Towards a loop space description of non-linear sigma model

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Abstract. Non-linear sigma model with target space M can be described as a single particle quantum mechanics in the corresponding free loop space LM. We first discuss a formal description of this loop space quantum mechanics (LSQM) using the general coordinates in LM. Then we consider a semi-classical limit where the string wavefunction is localized on the submanifold of vanishing loops. The semi-classical expansion is related to the tubular expansion of LSQM around this submanifold. We develop the mathematical framework required to compute the effective dynamics on the submanifold in the Born-Oppenheimer sense at leading order in α' expansion. In particular, we show that the linearized tachyon effective equation is reproduced correctly with divergent terms all proportional to the Ricci scalar of M.

1. Introduction and summary

This talk will be based on [1, 2, 3]. We will consider a closed bosonic string moving in a curved target space M[1]. The relevant classical Lagrangian in unit gauge is given by,

$$L \;\; = \;\; \frac{1}{2}\int_0^{2\pi} \frac{d\sigma}{2\pi}\, G_{\alpha\beta}(Z(\sigma))[\dot{Z}^\alpha(\sigma)Z^\beta(\sigma) - \partial Z^\alpha(\sigma)\partial Z^\beta(\sigma)] \; , \tag{1.1}$$

where $G_{\alpha\beta}$ is the metric on M. A $\dot{}$ and a ∂ denote time and space (σ) derivatives respectively. The configuration space of our system is given by the collection of all small loops which can be entirely contained in a single convex neighborhood [8] in M.

The worldsheet theory can be viewed as the worldline description of a single-particle mechanical system where the particle moves in the free loop space LM corresponding to M,

$$LM = C^\infty(S^1, M) \; . \tag{1.2}$$

In §2 we first discuss the formal structure of the corresponding quantum mechanics [1, 2]. Then in §3 we discuss a semi-classical limit [3] where the string wavefunction is localized on the submanifold of vanishing loops (which is isomorphic to M) in LM. The corresponding semi-classical expansion is related to the tubular expansion [9, 10] of the theory around this submanifold. We also discuss the linearized tachyon effective equation at leading order and the underlying mathematical framework for our computation. Finally, we conclude in §4 with future directions.

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[1] See [4, 5] for the study of non-linear sigma model in Lagrangian framework using background field method. We wish to study a loop space description which is a natural set-up for Hamiltonian framework [6, 7].

2. Loop space quantum mechanics

A string embedding $Z^\alpha(\sigma)$ corresponds to a point in LM. The general coordinates of this point is given by[2],

$$z^a = \int_0^{2\pi} \frac{d\sigma}{2\pi} \, Z^\alpha(\sigma) e^{-ia\sigma} \ . \tag{2.3}$$

The Lagrangian in (1.1) takes the following form in the above general coordinates,

$$L \;=\; \frac{1}{2} g_{ab}(z) \dot{z}^a \dot{z}^b - V \ , \quad V = \frac{1}{2} g_{ab}(z) v^a(z) v^b(z) \ , \tag{2.4}$$

where the metric g and a vector field v in LM are given by,

$$g_{ab}(z) \;=\; \int_0^{2\pi} \frac{d\sigma}{2\pi} \, G_{\alpha\beta}(Z(\sigma)) e^{i(a+b)\sigma} \ , \quad v^a(z) = \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \partial Z^\alpha(\sigma) e^{-ia\sigma} \ . \tag{2.5}$$

Such relations, which relate geometric quantities in LM to those in M taken as local fields on S^1, follow certain systematics which is discussed in detail in [1, 3][3]. The above Lagrangian is the standard non-linear system of a particle in an arbitrary curved space which can be quantized following DeWitt in [12]. In particular, generally covariant position space representation of operators can be constructed in scalar states. For example, the Hamiltonian is given by: $\langle \chi | H | \psi \rangle = \int dw \, \chi^*(z) [-\frac{\hbar^2}{2} \mathcal{D}^2 + V] \psi(z)$, where $dw = dz\sqrt{g}$ is the invariant measure, $\hbar = \alpha'$ and $\mathcal{D}^2$ is the Laplacian in loop space. In our case, such a quantization is formal because of the infinite dimensionality of LM (UV divergences are present in the form of infinite dimensional traces). Nonetheless, an interesting result is obtained by applying the above procedure to the Virasoro generators and then computing their algebra [1]. The final result is,

$$\langle \chi | \left\{ \begin{array}{l} [L_a, L_b] = (a-b)\hbar L_{a+b} \\ [\tilde{L}_a, \tilde{L}_b] = (a-b)\hbar \tilde{L}_{a+b} \\ [L_a, \tilde{L}_b] = \hbar^2 \mathcal{A}_{ab} \end{array} \right\} | \psi \rangle \ , \quad \forall a, b \in \mathbb{Z} \ , \tag{2.6}$$

where $\mathcal{A}_{ab}$ is an *operator anomaly* term which is linear in the Ricci tensor of M. A generalization of DeWitt's analysis in [12] to higher rank tensor states has been obtained in [2]. As consistency would require, the analysis leads to the same result for the above algebra, with scalar states replaced by tensorial ones.

3. A semi-classical limit

It is of interest to understand how to make sense of the infinite dimensional structure of LSQM as discussed in the previous section. As a first step towards this direction we study a semi-classical limit ($\hbar = \alpha' \to 0$) where the wavefunction localizes on $M \hookrightarrow LM$. The general idea,

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[2] We adopt the following convention for an infinite-dimensional coordinate index. It is given by a lower case Latin alphabet, which is associated to a pair containing the corresponding Greek alphabet (a target space index) and an integer, denoted by the same Latin alphabet in text format. For example, $a \to (\alpha, a)$, $b \to (\beta, b)$. We will also adopt a similar association between such a pair and the corresponding upper case Latin alphabet when the integer is non-zero, i.e. $A \to (\alpha, a)$, $B \to (\beta, b)$ etc. only when $a, b \neq 0$.

[3] Some geometric facts related to v are as follows: it is a Killing vector field. This induces an isometry of LM, which corresponds to the reparametrization invariance and therefore, is present irrespective of the property of M. Notice that v vanishes on the submanifold $M \hookrightarrow LM$ of vanishing loops. This situation is similar to the consideration of Kobayashi's theorem in [11] (in finite dimensions), which claims that the space of fixed points of an isometry is a totally geodesic submanifold of even co-dimension. We will find in next section that $M \hookrightarrow LM$ is indeed totally geodesic. Although, this has infinite number of transverse directions, from the discussion below eq.(2.3), it is clear that for every transverse index $A \to (\alpha, a)$, there is a pair $\bar{A} \to (\alpha, -a)$.

which goes along the line of what is called *constrained quantum systems* in the literature [13], is as follows. Given that V vanishes, more importantly, minimizes on the submanifold[4], one finds a suitable $\hbar$-dependent rescaling of the theory, in particular, of the transverse coordinates such that in the semi-classical limit the potential deepens heavily causing the localization. Moreover the transverse coordinates become *fast* with respect to the longitudinal ones in the Born-Oppenheimer sense. The goal is to find the effective theory of the longitudinal fluctuations on the submanifold order by order in α'-expansion. The technical/conceptual challenges are as follows: (1) Finding the precise definition of the rescaled theory. Once this is understood, each term in the semi-classical expansion of, say, the Hamiltonian can be written down in terms of the tubular expansion coefficients of the loop space metric. (2) To understand tubular expansion of the loop space metric. (3) Divergences are still present in general. How to interpret them?

3.1. Semi-classical expansion
In [3] the first question was answered by demanding the correct flat space limit. The rescaled Hamiltonian (up to next to leading order in $\hbar$) tunrs out to be,

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}^{\perp}_{flat} + \hbar\Delta\mathcal{H} + O(\hbar^{3/2}) \,, \\
\Delta\mathcal{H} &= -\frac{1}{2}(\nabla^{\alpha} + i\bar{\omega}^{\alpha AB}\,\Lambda_{AB})(\nabla_{\alpha} + i\bar{\omega}_{\alpha}{}^{CD}\,\Lambda_{CD}) - \frac{1}{4}\bar{r}_{\parallel} - \frac{1}{12}\bar{r}_{\perp} \\
&\quad + \frac{1}{6}\bar{r}^{ABCD}\,\Lambda_{AB}\,\Lambda_{CD} + \frac{1}{6}\sum_{A,B,C,D}\epsilon_{ABCD}\,\bar{r}_{ACDB}\mathrm{y}^{A}\mathrm{y}^{C}\mathrm{y}^{D}\mathrm{y}^{B} \,,
\end{aligned} \tag{3.7}
$$

where $\mathcal{H}^{\perp}_{flat}$ is precisely the oscillator part of the flat space Hamiltonian with the correct normal ordering constant, ∇ is the covariant derivative in M, ω and r_{abcd} are spin connection and Riemann tensor of LM respectively with a bar indicating that the quantities are being computed on the submanifold. ϵ_{ABCD} is a number involving $|a|, \cdots, |d|$ and y^{A} is the rescaled transverse coordinate[5]: $\mathrm{y}^{A} = \sqrt{\frac{|\mathrm{a}|}{\hbar}}y^{A}$, y^{A} being the Fermi normal coordinate (FNC) [9, 10]. Finally,

$$
\bar{r}_{\parallel} = \bar{r}^{\alpha B}{}_{\alpha B} \,, \quad \bar{r}_{\perp} = \bar{r}^{AB}{}_{AB} \,, \quad \Lambda_{AB} = -\frac{i}{2}(\eta_{AC}y^{C}\frac{\partial}{\partial y^{B}} - \eta_{BC}y^{C}\frac{\partial}{\partial y^{A}}) \,, \tag{3.8}
$$

where,

$$
\eta_{AB} = \eta_{\alpha\beta}\delta_{\mathrm{a+b},0} \,. \tag{3.9}
$$

Notice that Λ_{AB} is the angular momentum operator in the transverse space and $\bar{\omega}_{\alpha AB}$ is analogous to a non-abelian Berry connection [14].

The quadratic tachyon effective action is obtained by computing the expectation value of the above Hamiltonian in the state[6]: $\psi(z) = T(x)\chi(y)$, where $T(x)$ is the tachyon field in M and $\chi(y)$ is the wavefunction for the oscillator ground state in flat space. This gives the following linearized effective equation at leading order: $[-\nabla^{2} + m_{T}^{2} + V_{eff}]T = 0$, where m_{T} is the correct tachyon mass and,

$$
V_{eff} = -\frac{1}{2}\bar{r}_{\parallel} - \frac{1}{6}\bar{r}_{\perp} + \sum_{B,D}(\beta_{BD}\bar{r}^{BD}{}_{BD} + \gamma_{BD}\bar{\omega}^{\alpha BD}\bar{\omega}_{\alpha BD}) \,, \tag{3.10}
$$

β_{BD}, γ_{BD} being certain numbers involving $|\mathrm{b}|, |\mathrm{d}|$.

[4] This is not true when M is Lorentzian. However, this is the standard problem of negative norm states and should be solved with the help of ghosts in the usual manner.
[5] Notice that non-zero transverse coordinates correspond to non-zero loops and therefore should have upper case Latin indices according to the rules mentioned in footnote 2.
[6] The coordinate system in LM, which was implicit in eqs(3.7), is given by $z^{a} = (x^{\alpha}, y^{A})$, x being the general coordinates on the submanifold.

3.2. Tubular expansion in loop space and divergence in tachyon effective equation

The final results have so far been written in terms of certain tensors of LM evaluated on the submanifold. How do we write them entirely in terms of intrinsic geometric data of M? The above results are obtained from the FNC expansion of g up to quadratic order which, for a general embedding, is given by [3][7],

$$
\begin{aligned}
g_{\alpha\beta} &= G_{\alpha\beta} + \bar{s}_{\alpha\beta C}y^C + (\bar{\omega}_\alpha{}^\gamma{}_C\bar{\omega}_{\beta\gamma D} + \bar{\omega}_\alpha{}^B{}_C\bar{\omega}_{\beta BD} + \bar{r}_{\alpha CD\beta})y^C y^D + O(y^3) \ , \\
g_{\alpha B} &= \bar{\omega}_{\alpha BC}y^C + \frac{2}{3}\bar{r}_{\alpha CDB}y^C y^D + O(y^3) \ , \\
g_{AB} &= \eta_{AB} + \frac{1}{3}\bar{r}_{ACDB}y^C y^D + O(y^3) \ ,
\end{aligned}
\tag{3.11}
$$

where $\bar{s}$ is the second fundamental form [15]. The expansion coefficients (barred objects), which are tensors of the ambient space evaluated on the submanifold, carry information about the extrinsic properties of the embedding. In a generic situation they can be chosen somewhat independently of the intrinsic properties of the submanifold. However, the only independent geometric data that are used to construct LM are those of M. Therefore understanding the tubular expansion in loop space means knowing all the expansion coefficients[8] in terms of the intrinsic geometric data of M. This has been determined (up to an arbitrary real constant q) in [3] for the metric expansion in (3.11) up to quadratic level. The results are given by eq.(3.9) and,

$$
\begin{aligned}
\bar{\omega}_{\alpha\beta D} &= 0(\Rightarrow \bar{s}_{\alpha\beta D} = 0) \ , \quad \bar{\omega}_{\alpha BD} = E_{(\beta)\gamma}(x)\partial_\alpha E_{(\delta)}{}^\gamma(x)\delta_{\text{b+d},0} \ , \\
\bar{r}_{\alpha DE\beta} &= 2qR_{\alpha(\delta\eta)\beta}(x)\delta_{\text{d+e},0} \ , \quad \bar{r}_{\alpha BDE} = 0 \ , \quad \bar{r}_{ABDE} = R_{(\alpha\beta\delta\eta)}(x)\delta_{\text{a+b+d+e},0} \ ,
\end{aligned}
\tag{3.12}
$$

where $R_{\alpha\beta\gamma\delta}$ and $E^{(\alpha)}{}_\beta$ are the Riemann tensor and the vielbein of M respectively. The first equation implies [15] that $M \hookrightarrow LM$ is totally geodesic as expected (see footnote 3). Using eqs(3.9, 3.12) in (3.10) one finds,

$$
V_{eff} \propto R(x) \ ,
\tag{3.13}
$$

where R is the Ricci scalar of M and the proportionality constant is divergent. This shows that *the tachyon effective equation at leading order is correctly reproduced up to divergent terms all proportional to the equation of motion for the background metric.*

3.3. The mathematical argument

The general procedure of arriving at eqs.(3.12) uses a basic mathematical structure that is relevant for multi-particle dynamics in curved space. The method of finding the centre of mass (CM) would be as follows. Given a multi-particle configuration, erect a tangent space T_xM such that x lies in the neighborhood of that configuration. Then find the preimage of the configuration in T_xM under the exponential map $\exp_x$. If the sum of the position vectors in T_xM vanishes, then x is the CM. Therefore all possible configurations can be described on the tangent bundle TM in the following way. The CM always lies on the zero section $TM_0(\cong M)$ whereas any given configuration lies entirely on the corresponding fibre with one constraint that the average position on the fibre is at the origin. The exponential map being a diffeomorphism, this configuration in TM is actually diffeomorphic to the original configuration whose correct description is given in $M \times M$ where the CM lies on the diagonal submanifold $\Delta(\cong M)$. If $\Phi : TM \to M \times M$ is the relevant bundle map, then we may write,

$$
\Phi : (x, \hat{y}) \to (\exp_x(\hat{y}), \exp_x(-\hat{y})) \ .
\tag{3.14}
$$

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[7] Lower case symbols are used to denote tensors of the ambient space at a generic point $z = (x, y)$.

[8] Closed form expressions for the all-order results for vielbein have been computed in [10] for a generic embedding.

This induces a Riemannian structure from $M \times M$, which has a simple direct product structure, to TM such that $TM_0 \hookrightarrow TM$ admits a tubular neighborhood which is diffeomorphic to that of $\Delta \hookrightarrow M \times M$. The relevant tubular expansion coefficients are all related to intrinsic geometric data of M which can be calculated by exploiting the coordinate transformation in (3.14).

In the context of a string this overall picture remains the same with the multi-particle configuration on a fibre being replaced by a loop $\hat{Y}^\alpha(\sigma)$ with the constraint: $\int_0^{2\pi} \hat{Y}^\alpha(\sigma) = 0$. The submanifold $M \hookrightarrow LM$ is the same as TM_0 and the transverse coordinates (FNC) y^A are given by the Fourier modes of $\hat{Y}^\alpha(\sigma)$ following the definition similar to (2.3). Therefore the Riemannian structure in LM is directly related to that of TM as discussed above.

4. Future directions

Many avenues need to be explored with the new approach described here and its supersymmetric counterpart. The study of the semi-classical limit discussed here needs to be extended to the massless modes. Among other things it is important to understand the pattern of leading order divergences. The implication of the Born-Oppenheimer approach to the study of low energy effective theory in general should be explored. The leading order analysis of the DeWitt-Virasoro algebra in (2.6) and its possible relation to the tensor representation worked out in [2] should also be investigated.

What is the best way to handle the divergences at sub-leading order is an important question. This may require one to understand the tubular expansion in loop space in the sense described below eq.(3.11) more completely.

It may be interesting to explore if the string path integral can be understood as a tubular expansion around particle worldline in M. Perhaps the question of interest is if the interpretation of this type of tubular geometry can in any way be helpful to understand α'-corrections in string theory.

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