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Convex combination of quantum states and characterization of correlation

Guangrong Liu*, Yang Liu and Xiangrong Li

Fundamental Department, Air Force Engineering University, Xi'an, Shaanxi, 710051, China

*Corresponding author's e-mail: fclgr@163.com

Abstract: In the study of optical quantum information, quantum teleportation can be realized by preparing multiphoton entangled states. Entanglement is a special quantum correlation, and there is also a quantum correlation in separable quantum States, which comes from the measurement of quantum states. Convex combinations of arbitrary quantum states are still quantum states. From the spectral decomposition of a quantum state, an ensemble formed by a complete eigenvector system can be determined, which corresponds to an orthogonal projection operator family with rank 1, and the convex combination of the operator family will correspond to a quantum state. The characterization of quantum correlation is an important issue in the study of quantum information theory. A Quantum state that remains unchanged under local orthogonal projection is a classical correlation state, which is not affected by decoherence. Based on the standard orthogonal basis, this paper considers the correlation of the quantum states formed by the convex combination of the operator family. According to the separability and entanglement of the ground state and the mutual dissimilarity of the corresponding non-zero eigenvalues, it is concluded that the convex combination of the operator family formed by the separable standard orthogonal basis in the composite system must be a classical correlation state. When the entangled pure state is an eigenvector, it is a quantum correlation state if the corresponding eigenvalue is a non-zero single value. In the case of multiple roots, we can illustrate that both are possible. For the correlation of unitary evolution of quantum states under different standard orthogonal bases, the evolution law is revealed by considering important quantum gates.

1. Introduction

With the development of optical quantum information technology, the experimental demonstration of multi-qubit operation has entered a new stage of high complexity and high-performance optical quantum processor prototypes. The preparation of quantum correlation state, the basic operation and algorithm of optical quantum have been initially realized, and the characterization of quantum state correlation is also the key problem of quantum correlation theory. The characterization of quantum state correlation [1-3] is the key problem of quantum correlation theory. Relative to separability, there is quantum entanglement [4-18] in composite systems. The entangled quantum state must be a quantum correlation, and there is also a quantum correlation in the separable quantum state. Using the standard orthogonal basis of the composite system, the local orthogonal projection measurement of the quantum state can be given. On this basis, Guo [1] gives the definition of the classical associated state, which is a quantum state that remains unchanged in the presence of some local orthogonal projection measurement. However, with this definition, it is not easy to judge whether a quantum state is a classical correlation state or a quantum correlation state by calculation. Combined with Theorem 2.1.1 in [1], we can see that the classical associated state must be a quantum state diagonalized under a separable standard orthogonal basis. Taking this as a starting point, we can construct classical correlation states and quantum correlation states by convex combination. At the same time, the correlation can be measured by the commutativity of the operator family generated in the standard orthogonal basis and the distance between operators [1], but the calculation is also complicated.

2. Construction of classical correlation state and quantum associated state

We make $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces, which are used to represent the state spaces of quantum systems A and B according to the hypothesis of quantum mechanics. The state space of composite systems A and B can be written as $\mathcal{H}_A \otimes \mathcal{H}_B$ and abbreviated as $\mathcal{H}_{AB}$, and its separable orthonormal basis is $|e_i\rangle|f_j\rangle$, where $\{e_i\}$ and $\{f_j\}$ are respectively orthonormal basis, i=1, 2, …,d_A,*j*= 1, 2, …,d_B. From this, we can get the local orthogonal projection measurement $|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|$, which is a family of one-rank projection operators. By using the convex combination of joint probability density and operator family, the following definition can be given from Theorem 2.1.1 in [1].

Definition 1 If the quantum state ρ in the two-body quantum system $\mathcal{H}_{AB} \triangleq \mathcal{H}_A \otimes \mathcal{H}_B$ has the following form

$$\rho = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} p_{ij} |e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j| \tag{1}$$

then, it is called a classical correlation state, where $\{p_{ij}\}$ is the joint probability distribution, *i*=1, 2, …, d_A, *j*= 1, 2, …, d_B. $\{e_i\}$ and $\{f_j\}$ are the standard orthogonal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Otherwise, it is a quantum correlation state.

It can be seen from the definition that a joint probability distribution and the standard orthogonal basis correspond to a classical correlation state, and conversely, a classical correlation state can have different representations, in which $\{p_{ij}\}$ is the eigenvalue of the quantum state, and its eigenvector system corresponds to an ensemble of separable pure states. Especially, when the non-zero eigenvalues are all single roots, it is known from Theorem 2.1.2 in [1] that the expression is unique in a sense. If there are non-zero multiple roots, the expression may not be unique, because different eigenvectors can be selected to form standard orthogonal bases. So how to construct quantum correlation states? Therefore, the following theorem is given.

Theorem 1 For the quantum system $\mathcal{H}_{AB}$, given the standard orthogonal basis $\{e_i\}$, its corresponding probability density $\{p_i\}$, *i*=1, 2, …, $d_A\ d_B$, considers the convex combination

$$\mathrm{P} = \sum_{i=1}^{d_A d_B} p_i\ |e_i\rangle\langle e_i| \tag{2}$$

where p_i is a non-zero characteristic single root, $|e_i\rangle$ is an entangled state, and ρ is a quantum correlation state. When p_i is a non-zero characteristic multiple roots, even if the corresponding characteristic state is entangled, ρ may be a classical correlation state.

Proof It can be seen that Formula (2) is a spectral decomposition of ρ and a positive operator with a trace of 1, so it is a quantum state. When p_i is a non-zero characteristic root, if it is a classical correlation state, then p_i is an eigenvalue, and the corresponding | $|e_i\rangle$ is a separable pure state, which contradicts the hypothesis, so it is a quantum correlation state. When p_i is a non-zero characteristic multiple roots, if we choose bell state

$$|e_i\rangle = \left\{\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\right\}, \tag{3}$$

the probability densities are $\left\{\frac{1}{2}, \frac{1}{2}, 0, 0\right\}$ *and* $\left\{\frac{1}{2}, 0, \frac{1}{2}, 0\right\}$, respectively. Then

$$\rho_1 = \frac{1}{4}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + \frac{1}{4}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)$$
$$= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

$$=\frac{1}{2}\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&1\end{pmatrix} \tag{4}$$

$$\rho_2=\frac{1}{4}(|00\rangle+|11\rangle)(\langle 00|+\langle 11|)+\frac{1}{4}(|01\rangle+|10\rangle)(\langle 01|+\langle 10|)$$
$$=\frac{1}{4}|00\rangle\langle 00|+\frac{1}{4}|11\rangle\langle 11|+\frac{1}{4}|00\rangle\langle 11|+\frac{1}{4}|11\rangle\langle 00|$$
$$+\frac{1}{4}|01\rangle\langle 01|+\frac{1}{4}|01\rangle\langle 10|+\frac{1}{4}|10\rangle\langle 01|+\frac{1}{4}|10\rangle\langle 10|$$
$$=\frac{1}{4}\begin{pmatrix}1&&&1\\&1&1&\\&1&1&\\1&&&1\end{pmatrix}. \tag{5}$$

It can be seen that ρ_1 is a convex combination of separable orthogonal pure States, and the matrix under separable orthogonal basis is a diagonal matrix, which is a classical correlation state. Although the surface of ρ_2 looks like a combination of separable orthogonal pure states, the sum of weights is not 1, it is not a probability distribution, and it is not a diagonal matrix, but a quantum correlation state. The theorem is proved.

Next, the measurement method of quantum correlation introduced in [1] is further illustrated.

Example 1 For Werner state,

$$w_\lambda=\lambda|\Psi^-\rangle\langle\Psi^-|+\frac{1-\lambda}{3}(|\Psi^+\rangle\langle\Psi^+|+|\Phi^-\rangle\langle\Phi^-|+|\Phi^+\rangle\langle\Phi^+|) \tag{6}$$

where $\lambda\in[0,1]$, $|\Psi^\pm\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$, $|\Phi^-\rangle=|11\rangle$, $|\Phi^+\rangle=|00\rangle$. According to the Peres-Horodeckes criterion, which is the judgment method of separability and entanglement, it can be concluded that it is separable when $0\le\lambda\le 0.5$, and entangled when $0.5<\lambda\le 1$. $Q(w_\lambda)=\frac{\sqrt{2}(1-4\lambda)}{18}$ is further calculated by the measurement method of quantum correlation in [1], and we get that when $\lambda=0.25$, w_λ is a classical correlation state; otherwise, it is a quantum correlation state. The above judgment method is complicated in calculation. According to the conclusion of Theorem 1, when $\lambda\neq\frac{1-\lambda}{3}$, that is, $\lambda\neq 0.25$, w_λ is a quantum correlation state. When $\lambda=\frac{1-\lambda}{3}\lambda$, that is, $\lambda=0.25$, its corresponding characteristic subspace happens to be a two-dimensional composite system, and a separable natural basis can be selected, so w_λ is a classical correlation state.

Example 2 For two-dimensional quantum systems $\mathcal{H}_A$ and $\mathcal{H}_B$, the natural basis is selected $|0\rangle=\binom{1}{0}$, $|1\rangle=\binom{0}{1}$, and

$$\rho_1=\frac{1}{2}(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)\otimes|0\rangle\langle 0|=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}, \tag{7}$$

$$\rho_2=|0\rangle\langle 0|\otimes|0\rangle\langle 0|=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}. \tag{8}$$

Obviously, the product states ρ_1 and ρ_2 are both classical correlation states. Considering their convex combination $\rho=\lambda\rho_1+(1-\lambda)\rho_2$, for $0<\lambda<1$, $\mathcal{H}_A$ can't choose $|0\rangle,|1\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ at the same time, so it is a quantum correlation state. On the other hand, it can be obtained by

$$\rho=\begin{pmatrix}1-\frac{1}{2}\lambda&0&0&\frac{1}{2}\lambda\\&0&0&0\\&0&0&0\\\frac{1}{2}\lambda&0&0&\frac{1}{2}\lambda\end{pmatrix}. \tag{9}$$

According to Theorem 1, because the eigenvalues of $\rho, 1-\frac{1}{2}\lambda, \frac{1}{2}\lambda, 0, 0$, in which the eigenvector corresponding to the characteristic single root $\frac{1}{2}\lambda$ is $\left(\frac{\lambda}{2(1-\lambda)}, 0,0,1\right)^T$, normalization can only be an entangled pure state, so ρ is a quantum correlation state. This example can also be judged by using the commutativity of operator families with the help of theorem 2.2.6 in [1].

3. Quantum states and quantum unitary gate

Generally speaking, two different quantum states correspond to an ensemble composed of different eigenvectors. Unless these two quantum States are commutative, they can be diagonalized under the same ensemble. So, how does the correlation of quantum States evolve under different standard orthogonal bases?

Theorem 2 For the quantum system $\mathcal{H}_{AB}$, given the standard orthogonal basis $\{e_i\}$ and $\{f_i\}$ and the probability density $\{p_i\}$, i=1, 2, …, $d_A\, d_B$, two quantum states are

$$\rho = \sum_{i=1}^{d_A d_B} p_i\, |e_i\rangle\langle e_i|, \sigma = \sum_{i=1}^{d_A d_B} p_i\, |f_i\rangle\langle f_i|. \tag{10}$$

There is a unitary matrix U, which makes $\sigma = U\rho U^\dagger$. When the standard orthogonal basis $\{f_i\}$ is divisible, the unitary matrix U evolves the quantum state into a classical correlation state. When the standard orthogonal basis $\{f_i\}$ has an entangled state $|f_{i_0}\rangle$, and when $i_0 \neq i, p_{i_0 \neq} p_i \neq 0$, the unitary matrix u evolves the quantum state into a quantum correlation state.

Proof Because the transition matrix between orthogonal bases of different standards is a unitary matrix, there exists a unitary matrix U, which makes $|f_i\rangle = U|e_i\rangle$, $i = 1, 2, \dots, d_A\, d_B$, thus further satisfying $\sigma = U\rho U^\dagger$. Theorem 1 shows that the ensemble of quantum states depends on image states $\{f_i\}$. When $\{f_i\}$ is divisible, the quantum state σ is a classical correlation state; when $\{f_i\}$ has an entangled state $|f_{i_0}\rangle$, and when $i_0 \neq i, p_{i_0 \neq} p_i \neq 0$. The theorem is proved.

Example 3 For two-dimensional quantum systems $\mathcal{H}_A$ and $\mathcal{H}_B$, and composite system $\mathcal{H}_{AB}$, we consider single quantum bit gates Pauli-X gate and Hadamard-gate H as shown in Figure 1, in which

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{11}$$

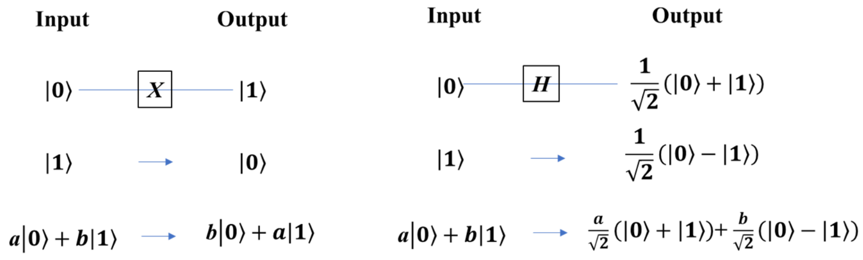

Figure 1. Single quantum bit gates: Pauli-X gate and Hardamard- gate.

Using the Pauli-X gate, we can get the controlled NOT-gate CNOT gate X as shown in Figure 2 in the double quantum bit gate, and its quantum circuit and matrix are shown as follows.

$$\text{CNOT}X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It maps base 1: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ into base 2: $|00\rangle, |01\rangle, |11\rangle, |10\rangle$. Thus, the classical correlation state diag (p_1, p_2, p_3, p_4) under base 1 is mapped to the classical correlation state $\text{diag}(p_1, p_2, p_4, p_3)$ under base 2.

For base 1 $|00\rangle, |01\rangle, |11\rangle, |10\rangle$, Standard orthogonal basis formed by mapping to Bell state, basis 2

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \tag{12}$$

The transition matrix is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

Furthermore, by combining the quantum circuits that generate bell states, we can get the quantum gate U as shown in Figure 3, in which the classical correlated states and quantum correlated states evolve mutually.

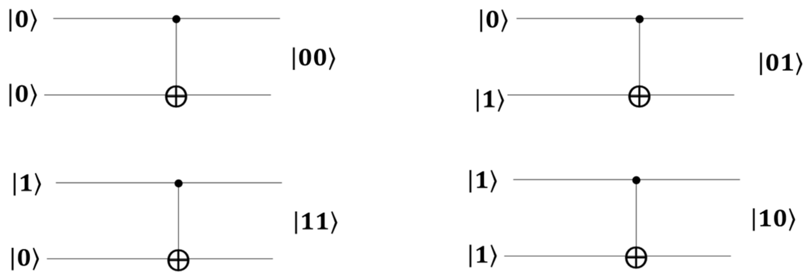

Figure 2. Controlled NOT gate and evolution of ground state under double qubits.

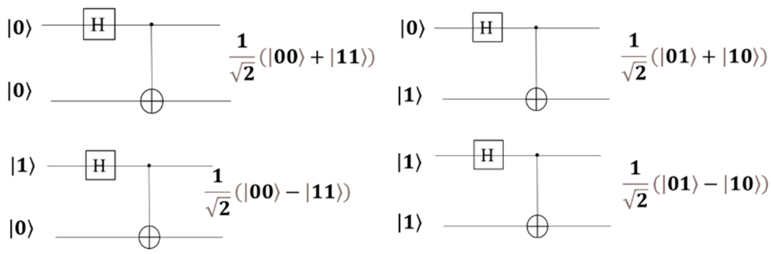

Figure 3. Quantum circuit of Bell state generation.

4. Conclusion

In a word, in the multiphoton composite system, not only quantum correlated states can be prepared, but also quantum classical correlated states can be prepared. The classical correlation state can be constructed by using the separable standard orthogonal basis, and the quantum correlation state can be constructed by using the standard orthogonal basis if there is an entangled pure state. It can be seen that quantum correlation is closely related to the multiplicity of eigenvalues and the separability of eigenvectors. The quantum state constructed by a probability distribution and one-rank projection orthogonal operator family is easy to control, and the evolution of the quantum state can be further discussed by using the quantum unitary gate.

Acknowledgments

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