{T_B} = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$, and noting that $\mathcal{N}(\rho) = (\sum_i |\lambda_i| - 1)/2$. Now since the trace is invariant under partial transposition we have that $\sum_i \lambda_i = 1$, and hence $\mathcal{N}(\rho) = \sum_{\lambda_i<0} |\lambda_i|$. The negativity as defined has been shown to be an entanglement monotone [122, 123].

There are a multitude of entanglement monotones and measures (see [107] for a review) and we tend to choose those which are most relevant or convenient to the particular setting of interest. We shall only discuss two more examples here. These shall be an entanglement measure and a related entanglement monotone.

Our example of an entanglement measure is the entanglement of formation. It is the minimal possible average entanglement over all pure state decompositions. Mathematically we write this as

$$E_F(\rho) = \inf \sum_j p_j E(|\psi_j\rangle\langle\psi_j|), \tag{2.47}$$

where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$.

The main reason we mention this measure is that it motivated the introduction of the concurrence [124, 125], which is an entanglement monotone that we shall make use of later. The concurrence $C(\rho)$ can be used to give an explicit expression for the entanglement of formation for a two qubit state via

$$E_F(\rho) = f(C(\rho)), \tag{2.48}$$

where

$$f(x) = -\frac{1+\sqrt{1-x^2}}{2}\log\frac{1+\sqrt{1-x^2}}{2} - \frac{1-\sqrt{1-x^2}}{2}\log\frac{1-\sqrt{1-x^2}}{2}. \tag{2.49}$$

The concurrence of a two qubit mixed state is efficiently calculated as

$$C(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{2.50}$$

where $\lambda_1, ..., \lambda_4$ are the eigenvalues in decreasing order of $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, where $\tilde{\rho} = (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$, and ρ^* denotes complex conjugation in the computational basis. It is equivalent to work with the square roots of the eigenvalues of $\rho\tilde{\rho}$.

We highlight the fact that this expression for concurrence is equivalent to a definition via the convex roof extension

$$C(\rho) = \inf \sum_j p_j C(|\psi_j\rangle), \tag{2.51}$$

where the infimum is taken over all such ensemble decompositions of the form $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, and where for a pure qubit state in Schmidt decomposed form $|\psi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$ the concurrence is simply $C(|\psi\rangle) = 2\sqrt{\lambda_0\lambda_1}$. We note that this can also be viewed as the convex-roof extended negativity [126].

Though the initial focus for concurrence was centred on the entanglement of formation, it has since been studied, generalised and measured in various contexts [127, 128, 129, 130, 131, 132, 133]. As mentioned, we shall make particular use of this monotone later.

There is a large quantity of material on entanglement in mixed systems and we cannot cover it all here. For completeness we close by describing the modern view of entanglement as a quantum resource theory [62]. To define a resource theory one needs to specify two things: a set of free operations and a set of free states. The free operations are quantum operations that we consider free to perform, in the sense that we do not need to use any of the resource to enact them. The operations are chosen to respect some physically motivated restriction. The free states are the set of states that we can prepare freely, so are those states which can be prepared under the same physical restriction as used to define the free operations. We can if we want consider preparing the free states as special cases of the free operations, then specifying the free operations fully defines the resource theory. Naturally to have a consistent resource theory the free operations acting on free states must not give non-free states, since this would mean we could create a resource state for free.

For the resource theory of entanglement, the physical scenario considered is spatially separate labs that can exchange classical information. The set of free operations are all those that can be achieved via LOCC. The free states are the separable states as defined in Eq. (2.43), since they can be produced via local operations and shared classical randomness. In this setting entangled states are resource states that can allow one to perform operations outside of the set LOCC such as teleportation

of an unknown quantum state. These choices are good because entanglement cannot be increased under LOCC, and the physical restriction is well motivated. Indeed it is the situation most labs in the world are currently in, they have reliable classical channels between them but not quantum channels. Note that one could take a larger set of free operations by considering all operations that map separable states to separable states. This would include Alice and Bob swapping their systems, which does not produce entanglement but is clearly not in LOCC. Working with this larger set can be useful for mathematical convenience but a true resource theory should be physically motivated.

In summary, we tend to view entanglement as a resource and have a set of tools for quantifying it, which we select from as appropriate to the problem of interest. With this all said we have presented all the key entanglement concepts that we shall call upon later, so we leave this topic here and move to discussing the other important types of quantum correlations.

2.4.3 Bell Non-Locality and Steering

Here we shall present the concepts of Bell non-locality and steering. In doing this we closely follow the presentation in the seminal work of Wiseman et al. [134]. We begin by giving the simple operational definitions for Bell non-local (BNL) and steerable states, and then move to mathematical definitions.

We start with Bell non-locality. Consider a setting where Alice and Bob share copies of a bipartite quantum state where Alice holds one subsystem and Bob the other. They are not allowed to communicate. Their task is to persuade Charlie that they have entanglement by making measurements and passing him results. Charlie is a skeptic so if he can explain the correlations between their results using a local hidden variable (LHV) model then he will, and if he can do this he will claim they do not have entanglement. The set of states that Alice and Bob can use to succeed are the Bell non-local states.

For the steerable states we change the setup so that Alice is preparing the bipartite quantum states, keeping one subsystem and sending the other over to Bob. They make measurements on their parts and communicate classically, and Alice's task is to persuade Bob that she has been producing copies of an entangled state. Bob is skeptical and if he can explain the results with a local hidden state (LHS) model then he will. The states that Alice can use to succeed in this case are the steerable states. The set of Bell non-local states is strictly larger than the set of steerable states which is strictly larger than the set of entangled states.

Before giving the mathematical definitions of Bell non-locality and steering it is informative to reformulate the definition of entanglement in terms of the probabilities

of results from measurements. First we establish terminology. We denote the set of all observables on Alice's system as $\mathcal{D}_\alpha$. For $A \in \mathcal{D}_\alpha$, we denote its set of eigenvalues $\{a\}$ by $\lambda(A)$. The probability that Alice obtains result $a \in \lambda(A)$, given that she measured $A \in \mathcal{D}_\alpha$, on the quantum state W, is denoted $P(a|A;W)$. Similar notation is used for Bob's measurements.

A state is entangled if it is not separable. A state W is separable if it is not of the form given in Eq. (2.43), which we write here as $W = \sum_\xi p_\xi \sigma_\xi \otimes \rho_\xi$. Equivalently we can say a state is separable if for all $a \in \lambda(A)$, $b \in \lambda(B)$, for all $A \in \mathcal{D}_\alpha$, $B \in \mathcal{D}_\beta$, we can find some set of $p_\xi, \sigma_\xi, \rho_\xi$ such that

$$P(a,b|A,B;W) = \sum_\xi p_\xi P(a|A;\sigma_\xi) P(b|B;\rho_\xi), \tag{2.52}$$

where $P(a|A;\sigma_\xi), P(b|B;\rho_\xi)$ are given by the Born rule. Equivalently a state is entangled if this is not the case.

With this formulation in hand we now return to mathematically define Bell non-locality. A state is Bell non-local if we can find measurement sets $\mathcal{M}_\alpha \subseteq \mathcal{D}_\alpha$, $\mathcal{M}_\beta \subseteq \mathcal{D}_\beta$, such that the correlations of the results cannot be explained by a LHV model. The results can be explained by a LHV model if for all $a \in \lambda(A)$, $b \in \lambda(B)$, for all $A \in \mathcal{M}_\alpha$, $B \in \mathcal{M}_\beta$, we find probability distributions $p_\xi, P(a|A,\xi), P(b|B,\xi)$ such that

$$P(a,b|A,B;W) = \sum_\xi p_\xi P(a|A,\xi) P(b|B,\xi), \tag{2.53}$$

and a state is Bell non-local if there exists a measurement set $\mathcal{M}_\alpha \times \mathcal{M}_\beta$, for which this is not the case.

We now turn to the notion of steering which, unlike entanglement and Bell non-locality, is asymmetric. Alice makes a choice of measurement A, and obtains the result a, which means she knows the state Bob has, the options for which are defined by the ensemble $E^A \equiv \{\tilde{\rho}_a^A : a \in \lambda(A)\}$. Here we write Bob's unnormalized states $\tilde{\rho}_a^A \equiv \mathrm{Tr}_\alpha[(\Pi_a^A \otimes \mathbb{1})W]$, where Π_a^A is the projector onto the eigenspace of A, with eigenvalue a.

The test for steering proceeds as follows. Alice announces the ensembles $E_A : A \in \mathcal{M}_\alpha$, that she can steer Bob's state into. Bob picks an ensemble E_A and asks Alice to steer his state into it. Alice measures A and tells Bob the result obtained such that he knows his state should be ρ_a^A. They repeat this many times and Bob determines whether he is actually getting the states ρ_a^A, and whether they are occurring at the correct frequency $\mathrm{Tr}(\tilde{\rho}_a^A)$.

If Alice is trying to deceive Bob then she will be giving him a known LHS ρ_ξ, chosen from an ensemble $F = \{p_\xi \rho_\xi\}$. Alice knows the value of ξ, so would announce a, using a stochastic map that defines $p(a|A,\xi)$. Hence if Bob can find an ensemble

F, and a probability distribution $p(a|A,\xi)$, such that

$$\tilde{\rho}_a^A = \sum_\xi p(a|A,\xi)p_\xi\rho_\xi, \tag{2.54}$$

then Bob will not be convinced Alice is steering his state. If Bob cannot describe the results in this manner then he concludes there is steering.

Steering can be recast with a condition that sits between those given in Eq. (2.52) and Eq. (2.53). Alice has not demonstrated steering if for all $a \in \lambda(A)$, $b \in \lambda(B)$, for all $A \in \mathcal{M}_\alpha$, $B \in \mathcal{D}_\beta$, we can find $p_\xi, P(a|A,\xi), \rho_\xi$ such that

$$P(a,b|A,B;W) = \sum_\xi p_\xi P(a|A,\xi)P(b|B;\rho_\xi), \tag{2.55}$$

where $P(b|B;\rho_\xi)$ is given by the Born rule. If this is not the case then she has demonstrated steering and if there exists $\mathcal{M}_\alpha$, that demonstrates steering for state W, then we say W is steerable. Comparing the forms of Eq. (2.52), Eq. (2.53), and Eq. (2.55) shows that Bell non-locality is stronger than steering which is stronger than entanglement. As we discuss in the next section all three of these relations are strictly stronger i.e. the corresponding sets of states are proper subsets.

A resource theory of steering has been formulated [135] where the free operations are those that can be performed via local operations and one way classical communication (1W-LOCC), from Bob to Alice for the case where we consider Alice steering Bob's state. Note that, since we know LOCC cannot produce entanglement from separable states and all steerable states are entangled, the restriction to only one way classical communication is to prevent free operations taking us from entangled states to steerable states.

Non-locality is viewed as an information theoretic resource [136]. In particular it is a resource for device independent quantum information processing (DQIP), where all devices are viewed as black boxes, and has been formulated as a resource theory [137]. The free operations are those that can be performed via local operations and communication that takes place before the inputs are known [138], otherwise the communication could be used to create nonlocal correlations.

2.4.4 Discord

We now turn to the weakest form of non-classical correlation. Quantum discord was introduced relatively recently [70, 69], and it is aptly named, as it has certainly caused a fair bit of discord in the quantum community. The criticisms tend to focus on it being too prevalent (almost all states have non-zero quantum discord [139]) and not being practically useful for anything. My own view is that regardless of these

issues, its study has still proved interesting and that it has helped us improve our general understanding of quantum theory and in particular prompts us to sharpen and refine what we mean when we talk about quantum correlations.

Quantum discord has been investigated in numerous areas of quantum information [140, 141, 110]. Operational connections have been proposed in quantum state merging [73, 74], quantum communication [142], and bounding entanglement distribution [143] (which followed on from the result that one can distribute entanglement with separable states [144] which has been experimentally demonstrated [145, 146, 147]). Discord has also been investigated in metrology [85], and it has been claimed interferometric power can be quantified by discord [86, 87]. There has also been work on discord's role in noisy quantum computing [82]. In particular for DCQ1 [148] it was claimed that discord could account for the quantum advantage where entanglement could not [80]. However it was subsequently shown that the speedup is present in a case where there is vanishing discord [81], throwing significant doubt on its role as the responsible resource.

Discord can be a fairly subtle concept to get across. Essentially states with quantum discord are those whose quantum mutual information is larger than the maximum classical mutual information one could obtain via POVMs. Therefore to present an explicit definition we must first give the definition of classical mutual information

$$J(A:B) = H(A) + H(B) - H(AB), \tag{2.56}$$

where $H(A)$ is the Shannon entropy of the random variable A.

Similarly the definition of quantum mutual information is

$$I_\rho(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{2.57}$$

where $S(\rho_A)$ is the von-Neumann entropy of state ρ_A. Note that these are both symmetric quantities.

The most general quantum measurement we can perform is a POVM which is defined by a set of operators $\{F_i\}$ that are Hermitian $F_i^\dagger = F_i$, positive semi-definite $F_i \geq 0$, and sum to unity $\sum_i F_i = \mathbb{1}$. We now write the mutual information obtained via a POVM $\{A_a\}$ on A as

$$J_\rho(B|\{A_a\}) = S(\rho_B) - \sum_a p_a S(\rho_{B|a}), \tag{2.58}$$

where $p_a = \mathrm{Tr}_{AB}(A_a \rho_{AB})$, and $\rho_{B|a} = \mathrm{Tr}_A(A_a \rho_{AB})/p_a$. Note we use the subscript ρ to distinguish this from the classical mutual information. This mutual information is not a symmetric quantity.

We write the maximization of this quantity over all possible POVMs as

$$J_\rho(B|A) = \max_{\{A_a\}} J_\rho(B|\{A_a\}). \tag{2.59}$$

Now the asymmetric quantum discord is defined as

$$\delta_\rho(B|A) = I_\rho(A:B) - J_\rho(B|A). \tag{2.60}$$

We can understand this equation as defining the discord to be the difference between the total correlations and the maximum classical correlations obtainable via a POVM. Viewed another way we are essentially dividing up the total correlations into a quantum and classical part.

Note that Eq. (2.60) is asymmetric. To define the symmetric quantum discord we first write

$$J_\rho(A:B) = \max_{\{A_a\},\{B_b\}} J_\rho(\{A_a\}:\{B_b\}), \tag{2.61}$$

and then we have the symmetric discord

$$\delta_\rho(A:B) = I_\rho(A:B) - J_\rho(A:B). \tag{2.62}$$

To understand more clearly what these definitions capture it is useful to examine the states that do not have discord. To this end, one defines Classical Classical (CC), Classical Quantum (CQ), and Quantum Classical (QC) states, as states that can be written in the respective forms

$$\rho_{AB} = \begin{cases} \sum_{i,j} p_{ij} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B, & \text{CC}, \\ \sum_i p_i |i\rangle\langle i|_A \otimes |\rho_B^{(i)}, & \text{CQ}, \\ \sum_j p_j \rho_A^{(j)} \otimes |j\rangle\langle j|_B, & \text{QC}, \end{cases} \tag{2.63}$$

where the sets of states $\{|i\rangle\langle i|\}$, and $\{|j\rangle\langle j|\}$, are orthonormal sets.

Now a state has $\delta_\rho(B|A) = 0$ if and only if the state is CQ. Similarly we have $\delta_\rho(A|B) = 0$ if and only if the state is QC and $\delta_\rho(A:B) = 0$, if and only if the state is CC. The states that are not in any of these three sets have quantum discord and are sometimes referred to as quantum correlated states [149].

Since CC, CQ, and QC states are all separable it is clear all entangled states have discord, however not all discordant states are entangled. A simple example of a state that has discord but is not entangled is $|00\rangle\langle 00| + |++\rangle\langle ++|$.

There is as yet no resource theory of discord. It is known that the set of operations that do not create quantum correlations are the commutativity preserving operations [150], but these do not have a practical motivation and there are ad-

ditional difficulties such as the set of free states not being convex. This lack of a physically motivated resource view may indicate that discord captures too large a set of states to be in general viewed as a useful quantity.

This brings an end to our presentation on the fundamentals of quantum correlations, and with it an end to this chapter on fundamentals. We now turn to our first application of these tools.

Chapter 3

Anonymous Metrology

Anonymity is no excuse for stupidity

– Albert Einstein

3.1 Overview

Amongst their many uses, quantum correlations can be used to hide information non-locally [151, 152, 153, 154, 155]. Recent studies have investigated non-locally hiding computations [156], and when it is possible to mask quantum information [157]. In this chapter, we present the task of anonymous metrology, which involves encoding an initially unknown continuous parameter in a state whilst hiding where the encoding happened. We identify the quantum states that enable the task and separately treat the two cases of having a trustworthy or untrustworthy source of states. We term the resourceful states as weakly anonymous (WA) and strongly anonymous (SA) respectively, and give physical intuition for the distinction by demonstrating how SA states allow the encoding's location to be not just hidden but quantum mechanically delocalised. We derive general forms for the WA and SA states, using modes of translational asymmetry [158, 159] for the former, and for the latter showing equivalence to the entangled "maximally correlated states" [160] extended by degeneracy. We then determine the nature of their quantum correlations. As discussed in Chapter 2, quantum correlations exist in different "strengths", from discord [140] to full Bell nonlocality [161], and understanding their respective utilities remains to be fully explored. Whilst there are several works that demonstrate an operational use for discord [162, 73, 87, 85], our results additionally reveal an operational distinction between different types of quantum correlations. We find that WA states require a form of discord that we term *aligned discord*, while SA states require a stronger type of correlation, correspondingly termed *aligned entanglement*. This chapter draws from [163].

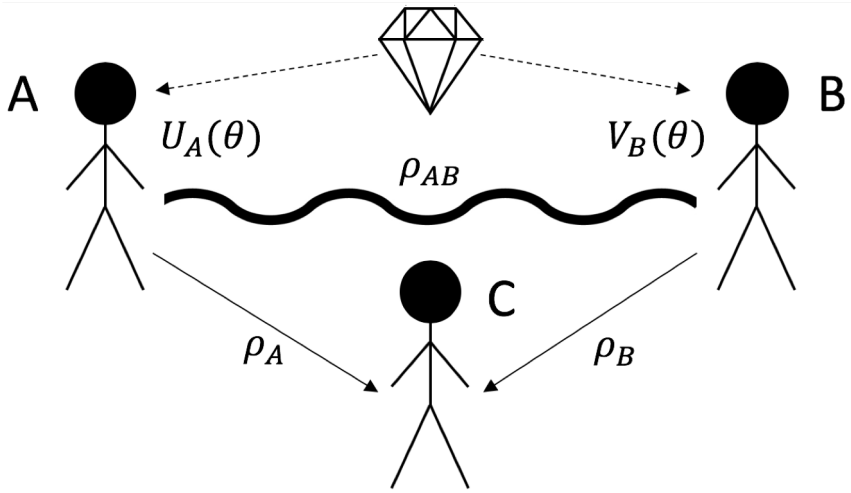

Figure 3.1: Illustration of the anonymous metrology task. Initially A and B share the state ρ_{AB}, and one of them is given the system to hide. They engineer U_A, or V_B, to encode θ into the shared state, then both halves are sent to C.

3.2 Defining anonymous metrology

We introduce the task of anonymous metrology with an example, illustrated in Fig. 3.1. Alice and Bob are in spatially separated laboratories, and one of them receives a system, the location of which they must keep hidden (e.g. some valuable diamond). Charlie wants them to probe it with a (finite-dimensional) quantum system to give him information about some initially unknown continuous parameter θ (such as a refractive index).

We make the following assumptions:

- If the object is in Alice's lab, it interacts with her system such that the latter undergoes a unitary transformation $U_A(\theta) = e^{-i\theta H_A}$ with some parameter-independent Hamiltonian H_A. (Otherwise, Alice's system is unchanged.) Similarly, Bob's system undergoes $V_B(\theta) = e^{-i\theta G_B}$ if the object is with him.

- Alice and Bob may use classical communication freely, but this is not secure. Charlie is also allowed to know the set-up of their labs, so he knows the quantum operations they apply.

- Their devices are unsecure, in that any measurement outcomes obtained are assumed to be available to Charlie. Note that they may implement any parameter-independent local unitaries without loss of security. Hence it is possible to effectively change the eigenbasis of H_A, G_B arbitrarily.

The task is to enable Charlie to estimate θ, but prevent him from learning where the hidden system is, i.e. the one who actually encoded θ into the quantum system must remain anonymous.

At first glance this task may seem impossible, since θ is initially unknown and Alice and Bob can only learn its value via untrusted local measurements. With Charlie accessing their measurement results we might fear that he will always be able to determine where the encoding is happening and thus the system's location. However it turns out that quantum resources allow them to succeed in this task.

We formalize this statement later, but we shall start by illustrating with a simple example using an entangled state. We allow Alice and Bob to request copies of a bipartite quantum state ρ_{AB} to use as a resource. Consider the situation when they choose the Bell state $|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$. Alice could apply $U_A(\theta) = e^{-i\theta|1\rangle_A\langle 1|}$, to produce $|\psi(\theta)\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + e^{-i\theta}|11\rangle_{AB})$, but similarly Bob could apply $V_B(\theta) = e^{-i\theta|1\rangle_B\langle 1|}$, and for all θ he produces the same state. This indicates the solution to their problem. The one who has the hidden system interacts their half of ρ_{AB} with it, to realize the relevant encoding unitary (they may need a rescaling such that $\theta \in [0, 2\pi)$), and then they both send their halves of $|\psi(\theta)\rangle_{AB}$ to Charlie. Given multiple copies Charlie can determine θ to arbitrary precision but cannot tell if it was U_A or V_B that changed the state, so cannot learn the system's location[1]. Clearly the Bell state is a resource for this task, but we shall show that a number of quantum states are, and in some cases they are not entangled.

3.2.1 Anonymity and encoding conditions

For anonymous metrology there are two relevant sets of useful states, and the choice between them depends on who provides the states for Alice and Bob.

First consider the situation where a trustworthy fourth party, who will never share information with Charlie, is sending quantum states to them. Alice and Bob should request copies of a state ρ_{AB}, that satisfies two conditions. First that they can find continuously parameterised unitaries $U_A(\theta), V_B(\theta)$ such that, for θ in some interval,

$$U_A(\theta)\rho_{AB}U_A^\dagger(\theta) = V_B(\theta)\rho_{AB}V_B^\dagger(\theta). \tag{3.1}$$

This is termed the *weak anonymity condition*, since it ensures that for a given parameter the same state is produced no matter who encoded it. The second condition is that from the same interval, different phases produce different states, i.e. given $\theta \neq \phi$ we have

$$U_A(\theta)\rho_{AB}U_A^\dagger(\theta) \neq U_A(\phi)\rho_{AB}U_A^\dagger(\phi). \tag{3.2}$$

We term this the *encoding condition* since it ensures that different parameters are mapped to different states, so in principle Charlie learns at least some information

[1]Note instead of sending states to Charlie they could perform a set of informationally complete local measurements, tomographically reconstruct the state and extract θ from it, but this is unnecessarily complicated and would generally be less efficient for learning θ.

about the parameter.

If however it is Charlie who is sending the states (or the fourth party is untrustworthy) then anonymity with these states is not assured. The most dangerous situation is when Charlie holds a third system and knows the pure state $|\psi\rangle_{ABC}$, but Alice and Bob only know the state $\rho_{AB} = \mathrm{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$. They can verify that they have been sent copies of ρ_{AB} by using a subset of the states they receive to perform metrology, but by doing this they cannot learn what Charlie holds. We also note that using a finite number of copies for metrology will inevitably lead to some finite error, and we address this later in section 3.5.1 where we show robustness for the protocol.

To maintain anonymity when Charlie could be holding a purification, they must be able to find unitaries such that $U_A(\theta)|\psi\rangle_{ABC} = V_B(\theta)|\psi\rangle_{ABC}$, up to an irrelevant global phase. We use this to derive a condition on the states ρ_{AB} that they can choose. We expand using the Schmidt decomposition to write the full state as $|\psi\rangle_{ABC} = \sum_j \lambda_j|\phi_j\rangle_{AB} \otimes |\chi_j\rangle_C$, in terms of an orthogonal product basis. We then project onto $|\chi_j\rangle_C$, and have $U_A(\theta)|\phi_j\rangle_{AB} = V_B(\theta)|\phi_j\rangle_{AB}$. Writing $\rho_{AB} = \sum_j |\lambda_j|^2|\phi_j\rangle_{AB}\langle\phi_j|$, and acting from the left with $U_A(\theta)$, we arrive at

$$U_A(\theta)\rho_{AB} = V_B(\theta)\rho_{AB}. \tag{3.3}$$

This is termed the *strong anonymity condition*. A more direct way to arrive at this is simply to start from $U_A(\theta)|\psi\rangle_{ABC} = V_B(\theta)|\psi\rangle_{ABC}$, act from the right with $\langle\psi|_{ABC}$, and then apply Tr_C . However, reversing the argument is most straightforwardly done by working with the kets from the Schmidt decomposition. We start from Eq. (3.3), write out $\rho_{AB} = \sum_j |\lambda_j|^2|\phi_j\rangle_{AB}\langle\phi_j|$, and act from the right with $|\phi_j\rangle_{AB}$ to find that $U_A(\theta)|\phi_j\rangle_{AB} = V_B(\theta)|\phi_j\rangle_{AB}$. Using this together with the fact that all purifications of $\rho_{AB} = \sum_j |\lambda_j|^2|\phi_j\rangle_{AB}\langle\phi_j|$, can be written as $|\psi\rangle_{ABC} = \mathcal{U}_{C'\to C}\sum_i \lambda_j|\phi_i\rangle_{AB}|i\rangle_{C'}$, where $\mathcal{U}_{C'\to C}$ is an isometry, we arrive at $U_A(\theta)|\psi\rangle_{ABC} = V_B(\theta)|\psi\rangle_{ABC}$.

Hence the state $|\psi\rangle_{ABC}$ has unitaries that satisfy $U_A(\theta)|\psi\rangle_{ABC} = V_B(\theta)|\psi\rangle_{ABC}$, if and only if $\rho_{AB} = \mathrm{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$ has unitaries that satisfy Eq. (3.3). It is clear that Eq. (3.3) implies Eq. (3.1), but not vice versa, so the condition (3.3) is stronger. With this we now formally establish appropriate terminology.

Definitions: A state ρ_{AB} is a *weakly anonymous* (WA) state if there exist unitaries $U_A(\theta) = e^{-i\theta H_A}$, and $V_B(\theta) = e^{-i\theta G_B}$ that satisfy the conditions given by Eq. (3.1) and (3.2). The subset of these that also satisfy the condition of Eq. (3.3) are *strongly anonymous* (SA) states.

For pure states the WA and SA conditions coincide, and furthermore a pure state is WA/SA if and only if it is entangled. To prove sufficiency we use the Schmidt decomposition to write $|\psi\rangle_{AB} = \sum_j \lambda_j|\phi_j\rangle_A \otimes |\chi_j\rangle_B$. Entangled states have

a Schmidt number of at least two, so without loss of generality we take $\lambda_0 \neq 0$ and $\lambda_1 \neq 0$. Now the unitaries $U_A(\theta) = e^{-i\theta|\phi_1\rangle\langle\phi_1|_A}$, and $V_B(\theta) = e^{-i\theta|\chi_1\rangle\langle\chi_1|_B}$, satisfy the conditions. Hence all pure entangled states are WA/SA. To prove entanglement is necessary consider a separable state $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$. The anonymity condition becomes $U_A(\theta)|\phi\rangle_A \otimes |\chi\rangle_B = |\phi\rangle_A \otimes V_B(\theta)|\chi\rangle_B$. Applying the ket $\langle\phi|_A$ we get $\langle\phi|U_A(\theta)|\phi\rangle|\chi\rangle_B = V_B(\theta)|\chi\rangle_B$, and projecting this equation onto itself we arrive at $|\langle\phi|U_A(\theta)|\phi\rangle| = 1$. Hence $U_A(\theta)$ only imparts an unobservable global phase, so violates the encoding condition. Therefore entanglement is necessary for pure states to be WA/SA.

For mixed states things are more complicated. We find that a resource distinction between WA and SA opens up, and we shall fully address this shortly. Before getting to this it is useful to present a different non-local task that helps to illustrate the significant operational distinction between the WA and SA cases.

Hiding the location of a system from Charlie somewhat resembles hiding which-path information. We can think of where the diamond goes as similar to where a particle goes in some interference experiment. We can formalise this intuition with the following setup. Consider Alice and Bob are tasked with measuring a system that is put in a superposition of going to Alice and Bob. Can they perform measurements without acquiring which-path information and thus without decohering the spatial superposition?

Formalizing this, we consider quantizing the path degree of freedom P of the system to be measured; it is put into some superposition $a|L\rangle_P + b|R\rangle_P$ of going left to Alice and right to Bob. They then want to use ρ_{AB} to measure this system that has been put into a spatial superposition, but without decohering said superposition. We consider the unitary that they jointly perform. Alice sets up her lab such that if the particle comes to her then she performs the controlled unitary U_A, and Bob does similarly with V_B (we leave the θ-dependence implicit). Together this gives the full unitary as

$$W = |L\rangle_P\langle L| \otimes U_A \otimes \mathbb{1}_B + |R\rangle_P\langle R| \otimes \mathbb{1}_A \otimes V_B, \tag{3.4}$$

We act with this on the initial state

$$\rho = (a|L\rangle + b|R\rangle)_P(a^*\langle L| + b^*\langle R|) \otimes \rho_{AB}. \tag{3.5}$$

Writing the new state as a matrix in the L, R basis we have

$$\rho' = \begin{bmatrix} |a|^2 U_A\rho_{AB}U_A^\dagger & ab^* U_A\rho_{AB}V_B^\dagger \\ a^*bV_B\rho_{AB}U_A^\dagger & |b|^2 V_B\rho_{AB}V_B^\dagger \end{bmatrix}. \tag{3.6}$$

We see that if

$$U_A \rho_{AB} U_A^\dagger = V_B \rho_{AB} V_B^\dagger = V_B \rho_{AB} U_A^\dagger, \tag{3.7}$$

then we can factor out and write our state in the product form

$$\rho' = (a|L\rangle + b|R\rangle)_P (a^*\langle L| + b^*\langle R|) \otimes U_A \rho_{AB} U_A^\dagger. \tag{3.8}$$

Factorisation is also possible with any phase factor on the right-hand side of Eq. (3.7), resulting in a relative phase appearing in system P. This phase may be absorbed into the definition of U_A.

The conditions of Eq. (3.7) are seen to be equivalent to the SA condition of Eq. (3.3). This emphasises the fact that the SA condition ensures no information exists on where the interaction took place. The interaction was delocalised by the correlations. We shall explore this phenomenon in much greater depth in the next chapter. For now we return to the task at hand and move from operational considerations to investigate the form of the resourceful states.

3.3 Form of useful states

3.3.1 Form of WA states

In order to arrive at a form for the WA states, it is useful to employ modes of translational asymmetry [158, 159]. Given our unitary action $\mathcal{U}_{A,\theta}(.) = U_A(\theta)(.)U_A^\dagger(\theta)$, a given state can be decomposed into modes as $\rho_A = \sum_\omega \rho_A^{(\omega)}$, where $\mathcal{U}_{A,\theta}(\rho_A^{(\omega)}) = e^{i\omega\theta}\rho_A^{(\omega)}$. This is akin to Fourier decomposition of a function. We can select out modes with the twirling superoperator

$$\mathcal{P}_A^\omega = \lim_{\theta_0\to\infty} \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} d\theta e^{-i\omega\theta} \mathcal{U}_{A,\theta}. \tag{3.9}$$

One can verify that $\mathcal{P}_A^\omega(\rho_A) = \rho_A^{(\omega)}$. We similarly define $\mathcal{P}_B^\omega$, using the twirling superoperator $\mathcal{V}_{B,\theta}(.) = V_B(\theta)(.)V_B^\dagger(\theta)$. Both of these twirling superoperators satisfy a relation of the form $\mathcal{U}_{A,\theta}\mathcal{P}_A^\omega = e^{i\omega\theta}\mathcal{P}_A^\omega$, together with a completeness relation $\sum_\omega \mathcal{P}_A^\omega = \mathbb{1}$. We can see the first of these holds by writing out

$$\begin{aligned}
\mathcal{U}_{A,\theta}\mathcal{P}_A^\omega &= \lim_{\theta_0\to\infty} \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} d\theta' e^{-i\omega\theta'} \mathcal{U}_{A,\theta}\mathcal{U}_{A,\theta'}, \\
&= \lim_{\theta_0\to\infty} \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} d\theta' e^{-i\omega\theta'} \mathcal{U}_{A,\theta'+\theta}, \\
&= e^{i\omega\theta} \lim_{\theta_0\to\infty} \frac{1}{2\theta_0} \int_{-\theta_0+\theta}^{\theta_0+\theta} d\theta' e^{-i\omega\theta'} \mathcal{U}_{A,\theta'} = e^{i\omega\theta}\mathcal{P}_A^\omega.
\end{aligned} \tag{3.10}$$

Additionally, the completeness relation follows by first noting that we can write $U_A(\theta) = \sum_n e^{in\theta} \sum_\alpha |n,\alpha\rangle\langle n,\alpha|$, and the space of linear maps will be spanned by $\{|n,\alpha\rangle\langle m,\beta|\}$. Then mode decomposition is simply viewed as separating out subspaces $\{|n+\omega,\alpha\rangle\langle n,\beta|\}$, labeled by a fixed ω. Since $\mathcal{P}_A^\omega$ picks out one of these subspaces, the sum over all ω returns the full space, hence $\sum_\omega \mathcal{P}_A^\omega = \mathbb{1}$.

Returning to the task at hand, we can rewrite the WA condition of Eq. (3.1) in terms of superoperators by simply multiplying by $\frac{e^{-i\omega\theta}}{2\theta_0}$, integrating $\int_{-\theta_0}^{\theta_0} d\theta$, and taking the limit $\theta_0 \to \infty$, to get

$$\mathcal{P}_A^\omega \rho_{AB} = \mathcal{P}_B^\omega \rho_{AB}. \tag{3.11}$$

We prove the converse by acting on this equation with $\mathcal{U}_{A,\theta}\mathcal{V}_{B,\theta}$, to get $e^{i\omega\theta}\mathcal{V}_{B,\theta}\mathcal{P}_A^\omega\rho_{AB} = e^{i\omega\theta}\mathcal{U}_{A,\theta}\mathcal{P}_B^\omega\rho_{AB}$. The $e^{i\omega\theta}$ terms cancel, and then summing over ω using the completeness relations we return to $\mathcal{V}_{B,\theta}\rho_{AB} = \mathcal{U}_{A,\theta}\rho_{AB}$. Hence Eq (3.11) captures the weak anonymity condition. Note that the encoding condition Eq. (3.2) now becomes the statement that there must be some $\omega \neq 0$ for which $\mathcal{P}_A^\omega \rho_{AB} \neq 0$.

From Eq. (3.11) we can explicitly write the form of the WA states. First we define

$$\rho_{AB}^{(\omega_1,\omega_2)} = \sum_{\substack{i,i',j,j',\\ E_{i'}=E_i+\omega_1,\\ E_{j'}=E_j+\omega_2}} c_{ii'jj'} |i\rangle_A\langle i'| \otimes |j\rangle_B\langle j'|, \tag{3.12}$$

where $H_A|i\rangle = E_i|i\rangle$, with H_A the Hamiltonian generator of $\mathcal{U}_{A,\theta}$, and similarly for B. Then the WA states are of the form

$$\rho_{AB} = \sum_\omega \rho_{AB}^{(\omega,\omega)}, \tag{3.13}$$

where we require non-zero terms for $\omega \neq 0$ so that encoding is possible.

This shows that WA states are those with correlated modes of asymmetry, which indicates a connection with the resource of quantum coherence [164]. We can view WA states as having correlated coherence in the eigenbasis of the unitaries. There is a formal similarity with the correlated coherence defined in [165, 166, 167, 168].

3.3.2 Form of SA states

We now derive the form of SA states. Here, working with modes of asymmetry is not as straightforward, which for completeness we now demonstrate.

Starting from the SA condition $U_A(\theta)\rho_{AB} = V_B(\theta)\rho_{AB}$, we act from the right with $V_B^\dagger(\theta)$. We then multiply by $\frac{e^{-i\omega\theta}}{2\theta_0}$, integrate $\int_{-\theta_0}^{\theta_0} d\theta$, and take the limit of $\theta_0 \to \infty$. Writing the superoperator $\mathcal{W}_{AB,\theta}(.) = U_A(\theta)(.)V_B^\dagger(\theta)$, we then define the

split twirling superoperator

$$\mathcal{P}^{\omega}_{AB} = \lim_{\theta_0 \to \infty} \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} d\theta e^{-i\omega\theta} \mathcal{W}_{AB,\theta}, \tag{3.14}$$

and we arrive at

$$\mathcal{P}^{\omega}_{AB}\rho_{AB} = \mathcal{P}^{\omega}_{B}\rho_{AB}. \tag{3.15}$$

To go the other way we first note that by summing over ω in this equation we find $\sum_{\omega} \mathcal{P}^{\omega}_{AB}\rho_{AB} = \rho_{AB}$, even though $\mathcal{P}^{\omega}_{AB}$ does not satisfy a completeness relation in general (note that we can have a completeness relation if we restrict the unitaries to share the same spectrum). This allows us to perform essentially the same argument as in the WA case. We note that $\mathcal{W}_{AB,\theta}\mathcal{P}^{\omega}_{AB} = e^{i\omega\theta}\mathcal{P}^{\omega}_{AB}$, and then act on Eq. (3.15) with $\mathcal{W}_{AB,\theta}\mathcal{V}_{B,\theta}$, cancel the $e^{i\omega\theta}$ terms and sum over ω to arrive back at the original SA conditions. However, the split twirling superoperator $\mathcal{P}^{\omega}_{AB}$ is not an established tool, and as such it is not as straightforward to go from these expressions to a form for the useful states, therefore we adopt a different approach.

Rearranging the anonymity condition of Eq. (3.3) to $(U_A(\theta) - V_B(\theta))\rho_{AB} = 0$, and taking matrix elements in the eigenbasis of the local unitaries, we obtain $(u_i(\theta) - v_{i'}(\theta))\langle ii'|\rho_{AB}|jj'\rangle = 0$. The non-zero matrix elements are those for which $u_i(\theta) = v_{i'}(\theta)$. Initially it is simplest to consider the non-degenerate case so that $u_i(\theta) \neq u_j(\theta), \forall i \neq j$ and similarly for the v_i. With this the largest set of non-zero matrix elements is achieved by pairing every $u_i(\theta)$ with a $v_{i'}(\theta)$ such that $u_i(\theta) = v_{i'}(\theta)$. Since relabeling is physically irrelevant we can write the non-zero matrix elements as $\langle ii|\rho_{AB}|jj\rangle$, and so we write the state as $\rho_{AB} = \sum_{i,j} \rho_{ij}|ii\rangle\langle jj|$. This is the form of so-called "maximally correlated" states [160]. Note that we need at least one non-zero off-diagonal $\rho_{ij} = \rho^*_{ji}$, to ensure that the encoding condition of Eq. (3.2) is satisfied.

We can lift the non-degeneracy restriction by introducing a new label, such that we write states that are degenerate under U_A as $|i\lambda\rangle$. We then write the form of SA states as

$$\rho_{AB} = \sum_{i,j,\lambda,\lambda',\mu,\mu'} \rho_{ij\lambda\lambda'\mu\mu'}|i\lambda, i\lambda'\rangle_{AB}\langle j\mu, j\mu'|. \tag{3.16}$$

Hence the SA states are a generalisation of the "maximally correlated" states, where we only include the entangled ones.

For completeness we can also generalise both cases to the multipartite case. To generalise the WA states is straightforward. We consider the case of n subsystems labeled $1, 2, ..., n$ and demand that the anonymity condition holds between every pair of subsystems. Now the WA condition in terms of the twirling superoperators

becomes

$$\mathcal{P}^{\omega}_{\alpha}\rho_{12...n} = \mathcal{P}^{\omega}_{\beta}\rho_{12...n}, \tag{3.17}$$

for all α, β in the set $1, 2, ..., n$. In direct analogy to the bipartite case we have

$$\rho^{(\omega_1,...,\omega_n)}_{1...n} = \sum_{\substack{i,i',...,k,k',\\ E_{i'}=E_i+\omega_1,...,\\ E_{k'}=E_k+\omega_n}} c_{ii'...kk'}|i\rangle_1\langle i'| \otimes ... \otimes |k\rangle_n\langle k'|, \tag{3.18}$$

and now we can write the WA states as

$$\rho_{12...n} = \sum_{\omega} \rho^{(\omega,\omega,...,\omega)}_{12...n}, \tag{3.19}$$

where as before we require non-zero terms for $\omega \neq 0$ so that encoding is possible.

For the SA state we generalise the condition to be $U_\alpha\rho_{12...n} = U_\beta\rho_{12...n}$, for all α, β in the set $1, 2, ..., n$. We apply the same approach as before, taking matrix elements in the eigenbasis of the unitaries. Working in the non-degenerate case with the same approach as before we find each of these equations gives us an expression for the allowed non-zero terms like $|iikl...n\rangle\langle jjk'l',,,n|$, and $|ikil...n\rangle\langle jk'jl',,,n|$. Taking all such conditions together we see the only non-zero terms allowed are of the form $|iii...i\rangle\langle jjj...j|$. And so we have the form

$$\rho_{12...n} = \sum_{i,j} \rho_{ij}|ii...i\rangle\langle jj...j|. \tag{3.20}$$

One can then extend to allow for degeneracy in the same way as before by introducing degeneracy labels.

This concludes our discussion on the form of the useful states. Having established these expressions for WA and SA states, we can now turn to discuss their quantum correlations.

3.4 Quantum correlations required

For the anonymity tasks, we shall show that information is being hidden by the quantum correlations of the states. The main correlations that prove relevant are entanglement and discord, as described in detail in Section 2.4. To briefly recap the important points, both entanglement and discord can be defined mathematically by specifying forms for the correlated states. A bipartite state is entangled if it cannot be written in the separable form $\rho_{AB} = \sum_i p_i\rho^{(i)}_A \otimes \rho^{(i)}_B$. A bipartite state is deemed discordant if, for some local basis, it cannot be written in any of the following three forms $\rho_{AB} = \sum_{i,j} p_{ij}|i\rangle\langle i|_A \otimes |j\rangle\langle j|_B$, $\rho_{AB} = \sum_i p_i|i\rangle\langle i|_A \otimes \rho_{B|i}$,

and $\rho_{AB} = \sum_j p_j \rho_{A|j} \otimes |j\rangle\langle j|_B$, termed Classical-Classical (CC), Classical-Quantum (CQ), and Quantum-Classical (QC) respectively. It follows from these definitions that entangled states are a subset of discordant states.

As we shall show, the WA and SA states form subsets of the known sets of correlated states. We therefore use the terms aligned discord and aligned entanglement for the WA and SA resources respectively. For aligned discord we establish that discord is necessary but not sufficient, and entanglement is neither necessary nor sufficient. For aligned entanglement we show entanglement is necessary but not sufficient. This is illustrated in Fig. 3.2.

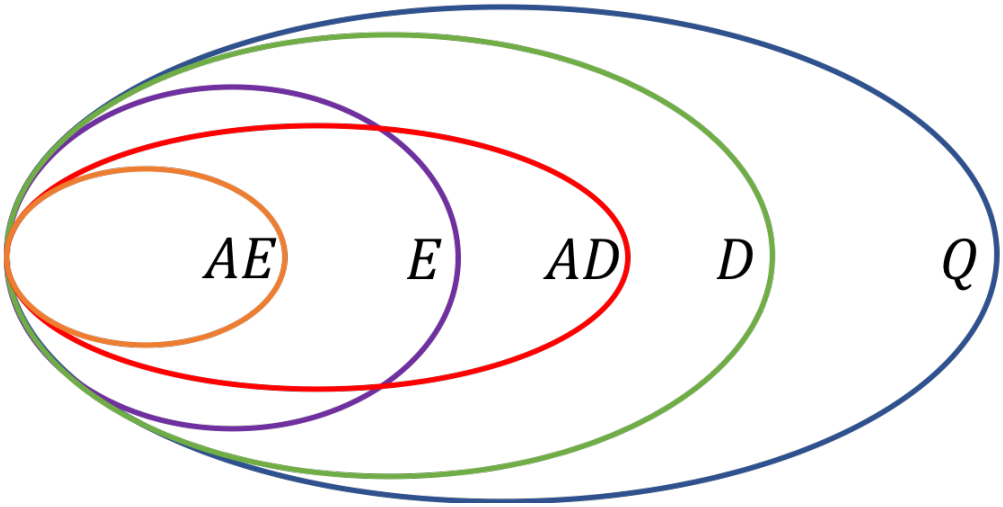

Figure 3.2: Schematic summary of the relations between all quantum states (Q), and those that are discordant (D), aligned discordant (AD), entangled (E), and aligned entangled (AE).

3.4.1 WA Hamiltonian condition

Before proving these results, we recast the WA conditions in terms of Hamiltonians, to describe families of unitaries that encode a continuous parameter. If we were to demand Eq. (3.1) and (3.2) without enforcing the continuous parameter requirement, then an anonymous encoding would be given by the classically correlated state $\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, with bit flip unitaries $U_A = \sigma_A^x$, and $V_B = \sigma_B^x$. Note that for the SA case there is no such distinction.

Writing $U_A(\theta) = e^{-i\theta H_A}$, and $V_B(\theta) = e^{-i\theta G_B}$, we see that the weak anonymity condition of Eq. (3.1) is equivalent to requiring that there exist local Hermitian operators H_A, G_B, for which

$$[H_A - G_B, \rho_{AB}] = 0. \tag{3.21}$$

Similarly the encoding condition of Eq. (3.2), becomes

$$[H_A, \rho_{AB}] \neq 0. \tag{3.22}$$

We can work with these conditions to intrinsically restrict to continuous parameter encodings.

3.4.2 Aligned discord

To prove discord is necessary we start with a CQ state which we write out as $\rho_{AB} = \sum_a \lambda_a |\psi_a\rangle\langle\psi_a| \otimes \rho_{B|a}$, and take that for some choice of Hermitian operators H_A and G_B we have $[H_A, \rho_{AB}] = [G_B, \rho_{AB}]$. We project A onto $|\psi_c\rangle\langle\psi_c|$ and use the fact that $\langle\psi_c|[H_A, |\psi_a\rangle\langle\psi_a|]|\psi_c\rangle = 0$ to get $\lambda_c[G_B, \rho_{B|c}] = 0$, $\forall c$. From this we find that $[H_A, \rho_{AB}] = 0$. Hence we can only satisfy the anonymity condition if we violate the encoding condition. Essentially the same argument works for a QC state, so it is true for all non-discordant states. This proves that discord is necessary for WA states.

The fact entanglement is not necessary is proved with the Werner state [68] $\rho_W = a|\psi^-\rangle\langle\psi^-| + \frac{1-a}{4}\mathbb{1}$. The WA conditions can always be satisfied using $e^{-i\theta|1\rangle\langle 1|}$, except when $a = 1$, but the state is not entangled for values of $a \leq \frac{1}{3}$.

To prove that neither entanglement nor discord are sufficient we use a two-qubit example. We construct a state that is entangled and discordant but is not of the appropriate form. We therefore first need to discuss the explicit forms of WA states for two qubits.

For a two qubit bipartite state ρ_{AB}, where the eigenvalues of $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, are non-degenerate and similarly for ρ_B, such that they have unique eigenvectors, then the WA states are those that, when written in the eigenbasis of their reduced density matrices, have the form

$$\rho = \begin{bmatrix} \alpha & 0 & 0 & \epsilon \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \epsilon^* & 0 & 0 & \delta \end{bmatrix} \text{ or } \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & \epsilon & 0 \\ 0 & \epsilon^* & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix}, \tag{3.23}$$

where $\epsilon \neq 0$. These two forms are related by a relabelling of the eigenbasis of ρ_A.

We can arrive at this by considering the general form given in Eq. (3.11), and here we present a constructive proof from the Hamiltonian WA conditions of Eq. (3.21) and (3.22). First note the local Hamiltonian H_A must share an eigenbasis with ρ_A and similarly for G_B. Using this local eigenbasis we take matrix elements of Eq. (3.21) to get $(h_i - g_j - h_{i'} + g_{j'})\langle ij|\rho_{AB}|i'j'\rangle = 0$, where h, g are the local Hamiltonian eigenvalues. From this equation we see that the diagonal terms $\langle ij|\rho_{AB}|ij\rangle$ are

unconstrained. To see what other terms are free to be non-zero we need to consider when we can make $(h_i - g_j - h_{i'} + g_{j'}) = 0$.

Since we have 2 qubits we have 4 eigenvalues to set: h_0, h_1, g_0, g_1. The encoding condition Eq. (3.22) enforces $h_0 \neq h_1$ and $g_0 \neq g_1$, and that ρ_{AB} has at least one off-diagonal term, since $H_A \otimes \mathbb{1}_B$ is diagonal and diagonal matrices commute with each other. We now have two options, choose $h_0 = g_0$, and $h_1 = g_1$, or $h_0 = g_1$, and $h_1 = g_0$. The first case allows the terms $\langle 00|\rho_{AB}|11\rangle$, and $\langle 11|\rho_{AB}|00\rangle$, to be non-zero and the second case allows $\langle 01|\rho_{AB}|10\rangle$, and $\langle 10|\rho_{AB}|01\rangle$. Putting this all together we arrive at the forms stated in Eq. (3.23).

Purely for interest's sake, there is also a nice alternative way to arrive at Eq. (3.23), using tools from asymmetry theory [169]. First we note that we can write the WA anonymity condition as a symmetry constraint by using the G-twirling superoperator, which we define as

$$\mathcal{G}(\rho_{AB}) = \lim_{\theta_0 \to \infty} \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} d\theta \, \mathcal{U}_{A,\theta} \otimes \mathcal{V}_{B,\theta}^{\dagger} \rho_{AB}. \tag{3.24}$$

With this we can write the WA condition as

$$\mathcal{G}(\rho_{AB}) = \rho_{AB}, \tag{3.25}$$

Taking the two-qubit case we write out the unitaries

$$U_A(\theta) \otimes V_B(\theta)^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{ia\theta} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{-ib\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-ib\theta} & 0 & 0 \\ 0 & 0 & e^{ia\theta} & 0 \\ 0 & 0 & 0 & e^{i(a-b)\theta} \end{bmatrix}. \tag{3.26}$$

We then find

$$\mathcal{U}_{A,\theta} \otimes \mathcal{V}_{B,\theta}^{\dagger} \rho_{AB} = \begin{bmatrix} 1 & e^{ib\theta} & e^{-ia\theta} & e^{-i(a-b)\theta} \\ e^{-ib\theta} & 1 & e^{-i(a+b)\theta} & e^{-ia\theta} \\ e^{ia\theta} & e^{i(a+b)\theta} & 1 & e^{ib\theta} \\ e^{i(a-b)\theta} & e^{ia\theta} & e^{-ib\theta} & 1 \end{bmatrix} * \rho_{AB}, \tag{3.27}$$

where $*$ denotes the entrywise product in the computational basis. When we integrate to perform the G-twirling, the two choices of either $a = b$, or $a = -b$, give the two forms of viable density matrix, as in Eq. (3.23).

The facts that entanglement and discord are not sufficient are proved by a two qubit example that is not of the form given in Eq. (3.23) but is entangled (and

therefore discordant). First we define

$$\begin{aligned}\rho_1 &= (\sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle)(\sqrt{a}\langle 00| + \sqrt{1-a}\langle 11|) \\ \rho_2 &= (\sqrt{b}|01\rangle + \sqrt{1-b}|10\rangle)(\sqrt{b}\langle 01| + \sqrt{1-b}\langle 10|)\end{aligned} \tag{3.28}$$

Now we can form $\rho = m\rho_1 + (1-m)\rho_2$. Choosing $a, b \neq \frac{1}{2}$ ensures we have unique local eigenvectors. Picking the fairly arbitrary values $a = 0.45, b = 0.4, m = 0.35$, we find the matrix in its local eigenbasis is approximately

$$\begin{bmatrix} 0.2 & 0 & 0 & 0.2 \\ 0 & 0.3 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0 \\ 0.2 & 0 & 0 & 0.1 \end{bmatrix}, \tag{3.29}$$

where we are quoting values only to one significant figure. This is not in one of the viable forms given in Eq. (3.23). Taking the partial transpose it has a negative eigenvalue. Thus by the Peres-Horodecki criterion [120, 121] the state is entangled. Since entangled states are always discordant, this example proves neither discord nor entanglement are sufficient. Furthermore, if one wanted to show this result just for discord, the simpler example $\frac{1}{2}(|00\rangle\langle 00| + |++\rangle\langle ++|)$ suffices, since this state is discordant but in the local eigenbasis can be written as

$$\begin{bmatrix} 0.729 & 0 & 0 & 0.125 \\ 0 & 0.125 & 0.125 & 0 \\ 0 & 0.125 & 0.125 & 0 \\ 0.125 & 0 & 0 & 0.021 \end{bmatrix}, \tag{3.30}$$

where here we are quoting to three significant figures to keep the trace normalization apparent. As in the last case, this is not in one of the viable forms given by Eq. (3.23). This concludes the proof.

3.4.3 Aligned entanglement

The fact entanglement is necessary for SA states follows from applying the Peres-Horodecki criterion [120, 121] to states of the form presented in Eq. (3.16). The fact that it is not sufficient also follows, since not all entangled states can be written in this form, for example the Werner state. We now present a detailed account of this.

Starting from the form given in Eq. (3.16) we find that entanglement is necessary for aligned entanglement. To see this we write $\rho_{AB} = \sum_{i,j} \alpha_{ij}|ii\rangle_{AB}\langle jj|$, where we have dropped the degeneracy labels. In order to not violate the encoding condition of Eq. (3.2) we must have a k and l for which $k \neq l$, and $\alpha_{kl} = e^{i\phi_{kl}}|\alpha_{kl}| \neq 0$.

We can apply the local unitary $e^{-i\phi_{kl}|k\rangle_A\langle k|}$, without affecting the entanglement, and we absorb the phases into the α_{kl}, and α_{lk}, such that now they are both real and positive. We then perform the partial transpose to get $\rho_{AB}^{T_B} = \sum_{i,j} \alpha_{ij}|ij\rangle_{AB}\langle ji|$, and see that the state $|kl\rangle - |lk\rangle$, is an eigenvector, with the negative eigenvalue $-|\alpha_{kl}|$. Hence by the Peres-Horodecki criterion [120, 121], aligned entangled states are always entangled.

We can also prove this result that entanglement is necessary for SA states without relying on the form given in Eq. (3.16). To do this we first state and prove the following useful lemma.

Given a probability distribution $\{p_j\}$, and a set of complex numbers $\{Z_j\}$, where $|Z_j| \leq 1$. Then $\sum_j p_j Z_j = e^{i\xi}$, if and only if $Z_j = e^{i\xi}, \forall j$ where we only consider j for which $p_j \neq 0$. To see this is true, note any complex number on the boundary of the unit disk cannot be expressed as a convex combination of other complex numbers in the unit disk. This should be convincing but for completeness we give a more formal proof.

If $Z_j = e^{i\xi}$, then clearly $\sum_j p_j Z_j = e^{i\xi}$, so we now just need to prove the other direction. For a set of non-zero complex numbers ζ_j, we have $|\sum_j \zeta_j| \leq \sum_j |\zeta_j|$, with equality if and only if $\arg(\zeta_j) = \arg(\zeta_k), \forall j, k$. This is just a restatement of the polygon inequality (generalisation of the triangle inequality) for complex numbers. We now write $\sum_j p_j Z_j = \sum_j \zeta_j$, and from this we have

$$\sum_j |\zeta_j| = \sum_j p_j |Z_j| \leq \sum_j p_j = 1, \tag{3.31}$$

where we used $|Z_j| \leq 1$, so the equality holds when $|Z_j| = 1 \forall j$. We now have

$$|\sum_j \zeta_j| \leq \sum_j |\zeta_j| \leq 1. \tag{3.32}$$

However, we know $|\sum_j \zeta_j| = |e^{i\xi}| = 1$, and so these three terms must in fact all be equal. The conditions on the inequalities then tell us every Z_j has unit magnitude and the same fixed argument. This shared argument must be ξ and therefore we have the $Z_j = e^{i\xi}$ as claimed.

We can now use this lemma to prove entanglement is necessary for SA states. From the anonymity condition of Eq. (3.3) we obtain $\mathrm{Tr}(U_A(\theta)V_B^\dagger(\theta)\rho_{AB}) = 1$. With a separable state $\rho_{AB} = \sum_j p_j \rho_A^{(j)} \otimes \rho_B^{(j)}$, the condition becomes

$$\sum_j p_j \mathrm{Tr}(U_A(\theta)\rho_A^{(j)})\mathrm{Tr}(V_B^\dagger(\theta)\rho_B^{(j)}) = 1. \tag{3.33}$$

Since $|\mathrm{Tr}(U_A(\theta)\rho_A^{(j)})| \leq 1$, and $|\mathrm{Tr}(V_B(\theta)\rho_B^{(j)})| \leq 1$, we can use the above stated

lemma to conclude that $\mathrm{Tr}(U_A(\theta)\rho_A^{(j)})\mathrm{Tr}(V_B^\dagger(\theta)\rho_B^{(j)}) = 1$. This can only be true if $|\mathrm{Tr}(U_A(\theta)\rho_A^{(j)})| = 1$, so $\mathrm{Tr}(U_A(\theta)\rho_A^{(j)}) = e^{i\phi(j)}$. Using the eigendecomposition $\rho_A^{(j)} = \sum_k q_k^{(j)}|\psi_k^{(j)}\rangle\langle\psi_k^{(j)}|$, this becomes $\sum_k q_k^{(j)}\langle\psi_k^{(j)}|U_A(\theta)|\psi_k^{(j)}\rangle = e^{i\phi_A(j)}$. Applying the same lemma again we have that $\langle\psi_k^{(j)}|U_A(\theta)|\psi_k^{(j)}\rangle = e^{i\phi_A(j)}$. This implies $U_A(\theta)|\psi_k^{(j)}\rangle = e^{i\phi_A(j)}|\psi_k^{(j)}\rangle$. We now see that this violates the encoding condition since

$$U_A(\theta)\rho_{AB}U_A^\dagger(\theta) = \sum_{j,k} p_j q_k^{(j)} e^{i\phi_A(j)}|\psi_k^{(j)}\rangle\langle\psi_k^{(j)}|e^{-i\phi_A(j)} \otimes \rho_B^{(j)} = \rho_{AB}. \tag{3.34}$$

We have therefore shown that all separable states cannot satisfy the SA conditions and hence entanglement is necessary.

To prove that entanglement is not sufficient we use the Werner state example $\rho_W = a|\psi^-\rangle\langle\psi^-| + \frac{1-a}{4}\mathbb{1}$, and show the only way that this can satisfy the strong anonymity condition of Eq. (3.3) is if it violates the encoding condition. The anonymity condition becomes

$$aU_A(\theta)|\psi^-\rangle\langle\psi^-| + \frac{1-a}{4}U_A(\theta) = aV_B(\theta)|\psi^-\rangle\langle\psi^-| + \frac{1-a}{4}V_B(\theta). \tag{3.35}$$

We now act from the right with $|\psi^-\rangle$, and we get through to $U_A(\theta)|\psi^-\rangle = V_B(\theta)|\psi^-\rangle$. Substituting this into the original condition we have $U_A(\theta) \otimes \mathbb{1}_B = I_A \otimes V_B(\theta)$. This implies $U_A(\theta) = \mathbb{1}_A$, and hence we must violate the encoding condition. This is true for $0 < a < 1$, and thus is true for values of a for which the state is entangled. Hence entanglement is not sufficient for aligned entanglement.

Note that the Werner state also reveals that SA states are not simply the entangled WA states, but a strict subset of them. It also proves that steerability and Bell non-locality are not sufficient for aligned entanglement since for $a > \frac{1}{2}$ the state is steerable [134], and for $a > \frac{1}{\sqrt{2}}$ it is known to be Bell non-local [170].

3.5 Using non-ideal states

3.5.1 Robustness

The anonymous metrology protocol has robust anonymity. If Alice and Bob verify that their state ρ_{AB} is close to a WA/SA state σ_{AB}, in terms of trace distance $T(\rho_{AB}, \sigma_{AB}) \leq \epsilon$, then this bounds Charlie's ability to correctly guess who applied the unitary. For the WA case we have $T(U_A\rho_{AB}U_A^\dagger, V_B\rho_{AB}V_B^\dagger) \leq 2\epsilon$, and for the SA case we have $T(U_A|\psi\rangle_{ABC}, V_B|\psi\rangle_{ABC}) \leq 2\sqrt{\epsilon - \epsilon^2}$. We now prove these results.

First we consider the WA case. Alice and Bob can use a random selection of the copies they receive to verify that the state they are using ρ_{AB} is close, in terms of

trace distance, to a WA state σ_{AB}. Formally we say they verify that

$$T(\rho_{AB}, \sigma_{AB}) \leq \epsilon. \tag{3.36}$$

We now show that this leads to a bound on $T(U_A \rho_{AB} U_A^\dagger, V_B \rho_{AB} V_B^\dagger)$. This quantifies Charlie's ability to distinguish whether Alice or Bob applied their unitary, since the maximal probability of correctly guessing the state is

$$P_{WA} = \frac{1}{2}(1 + T(U_A \rho_{AB} U_A^\dagger, V_A \rho_{AB} V_A^\dagger)), \tag{3.37}$$

as discussed in Section 2.3.6.

Starting from Eq. (3.36) we use the fact that trace distance is preserved under unitaries to write

$$T(U_A \rho_{AB} U_A^\dagger, \sigma'_{AB}) \leq \epsilon, \tag{3.38}$$

$$T(V_B \rho_{AB} V_B^\dagger, \sigma'_{AB}) \leq \epsilon, \tag{3.39}$$

where we have defined $\sigma'_{AB} = U_A \sigma_{AB} U_A^\dagger = V_B \sigma_{AB} V_B^\dagger$, using the fact σ_{AB} is a WA state. We now use the triangle inequality $T(A, C) \leq T(A, B) + T(B, C)$, to arrive at

$$T(U_A \rho_{AB} U_A^\dagger, V_B \rho_{AB} V_B^\dagger) \leq 2\epsilon. \tag{3.40}$$

We now turn to the SA case. Again Alice and Bob use some of their states to establish Eq. (3.36). However, for the SA case we need to consider distinguishability for the fully purified states, so we need to bound $T(U_A(\theta)|\psi\rangle_{ABC}, V_B(\theta)|\psi\rangle_{ABC})$.

First consider the fidelity between the two states $U_A|\psi\rangle_{ABC}$, and $V_B|\psi\rangle_{ABC}$. This can be written out as

$$F(U_A|\psi\rangle_{ABC}, V_B|\psi\rangle_{ABC}) = |\langle\psi|_{ABC} V_B^\dagger U_A |\psi\rangle_{ABC}| = |\mathrm{Tr}(V_B^\dagger U_A \rho_{AB}))|. \tag{3.41}$$

We then can proceed to write

$$\begin{aligned}
1 - |\mathrm{Tr}(V_B^\dagger U_A \rho_{AB})| &\leq |1 - \mathrm{Tr}(V_B^\dagger U_A \rho_{AB})|, \\
&= |\mathrm{Tr}(\sigma_{AB} - V_B^\dagger U_A \rho_{AB})|, \\
&= |\mathrm{Tr}(V_B^\dagger U_A (\sigma_{AB} - \rho_{AB}))|, \\
&\leq \mathrm{Tr}(|V_B^\dagger U_A (\sigma_{AB} - \rho_{AB})|), \\
&= \mathrm{Tr}(|\sigma_{AB} - \rho_{AB}|), \\
&= 2T(\rho_{AB}, \sigma_{AB}),
\end{aligned}$$

where in the third line we used the fact that σ_{AB} is an SA state.

We now have

$$|\mathrm{Tr}(V_B^\dagger U_A \rho_{AB})| \geq 1 - 2T(\rho_{AB}, \sigma_{AB}) \geq 1 - 2\epsilon, \tag{3.42}$$

and taking this together with Eq. (3.41) shows that we have a bound on the fidelity of

$$F(U_A|\psi\rangle_{ABC}, V_B|\psi\rangle_{ABC}) \geq 1 - 2\epsilon. \tag{3.43}$$

We now change this to an inequality in terms of the trace distance by using the fact that for pure states $T(|\psi\rangle, |\phi\rangle) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)^2}$, to arrive at

$$T(U_A|\psi\rangle_{ABC}, V_B|\psi\rangle_{ABC}) \leq 2\sqrt{\epsilon - \epsilon^2}. \tag{3.44}$$

Pulling this all together we have shown that given a single copy, the probabilities for Charlie to correctly guess where the encoding happened are bounded [171], as $P_{WA} \leq \frac{1}{2} + \epsilon$, and $P_{SA} \leq \frac{1}{2} + \sqrt{\epsilon - \epsilon^2}$.

It is worth noting that in general Alice and Bob send multiple copies, which Charlie could use to improve his guess. However using the property of the fidelity that $F(\rho_1^{\otimes n}, \rho_2^{\otimes n}) = F(\rho_1, \rho_2)^n$, and the Fuchs-van de Graff inequality $1 - F \leq T \leq \sqrt{1 - F^2}$ [105], we find that

$$T(\rho_1^{\otimes n}, \rho_2^{\otimes n}) \leq \sqrt{1 - [1 - T(\rho_1, \rho_2)]^{2n}}. \tag{3.45}$$

Hence robustness for many copies follows, since a bound for the single copy case implies a bound for the multiple copy case.

3.5.2 General figure of merit

Following similar considerations, we now define a general figure of merit for any bipartite state ρ used for anonymous metrology. For given Hamiltonians H, G, we can bound the increase in Charlie's guessing probability to δ by limiting the number of copies sent. Using Fuchs-van de Graff we bound the probability gain with $\delta \leq \frac{1}{2}\sqrt{1 - \min_\theta F(\rho_H(\theta), \rho_G(\theta))^{\otimes n}}$, where $\rho_H(\theta)$ and $\rho_G(\theta)$ are the states Charlie is trying to discriminate between and the minimisation over θ is to ensure anonymity for the whole range of parameter values. From this we then see that to maintain this level of anonymity, Alice and Bob must keep the number of copies they send to be no more than

$$n_\delta = \frac{\log(1 - (2\delta)^2)}{2 \log \min_\theta F(\rho_H(\theta), \rho_G(\theta))}. \tag{3.46}$$

The usefulness of a state in parameter estimation may be quantified by the quantum Fisher information (QFI) $\mathcal{F}$ [106], which sets a lower limit on the uncer-

tainty $\Delta\theta$ with which a parameter θ can be estimated, via the quantum Cramér-Rao bound: $\Delta\theta \geq (n\mathcal{F})^{-1/2}$ for n measurements. We are considering the unitary encoding $\rho_H(\theta) = U_A(\theta)\rho U_A^\dagger(\theta)$, so we denote the QFI as $\mathcal{F}(\rho; H_A)$. Since the QFI can depend on which party does the encoding, we define the average

$$\bar{\mathcal{F}}(\rho; H, G) = \frac{1}{2}(\mathcal{F}(\rho; H_A) + \mathcal{F}(\rho; G_B)). \tag{3.47}$$

We combine $\bar{\mathcal{F}}$, and n_δ to form the figure of merit $n_\delta\bar{\mathcal{F}}$. This captures the amount of parameter information that can be transferred to Charlie with δ anonymity. We identify the state-dependent part as the figure of merit

$$M(\rho; H, G) = \frac{\bar{\mathcal{F}}(\rho; H, G)}{-\log \min_\theta F(\rho_H(\theta), \rho_G(\theta))}. \tag{3.48}$$

The larger $M(\rho; H, G)$, the better a state ρ is for anonymous metrology with Hamiltonians H, G.

For a function purely of the state, we must maximize over all possible choices of Hamiltonian. In order for this to be well-defined, we need to bound the spectra – therefore we define

$$M(\rho) = \min_{H,G,\, \|H\|_{op}=\|G\|_{op}=1} M(\rho; H, G), \tag{3.49}$$

where $\|\cdot\|_{op}$ is the standard operator norm, defined as

$$\|A\|_{op} = \inf\{c \geq 0 : \|Av\| \leq c\|v\| \quad \forall v \in V\}. \tag{3.50}$$

3.6 Conclusions

We have shown that quantum mechanics enables a metrology protocol whereby a continuous parameter may be determined whilst hiding the location where it was encoded. We established the nature of the quantum correlations responsible for this phenomenon, according to the level of privacy required. With a trusted source of states, discord is needed, while entanglement provides privacy with an untrusted source. The useful correlations have a particular symmetry, and are named aligned discord and aligned entanglement respectively.

We note that the difference between the WA and SA tasks resembles device-dependent versus device-independent cryptography, where the former requires discord and the latter entanglement [172], specifically Bell non-locality. However the tasks are clearly distinct, and importantly we note that they do not produce the same sets of resourceful states. This is most readily seen by considering the aligned

discord states that are useful for the WA task. For these states discord and entanglement are not sufficient, whereas for the device-dependent quantum key distribution task entanglement is sufficient.

The fact the WA case only requires discord reduces the technological challenge in realizing protocols, with discord being relatively robust to noise [173, 162]. The SA states are more practically challenging, but bring greater operational power. It is also noteworthy that there is a connection between aligned discord/entanglement and quantum coherence. This is indicated by the redefinition of the WA states in terms of modes of asymmetry given in Eq. (3.11). This suggests a potential link between the anonymity resources and the resource of quantum coherence [174, 175, 176, 177]. The anonymity resources should arguably be viewed as a hybrid of coherence and correlation. One could describe it as correlated coherence, though this appears distinct from the correlated coherence of [165, 166, 167, 168].

Our results highlight an operational boundary within the hierarchy of quantum correlations, providing a novel nonclassical task whereby different correlations are at play depending on the desired level of anonymity. With this established, we now proceed to depart on a tangent, namely the noted point of further interest that SA states allowed measurements to be delocalised as demonstrated by the lack of decoherence of spatial superpositions. In the next chapter we shall follow this thread, reframe the problem in a more general manner which allows the consideration of all states, and develop a deeper understanding of this highly non-classical phenomenon.

Chapter 4

Quantum Delocalised Interactions

Location, location, location.
– Incorrectly attributed to Lord Harold Samuel

4.1 Overview

In this chapter, we study quantum delocalised-interactions, whereby information encoded using non-locally superposed quantum states, is recorded via local interactions whilst causing less disturbance than would be classically possible. This indicates that such interactions cannot be said to happen at a single location. This stands in stark contrast to our classical intuition that interactions happen at unique places, we just might not know where. This non-classical phenomenon has in fact been instrumental in enabling certain quantum protocols [178, 163]. In order to characterise delocalised interactions quantitatively, we formulate quantum games and study two particular instances. We establish that the win probabilities of these games are upper bounded in terms of the concurrence for two-qubit states [124, 125], and the bounds can be saturated for any pure state and a broad class of mixed states. This provides an operational meaning of the concurrence, which has been a widely studied measure of entanglement but is often viewed as a mathematical device. We find that the capacity for non-classical teleportation fidelity [47] guarantees the capacity for non-classical performance in a delocalised-interaction game. Furthermore, we consider the possibility of establishing an information theoretic notion of delocalised-interactions by utilising the trace distance, and compare this with the formulated games. There are connections between the two in limiting cases but numerics indicate that in general the derived inequality does not always capture delocalised-interactions as established by the games. We also demonstrate the games using an IBM quantum processor, achieving non-classical performance. This chapter draws from [179].

4.2 Double slit illustration

First we illustrate what we mean by delocalised-interactions using the familiar double slit thought experiment. Suppose a game where Charlie (C) either sends a particle through the double slit or does not. Alice (A) standing at one slit together with Bob (B) standing at the other, team up to guess whether C sent the particle or not, without destroying the interference pattern.

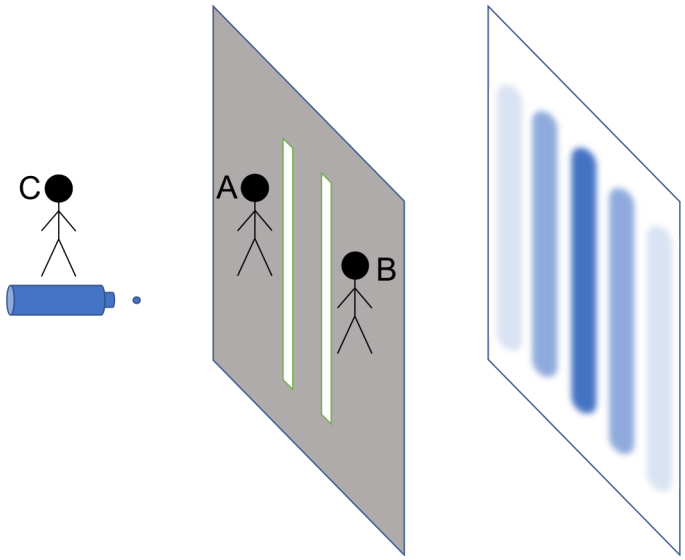

Figure 4.1: Illustration of the imagined double-slit setup. A and B must try to record when C sends through particles, whilst attempting to not damage the interference pattern.

To win this game, A and B should be able to distinguish between two different states passing through the double-slit, namely a vacuum state $|0\rangle$ and a superposition between spatially separated states $|\psi_L\rangle + |\psi_R\rangle$, by locally interacting with the particle. Note these states can also be written as $|00\rangle_{A_pB_p}$ and $\frac{1}{\sqrt{2}}(|10\rangle_{A_pB_p} + |01\rangle_{A_pB_p})$, where A_p and B_p are the particle Fock spaces at A and B's locations. If A and B only share classical resources (non-entangled states), then within the game a perfect record of the existence of the particle is impossible due to the complementarity principle of quantum mechanics. There will be a trade-off, the more information A and B locally record on whether a particle is present, the more they destroy the interference between the different paths by disturbing the superposition state $|\psi_L\rangle + |\psi_R\rangle$ [180]. On the other hand, if A and B share copies of a Bell state, for example, $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$, then they can produce a perfect record of when there were particles without affecting the interference pattern. To do this, A and B set up their local interactions such that the particle flips the local state as $|0\rangle_{A(B)}|1\rangle_{A_p(B_p)} \to |1\rangle_{A(B)}|1\rangle_{A_p(B_p)}$ and $|1\rangle_{A(B)}|1\rangle_{A_p(B_p)} \to |0\rangle_{A(B)}|1\rangle_{A_p(B_p)}$, while the local states remain the same when the particle is not present. Under this interaction, the resulting joint state evolves as $|\Phi^+\rangle_{AB}(|\psi_L\rangle + |\psi_R\rangle) \to$

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{AB}(|\psi_L\rangle + |\psi_R\rangle) = |\Psi^+\rangle_{AB}(|\psi_L\rangle + |\psi_R\rangle)$ when C sent the particle or $|\Phi^+\rangle_{AB}|0\rangle \rightarrow |\Phi^+\rangle_{AB}|0\rangle$ when C does not send the particle. The interference patterns of the particle have not been disturbed and A and B will have a perfect record of the existence of the particle as their shared outcome states $|\Phi^+\rangle_{AB}$ and $|\Psi^+\rangle_{AB}$ are perfectly distinguishable.

Note that the key property is not simply that we end up in the same state with either unitary, or else the classically correlated state $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ would exhibit the same behaviour, and it does not. One can appreciate this better by considering the purification $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, which is transformed to orthogonal states by σ_A^x, σ_B^x, and hence even though there is no difference in the original state, the information as to which side the unitary happened still exists in the universe (adhering to the "church of the larger Hilbert space" as coined by John Smolin, whereby every mixed state can be viewed as part of some larger pure state).

As illustrated in the double-slit experiment, entanglement allows us to overcome the trade-off between "information gain via local interaction" and "disturbance in non-local superposition" i.e., recording information encoded using non-locally superposed quantum states, via local interactions whilst causing less disturbance than would be classically possible. We term this phenomenon delocalised-interactions, as the interaction cannot be known to have definitely happened at either A or B's location, since this would destroy the non-local superposition. We proceed to construct a formal quantum game to quantitatively capture the advantage of sharing entanglement between A and B when demonstrating delocalised-interactions.

4.3 Formulation of quantum games

4.3.1 Background on quantum games

A key method for studying particular aspects of entanglement is to consider non-local games, where entanglement can provide a non-classical advantage. The archetypal example is the game constructed from the Clauser-Horne-Shimony-Holt (CHSH) test [20]. In this CHSH game, Charlie passes two random classical bits $x, y \in \{0, 1\}$ to Alice and Bob respectively. Without communicating to each other, Alice and Bob must select and send back bits $a, b \in \{0, 1\}$ respectively, and they win the game if $a \oplus b = x \cdot y$, where $\oplus$ denotes addition modulo 2. The best classical strategy gives a win probability of 0.75, but using entangled quantum resources they can win with the maximum probability $\frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \approx 0.85$.

This game constructed from Clauser-Horne-Shimony-Holt test is an example of a general class of games called XOR games [181]. Further games have been considered where the questions are asked with quantum states, as in semi-quantum

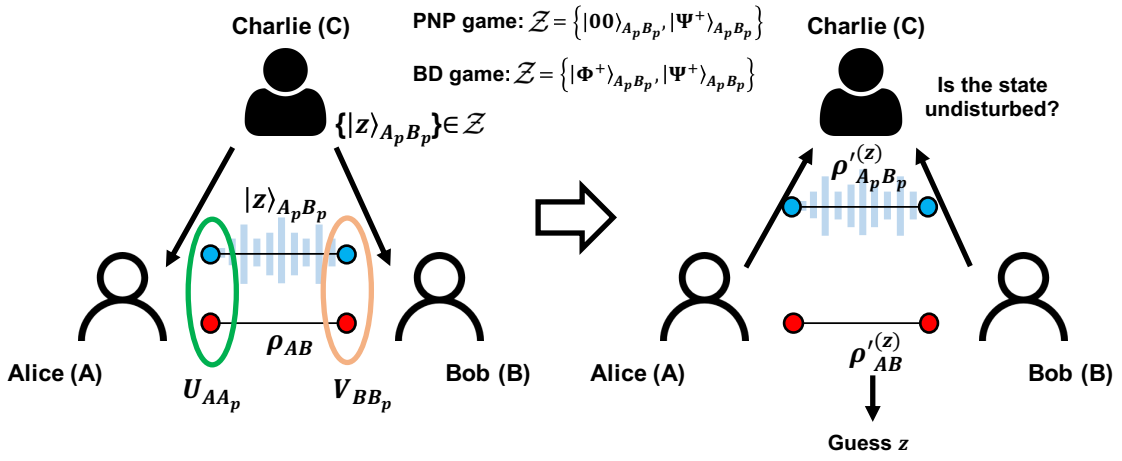

Figure 4.2: Schematic illustration of the quantum delocalised-interaction games as described in the main text, with $\rho'_{A_pB_p}$ denoting the final state returned to C, and $\rho'^{(z)}_{AB}$ denoting the final state obtained by A and B, which they measure to determine their guess for z. The sets of question states used for the PNP game and the BD game are presented at the top.

non-local games [182, 183] that can witness all forms of entanglement, and includes as a subclass the quantum XOR games [184]. There are also extended non-local games (sometimes referred to as bipartite steering games) [185, 186, 187] in which the referee also holds a quantum system that is provided at the start of the game by the players, and numerous other interesting setups and variations have been studied [188, 189, 190, 191, 192, 193].

4.3.2 Quantum delocalised-interaction games

We formulate general quantum delocalised-interaction games as follows (illustrated in Fig. 4.2)

1. C prepares a state $|z\rangle_{A_pB_p}$ selected from some finite set of question states $\mathcal{Z}$ with non-zero probability P_z, and sends the subsystems A_p and B_p to A and B, respectively.

2. A and B attempt to record the information z onto their shared state ρ_{AB} via local controlled unitaries U_{AA_p} and V_{BB_p}, then return the subsystems A_p and B_p to C.

3. C checks whether the returned subsystems A_pB_p have been disturbed by performing a projective measurement onto the initial state $|z\rangle_{A_pB_p}$.

4. A and B perform joint measurements $\Pi_{AB}^{(z_a)}$ to determine their answer z_a.

5. A and B win the game if their answer is correct $z_a = z$, and C's projective measurement returns the initial state $|z\rangle_{A_pB_p}$.

The question states must not be chosen such that A and B cannot distinguish them, and at least one $|z\rangle_{A_pB_p} \in \mathcal{Z}$ must be entangled. This condition ensures that the games capture the classical trade-off which a quantum delocalised-interaction circumvents.

The probability that A and B win the game is given as

$$p(\rho_{AB}) = \sum_z P_z \mathrm{Tr}\big[(\Pi_{AB}^{(z)} \otimes |z\rangle\langle z|)W(\rho_{AB} \otimes |z\rangle\langle z|)W^\dagger\big], \tag{4.1}$$

where $W = U_{AA_p} \otimes V_{BB_p}$. We shall use the superscript form p^{m} to denote the maximum of this quantity over all choices of measurements Π_{AB} and controlled unitaries U_{AA_p}, V_{BB_p}, and we shall use subscripts to distinguish specific instances.

4.4 Particle/No-Particle game

The double-slit scenario can now be simplified into an example of a quantum delocalised-interaction game. In this case, $\mathcal{Z} = \{|\mathrm{p}\rangle, |\mathrm{np}\rangle\}$ with $P_{\mathrm{p}} = 1/2 = P_{\mathrm{np}}$ and we take $|\mathrm{p}\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{A_pB_p} + |10\rangle_{A_pB_p})$, and $|\mathrm{np}\rangle = |00\rangle_{A_pB_p}$, which represent the states after passing the double-slit depending on whether C sends (p) or does not send (np) the particle.

We also choose to work with the interaction only happening if the particle exists in the local subsystem, since a unitary in the absence of a particle physically corresponds to free evolution which we can simply factor out. Hence $U_{AA_p} = \mathbb{1}_A \otimes |0\rangle_{A_p}\langle 0| + U_A \otimes |1\rangle_{A_p}\langle 1|$ and $V_{BB_p} = \mathbb{1}_B \otimes |0\rangle_{B_p}\langle 0| + V_B \otimes |1\rangle_{B_p}\langle 1|$. The overall interaction then can be written as

$$\begin{aligned} W = {} & \mathbb{1}_{AB} \otimes |00\rangle_{A_pB_p}\langle 00| + U_A \otimes \mathbb{1}_B \otimes |10\rangle_{A_pB_p}\langle 10| \\ & + \mathbb{1}_A \otimes V_B \otimes |01\rangle_{A_pB_p}\langle 01| + U_A \otimes V_B \otimes |11\rangle_{A_pB_p}\langle 11|. \end{aligned} \tag{4.2}$$

We refer to this game as the Particle/No-Particle (PNP) game. The total win probability after maximization over the choice of POVMs can be written as

$$p_{\mathrm{pnp}}(\rho_{AB}) = \frac{1}{2} + \frac{1}{2}\mathrm{eigs}_+(\tilde{\sigma}_{AB} - \rho_{AB}), \tag{4.3}$$

where $\tilde{\sigma}_{AB} = \frac{1}{4}(U_A + V_B)\rho_{AB}(U_A + V_B)^\dagger$, eigs_+ sums the positive eigenvalues.

4.4.1 Pure states

We shall now show that for any pure two-qubit state, the maximum obtainable win probability for the PNP game is given by

$$p^{\mathrm{m}}_{\mathrm{pnp}}(|\psi\rangle_{AB}) = \frac{3}{4} + \frac{1}{4}C(|\psi\rangle_{AB}), \tag{4.4}$$

where $C(|\psi\rangle_{AB}) = 2\sqrt{\lambda_0\lambda_1}$ with λ_i denoting the Schmidt coefficients, is the well-known concurrence entanglement monotone [124, 125] which is zero for all separable states, giving the classical bound as $\frac{3}{4}$. We can therefore view the game as providing a direct operational meaning of pure state concurrence.

Before proving this result we should note that it has interesting implications, for instance one might have thought that A and B would be helped by allowing a pre-processing step where they have temporary access to all the states they will use, and can apply entanglement distillation. However, using the concurrence result we can show that this would not increase their win probability. Consider A and B granted pre-processing access to N copies of the qubit state $|\psi\rangle$, from which they distil m copies of the maximally entangled state and $N-m$ pure separable states. Then when the game starts they use these new states one by one, and win m cases with probability 1 and $N-m$ cases with the maximum classical win probability $\frac{3}{4}$. It is known that in the asymptotic limit of large N we have $m = NE(|\psi\rangle)$, where $E(|\psi\rangle)$ is the entanglement entropy [112]. This means the win probability for the outlined distillation strategy will be bounded by $\frac{1}{N}[NE(|\psi\rangle)+\frac{3}{4}(N-NE(|\psi\rangle)] = \frac{3}{4}+\frac{1}{4}E(|\psi\rangle)$. However by using the original states they would obtain $\frac{3}{4}+\frac{1}{4}C(|\psi\rangle)$, and it is known that $C(|\psi\rangle) \geq E(|\psi\rangle)$. Therefore the distillation does not provide improvement.

We now turn to the proof of Eq. (4.4). We proceed by deriving a more general result for the maximum win probability of any pure state. Using Eq. (4.3), which already maximises over the choice of measurements, the win probability for a pure state can be written as

$$p_{\mathrm{pnp}}(|\psi\rangle_{AB}\langle\psi|) = \frac{1}{2}[1 + \mathrm{eigs}_{+}(M)], \tag{4.5}$$

where $M = \frac{1}{4}(|\psi_u\rangle + |\psi_v\rangle)(\langle\psi_u| + \langle\psi_v|) - |\psi\rangle\langle\psi|$, and we are writing $U_A|\psi\rangle = |\psi_u\rangle$, $V_B|\psi\rangle = |\psi_v\rangle$.

We now need to maximise over the choice of unitaries. Note that if $|\psi_u\rangle + |\psi_v\rangle$ is parallel with $|\psi\rangle$, then it follows that $M \leq 0$, and so $p_{\mathrm{pnp}} \leq \frac{1}{2}$. This means we just need to consider the case where this is not true, such that M has rank 2. The two non-zero eigenvalues must sum to $\mathrm{Tr}(M)$, so we write $m_1+m_2 = \frac{1}{2}(1+\mathrm{Re}\langle\psi_u|\psi_v\rangle)-1$, from which we have $-1 \leq m_1 + m_2 \leq 0$. This means that we can have at most one positive eigenvalue, and the maximization will choose unitaries that maximise the

magnitude of this positive eigenvalue.

We now consider an operator of the general form $M = K_1|\psi\rangle\langle\psi|K_1^\dagger - K_2|\psi\rangle\langle\psi|K_2^\dagger$, (we are performing a more general treatment as the result shall prove useful later as well). For this we derive an expression for its eigenvalues. We write

$$m(\alpha K_1|\psi\rangle + \beta K_2|\psi\rangle) = (K_1|\psi\rangle\langle\psi|K_1^\dagger - K_2|\psi\rangle\langle\psi|K_2^\dagger)(\alpha K_1|\psi\rangle + \beta K_2|\psi\rangle)$$
$$= (\alpha\langle\psi|K_1^\dagger K_1|\psi\rangle + \beta\langle\psi|K_1^\dagger K_2|\psi\rangle)K_1|\psi\rangle - (\alpha\langle\psi|K_2^\dagger K_1|\psi\rangle + \beta\langle\psi|K_2^\dagger K_2|\psi\rangle)K_2|\psi\rangle.$$

Under the assumption that $K_1|\psi\rangle$ and $K_2|\psi\rangle$ are not proportional, and defining $k_{ij} = \langle\psi|K_i^\dagger K_j|\psi\rangle$, we obtain the following two equations

$$m\alpha = (\alpha k_{11} + \beta k_{12}),$$

$$m\beta = -(\alpha k_{21} + \beta k_{22}).$$

Combining these to eliminate α and β we find

$$m^2 + (k_{22} - k_{11})m - (k_{11}k_{22} - k_{12}k_{21}) = 0.$$

The solutions to this quadratic equation are then found to be

$$m = \frac{1}{2}\big(k_{11} - k_{22} \pm \sqrt{(k_{22} + k_{11})^2 - 4k_{12}k_{21}}\big). \tag{4.6}$$

Applying this result to the case at hand we find that the largest eigenvalue is given by

$$m(k, \kappa) = \frac{(k-1) + \sqrt{(k+1)^2 - 4\kappa}}{2},$$

where $k = \langle\psi|K^\dagger K|\psi\rangle$, and $\kappa = |\langle\psi|K|\psi\rangle|^2$, and $K = (U_A + V_B)/2$. We write these out explicitly as

$$k = \frac{1}{2}(1 + \mathrm{Re}\langle\psi|U_A^\dagger V_B|\psi\rangle),$$

$$\kappa = \frac{1}{4}(u^2 + v^2 + 2uv\cos\Delta\phi),$$

where we are using $\langle\psi|U_A|\psi\rangle = e^{i\phi_A}u$ and $\langle\psi|V_B|\psi\rangle = e^{i\phi_B}v$ with $u, v \in \mathbb{R}_{\geq 0}$, and $\Delta\phi \equiv \phi_A - \phi_B$.

We now insert a resolution of the identity $\mathbb{1} = |\psi\rangle\langle\psi| + P_\perp$ to rewrite k as

$$k = \frac{1}{2}(1 + uv\cos(\Delta\phi) + \mathrm{Re}\langle\psi|U_A^\dagger P_\perp V_B|\psi\rangle).$$

From this we write

$$k \leq \frac{1}{2}[1 + uv\cos\Delta\phi + G(|\psi\rangle)] \equiv k_b,$$

where

$$G(|\psi\rangle) \equiv \max_{U_A,V_B} |\langle\psi|U_A^\dagger P_\perp V_B|\psi\rangle|,$$

is a state dependent constant.

We shall only consider values of $G(|\psi\rangle) < 1$ since the maximum win probability is 1 for $G = 1$. Now since increasing k can only increase m we can write $m(k,\kappa) \leq m(k_b,\kappa)$. This is important as by construction $m(k_b,\kappa)$, only depends on the three variables $u, v, \Delta\phi$, and this enables us to maximise via taking partial derivatives.

We start by writing

$$\tilde{m}(k,\kappa) \equiv m(k_b,\kappa) = \frac{(k_b-1)+\sqrt{(k_b+1)^2-4\kappa}}{2},$$

where we have $k_b = \frac{1}{2}[1 + uv\cos\Delta\phi + G(|\psi\rangle)]$, and $\kappa = \frac{1}{4}(u^2+v^2+2uv\cos\Delta\phi)$.

Taking partial derivatives w.r.t. $\Delta\phi, u, v$ via the chain rule we find

$$\frac{\partial\tilde{m}}{\partial\Delta\phi} = -\frac{1}{4}uv\sin\Delta\phi(1+\frac{k_b-1}{\sqrt{(k_b+1)^2-4\kappa}}),$$

$$\frac{\partial\tilde{m}}{\partial u} = \frac{1}{4}(1+\frac{k_b-1}{\sqrt{(k_b+1)^2-4\kappa}})(v\cos\Delta\phi) - \frac{1}{2\sqrt{(k_b+1)^2-4\kappa}}u,$$

$$\frac{\partial\tilde{m}}{\partial v} = \frac{1}{4}(1+\frac{k_b-1}{\sqrt{(k_b+1)^2-4\kappa}})(u\cos\Delta\phi) - \frac{1}{2\sqrt{(k_b+1)^2-4\kappa}}v.$$

Setting these expressions equal to zero we obtain the three equations

$$0 = Xuv\sin\Delta\phi, \tag{4.7}$$

$$0 = Xv\cos\Delta\phi - Yu, \tag{4.8}$$

$$0 = Xu\cos\Delta\phi - Yv, \tag{4.9}$$

where $X \equiv \frac{1}{2}(1+\frac{k_b-1}{\sqrt{(k_b+1)^2-4\kappa}})$, and $Y \equiv \frac{1}{\sqrt{(k_b+1)^2-4\kappa}}$.

First we observe that X cannot equal zero. To see this we set it to zero and solve by squaring it to arrive at $k_b = \kappa$, but this does not make the original term zero. We also note that $Y \equiv \frac{1}{\sqrt{(k_b+1)^2-4\kappa}}$ cannot equal zero either.

It follows from this that to satisfy Eq. (4.7) we must take $u = 0$, $v = 0$, or $\sin\Delta\phi = 0$. If we consider $u = 0$ then Eq. (4.9) implies $v = 0$. Similarly if we consider $v = 0$, then Eq. (4.8) implies $u = 0$. So in both cases we have $u = v = 0$, and this naturally makes the choice of $\Delta\phi$ irrelevant. From this we see that the only choice we now have to consider is $\sin\Delta\phi = 0$.

Taking this case we write Eq. (4.8) and Eq. (4.9) as $\pm Xv = Yu$ and $\pm Xu = Yv$ respectively. Now if either $\pm Xv = Yu = 0$ or $\pm Xu = Yv = 0$, then we quickly see

this implies $u = v = 0$, so we just need to consider the option of $\pm Xv = Yu \neq 0$ and $\pm Xu = Yv \neq 0$. In this case we can divide through and get $u^2 = v^2$ so $u = \pm v$. Substituting back we have $(X \pm Y)u = 0$ but we find that $(X \pm Y) \neq 0$, so we again arrive back at $u = 0$, $v = 0$, which means $\pm Xv = Yu = 0$ and $\pm Xu = Yv = 0$, so we have a contradiction.

Putting all this together we have found that the only turning point solution is given by $u = v = 0$. This gives $\kappa = 0$ and $k_b = \frac{1}{2}(1 + G(|\psi\rangle))$. And plugging this into the original equation we arrive at the result

$$\tilde{m}^{\max} = \frac{1}{2}(1 + G(|\psi\rangle)).$$

To verify that this is a maximum we first note that because we are interested in the line $u = v = 0$ for all $\Delta\phi$ then we simply have a 2D problem in each plane defined by a fixed value of $\Delta\phi$. Thus we apply the two dimensional second partial derivative test.

The required second-order partial derivatives are found to be

$$\frac{\partial^2 \tilde{m}(u = v = 0)}{\partial u^2} = -\frac{1}{2 + G},$$

$$\frac{\partial^2 \tilde{m}(u = v = 0)}{\partial v^2} = -\frac{1}{2 + G},$$

$$\frac{\partial^2 \tilde{m}(u = v = 0)}{\partial u \partial v} = \frac{\partial^2 \tilde{m}(u = v = 0)}{\partial v \partial u} = \big(\frac{1}{4} + \frac{G - 1}{G + 3}\big) \cos \Delta\phi.$$

From which one finds the determinant of the Hessian matrix as

$$D(u, v) = \frac{1}{(2 + G)^2} - \big(\frac{1}{4} + \frac{G - 1}{G + 3}\big)^2 \cos^2 \Delta\phi.$$

Now since $D(u, v) > 0$ and $\frac{\partial^2 \tilde{m}(u=v=0)}{\partial u^2} < 0$ for all valid values of $\Delta\phi$ and G, the second partial derivative test informs us that we have found a maximum.

When we do this we find

$$m(k, \kappa) \leq m(k_b, \kappa) \leq \frac{1}{2}[1 + G(|\psi\rangle)].$$

Using this with Eq. (4.5) we arrive at

$$p_{\text{pnp}}(|\psi\rangle_{AB}) \leq \frac{3}{4} + \frac{1}{4} G(|\psi\rangle_{AB}).$$

This bound is obtainable for all pure states. In order to see this we note that if we only consider unitaries that map the initial state to an orthogonal state such that $\langle\psi|U_A|\psi\rangle = 0$ and $\langle\psi|V_B|\psi\rangle = 0$. Then the win probability becomes

$\frac{3}{4} + \frac{1}{4}\mathrm{Re}[\mathrm{Tr}(U_A|\psi\rangle_{AB}\langle\psi|V_B^\dagger)]$, which by inserting the previous resolution of identity becomes $\frac{3}{4} + \frac{1}{4}\mathrm{Re}[\langle\psi|U_A P_\perp V_B^\dagger|\psi\rangle]$. We can always choose phases such that $\langle\psi|U_A P_\perp V_B^\dagger|\psi\rangle$ is real, and hence we see that all we need to do is find the best choice of U_A, V_B subject to the orthogonality constraint, and we thus obtain the optimal win probability

$$p_{\mathrm{pnp}}^{\mathrm{m}}(|\psi\rangle_{AB}) = \frac{3}{4} + \frac{1}{4}G(|\psi\rangle_{AB}),$$

where we use the superscript m to denote the maximum obtainable value.

Now to complete the proof of Eq. (4.4) we prove that for qubit states $G(\rho_{AB}) = C(\rho_{AB})$. We start by defining the orthogonal basis states

$$\begin{aligned}
|\psi\rangle &= \sqrt{r}|00\rangle + \sqrt{1-r}|11\rangle, \\
|\psi_1\rangle &= \sqrt{r}|10\rangle + \sqrt{1-r}|01\rangle, \\
|\psi_2\rangle &= \sqrt{1-r}|10\rangle - \sqrt{r}|01\rangle. \\
|\psi_3\rangle &= \sqrt{1-r}|00\rangle - \sqrt{r}|11\rangle,
\end{aligned}$$

We can then write $P_\perp = \sum_i |\psi_i\rangle\langle\psi_i|$.

We parametrize the unitaries in terms of the identity and Pauli operators as $U_A = e^{i\phi_A}\mathbf{a}\cdot\boldsymbol{\sigma}_A$, where $\boldsymbol{\sigma} = (\mathbb{1}, \sigma x, \sigma y, \sigma z)^T$, $\mathbf{a} = (a_0, ia_1, ia_2, ia_3)^T$, $a_i \in \mathbb{R}$ and $\sum_i a_i^2 = 1$. Similarly we write $V_B = e^{i\phi_B}\mathbf{b}\cdot\boldsymbol{\sigma}_B$. We also define the three vectors $\vec{a} = (a_1, a_2, a_3)^T$, $\vec{b} = (b_1, b_2, b_3)^T$, and $\vec{\sigma} = (\sigma x, \sigma y, \sigma z)^T$,

We wish to evaluate

$$\begin{aligned}
\langle\psi|U_A^\dagger P_\perp V_B|\psi\rangle &= e^{i(\phi_B-\phi_A)}\langle\psi|\vec{a}\cdot\vec{\sigma}P_\perp\vec{b}\cdot\vec{\sigma}|\psi\rangle, \\
&= e^{i(\phi_B-\phi_A)}(\langle\psi|\vec{a}\cdot\vec{\sigma}|\psi_1\rangle\langle\psi_1|\vec{b}\cdot\vec{\sigma}|\psi\rangle \\
&+ \langle\psi|\vec{a}\cdot\vec{\sigma}|\psi_2\rangle\langle\psi_2|\vec{b}\cdot\vec{\sigma}|\psi\rangle + \langle\psi|\vec{a}\cdot\vec{\sigma}|\psi_3\rangle\langle\psi_3|\vec{b}\cdot\vec{\sigma}|\psi\rangle).
\end{aligned} \tag{4.10}$$

To do this we calculate

$$\begin{aligned}
\langle\psi_1|\vec{a}\cdot\vec{\sigma}|\psi\rangle &= a_1 + ia_2(2r-1)), \\
\langle\psi_2|\vec{a}\cdot\vec{\sigma}|\psi\rangle &= 2ia_2\sqrt{r(1-r)}, \\
\langle\psi_3|\vec{a}\cdot\vec{\sigma}|\psi\rangle &= 2a_3\sqrt{r(1-r)}, \\
\langle\psi_1|\vec{b}\cdot\vec{\sigma}|\psi\rangle &= 2b_1\sqrt{r(1-r)}, \\
\langle\psi_2|\vec{b}\cdot\vec{\sigma}|\psi\rangle &= -ib_2 + b_1(1-2r), \\
\langle\psi_3|\vec{b}\cdot\vec{\sigma}|\psi\rangle &= 2b_3\sqrt{r(1-r)}.
\end{aligned}$$

We use these to evaluate Eq. (4.10) and find

$$\begin{aligned}\langle\psi|U_A^\dagger P_\perp V_B|\psi\rangle &= e^{i(\phi_B-\phi_A)}\big[2b_1\sqrt{r(1-r)}(a_1 - ia_2(2r-1)) \\ &\quad + (-ib_2 + b_1(1-2r))(-2ia_2\sqrt{r(1-r)}) + 4a_3b_3r(1-r)\big], \\ &= 2\sqrt{r(1-r)}e^{i(\phi_B-\phi_A)}(a_1b_1 - a_2b_2 + 2a_3b_3\sqrt{r(1-r)}).\end{aligned}$$

Taking the modulus of this gives

$$|\langle\psi|V_B^\dagger P_\perp U_A|\psi\rangle| = |2\sqrt{r(1-r)}(a_1b_1 - a_2b_2 + 2a_3b_3\sqrt{r(1-r)})|.$$

By writing $\vec{v}_1 = (a_1, a_2, a_3)^T$, $\vec{v}_2 = (b_1, -b_2, 2\sqrt{r(1-r)}b_3)^T$, we can rewrite the right-hand side as $2\sqrt{r(1-r)}|\vec{v}_1\cdot\vec{v}_2|$. The Cauchy-Schwarz inequality gives $|\vec{v}_1\cdot\vec{v}_2| \leq |\vec{v}_1||\vec{v}_2|$, and it is straightforward to show $|\vec{v}_1| \leq 1$ and $|\vec{v}_2| \leq 1$, which leads us to $|\langle\psi|V_B^\dagger P_\perp U_A|\psi\rangle| \leq 2\sqrt{r(1-r)}$. Since this maximum is clearly obtainable by for instance setting $a_1 = b_1 = 1$ and the other terms to zero we conclude

$$G(|\psi\rangle_{AB}) = \max_{U_A,V_B} |\langle\psi|U_A^\dagger P_\perp V_B|\psi\rangle| = 2\sqrt{r(1-r)} = C(|\psi\rangle_{AB}),$$

where we have identified $2\sqrt{r(1-r)}$ as the pure state concurrence [124, 125], (which for pure states coincides with the negativity [123]). This concludes the proof of Eq. (4.4).

4.4.2 Mixed state bounds

To generalise Eq. (4.4) to mixed states we use the fact that the maximum win probability is a convex function $p^{\mathrm{m}}(\sum_i r_i\rho^{(i)}) \leq \sum_i r_i p^{\mathrm{m}}(\rho^{(i)})$, which can be intuitively understood as follows. Consider A and B being either given copies of a known state $\sum_i r_i\rho^{(i)}$, or given labelled copies of known states $\rho^{(i)}$ where the number of each is in proportion to r_i. From the second case they can reproduce the first case by simply ignoring the labels, therefore in the second case they must be able to obtain at least as high a win probability as in the first case, hence the convexity result. Using this, combined with the fact that $C(\rho_{AB}) = \inf\sum_i q_i C(|\psi_i\rangle_{AB})$, we extend Eq. (4.4) to a bound for mixed states, giving

$$p^{\mathrm{m}}_{\mathrm{pnp}}(\rho_{AB}) \leq \frac{3}{4} + \frac{C(\rho_{AB})}{4}. \tag{4.11}$$

Since the concurrence has an analytic closed form, we can now easily calculate a bound on the win probability gain for any two-qubit state.

From this we can also view the game as providing a direct operational meaning of concurrence for mixed states that saturate the bound. It is therefore natural

to ask whether the bound can be tight for mixed states. We shall answer this with detailed examples in the next section, but the answer turns out to be yes, as demonstrated by mixtures of two Bell states. However, we shall also show that not all mixed states saturate the bound as demonstrated by Werner-like states [194] $\rho_{AB} = a|\psi^k\rangle_{AB}\langle\psi^k| + \frac{1-a}{4}\mathbb{1}_{AB}$, where $0 \leq a \leq 1$ and $|\psi^k\rangle$ is chosen as one of the four Bell states.

We can however, understand this behaviour as the mixedness of states degrading the record quality. Consider the extreme example of the maximally mixed state $\mathbb{1}_{AB}/n$. It is clear that if A and B try to unitarily encode the presence of a particle in this state then they will not gain information. This inability of the state to acquire information is what we intuitively mean when we say it has bad record quality.

We capture this general effect via a bound we term the record quality bound. This both formalises the above observation and will enable us to analytically prove that certain states cannot provide advantage in the PNP game. We write the record quality bound as

$$p^{\mathrm{m}}_{\mathrm{pnp}}(\rho_{AB}) \leq \frac{1}{2} + \frac{1}{2}T_c(\lambda^{\uparrow}, \lambda^{\downarrow}), \tag{4.12}$$

where we denote the classical trace distance $T_c(p,q) = \frac{1}{2}\sum_i |p_i - q_i|$ for probability vectors p, q defined over the same index set, and $\lambda^{\uparrow}$ is the vector of eigenvalues of ρ arranged in ascending order and including any zero values. For the $\mathbb{1}_{AB}/n$ example, we see that the win probability cannot exceed $\frac{1}{2}$, i.e., the best they can do is just guess.

The proof proceeds as follows. First we note that one can rewrite the win probability in terms of a trace distance but care is needed on account of $\tilde{\sigma}_{AB}$ not being normalised. To see this we perform the standard separation of positive and negative eigenvalues by writing $\tilde{\sigma} - \rho = Q - S$, so that $|\tilde{\sigma} - \rho| = Q + S$, and therefore $T(\tilde{\sigma}, \rho) = \frac{1}{2}\mathrm{Tr}|\tilde{\sigma} - \rho| = \frac{1}{2}(\mathrm{Tr}Q + \mathrm{Tr}S)$. We now use $\mathrm{Tr}(\tilde{\sigma} - \rho) = \mathrm{Tr}\tilde{\sigma} - 1 = \mathrm{Tr}Q - \mathrm{Tr}S$, to find that $\mathrm{Tr}Q = T(\tilde{\sigma}, \rho) - \frac{1}{2} + \frac{1}{2}\mathrm{Tr}\tilde{\sigma}$. Finally the fact that $\mathrm{Tr}Q = \mathrm{eigs}_{+}(\tilde{\sigma} - \rho)$, leads through to

$$p_{\mathrm{pnp}}(\rho_{AB}) = \frac{1}{4} + \frac{1}{2}T(\tilde{\sigma}_{AB}, \rho_{AB}) + \frac{1}{4}\mathrm{Tr}[\tilde{\sigma}_{AB}]. \tag{4.13}$$

With this we now write $\tilde{\sigma}_{AB} = \sigma_{AB} - \tilde{\sigma}^{(-)}_{AB}$, where $\sigma_{AB} = \frac{1}{2}(U_A\rho_{AB}U_A^{\dagger} + V_B\rho_{AB}V_B^{\dagger})$, and $\tilde{\sigma}^{(-)}_{AB} = \frac{1}{4}(U_A - V_B)\rho_{AB}(U_A - V_B)^{\dagger}$. Since $T(\tilde{\sigma}_{AB}, \rho_{AB}) = \frac{1}{2}||\tilde{\sigma}_{AB} - \rho_{AB}||_1$, we can apply the triangle inequality $||A + B||_1 \leq ||A||_1 + ||B||_1$, to get $T(\tilde{\sigma}_{AB}, \rho_{AB}) \leq T(\sigma_{AB}, \rho_{AB}) + \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^{(-)}_{AB})$. Using this with Eq. (4.13), and noting that $\mathrm{Tr}(\tilde{\sigma}^{(-)}_{AB}) + \mathrm{Tr}(\tilde{\sigma}_{AB}) = 1$, we arrive at

$$p_{\mathrm{pnp}}(\rho_{AB}) \leq \frac{1}{2} + \frac{1}{2}T\big(\rho_{AB}, \frac{U_A\rho_{AB}U_A^{\dagger} + V_B\rho_{AB}V_B^{\dagger}}{2}\big). \tag{4.14}$$

Before progressing further, we can understand this more intuitively by considering a variation on the game, where C no longer performs a measurement afterwards to see if they have decohered the state. This means we are just focusing on A and B's ability to record the presence of a particle. The win probability for this game is derived similarly to before, and we find $\tilde{p}_{\text{pnp}}(\rho_{AB}) = \frac{1}{2} + \frac{1}{2}T(\rho_{AB}, \sigma_{AB})$, where we have already performed the maximization over the choice of POVMs.

This is an easier game by construction, so the maximum win probability for it must upper bound the maximum win probability of the original game. This therefore gives the same bound that we arrived at via the triangle inequality, given in Eq. (4.14).

In order to proceed, we make use of convexity of the trace distance which gives

$$T\big(\rho_{AB}, \frac{U_A\rho_{AB}U_A^\dagger + V_B\rho_{AB}V_B^\dagger}{2}\big) \leq \frac{1}{2}(T(\rho_{AB}, U_A\rho_{AB}U_A^\dagger) + T(\rho_{AB}, V_B\rho_{AB}V_B^\dagger)). \tag{4.15}$$

We then use the fact that a maximization over all possible global unitaries will always give a value greater than or equal to that obtained by maximization over locally restricted unitaries to write

$$\frac{1}{2}\max_{U_A,V_B}(T(\rho_{AB}, U_A\rho_{AB}U_A^\dagger) + T(\rho_{AB}, V_B\rho_{AB}V_B^\dagger)) \leq \max_{U_{AB}} T(\rho_{AB}, U_{AB}\rho_{AB}U_{AB}^\dagger). \tag{4.16}$$

These inequalities combined with Eq. (4.14), allow us to bound the win probability for the PNP game as

$$p_{\text{pnp}}(\rho_{AB}) \leq \frac{1}{2} + \frac{1}{2}\max_{U_{AB}} T(\rho_{AB}, U_{AB}\rho_{AB}U_{AB}^\dagger). \tag{4.17}$$

We now state and prove the following Lemma. For two density matrices ρ and σ defined on the same Hilbert space $\mathcal{H}$ of dimension n. It holds that

$$T(\rho, \sigma) \leq T_c(\lambda^\uparrow, \mu^\downarrow),$$

where T is the quantum trace distance, T_c the Kolmogorov (classical trace) distance, $\lambda^\uparrow$ is the vector of n eigenvalues of ρ arranged in ascending order and $\mu^\downarrow$ is the vector of n eigenvalues of σ arranged in descending order, where these vectors of eigenvalues include any zero values.

To prove this we start by writing $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$, $\sigma = \sum_i \mu_i|\phi_i\rangle\langle\phi_i|$, and $\rho - \sigma = \sum_i a_i|\alpha_i\rangle\langle\alpha_i|$, with all the eigenvalues arranged in ascending order such that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and similarly for the others. The trace distance $\frac{1}{2}\text{Tr}|\rho - \sigma|$, is given by the sum of the positive eigenvalues of $\rho - \sigma$, i.e. $T(\rho, \sigma) = \sum_{i,a_i \geq 0} a_i$.

We now shall make use of the Min-max theorem which we state as follows.

Consider a Hermitian operator H, defined on a Hilbert space $\mathcal{H}$ of dimension n. We denote k dimensional subspaces as $\mathcal{H}_k$, i.e. this denotes any Hilbert space that satisfies $\mathcal{H}_k \subseteq \mathcal{H}$, and $\dim(\mathcal{H}_k) = k$. Now working with normalised vectors $|\langle\chi|\chi\rangle| = 1$, the Min-max theorem states that the eigenvalues of H, arranged such that $h_1 \leq h_2 \leq ... \leq h_n$, satisfy

$$h_k = \min_{\mathcal{H}_k} \max_{|\chi\rangle \in \mathcal{H}_k} \langle\chi|H|\chi\rangle.$$

A simple corollary of this is that

$$h_k \leq h_{k+l} \leq \max_{|\chi\rangle \in \mathcal{H}_{k+l}} \langle\chi|H|\chi\rangle.$$

We apply this Corollary to $\rho - \sigma$ to get

$$a_k \leq \max_{|\chi\rangle \in \mathcal{H}_{k+l}} \sum_{i=1}^{n} (\lambda_i |\langle\chi|\psi_i\rangle|^2 - \mu_i |\langle\chi|\phi_i\rangle|^2). \tag{4.18}$$

This is true for any choice of $\mathcal{H}_{k+l}$. We shall proceed by defining a particular case. To this end we define the linear operator

$$L = \sum_{i=1}^{n-k} |e_i\rangle(\sqrt{\lambda_{n-i+1}}\langle\psi_{n-i+1}| + \sqrt{\mu_i}\langle\phi_i|),$$

where the $|e_i\rangle$ form some set of orthonormal vectors.

We now consider the kernel of L. We know that the kernel of a linear operator is a vector space, and $L|\chi\rangle = 0$ implies the following $n - k$ constraints

$$\sqrt{\lambda_{n-i+1}}\langle\psi_{n-i+1}|\chi\rangle + \sqrt{\mu_i}\langle\phi_i|\chi\rangle = 0. \tag{4.19}$$

This specifies a $k + l$ dimensional subspace with $l = 0$ if the constraints are independent and $l \neq 0$ otherwise. From this we see that we can choose to define $\mathcal{H}_{k+l}$ as the kernel of L.

We now rewrite Eq. (4.18) as

$$a_k \leq \max_{|\chi\rangle \in \mathcal{H}_{k+l}} \Bigg(\sum_{i=1}^{k} \lambda_i |\langle\chi|\psi_i\rangle|^2 - \sum_{i=n-k+1}^{n} \mu_i |\langle\chi|\phi_i\rangle|^2 + \sum_{i=1}^{n-k} (\lambda_{n-i+1} |\langle\psi_{n-i+1}|\chi\rangle|^2 - \mu_i |\langle\phi_i|\chi\rangle|^2 \Bigg).$$

Substituting in for the constraints of Eq. (4.19) we have

$$a_k \leq \max_{|\chi\rangle} \Big(\sum_{i=1}^{k} \lambda_i |\langle\chi|\psi_i\rangle|^2 - \sum_{i=n-k+1}^{n} \mu_i |\langle\chi|\phi_i\rangle|^2 \Big).$$

From this we find

$$a_k \leq \lambda_k - \mu_{n-k+1}.$$

Now we use this to write

$$\begin{aligned} T(\rho,\sigma) &= \sum_{i,a_i\geq 0} a_i, \\ &\leq \sum_{i,a_i\geq 0} (\lambda_i - \mu_{n-i+1}), \\ &\leq \sum_{i,(\lambda_i-\mu_{n-i+1})\geq 0} (\lambda_i - \mu_{n-i+1}), \\ &= \sum_{j,(\lambda_{n-j}-\mu_{j+1})\geq 0} (\lambda_{n-j} - \mu_{j+1}). \end{aligned}$$

By writing the last term in this way it is now clear that this is equal to the Kolmogorov (classical trace) distance $T_c(\lambda^{\uparrow}, \mu^{\downarrow})$, therefore we have the stated result.

We now apply this lemma to Eq. (4.17). Since ρ_{AB} and $U_{AB}\rho_{AB}U_{AB}^{\dagger}$ have the same eigenvalues we have

$$T(\rho_{AB}, U_{AB}\rho_{AB}U_{AB}^{\dagger}) \leq T_c(\lambda^{\uparrow}, \lambda^{\downarrow}), \tag{4.20}$$

and using this we arrive at the final form for the record quality bound

$$p_{\text{pnp}}^{\text{m}}(\rho_{AB}) \leq \frac{1}{2} + \frac{1}{2} T_c(\lambda^{\uparrow}, \lambda^{\downarrow}).$$

With this bound and the concurrence bound of Eq. (4.11) in hand, we turn to examine explicit example states.

4.4.3 Mixed state examples

For pure states we have the equality Eq. (4.4), whereas for mixed states we have only a bound Eq. (4.11). As mentioned above, the question of whether this bound can be tight for mixed states is answered in the affirmative with mixtures of two Bell states as a straightforward example.

To see this, consider the specific example $\rho = a|\psi^+\rangle\langle\psi^+| + (1-a)|\psi^-\rangle\langle\psi^-|$. We find $\tilde{\rho} = (Y \otimes Y)\rho^*(Y \otimes Y) = \rho$, so $\rho\tilde{\rho} = \rho^2$, and from this we find $C(\rho) = 1 - 2a$ for $a \leq \frac{1}{2}$, and $C(\rho) = 2a - 1$ for $a \geq \frac{1}{2}$. Now for $a \leq \frac{1}{2}$, we calculate the win

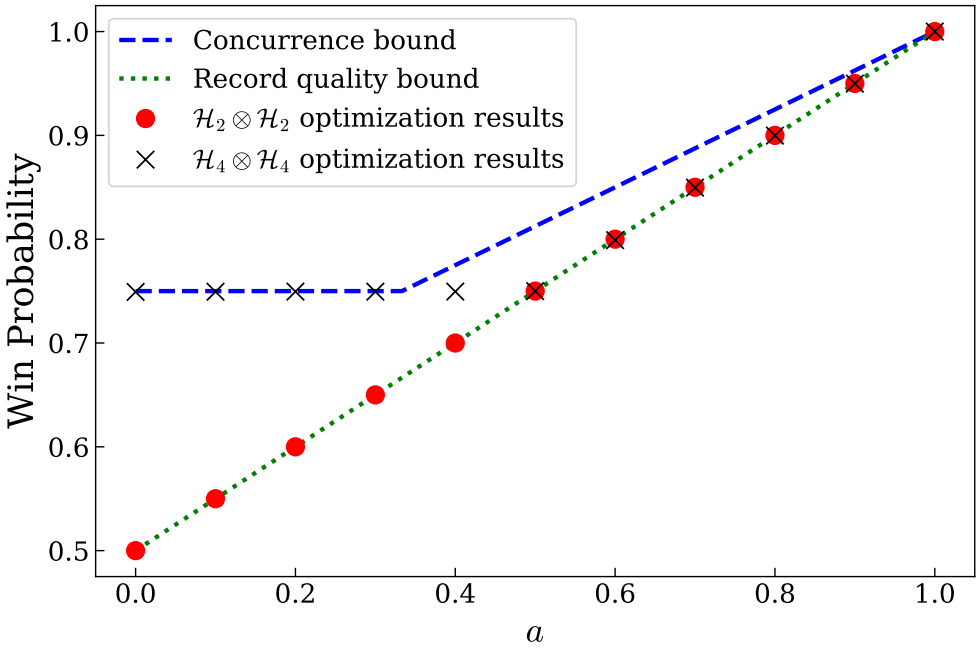

Figure 4.3: Plot showing the numerically optimized PNP win probabilities and bounds for 2-qubit states of the form $a|\psi^k\rangle\langle\psi^k| + \frac{1-a}{4}\mathbb{1}$, for different values of a, as described in the main text.

probability from Eq. (4.3) with the choices $U_A = X_A$, and $V_B = -X_B$, noting that this choice maps ρ to an orthogonal state. For this choice of unitaries we find $p_{\text{w}}(\rho) = 1 - \frac{a}{2} = \frac{3}{4} + \frac{1-2a}{4}$, and for $a \geq \frac{1}{2}$, we take $U_A = X_A$, and $V_B = X_B$, for which we get $p_{\text{w}}(\rho) = \frac{1}{2}(1+a) = \frac{3}{4} + \frac{2a-1}{4}$. We see that in both instances we are exactly saturating the concurrence bound.

However, not all mixed states can saturate the bound. An informative example is given by states of the form $\rho_{AB} = a|\psi^k\rangle_{AB}\langle\psi^k| + \frac{1-a}{4}\mathbb{1}_{AB}$, where $0 \leq a \leq 1$ and $|\psi^k\rangle$ is chosen as one of the four Bell states (e.g. the Werner state [194]). The $\mathcal{H}_2 \otimes \mathcal{H}_2$ data points in Fig. 4.3 show the results of numerical optimization over choices of unitaries for these qubit states (our code made use of qutip [195, 196]). This indicates that the record quality bound can be saturated. This can be demonstrated analytically with $U_A = \sigma_A^x$, and $V_B = \pm\sigma_B^x$, where the sign is chosen to match the sign of $\langle\psi^k|\sigma_A^x\sigma_B^x|\psi^k\rangle$. This gives $p_{\text{w}} = \frac{1}{2}(1+a)$, which exactly saturates the bound and is therefore an optimal tactic.

Since the Werner-like state is entangled for $a > \frac{1}{3}$, these results indicate that entanglement is not sufficient to observe nonclassical advantage in the PNP quantum game. Additionally we note that the capacity for Bell non-locality is not necessary for a state to demonstrate non-classical performance, since there is a local model for projective measurements for $a \lesssim 0.66$ [197].

The limiting factor for these states is poor record quality, but this issue could in principle be dealt with if we change the rules and allow A and B unlimited access to additional pure separable resources. This introduction of additional classical resources is seen to be equivalent to embedding in a higher dimensional Hilbert space, which then zero-pads the vector of eigenvalues such that $T_c(\lambda^{\uparrow}, \lambda^{\downarrow}) = 0$. If A and B share a Werner state and are given access to the additional pure qubit state $|00\rangle_{A'B'}\langle 00|$, this is equivalent to allowing them to optimize their unitaries U_A, V_B over the group $U(4)$ as opposed to the previous case where we used $U(2)$. The $\mathcal{H}_4 \otimes \mathcal{H}_4$ data points in Figure 4.3 illustrate the numerical optimization results for this. As expected the win probability is never significantly below the $\frac{3}{4}$ classical limit. However, we note the striking feature that the win probability still does not get above the classical limit until $a > \frac{1}{2}$, and that when it does it follows the $\mathcal{H}_2 \otimes \mathcal{H}_2$ record quality bound. This indicates that entanglement is not sufficient to demonstrate non-classical performance even when we change the rules to grant access to additional pure classical resources.

4.4.4 Different sending probabilities

As we have seen, the record quality factor complicates the situation. It would be cleaner to have a game where improvements come from getting more entangled states, but for the PNP game if we go from the maximally mixed state to a pure separable state then we have increased the maximum win probability by $\frac{1}{4}$ without any increase in entanglement.

A key problem is that the $|\mathrm{np}\rangle_{A_pB_p} = |00\rangle_{A_pB_p}$ does not get disturbed when measured, so can be measured using a separable state without consequence. This suggests two approaches. The first is to reduce the probability of sending this state, and the second is to replace this state completely. In this section we consider the former which turns out to be a less promising approach, then in the next section we turn to the latter.

Accounting for different probabilities for C sending a particle P_{p} and no particle P_{np}, the win probability can be written as

$$p_{\mathrm{pnp}}(\rho_{AB}) = P_{\mathrm{np}} + \mathrm{eigs}_{+}(P_{\mathrm{p}}\tilde{\sigma}_{AB} - P_{\mathrm{np}}\rho_{AB}),$$

For pure states, using the same logic and notation as before via Eq. (4.6) we arrive at needing to maximise

$$n = P_{\mathrm{p}}k - P_{\mathrm{np}} + \sqrt{(P_{\mathrm{p}}k + P_{\mathrm{np}})^2 - 4P_{\mathrm{p}}P_{\mathrm{np}}\kappa}$$

As before we instead maximise an upper bound on this given by

$$\tilde{n} = P_{\rm p}k_b - P_{\rm np} + \sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa},$$

where we have $k_b = \frac{1}{2}[1 + uv\cos\Delta\phi + G(|\psi\rangle)]$, and $\kappa = \frac{1}{4}(u^2 + v^2 + 2uv\cos\Delta\phi)$. Calculating the partial derivatives we find

$$\frac{\partial\tilde{n}}{\partial\Delta\phi} = -\frac{1}{2}uv\sin\Delta\phi(P_{\rm p} + \frac{P_{\rm p}^2k_b - P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}}),$$

$$\frac{\partial\tilde{n}}{\partial u} = \frac{1}{2}(P_{\rm p}+\frac{P_{\rm p}^2k_b - P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}})(v\cos\Delta\phi) - \frac{P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}}u,$$

$$\frac{\partial\tilde{n}}{\partial v} = \frac{1}{2}(P_{\rm p}+\frac{P_{\rm p}^2k_b - P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}})(v\cos\Delta\phi) - \frac{P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}}v,$$

Setting these expressions equal to zero we obtain the three equations

$$0 = \tilde{X}uv\sin\Delta\phi, \tag{4.21}$$

$$0 = \tilde{X}v\cos\Delta\phi - \tilde{Y}u, \tag{4.22}$$

$$0 = \tilde{X}u\cos\Delta\phi - \tilde{Y}v, \tag{4.23}$$

where $\tilde{X} \equiv (P_{\rm p} + \frac{P_{\rm p}^2k_b - P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}})$ and $\tilde{Y} = \frac{2P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}}$. Note that setting $P_{\rm p} = P_{\rm np} = 1/2$, recovers the equations we had before.

For $P_{\rm p} \neq 0$ and $P_{\rm np} \neq 0$, we as before find that neither $\tilde{X}$ nor $\tilde{Y}$ can equal zero. It follows that to satisfy Eq. (4.21) we must take $u = 0$, $v = 0$, or $\sin\Delta\phi = 0$. If we consider $u = 0$ then Eq. (4.23) implies $v = 0$. Similarly if we consider $v = 0$, then Eq. (4.22) implies $u = 0$. So in both cases we have $u = v = 0$, and this naturally makes the choice of $\Delta\phi$ irrelevant. Therefore the only choice we now have to consider is $\sin\Delta\phi = 0$.

Taking this case we write Eq. (4.22) and Eq. (4.23) as $\pm\tilde{X}v = \tilde{Y}u$ and $\pm\tilde{X}u = \tilde{Y}v$ respectively. Now if either $\pm\tilde{X}v = \tilde{Y}u = 0$ or $\pm\tilde{X}u = \tilde{Y}v = 0$, then this implies $u = v = 0$, so we just need to consider the option of $\pm\tilde{X}v = \tilde{Y}u \neq 0$ and $\pm\tilde{X}u = \tilde{Y}v \neq 0$. In this case we can divide through and get $u^2 = v^2$ so $u = \pm v$. Substituting back we have $(\tilde{X} \pm \tilde{Y})u = 0$. We have

$$\tilde{X} \pm \tilde{Y} = P_{\rm p} + \frac{P_{\rm p}^2k_b - P_{\rm p}P_{\rm np} \pm 2P_{\rm p}P_{\rm np}}{\sqrt{(P_{\rm p}k_b + P_{\rm np})^2 - 4P_{\rm p}P_{\rm np}\kappa}},$$

and see that for the relevant ranges only $\tilde{X} - \tilde{Y}$ could be equal to zero so we write

$$P_{\mathrm{p}} + \frac{P_{\mathrm{p}}^2 k_b - 3P_{\mathrm{p}}P_{\mathrm{np}}}{\sqrt{(P_{\mathrm{p}}k_b + P_{\mathrm{np}})^2 - 4P_{\mathrm{p}}P_{\mathrm{np}}\kappa}} = 0.$$

The solution to this equation is $k_b = \frac{P_{\mathrm{np}}}{P_{\mathrm{p}}} + \frac{\kappa}{2}$. Plugging in the definitions of k_b and κ and using that $\sin\Delta\phi = 0$, and $u^2 = v^2$, we have

$$u^2 \mp u^2 = 2(1 + G(|\psi\rangle)) - \frac{4P_{\mathrm{np}}}{P_{\mathrm{p}}}.$$

We can check for our previous result by setting $P_{\mathrm{np}} = P_{\mathrm{p}} = \frac{1}{2}$, we then have that the right hand side is always negative for $G(|\psi\rangle) < 1$, therefore there is no real solution. For $P_{\mathrm{np}} \geq P_{\mathrm{p}}$ this argument holds, but for $P_{\mathrm{np}} < P_{\mathrm{p}}$ things are more complicated. However, we can extract a useful extra result for the case of separable states $G(|\psi\rangle) = 0$. For these we find that the right hand side is negative for $P_{\mathrm{p}} < \frac{2}{3}$, and for this by similar arguments to before we have that the turning point is given by $u = v = 0$. This gives the maximum separable win probability in this range as $P_{\mathrm{np}} + \frac{1}{2}P_{\mathrm{p}}$. It makes sense that this regime breaks down at $P_{\mathrm{p}} = \frac{2}{3}$ because this is when doing nothing and simply always guessing that there is a particle, will give equal win probability to measuring all the time and losing half the time when there is a particle.

This approach is valid but complicated. As we shall show, replacing the question state $|\mathrm{np}\rangle_{A_pB_p} = |00\rangle_{A_pB_p}$ results in a much more fruitful setting.

4.5 Bell distinguishing game

Here we study a modified game that indicates an even stronger connection with concurrence. In the PNP game considered above, the no particle state $|\mathrm{np}\rangle = |00\rangle_{A_pB_p}$, has no spatial superposition which can be damaged by the local measurements. To move away from this, we can consider replacing $|00\rangle_{A_pB_p}$, with the Bell state $|\phi^+\rangle_{A_pB_p} = \frac{1}{\sqrt{2}}(|00\rangle_{A_pB_p} + |11\rangle_{A_pB_p})$. So Alice and Bob are now tasked with distinguishing two Bell states $|\psi^+\rangle$ and $|\phi^+\rangle$ whilst trying to return them undamaged. We shall refer to this as the Bell-Distinguishing (BD) game. It is noteworthy that this task can be viewed as detecting local bit-flip errors, where in contrast to a conventional syndrome measurement [198] one is using two ancilla modes, each of which can only interact with its local part of the system.

4.5.1 Concurrence bound

Unlike for the PNP game we shall just focus on the two-qubit case. For these states we again find that the concurrence quantifies the maximum obtainable win probability, via

$$p_{\rm bd}^{\rm m}(|\psi\rangle_{AB}) = \frac{1}{2} + \frac{1}{2}C(|\psi\rangle_{AB}), \tag{4.24}$$

and thus we have the general bound

$$p_{\rm bd}^{\rm m}(\rho_{AB}) \leq \frac{1}{2} + \frac{1}{2}C(\rho_{AB}). \tag{4.25}$$

We prove this as follows. The win probability for a pure state $|\psi\rangle$, can be written as

$$p_{\rm bd} = \frac{k_{11} + k_{22} + \sqrt{(k_{11} + k_{22})^2 - 4k_{12}k_{21}}}{4}.$$

where $k_{ij} = \langle\psi|K_i^\dagger K_j|\psi\rangle$, with $K_1 = (U_A + V_B)/2$ and $K_1 = (U_A V_B + \mathbb{1})/2$

One can rewrite the terms as

$$k_{11} + k_{22} = 1 + \langle\psi|AB|\psi\rangle,$$

$$k_{12} = \frac{1}{2}\langle\psi|(A + B)|\psi\rangle,$$

where we have defined the local operators $A = (U_A + U_A^\dagger)/2$, and $B = (V_B + V_B^\dagger)/2$.

We can then re-express the term under the square root as

$$\begin{aligned}(1 + \langle\psi|AB|\psi\rangle)^2 - \langle\psi|(A + B)|\psi\rangle^2 &= \langle\psi|(\mathbb{1} + AB)|\psi\rangle^2 - \langle\psi|(A + B)|\psi\rangle^2, \\ &= \langle\psi|(\mathbb{1} + AB + A + B)|\psi\rangle\langle\psi|(\mathbb{1} + AB - A - B)|\psi\rangle, \\ &= 2^4\langle\psi|\Pi_A^+\Pi_B^+[|\psi\rangle\langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle.\end{aligned}$$

where we define the positive semi-definite operators $\Pi_A^\pm = (\mathbb{1} \pm A)/2$, and similarly for B.

We also write

$$\begin{aligned}k_{11} + k_{22} &= \langle\psi|(\mathbb{1} + AB)|\psi\rangle, \\ &= \frac{1}{2}\langle\psi|(\mathbb{1} + A)(\mathbb{1} + B) - (\mathbb{1} - A)(\mathbb{1} - B)|\psi\rangle, \\ &= 2(\langle\psi|\Pi_A^+\Pi_B^+|\psi\rangle + \langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle).\end{aligned}$$

Putting these together we have

$$\begin{aligned}p_{\rm bd} &= \frac{\langle\psi|\Pi_A^+\Pi_B^+|\psi\rangle + \langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle + 2\sqrt{\langle\psi|\Pi_A^+\Pi_B^+|\psi\rangle\langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle}}{2}, \\ &= \frac{1}{2}\big(\sqrt{\langle\psi|\Pi_A^+\Pi_B^+|\psi\rangle} + \sqrt{\langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle}\big)^2.\end{aligned}$$

Now considering qubit states we write $|\psi\rangle = \sqrt{r}|00\rangle + \sqrt{1-r}|11\rangle$, and we denote $a_{i,j} = \langle i|\Pi_A^+|j\rangle$, and $b_{i,j} = \langle i|\Pi_B^+|j\rangle$. Since $\Pi_A^+ \geq 0$, we have $a_{00}a_{11} \geq |a_{01}|^2$, and since $\mathbb{1} - \Pi_A^+ \geq 0$, we have $(1-a_{00})(1-a_{11}) \geq |a_{01}|^2$. Using these we write

$$\begin{aligned}\langle\psi|\Pi_A^+\Pi_B^+|\psi\rangle &\leq ra_{00}b_{00} + (1-r)a_{11}b_{11} + 2\sqrt{r(1-r)}\sqrt{a_{00}a_{11}b_{00}b_{11}}, \\ &= (\sqrt{ra_{00}b_{00}} + \sqrt{(1-r)a_{11}b_{11}})^2,\end{aligned}$$

and similarly

$$\langle\psi|\Pi_A^-\Pi_B^-|\psi\rangle \leq= (\sqrt{r(1-a_{00})(1-b_{00})} + \sqrt{(1-r)(1-a_{11})(1-b_{11})})^2.$$

This now gives us that

$$\begin{aligned}p_{\text{bd}}^{\text{m}} &\leq \frac{1}{2}[\sqrt{r}(\sqrt{a_{00}b_{00}} + \sqrt{(1-a_{00})(1-b_{00})} + \sqrt{1-r}(\sqrt{a_{11}b_{11}} + \sqrt{(1-a_{11})(1-b_{11})})]^2, \\ &\leq \frac{1}{2}[\sqrt{r} + \sqrt{1-r}]^2, \\ &= \frac{1}{2}[1 + C(|\psi\rangle)].\end{aligned}$$

The bit flip strategy saturates this bound thus proving the equality in Eq. (4.24). We can then extend to mixed states in the usual manner to obtain Eq. (4.25).

4.5.2 Bound saturation

Unlike for the PNP game, Werner-like states can saturate the concurrence bound, and in fact we find that the well studied Bell diagonal states [199, 200, 201, 202, 203, 204] can all saturate the bound. In order to prove this, we note that there exists a tactic with win probability at least equal to the fully entangled fraction (singlet fraction) $\mathcal{F}(\rho) = \max_\psi \langle\psi|\rho|\psi\rangle$, where the maximum is taken over all maximally entangled states of the system. A and B can achieve this by adopting the optimal tactic for the maximally entangled pure state $|\psi^*\rangle = \arg\max_\psi \langle\psi|\rho|\psi\rangle$. Since all entangled Bell diagonal states have concurrence $C(\rho) = 2\mathcal{F}(\rho) - 1$, so the tactic outlined above leads to $p_{\text{bd}}(\rho) = \frac{1}{2} + \frac{1}{2}C(\rho)$ thus saturating the bound.

The above outlined tactic also produces an interesting corollary regarding quantum teleportation [47], namely that all entangled two-qubit states capable of non-classical teleportation fidelity are also capable of non-classical performance in the BD game, since it is known that a two-qubit state can achieve non-classical teleportation fidelity if and only if $\mathcal{F} > \frac{1}{2}$ [205, 206]. It would be an interesting open question to study whether the converse statement is true. We conjecture that this might be the case by numerically verifying that examples of entangled two-qubit states with $\mathcal{F} < \frac{1}{2}$ [207] do not show non-classical BD performance.

4.6 Information theoretic approach

It is interesting to consider a different approach to studying the phenomenon of delocalised interactions. In particular, it may prove insightful to ask whether there are information theoretic quantities that can be used to intuitively capture the capacity for states to perform delocalised-interactions. Here we consider one such formulation and compare this with the above studied games.

4.6.1 Trace distance inequality

For any separable state ρ_{AB}, with any purification state $|\Psi\rangle_{ABC}$ such that $\rho_{AB} = \mathrm{Tr}_C|\Psi\rangle_{ABC}\langle\Psi|$, then for all unitaries U_A, and V_B, acting locally on A, and B, we have that

$$T(U_A\rho_{AB}U_A^\dagger, \rho_{AB}) \leq T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC}), \tag{4.26}$$

where $T(\rho,\sigma) \equiv \frac{1}{2}||\rho-\sigma||_1$ is the standard quantum mechanical trace distance.

The proof is as follows . First we prove that for the special case of pure separable states (product states) we have

$$T(U_A|\psi\rangle_{AB}, |\psi\rangle_{AB}) \leq T(U_A|\psi\rangle_{AB}, V_B|\psi\rangle_{AB}). \tag{4.27}$$

To prove this we first note that for product states $|\langle\psi|V_B^\dagger U_A|\psi\rangle| = |\langle\psi|V_B^\dagger|\psi\rangle\langle\psi|U_A|\psi\rangle|$. It then follows that

$$\begin{aligned} T(U_A|\psi\rangle_{AB}, V_B|\psi\rangle_{AB}) &= \sqrt{1-|\langle\psi|V_B^\dagger U_A|\psi\rangle|^2} \\ &= \sqrt{1-|\langle\psi|V_B^\dagger|\psi\rangle\langle\psi|U_A|\psi\rangle|^2} \\ &\geq \sqrt{1-|\langle\psi|U_A|\psi\rangle|^2} \\ &= T(U_A|\psi\rangle_{AB}, |\psi\rangle_{AB}) \end{aligned}$$

We use this result to prove the general statement as follows. Using the fact that a separable state can be decomposed into a convex mixture of product states $\rho_{AB} = \sum_i q_i|\psi_i\rangle_{AB}\langle\psi_i|$, we write

$$\begin{aligned} T(U_A\rho_{AB}U_A^\dagger, \rho_{AB}) &= T(\sum_i q_i U_A|\psi_i\rangle_{AB}\langle\psi_i|U_A^\dagger, \sum_i q_i|\psi_i\rangle_{AB}\langle\psi_i|) \\ &\leq \sum_i q_i T(U_A|\psi_i\rangle_{AB}, |\psi_i\rangle_{AB}) \\ &\leq \sum_i q_i T(U_A|\psi_i\rangle_{AB}, V_B|\psi_i\rangle_{AB}) \\ &\leq T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC}). \end{aligned}$$

To get to the second line we use convexity of the trace distance and to get to

the third we use Eq. (4.27). To arrive at the last line we consider doing state discrimination between the two states $U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC}$. All purifications of $\rho_{AB} = \sum_i q_i|\psi_i\rangle\langle\psi_i|$ can be written as $|\Psi\rangle_{ABC} = \mathcal{U}_{C'\to C}\sum_i |\psi_i\rangle_{AB}|i\rangle_{C'}$, where $\mathcal{U}_{C'\to C}$ is an isometry. A strategy to discriminate between $U_A|\Psi\rangle_{ABC}$, and $V_B|\Psi\rangle_{ABC}$, is given by first undoing any isometry, then measuring in the $|i\rangle_C$ basis, and finally performing the optimally discriminating measurement between $U_A|\psi_i\rangle_{AB}$ and $V_B|\psi_i\rangle_{AB}$. This will correctly discriminate between the two states with probability $\sum_i \frac{q_i}{2}(1 + T(U_A|\psi_i\rangle_{AB}, V_B|\psi_i\rangle_{AB})$, however the maximal discrimination probability is given by $\frac{1}{2}(1 + T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC})$, hence $\sum_i q_i T(U_A|\psi_i\rangle_{AB}, V_B|\psi_i\rangle_{AB}) \leq T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC})$. This concludes the proof.

This result has a nice operational interpretation. The left-hand side is the trace distance $T(U_A\rho_{AB}U_A^\dagger, \rho_{AB})$, which quantifies the probability that one can tell the local unitary U_A has been applied to ρ_{AB}. The right-hand side $T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC})$, quantifies the probability that someone given access to the full purified state can distinguish the application of U_A from a unitary action V_B on the other subsystem.

4.6.2 Comparison with games

Despite the operational meaning, violation of this inequality does not appear to correspond directly with non-classical performance in the considered delocalised-interaction games. For pure states it does, but we can see this does not extend to mixed states with the Werner state example. For the PNP and BD games we have non-classical performance for $a > \frac{1}{2}$, and $a > \frac{1}{3}$ respectively. However, numerically we find violation of the trace distance inequality only for $a > \frac{7}{10}$. Despite this, it is interesting to note that if we add the additional condition that A and B must keep a perfect record of what they received, or that the Bell states must not be decohered at all, then non-classical performance implies violation of the inequality for both games. These correspond to the extreme cases, where keeping a perfect record means having the left-hand side of the trace distance inequality equal to 1, and no decoherence requires the right-hand side to equal 0.

To show this we first consider the PNP game subject to one of the following conditions

1. A and B must always correctly record whether there was a particle and they win if C projects onto his original state.

2. A and B must ensure that C always projects onto his original state and they win when they correctly record whether there was a particle.

Note that these can both be considered tactical choices for the general PNP game.

Taking condition 1, A and B aim to achieve the largest possible win probability p_{pnp1} subject to a perfect record condition. They must maximise

$$\begin{aligned} &p_{\text{pnp1}}(\rho_{AB}) = \frac{3}{4} + \frac{1}{4}\text{Re}[\text{Tr}(U_A\rho_{AB}V_B^\dagger)], \\ &\text{s.t. } T(\rho_{AB}, \frac{1}{2}(U_A\rho_{AB}U_A^\dagger + V_B\rho_{AB}V_B^\dagger)) = 1. \end{aligned} \tag{4.28}$$

Note that in a finite-size Hilbert space condition 1 implies a restriction of the states ρ_{AB} that can be used to play this version of the game. In addition, we know that this game has the same bound as the general game, since we showed that for pure states the best tactic was to choose unitaries that map the initial state to an orthogonal state, in other words satisfying condition 1.

We can use our trace distance inequality to derive the classical bound for the win probability. Condition 1 and convexity of the trace distance implies that $T(\rho_{AB}, U_A\rho_{AB}U_A^\dagger) = 1$. If ρ_{AB} is a separable state then by the trace distance inequality we have $T(U_A|\Psi\rangle_{ABC}, V_B|\Psi\rangle_{ABC}) = 1$, which can be written as $(1 - |\text{Tr}(U_A\rho_{AB}V_B^\dagger)|^2)^{1/2} = 1$, and therefore $|\text{Tr}(U_A\rho_{AB}V_B^\dagger)| = 0$. Using this in the expression for the win probability given in Eq. (4.28) we have that for all separable states the maximum win probability is $\frac{3}{4}$. Therefore we can conclude that non-classical performance in the game implies violation of the trace distance inequality.

We now turn to consider condition 2. For this the entanglement bound is different from the general PNP game, so we proceed to derive it for qubits as follows. Taking condition 2 the problem is to maximise

$$\begin{aligned} &p_{\text{pnp2}}(\rho_{AB}) = \frac{1}{2}[1 + T(U_A\rho_{AB}U_A^\dagger, \rho_{AB})], \\ &\text{s.t. } \text{Re}[\text{Tr}(U_A\rho_{AB}V_B^\dagger)] = 1. \end{aligned} \tag{4.29}$$

Note that non-trivially satisfying the condition of Eq. (4.29) implies a restriction to states that can non-trivially satisfy $U_A\rho_{AB} = V_B\rho_{AB}$, which were termed strongly anonymous (SA) in the previous chapter, and were shown to be the "maximally correlated" states studied by Rains [208].

We make use of the fact that the form of states that can satisfy $U_A\rho_{AB} = V_B\rho_{AB}$ was derived earlier. We have that a general two qubit SA state can be written in the eigenbasis of the local unitaries U_A, V_B, as

$$\rho_{AB} = \rho_{00}|00\rangle\langle 00| + \rho_{01}|00\rangle\langle 11| + \rho_{01}^*|11\rangle\langle 00| + (1 - \rho_{00})|11\rangle\langle 11|.$$

Since this is in the eigenbasis of the local unitaries, the unitary action U_A can only

introduce off diagonal phases, so all possible $U_A\rho_{AB}U_A$ can be written as

$$U_A\rho_{AB}U_A = \rho_{00}|00\rangle\langle 00| + e^{i\phi}\rho_{01}|00\rangle\langle 11| + e^{-i\phi}\rho_{01}^*|11\rangle\langle 00| + (1-\rho_{00})|11\rangle\langle 11|.$$

The trace distance $T(U_A\rho_{AB}U_A, \rho_{AB})$ is then calculated via the standard formula $T(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}(\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)})$. Using an appropriate basis we can write the states in matrix form and have

$$U_A\rho_{AB}U_A - \rho_{AB} = \begin{bmatrix} 0 & \rho_{01}(e^{i\phi}-1) \\ \rho_{01}^*(e^{-i\phi}-1) & 0 \end{bmatrix}.$$

from which it follows that

$$T(U_A\rho_{AB}U_A, \rho_{AB}) = \sqrt{2(1-\cos\phi)}|\rho_{01}|.$$

As expected this is clearly maximised when $\cos\phi = -1$ and obtains a maximal value of $2|\rho_{01}|$.

We now calculate the concurrence, first writing out $\tilde{\rho} = (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$, as

$$\tilde{\rho}_{AB} = (1-\rho_{00})|00\rangle\langle 00| + \rho_{01}|00\rangle\langle 11| + \rho_{01}^*|11\rangle\langle 00| + \rho_{00}|11\rangle\langle 11|.$$

In matrix form we find

$$\rho_{AB}\tilde{\rho}_{AB} = \begin{bmatrix} \rho_{00}(1-\rho_{00}) + |\rho_{01}|^2 & 2\rho_{00}\rho_{01} \\ 2(1-\rho_{00})\rho_{01}^* & \rho_{00}(1-\rho_{00}) + |\rho_{01}|^2 \end{bmatrix}.$$

The eigenvalues are given by $\lambda_\pm = \rho_{00}(1-\rho_{00}) + |\rho_{01}|^2 \pm 2\sqrt{\rho_{00}(1-\rho_{00})|\rho_{00}|^2}$, and the square roots of these are $\sqrt{\lambda_\pm} = \sqrt{\rho_{00}(1-\rho_{00})} \pm |\rho_{01}|$. The fact that density matrices are positive semi-definite means that $\sqrt{\rho_{00}(1-\rho_{00})} \geq |\rho_{01}|$ and therefore $\sqrt{\lambda_+} \geq \sqrt{\lambda_-}$, so the concurrence is calculated as

$$\sqrt{\lambda_+} - \sqrt{\lambda_-} = 2|\rho_{01}|.$$

But this is precisely the maximum value of $T(U_A\rho_{AB}U_A, \rho)$, therefore we have shown that for qubit states that can be used to play this game we have maximum win probability

$$p^{\mathrm{m}}_{\mathrm{pnp2}}(\rho_{AB}) = \frac{1}{2}[1 + C(\rho_{AB})].$$

This is clearly reminiscent of the BD game concurrence bound, but note that not all Bell diagonal states are SA states, and therefore some states that saturate the BD game concurrence bound cannot even be used to play the PNP game with condition 2.

We could consider higher dimensional cases. The form for SA states (with no degeneracy) is $\sum_{i,j}\rho_{i,j}|ii\rangle\langle jj|$, and our local unitaries are of the form $U_A = \sum_k e^{i\phi_k}|k\rangle_A\langle k|$, $V_B = \sum_k e^{i\phi_k}|k\rangle_B\langle k|$. For the two qubit case we had two unitary phases to vary one relative phase in the state (the off-diagonal term). This made the maximization simple as we just arranged the relative phase to be π. However, we straight away can see that for high d dimensions we shall clearly run into a problem with the number of unitary phases scaling as only d, whereas the number of relative phases scales as $(d^2-d)/2$. Furthermore, even in the case $d=3$, where the scaling is not a problem since $(3^2-3)/2 = 3$, the result from the optimization appears to be a long and unenlightening expression.

Now that we know the classical bound, it is straightforward to demonstrate that non-classical performance implies violation of the trace inequality. Condition 2 implies that $T(U_A|\Psi\rangle, V_B|\Psi\rangle) = 0$, and hence for all separable states under this condition we have $T(U_A\rho_{AB}U_A^\dagger, \rho_{AB}) = 0$, giving a maximum win probability of $\frac{1}{2}$, i.e. without some form of entanglement all A and B can do is guess. Therefore we have the claimed result.

What of the BD game. For condition 1 the problem becomes the maximisation of

$$\begin{aligned} &p_{\text{bd1}}(\rho_{AB}) = \frac{1}{2} + \frac{1}{4}\text{Re}[\text{Tr}(U_A\rho_{AB}V_B^\dagger + U_A\rho_{AB}V_B)], \\ &\text{s.t. } T(\frac{1}{2}(\rho_{AB} + U_AV_B\rho_{AB}U_A^\dagger V_B^\dagger), \frac{1}{2}(U_A\rho_{AB}U_A^\dagger + V_B\rho_{AB}V_B^\dagger)) = 1. \end{aligned} \tag{4.30}$$

The classical bound for this game is the guessing probability of $\frac{1}{2}$, as we would expect. Applying the convexity of the trace distance to the condition of Eq. (4.30) leads through to $T(\rho_{AB}, U_A\rho_{AB}U_A^\dagger) = 1$. Then by applying the same arguments as above with the trace distance inequality we conclude that $|\text{Tr}(U_A\rho_{AB}V_B^\dagger)| = 0$ and $|\text{Tr}(U_A\rho_{AB}V_B)| = 0$, therefore the maximum win probability is $\frac{1}{2}$.

For the BD game under condition 2 we have to maximise

$$\begin{aligned} &p_{\text{bd2}}(\rho_{AB}) = \frac{1}{2}[1 + T(U_A\rho_{AB}U_A^\dagger, \rho_{AB})], \\ &\text{s.t. } \text{Re}[\text{Tr}(U_A\rho_{AB}V_B^\dagger)] = \text{Re}[\text{Tr}(U_A\rho_{AB}V_B)] = 1. \end{aligned} \tag{4.31}$$

Unlike in the PNP game, this time there is no need to derive a new bound, the guessing probability of $\frac{1}{2}$ is still the best we can do classically in accordance with Eq. (4.31) (since it is the do nothing strategy). By precisely the same argument as before we then see that the trace distance inequality implies a maximum win probability of $\frac{1}{2}$. Thus we have found that in all four cases, non-classical performance implies violation of the trace distance inequality.

These correspond to the extreme cases, where keeping a perfect record means

having the left-hand side of Eq. (4.26) equal to 1, and no decoherence requires the right-hand side to equal 0.

This shows that the trace distance inequality is part way successful in capturing the notion of delocalised-interactions, but that it is not strong enough. The satisfying operational interpretation cannot make up for this, and so it would be interesting if alternative approaches based on information theoretic quantities can be shown to produce a more complete picture in future research.

4.7 IBM machine demonstration

We implemented demonstrations for both the PNP and BD game, using different circuits to represent the various cases of different states prepared by Charlie. For both games we ran circuits for Alice and Bob having an entangled resource state $|\psi^+\rangle$ and only having a separable resource state $|00\rangle$. We designed the demonstrations to be implemented on a sequence of four qubits with linear connectivity. This is because it is a simple approach that requires only low depth circuits and is consistent with the qubit connectivity of the IBM device we selected, but naturally one could create more complex circuits using other architectures. We used *Paris* as it was the most recent device provided by IBM at the time of running.

The circuits for an initial entangled resource state are illustrated in Fig. 4.4. The circuits represent the cases where Charlie prepares $|\phi^+\rangle, |\psi^+\rangle$, and $|00\rangle$ respectively. The qubits q_0, q_3 are used to represent the resource state of Alice and Bob so correspond to A, B respectively. The qubits q_1, q_2 will represent the state Charlie sends and correspond to A_p, B_p, where this labelling ensures that A interacts with A_p and B with B_p. However, qubits q_1, q_2 have an extra use as before representing A_p, B_p, they will be used to distribute the initial entanglement between q_0 and q_3. We shall clarify the action of the circuits by explicitly describing their four stages.

The first stage is to prepare q_0, q_3 in the $|\psi^+\rangle_{q_0,q_3}$ Bell state, as this will be the entangled resource state. This is achieved by first preparing q_1, q_2 in the state $|\psi^+\rangle q_1, q_2$ by applying H_{q_1} followed by $\mathrm{CX}_{q_1 q_2}$, and then moving out this state by applying the swaps $\mathrm{SWAP}_{q_1 q_0}$ and $\mathrm{SWAP}_{q_2 q_3}$. This stage is the same for all three of the entangled cases, and when we do the separable cases we simply omit it and thereby Alice and Bob start with the unentangled $|00\rangle_{q_0,q_3}$

The second stage is to prepare Charlie's question state. This can be one of the three states $|\phi^+\rangle_{q_1,q_2}, |\psi^+\rangle_{q_1,q_2}, |00\rangle_{q_1,q_2}$, where for the BD game we use the first two and for the PNP game we use the latter two. The $|\phi^+\rangle$ state is prepared by applying $\mathrm{CX}_{q_1 q_2}\mathrm{H}_{q_1}$, the $|\psi^+\rangle$ state by applying $\mathrm{X}_{q_1}\mathrm{CX}_{q_1 q_2}\mathrm{H}_{q_1}$, and the $|00\rangle$ state requires no operations. This is the same for the separable cases because Alice and Bob's tactics have no bearing on the game as administered by Charlie.

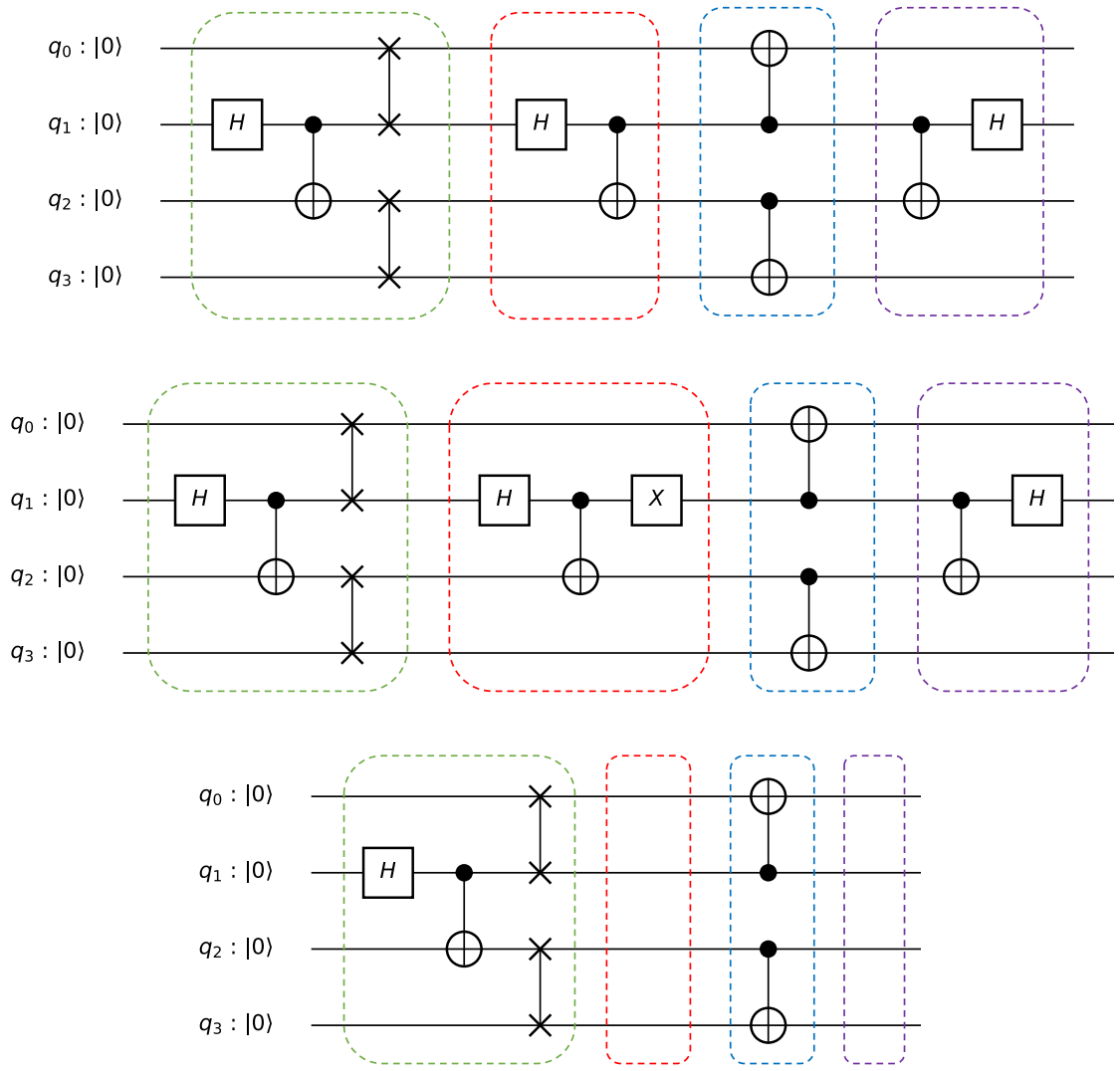

Figure 4.4: Circuit diagrams for A and B using the entangled resource state $|\phi^+\rangle$ and Charlie sending $|\phi^+\rangle$, $|\psi^+\rangle$, and $|00\rangle$, respectively. The circuits are partitioned into sections: AB State Preparation, C State Preparation, Interaction, and Measurement.

The third stage is the interaction stage and is the same in all cases. It is made up of the controlled unitaries $\mathrm{CX}_{q_1 q_0}$ and $\mathrm{CX}_{q_2 q_3}$. This is followed by the fourth and final stage which is the measurement stage. If Charlie has prepared an entangled state then he applies $\mathrm{CX}_{q_1 q_2}$ followed by H_{q_1}, and then all qubits are measured in the computational basis.

Each circuit was run with 8192 shots, with the win probabilities calculated from the measurement results, and these are presented in Fig. 4.5. The bars represent probabilities calculated from the data and the maximum total win probability for separable states is represented by the green dashed line. Note that it is the total probability that this line applies to, where the total probability is simply the average of the other two probabilities.

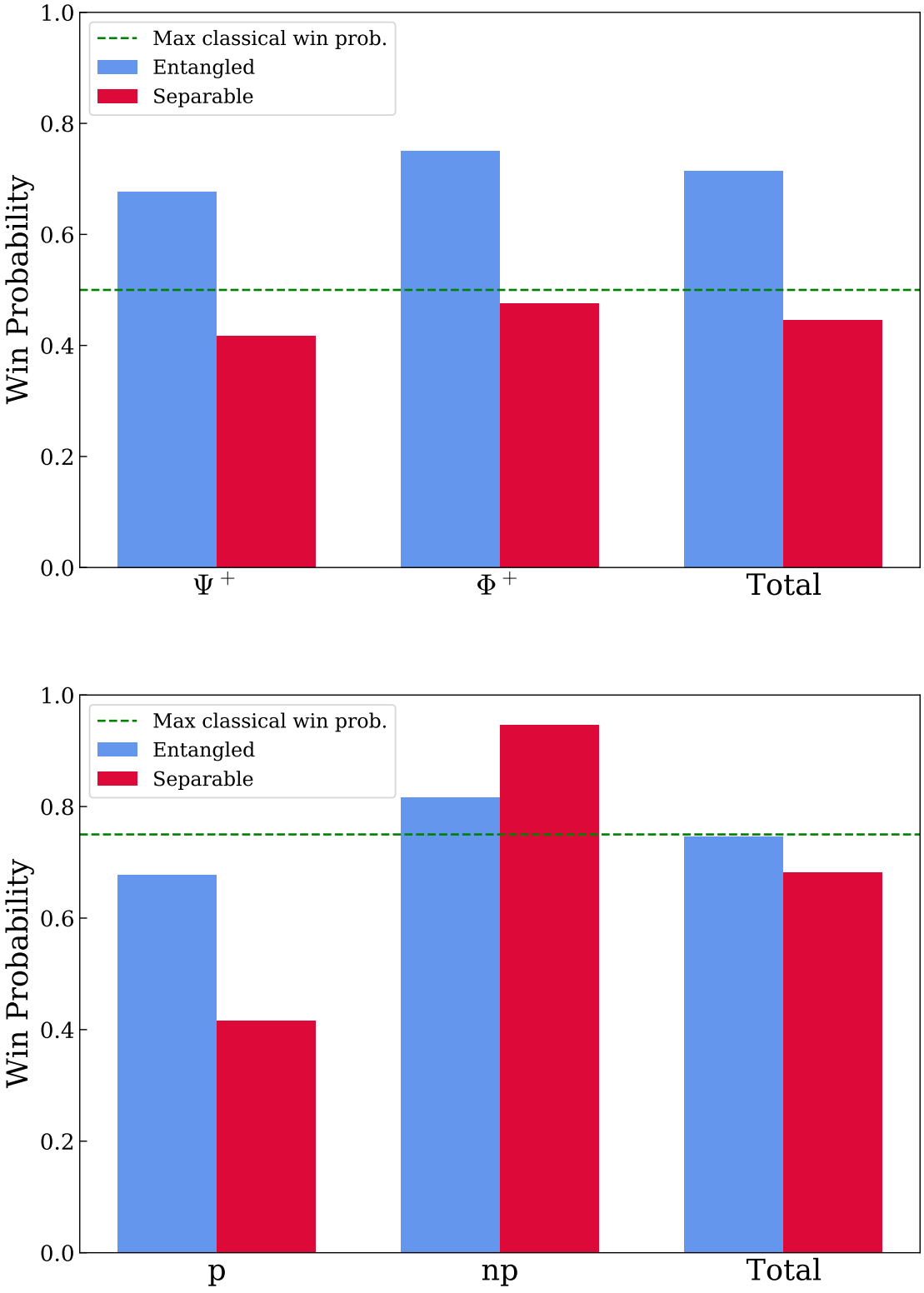

Figure 4.5: Plot of the results calculated from the Paris device measurements. The top plot is for the BD game and the bottom plot is for the PNP game. The total win probability is calculated for equal probability of sending either state. The blue bars are for the entangled initial resource state, the red for the separable, and the green line is to show the maximum classical win probabilities of $\frac{1}{2}$, and $\frac{3}{4}$, respectively.

Focusing first on the BD results, we see that the entangled state total win probability is higher than the classical limit of 0.5, from which we calculate that it demonstrates a usable concurrence of 0.42, and thus a convincing delocalised interaction. However, the realised win probability is far below the ideal case of 1. Additionally, we note the separable case is below the classical limit, demonstrating the problem with noise even for the classical strategy. For the standard PNP game we note that we were unable to convincingly demonstrate non-classical performance, as the entangled state total win probability approximately matches the classical limit of 0.75, it does not exceed it. Despite this, it should be noted that the entangled strategy still out performs the separable strategy as run on the device, which indicates the entanglement is still acting beneficially. Furthermore, if we use the different sending probabilities $P_{\mathrm{p}} = \frac{2}{3}$, $P_{\mathrm{np}} = \frac{1}{3}$ then we get total win probability 0.72 which is higher than this game's classical limit of $\frac{2}{3}$. Note however, that we currently do not have a way to directly relate this violation to an entanglement measure. Clearly the noise in the device is significant, but we still see non-classical behaviour and with the impressive rate of improvement in this field [209] one can envisage even better results in the near future.

Though the BD case apparently clearly demonstrates delocalised-interactions, as in Bell tests, there are a number of loopholes [210] one could consider to avoid the conclusion that a delocalised interaction took place. Chief amongst them is that A and B actually interacted during the game, which could be solved by keeping them space-like separated for the duration. The difficulty with this comes from having to reliably and quickly send quantum information, which for superconducting qubits is not currently feasible, therefore photonic qubits [211] could prove to be more appropriate.

4.8 Conclusions

In this chapter we studied the concept of delocalised-interactions. Information encoded using non-locally superposed quantum states, is recorded via local interactions whilst disturbing the superposition less than would be classically possible. This phenomenon has interesting foundational implications regarding events not requiring unique locations and has also been a key component for certain quantum protocols [178, 163].

In order to systematically study this quantum effect, we introduced and investigated quantum games for which non-classical performance demonstrates delocalised-interactions. We performed an in depth study of two particular instances, termed the Particle/No Particle game and the Bell Distinguishing game. This enabled us to prove a direct operational use of concurrence in bounding the non-classical win

probabilities, and a connection with quantum teleportation.

We also considered an information theoretic approach based on a trace distance inequality. This had a pleasing operational interpretation, but we found that though its violation corresponded with non-classical performance in the games for certain cases, in general it was not strong enough to fully capture the concept of delocalised interactions.

Finally, the delocalised-interaction games were demonstrated on an IBM superconducting quantum processor. Despite noise we were able to claim non-classical performance, though it must be noted that this was in no sense a a loophole free demonstration.

Our work can spur further research building from the tools and ideas introduced here, such as generalising to higher dimensions or multipartite settings, and establishing the exact nature of the connection with quantum teleportation. Additionally there may be improvements to understanding the phenomenon with information theoretic quantities, and it may prove interesting to compare the notions here with ideas regarding quantum mechanical time-delocalization [212].

With this chapter and the one preceding it we have performed detailed studies in operational settings for quantum correlations. In other words we have been looking at what quantum correlations, and in particular entanglement, make possible. In the next chapter we shift lenses and ask not what entanglement can do but how it behaves, in particular when we consider motion and relativity.

Chapter 5

Entanglement under motion

I do not define time, space, place and motion, as being well known to all.
– Isaac Newton in Principia

Just wait til you get a load of my definitions.
– Albert Einstein in Response

5.1 Overview

In this chapter we shall consider entanglement under motion. In particular the interesting effects when relativity is accounted for. We start by reviewing established results regarding internal spin degrees of freedom. We then show that analogous behaviour can be established more straightforwardly for the case of internal energy states. In doing this, we write down and examine an appropriate form for the required velocity boost. We then use this understanding to establish a distinction between classical and non-classical proper time for quantum clocks, with the velocity boost being the key to obtaining the classical behaviour. To do this we consider sequences of appropriately centred boosts and evolution operators to derive the different possible clock behaviours in a twin paradox scenario. From the classical observer's frame we show that the difference between the classical observer and quantum clock being set in motion is captured by translation operators, and that it is the transformation under translation operators that enables the velocity boost to describe both situations. In addition, we show how the necessary translation operators can be understood via considering the placement of the origin for the required potentials. We highlight that without an internal state dependent force one should expect additional effects. We demonstrate this for a theoretical clock model and in a more practical setting by deriving frequency shifts in ion traps, predicting the already observed shifts and an additional smaller shift. This chapter draws from [213].

5.2 Motion altering entanglement

5.2.1 Spin and motion

In the late 1990s and early 2000s, interesting work emerged on how the entanglement properties of spin systems can be affected by motion [214, 215, 216, 217]. The authors showed that relativistic physics indicates potentially unintuitive behaviour when considering changes of references frames, centred around the fact that boosts can vary the entanglement between spin and motional degrees of freedom.

Perhaps the most straightforward way to understand this is by first discussing the Thomas-Wigner rotation [218, 219]. This is the name given to a special relativistic phenomenon, whereby the combination of two non-collinear boosts results in not simply another boost, but a boost together with some additional rotation.

Writing the matrix for a boost of velocity $\mathbf{v}$ in block diagonal form we have

$$\Lambda_{\mathbf{v}} = \begin{pmatrix} \gamma_{\mathbf{v}} & -\frac{\gamma_{\mathbf{v}}}{c}\mathbf{v}^T \\ -\frac{\gamma_{\mathbf{v}}}{c}\mathbf{v} & \mathbf{I} + \frac{\gamma_{\mathbf{v}}^2}{c^2(\gamma_{\mathbf{v}}+1)}\mathbf{v}\mathbf{v}^T \end{pmatrix}, \tag{5.1}$$

where c denotes the speed of light and the Lorentz factor is $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\|\mathbf{v}\|^2/c^2}}$.

Now the combination of two boosts is found to be

$$\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u}} = \begin{pmatrix} \gamma & -\mathbf{a}^T \\ -\mathbf{b} & \mathbf{M} \end{pmatrix}, \tag{5.2}$$

where

$$\gamma = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}}(1 + \mathbf{v}^T\mathbf{u}/c^2), \tag{5.3}$$

$$\mathbf{a} = \frac{\gamma}{c}(\mathbf{u} \oplus_r \mathbf{v}), \tag{5.4}$$

$$\mathbf{b} = \frac{\gamma}{c}(\mathbf{v} \oplus_r \mathbf{u}), \tag{5.5}$$

$$\mathbf{M} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{\mathbf{v}\mathbf{u}^T}{c^2} + (\mathbf{I} + \frac{\gamma_{\mathbf{v}}^2}{c^2(\gamma_{\mathbf{v}}+1)}\mathbf{v}\mathbf{v}^T)(\mathbf{I} + \frac{\gamma_{\mathbf{u}}^2}{c^2(\gamma_{\mathbf{u}}+1)}\mathbf{u}\mathbf{u}^T), \tag{5.6}$$

and we are using $\oplus_r$ to denote relativistic velocity addition, which is defined as

$$\mathbf{u} \oplus_r \mathbf{v} = \frac{1}{1 + \mathbf{u}^T\mathbf{v}/c^2}\left[\left(1 + \frac{\gamma_{\mathbf{u}}}{c^2(1+\gamma_{\mathbf{u}})}\mathbf{u}^T\mathbf{v}\right)\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}\right]. \tag{5.7}$$

The key result is that $\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u}}$ cannot in general be written as a single boost $\Lambda_{\mathbf{w}}$ for some velocity $\mathbf{w}$. Instead it can be decomposed into a boost and a rotation as

$$\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u}} = R(\boldsymbol{\epsilon})\Lambda_{c\mathbf{a}/\gamma}, \tag{5.8}$$

where $R(\boldsymbol{\epsilon})$ is a rotation about the vector $\boldsymbol{\epsilon}$, which has direction $\hat{\boldsymbol{\epsilon}} = \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}$, and magnitude given by

$$\cos\|\boldsymbol{\epsilon}\| = \frac{(1+\gamma+\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}})^2}{(1+\gamma)(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})} - 1. \tag{5.9}$$

With this concept outlined we can now turn from classical special relativity to the consequences for quantum mechanical spin operators. We work with a particle which has some internal spin degree of freedom, such that the full Hilbert space is $\mathcal{H}_{\text{motional}} \otimes \mathcal{H}_{\text{spin}}$. Consider said particle in some lab frame where it has been prepared in a motional pure state $|\phi\rangle$ and a spin pure state $|s\rangle$, such that the full state can be written as the product state $|\phi\rangle|s\rangle$.

If we boost past this system with velocity $\mathbf{v}$ then what happens to our description of this state? First note that we can consider the motional state as a wavepacket of momentum states, i.e. $|\phi\rangle = \int d\mathbf{p}\phi(\mathbf{p})|\mathbf{p}\rangle$, where $\phi(\mathbf{p}) = \langle\mathbf{p}|\phi\rangle$. Each of these momentum states defines a different frame. To illustrate this we write $|\mathbf{p}\rangle|s\rangle$ in terms of a boost transformation $\Lambda_{\mathbf{u_p}}$ as $\Lambda_{\mathbf{u_p}}|\mathbf{0}\rangle|s\rangle$. We now act on the full state with $\Lambda_{\mathbf{v}}$ and thus write the state in the new frame as

$$\int d\mathbf{p}\phi(\mathbf{p})\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u_p}}|\mathbf{0}\rangle|s\rangle. \tag{5.10}$$

Now we use the fact that $\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u_p}}$ will in general not produce just a boost but also some rotation which will affect the spin state. Furthermore, this rotation will in general depend on $\mathbf{p}$. We therefore can heuristically write the action of this operator as $\Lambda_{\mathbf{v}}\Lambda_{\mathbf{u_p}}|\mathbf{0}\rangle|s\rangle = |\mathbf{q_p}\rangle R(\boldsymbol{\theta}_{\mathbf{p}})|s\rangle = |\mathbf{q_p}\rangle|s_{\mathbf{p}}\rangle$, where $\mathbf{q_p}$ is some resulting momentum for which we explicitly highlight the $\mathbf{p}$ dependence, and $R(\boldsymbol{\theta}_{\mathbf{p}})$ is some rotation applied to the spin state, where again we emphasise the $\mathbf{p}$ dependence. The full state can in this manner be written as

$$\int d\mathbf{p}\phi(\mathbf{p})|\mathbf{q_p}\rangle|s_{\mathbf{p}}\rangle. \tag{5.11}$$

It is now apparent that in general this state is not separable. By boosting to a different frame the internal and motional states can become entangled.

This phenomenon leads to interesting consequences. For instance, as noted by Gingrich and Adami [217], taking a pair of entangled spins and boosting to a different frame can result in the spin states becoming less entangled, the entanglement having shifted into their motional states. Understanding the phenomenon can result in a significant shift in perspective, indeed Peres, Scudo and Turno [216] went so far as to say that "we have shown that the notion "spin state of a particle" is meaningless if we don't specify its complete state, including the momentum variables."

5.2.2 Energy and motion

The results outlined above whereby boosting can affect the entanglement between an internal spin state and its motional state are highly interesting. However, the results are somewhat lengthy to derive and Wigner rotations are not an intuitively easy to grasp concept. We shall now show that the capacity for relativity to lead to such phenomena does not require such complexity, we simply need to consider a different consequence of relativity theory, perhaps the best known consequence.

As many a T-shirt will tell you, $E = Mc^2$. This is the well known concept Einstein introduced in the follow up paper [220] to his introduction of Special Relativity. Now consider the consequences of this for some quantum system which is prepared in an energy superposition, for instance a trapped ion prepared in a superposition of electronic excited states $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Relativity informs us that the ion must also be considered in a superposition of inertial masses. In the ground state we can write M_0 and in the excited state we have M_1. If we change frame by boosting past this atom at velocity v then this will impart different momenta depending on which branch of the superposition we are considering. We will respectively impart $M_0 v$ and $M_1 v$, and therefore by the same reasoning as for the rotations above, we shall have affected the entanglement between the internal and motional state.

We can formalise this simple reasoning by introducing the one-dimensional velocity boost operator

$$B_v(v_b) \equiv e^{i(m+H_0/c^2)v_b x/\hbar}, \tag{5.12}$$

where we have decomposed the total mass $M = m + H_0/c^2$, using the internal state Hamiltonian H_0, and we take m as the rest mass of the particle in the internal energy ground state $|0\rangle$, with $H_0|0\rangle = 0$.

An immediate consequence of using this velocity boost is precisely the fact described above that frame changes alter the entanglement between the motional and internal degrees of freedom. Repeating the above analysis but now with the boost operator we can write $B_v(v_b)|p\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|p + M_0 v_b\rangle|0\rangle + |p + M_1 v_b\rangle|1\rangle)$. The internal and motional states are separable and maximally entangled for the non-boosted and boosted frames respectively.

It is interesting to consider the size of this effect in a more practical setting. As such we shall consider a Gaussian motional state and calculate how much our description of the internal state changes when we boost to a new frame. To this end, assume that in the laboratory frame we have prepared our system to be in the state $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, where $|\phi\rangle$ is a Gaussian state, such that in the position basis we can write

$$\phi(x) = \langle x|\phi\rangle = \frac{e^{-x^2/4\sigma_x^2}}{\sigma_x^{1/2}(2\pi)^{1/4}}, \tag{5.13}$$

where we have chosen our origin as the centre of the wave-packet, and note that this is normalised such that $\int_{-\infty}^{\infty} dx|\phi(x)|^2 = 1$.

In the laboratory frame, the state of the internal energy system is naturally $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. However, if we now boost to a new frame our description of the internal state changes, as the tracing out of the motional state will have a decohering effect. The new state can be written in matrix form as

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \langle\phi|e^{-i(\Delta M)v_b x/\hbar}|\phi\rangle \\ \langle\phi|e^{i(\Delta M)v_b x/\hbar}|\phi\rangle & 1 \end{pmatrix}, \tag{5.14}$$

where $(\Delta M) = M_1 - M_0$.

We can evaluate the off-diagonal terms via

$$\begin{aligned} \langle\phi|e^{-i(\Delta M)v_b x/\hbar}|\phi\rangle &= \int_{-\infty}^{\infty} dx e^{-i(\Delta M)v_b x/\hbar}|\phi(x)|^2, \\ &= \frac{1}{\sigma_x(2\pi)^{1/2}}\int_{-\infty}^{\infty} dx e^{-x^2/2\sigma_x^2 - i(\Delta M)v_b x/\hbar}, \\ &= \frac{e^{-\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2}}{\sigma_x(2\pi)^{1/2}}\int_{-\infty}^{\infty} dx e^{-\frac{(x+i\sigma_x^2(\Delta M)v_b/\hbar)^2}{2\sigma_x^2}}, \\ &= e^{-\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2}. \end{aligned} \tag{5.15}$$

Here we have used the standard trick of completing the square in the exponent to evaluate the Gaussian integral. We therefore have that the new internal state can be written as

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & e^{-\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2} \\ e^{-\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2} & 1 \end{pmatrix}, \tag{5.16}$$

with exponential decay terms on the off-diagonal, which kill off the coherences more strongly as we increase (ΔM), v_b or σ_x. The dependence on (ΔM) and v_b is intuitive, and the dependence on σ_x can be understood by recalling that as we increase the uncertainty in position we are decreasing the uncertainty in momentum, and therefore getting closer to the extreme case of momentum eigenstates considered above, where even the smallest boost and mass difference will lead to orthogonal motional states.

In order to better appreciate the effect we now consider the magnitude of this decay in a day to day setting. Consider an experimentalist with a trapped ion[221], prepared with its internal state in an optical qubit superposition. If the experimentalist then walks across the room to get a cup of tea[1], what is the effect on their description of the state?

A human walks at roughly $v_b \approx 1.5ms^{-1}$ and the average experimentalist is

[1]Don't worry, I'm sure the experimentalist has no intention of drinking the tea in the lab as that would no doubt violate health and safety protocols.

human. An optical qubit has $\omega_0 \approx 10^{15}s^{-1}$ which gives $(\Delta M) = \hbar\omega_0/c^2 \sim 10^{-36}kg$. To estimate the position variance we shall assume that the ion has been prepared close to its ground state. The ground state is easy to solve for by using $a|0\rangle = 0$, and acting on this as $\langle x|a|0\rangle = 0$. Recall that $a = \sqrt{\frac{m\omega_m}{2\hbar}}(x + \frac{i}{m\omega_m}p)$ with ω_m the motional trapping frequency and m the ions mass (here we shall ignore the variation since we only care about orders of magnitude), and also that $\langle x|p|\psi\rangle = -i\hbar\frac{\partial\psi}{\partial x}$. Using these and writing $\langle x|0\rangle = \psi_0(x)$ we find the differential equation

$$(x + \frac{\hbar}{m\omega_m}\frac{\partial}{\partial x})\psi_0(x) = 0. \tag{5.17}$$

This can be solved to give the normalised ground state wave-function

$$\psi_0(x) = \left(\frac{m\omega_m}{\pi\hbar}\right)^{1/4} e^{-m\omega_m x^2/2\hbar}. \tag{5.18}$$

This gives us the variance $\sigma_x^2 = \frac{\hbar}{2m\omega_m}$, and using a trap frequency $\omega_m \approx 10^6 s^{-1}$ and ion mass $m \approx 10^{-26}kg$, we have $\sigma_x^2 \sim 10^{-15}m^2$.

Putting all of this together we have a $\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2 \sim 10^{-19}$, so we can write the off-diagonal coherences as $e^{-\sigma_x^2(\Delta M)^2 v_b^2/2\hbar^2} \approx 1 - 10^{-19}$. It is a very small effect, interesting but not something that should trouble us day to day. However, when we consider atomic clocks later on we shall show that the consequences of the mass difference between energy levels are not always so negligible.

Another interesting effect that suggests itself during this discussion, is the potential limits of our initial assumption, namely that the state is prepared in $|\phi\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. It is immediately apparent that if we want to use motional ground states then this is not possible, since the motional ground state depends on the mass so is different for the two branches of the superposition. We should then write the full state as $\frac{1}{\sqrt{2}}(|\phi_0\rangle|0\rangle + |\phi_1\rangle|1\rangle)$, and if we are just interested in the internal qubit state then the coherences depend on $\langle\phi_1|\phi_0\rangle$. For two motional ground states we can evaluate this as

$$\begin{aligned}
\langle\phi_1|\phi_0\rangle &= \left(\frac{M_0\omega_{M_0}}{\pi\hbar}\right)^{1/4}\left(\frac{M_1\omega_{M_1}}{\pi\hbar}\right)^{1/4}\int_{-\infty}^{\infty} dx e^{-(M_1\omega_{M_0}+M_0\omega_{M_1})x^2/2\hbar}, \\
&= \left(\frac{M_0\omega_{M_0}}{\pi\hbar}\right)^{1/4}\left(\frac{M_1\omega_{M_1}}{\pi\hbar}\right)^{1/4}\left(\frac{M_0\omega_{M_0}+M_1\omega_{M_1}}{2\pi\hbar}\right)^{-1/2}, \\
&= \frac{\sqrt{2}(M_0\omega_{M_0}M_1\omega_{M_1})^{1/4}}{\sqrt{M_0\omega_{M_0}+M_1\omega_{M_1}}}, \\
&= \frac{\sqrt{2}(M_0M_1)^{1/8}}{(\sqrt{M_0}+\sqrt{M_1})^{1/2}}.
\end{aligned} \tag{5.19}$$

Note that here we have had to account for the fact that ω will be mass dependent, as

can be appreciated by considering that for a harmonic oscillator we have $M\omega^2 = k$ and noting that this "spring constant" k should be held constant. In the final line we used this relation to write the expression entirely in terms of the masses.

This result implies a limit to the purity of the internal energy state taken on its own. We can calculate this using the same numbers as before and find that for this we have $\langle\phi_1|\phi_0\rangle \approx 1 - 10^{-16}$. As expected, the effect is very small, and would no doubt be dwarfed in experiments by other forms of noise, but again it is an interesting consequence of properly accounting for the entanglement induced via the relativistic treatment.

We shall move on from practical considerations for now and return to consider the form of the boost presented in Eq. (5.12). We shall now examine how we can arrive at this form more rigorously. We do this by beginning from the fully relativistic momentum transformation for a particle of rest mass M, and initial momentum p, for a new frame moving at velocity v. The Lorentz boost gives

$$p' = \sqrt{\gamma_v^2 - 1}\sqrt{M^2c^2 + p^2} + \gamma_v p, \tag{5.20}$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$. Now writing $\hat{M} = m + H_0/c^2$, and taking $\frac{v^2}{c^2}$, H_0/mc^2, and $\frac{p^2}{m^2c^2}$, as all $\ll 1$, then to first order we find

$$p' \approx p + mv(1 + \frac{H_0}{mc^2} + \frac{p^2}{2m^2c^2}) + \frac{v^2}{2c^2}p. \tag{5.21}$$

If we consider working in the limit where $H_0/mc^2 \ll 1$ but $H_0/mc^2 \gg \frac{p^2}{m^2c^2}$ and $H_0/mc^2 \gg \frac{v^2}{c^2}$, we can recover the momentum shift as used in the definition of the boost operator. In other words, this boost operator is valid when the relevant relativistic correction for the frames of interest is the mass-energy dependent correction and not the corrections resulting from high velocities.

One can also examine the boost by considering the limiting behaviour for the Lie algebra of the Poincaré group. The relevant commutation relations are

$$\begin{aligned} [H, p] &= 0, \\ [H, K] &= i\hbar cp, \\ [p, K] &= \frac{i\hbar}{c}H, \end{aligned} \tag{5.22}$$

where H is the generator of time translations, p the generator of spatial translations and K the generator of boosts.

For our case we are setting $H = Mc^2 + p^2/2M$, $K = -Mcx$, and $p = p$. We then find $[H, p] = 0$, $[H, K] = i\hbar cp$, and $[p, K] = \frac{i\hbar}{c}Mc^2$. Note the last one does not exactly satisfy the Poincaré commutation relation, for which we would need not Mc^2 but

$Mc^2 + p^2/2M$. This is exactly analogous to the situation in standard non-relativistic quantum mechanics, as one can see by replacing M with m. The difference then is that we are accounting for the term H_0/c^2 that is contained in M. In order for this term to be significant where the $p^2/2M$ term can still be neglected, we require $H_0/c^2 \gg p^2/2M$, which is satisfied if we are in the regime $H_0/mc^2 \gg p^2/m^2c^2$.

With the boost of Eq. (5.12) we have immediately seen similar behaviour for energy states as that found for spins, and we did not need the complications of Wigner rotations. As such the various interesting consequences found in the former case can also be considered for the latter. For instance, we could perform a boost relative to a pair of energy qubits that are entangled and this could shift the entanglement to their motional states. It could thus be debated whether we should choose to follow Peres, Scudo and Turno [216] and say that we have shown that the notion "energy state of a particle" is meaningless if we don't specify its complete state.

It is worth noting some of the differences between the energy and spin case, for though they have clear similarities, there are also important differences. The most obvious is that they depend on conjugate variables. For spin the boost includes a coupling of the internal state and the momentum, whereas for energy the boost couples the internal state and the position. As such, the momentum state should not lead to entangling between internal spin and motional states, but it leads to the maximal effect for internal energy and motional states, and vice versa for the position state.

Another difference is that in the energy case the boost operator cannot affect the states spanned by the energy state. For spin we can have some continuous rotation applied to the spin, so by boosting we could take it from $|\uparrow\rangle\langle\uparrow|$ to some state where $|\downarrow\rangle\langle\downarrow|$ is included in the span. For the energy case we do not have this as the boost operator is always diagonal in the energy basis.

A final difference worth noting here is that the spin effect requires multiple spatial dimensions. For a Wigner rotation the boosts have to be non-collinear. In the case of energy this is not so, one dimension is sufficient to witness the effect, and this contributes to the greater simplicity in calculating its consequences.

Despite these differences, the overarching conclusion is similar. When we account for relativistic effects we can end up with different frames observing differing entanglement properties for the various subsystems, as a consequence of boosts affecting the entanglement between internal and motional states.

The inclusion of mass-energy equivalence into quantum mechanics is not new. In particular, the mass-energy equivalence has been used to modify non-relativistic Hamiltonians in order to study quantum mechanical proper time [222]. With the velocity boost laid out in this chapter we are able to shed new light on this topic.

5.3 Consequences for proper time of quantum clocks

Proper time, ideal clocks, and boosts are well understood classically, but subtleties arise in quantum physics. We shall first give a brief overview of some of the ideas and research on this topic. We shall then show that a proper understanding of classical time dilation for quantum clocks requires use of the velocity boost defined above. We contrast this with the alternative uncoupled momentum effect and demonstrate that it is velocity boosts which lead to the ideal behaviour in both cases where the quantum clock and classical observer are set in motion.

Ideal clocks and proper time are key concepts in special and general relativity [223]. Full understanding of the union between relativity and quantum mechanics, must include how these ideas extend to the quantum realm. Recent work in this area can broadly be divided by whether the quantum clocks follow classical or quantum trajectories.

Adopting the former approach [224, 225, 226, 227] enables the utilization of techniques from quantum field theory in curved spacetime. In particular this has allowed explorations into consequences of the Unruh effect [228], and applications of techniques from relativistic quantum metrology [229, 230]. On the other hand, for quantum clocks following quantum mechanical trajectories [222, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240], most progress has been made investigating connections between proper time and mass superpositions [241]. This has necessitated the rejection of the Bargmann mass superselection rule [242], on the grounds that our universe is not Galilean. Notably this paradigm was used to investigate ideas for intrinsic time dilation decoherence caused by gravity [231].

We shall follow the second approach, where the clock's motion is described quantum mechanically. We show that a quantum clock set into motion by a force that does not depend on the internal state is not witnessing classical time dilation, since there is no unique Lorentz factor. This is because quantum clocks require coherence in some non-degenerate energy states [243, 244] but the inertial mass of this energy means that assigning an identical momentum to each branch of the superposition does not correspond to a well defined velocity. We therefore show that momentum boosts lead to a nonclassical dilation due to the lack of a unique Lorentz factor. On the other hand, by suitably coupling the motional and internal degrees of freedom, one can see that the velocity boost exactly recovers the expected classical time dilation results for the "twin paradox," in both cases where the observer and the quantum clock are respectively the ones set in motion.

We start from a Hamiltonian modified to account for the inertial mass of internal energy. We shall use this to point out the tempting but incorrect identification of classical time dilation and quantum clocks.

5.3.1 Modified Hamiltonian

We here present the simplest argument for the modified Hamiltonian. Note that this modification along with being introduced to study quantum mechanical proper time [222], has also been shown to resolve paradoxes in quantum optics [245, 246]. For a free composite particle of mass M, the non-relativistic Hamiltonian will consist of a kinetic energy term $\frac{p^2}{2M}$, and an internal energy term H_0. However the internal energy should contribute to the inertial mass since special relativity dictates that energy and inertial mass are equivalent. This leads to $M = m + H_0/c^2$, where we take m as the rest mass of the particle in its internal energy ground state $|0\rangle$, and set $H_0|0\rangle = 0$. Thus we have

$$\begin{aligned} H &= \frac{p^2}{2M} + H_0, \\ &= \frac{p^2}{2m} + H_0\left(1 - \frac{p^2}{2mMc^2}\right). \end{aligned} \tag{5.23}$$

To arrive at the second line we have used $1/(x+y) = 1/x - y/x(x+y)$. Note that since M is now operator valued, there is potential for ambiguity with operator ordering. However if the internal Hamiltonian commutes with the total momentum $[H_0, p] = 0$, then $[M^{-1}, p] = 0$. Fully accounting for relativity would strictly imply that the internal degrees of freedom should be described by a relativistic wave equation (or a quantum field theory). However the approach here is that regardless of the formalism, the effect on the centre of mass dynamics should only be via a mass change, otherwise we could not claim that energy contributes to inertial mass as dictated by mass-energy equivalence.

This Hamiltonian is often expanded in H_0/mc^2 neglecting higher order terms [231, 246] to write

$$H = \frac{p^2}{2m} + H_0\left(1 - \frac{p^2}{2(mc)^2}\right). \tag{5.24}$$

It is then tempting to claim that $(1 - \frac{p^2}{2m^2c^2})$ represents our familiar notion of time dilation, however, this is not correct as we shall show. It should be emphasised that this Hamiltonian is often a good approximation but that it can prove misleading.

It is also important to emphasise that we shall always be working in the limit where the energy $E = \sqrt{p^2c^2 + M^2c^4}$ is approximated by $E = p^2/2M + Mc^2$. In words, one can consider the regime we utilise as one where the mechanics can be treated in a "Newtonian" sense, but the rest mass of the internal energy is now accounted for. On a technical level one must appreciate (as noted in [234]) that there are two relevant small quantities: H_0/mc^2 and p^2/m^2c^2, where the first relates to the internal degrees of freedom and the second can be viewed as a motional v^2/c^2 term (note $p^2/m^2c^2 > p^2/M^2c^2$,). It is therefore not sufficient to merely think of

approximations in terms of how many factors of $1/c^2$, are present. One can consider the regime where $H_0/mc^2 \ll 1$, but $H_0/mc^2 \gg p^2/m^2c^2$. With this in mind one arrives at Eq. (5.23) (plus ground state rest mass energy) by expanding the full relativistic energy $E = \sqrt{M^2c^4 + p^2c^2} = \gamma Mc^2$, neglecting terms of $O(p^4/m^4c^4)$ in the Lorentz factor whilst retaining the H_0/mc^2 terms and translating the zero of energy by mc^2. Note that in practise the order of H_0/mc^2 terms kept would be dictated by the physical system under consideration and depends on the integer n for which $(H_0/mc^2)^n \gg p^2/m^2c^2$. However, as will become apparent, it will prove more straightforward for our initial theoretical study to work with the untruncated $(1 + H_0/mc^2)^{-1}$. We shall denote the unitary evolution generated by Eq. (5.23) over time t as $U(t)$.

5.3.2 Different boosts

We now explore the consequences of the modified Hamiltonian. Time is a complex topic in quantum mechanics [247, 248], but here we shall simply take the internal state (Hamiltonian H_0) to define some quantum clock, and consider the situation where it is boosted away and back, then measured to observe the motion's effect on the clock. To do this we use the following sequence of operations as illustrated in Fig. 5.1. First we apply some boost operator to the particle and let it freely evolve for some time t, then at a shifted position we apply the inverse boost twice and let it evolve for another time t, and finally we apply the original boost to stop the motion. Note the magnitude of all boosts must be chosen such that the state is kept within the approximation regime of our Hamiltonian.

We initially work with the boost that corresponds to the claim that the $(1-\frac{p^2}{2m^2c^2})$ in Eq. (5.24) is the correct classical time dilation. This is the standard quantum mechanical momentum boost, which centred at the origin we write as

$$B_p(p_b) \equiv e^{ip_b x/\hbar}. \tag{5.25}$$

This acts on momentum eigenstates as $B_p(p_b)|p\rangle = |p + p_b\rangle$. This represents the physical situation typically considered for use in the laboratory [249], with no internal state dependence as per the potentials typically used to move quantum systems (e.g. an ion moved via an Electromagnetic potential).

Using this boost, the translation operator $T(p_b t/m) = e^{-ipp_b t/m\hbar}$, and the free evolution under the Hamiltonian of Eq. (5.23), we have

$$B_p(p_b)U(t)T(p_b t/m)B_p(-2p_b)T(-p_b t/m)U(t)B_p(p_b) = e^{-\frac{i2t}{\hbar}\frac{p^2}{2M}} e^{\frac{2it}{\hbar}\frac{p_b^2}{2m}} e^{-\frac{2it}{\hbar}H_0(1-\frac{p_b^2}{2mMc^2})}. \tag{5.26}$$

The first exponential term is the unaltered motional evolution of the state that we

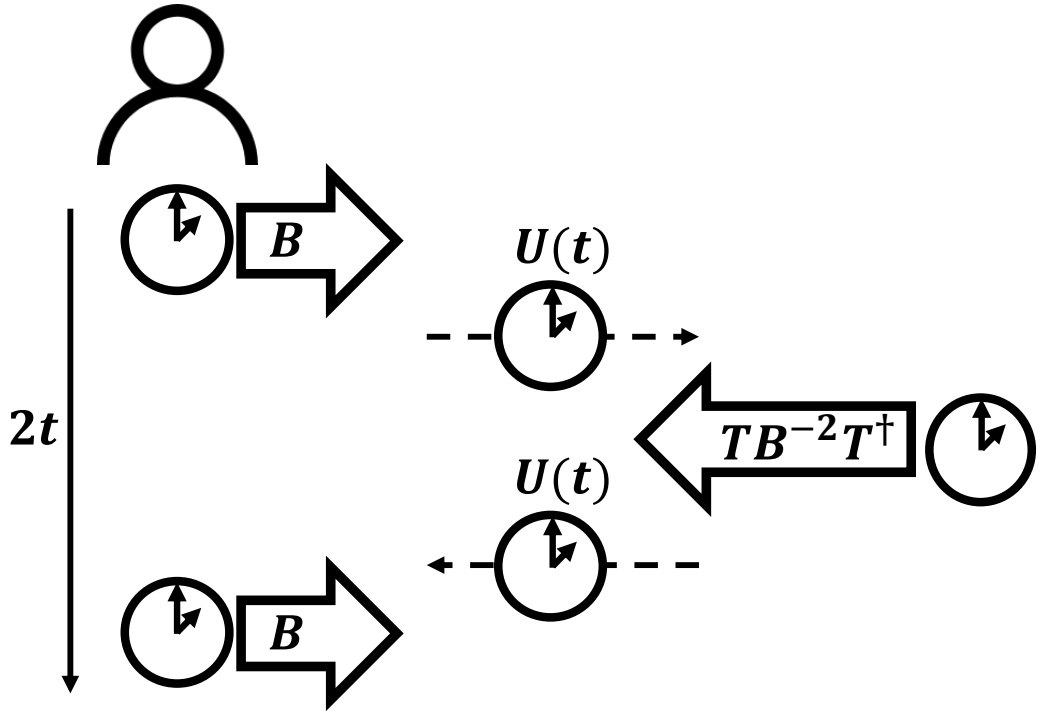

Figure 5.1: Schematic illustration of the boost and evolution sequence. Note that the second boost is twice the inverse of the first, as it has to stop the initial motion and then replicate it in the opposite direction. Note also that this stage is shifted away from the origin via appropriate translation operators.

would expect if we had not applied any of the boosts. This term has no p_b dependent relativistic corrections, which is not surprising since the adopted formalism does not include the relevant relativistic motional terms. The second term is a global phase that is connected to the choice of the translation operators, discussed in detail below. However, it is the final term that captures the effect on the evolution of the internal state and therefore is our primary focus.

The third exponential term in Eq. (5.26) has the internal Hamiltonian multiplied by a factor $(1 - \frac{p_b^2}{2mMc^2})$ that is less than unity. This is precisely the factor we would get by considering Eq. (5.23) acting on a momentum eigenstate $|p_b\rangle$. We see that if we have some coherence in the internal state and are using it as a clock, then it appears that the clock is running slower. Note it is not dilated by a constant inverse Lorentz factor as in classical relativity. We can see why this is so by asking what Lorentz factor we would expect. The clock has been given a momentum p_b but because our clock is in a superposition of energies E_n, we also have a superposition of different masses $M_n = m + E_n/c^2$. Hence we can write various velocities and thus Lorentz factors. For example, the N single shot values $\gamma_n = \sqrt{1 + p_b^2/M_n^2c^2}$, or the expectation value of an operator $\langle\gamma\rangle$. Furthermore, Eq. (5.26) indicates that none of these is correct. Instead the dilation of the phase factor between any two branches n, m is $1 - p_f^2/2M_nM_mc^2$, so is always bounded by the single shot inverse Lorentz

factors for the individual branches. We in particular note that this is *not equivalent* to what one would expect from the analogous classical mixture of Lorentz factors, as we now show.

First one must note that with a classical mixture we can no longer use the internal state as a clock, so we must consider that we have some idealized classical clock that follows along. For simplicity we illustrate with a two-level system, and take an equal classical mixture of the two masses M_0, M_1. Working with the momentum boost, the two branches of the mixture will be put into different frames defining different Lorentz factors γ_0, γ_1. After time t in the initial frame we measure the time elapsed on the classical clocks and would get t/γ_0 half of the time and t/γ_1 the other half. When we take many measurements, we would write our average time as $t(1/\gamma_0 + 1/\gamma_1)/2$. Working to first order in p^2/M^2c^2 we find that this gives us $t[1 - \frac{p^2}{4c^2}\frac{M_0^2+M_1^2}{M_0^2M_1^2}]$. This can now be contrasted with our result from the quantum analysis where we have shown the theory predicts our measurement of the time elapsed is $t[1 - \frac{p^2}{2M_0M_1c^2}]$, note this is the inverse Lorentz factor defined using the geometric mean mass $\sqrt{M_0M_1}$. We see the two are not equivalent. The classical case is unsurprisingly a simple average as we are just taking a statistical mixture of two situations with well-defined Lorentz factors. The quantum case is stranger, as the clock is not in a well-defined frame but a superposition across frames. Hence it is not a priori obvious how it should behave and indeed we show its behaviour does not correspond to the classical average.

Once we appreciate these general problems we can also see that there is ambiguity in the translation operations. It is natural to center the boost back at a distance from the origin that is equal to the relative velocity imparted multiplied by the time it has freely evolved, but if there are multiple velocities then there are multiple such distances. The clock will have been moved by $vt = (p_b/M)t$, (this can be seen in the boost sequence $B_p(-p_b)U(t)B_p(p_b)$ which generates the position shift operator $e^{-\frac{it}{\hbar}\frac{pp_b}{M}}$, together with the exponential of a kinetic term). Therefore each internal energy defines the shift $p_bt/(m + E_n/c^2)$, so we could justifiably choose to use $T(p_bt/(m + E_n/c^2)$ for any of the occupied n. This would make the second exponential term in Eq. (5.26) become $e^{-\frac{it}{\hbar}(\frac{p_b^2}{2m} - \frac{p_b^2}{(m+E_n/c^2)})}$, but as this is just altering a global phase it does not affect the clock.

Finally we point out that even working to first order in H_0/mc^2 we get the displacement operator $e^{-\frac{it}{\hbar}\frac{pp_b}{m}(1-\frac{H_0}{mc^2})}$, which is still dependent on the internal state. This is because, unlike the Lorentz factors, the velocities imparted on the different masses are still disparate at this level of approximation. Given sufficient time, and some reasonable localization, the different branches of the clock could in principle become completely spatially separate, which is clearly not in keeping with an interpretation that to this level of approximation we can view this as a clock moving

along a single trajectory. Note that when we move it away and back (as per usual in a twin paradox scenario), then we can cancel the shift effects and therefore not notice, but that does not remove the clear issue with the single trajectory interpretation. This shows that we cannot avoid the conceptual problems by simply working to lower order in H_0/mc^2.

In order to solve the above problems, we instead use the modified boost operator which we introduced earlier in the chapter in Eq. (5.12), namely

$$B_v(v_b) \equiv e^{i(m+H_0/c^2)v_b x/\hbar}. \tag{5.27}$$

The motivation for this is clear, we are trying to define a unique velocity and thereby a unique Lorentz factor. One can derive its form in the relevant non-relativistic limit as discussed above, and also relate it to the extended Galilean boost $G(v,t) = e^{iv(Mx-tp)/\hbar}$ where M is operator valued, via $G(v,t) = U(t)B_v(v)U^\dagger(t)$. The position shift for the clock is now uniquely defined to be $v_b t$, and using this we write

$$B_v(v_b)U(t)T(v_b t)B_v(-2v_b)T(-v_b t)U(t)B_v(v_b) = e^{-\frac{2it}{\hbar}\frac{p^2}{2M}} e^{\frac{2it}{\hbar}\frac{mv_b^2}{2}} e^{-\frac{2it}{\hbar}H_0(1-\frac{v_b^2}{2c^2})}. \tag{5.28}$$

As before the first term is the unaltered motional evolution and the second term is a global phase. It is the third term that interests us. From this we see that the clock has run slower by the inverse of the classical Lorentz factor $\gamma^{-1} \approx (1 - \frac{v_b^2}{2c^2})$. This is exactly as required for classical time dilation.

We can go further and consider the situation where the classical observer is the one that is set into motion. This means that the boosts must all be centred at the origin which gives us

$$B_v(-v_b)U(t)B_v(2v_b)U(t)B_v(-v_b) = e^{-\frac{2it}{\hbar}\frac{p^2}{2M}} e^{-\frac{2it}{\hbar}\frac{mv_b^2}{2}} e^{-\frac{2it}{\hbar}H_0(1+\frac{v_b^2}{2c^2})}. \tag{5.29}$$

Here the quantum clock is running faster by the classical Lorentz factor $\gamma \approx (1+\frac{v_b^2}{2c^2})$. This is again as expected as the classical observer is moving so their clock runs slower. It is satisfying and encouraging that the modified Hamiltonian produces the correct solution to the twin paradox when we use the velocity boost. The key difference in the two cases is caused by the manner in which the velocity boost transforms under translations $T^{-1}(s)B_v(v_b)T(s) = e^{i(m+H_0/c^2)v_b s/\hbar}B_v(v_b)$. It is important to realise that we cannot have the same interpretation with momentum boosts due to the fact that a classical observer cannot move in a superposition of velocities. Note also that the case of the observer moving was examined in a concomitant manner by Greenberger [241].

It is worth a further comment here on the motional term $e^{-\frac{2it}{\hbar}\frac{p^2}{2M}}$. As stated

above this has been left unaffected by the boosting which is due to the adopted approximations. The full relativistic algebra shown earlier would indicate that a correction term $e^{\pm\frac{2it}{\hbar}\frac{p^2v_b^2}{4Mc^2}}$ is missing. To obtain a fully consistent regime for these equations one requires this term to approximate the identity, thus restricting the wavepacket momentum spread and time t.

A final point of interest here is to note that with the velocity boost one can show consistency with the equivalence principle. We take fixed acceleration a, for time t broken into n time steps $\delta t = \frac{t}{n}$. For a single time step δt, we apply a boost $B_v(a\delta t)$, and the evolution $U(\delta t)$. This gives the unitary $(U(\delta t)B_v(a\delta t))^n$, for which we take $n \to \infty$, and reverse the Trotter expansion [250] to arrive at a new unitary which defines the Hamiltonian in the accelerating frame as $H = \frac{p^2}{2M} + H_0 + aMx$. This agrees with the result from an alternative derivation for the accelerating frame's Hamiltonian [251], and importantly is the same form as Hamiltonians to describe the composite particle in a gravitational potential [222, 231, 232, 233].

5.3.3 Hamiltonian description of translations

The translation operators play a key role in the above. To better understand them we can consider the Hamiltonians necessary to enact the boosts on the state. We start with the velocity case.

Consider classical observers Alice and Bob at rest in each other's frames, separated by a distance $v_b t$. Alice initially holds a quantum clock and she sends it to Bob by applying the potential $-\alpha(m + H_0/c^2)x$ for a short time Δt such that the full Hamiltonian in this time is

$$H = \frac{p^2}{2M} + H_0 - \alpha(m + H_0/c^2)x. \tag{5.30}$$

We choose α large and Δt small with $\alpha\Delta t = v_b$, such that the first two terms are irrelevant and we effectively generate $U(\Delta t) = e^{i(m+H_0/c^2)v_b x/\hbar}$.

After time t evolving under the free Hamiltonian, the clock reaches Bob who applies a potential to send it back. Viewed from Alice's frame this potential is $+2\alpha(m + H_0/c^2)(x - v_b t)$. So the full Hamiltonian is

$$H = \frac{p^2}{2M} + H_0 + 2\alpha(m + H_0/c^2)(x - v_b t). \tag{5.31}$$

This means that the operator generated in the appropriate limit is

$$e^{-2i(m+H_0/c^2)(x-v_b t)/\hbar} = T(v_b t)B_v(-2v_b)T(-v_b t). \tag{5.32}$$

One can do the same thing for the momentum boosts but there is now an extra

subtlety. Namely, that we do not have a uniquely defined position to centre Bob's potential from, but as we can see from $T(L)B_p(-2p_b)T(-L) = e^{-2ip_b(x-L)}$, this only alters a global phase. However, this is only true if we insist on a uniquely defined position shift. It is at least formally interesting to note that if we allow for the positioning of Bob's boost back to be dependent on the internal state, such that the translation operator is $T(p_b t/M)$, then we find the internal state evolution multiplied by $(1 + \frac{p_b^2}{2mMc^2})$. So the clock runs faster, analogously to the boosted classical observer case, but again not by a relevant classical Lorentz factor. It may be that this approach has some interpretation in the emerging topic of quantum reference frames [252].

5.4 The nonclassical behaviour

We have shown that the description of proper time for quantum clocks requires velocity boosts and therefore for cases without internal state dependent forces additional effects must arise. For instance we can consider the effect on proposed theoretical clock models and naturally find the emergence of non-ideal clock behaviour.

The example we consider here is the Salecker-Wigner-Peres (SWP) definition of a quantum clock [253, 254]. Taking the internal energy Hilbert space to be spanned by N non-degenerate energy eigenstates $|n\rangle$, $n = 0, 1, ..., N-1$, with equally spaced eigenvalues such that $H_0 = \sum_n n\hbar\omega_0 |n\rangle\langle n|$. The SWP clock is then defined by the N orthogonal states $|w_k\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi ikn/N} |n\rangle$. Initialised in $|w_0\rangle$, the clock will pass through successive states $|w_k\rangle$ at external times $t_k = k\tau$, where $\tau = \frac{2\pi}{N\omega_0}$. One then defines a clock operator

$$T_c = \tau \sum_k k \, |w_k\rangle \langle w_k| \,, \tag{5.33}$$

with variance $(\Delta T_c)^2 = 0$ at times t_k, and $(\Delta T_c)^2 \neq 0$ in-between.

For this setup we see that the ideal clock behaviour is broken by the non-linear H_0 dependence in the nonclassical dilation term $H_0(1 - \frac{p^2}{2m(m+H_0/c^2)c^2})$. The initial state $|w_0\rangle$ evolving under this will no longer in general reach perfect alignment with the later clock states $|w_k\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi ikn/N} |n\rangle$ and therefore there are no longer well defined ticks with $(\Delta T_c)^2 = 0$.

In addition to such theoretical considerations, the non-classical behaviour can have physical consequences. We now turn to demonstrate this point by deriving observed frequency shifts in ion trap atomic clocks and indicating a small additional shift.

5.4.1 Atomic clocks

The velocity boost is the relevant operator when dealing with questions of classical proper time for quantum clocks. However, setting a clock in motion in this manner requires an entangling force that couples the internal and motional degrees of freedom, but for physical situations this is often not the case. Under these circumstances the momentum boost behaviour is more relevant. There has been an experimental proposal [239] to use a trapped single electron to test for interference effects caused by the Hamiltonian of Eq. (5.23). As a different example we shall consider trapped ion optical atomic clock frequency shifts, and arguing they already provide corroboration for the modified Hamiltonian and potentially could provide more. Similar conclusions were also reached in [255]. Since current experimental values give $\frac{\langle H_0 \rangle}{mc^2} \approx 10^{-10} \gg \frac{\langle p^2 \rangle}{m^2c^2} \approx 10^{-19}$, these systems are a good candidate to study the modified Hamiltonian.

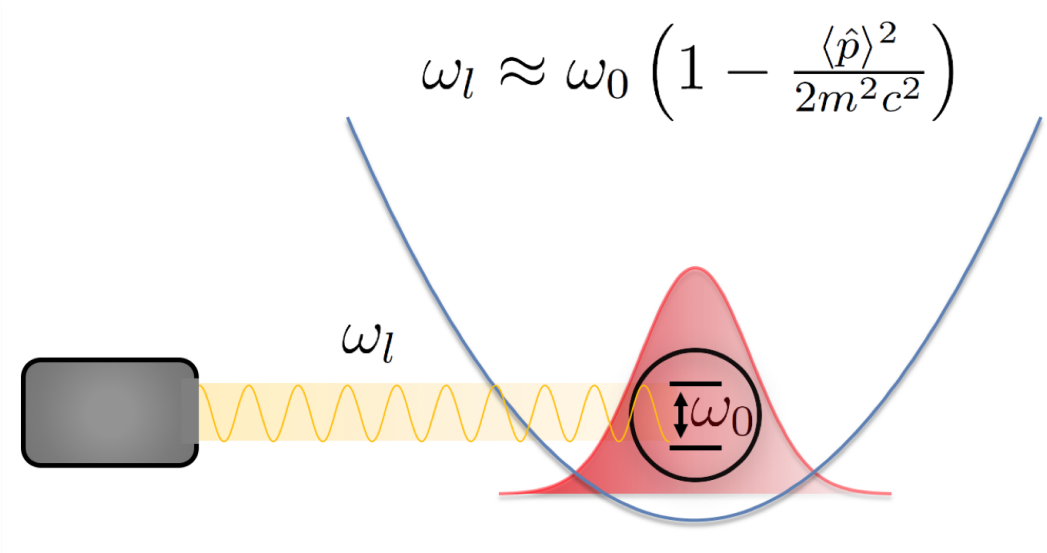

Figure 5.2: Atomic clock setup. A trapped ion is cooled to near its motional ground state. The internal qubit, with frequency gap ω_0, is initialised in the ground state. A laser is used to set up Rabi oscillations, and its frequency ω_l varied to optimise the transition probability.

First we briefly outline the basic operation of an ion trap atomic clock as illustrated in Fig. 5.2 (see [256] for a comprehensive review). An ion is trapped in a harmonic potential with trap frequency ω_m, and the clock reference frequency is obtained by tuning a laser to an electronic transition frequency ω_0 of the ion. The laser frequency is varied to maximise the probability of exciting a transition, which standard quantum mechanics predicts will occur when $\omega_l \approx \omega_0$. However, with relativity the ion's motion will lead to a dilation effect, which manifests in a frequency shift of the transition. The common approach for incorporating this is to apply the classical time dilation formula, substituting the expectation value of the momentum

squared to give $\omega_l \approx \omega_0(1-\frac{\langle p^2\rangle}{2m^2c^2})$. This is found to be in line with experiment [257].

The approach works well, however it is essentially a semi-classical analysis, because we are making no relativistic correction to the quantum mechanical description. A more natural method is to start from the Hamiltonian of Eq. (5.23). The interaction of an ion with a monochromatic classical laser field is a well documented problem [258], and adapting the standard approach we can derive a differential equation to describe the time evolution (details presented below). Under simplifying approximations we find that the frequency shift for an ion initially in the nth Fock state is

$$\omega_l \approx \omega_0(1-\frac{\hbar\omega_m(n+\frac{1}{2})}{2mc^2}+\frac{\hbar\omega_0\hbar\omega_m(n+\frac{1}{2})}{2(mc^2)^2}). \tag{5.34}$$

The first correction term is the same type of shift studied in [239] and is broadly in agreement with the semi-classical argument and the observations [257]. This provides empirical evidence for the modified Hamiltonian, since it gives a quantum mechanical description for a real world experiment up to the level of precision achieved. The second correction term captures the additional behaviour that we now expect, however it should be taken as illustrative rather than a concrete experimental prediction. Here we have not considered other effects that could be relevant at this precision, such as the higher order p^2/M^2c^2 term. To predict such new shifts one should perform full simulations, with all relevant physics, and using experimental parameters. However, we can make estimates based on the terms above and for a typical experiment the new shift would be a factor of $\frac{\hbar\omega_0}{mc^2} \sim 10^{-10}$ smaller than that observed. Thus state of the art experiments are far from observing these effects. While this is discouraging, the key point is that the modified Hamiltonian predicts effects that could lead to new observable consequences.

We now present the derivation of Eq. (5.34). Using the modified Hamiltonian we can alter the standard approach for the interaction of an ion with a monochromatic classical laser field [258]. For the general case the relativistic modification is complicated to work with, however when the laser is tuned close to resonance we can neglect certain fast oscillating terms via a rotating wave approximation and thus write our Hamiltonian as

$$H=\hbar\omega_m(a^\dagger a+\frac{1}{2})+\hbar\omega_0|e\rangle\langle e|(1-\frac{p^2}{2m(m+\frac{\hbar\omega_0}{c^2})c^2})+\frac{\hbar\Omega}{2}(|e\rangle\langle g|e^{-i\omega_l t}+|g\rangle\langle e|e^{i\omega_l t}), \tag{5.35}$$

where a is the motional annihilation operator, ω_m the trap frequency, $|g\rangle, |e\rangle$ are the ground and excited states of the internal energy qubit, and Ω is the Rabi frequency.

We now substitute the form $|\psi(t)\rangle = \sum_n(a_n|n\rangle|g\rangle + b_n|n\rangle|e\rangle)$ into the time

dependent Schrödinger equation, which gives

$$i\hbar\sum_n(\frac{\partial a_n}{\partial t}|n\rangle|g\rangle+\frac{\partial b_n}{\partial t}|n\rangle|e\rangle)=\sum_n\Bigg(\hbar\omega_m(n+\frac{1}{2})(a_n|n\rangle|g\rangle+b_n|n\rangle|e\rangle)$$
$$+\hbar\omega_0(1-\frac{p^2}{2m(m+\frac{\hbar\omega_0}{c^2})c^2})b_n|n\rangle|e\rangle+\frac{\hbar\Omega}{2}(e^{-i\omega_l t}a_n|n\rangle|e\rangle+e^{i\omega_l t}b_n|n\rangle|g\rangle)\Bigg). \quad (5.36)$$

Applying $\langle j|\langle e|$ to Eq. (5.36) gives

$$i\hbar\frac{\partial b_j}{\partial t}=(\hbar\omega_m(j+\frac{1}{2})+\hbar\omega_0)b_j-\frac{\hbar\omega_0}{2m(m+\frac{\hbar\omega_0}{c^2})c^2}\sum_n b_n\langle j|p^2|n\rangle+\frac{\hbar\Omega}{2}e^{-i\omega_l t}a_j, \quad (5.37)$$

and applying $\langle j|\langle g|$ to Eq. (5.36) gives

$$i\hbar\frac{\partial a_j}{\partial t}=\hbar\omega_m(j+\frac{1}{2})a_j+\frac{\hbar\Omega}{2}e^{i\omega_l t}b_j. \quad (5.38)$$

We now write $a_j(t)=e^{-i\omega_m t(j+\frac{1}{2})}\tilde{a}_j(t)$, and $b_j(t)=e^{-it(\omega_m(j+\frac{1}{2})+\phi(j))}\tilde{b}_j(t)$, where we define the function $\phi(j)=\omega_0-G(2j+1)$, and the constant $G=\frac{\omega_0}{2m(m+\frac{\hbar\omega_0}{c^2})c^2}\frac{\hbar m\omega_m}{2}$. We also explicitly evaluate $\langle j|p^2|n\rangle$, and adopt the convention that $\tilde{b}_{j<0}=0$. This produces the new differential equations

$$i\frac{\partial\tilde{b}_j}{\partial t}=\frac{\Omega}{2}e^{-it(\omega_l-\phi(j))}\tilde{a}_j+G(\sqrt{j(j-1)}\tilde{b}_{j-2}e^{it(2\omega_m+\phi(j)-\phi(j-2))}$$
$$+\sqrt{(j+2)(j+1)}\tilde{b}_{j+2}e^{-it(2\omega_m+\phi(j+2)-\phi(j))}), \quad (5.39)$$

$$i\frac{\partial\tilde{a}_j}{\partial t}=\frac{\Omega}{2}e^{it(\omega_l-\phi(j))}\tilde{b}_j. \quad (5.40)$$

Taking Eq. (5.39) we rearrange for $\frac{\partial\tilde{a}_j}{\partial t}$, substitute into Eq. (5.40), and find

$$\frac{\partial^2\tilde{b}_j}{\partial t^2}+i[\omega_l-\phi(j)]\frac{\partial\tilde{b}_j}{\partial t}+\frac{\Omega^2}{4}\tilde{b}_j=-iG\Bigg(\frac{\partial\tilde{b}_{j-2}}{\partial t}e^{it[2\omega_m+\phi(j)-\phi(j-2)]}$$
$$+i[\omega_l+2\omega_m-\phi(j-2)]\tilde{b}_{j-2}e^{it[2\omega_m+\phi(j)-\phi(j-2)]}$$
$$+\frac{\partial\tilde{b}_{j+2}}{\partial t}e^{-it[2\omega_m-\phi(j)+\phi(j+2)]}-i[2\omega_m+\phi(j+2)-\omega_l]\tilde{b}_{j+2}e^{-it[2\omega_m-\phi(j)+\phi(j+2)]}\Bigg), \quad (5.41)$$

where $\tilde{b}_j(t)=e^{it(\omega_m(j+\frac{1}{2})+\phi(j))}b_j(t)$, $\phi(j)=\omega_0-G(2j+1)$, $G=\frac{\hbar\omega_0\omega_m}{4(m+\frac{\hbar\omega_0}{c^2})c^2}$, and $\tilde{b}_j$ is set to zero for $j<0$.

In general these equations need to be solved numerically, however we can gain insight via simplifying approximations. First note that the left hand side of Eq. (5.41) is the differential equation for Rabi oscillations, where the timescale is defined

by Ω. The right hand side has terms that oscillate with frequencies $2\omega_m \pm \phi(j) \mp \phi(j \mp 2)$. With these significantly greater than Ω, the separation of time scales allows us to neglect the fast oscillating terms. Under this approximation one finds the differential equation $\frac{\partial^2 \tilde{b}_j}{\partial t^2} + i(\omega_l - \phi(j))\frac{\partial \tilde{b}_j}{\partial t} + \frac{\Omega^2}{4}\tilde{b}_j = 0$ which has the resonance condition $\omega_l = \phi(j)$. Taking the case where the ion is initially in the nth Fock state we find

$$\omega_l \approx \omega_0(1 - \frac{\hbar\omega_m(n+\frac{1}{2})}{2mc^2} + \frac{\hbar\omega_0\hbar\omega_m(n+\frac{1}{2})}{2(mc^2)^2}). \tag{5.42}$$

This is the result quoted above.

To understand the approximation regime one needs to consider the two relevant small quantities H_0/mc^2 and p^2/m^2c^2, (note $p^2/m^2c^2 \geq p^2/M^2c^2$ since m is the ground state mass). Starting from the relativistic expression for energy $\sqrt{M^2c^4 + p^2c^2}$, we can expand out and examine the following terms

$$H = mc^2 + H_0 + \frac{p^2}{2m}(1 - \frac{H_0}{mc^2} + (\frac{H_0}{mc^2})^2) - \frac{p^2}{2m}\frac{1}{4}\frac{p^2}{m^2c^2} \tag{5.43}$$

In order to arrive at the shift result as quoted we require $H_0/mc^2 \gg (p^2/m^2c^2)$, for the first shift and $(H_0/mc^2)^2 \gg (p^2/m^2c^2)$, for the second. For an ion trap we have typical values of $H_0/mc^2 \equiv 10^{-10}$ and $p^2/m^2c^2 \equiv 10^{-19}$ and so whilst we can say $H_0/mc^2 \gg (p^2/m^2c^2)$, we do not have $(H_0/mc^2)^2 \gg (p^2/m^2c^2)$. This is why we describe the second term as illustrative rather than a concrete experimental prediction. To experimentally investigate the consequences of the $(H_0/mc^2)^2$ term one would need to perform more complete simulations or find some alternative system to ion traps which can reach the required regime. We can however make estimates of the size of the new shift based on the terms above, and as stated above, for a typical ion trap the new shift would be a factor of $\frac{\hbar\omega_0}{mc^2} \sim 10^{-10}$ less than those observed.

5.5 Conclusions

We began this chapter by discussing established results on how the entanglement of internal spin systems and motional states can be affected by boosts when relativity is accounted for. We then demonstrated how similar behaviour can be obtained much more straightforwardly for energy states by using mass energy equivalence, and we compared and contrasted these.

We then considered the new velocity boosts in the context of proper time. We demonstrated that there are conceptual problems with viewing momentum boosts as leading to quantum clocks witnessing a classical time dilation. We found that the velocity boost recovers the expected classical behaviour and demonstrated the importance of translation operators in distinguishing the cases of the clock or the

observer being set in motion. We showed how this can be understood by considering the Hamiltonians necessary to realise the boosts on the quantum clock. We also showed the velocity boost also enables a simple demonstration for consistency with the gravitational equivalence principle.

We emphasised that moving the quantum clock without an internal state dependent force should present additional effects. We considered the effects of the non-classical dilation for the SWP clock, finding it removes the ideal clock behaviour. From a practical point of view we demonstrated the consequences of the non-classical dilation for ion trap atomic clocks, finding that the formalism predicts the already observed relativistic frequency shift and indicates an additional small correction.

This chapter shows the power of a simple idea. We were able to go from an interesting result focused on how entanglement is affected under motion, to developing a better understanding for the concept of proper time for quantum clocks. In the next chapter we shall consider another setting where one simple idea in entanglement theory has the potential for deep consequences.

Chapter 6

Witnessing non-classical gravity

Gravity is a contributing factor in nearly 73 percent of all accidents involving falling objects. And yet the so-called "federal government" does nothing!

– Dave Barry

6.1 Overview

In this chapter we examine the role entanglement could play in establishing the answer to arguably one of the most important questions in physics: is gravity non-classical? We critically revisit a spin witness proposal [259] aimed at answering this question, examining new theoretical considerations and improvements. We begin by summarising the original treatment which makes use of certain approximations which we show to be valid: the corrections are negligible for reasonable thermal states. We go on to demonstrate that by introducing an improved entanglement witness, we can reveal entanglement in a much shorter free fall time. We propose and demonstrate a likelihood ratio approach to rule out alternative interaction explanations and show that this allows one to reach conclusions on the non-classicality of gravity even in regimes dominated by non-gravitational forces, such as Casimir-Polder (CP) interactions. We show that this can still be true even with error in our knowledge of the non-gravitational coupling strength. Our approach allows revelation of entanglement in a 20 times shorter free-fall duration than in the original proposal. Finally, we point out that although witness experiments can provide convincing evidence, a fully rigorous certification of gravitationally induced entanglement would require more, such as the knowledge of an entanglement monotone. We illustrate a solution to this loophole using state tomography, and we find the number of repetitions required in different settings. This chapter draws from [260].

6.2 Non-Classicality of gravity

6.2.1 The problem

Gravity, was the first of the four known fundamental forces to be studied, but is now the one for which our understanding appears least complete. It is currently best described by Einstein's general relativity, which is a classical field theory [261]. This description has proven to be impressively robust to experimental evidence, once one allows for the existence of dark matter and dark energy, and up to this point there is still no convincing empirical evidence going against its predictions [262]. However, there are many cases where the coexistence of general relativity and quantum theory indicates problematic open questions [263, 264]. The most common view is that gravity cannot remain a classical theory, and therefore finding evidence for its non-classicality is a highly important research direction in modern physics.

The pursuit of evidence to help understand the interplay between gravity and quantum theory has given rise to novel ideas and experimental proposals [265, 266, 267, 268, 269, 270, 271]. In particular recent works suggesting tests for gravitationally generated entanglement [272, 259] have drawn significant attention [273, 274, 275, 276, 277, 278, 279]. The central idea behind this direction can be put relatively simply. Entanglement cannot be increased by local operations and classical communication, therefore if gravity can be used to increase entanglement between two quantum systems then gravity must be acting in a non-classical manner. Note that the original claim about witnessing a quantum feature of gravity [272, 259] has been subject to some debate [275], but it was pointed out that what is tested is a non-classicality of the gravitational field, in the sense that a successful test rules out any framework in which its state is described by (possibly probabilistic) unique (tensorial, vectorial, scalar, etc.) values at each space-time point [280, 281]. Regardless of terminology, the realization of such a proposal would be significant, as far as gravitationally induced entanglement would demonstrate a behaviour -superposition of space-time geometry - that is not predicted by GR [270, 276, 279].

In this chapter we focus on a spin witness approach [259], which we shall refer to as the spin witness protocol. The spin witness protocol proposes to use microdiamonds each with an embedded NV-center, and put them through two Mach-Zehnder setups that are positioned close to one another. If gravity is non-classical in nature then it should induce phase shifts between the spins and thus result in an entangled spin state which can be verified via measurements at the end. We shall give a more detailed account of the proposal in the next section. From an experimental point of view there are several difficulties in the splitting and refocusing operations, such as control pulse timing, particle rotation [282], radiation and spin decoherence [283], along with dipole-dipole interactions, diamagnetic properties of diamond, overheat-

ing and loading issues [284, 285, 286]. In working to overcome these issues it could prove necessary to adopt a significantly different approach, such as replacing the internal spin degree of freedom with a physically separate qubit system that is entangled to the motional state, as in [287]. These experimental issues are highly important but they are not the focus of this work. Instead, we address theoretical questions that underpin the protocol, independent of our ability to overcome the immense experimental challenges. As we shall show, this work makes any such experiment more viable.

6.2.2 The spin witness protocol

We here present the spin witness protocol and introduce notations that will be used throughout the rest of the chapter. We shall derive the results presented in [259] with a modification, namely without using position eigenstates. Instead we shall start from products of two arbitrary trapped motional states, and specialize to the case of two copies of the same motional state. We shall show that the resulting corrections to the position eigenstate approximation are negligible in a reasonable range of trapping frequencies and temperatures.

The reason we choose to perform the derivation in this more complicated setting is chiefly to rigorously establish more consistency for the conclusions in [259]. The original proposal implicitly adopts two theoretical simplifications: using position eigenstates throughout, and assuming that tracing out the motional degree of freedom after refocusing does not completely decohere the spin state. Although useful for illustrating the key ideas, it should be noted that these are in fact contradictory. If the states at the end could somehow be close to delta-distributed in position-space, then drift could certainly not be neglected. Therefore, for the sake of completeness, in our presentation we relax the first simplifying assumption and derive the second. This rigorously shows that the original position eigenstate approximation is unproblematic.

The setup is illustrated in Fig. 6.1, and consists of two adjacent Mach-Zehnder interferometers. The quantum system of interest is the motion of two particles, one for each interferometer, and their internal spin degrees of freedom. The full initial state ρ^i is comprised of two copies of local motional states separated by a distance d, with each particle's spin prepared in an equal superposition. The splitting stage is then a spin-controlled spatial displacement by $\pm\delta$, which puts the system into the state $\rho(0)$ with two spin dependent spatial superpositions. The free-fall duration is labelled by τ, after which the state is $\rho(\tau)$. This is the stage in which the particles are intended to interact gravitationally for a sufficiently long time such that some resulting entanglement can be subsequently measured. The refocusing stage merges

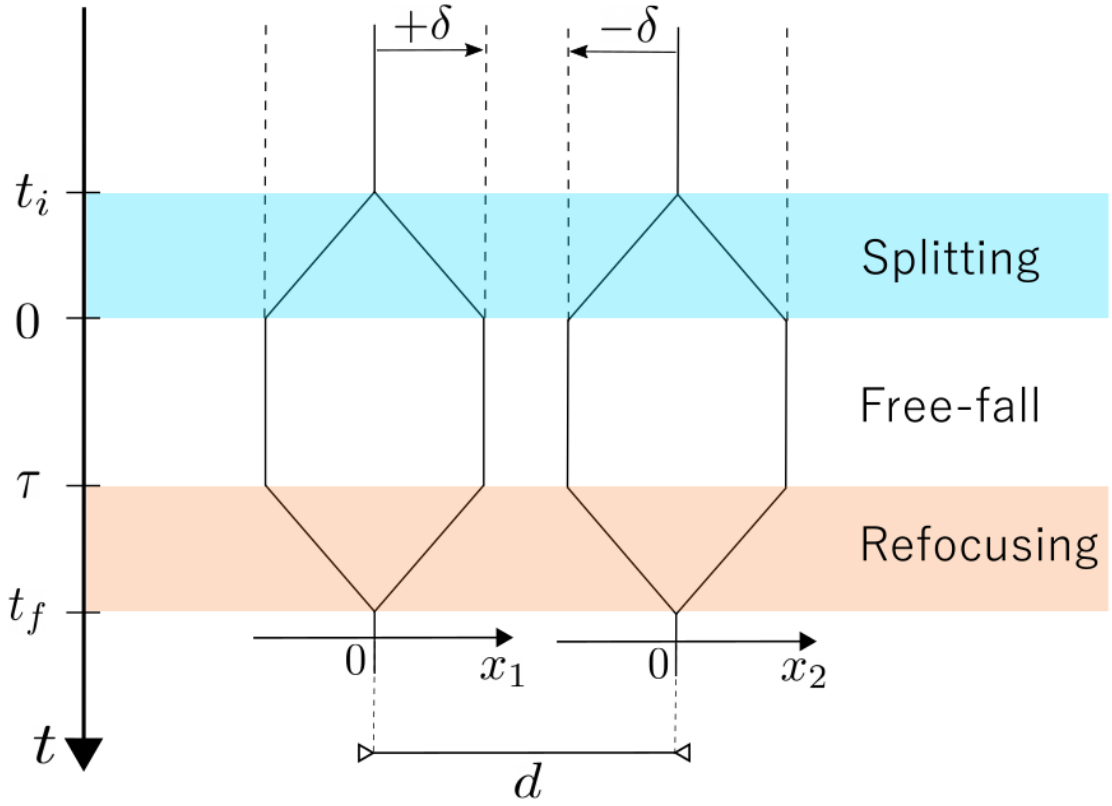

Figure 6.1: Illustration of the SWP protocol. The two systems have two position coordinates, the origins of which are separated by a distance d.

the positional superpositions, resulting in some final state that depends on the free-fall duration $\rho^f(\tau)$, and the system is thus prepared for spin state measurements. We shall take this merging to be non-adaptive so that it does not account for possible drift during free-fall.

We follow the lines of the original proposal in assuming that the splitting and refocusing operations can be done in a negligibly short amount of time compared to the free-fall time, and by using a Newtonian potential for gravity. Since $d/c \ll \tau$, this is a completely valid [274] static limit to the fully general relativistic description, which was proven to formulate the same predictions in the case of superposition of geometries [279]. It is worth emphasising that the protocol itself, in its conclusion on non-classicality through entanglement growth, is model agnostic [277, 288].

We now derive the final state $\rho^f(\tau)$ in detail. Consider two identical particles of mass m that are initially in a product of two arbitrary motional states and in a superposition of spin states $\rho^i = (\pi_1 \otimes \rho_+) \otimes (\pi_2 \otimes \rho_+)$. We have introduced $\rho_+ = \frac{1}{2}(|s_L\rangle + |s_R\rangle)(\langle s_L| + \langle s_R|)$, where this spin notation will provide labels for the displacements. As in the original proposal, we assume the spin-controlled spatial splitting can be done in a short time, and such that there is no mean momentum at the beginning of the free fall. That is, the splitting operation reads

$(D_L \otimes |s_L\rangle\langle s_L| + D_R \otimes |s_R\rangle\langle s_R|)^{\otimes 2}$, where $D_\mu = D(\kappa_\mu) = e^{\kappa_\mu a^\dagger - \kappa_\mu^* a}$ is the displacement operator, and $\kappa_R = -\kappa_L = \kappa \in \mathbb{R}_+$. We denote δ the physical distance by which the state is displaced, so $\kappa = \delta\sqrt{m\omega/2\hbar}$ where ω is some initial trap frequency. The refocusing is simply the conjugate operation.

For now we treat the free-fall stage as simply free evolution under the Hamiltonian

$$H_d = \frac{p_1^2 + p_2^2}{2m} - \frac{Gm^2}{d + x_2 - x_1}, \tag{6.1}$$

where G is the gravitational constant. We denote the unitary evolution generated by this acting for time τ as U_d.

Combining the appropriate operations for splitting, free-fall, and refocusing, we have the final state obtained in this noiseless case as

$$\rho^f(\tau) = \frac{1}{4}\sum_{\alpha\beta\mu\nu}(D_\alpha^\dagger \otimes D_\beta^\dagger)U_d(D_\alpha \otimes D_\beta)\rho^i(D_\mu \otimes D_\nu)U_d^\dagger(D_\mu^\dagger \otimes D_\nu^\dagger), \tag{6.2}$$

where the sum is performed over $(\alpha, \beta, \mu, \nu) \in \{L, R\}$.

The displacement operators around the unitaries simply amount to a shift of origins for the position operators, which can be absorbed into the separating distance d. Explicitly, $(D_\mu^\dagger \otimes D_\nu^\dagger)U_d(D_\mu \otimes D_\nu) = U_{d_{\mu\nu}}$ where $d_{\mu\nu} = d - \delta_\mu + \delta_\nu \in \{d-2\delta, d, d+2\delta\}$. Thus, instead of having three distinct relative displacements, we deal with three distinct separations and propagators acting on the same initial state. Then, the matrix elements $s_{\alpha\beta\mu\nu}$ of the reduced spin state $\mathrm{Tr}_{\mathrm{motion}}[\rho^f(\tau)] = \sum_{\alpha\beta\mu\nu} s_{\alpha\beta\mu\nu} |s_\alpha s_\beta\rangle\langle s_\mu s_\nu|$ boil down to the bipartite vacuum expectation values

$$s_{\alpha\beta\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\Big[U_{\alpha\beta}^\dagger U_{\mu\nu}(\pi_1 \otimes \pi_2)\Big]. \tag{6.3}$$

Pulling out the order zero potential term $-Gm^2/d_{\mu\nu}$ from the Hamiltonian, we transform Eq. (6.3) to obtain

$$s_{\alpha\beta\mu\nu} = \frac{1}{4}\exp\left[\frac{-iGm^2\tau}{\hbar}Q^{(1)}_{\alpha\beta\mu\nu}\right]\mathrm{Tr}\Big[\Psi_{\alpha\beta}^\dagger \Psi_{\mu\nu}(\pi_1 \otimes \pi_2)\Big]. \tag{6.4}$$

where $\forall n \in \mathbb{N}$ we define

$$Q^{(n)}_{\alpha\beta\mu\nu} = \frac{1}{d^n_{\alpha\beta}} - \frac{1}{d^n_{\mu\nu}}, \tag{6.5}$$

and we use Ψ to denote unitaries with the zero-order phase factored out, or equivalently the evolution generated by Hamiltonians modified to remove the order zero potentials. Essentially, the propagation of the full system is equivalent, up to a shift of position operator origins, to a sum of four pairwise evolutions, three of which are distinct. The position eigenstate approximation, which was adopted in the original proposal [259] equates to discarding the remaining trace term. This then answers

the question of when the conclusions are valid, we need this trace term to be a good approximation to unity,

$$\mathrm{Tr}\left[\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}(\pi_1\otimes\pi_2)\right]\approx 1. \tag{6.6}$$

To investigate how reasonable the approximation is we consider restricting to the case where $|x_2-x_1|\ll d-2\delta$ and inspect results that can thus be obtained with a truncated potential. If the truncation is made to first order then the Baker-Campbell-Hausdorff identity gives

$$\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}=e^{-i\tau^3G^2m^3Q^{(4)}_{\alpha\beta\mu\nu}/6\hbar}\exp\left[\frac{-i\tau GmQ^{(2)}_{\alpha\beta\mu\nu}}{\hbar}\left(m(x_1-x_2)+\frac{\tau}{2}(p_1-p_2)\right)\right]. \tag{6.7}$$

As expected from classical mechanics, the two particles are displaced towards one another and acquire opposite momenta, that is, we can rewrite Eq. (6.7) as

$$\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}=e^{-i\tau^3G^2m^3Q^{(4)}_{\alpha\beta\mu\nu}/6\hbar}D\left(\theta_{\alpha\beta\mu\nu}\right)\otimes D\left(-\theta_{\alpha\beta\mu\nu}\right), \tag{6.8}$$

where we write

$$\theta_{\alpha\beta\mu\nu}=\frac{GmQ^{(2)}_{\alpha\beta\mu\nu}\tau}{\sqrt{2}}\left[\frac{\tau}{2}\sqrt{\frac{m\omega}{\hbar}}-i\sqrt{\frac{m}{\hbar\omega}}\right]. \tag{6.9}$$

We now consider the two motional states to be identical $\pi_1=\pi_2=\pi$, and we write the initial motional state in the Glauber-Sudarshan P-representation as

$$\pi^{\otimes 2}=\iint d^2\varepsilon d^2\zeta P(\varepsilon)P(\zeta)\left|\varepsilon\zeta\right\rangle\!\left\langle\varepsilon\zeta\right|, \tag{6.10}$$

the relevant trace term is then

$$\mathrm{Tr}\left[\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}\pi^{\otimes 2}\right]=\iint d^2\varepsilon d^2\zeta P(\varepsilon)P(\zeta)\,\mathrm{Tr}\left[\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}\left|\varepsilon\zeta\right\rangle\!\left\langle\varepsilon\zeta\right|\right]. \tag{6.11}$$

Taking this together with Eq. (6.8) we find that

$$\mathrm{Tr}\left[\Psi^{\dagger}_{\alpha\beta}\Psi_{\mu\nu}\pi^{\otimes 2}\right]=e^{-iG^2m^3\tau^3Q^{(4)}_{\alpha\beta\mu\nu}/6\hbar}I(\theta_{\alpha\beta\mu\nu}), \tag{6.12}$$

where

$$I(\theta_{\alpha\beta\mu\nu})=\iint d^2\varepsilon d^2\zeta P(\varepsilon)P(\zeta)\left\langle\varepsilon\right|\left.\varepsilon+\theta_{\alpha\beta\mu\nu}\right\rangle\left\langle\zeta\right|\left.\zeta-\theta_{\alpha\beta\mu\nu}\right\rangle. \tag{6.13}$$

This double integral can be simplified as

$$\begin{aligned} I(\theta) &= e^{-|\theta|^2}\left(\int d^2\varepsilon P(\varepsilon) e^{\frac{1}{2}(\varepsilon^*\theta-\theta^*\varepsilon)}\right)\left(\int d^2\zeta P(\zeta) e^{\frac{1}{2}(\theta^*\zeta-\zeta^*\theta)}\right), \\ &= e^{-|\theta|^2}\tilde{P}\left(\frac{\theta}{2}\right)\tilde{P}\left(-\frac{\theta}{2}\right), \\ &= e^{-|\theta|^2}\left(C_N\left(\frac{\theta}{2}\right)\right)^2, \end{aligned} \tag{6.14}$$

where we have dropped the indices for θ and used the fact that the P-function is real and so has even Fourier transform $\tilde{P}$, and where C_N is the normally ordered characteristic function [289] of the initial local state π (that we have two copies of). Explicitly we write $C_N(\lambda) = \mathrm{Tr}\left[\pi e^{\lambda a^\dagger} e^{-\lambda^* a}\right]$.

Taking all this together we now have

$$\mathrm{Tr}\left[\Psi^\dagger_{\alpha\beta}\Psi_{\mu\nu}\pi^{\otimes 2}\right] = e^{-i\tau^3 G^2 m^3 Q^{(4)}_{\alpha\beta\mu\nu}/6\hbar} e^{-|\theta_{\alpha\beta\mu\nu}|^2}\left(C_N\left(\frac{\theta_{\alpha\beta\mu\nu}}{2}\right)\right)^2. \tag{6.15}$$

With this we can assess the approximation for specific initial states. Here we take the initial local state as thermal with $\langle N\rangle = \overline{n}$. For this the characteristic function is $C_N(\lambda) = e^{-\overline{n}|\lambda|^2}$. This can be straightforwardly shown, starting from the familiar $P(\varepsilon) = \frac{e^{-|\varepsilon|^2/\overline{n}}}{\pi\overline{n}}$, and rewriting the integral with $\varepsilon = x + iy$ and $\theta = a + ib$ as follows

$$\begin{aligned} C_N\left(\frac{\theta}{2}\right) &= \frac{1}{\pi\overline{n}}\int d^2\varepsilon e^{-|\varepsilon|^2/\overline{n} + \frac{1}{2}(\varepsilon^*\theta - \theta^*\varepsilon)}, \\ &= \frac{1}{\pi\overline{n}}\int dx e^{-x^2/\overline{n} + ixb}\int dy e^{-y^2/\overline{n} + iya}, \\ &= \frac{1}{\pi\overline{n}}\int dx e^{-(x - i\overline{n}b/2)^2/\overline{n}} e^{-\overline{n}b^2/4}\int dy e^{-(y - i\overline{n}a/2)^2/\overline{n}} e^{-\overline{n}a^2/4}, \\ &= e^{-\overline{n}|\theta|^2/4}. \end{aligned} \tag{6.16}$$

For the second line we have completed the square in the exponents and for the third we used the result for the standard Gaussian integral.

From this, the resulting trace term for an initial thermal state with $\langle\hat{N}\rangle = \overline{n}$ is

$$\mathrm{Tr}\left[\Psi^\dagger_{\alpha\beta}\Psi_{\mu\nu}\pi^{\otimes 2}\right] = e^{-iG^2 m^3\tau^3 Q^{(4)}_{\alpha\beta\mu\nu}/6\hbar} e^{-(\frac{\overline{n}}{2}+1)|\theta_{\alpha\beta\mu\nu}|^2}. \tag{6.17}$$

We see that this consists of a first order phase correction and a first order decoherence effect due to drift.

To facilitate comparison, we work with the parameters of the original proposal [259], ($d = 400\ \mu$m, $\delta = 125\ \mu$m, $m = 10^{-14}$ kg) and with a sensible trapping frequency $\omega = 10^3$ Hz [290]. Note that the experiment needs to be performed for a

time such that the original phase term $\exp\left[\frac{-iGm^2\tau}{\hbar}Q^{(1)}_{\alpha\beta\mu\nu}\right]$ becomes sufficiently large. For these parameters we find $|Q^{(1)}| \sim 3.6 \times 10^3$ m^{-1}, and thus the desired phase reaches unit radian after a characteristic free-fall duration $\tau \sim 4.4$ s. We can now calculate the corrections. We find $|Q^{(4)}| \sim 6.2 \times 10^{14}$ m^{-4}, such that after 10 seconds of free-fall the phase correction is approximately 4×10^{-12} rad and the decoherence factors are $\exp(-2.8 \times 10^{-8})$ and $\exp(-0.014) \approx 0.99$ respectively for zero temperature and $T = 7.6$ mK ($\overline{n} = 10^6$). This shows that the original position eigenstate approximation is valid for the considered regime, and the main factor of decoherence will originate from noise in the device and the environment, rather than from the limits of this simplifying approximation.

With this established we shall for the most part take the form posited in the original proposal [259] as our noiseless state. We shall include noise as discussed later, but the noiseless case for the final reduced spin state will be considered to be

$$|\psi_s(\tau)\rangle = \frac{1}{2}(|00\rangle + e^{i\Delta\phi_{LR}}|01\rangle + e^{i\Delta\phi_{RL}}|10\rangle + |11\rangle), \tag{6.18}$$

where $\Delta\phi_{\mu\nu} = Gm^2\tau(\frac{1}{d} - \frac{1}{d_{\mu\nu}})$ and we have discarded a global phase term.

Before moving to consider measurements and our proposed improvements to the protocol, we need a final discussion on the Casimir-Polder (CP) effect. It was argued that the CP interactions could be neglected with the original parameters [259]. One can write down the potential from [291] as $\hat{V}^C_{\mu\nu} = -\alpha(R,\epsilon)/(d_{\mu\nu} + x_2 - x_1)^7$, where $\alpha(R,\epsilon) = \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \frac{23\hbar c R^6}{4\pi}$ depends on the radius R of the particles (taken to be spheres) and their relative permittivity ϵ. The spin density matrix elements are then

$$s_{\alpha\beta\mu\nu} = \frac{1}{4}\exp\left(\frac{-it}{\hbar}\left(Gm^2Q^{(1)}_{\alpha\beta\mu\nu} + \alpha Q^{(7)}_{\alpha\beta\mu\nu}\right)\right)\mathrm{Tr}\left[\Psi^\dagger_{\alpha\beta}\Psi_{\mu\nu}(\pi_1 \otimes \pi_2)\right]. \tag{6.19}$$

With $R \approx 10^{-4}$ m, which roughly corresponds to a diamond microsphere of mass 10^{-14} kg and $\epsilon \approx 5.7$, the most rapidly evolving terms have a gravity frequency of 0.226 Hz and a CP frequency 0.016 Hz. Therefore the CP interaction is negligible compared to the gravitational effect if the closest approach is above 200 μm.

Later we shall show how to overcome the CP closest approach limit given in [259], so it is worth mentioning that the position eigenstate approximation is still valid in a smaller setup separation. If we decrease the separation distance d from 450 μm to 350 μm, the closest approach is then 100 μm such that for the $|RL\rangle$ pair the regime is dominated by CP coupling, which becomes roughly 4 times as strong as the gravitational coupling. The characteristic free-fall duration is lowered to $\tau \approx 54$ ms. After 1 second of free-fall, the phase correction is approximately 7.03×10^{-14} rad and the decoherence factors are $\exp(-5 \times 10^{-11})$ and $\exp(-2.5 \times 10^{-6})$ respectively for zero temperature and $T = 7.6$ mK ($\overline{n} = 10^6$). These corrections are still negligible.

6.3 Improving the protocol

6.3.1 Witnesses for gravitationally induced entanglement

We now present the non-separability condition from [259], and show its corresponding entanglement witness is not optimal. We propose an optimal entanglement witness in the sense that it theoretically can detect entanglement for arbitrarily short free-fall durations τ.

From the pure spin state resulting from the position eigenstate approximation, one can read off $\Delta\phi_{LR}+\Delta\phi_{RL} \in \{2n\pi | n \in \mathbb{Z}\}$ as a necessary and sufficient condition for separability. In [259], the condition $|\langle\sigma^x \otimes \sigma^z\rangle + \langle\sigma^y \otimes \sigma^y\rangle| > 1$ is proposed to certify the entanglement. This formally corresponds to selecting

$$W_0 = I \otimes I + \sigma^x \otimes \sigma^z + \sigma^y \otimes \sigma^y, \tag{6.20}$$

as an entanglement witness [292], as for any separable two qubit state ρ, $\mathrm{Tr}(W_0\rho) \geq 0$. In the noiseless case and using the same parameters as above, entanglement is revealed after roughly 8 seconds of free-fall, as shown in Fig. 6.2.

Although the order of magnitude for the required free-fall time is promising, it would still correspond to a falling distance of a few 10^2 meters on Earth, and is still 3 orders of magnitude above the coherence times observed in cutting edge matter-wave interferometry with much less massive particles [293]. Adapting the protocol to work with shorter free-fall times makes it not only more feasible on Earth, but also more robust to decoherence. To illustrate the effect of decoherence, we choose a scattering term that induces an exponential dephasing of local motional states [294]. We shall denote the off-diagonal damping rate γ. Explicitly in the local position eigenstate basis $\{|L\rangle, |R\rangle\}$, the decoherence after duration τ acts as a dephasing channel $\pi \longmapsto (1-p)\pi + p\sigma^z\pi\sigma^z$ where $p = (1-e^{-\gamma\tau})/2$. The original witness W_0 fails to detect any entanglement for $\gamma \geq 0.03\ s^{-1}$. The failure of this witness is primarily due to the fact that even in an ideal zero-temperature noiseless scenario, it requires over 8 seconds of free-fall time for revelation. This means that when there is noise it is being allowed too long to destroy the entanglement. It is also not obvious that we have to accept these long free-fall times, since state negativity is achieved immediately, as shown in Fig. 6.3.

To shorten the required interaction time, we construct another entanglement witness using few local Pauli measurements in the spirit of [295], using the PPT-criterion [296]. For the purposes of constructing a new witness we consider the case where $\phi = \Delta\phi_{LR} \gg \Delta\phi_{RL}, \Delta\phi$, which amounts to neglecting all but the phase induced by the strongest interacting couple of states. In the position eigenstate approximation, the final spin state would then read $|\psi_s(\phi)\rangle = \frac{1}{2}(|00\rangle + |01\rangle + e^{i\phi}|10\rangle + |11\rangle)$

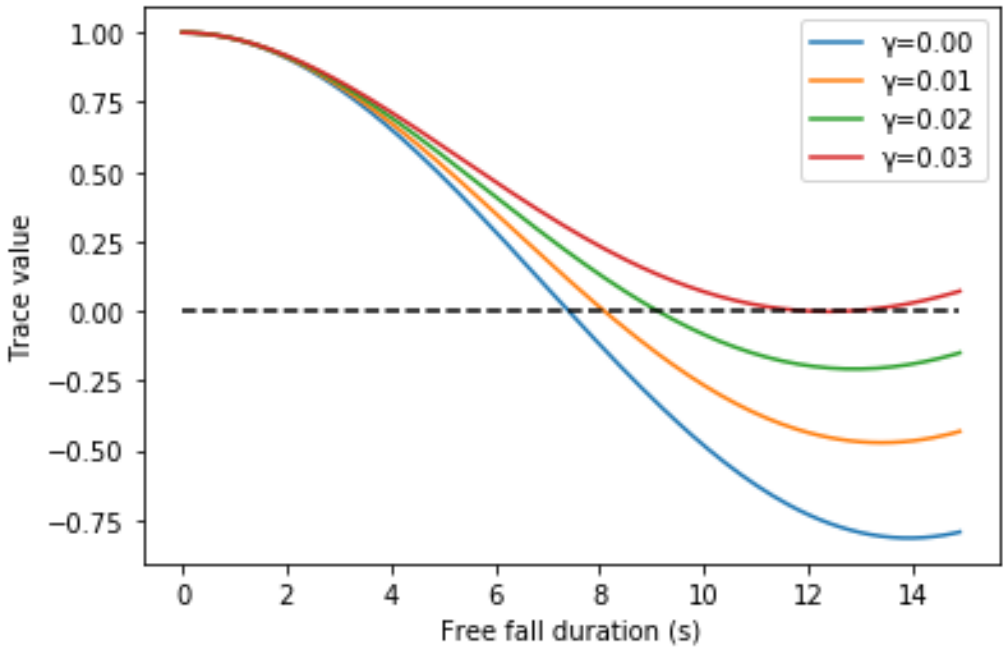

Figure 6.2: $\mathrm{Tr}(W_0 \left|\psi_s\right\rangle\!\left\langle\psi_s\right|)$, where W_0 is the entanglement witness put forward in [259], as a function of free-fall time with decoherence rates $\gamma \in \{0, 0.01, 0.02, 0.03\}$. At $\gamma = 0.03$ the witness can no longer reveal entanglement.

with density operator $\rho_s(\phi) = |\psi_s(\phi)\rangle\langle\psi_s(\phi)|$. We take the partial transpose $\rho_s^{\Gamma_2}(\phi)$, and find that the eigenstate associated with the negative eigenvalue is $|\chi_-(\phi)\rangle = \frac{1}{2}(|00\rangle + ie^{-i\phi/2}|01\rangle - ie^{i\phi/2}|10\rangle - e^{-i\phi}|11\rangle)$. Taking $\phi = 0$ (this is to reduce the number of Pauli operators involved) one has $4\left|\chi_-\right\rangle\!\left\langle\chi_-\right| = I\otimes I - \sigma^x\otimes\sigma^x + \sigma^z\otimes\sigma^y - \sigma^y\otimes\sigma^z$, and we propose this as our new witness

$$W_1 = 4\left|\chi_-\right\rangle\!\left\langle\chi_-\right|^{\Gamma_2} = I\otimes I - \sigma^x\otimes\sigma^x - \sigma^y\otimes\sigma^z - \sigma^z\otimes\sigma^y. \tag{6.21}$$

It transpires that this new witness reveals entanglement immediately after the start of the free-fall, as shown in Fig. 6.4, as long as the decoherence rate γ satisfies $\gamma < (\omega_{RL} + \omega_{LR})/2$ where the ω are the respective coupling strengths $\omega_{\mu\nu}t = \Delta\phi_{\mu\nu}$. This can be proved as follows. In terms of density matrix components we write

$$\begin{aligned}\mathrm{Tr}(W_1\rho) = \mathrm{Tr}(\rho) + 2\,\mathrm{Im}\{\rho_{12}\} + 2\,\mathrm{Im}\{\rho_{13}\} - 2\,\mathrm{Re}\{\rho_{14}\} - 2\,\mathrm{Re}\{\rho_{23}\} \\ -2\,\mathrm{Im}\{\rho_{24}\} - 2\,\mathrm{Im}\{\rho_{34}\}.\end{aligned} \tag{6.22}$$

The pure spin state from the position eigenstate approximation is

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & e^{-i\Delta\phi_{LR}} & e^{-i\Delta\phi_{RL}} & 1 \\ e^{i\Delta\phi_{LR}} & 1 & e^{i(\Delta\phi_{LR}-\Delta\phi_{RL})} & e^{i\Delta\phi_{LR}} \\ e^{i\Delta\phi_{RL}} & e^{i(\Delta\phi_{RL}-\Delta\phi_{LR})} & 1 & e^{i\Delta\phi_{RL}} \\ 1 & e^{-i\Delta\phi_{LR}} & e^{-i\Delta\phi_{RL}} & 1 \end{pmatrix}. \tag{6.23}$$

Our decoherence model is a damping of off diagonal terms of local states with exponential decay rate γ. This amounts to an entrywise multiplication of the spin density matrix by

$$\begin{pmatrix} 1 & e^{-\gamma t} & e^{-\gamma t} & e^{-2\gamma t} \\ e^{-\gamma t} & 1 & e^{-2\gamma t} & e^{-\gamma t} \\ e^{-\gamma t} & e^{-2\gamma t} & 1 & e^{-\gamma t} \\ e^{-2\gamma t} & e^{-\gamma t} & e^{-\gamma t} & 1 \end{pmatrix} \tag{6.24}$$

Taking the components of the full decohered spin state into Eq. (6.22) we can write the witness expectation value as

$$\mathrm{Tr}(W_1\rho) = 1 - e^{-\gamma t}\left(\sin(\Delta\phi_{LR}) + \sin(\Delta\phi_{RL})\right) - \frac{1}{2}e^{-2\gamma t}(1 + \cos(\Delta\phi_{LR} - \Delta\phi_{RL})). \tag{6.25}$$

Writing $\omega_{\mu\nu}t = \Delta\phi_{\mu\nu}$ a first order expansion around $t = 0$ gives

$$\mathrm{Tr}(W_1\rho) = (2\gamma - (\omega_{LR} + \omega_{RL}))t. \tag{6.26}$$

We see that for short times this will be negative so long as $\gamma < (\omega_{RL} + \omega_{LR})/2$. Alternatively we can view this as a condition on the decoherence rate, in that we will not be able to immediately witness entanglement for decoherence rates greater than the average of the two path frequencies $\gamma \geq (\omega_{RL} + \omega_{LR})/2$.

Taking the original parameter values, the witness works in principle for $\gamma < 0.0627\ \mathrm{s}^{-1}$. Numerical tests reveal that in fact the state is not entangled for any higher decoherence rates, showing that in this sense our new witness is optimal.

Having a witness that can detect entanglement in theory with arbitrarily minute phase accumulation is advantageous, as it decreases the required free-fall duration. However, it raises the question of residual interactions, such as CP coupling. Even if gravity were to be the dominant interaction, small CP couplings would still induce entanglement that will be detected by the new witness. Empirical results are statistical statements, and the presence of additional interactions also should afflict experimental data on the original witness W_0. If measurements are performed after a free-fall duration τ when $\langle W_0\rangle$ is barely negative, then one must be able to make statistical statements on the impact of non-gravitational interactions on the observed entanglement. The consequences of such an experiment being realized are important enough to justify a more rigorous approach to the significance of these empirical results. In the same spirit as high-energy physics experiments are processed, the effect of a negligible but still existing CP effect on the entanglement witness and more precisely on the resulting statistics and probability of false positives, deserve closer inspection.

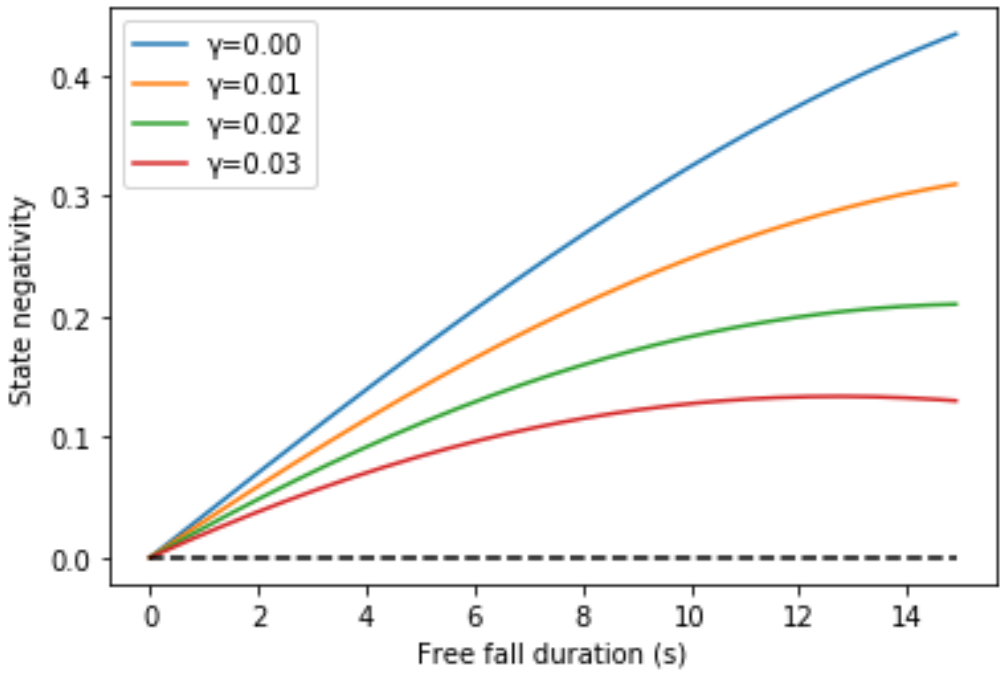

Figure 6.3: Negativity of the spin state $\mathcal{N}(\rho_s) = \sum_{\lambda \in \mathrm{Sp}(\rho_s) \cap \mathbb{R}_-} |\lambda|$ with respect to free-fall time with decoherence rates $\gamma \in \{0, 0.01, 0.02, 0.03\}$. Negativity is an entanglement monotone, and the state is entangled when the negativity is positive [122].

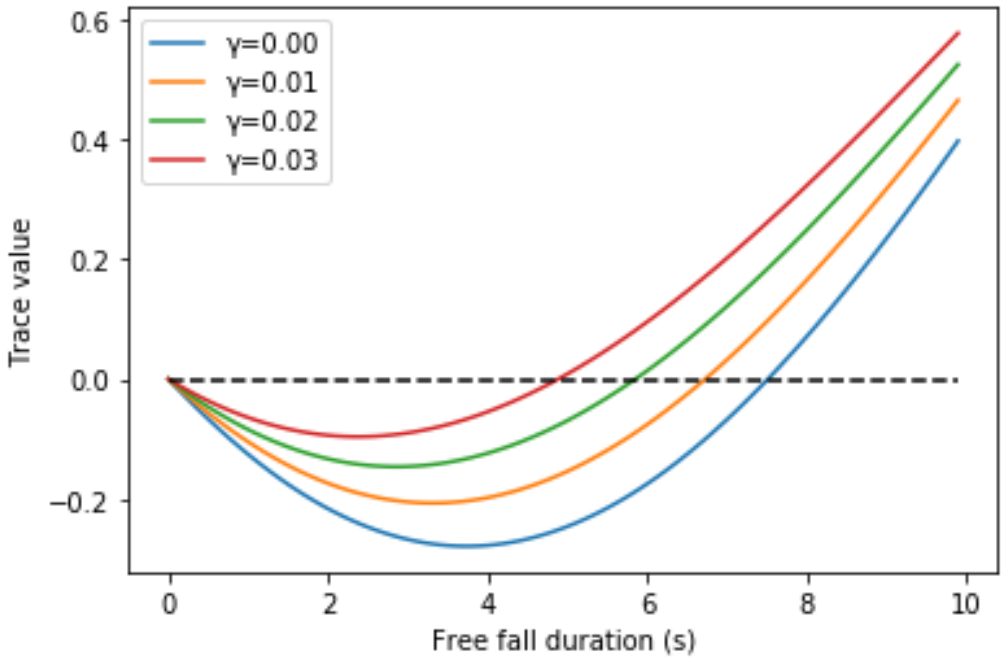

Figure 6.4: $\mathrm{Tr}(W_1 |\psi_s\rangle\langle\psi_s|)$ as a function of free-fall time with decoherence rates $\gamma \in \{0, 0.01, 0.02, 0.03\}$. The witness in theory can reveal entanglement even for strong decoherence rates if the free-fall time is kept short. This comes at the expense of expectation values that are closer to zero.

6.3.2 Hypothesis testing the gravitational coupling

Here we shall show how it is possible to infer the presence of a gravitational interaction from the witness measurements, in a manner which allows one to rule out the possibility that the observed entangled state could have been obtained only from the CP interaction. We do this by making use of statistical hypothesis testing, and with this we are then able to show that when the two Mach-Zehnder setups are brought closer to one another one can still certify the presence of the gravitational interaction, even at distances where gravity is not the dominant interaction.

In order to affirm that the final spin state was induced by a gravitational propagator in the presence of other interactions we adapt the approach developed in [297] for entanglement verification. In using such statistical methods, we assume the experiment can be repeated, for instance with particle recycling as outlined in [266]. For now we shall assume that we have good knowledge of the non-gravitational interactions, so for this section we limit ourselves to a CP interaction with known coupling strength. In the next section we shall show how the same techniques can work even when this is not the case.

First we need to discuss the statistical method we use, which is termed the likelihood ratio test. This test is the most powerful test for a given confidence level, a result known as the Neyman-Pearson lemma [298], which we shall prove as part of our presentation of the test.

Suppose we have two hypotheses: a null hypothesis $H_0 : \theta = \theta_0$ and an alternative hypothesis $H_1 : \theta = \theta_1$. The likelihood function for θ given that some discrete random variable X has been observed as $X = x$ is written as $\mathcal{L}(\theta|x) = P_\theta(X = x)$. For a continuous random variable we would use the probability density function. To perform a likelihood ratio test we take the ratio $\Lambda(x) = \frac{\mathcal{L}(\theta_0|x)}{\mathcal{L}(\theta_1|x)}$, and then if $\Lambda(x) > c$ we do not reject H_0, if $\Lambda(x) = c$ we reject with probability q, and if $\Lambda(x) < c$ we reject. The values are chosen so that $qP(\Lambda = c|H_0) + P(\Lambda < c|H_0) = \alpha$, in other words the probability of incorrectly rejecting the null hypothesis is α, this is termed the significance level. The Neyman-Pearson lemma states that the likelihood ratio test is the most powerful statistical test at significance level α. The power of a binary hypothesis test is the probability that the test rejects the null hypothesis H_0 when a specific alternative hypothesis H_1 is true. This lemma can be proved as follows.

First define the rejection region of the Neyman-Pearson ratio test

$$R_{NP} = \left\{ x : \frac{\mathcal{L}(\theta_0|x)}{\mathcal{L}(\theta_1|x)} \leq c \right\}, \tag{6.27}$$

where we are using $q = 1$ and c is chosen so that $P(R_{NP}|\theta_0) = \alpha$, with the probability

of data falling in region R given parameter θ given by

$$P(R|\theta) = \int_R \mathcal{L}(\theta|x)dx. \tag{6.28}$$

Consider a different rejection region R_A for which the significance level is at least α so

$$P(R_A|\theta_0) \leq P(R_{NP}|\theta_0). \tag{6.29}$$

We now note that we can break up these probabilities in terms of intersections with the complement regions as $P(R_{NP}) = P(R_{NP} \cap R_A) + P(R_{NP} \cap R_A^c)$ and also $P(R_A) = P(R_{NP} \cap R_A) + P(R_{NP}^c \cap R_A)$, which allows us to then rewrite the above inequality of Eq. (6.29) as

$$P(R_{NP}^c \cap R_A|\theta_0) \leq P(R_{NP} \cap R_A^c|\theta_0). \tag{6.30}$$

The powers of the tests are $P(R_{NP}|\theta_1)$ and $P(R_A|\theta_1)$. We want to prove that the ratio test is of equal or greater power so $P(R_{NP}|\theta_1) \geq P(R_A|\theta_1)$, which we can rewrite using the same trick as before to be

$$P(R_{NP} \cap R_A^c|\theta_1) \geq P(R_A \cap R_{NP}^c|\theta_1). \tag{6.31}$$

This can then be proved as follows

$$\begin{aligned}
P(R_{NP} \cap R_A^c|\theta_1) &= \int_{R_{NP} \cap R_A^c} \mathcal{L}(\theta_1|x)dx, \\
&\geq \frac{1}{c}\int_{R_{NP} \cap R_A^c} \mathcal{L}(\theta_0|x)dx, \\
&= \frac{1}{c}P(R_{NP} \cap R_A^c|\theta_0), \\
&\geq \frac{1}{c}P(R_{NP}^c \cap R_A|\theta_0), \\
&= \frac{1}{c}\int_{R_{NP}^c \cap R_A} \mathcal{L}(\theta_0|x)dx, \\
&> \int_{R_{NP}^c \cap R_A} \mathcal{L}(\theta_1|x)dx, \\
&= P(R_{NP}^c \cap R_A|\theta_1).
\end{aligned} \tag{6.32}$$

To get to the second line we have used the relation in Eq. (6.27), to get to the forth line we used Eq. (6.30), and to get to the sixth line we used the complementary relation to Eq. (6.27), since the integral is entirely contained within the region R_{NP}^c. This concludes the proof of the lemma, since we have shown that the likelihood ratio test is at least as powerful as any other test at the same significance level.

Having described the statistical test that we plan to use, we now lay out the null and alternative hypotheses that we shall be testing.

- The null hypothesis H_0 is: "*The observed entangled state is the result of CP interactions without gravity.*"

- The alternative hypothesis H_a is: "*The observed entangled state results not only from CP coupling but also from the gravitational interaction.*"

It should be stressed that the test of these hypotheses is the second of two distinct ways in which the approach outlined here uses the measurement data. The first is to straightforwardly confirm that the state is entangled via a negative expectation value for the witness given in Eq. (6.21). The second is then to rule out H_0 in favour of H_a, via a likelihood ratio test, or in other words establishing that gravity played a part in obtaining the observed entangled state.

In the original presentation [259] this distinction is not made, both tests are considered to be contained within the use of the sub-optimal witness. This is because the distances are set such that the CP interaction alone should not be able to demonstrate entanglement with this witness, and therefore a negative witness expectation value is taken to demonstrate both entanglement and that the gravitational interaction was responsible. As we shall show, separating these tasks brings advantages.

In order to obtain the likelihood ratios, we consider settings where one can choose to measure a list of bipartite Pauli observables $\underline{\sigma} = [\sigma_1, ..., \sigma_l]$ a number $N \in \mathbb{N}^*$ times each. Every bipartite observable has 4 eigenstates, and we denote the full list of eigenstates as $\underline{e} = [|e_{11}\rangle, ..., |e_{14}\rangle, |e_{21}\rangle, ..., |e_{l4}\rangle]$. This defines a $4l$-dimensional probability vector

$$\underline{p} = [p_{ij}]_{1\leq i\leq l, 1\leq j\leq 4} = [\mathrm{Tr}(\rho\, |e_{ij}\rangle\langle e_{ij}|)]. \tag{6.33}$$

The data $\mathcal{D}$ is a list of numbers of measurement outcomes, each corresponding to an obtained eigenstate, $\underline{n} = [n_{11}, ..., n_{14}, n_{21}, ..., n_{l4}]$. The probability of having obtained the empirical vector $\underline{n}$ given state ρ is the joint probability distribution

$$\mathbb{P}(\underline{n}|\rho) = \prod_{ij} p_{ij}^{n_{ij}} \overset{\text{def.}}{=} \mathcal{L}(\rho|\mathcal{D}). \tag{6.34}$$

The likelihood ratio to test whether we should reject our null hypothesis in favour of the alternative is

$$\Lambda_a = \frac{\mathcal{L}(\rho_{\mathrm{C}}(\gamma, t)|\mathcal{D}_a)}{\mathcal{L}(\rho_{\mathrm{CG}}(\gamma, t)|\mathcal{D}_a)}, \tag{6.35}$$

where ρ_{C} is the spin density matrix obtained with an exclusively CP induced evolution (null hypothesis state), and ρ_{CG} is the state obtained under full CP and

gravitational propagator (alternative hypothesis state). We shall evaluate this ratio at the free-fall duration τ, which is chosen to correspond to the duration after which the witness expectation value is minimal in the alternative hypothesis so as to maximize the probability of certifying entanglement, and is thus made dependent on γ.

For computational reasons we shall make the standard switch to using the negative logarithmic likelihood ratio $\lambda_a = -2\log(\Lambda_a)$. High values of λ_a strongly support the alternative hypothesis. This choice is convenient since the logarithmic likelihood ratio reduces to the scalar product

$$\lambda_a = 2\underline{n}_a.\left(\log\left(\underline{p}_a\right) - \log\left(\underline{p}_0\right)\right). \tag{6.36}$$

For the adopted witness presented in Eq. (6.21), we have $l = 3$ and we take the data $\mathcal{D}_a$ as a set of $3N$ empirical measurement outcomes, N for each bipartite Pauli observable $(\sigma^x \otimes \sigma^x, \sigma^y \otimes \sigma^z, \sigma^z \otimes \sigma^y)$ on ρ_{CG}. It is possible to predict values of λ_a that are typically obtained, but we also need to establish what a sufficiently good value of λ_a is.

Our overall aim is to achieve the minimum rate of false positives, i.e. we want to have a small significance level α or equivalently a high confidence level $1-\alpha$. Given we have some desired significance level α, we can then define the minimum value λ_{min} via

$$\mathbb{P}(\lambda_a \geq \lambda_{\text{min}}|H_0) = \alpha. \tag{6.37}$$

In order to find values for λ_{min} we simply generate multiple sets of data $\mathcal{D}_0$ assuming the null hypothesis is true, and then make use of the distribution of the resulting λ_0 for different values of N. For $\alpha = 1\%$, λ_{min} is then the 99-th percentile of the obtained λ_0.

With the $\lambda_{min}(N, \gamma, \tau)$ determined, we return to calculating values of λ_a from sets of data generated with the alternative hypothesis assumed as true. We can then examine the probability of obtaining $\lambda_a \geq \lambda_{min}$. This frequency is termed state distinction success rate, and is what has been plotted in Fig. 6.5 for a confidence level of 99%. This has been performed both for the original parameter settings and for a closer separation $d = 350$ μm.

From Fig. 6.5, we can see that for the original separation $d = 450$ μm, in the noiseless case as well as with strong decoherence, 10^2 measurements of W_1, obtained with 3×10^2 repetitions of the experiment is enough to almost certainly reject the null hypothesis state in favour of the alternative state with a false positive probability of less than 1%. We also observe that for the closer separation setting $d = 350$ μm certifying the alternative state reliably requires around 10^3 repetitions. These number of repetitions seem reasonable to expect from a reproducible experimental setup.

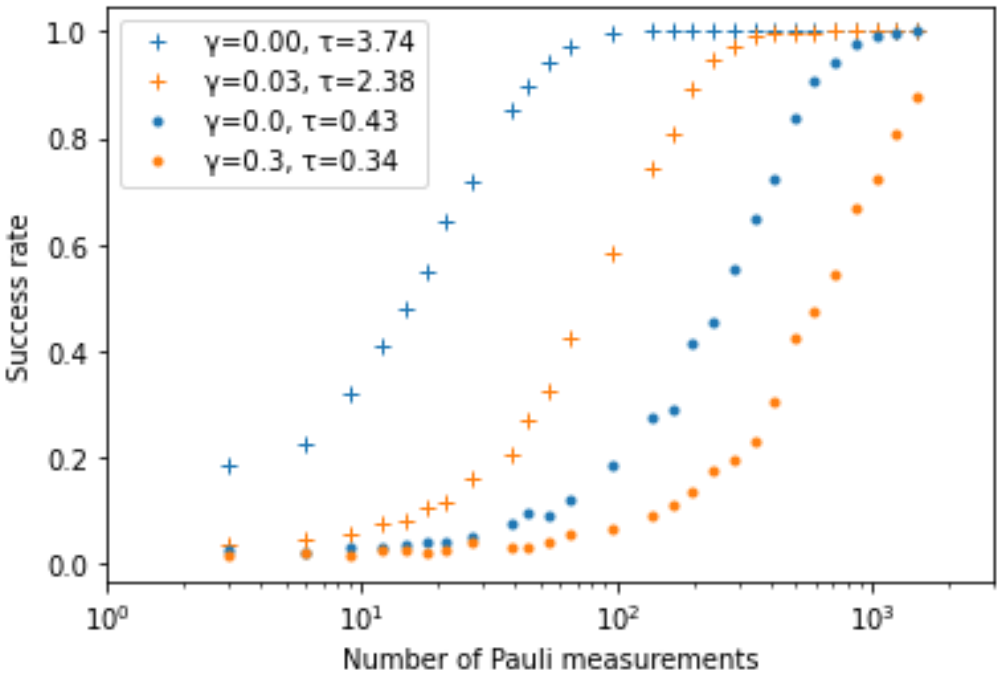

Figure 6.5: W_1 measurement state distinction success rates for a 99% confidence threshold, with respect to the number of bipartite Pauli measurements, for noiseless and strong decoherence scenarios. The crosses correspond to the original separation distance $d = 450\mu$m while the dots correspond to the closer separation $d = 350\mu$m. We observe that for decoherence rate $\gamma = 0.03$ and separating distance $d = 450$ μm, measuring W_1 10^2 times is enough to consistently tell the two states apart with a false positive probability of 1%.

As stated previously, one must also be able to certify entanglement from the witness empirical data. We see from Fig. 6.6 that even though in the original $d = 450$ μm separation 10^2 witness measurements is sufficient to certify the alternative state, there is only a 70% chance that entanglement will be certified in the strong decoherence scenario. Conversely, in the closer separation setup $d = 350$ μm the resulting state is more entangled, which makes entanglement certification more likely to succeed, and it is then the rejection of the null hypothesis in favour of the alternative hypothesis that is more demanding.

This shows that there is a trade-off present, and one should carefully consider how to optimise the strategy adopted depending on the experimental setup and expected noise. The important point is that we have demonstrated that the protocol does not need to be limited to negligible CP coupling. As such the distance can be reduced as part of improving our approach to witnessing the desired non-classical effect.

Hence, from the repeated measurement of a single entanglement witness, provided good enough knowledge of the non-gravitational interactions, one can confirm the presence of an entangled state that could not have been obtained without gravity. On its own, obtaining such experimental conclusions would be quite convincing

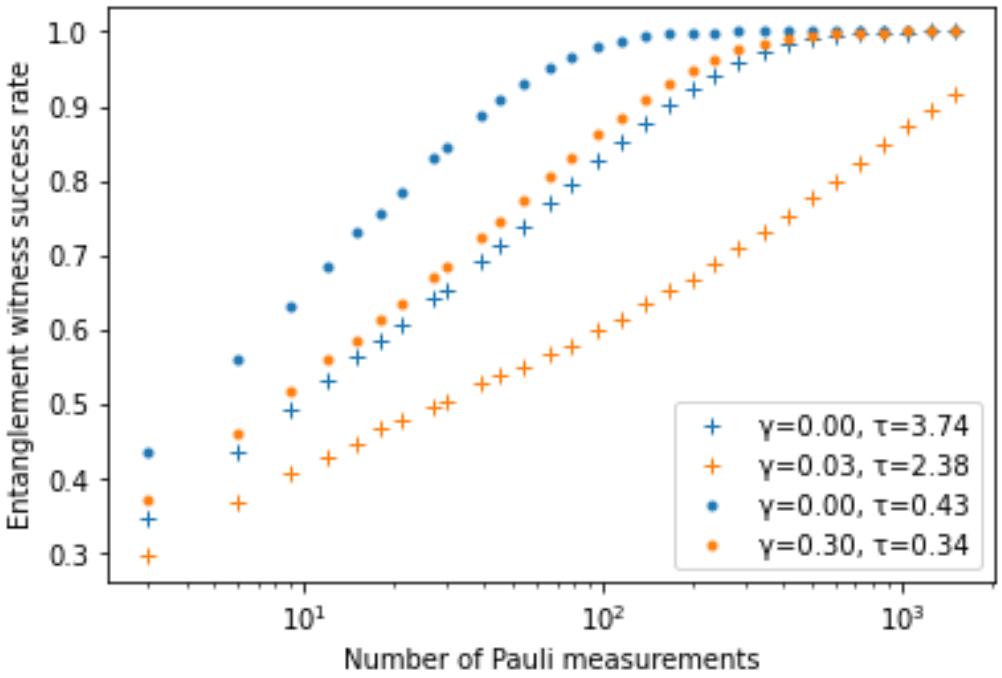

Figure 6.6: Probability of observing a negative empirical W_1 witness average, with respect to the number of bipartite Pauli measurements, for noiseless and strong decoherence scenarios. The crosses correspond to the original separation distance $d = 450\mu$m while the dots correspond to $d = 350\mu$m.

regarding the entangling ability of gravity, especially if gravity is the dominant interaction.

One could ask whether quantum state tomography could be more reliable for state distinction. In fact this turns out not to be the case. To rule out H_0 in favour of H_a, the witness measurement and full tomography are equivalently efficient even in a regime dominated by CP interactions, that is, with the separation d brought down to 350 μm. This is illustrated on the dotted plots in Fig. 6.5 and Fig. 6.7. In both the cases of tomographic and witness measurements, the CP limit can be overcome, and distinguishing the two states reliably requires around 10^3 bipartite Pauli measurements, that is, around 3×10^2 witness measurements, or 10^2 state tomographies. It seems the witness measurement works well enough not to need tomography. We shall see in Sec. 6.4 why it can still be interesting to use tomographic data.

A final point here is that this does not rule out the possibility for better sets of measurements than those provided by our new witness. Some alternative fixed set of Pauli measurements could be found to perform better, or perhaps an adaptive scheme [299] could provide improvements. Alternatively, an experiment for which joint measurements are easily implemented would bring new possibilities into play.

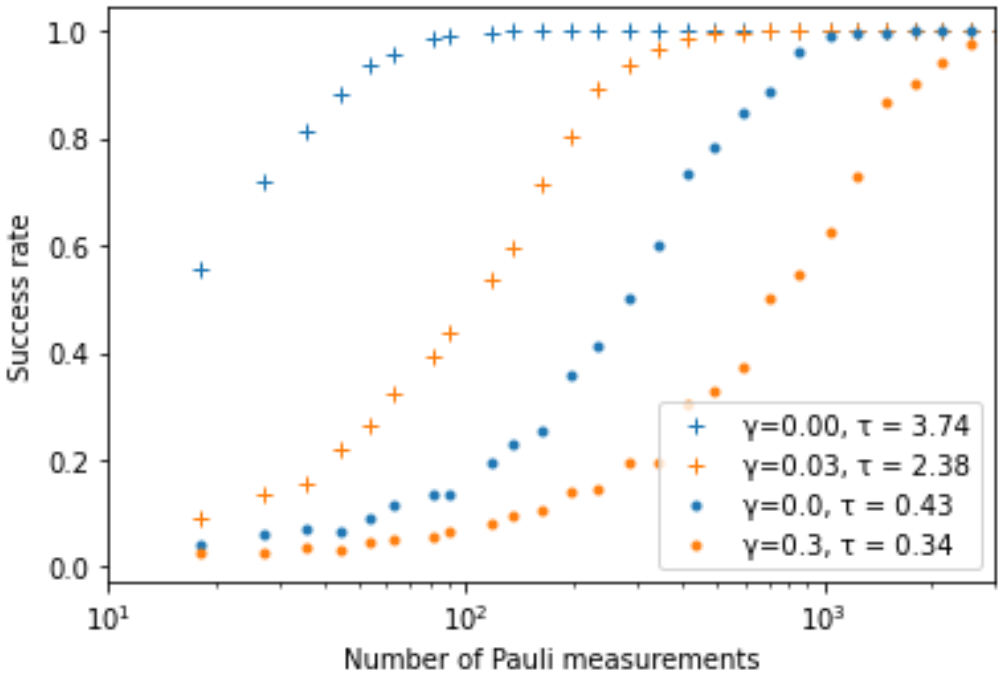

Figure 6.7: Tomographic state distinction success rates, for a 99% confidence threshold, with respect to the number of bipartite Pauli measurements, for noiseless and strong decoherence scenarios. The crosses correspond to the original separation distance $d = 450$ μm while the dots correspond to the closer separation $d = 350$ μm.

6.3.3 Uncertainty in non-gravitational interactions

Having shown how to account for known non-gravitational interactions, we now investigate the impact on our approach of uncertainty in the non-gravitational interaction strength. As a method to rule out H_0, we demonstrated the use of a likelihood ratio test, and in doing so we assumed good knowledge of the non-gravitational interactions. Here, we shall show that the protocol and the likelihood ratio method are robust even when there is uncertainty in the non-gravitational coupling constants, with no need for modifications such as performing the experiment with two different separation distances. This is essentially due to the fact that a difference measurement is already included in the density matrix state that is a result of the double interferometry, and that our proposed witness measurement relies on multiple independent Pauli measurements.

The CP coupling constant $\alpha(R, \epsilon)$ is a good example of a quantity that is not precisely known, as some uncertainty may simply arise from the geometry of the not strictly spherical microdiamonds. From first glance at Eq. (6.19) it seems that an uncertainty in the CP interaction could potentially account for the observed data we could obtain given H_a is true, and thus ruin all hopes of ruling out modified but plausible versions of H_0, in which the value of α is allowed a plausible degree of uncertainty. The immediately apparent potential issue is that the witness expectation

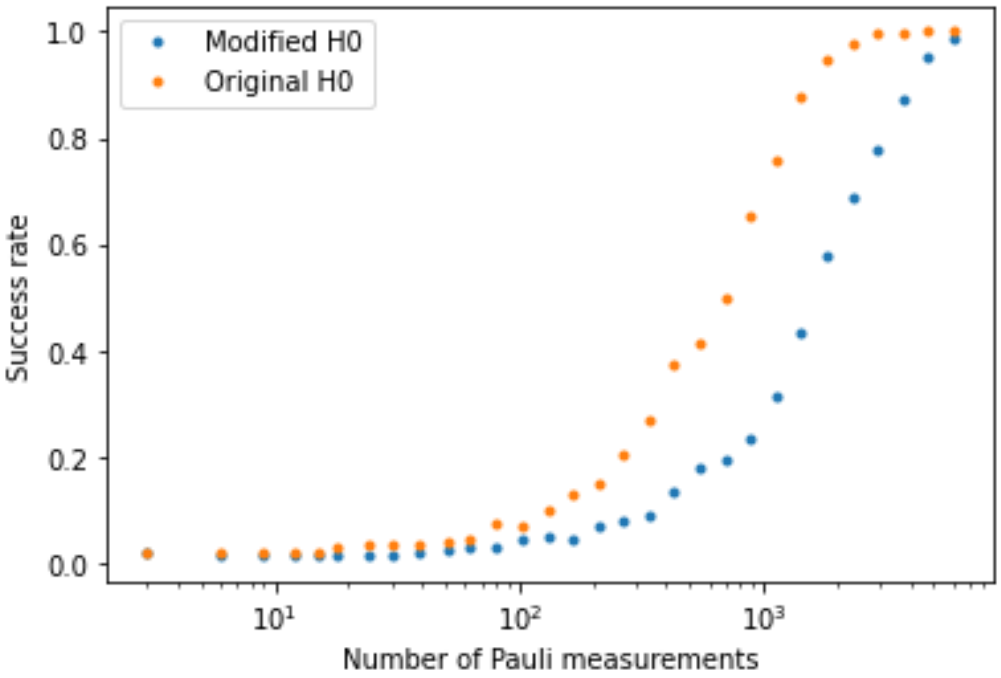

Figure 6.8: Witness measurement success rates with $d = 350\ \mu\text{m}, \gamma = 0.3\ \text{s}^{-1}$ of ruling out H_0 or the modified H_0 in favour of H_a, with respect to the number of bipartite Pauli measurements.

values $\langle W_1 \rangle_0$ and $\langle W_1 \rangle_a$ measured on the two possible states after a fixed free-fall duration τ may coincide up to the error in α. In other words we might think gravity is responsible for the witness expectation value but it could also be accounted for by a larger α.

Although for the original separation distance $d = 450\ \mu$m, this would require a relative uncertainty on α of over 500%, not having precise knowledge of α becomes more problematic at the proposed smaller separation setting $d = 350\ \mu$m. With this closer separation setting and a decoherence rate of $\gamma = 0.3\ \text{s}^{-1}$ the proposed optimal free-fall duration for witnessing entanglement is $\tau = 0.34$ s. Numerics show that for this same free-fall duration, a modified null hypothesis state where $\alpha' = 1.087\alpha$ will yield the same witness expectation value. This means a less than 9% uncertainty on α is sufficient for witness measurement values in the null-hypothesis state to converge to the same value as in the alternative hypothesis. One known solution to solve this issue is to use the fact that the CP potential follows a $1/r^7$ law, whereas the gravitational follows $1/r$, which should show if we repeat the experiment but slightly increase the separation distance, sometimes referred to as a differential measurement.

However, we in fact do not need to change the separation at all. Interestingly, the protocol is already itself a differential measurement, and although one can think of a modified α that could make the null and alternative witness expectation values coincide, differences will still show in the individual Pauli observables involved in

the witness measurement. This is corroborated by the plot shown on Fig. 6.8 where we observe that the success rate still increases with the number of measurements, despite having performed $\alpha \to 1.087\alpha$ for the null hypothesis state, such that the witness expectation values are equal in both hypotheses. This modification of α in the null hypothesis just makes it more demanding to rule out H_0 but not impossible. The number of bipartite Pauli measurements for a good state distinction success rate go from a few 10^3 to a little less than 10^4.

In fact, the spin density matrices in the null and alternative hypotheses can never be made equal by a simple change in the coupling constants. This is because in Eq. (6.19) the $Q^{(1)}$ and $Q^{(7)}$ quantities are non-proportional tensors, they are differences between several proximities to different powers. Let us note that the same observation applies if one wishes to include dipole-dipole interactions, which would involve a $Q^{(3)}$ quantity that is linearly independent of $Q^{(1)}$ and $Q^{(7)}$. The only term that could not be separated would be one that varies with distance as gravity does and thus contributes some $Q^{(1)}$ term. This emphasises the power of our approach based on Pauli measurement likelihood ratios, rather than comparisons of only the witness expectation values. The latter case cannot distinguish these different interactions, whereas the former can.

6.4 Monotonicity loophole

6.4.1 Alternative states

Finally, we address a potential loophole for the witness-based approach and describe how full tomography could be used to provide a solution. We present results from tomographic simulations which provide an order of magnitude estimate for the number of measurements required for this approach.

We have shown that by analyzing the witness' Pauli measurements, we are able to state with a high degree of statistical confidence not only that the state is entangled but also that the state was produced by a gravitational interaction as opposed to merely a CP interaction. This would be a highly significant observation, but there remains a potential loophole. Since entanglement witnesses are not entanglement monotones, a skeptic could argue that we have not explicitly shown that gravity has *increased* the entanglement. Indeed, in our model the null hypothesis state is already entangled, with negativity $\mathcal{N}(\rho_{\mathrm{C}}) = \mathcal{N}_0 > 0$, and there exist other valid quantum states ρ'_{CG} that are indistinguishable from ρ_{CG} in the witness statistics, for which the negativities $\mathcal{N}_a$ may satisfy $\mathcal{N}_0 > \mathcal{N}_a > 0$. One such state can be found explicitly as follows.

We have simulated a 3×10^3 element string of Pauli outcome data obtained

from 10^3 measurements of W_1 on the alternative hypothesis state ρ_{CG}. Given this data, we have constructed a state ρ_{lh} that is at least as likely as ρ_{CG} given the data but has a negativity that is lower than the null hypothesis state ρ_C. Explicitly for separation distance $d = 350$ μm, decoherence rate $\gamma = 0.3$ s^{-1} with corresponding free-fall duration $\tau = 0.34$ s, one has $\mathcal{N}(\rho_{\mathrm{C}}) \approx 0.108 > \mathcal{N}(\rho_{\mathrm{lh}}) \approx 0.104$, where quoting values to three decimal places we have

$$\rho_{\mathrm{lh}} \approx \begin{pmatrix} 0.257 & 0.009+0.012i & 0.042-0.174i & 0.211+0.011i \\ 0.009-0.0120i & 0.244 & 0.108-0.023i & -0.008+0.004i \\ 0.042+0.174i & 0.108+0.023i & 0.245 & 0.017+0.161i \\ 0.211-0.011i & -0.008-0.004i & 0.017-0.161i & 0.254 \end{pmatrix}. \tag{6.38}$$

This state was obtained via a constrained optimization method (sequential least squares programming) on the surface of a 15-dimensional sphere representing the valid 2-qubit quantum state space, parameterized by 15 real angles. The function to minimize was the negative logarithmic likelihood ratio given the data, under the constraint that the state should be less negative than the null hypothesis state.

However contrived the argument would have to be to justify such a state, an answer to eliminate this loophole can be provided by using full tomography and using complete data to calculate an entanglement monotone, as we shall demonstrate.

6.4.2 Closing loophole with state reconstruction

Quantum state tomography is a method to estimate quantum states from a complete set of measurements on many copies of the same state. It has been widely studied in general [103, 300] as well as in its application to entanglement verification [301]. Tomographic data allows one to perform state reconstruction and to formulate rigorous and more general statistical statements on negativity distributions. We construct state estimators using the well known method of maximum likelihood estimation [302]. It may not always be the most accurate estimation method [303], but it is sufficient for our purpose, and indeed has been extensively used in experiments [304, 305, 306].

We seek to predict how reliably the experiment with tomographic data can certify gravitationally induced entanglement growth. To this end, we simulate a series of full tomographies on the null and alternative hypothesis states ρ_{C} and ρ_{CG}, and reconstruct their corresponding maximum likelihood states ρ_{ML} following a fixed-point iterative method [307]. Explicitly, the maximum likelihood state can be obtained from the empirical data vector $\underline{n} = [n_i]_{1 \leq i \leq 4N}$ containing the number of occurrence

Setting \ Repetitions		1	10	10^2	10^3
$d = 450\mu$m	$\gamma = 0$	1.8%	40.7%	$> 99.9\%$	$> 99.9\%$
	$\gamma = 0.03$	2.6%	2.8%	66.3%	$> 99.9\%$
$d = 350\mu$m	$\gamma = 0$	1.2%	7.5%	53.2%	$> 99.9\%$
	$\gamma = 0.3$	1.7%	4.1%	10.2%	88.7%

Figure 6.9: Probability for the reconstructed alternative hypothesis state to have a higher negativity than the 99-th percentile most entangled reconstructed null hypothesis state, with respect to the number of tomographic trials. Results are shown for the original and shorter separation distances, and for the noiseless and strongest decoherence rate cases.

of the Pauli-measurement outcome $|e_i\rangle\langle e_i|$ as it solves $\rho_{\mathrm{ML}} = R(\rho_{\mathrm{ML}})\rho_{\mathrm{ML}}$, where

$$R : \rho \longmapsto \frac{1}{||\underline{n}||_1} \sum_i \frac{n_i}{\mathrm{Tr}(\rho\,|e_i\rangle\langle e_i|)}\,|e_i\rangle\langle e_i|\,. \tag{6.39}$$

Then the sequence of two-qubit density matrices defined by $\rho_0 = I_4/4$ and $\forall k \in \mathbb{N}, \rho_{k+1} = \mathcal{N}_{\mathrm{Tr}}\left(R(\rho_k)\rho_k R(\rho_k)\right)$ where $\mathcal{N}_{\mathrm{Tr}}$ designates trace normalization, converges heuristically to the maximum-likelihood state. In our simulations, we end the algorithm at the 100th iteration. Among 10^3 trials of 10 full tomographic data generations and reconstructions, the fidelity $\mathcal{F}(\rho_{100}, \rho)$ between the simulated states is at least 90% and on average 95%. Among 10^3 trials of 10^3 tomographies, the fidelity is systematically over 99%.

From the maximum likelihood reconstructed states, we find different negativity distributions in the two hypotheses. The results are summed up in Fig. 6.9, and show that in the original separation $d = 450$ μm a few 10^2 full tomographies is enough to consistently reject the null hypothesis in the strongest considered decoherence rate, and for the closer setup $d = 350$ μm where the CP interaction becomes significant, 10^3 full tomographies (so 9×10^3 bipartite Pauli measurements) is sufficient to reject the null hypothesis almost 90% of the time, in the strongest decoherence scenario with $\gamma = 0.3$ s^{-1}.

It is therefore possible, in a relatively large but not unreasonable number of state tomographies, to obtain reliable proof of entanglement growth by gravitational interaction even in the presence of other stronger known couplings, and some unknown interactions that contribute to the dynamics as a decoherence rate.

It should be emphasised that an experiment that adopts the witness approach and finds positive results will be more than convincing enough for most physicists. The tomographic approach should be viewed as an extra piece that could follow, in order to make the empirical evidence more rigorous and complete.

6.5 Conclusion

The recent proposal [259] for a test of the non-classical nature of gravity features a promising protocol. Although the experimental requirements are futuristic, some important theoretical questions, such as the adopted approximations, the possibility of a better witness, the necessity of a CP closest approach limit, and the closing of possible loopholes, were left open. In this chapter, we have addressed those questions.

We showed that the approximation of the initial work are valid for a good range of thermal states in the range of proposed experiment durations. By introducing a new entanglement witness, we were able to show that entanglement revelation is possible with dramatically less free-fall time. We have furthermore pointed out that the result of the experiment should be analysed statistically and that this enables closer distances where the CP interaction is no longer negligible, i.e. we have enabled a relaxing of the original proposals requirement that gravity be the dominant interaction. We also showed that our statistical approach allows one to deal with some uncertainty in the coupling constants of the other significant interactions, illustrated again with the CP interaction. In brief our work enables the spin witness protocol to work better with noisy dynamics, in sub-second interaction times.

Finally, we pointed out a potential loophole with the witness approach based on the fact witnesses are not entanglement monotones and illustrated how this can be rigorously closed using state tomography. An interesting future research direction would be to see if the same result can be achieved more efficiently, through some adaptive approach or a hybrid scheme where witness measurements are supplemented by a limited number of measurements in other bases.

Chapter 7

Conclusions and Outlook

> *He knew that all the hazards and perils were now drawing together to a point: the next day would be a day of doom, the day of final effort or disaster, the last gasp.*
>
> – J. R. R. Tolkien, *The Return of the King*

In this thesis we have presented work on quantum correlations. We began by providing historical context and motivation, tracing the story back from the early 20th century through to the modern understanding within the field of quantum information theory. This was followed by a chapter laying out the fundamentals, taking in the necessary mathematics, quantum theory and foundations of quantum correlations. We then reported results from research on various topics in the field of quantum correlations. These have ranged from points of foundational interest such as understanding entanglement's capacity to enable interactions to be delocalised and how entanglement can be affected by relativistic motion, through to more practical uses as a resource for sharing information whilst maintaining anonymity and as a means to experimentally test for non-classicality of gravity. This has hopefully demonstrated some of the impressive breadth of ideas and new directions within the field. In this final chapter we shall briefly draw together the main conclusions and look forward to future research directions for each of the topics considered.

In Chapter 3 we characterised the quantum correlations required for maintaining anonymity in a metrology task. The most interesting result was the demonstrated operational distinction between subsets of discord and entanglement. It also captured anonymity resources which appear to resemble some hybrid of coherence and correlation which could prove an interesting topic for further study. Experimental realisations would probably be possible, but the protocol does not have any obvious commercial applications, instead it helps clarify the strengths of the various quantum correlations. Furthermore, results on the strongly anonymous case were the direct inspiration for the developments in formulating and understanding delocalised in-

teractions. More generally, the use of quantum correlations to hide information is still a topic of interest [308, 309, 310], and the related topic of quantum cryptography may be close to significant commercial fruition, with companies already selling actual devices [311].

In Chapter 4 we formulated and studied delocalised interactions. The concept of delocalised interactions is physically interesting in large part by virtue of its classically unintuitive nature, demonstrating that interactions can happen without a unique location and that entanglement is key to this departure from classical logic. The strong relation with concurrence was an unexpected discovery, together with the connection to teleportation which could be further explored in future work. The attempt to encapsulate the concept with an information theoretic inequality did not fully succeed but remains an interesting direction for further study. Perhaps entropic quantities could prove superior to the trace distance in this regard. Additional research could also generalise the games to multipartite settings, where it is known that entanglement can have a more complicated structure, even for just three qubits [312]. Further generalisation could be found by exploring the use of higher dimensional quantum systems and investigating games with more question states. From an experimental point of view it was pleasing to be able to demonstrate the effect using the IBM device, but a less noisy demonstration with attention paid to closing loopholes could provide a more convincing demonstration.

In Chapter 5 we studied entanglement under motion when relativity is accounted for. We showed how the capacity for boosts to change the entanglement between internal spin and motional states is also present for internal energy levels, and indeed is relatively more straightforward to work with. We found that this could lead to small but interesting effects such as reducing the purity of energy-level based qubits. We then applied our understanding of these boosts to the question of proper time for quantum clocks and recovered the full behaviour for both cases in a twin-paradox scenario. We contrasted this with the non-classical behaviour caused by momentum boosts, in particular deriving the experimentally observed behaviour for atomic clocks together with an additional frequency shift. This situation where clocks are defined despite occupying superpositions of frames might prove interesting to study through the lens of quantum reference frames [252]. The topic of proper time for quantum clocks is an active area of study, although unfortunately many of the new effects are not easy to demonstrate experimentally. Perhaps technological improvements or novel quantum control techniques will help move this forward. From a theoretical point of view there is still a lack of consensus on the importance of entanglement between internal and motion degrees of freedom in order to claim classical time dilation [249, 313, 314]. Hopefully going forward a clear understanding and terminology can be agreed upon across the community.

In Chapter 6 we introduced improvements to a proposed protocol for testing non-classicality of gravity. This helped bring the experiment closer to feasibility, though it is still a highly challenging project. However there is a continuing effort to improve, such as recent work addressing important aspects for noise mitigation [315]. It could be interesting to see if we can build further on our approach, for instance by considering non-linear entanglement witnesses [316], or by trying to close the monotonicity loophole with some minimal introduction of new measurement settings or an adaptive scheme rather than full tomography. There can be other approaches to the problem of witnessing non-classical gravity such as using non-Gaussianity [317] and it could be fruitful to see if such proposals can be enhanced by the tools we made use of here. It would also be worthwhile to understand whether different proposals can be robust to non-gravitational interactions of unknown strength, which we demonstrated in the spin witness protocol and is essentially a consequence of it being a form of difference measurement. There has also been a proposal to use a similar setup to the spin witness protocol but to test for discreteness of time [318], so there could be opportunities to build on this line of enquiry. Additionally, it could prove beneficial to consider the spin witness protocol but refocused to observing the effects of the Casimir-Polder interaction. Since at short distances this can be significantly stronger than the gravitational interaction and it does not require as large masses, it might prove an easier first experiment to realise. This would demonstrate a novel phenomenon in creating and verifying entanglement generated by the Casimir-Polder force, and would also serve as a useful testing ground to develop and improve ideas with the aim of an eventual experiment to test for non-classicality of gravity.

The overall outlook for quantum correlations as a field of study is certainly positive. We have only touched on a handful of the many directions in which this field is advancing. In the 85 years since its inception, many important questions have been answered, but we are still uncovering new facets and deepening our understanding of this most enigmatic of topics. Conceptual questions in this area are naturally enticing to theorists, but with the ever impressive improvements in quantum technology we are arguably in a golden age for quantum research. The much touted quantum internet [319, 320] seems ever nearer, with a growing number of demonstration networks [321, 322, 323, 324] and realisations of various quantum protocols using satellites [325, 326, 327, 328, 329, 330, 331]. Additionally the field of quantum computation keeps advancing [332, 333, 334, 335]. Researchers can now construct devices with which they claim to demonstrate the dramatically named quantum supremacy [336], and though this is a controversial assertion, it would have seemed completely outlandish not that long ago. With large networks of distributed entanglement and such well controlled multi-qubit quantum systems, the ability to

manipulate entanglement and realise protocols that test or make use of quantum correlations is ever expanding. Just to take one example, so much past study of quantum correlations has focused on the bipartite setting, but the preponderance of large scale quantum systems could spur a new wave of improvements to our understanding of the multipartite settings. In addition to this direction of progress, we should also highlight that quantum control of increasingly massive systems has also shown great advancements [337, 338, 339, 340, 341, 342, 343]. Not only do some of these bring practical uses such as in quantum sensing [344], but they also provide the potential for probing new regimes together with various exciting tests of fundamental physics, in which as we have seen, entanglement can play a key role.

As a final point to bear in mind, it is always difficult to fully envisage where science and technology will take us. In 1952 Schrödinger wrote that "we never experiment with just one electron or atom or (small) molecule. In thought-experiments we sometimes assume that we do; this invariably entails ridiculous consequences" [345]. The 2012 Nobel prize was awarded for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems. The lesson is that even great scientists can be blindsided by the advancements in our ability to control and investigate nature. As such, new experimental breakthroughs could open entirely unexpected areas for quantum research, but one would not bet against the importance of quantum correlations in aiding our understanding for whatever direction the field moves. Quantum correlations will always remain one of the most counter-intuitive and foundationally important aspects of quantum theory.

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