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Opérateurs différentiels linéaires $\mathfrak{n}(1|N)$*–invariants et cohomology* $\mathfrak{n}(1|N)$*–relative*

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Abstract. Over the $(1,N)$-dimensional supercircle $S^{1|N}$, we classify $\mathfrak{n}(1|N)$-invariant linear differential operators acting on the superspaces of weighted densities on $S^{1|N}$, where $\mathfrak{n}(1|N)$ is the Heisenberg Lie superalgebra. This result allows us to compute the first differential $\mathfrak{n}(1|N)$-relative cohomology of the Lie superalgebra $\mathscr{K}(N)$ of contact vector fields with coefficients in the superspace of weighted densities. For $N = 0,1,2$, we investigate the first $\mathfrak{n}(1|N)$-relative cohomology space associated with the embedding of $\mathscr{K}(N)$ in the superspace of the supercommutative algebra $\mathscr{SP}(N)$ of pseudodifferential symbols on $S^{1|N}$ and in the Lie superalgebra $\mathscr{S}\Psi\mathscr{DO}(S^{1|N})$ of superpseudodifferential operators with smooth coeffcients. We explicity give 1-cocycles spanning these cohomology spaces.

Résumé. Sur le supercercle $(1,N)$-dimensionnel $S^{1|N}$, nous classifions les opérateurs différentiels linéaires $\mathfrak{n}(1|N)$-invariant agissant sur les densités tensorielles sur $S^{1|N}$, où $\mathfrak{n}(1|N)$ est la superalgèbre de Lie de Heisenberg. Ce résultat permet de calculer le premier espace de cohomologie différentiels $\mathfrak{n}(1|N)$-relative de la superalgèbre de Lie des champs de vecteurs de contact $\mathscr{K}(N)$ à coefficients dans le superespace des densités tensorielles. Pour $N = 0,1,2$, nous etudions le premier espace de cohomologie $\mathfrak{n}(1|N)$-relative de $\mathscr{K}(N)$ dans le superespace de l'algèbre supercommutative $\mathscr{SP}(N)$ des symboles pseudodifférentiels sur $S^{1|N}$ et dans la superalgèbre de Lie $\mathscr{S}\Psi\mathscr{DO}(S^{1|N})$ des opérateurs superpseudodifférentiels. Nous donnons explicitement les 1-cocycles engendrent ces espaces de cohomologie.

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1. Introduction

Let $\mathrm{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle S^1. Consider the 1-parameter deformation of the $\mathrm{Vect}(S^1)$-action on $C^\infty_{\mathbb{C}}(S^1)$:

$$L^\lambda_{X\frac{\mathrm{d}}{\mathrm{d}x}}(f) = Xf' + \lambda X' f,$$

where $X, f \in C^\infty_{\mathbb{C}}(S^1)$ and $X' := \frac{\mathrm{d}X}{\mathrm{d}x}$. Denote by $\mathscr{F}_\lambda$ the $\mathrm{Vect}(S^1)$-module structure on $C^\infty_{\mathbb{C}}(S^1)$ defined by L^λ for a fixed λ. Geometrically, $\mathscr{F}_\lambda = \left\{ f\,\mathrm{d}x^\lambda \mid f \in C^\infty_{\mathbb{C}}(S^1)\right\}$ is the space of weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathscr{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1$, 0 and 1, respectively.

Denote by $\mathrm{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathscr{F}_\lambda, \mathscr{F}_\mu)$ the $\mathrm{Vect}(S^1)$-module of linear differential operators with the natural $\mathrm{Vect}(S^1)$-action denoted $L^{\lambda,\mu}_X(A)$. Each module $\mathrm{D}_{\lambda,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathscr{S}_{\lambda,\mu} := \mathrm{gr}\,\mathrm{D}_{\lambda,\mu}$ is called the *space of symbols*. The quotient-module $\mathrm{D}^k_{\lambda,\mu}/\mathrm{D}^{k-1}_{\lambda,\mu}$ is isomorphic to the module of weighted densities $\mathscr{F}_{\mu-\lambda-k}$; the isomorphism is provided by the principal symbol map σ_{r} defined by:

$$A = \sum_{i=0}^{k} a_i(x)\left(\frac{\partial}{\partial x}\right)^i \mapsto \sigma_{\mathrm{pr}}(A) = a_k(x)(\mathrm{d}x)^{\mu-\lambda-k},$$

We study the classification of $\mathfrak{n}(1|N)$-invariant linear differential operators on $S^{1|N}$ acting in the spaces $\mathfrak{F}^N_\lambda$. Ovsienko and Roger [11] calculated the space $\mathrm{H}^1(\mathrm{Vect}(S^1), \Psi\mathscr{D}\mathscr{O}(S^1))$, where $\mathrm{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle S^1 and $\Psi\mathscr{D}\mathscr{O}(S^1)$ is the space of pseudodifferntial operators. The action is given by the natural embedding of $\mathrm{Vect}(S^1)$ in $\Psi\mathscr{D}\mathscr{O}(S^1)$. They used the results of D. B. Fuks [5] on the cohomology of $\mathrm{Vect}(S^1)$ with coefficients in tensor densities to determine the cohomology with coefficients in the graded module $\mathrm{Grad}(\Psi\mathscr{D}\mathscr{O}(S^1))$, namely $\mathrm{H}^1(\mathrm{Vect}(S^1), \mathrm{Grad}^p(\Psi\mathscr{D}\mathscr{O}(S^1)))$; here $\mathrm{Grad}^p(\Psi\mathscr{D}\mathscr{O}(S^1))$ is isomorphic, as $\mathrm{Vect}(S^1)$-module, to the space of tensor densities $\mathscr{F}_p$ of degree p on S^1. To compute $\mathrm{H}^1(\mathrm{Vect}(S^1), \Psi\mathscr{D}\mathscr{O}(S^1))$, V. Ovsienko and C. Roger applied the theory of spectral sequences to a filtered module over a Lie algebra.

In this paper we consider the superspace $S^{1|N}$ equipped with the contact structure determined by a 1-form α_N, and the Lie superalgebra $\mathscr{K}(N)$ of contact vector fields on $S^{1|N}$. We introduce the $\mathscr{K}(N)$-module $\mathfrak{F}^N_\lambda$ of λ-densities on $S^{1|N}$ and the $\mathscr{K}(N)$-module of linear differential operators, $\mathfrak{D}^N_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}^N_\lambda, \mathfrak{F}^N_\mu)$, which are super analogues of the spaces $\mathscr{F}_\lambda$ and $\mathrm{D}_{\lambda,\mu}$, respectively. We classify all $\mathfrak{n}(1|N)$-invariant linear differential operators on $S^{1|N}$ acting in the spaces $\mathfrak{F}^N_\lambda$. We use the result to compute $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N), \mathfrak{F}^N_\lambda)$. We show that, the non-zero cohomology only appear for resonant values of weights. Moreover, we give explicit bases of these cohomology spaces. For $N = 0, 1, 2$, we follow again the same methods by V. Ovsienko and C. Roger [11] to compute the $\mathfrak{n}(1|N)$-relative cohomology $\mathrm{H}^1(\mathscr{K}(N), \mathfrak{n}(1|N), \mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N}))$, where $\mathfrak{n}(1|N)$ is the Heisenberg Lie superalgebra, and $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ is the space of superpseudodifferential operators on $S^{1|N}$. Moreover, we give explicit bases of these cohomology spaces.

2. Definitions and notations

In this section, we recall the main definitions and facts related to the geometry of the superspace $S^{1|N}$; for more details, see [6, 7, 8, 9, 10].

2.1. *The Lie superalgebra of contact vector fields on* $S^{1|N}$

We define the supercircle $S^{1|N}$ in terms of its superalgebra of functions, denoted by $C^\infty_{\mathbb{C}}(S^{1|N})$ and consisting of elements of the form:

$$F=\sum_{s=0}^{N}\sum_{1\le i_1<i_2<\cdots<i_s\le N} f_{i_1i_2\ldots i_s}(x)\theta_{i_1}\ldots\theta_{i_s},$$

where $f_{i_1i_2\ldots i_s}\in C^\infty_{\mathbb{C}}(S^1)$, and where x is the even indeterminate, $\theta_1,\ldots,\theta_N$ are the odd indeterminates, i.e., $\theta_i\theta_j=-\theta_j\theta_i$. Consider the standard contact structure given by the following 1-form:

$$\alpha_N=\mathrm{d}x+\sum_{i=1}^{N}\theta_i\mathrm{d}\theta_i.$$

On the space $C^\infty_{\mathbb{C}}(S^{1|N})$, we consider the contact bracket

$$\{F,G\}=FG'-F'G-\frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{N}\overline{\eta}_i(F)\cdot\overline{\eta}_i(G),$$

where $\overline{\eta}_i=\frac{\partial}{\partial\theta_i}-\theta_i\frac{\partial}{\partial x}$ and $p(F)$ is the parity of F. Let $\mathrm{Vect}_{\mathbb{C}}(S^{1|N})$ be the superspace of vector fields on $S^{1|N}$:

$$\mathrm{Vect}_{\mathbb{C}}(S^{1|N})=\left\{F_0\partial_x+\sum_{i=1}^{N}F_i\partial_i\;\middle|\;F_i\in C^\infty_{\mathbb{C}}(S^{1|N})\right\},$$

where $\partial_i=\frac{\partial}{\partial\theta_i}$ and $\partial_x=\frac{\partial}{\partial x}$, and consider the superspace $\mathscr{K}(N)$ of contact vector fields on $S^{1|N}$:

$$\mathscr{K}(N)=\left\{X\in\mathrm{Vect}_{\mathbb{C}}(S^{1|N})\;\middle|\;\text{there exists }F\in C^\infty_{\mathbb{C}}(S^{1|N})\text{ such that }\mathfrak{L}_X(\alpha_N)=F\alpha_N\right\},$$

The Lie superalgebra $\mathscr{K}(N)$ is spanned by the fields of the form:

$$X_F=F\partial_x-\frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{N}\overline{\eta}_i(F)\overline{\eta}_i,\ \text{ where } F\in C^\infty_{\mathbb{C}}(S^{1|N}).$$

In particular, we have $\mathscr{K}(0)=\mathrm{Vect}_{\mathbb{C}}(S^1)$. The bracket in $\mathscr{K}(N)$ can be written as:

$$[X_F,\,X_G]=X_{\{F,G\}}.$$

The Lie superalgebra $\mathscr{K}(N-1)$ can be realized as a subalgebra of $\mathscr{K}(N)$:

$$\mathscr{K}(N-1)=\{X_F\in\mathscr{K}(N)\,|\,\partial_N F=0\}.$$

Note also that, for any i in $\{1,2,\ldots,N\}$, $\mathscr{K}(N-1)$ is isomorphic to

$$\mathscr{K}(N-1)^i=\{X_F\in\mathscr{K}(N)\,|\,\partial_i F=0\}.$$

2.2. *The Heisenberg subalgebra* $\mathfrak{n}(1|N)$

The Heisenberg Lie superalgebra $\mathfrak{n}(1|N)$ can be realized as a subalgebra of $\mathscr{K}(N)$:

$$\mathfrak{n}(1|N)=\mathrm{Span}\left(X_1,\,X_{\theta_i}\right),\quad 1\le i\le N.$$

We easily see that $\mathfrak{n}(1|N-1)$ is a subalgebra of $\mathfrak{n}(1|N)$:

$$\mathfrak{n}(1|N-1)=\{X_F\in\mathfrak{n}(1|N-1)\,|\,\partial_N F=0\}.$$

Note also that, for any i in $\{1,2,\ldots,N-1\}$, $\mathfrak{n}(1|N-1)$ is isomorphic to

$$\mathfrak{n}(1|N-1)^i=\{X_F\in\mathfrak{n}(1|N-1)\,|\,\partial_i F=0\}.$$

2.3. *Modules of weighted densities*

For every contact vector field X_F, define a one-parameter family of first-order differential operators on $C^\infty_{\mathbb{C}}(S^{1|N})$:

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{C}.$$

We easily check that

$$\left[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}\right] = \mathfrak{L}^\lambda_{X_{\{F,G\}}}.$$

We thus obtain a one-parameter family of $\mathscr{K}(N)$-modules on $C^\infty_{\mathbb{C}}(S^{1|N})$ that we denote $\mathfrak{F}^N_\lambda$, the space of all weighted densities on $S^{1|N}$ of weight λ with respect to α_N:

$$\mathfrak{F}^N_\lambda = \left\{F\alpha^\lambda_N \,\middle|\, F \in C^\infty(S^{1|N})\right\}.$$

2.4. *Differential operators on weighted densities*

A differential operator on $S^{1|N}$ is an operator on $C^\infty_{\mathbb{C}}(S^{1|N})$ of the form:

$$A = \sum_{k=0}^{M} \sum_{\varepsilon=(\varepsilon_1,\dots,\varepsilon_N)} a_{k,\varepsilon}(x,\theta)\partial_x^k \partial_1^{\varepsilon_1}\dots\partial_N^{\varepsilon_N};\ \varepsilon_i = 0,1;\ M \in \mathbb{N}.$$

Of course any differential operator defines a linear mapping $F\alpha^\lambda_N \mapsto (AF)\alpha^\mu_N$ from $\mathfrak{F}^N_\lambda$ to $\mathfrak{F}^N_\mu$ for any λ, $\mu \in \mathbb{R}$, thus the space of differential operators becomes a family of $\mathscr{K}(N)$-modules $\mathfrak{D}^N_{\lambda,\mu}$ for the natural action:

$$X_F \cdot A = \mathfrak{L}^\mu_{X_F} \circ A - (-1)^{p(A)p(F)} A \circ \mathfrak{L}^\lambda_{X_F}.$$

Every differential operator $A \in \mathfrak{D}^N_{\lambda,\mu}$ can be expressed in the form

$$A(F\alpha^\lambda_N) = \sum_{\ell=(\ell_1,\dots,\ell_N)} a_\ell(x,\theta)\overline{\eta}_1^{\ell_1}\dots\overline{\eta}_N^{\ell_N}(F)\alpha^\mu_N,$$

where the coefficients $a_\ell(x,\theta)$ are arbitrary functions.

Lemma 1 ([2])**.** *As a $\mathscr{K}(N-1)$-module, we have*

$$\mathfrak{D}^N_{\lambda,\mu} \simeq \mathfrak{D}^{N-1}_{\lambda,\mu} \oplus \mathfrak{D}^{N-1}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \oplus \Pi\left(\mathfrak{D}^{N-1}_{\lambda,\mu+\frac{1}{2}} \oplus \mathfrak{D}^{N-1}_{\lambda+\frac{1}{2},\mu}\right), \tag{1}$$

where Π is the change of parity operator.

2.5. *Pseudodifferential operators on $S^{1|N}$*

Let $T^*S^{1|N}$ be the cotangent bundle on $S^{1|N}$ with local coordinates $(x,\theta_1,\dots,\theta_N,\xi,\bar\theta_1,\dots,\bar\theta_N)$, where $p(\bar\theta_i) = 1$. The superspace of the supercommutative algebra $\mathscr{SP}(N)$ of pseudodifferential symbols on $S^{1|N}$ with its natural multiplication is spanned by the series

$$\mathscr{SP}(N) = \left\{\sum_{k=-M}^{\infty} \sum_{\epsilon=(\epsilon_1,\dots,\epsilon_N)} a_{k,\epsilon}(x,\theta)\xi^{-k}\bar\theta_1^{\epsilon_1}\dots\bar\theta_N^{\epsilon_N} \,\middle|\, a_{k,\epsilon} \in C^\infty_{\mathbb{C}}(S^{1|N});\ \epsilon_i = 0,1;\ M \in \mathbb{N}\right\}.$$

This space has a structure of the Poisson Lie superalgebra given by the following bracket:

$$\{A,B\} = \partial_\xi A \partial_x B - \partial_x A \partial_\xi B - (-1)^{p(A)} \sum_{i=1}^{N} \left(\partial_i A \partial_{\bar\theta_i} B + \partial_{\bar\theta_i} A \partial_i B\right),$$

where $\partial_x = \frac{\partial}{\partial_x}$, $\partial_\xi = \frac{\partial}{\partial_\xi}$, $\partial_i = \frac{\partial}{\partial_{\theta_i}}$ and $\partial_{\overline{\theta}_i} = \frac{\partial}{\partial_{\overline{\theta}_i}}$. Of course $\mathscr{SP}(0)$ is the classical space of symbols, usually denoted $\mathscr{P}$.

The associative superalgebra of pseudodifferential operators $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ on $S^{1|N}$ has the same underlying vector space as $\mathscr{S}\mathscr{P}(N)$, but the multiplication is now defined by the following rule:

$$A\circ B=\sum_{\alpha\geq 0,\,\nu_i=0,1}\frac{(-1)^{p(A)+1}}{\alpha!}\left(\partial_\xi^\alpha\partial_{\bar\theta_i}^{\nu_i}A\right)\left(\partial_x^\alpha\partial_i^{\nu_i}B\right).$$

Denote by $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})_{SL}$ the Lie superalgebra with the same superspace as $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ and the supercommutator defined on homogeneous elements by:

$$[A,B]=A\circ B-(-1)^{p(A)p(B)}B\circ A.$$

In particular, we have $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|0})=\Psi\mathscr{D}\mathscr{O}(S^1)$.

3. The structure of $\mathscr{S}\mathscr{P}(N)$ as a $\mathscr{K}(N)$–module

The natural embedding of $\mathscr{K}(N)$ into $\mathscr{S}\mathscr{P}(N)$ defined by

$$\pi(X_F)=F\xi-\frac{(-1)^{p(F)}}{2}\sum_{i=1}^N\overline{\eta}_i(F)\zeta_i,\quad\text{where }\zeta_i=\bar\theta_i-\theta_i\xi,$$

induces a $\mathscr{K}(N)$-module structure on $\mathscr{S}\mathscr{P}(N)$.

Setting $\deg x=\deg\theta_i=0$, $\deg\xi=\deg\bar\theta_i=1$ for all i, we endow the Poisson superalgebra $\mathscr{S}\mathscr{P}(N)$ with a $\mathbb{Z}$-grading:

$$\mathscr{S}\mathscr{P}(N)=\widetilde{\bigoplus_{n\in\mathbb{Z}}}\mathscr{S}\mathscr{P}_n(N),$$

where $\widetilde{\bigoplus}_{n\in\mathbb{Z}}=(\bigoplus_{n<0})\bigoplus\prod_{n\geq 0}$ and

$$\mathscr{S}\mathscr{P}_n(N)=\left\{F\xi^{-n}+G_1\xi^{-n-1}\bar\theta_1+G_2\xi^{-n-1}\bar\theta_2+\cdots+H_{1,2}\xi^{-n-2}\bar\theta_1\bar\theta_2+\ldots\,\middle|\,F,G_i,H_{i,j}\in C^\infty_{\mathbb{C}}(S^{1|N})\right\}$$

is the homogeneous subspace of degree $-n$.

Note that each element of $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ can be expressed as

$$A=\sum_{k\in\mathbb{Z}}(F_k+G_k^1\xi^{-1}\bar\theta_1+\cdots+H_k^{1,2}\xi^{-2}\bar\theta_1\bar\theta_2+\ldots)\xi^{-k},$$

where $F_k, G_k^i, H_k^{i,j}\in C^\infty_{\mathbb{C}}(S^{1|N})$. We define the *order* of A to be

$$\mathrm{ord}(A)=\sup\left\{k\,\middle|\,F_k\neq 0\text{ or }G_k^i\neq 0\text{ or }H_k^{i,j}\neq 0\right\}.$$

This definition of order equips $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ with a decreasing filtration as follows: set

$$\mathbf{F}_n=\left\{A\in\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})\,\middle|\,\mathrm{ord}(A)\leq -n\right\},$$

where $n\in\mathbb{Z}$. So we have

$$\cdots\subset\mathbf{F}_{n+1}\subset\mathbf{F}_n\subset\ldots.$$

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for $A\in\mathbf{F}_n$ and $B\in\mathbf{F}_p$, one has $A\circ B\in\mathbf{F}_{n+p}$ and $\{A,B\}\in\mathbf{F}_{n+p-1}$. This filtration makes $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$ an associative filtered superalgebra. Moreover, this filtration is compatible with the natural $\mathscr{K}(N)$-action on $\mathscr{S}\Psi\mathscr{D}\mathscr{O}(S^{1|N})$. Indeed,

$$X_F(A)=[X_F,A]\in\mathbf{F}_n\text{ for any }X_F\in\mathscr{K}(N)\text{ and }A\in\mathbf{F}_n.$$

The induced $\mathscr{K}(N)$-module structure on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to that of the $\mathscr{K}(N)$-module $\mathscr{S}\mathscr{P}_n(N)$. Therefore,

$$\mathscr{S}\mathscr{P}(N)\simeq\widetilde{\bigoplus_{n\in\mathbb{Z}}}\mathbf{F}_n/\mathbf{F}_{n+1}.$$

4. $\mathfrak{n}(1|N)$–invariant linear differential operators

Now, we describe the spaces of $\mathfrak{n}(1|N)$-invariant linear differential operators $\mathfrak{F}_{\lambda}^{N} \to \mathfrak{F}_{\mu}^{N}$ for $N \in \mathbb{N}$. Our main result of this section is the following:

Theorem 2. *Let $\mathcal{N}_{\lambda,\mu}^{N} : \mathfrak{F}_{\lambda}^{N} \to \mathfrak{F}_{\mu}^{N}, (F\alpha_N^{\lambda}) \mapsto \mathcal{N}_{\lambda,\mu}^{N}(F)\alpha_N^{\mu}$ be a non-zero $\mathcal{N}(1|N)$-invariant linear differential operator. Then, up to a scalar factor, the map $\mathcal{N}_{\lambda,\mu}^{N}$ is given by:*

$$\mathcal{N}_{\lambda,\mu}^{N}(F) = \begin{cases} \sum_{k\geq 0} \gamma_k F^{(k)}, & \text{for } N \in \mathbb{N} \\ \sum_{k\geq 0} \gamma_k \overline{\eta}_1 \overline{\eta}_2 \dots \overline{\eta}_N (F^{(k)}), & \text{for } N \geq 1, \end{cases} \tag{2}$$

where $\gamma_k \in \mathbb{R}$.

Proof. (i). For $N = 0$, the generic form of any such a differential operator is

$$\mathcal{N}_{\lambda,\mu}^{0} : \mathfrak{F}_{\lambda}^{0} \to \mathscr{F}_{\mu}^{0}, F\mathrm{d}x^{\lambda} \mapsto \sum_{i=0}^{m} \gamma_i F^{(i)} \mathrm{d}x^{\mu},$$

where $\gamma_i \in C^{\infty}(S^1)$ are arbitrary functions and $F^{(i)}$ stands for $\frac{\mathrm{d}^i F}{\mathrm{d}x^i}$. The invariance property with respect to the vector field $X = \frac{\mathrm{d}}{\mathrm{d}x}$ implies that $\frac{\mathrm{d}\gamma_i}{\mathrm{d}x} = 0$.

(ii). By induction on N. For $N = 1$, let $\mathcal{N}_{\lambda,\mu}^{1} : \mathfrak{F}_{\lambda}^{1} \to \mathfrak{F}_{\mu}^{1}$ be an $\mathfrak{n}(1|1)$-invariant linear differential operator. The $\mathfrak{n}(1|1)$-invariance of $\mathcal{N}_{\lambda,\mu}^{1}$ is equivalent to invariance with respect just to the subalgebra $\mathfrak{n}(1|0)$ and the vector fields X_{θ_1}. Using the fact that, as $\mathfrak{vect}(S^1)$-modules,

$$\mathfrak{F}_{\lambda}^{1} \simeq \mathfrak{F}_{\lambda}^{0} \oplus \Pi\left(\mathscr{F}_{\lambda+\frac{1}{2}}^{0}\right), \tag{3}$$

we can deduce, by induction hypothesis, the restriction of $\mathcal{N}_{\lambda,\mu}^{1}$ to each component of the right-hand side of (3). The invariance of $\mathcal{N}_{\lambda,\mu}^{1}$ with respect X_{θ_1} determine thus completely the space of $\mathfrak{n}(1|1)$-invariant linear differential operator $\mathfrak{F}_{\lambda}^{1} \to \mathfrak{F}_{\mu}^{1}$.

Now, assume that the result holds for $N > 1$. Observe that the $\mathfrak{n}(1|N)$-invariance of any linear differential operators $\mathcal{N}_{\lambda,\mu}^{N} : \mathfrak{F}_{\lambda}^{N} \to \mathfrak{F}_{\mu}^{N}$ is equivalent to invariance with respect just to the subalgebras $\mathfrak{n}(1|N-1)$ and $\mathfrak{n}(1|N-1)^i$, $i = 1,\dots,N-1$, and that $\mathcal{N}_{\lambda,\mu}^{N}$ is decomposed into four $\mathfrak{n}(1|N-1)$-invariant maps:

$$\Pi^{\imath}\left(\mathfrak{F}_{\lambda+\frac{\imath}{2}}^{N-1}\right) \longrightarrow \Pi^{\jmath}\left(\mathfrak{F}_{\mu+\frac{\jmath}{2}}^{N-1}\right), \quad \imath, \jmath = 0, 1. \tag{4}$$

Thus, by induction assumption, we exhibit the $\mathfrak{n}(1|N-1)$-invariant linear differential operators $\mathfrak{F}_{\lambda}^{N} \to \mathfrak{F}_{\mu}^{N}$. More precisely, any $\mathfrak{n}(1|N-1)$-invariant binary differential operators $\mathcal{N}_{\lambda,\mu}^{N} : \mathfrak{F}_{\lambda}^{N} \to \mathfrak{F}_{\mu}^{N}$ can be expressed as:

$$\begin{aligned} \mathcal{N}_{\lambda,\mu}^{N}(F) &= \Xi_{\lambda,\mu}(1-\theta_N\partial_{\theta_N})(\mathcal{N}_{\lambda,\mu}^{N-1}) - \Theta_{\lambda,\mu}(-1)^{p(F)}\partial_{\theta_N}(\mathcal{N}_{\lambda,\mu}^{N-1})\theta_N, \\ \widetilde{\mathcal{N}}_{\lambda,\mu}^{N}(F) &= (-1)^{p(F)}\Omega_{\lambda,\mu}(1-\theta_i\partial_{\theta_i})(\widetilde{\mathcal{N}}_{\lambda,\mu}^{N-1})\theta_N + \Gamma_{\lambda,\mu}(\partial_{\theta_i}(\widetilde{\mathcal{N}}_{\lambda,\mu}^{N-1}), \end{aligned}$$

where the coefficients $\Omega_{\lambda,\mu}, \Gamma_{\lambda,\mu}, \Xi_{\lambda,\mu}$ and $\Theta_{\lambda,\mu}$ are, a priori, arbitrary constants, but the invariance of $\mathcal{N}_{\lambda,\mu}^{N}$ with respect $\mathfrak{n}(1|N-1)^i$, $i = 1,\dots,N-1$, shows that

$$\Gamma_{\lambda-\frac{N}{2},\lambda+k} = -\Omega_{\lambda-\frac{N}{2},\lambda+k}, \quad \Xi_{\lambda,\lambda+k} = \Theta_{\lambda,\lambda+k}.$$

Therefore, we easily check that $\mathcal{N}_{\lambda,\mu}^{N}$ is expressed as in Theorem 2. This completes the proof of Theorem 2. □

5. Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [4]). Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra acting on a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it assumed to be $\{0\}$). The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g},\mathfrak{h};V) := \mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h});V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g},\mathfrak{h};V) \longrightarrow C^{n+1}(\mathfrak{g},\mathfrak{h};V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted $Z^n(\mathfrak{g},\mathfrak{h};V)$, is the space of $\mathfrak{h}$-relative *n-cocycles*, among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$-relative *n-coboundaries*. We denote $B^n(\mathfrak{g},\mathfrak{h};V)$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{h}$-relative cohomolgy space is the quotient space

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{h};V) = Z^n(\mathfrak{g},\mathfrak{h};V)/B^n(\mathfrak{g},\mathfrak{h};V).$$

5.1. *The spaces* $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N),\mathfrak{F}^N_\lambda)$

In this subsection, we will compute the first differential cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N),\mathfrak{F}^N_\lambda)$. Our main result is the following:

Theorem 3. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N),\mathfrak{F}^N_\lambda)$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N),\mathfrak{F}^N_\lambda) = \begin{cases} \mathbb{R}^2 & \text{if } N=2 \text{ and } \lambda=0 \\ \mathbb{R} & \text{if } \begin{cases} N=0 \text{ and } \lambda=0,1,2 \\ N=1 \text{ and } \lambda=0,\frac{1}{2},\frac{3}{2} \\ N=2 \text{ and } \lambda=1 \\ N=3 \text{ and } \lambda=0,\frac{1}{2} \\ N\geq 4 \text{ and } \lambda=0 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

The following 1*-cocycles* Υ^N_λ *span the corresponding cohomology spaces:*

$$\begin{array}{ll} \Upsilon^N_0(X_F) = F';\ N\in\mathbb{N}, & \Upsilon^1_{\frac{3}{2}}(X_F) = \bar\eta_1(F'')\alpha_1^{\frac{3}{2}}, \\ \Upsilon^0_1(X_F) = F''\mathrm{d}x^1, & \Upsilon^2_0(X_F) = \bar\eta_1\bar\eta_2(F)\alpha_2, \\ \Upsilon^0_2(X_F) = F'''\mathrm{d}x^2, & \Upsilon^2_1(X_F) = \bar\eta_1\bar\eta_2(F')\alpha_2, \\ \Upsilon^1_{\frac{1}{2}}(X_F) = \overline{\eta}_1(F')\alpha_1^{\frac{1}{2}}, & \Upsilon^3_{\frac{1}{2}}(X_F) = \bar\eta_1\bar\eta_2\bar\eta_3(F)\alpha_3^{\frac{1}{2}}. \end{array} \tag{5}$$

The proof of Theorem 3 will be the subject of subsection 5.2. In fact, we need first the following classical fact:

Lemma 4 ([3])**.** *Any 1-cocycle* Υ *on* $\mathcal{K}(N)$ *vanishing on* $\mathfrak{n}(1|N)$*, with values in* $\mathfrak{F}^N_\lambda$*, the linear differential operator* $\mathcal{N} : \mathcal{K}(N) \to \mathfrak{F}^N_\lambda$ *defined by*

$$\mathcal{N}(X) = \Upsilon(X),$$

is $\mathfrak{n}(1|N)$*-invariant.*

5.2. *Proof of the Theorem 3*

Let $\Upsilon^N_{-1,\mu}$ be a 1-cocycle on $\mathscr{K}(N)$ vanishing on $\mathfrak{n}(1|N)$, with values in $\mathfrak{F}^N_\mu$. By Lemma 4, up to a scalar factor, $\Upsilon^N_{-1,\mu}$ is a linear differential operator $\mathfrak{n}(1|N)$-invariant $\mathscr{N}^N_{-1,\mu}: \mathfrak{F}^N_{-1} \to \mathfrak{F}^N_\mu$. Thus, by Theorem 2, we get the explicit formulae for $\mathscr{N}^N_{-1,\mu}$:

$$\text{For } N=0, \left\{\mathscr{N}^0_{-1,\mu}(X_F) = \textstyle\sum_{k\geq 0}\gamma_k F^{(k)}\mathrm{d}x^\mu\right.$$

$$\text{For } N\geq 1, \begin{cases}\mathscr{N}^N_{-1,\mu}(X_F) = \sum_{k\geq 0}\gamma_k F^{(k)}\alpha^\mu_N\\ \mathscr{N}^N_{-1,\mu}(X_F) = \sum_{k\geq 0}\gamma_k \overline{\eta}_1\overline{\eta}_2\ldots\overline{\eta}_N(F^{(k)})\alpha^\mu_N.\end{cases}$$

Now let us check if each of the maps $\mathscr{N}^N_{-1,\mu}$ are 1-cocycles. If the maps $\mathscr{N}^N_{-1,\mu}$ are 1-cocycles one has to check the 1−cocycles one has to check the 1-cocycle relation. It reads as follows:

$$\begin{aligned}\delta(\mathscr{N}^N_{-1,\mu}) &= (-1)^{p(X)p(\mathscr{N}^N_{-1,\mu})}\mathfrak{L}^\mu_X(\mathscr{N}^N_{-1,\mu}(Y)) - (-1)^{p(Y)(p(X)+p(\mathscr{N}^N_{-1,\mu}))}\mathfrak{L}^\mu_Y(\mathscr{N}^N_{-1,\mu}(X)) - \mathscr{N}^N_{-1,\mu}([X,Y])\\ &= 0,\end{aligned}$$

where $X, Y \in \mathscr{K}(N)$. By direct computation, we can see that only the operators $\mathscr{N}^N_{-1,\mu} = \Upsilon^N_\mu$ expressed as in (5) are 1-cocycles vanishing on $\mathfrak{n}(1|N)$.

Finally, we study the non-triviality of these 1-cocycles $\mathscr{N}^N_{-1,\lambda}$. For instance, assume that the 1-cocycle $\mathscr{N}^0_{-1,2}$ is trivial, then there exists a density $\varphi(x)\mathrm{d}x^2 \in \mathfrak{F}^0_2$ such that

$$\mathscr{N}^0_{-1,2}(X_F) = L^2_{X_F}\varphi(x)\mathrm{d}x^2. \tag{6}$$

The coefficient of F''' is zero in the expression of the coboundary and the coefficient of F''' is 1 in the expression of 1-cocycle $\mathscr{N}^0_{-1,2}$. Thus, the relation (6) implies 1 = 0 which is absurd. With the same arguments, we prove the non-triviality of 1-cocycles $\mathscr{N}^N_{-1,0}$, $\mathscr{N}^0_{-1,1}$, $\mathscr{N}^0_{-1,2}$, $\mathscr{N}^1_{-1,\frac{1}{2}}$, $\mathscr{N}^1_{-1,\frac{3}{2}}$, $\mathscr{N}^2_{-1,0}$, $\mathscr{N}^2_{-1,1}$ and $\mathscr{N}^3_{-1,\frac{1}{2}}$. Therefore, we easily check that Υ^N_λ is expressed as in (5). This completes the proof of Theorem 3.

6. $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{SP}_n(N))$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{S}\Psi\mathscr{DO}(S^{1|N}))$

6.1. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{SP}_n(N))$

The space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{SP}_n(N))$ inherits the grading (3) of $\mathscr{SP}_n(N)$, so it suffices to compute it in each degree. The main result of this section for $N = 0, 1, 2$, is the following.

Theorem 5. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{SP}_n(N))$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N), \mathfrak{n}(1|N); \mathscr{SP}_n(N)) \simeq \begin{cases}\mathbb{R} & \textit{if} \begin{cases} N=2 \ \textit{and} \ n=1\\ N=0 \ \textit{and} \ n=0,1,2\\ N=1 \ \textit{and} \ n=1\end{cases}\\ \mathbb{R}^2 & \textit{if} \begin{cases} N=2 \ \textit{and} \ n=-1\\ N=1 \ \textit{and} \ n=0\end{cases}\\ \mathbb{R}^5 & \textit{if } N=2 \ \textit{and} \ n=0\\ 0 & \textit{otherwise.}\end{cases} \tag{7}$$

The following 1*-cocycles* χ_n^N *span the corresponding cohomology spaces:*

$$\begin{array}{ll}
\chi_0^N = F', \text{ for } N=0,2, & \chi_{-1}^2 = \overline{\eta}_1\overline{\eta}_2(F)\xi^{-1}\zeta_1\zeta_1,\\
\chi_1^0 = F''\xi^{-1}, & \widetilde{\chi}_{-1}^2 = F'\xi^{-1}\zeta_1\zeta_1,\\
\chi_2^0 = F'\xi^{-2}, & \widetilde{\chi}_0^2 = \overline{\eta}_1\overline{\eta}_2(F),\\
\chi_0^1 = (1+(-1)^{p(F)})F' + \overline{\eta}_1(F')\xi^{-1}\zeta_1, & \widehat{\chi}_0^2(X_F) = (-1)^{p(F)}\left(\overline{\eta}_1(F')\zeta_1 + \overline{\eta}_2(F')\zeta_2\right)\xi^{-1},\\
\widetilde{\chi}_0^1 = \overline{\eta}_1(F')\xi^{-1}\zeta_1 - 2\theta_1\overline{\eta}_1(F'), & \overline{\chi}_0^2(X_F) = F''\xi^{-2}\zeta_1\zeta_2 + -(1)^{p(F)}\left(\overline{\eta}_2(F')\zeta_1 - \overline{\eta}_1(F')\zeta_2\right)\xi^{-1},\\
\chi_1^1 = \overline{\eta}_1(F'')\xi^{-2}\zeta_1 - 2\theta_1\overline{\eta}_1(F'')\xi^{-1}, & \underline{\chi}_0^2(X_F) = \overline{\eta}_1\overline{\eta}_2(F')\xi^{-2}\zeta_1\zeta_2,
\end{array}$$

$$\chi_1^2(X_F) = \frac{2}{3}F^{(3)}\xi^{-3}\zeta_1\zeta_2 + -(1)^{p(F)}\left(\overline{\eta}_2(F'')\zeta_1 - \overline{\eta}_1(F'')\zeta_2\right)\xi^{-2} + 2\overline{\eta}_1\overline{\eta}_2(F')\xi^{-1}. \tag{8}$$

Proof. The case where $N=0$**.** In this case, we can see that the map $\phi : \mathscr{F}_n \longrightarrow \mathscr{P}_n$ defined by $\phi(Fdx^n) = F\xi^{-n}$ provide us with an isomorphism of $\mathrm{Vect}(S^1)$-modules. So, we can deduce the structure of $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(S^1), \mathfrak{n}(1|0); \mathscr{P}_n)$ from $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(S^1), \mathfrak{n}(1|0); \mathscr{F}_n)$ given in Theorem 3.

The case where $N=1$**.** In this case, as a $\mathscr{K}(1)$-module, we have

$$\mathscr{S}\mathscr{P}_n(1) = \mathscr{S}\mathscr{P}_n^1 \oplus \mathscr{S}\mathscr{P}_n^2,$$

where

$$\mathscr{S}\mathscr{P}_n^1 = \{(1+(-1)^{p(F)})F\xi^{-n} + \overline{\eta}_1(F)\xi^{-n-1}\overline{\zeta}_1,\ F \in C^\infty_{\mathbb{C}}(S^{1|1})\},$$

$$\mathscr{S}\mathscr{P}_n^2 = \{F\xi^{-n-1}\overline{\zeta}_1 - 2\theta_1 F\xi^{-n},\ F \in C^\infty_{\mathbb{C}}(S^{1|1})\}.$$

The natural maps

$$\begin{array}{lll}
\varphi_1 : & \mathfrak{F}_n^1 & \longrightarrow \mathscr{S}\mathscr{P}_n^1\\
& F\alpha_1^n & \longmapsto (1+(-1)^{p(F)})F\xi^{-n} + \overline{\eta}_1(F)\xi^{-n-1}\overline{\zeta}_1,\\
\varphi_2 : & \Pi\left(\mathfrak{F}^1_{n+\frac{1}{2}}\right) & \longrightarrow \mathscr{S}\mathscr{P}_n^1\\
& \Pi\left(F\alpha_1^{n+\frac{1}{2}}\right) & \longmapsto F\xi^{-n-1}\overline{\zeta}_1 - 2\theta_1 F\xi^{-n},
\end{array}$$

provide us with isomorphisms of $\mathscr{K}(1)$-modules. Hence, as $\mathscr{K}(1)$-modules, we have $\mathscr{S}\mathscr{P}_n(1) \simeq \mathfrak{F}_n^1 \oplus \Pi(\mathfrak{F}^1_{n+\frac{1}{2}})$. This isomorphism induces the following isomorphism between cohomology spaces:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1), \mathfrak{n}(1|1); \mathscr{S}\mathscr{P}_n(1)) \simeq \mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1), \mathfrak{n}(1|1); \mathfrak{F}_n^1) \oplus \mathrm{H}^1_{\mathrm{diff}}\left(\mathscr{K}(1), \mathfrak{n}(1|1); \Pi\left(\mathfrak{F}^1_{n+\frac{1}{2}}\right)\right).$$

We deduce from this isomorphism and Theorem 3, the 1-cocycles (8).

The case where $N=2$**.** To prove Theorem 5 in this case, we need first the following:

Proposition 6. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i, \mathfrak{F}^2_\lambda)$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i, \mathfrak{F}^2_\lambda) = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0\\ \mathbb{R} & \text{if } \lambda = -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}\\ 0 & \text{otherwise.}\end{cases}$$

The following 1-cocycles γ^i_λ span the corresponding cohomology spaces:

$$\begin{array}{lll}
\gamma_0^i = F', & \gamma^i_{\frac{3}{2}} = \overline{\eta}_1(F''), & \gamma^i_{\frac{1}{2}} = \overline{\eta}_1(F'),\\
\widetilde{\gamma}_0^i = (-1)^{p(F)}\overline{\eta}_{3-i}(F')\theta_i, & \gamma_1^i = (-1)^{p(F)}\overline{\eta}_{3-i}(F'')\theta_i, & \gamma^i_{-\frac{1}{2}} = F'\theta_i.
\end{array} \tag{9}$$

Proof of Proposition 6. Let $F\alpha_2^{\lambda} = (f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2)\alpha_2^{\lambda} \in \mathfrak{F}_{\lambda}^2$. The map

$$\begin{aligned}\Phi :\ \mathfrak{F}_{\lambda}^2 &\longrightarrow \mathfrak{F}_{\lambda}^{1,i} \oplus \Pi\left(\mathfrak{F}_{\lambda+\frac{1}{2}}^{1,i}\right)\\ F\alpha_2^{\lambda} &\longmapsto \left((1-\theta_i\partial_{\theta_i})(F)\alpha_1^{\lambda}, \Pi\left((-1)^{p(F)+1}\partial_{\theta_i}(F)\alpha_1^{\lambda+\frac{1}{2}}\right)\right),\end{aligned}$$

provides us with an isomorphism of $\mathcal{K}(1)^i$-modules. This map induces the following isomorphism between cohomology spaces:

$$\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \mathfrak{F}_{\lambda}^2\right) \simeq \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \mathfrak{F}_{\lambda}^{1,i}\right) \oplus \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \Pi\left(\mathfrak{F}_{\lambda+\frac{1}{2}}^{1,i}\right)\right). \tag{10}$$

Of course, we can deduce the structure of

$$\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \Pi\left(\mathfrak{F}_{\lambda}^{1,i}\right)\right) \quad \text{from} \quad \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \mathfrak{F}_{\lambda}^{1,i}\right).$$

Indeed, to any $\Upsilon \in \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \mathfrak{F}_{\lambda}^{1,i}\right)$ corresponds $\widetilde{\Upsilon} \in \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(1)^i, \mathfrak{n}(1|1)^i; \Pi\left(\mathfrak{F}_{\lambda}^{1,i}\right)\right)$ where $\widetilde{\Upsilon}(X_F) = \Pi(\sigma \circ \Upsilon(X_F))$ with $\sigma(F) = (-1)^{p(F)}F$. Obviously,Υ is a coboundary if and anly if $\widetilde{\Upsilon}$ is a coboundary. We deduce from isomorphism (10) and formula (5), the 1-cocycles (9). □

Lemma 7. *For $n \in \mathbb{Z}$, any element of $Z^1(\mathcal{K}(2), \mathfrak{n}(1|2); \mathcal{SP}_n(2))$ is a $\mathfrak{n}(1|2)$-relative coboundary over $\mathcal{K}(2)$ if and only if its restriction to the subalgebra $\mathcal{K}(1)^i$ is $\mathfrak{n}(1|1)^i$-relative coboundary for i = 1 and 2.*

Proof of Lemma 7. It is easy to see that if C is a $\mathfrak{n}(1|2)$-relative coboundary over $\mathcal{K}(2)$, then $\mathcal{C}_{|\mathcal{K}(1)^i}$ is a $\mathfrak{n}(1|1)^i$-relative coboundary of $\mathcal{K}(1)^i$. Now, assume that $\mathcal{C}_{|\mathcal{K}(1)^i}$ is a $\mathfrak{n}(1|1)^i$-relative coboundary of $\mathcal{K}(1)^i$ for i = 1 and 2. Using the condition of a 1-cocycle, we prove that there exists an element $\mathfrak{n}(1|1)^i$-invariant $G \in \mathcal{SP}_n(2)$ such that

$$\begin{aligned}\mathcal{C}\left(X_{f_0+f_i\theta_i}\right) &= \left\{\rho_0\left(X_{f_0+f_i\theta_i}\right), G\right\} \quad \text{for any } f_0, f_i \in C^{\infty}_{\mathbb{C}}\left(S^1\right), \quad i = 1,2\\ \mathcal{C}\left(X_{f_{12}\theta_1\theta_2}\right) &= \left\{\rho_0\left(X_{f_{12}\theta_1\theta_2}\right), G\right\} \quad \text{for any } f_{12} \in C^{\infty}_{\mathbb{C}}\left(S^1\right).\end{aligned}$$

We deduce that $\mathcal{C}(X_F) = \left\{\rho_0(X_F), G\right\}$, for any $F \in C^{\infty}_{\mathbb{C}}\left(S^{1|2}\right)$, and therefore $\mathcal{C}$ is a $\mathfrak{n}(1|2)$ -relative coboundary of $\mathcal{K}(2)$. □

We also need the following:

Proposition 8 ([1])**.**

(1) *As a $\mathcal{K}(1)^i$-module, $i = 1,2$, we have*

$$\mathcal{SP}_n(2) \simeq \mathfrak{F}_n^2 \oplus \Pi\left(\mathfrak{F}_{n+\frac{1}{2}}^2 \oplus \mathfrak{F}_{n+\frac{1}{2}}^2\right) \oplus \mathfrak{F}_{n+1}^2, \text{ for } n = 0, -1. \tag{11}$$

(2) *For $n \neq 0, -1$:*

(a) *The following subspace of $\mathcal{SP}_n(2)$:*

$$\mathcal{SP}_{n,i} = \left\{ B_F^{(n,i)} = F\theta_i\bar{\theta}_i\xi^{-n-1} + \theta_i\left(\overline{\eta}_i - \frac{1}{2}\overline{\eta}_{3-i}\right)(F)\zeta_{3-i}\zeta_i\xi^{-n-2} \,\middle|\, F \in C^{\infty}_{\mathbb{C}}(S^{1|2}) \right\} \tag{12}$$

is a $\mathcal{K}(1)^i$- module, $i = 1,2$, isomorphic to $\mathfrak{F}_{n+1}^2$.

(b) *As a $\mathcal{K}(1)^i$-module we have*

$$\mathcal{SP}_n(2)/\mathcal{SP}_{n,i} \simeq \mathfrak{F}_n^2 \oplus \Pi\left(\mathfrak{F}_{n+\frac{1}{2}}^2 \oplus \mathfrak{F}_{n+\frac{1}{2}}^2\right), \quad i = 1,2. \tag{13}$$

Moreover, in [1] it was proved that the natural maps

$$\begin{array}{ll} \psi^i_{n,0} : \begin{array}{rcl}\mathfrak{F}^2_n & \longrightarrow & \mathscr{S}\mathscr{P}_{(n,0,i)} \\ F\alpha_2^n & \longmapsto & A_F^{(n,0,i)}\end{array}, & \psi^i_{n,1} : \begin{array}{rcl}\mathfrak{F}^2_{n+1} & \longrightarrow & \mathscr{S}\mathscr{P}_{(n,1,i)} \\ F\alpha_2^n & \longmapsto & A_F^{(n,1,i)}\end{array} \\ \psi^i_{n,\frac12} : \begin{array}{rcl}\Pi\left(\mathfrak{F}^2_{n+\frac12}\right) & \longrightarrow & \mathscr{S}\mathscr{P}_{(n,\frac12,i)} \\ \Pi\left(F\alpha_2^{n+\frac12}\right) & \longmapsto & A_F^{(n,\frac12,i)}\end{array} \quad \text{and} & \widetilde{\psi}^i_{n,\frac12} : \begin{array}{rcl}\Pi\left(\mathfrak{F}^2_{n+\frac12}\right) & \longrightarrow & \widetilde{\mathscr{S}\mathscr{P}}_{(n,\frac12,i)} \\ \Pi\left(F\alpha_2^{n+\frac12}\right) & \longmapsto & \widetilde{A}_F^{(n,\frac12,i)}\end{array}\end{array} \tag{14}$$

provide us with isomorphisms of $\mathscr{K}(1)$-modules.

Now, according to Lemma 7, the restriction of any nontrivial $\mathfrak{n}(1|2)$-relative 1-cocycle of $\mathscr{K}(2)$ with coefficients in $\mathscr{S}\mathscr{P}_n(2)$ to $\mathscr{K}(1)^i$ is a nontrivial $\mathfrak{n}(1|1)^i$-relative 1-cocycle. Using Proposition 6 and Propositions 8, we obtain:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i; \mathscr{S}\mathscr{P}_n(2)) \simeq \begin{cases} \mathbb{R}^4 & \text{if } n=-1 \\ \mathbb{R}^5 & \text{if } n=0 \\ \mathbb{R}^3 & \text{if } n=1 \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

In the case $n=-1$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i; \mathscr{S}\mathscr{P}_{-1}(2))$ is spanned by the following 1-cocyles:

$$C^{1,i}_{-1}(X_F) = \psi^i_{-1,1}\circ\gamma^i_0(X_F), \qquad C^{3,i}_{-1}(X_F) = \psi^i_{-1,\frac12}\circ\Pi\left(\gamma^i_{-\frac12}(X_F)\right),$$

$$C^{2,i}_{-1}(X_F) = \psi^i_{-1,1}\circ\widetilde{\gamma}^i_0(X_F), \qquad C^{4,i}_{-1}(X_F) = \widetilde{\psi}^i_{-1,\frac12}\circ\Pi\left(\gamma^i_{-\frac12}(X_F)\right).$$

In the case $n=0$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i; \mathscr{S}\mathscr{P}_0(2))$ is spanned by the following 1-cocyles:

$$C^{1,i}_0(X_F) = \psi^i_{0,0}\circ\gamma^i_0(X_F), \qquad C^{4,i}_0(X_F) = \widetilde{\psi}^i_{0,\frac12}\circ\Pi\left(\gamma^i_{\frac12}(X_F)\right),$$

$$C^{2,i}_0(X_F) = \psi^i_{0,0}\circ\widetilde{\gamma}^i_0(X_F), \qquad C^{3,i}_0(X_F) = \psi^i_{0,\frac12}\circ\Pi\left(\gamma^i_{\frac12}(X_F)\right),$$

$$C^{5,i}_0(X_F) = \psi^i_{0,1}\circ\gamma^i_1(X_F).$$

In the case $n=1$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i; \mathscr{S}\mathscr{P}_1(2))$ is spanned by the following 1-cocyles:

$$C^{1,i}_1(X_F) = \psi^i_{1,0}\circ\gamma^i_1(X_F),$$

$$C^{2,i}_1(X_F) = \psi^i_{1,\frac12}\circ\Pi\left(\gamma^i_{\frac32}(X_F)\right),$$

$$C^{3,i}_1(X_F) = \widetilde{\psi}^i_{1,\frac12}\circ\Pi\left(\gamma^i_{\frac32}(X_F)\right),$$

where the cocycles $\gamma^i_0, \widetilde{\gamma}^i_0, \gamma^i_{\frac12}, \gamma^i_{-\frac12}, \gamma^i_{\frac32}$ and γ^i_1 are defined by the formulae (9) and $\psi^i_{n,j}, \widetilde{\psi}^i_{n,j}$ are as in (14).

Now, note that any nontrivial $\mathfrak{n}(1|2)$-relative 1-cocycle of $\mathscr{K}(2)$ with coefficients in $\mathscr{S}\mathscr{P}_n(2)$ should retain the following general form $\Upsilon = \Upsilon^1+\Upsilon^2+\Upsilon^3+\Upsilon^4$, where

$$\begin{cases} \Upsilon^1 & : \mathfrak{vect}(1) \longrightarrow \mathscr{S}\mathscr{P}_n(2), \\ \Upsilon^2, \Upsilon^3 & : \mathscr{F}_{-\frac12} \longrightarrow \mathscr{S}\mathscr{P}_n(2), \\ \Upsilon^4 & : \mathscr{F}_0 \longrightarrow \mathscr{S}\mathscr{P}_n(2),\end{cases}$$

are linear maps. The space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(1)^i, \mathfrak{n}(1|1)^i, \mathscr{S}\mathscr{P}_n(2))$, $i=1,2$, determines the linear maps Υ^1, Υ^2 and Υ^3. The 1-cocycle conditions determines Υ^4. More precisely, we get:

For $n=-1$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(2),\mathfrak{n}(1|2),\mathscr{SP}_{-1}(2))$ is generated by the nontrivial $\mathfrak{n}(1|2)$-relative cocycles χ^2_{-1} and $\widetilde{\chi}^2_{-1}$ corresponding to the $\mathfrak{n}(1|1)^i$-relative cocycles $C^{2,i}_{-1}$ and $C^{3,i}_{-1}$ respectively, via their restrictions to $\mathscr{K}(1)^i$.

For $n=0$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(2),\mathfrak{n}(1|2),\mathscr{SP}_0(2))$ is generated by the nontrivial $\mathfrak{n}(1|2)$-relative cocycles $\chi^2_0,\widehat{\chi}^2_0,\widetilde{\chi}^2_0,\overline{\chi}^2_0$ and $\underline{\chi}^2_0$ corresponding to the $\mathfrak{n}(1|1)^i$-relative cocycles $C^{1,i}_0, C^{2,i}_0, C^{3,i}_0, C^{4,i}_0$ and $C^{5,i}_0$ respectively, via their restrictions to $\mathscr{K}(1)^i$.

For $n=1$, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(2),\mathfrak{n}(1|2),\mathscr{SP}_1(2))$ is generated by the nontrivial $\mathfrak{n}(1|2)$-relative cocycles χ^2_1 corresponding to the $\mathfrak{n}(1|1)^i$-relative cocycles $C^{1,i}_1$, via their restrictions to $\mathscr{K}(1)^i$. Theorem 5 is proved. □

6.2. *The spectral sequence for a filtered module over a Lie (super)algebra*

The reader should refer to [12], for the details of the homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module M with decreasing filtration $\{M_n\}_{n\in\mathbb{Z}}$ over a Lie (super)algebra $\mathfrak{g}$ so that $M_{n+1}\subset M_n, \bigcup_{n\in\mathbb{Z}} M_n = M$ and $\mathfrak{g}M_n\subset M_n$.

Consider the natural filtration induced on the space of cochains by setting:

$$F^n(C^*(\mathfrak{g},M)) = C^*(\mathfrak{g},M_n),$$

then we have:

$$dF^n(C^*(\mathfrak{g},M))\subset F^n(C^*(\mathfrak{g},M)) \text{ (i.e., the filtration is preserved by } d);$$
$$F^{n+1}(C^*(\mathfrak{g},M))\subset F^n(C^*(\mathfrak{g},M)) \text{ (i.e. the filtration is decreasing).}$$

Then there is a spectral sequence $(E^{*,*}_r, d_r)$ for $r\in\mathbb{N}$ with d_r of degree $(r,1-r)$ and

$$E^{p,q}_0 = F^p(C^{p+q}(\mathfrak{g},M))/F^{p+1}(C^{p+q}(\mathfrak{g},M)) \;\text{ and }\; E^{p,q}_1 = H^{p+q}(\mathfrak{g},\mathrm{Grad}^p(M)).$$

To simplify the notations, we have to replace $F^n(C^*(\mathfrak{g},M))$ by F^nC^*. We define

$$Z^{p,q}_r = F^pC^{p+q}\cap d^{-1}(F^{p+r}C^{p+q+1}),$$
$$B^{p,q}_r = F^pC^{p+q}\cap d(F^{p-r}C^{p+q-1}),$$
$$E^{p,q}_r = Z^{p,q}_r/(Z^{p+1,q-1}_{r-1}+B^{p,q}_{r-1}).$$

The differential d maps $Z^{p,q}_r$ into $Z^{p+r,q-r+1}_r$, and hence includes a homomorphism

$$d_r : E^{p,q}_r \longrightarrow E^{p+r,q-r+1}_r$$

The spectral sequence converges to $H^*(C,d)$, that is

$$E^{p,q}_\infty \simeq F^pH^{p+q}(C,d)/F^{p+1}H^{p+q}(C,d),$$

where $F^pH^*(C,d)$ is the image of the map $H^*(F^pC,d)\to H^*(C,d)$ induced by the inclusion $F^pC\to C$.

6.3. *Computing* $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N),\mathfrak{n}(1|N),\mathscr{S}\Psi\mathscr{DO}(S^{1|N}))$

Since the cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N),\mathfrak{n}(1|N);\mathscr{S}\Psi\mathscr{DO}(S^{1|N}))$ is upper bounded by cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathscr{K}(N),\mathfrak{n}(1|N);\mathscr{SP}(N))$, we can check the behavior of the cocycles with values in $\mathscr{S}\Psi\mathscr{DO}(S^{1|N})$ under the successive differentials of the spectral sequence. More precisely we consider a cocycle with values in $\mathscr{SP}(N)$; but we compute its boundary as it was in $\mathscr{S}\Psi\mathscr{DO}(S^{1|N})$ for $N=0,1,2$, and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one. We iterate this procedure, we establish a recurrence formula between successive terms. A straightforward computations leads to the following result:

Theorem 9. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N);\mathcal{S}\Psi\mathcal{D}\mathcal{O}(S^{1|N}))$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(N),\mathfrak{n}(1|N);\mathcal{S}\Psi\mathcal{D}\mathcal{O}(S^{1|N}))\simeq\begin{cases}\mathbb{R}^3 & \text{if } N=0,1\\ \mathbb{R}^8 & \text{if } N=2\\ 0 & \text{otherwise.}\end{cases}\tag{16}$$

The following 1*-cocycles* Ξ_i^N *span the corresponding cohomology spaces:*

$$\begin{aligned}
&\Xi_1^N(X_F)=F',\quad \text{for } N=0,1,2, \qquad \Xi_4^2(X_F)=\eta_1\eta_2(F),\\
&\Xi_2^2(X_F)=F'\xi^{-1}\zeta_1\zeta_2, \qquad \Xi_2^0(X_F)=\sum_{n=2}^{\infty}(-1)^{n-1}\frac{2(n-3)}{n}F^{(n)}(x)\xi^{-n+1},\\
&\Xi_3^2(X_F)=\eta_1\eta_2(F)\xi^{-1}\zeta_1\zeta_2, \qquad \Xi_3^0(X_F)=\sum_{n=2}^{\infty}(-1)^{n}\frac{3(n-1)}{n+1}F^{(n+1)}(x)\xi^{-n},\\
&\Xi_2^1(X_F)=\sum_{n=1}^{\infty}(-1)^n\left(\frac{n-2}{n}(-1)^{p(F)}(\overline{\eta}_1(F^{(n)})\xi^{-n}\overline{\eta}_1-\frac{n-3}{n+1}F^{n+1}\xi^{-n}\right),\\
&\Xi_3^1(X_F)=\sum_{n=2}^{\infty}(-1)^n\left(\frac{n-1}{n}(-1)^{p(F)}(\overline{\eta}_1(F^{(n)})\xi^{-n}\overline{\eta}_1-\frac{n-1}{n+1}F^{n+1}\xi^{-n}\right),\\
&\Xi_5^2(X_F)=\sum_{n=0}^{\infty}\frac{(-1)^{p(F)+n}}{n+1}\Big(\eta_1(F^{(n+1)})\zeta_1+\eta_2(F^{(n+1)})\zeta_2\Big)\xi^{-n-1}\\
&\qquad+\sum_{n=0}^{\infty}\frac{2(-1)^n}{n+2}F^{(n+2)}\xi^{-n-1},\\
&\Xi_6^2(X_F)=\sum_{n=0}^{\infty}(-1)^{p(F)+n}\Big(\eta_2(F^{(n+1)})\zeta_1-\eta_1(F^{(n+1)})\zeta_2\Big)\xi^{-n-1}\\
&\qquad+\sum_{n=0}^{\infty}(-1)^nF^{(n+2)}\xi^{-n-2}\zeta_1\zeta_2+\sum_{n=1}^{\infty}(-1)^n\eta_1\eta_2(F^{(n)})\xi^{-n},\\
&\Xi_7^2(X_F)=\sum_{n=0}^{\infty}(-1)^n\eta_1\eta_2(F^{(n+1)})\xi^{-n-2}\zeta_1\zeta_2\\
&\qquad+\sum_{n=1}^{\infty}(-1)^{p(F)+n}\frac{n}{n+1}\Big(\eta_1(F^{(n+1)})\zeta_1+\eta_2(F^{(n+1)})\zeta_2\Big)\xi^{-n-1}\\
&\qquad+\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}F^{(n+2)}\xi^{-n-1},\\
&\Xi_8^2(X_F)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{n+2}F^{(n+2)}\xi^{-n-2}\zeta_1\zeta_2\\
&\qquad+\sum_{n=1}^{\infty}(-1)^{p(F)+n}\frac{2n}{n+1}\eta_1(F^{(n+1)})\xi^{-n-1}\zeta_2\\
&\qquad+\sum_{n=1}^{\infty}(-1)^{p(F)+n+1}\frac{2n}{n+1}\eta_2(F^{(n+1)})\xi^{-n-1}\zeta_1\\
&\qquad+\sum_{n=1}^{\infty}2(-1)^{n+1}\eta_1\eta_2(F^{(n)})\xi^{-n}.
\end{aligned}$$

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