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Casting more light in the shadows: dual Somos-5 sequences

J W E Harrow and A N W Hone*

School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury CT2 7NF, United Kingdom

E-mail: A.N.W.Hone@kent.ac.uk

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Abstract

Motivated by the search for an appropriate notion of a cluster superalgebra, incorporating Grassmann variables, Ovsienko and Tabachnikov considered the extension of various recurrence relations with the Laurent phenomenon to the ring of dual numbers. Furthermore, by iterating recurrences with specific numerical values, some particular well-known integer sequences, such as the Fibonacci sequence, Markoff numbers, and Somos sequences, were shown to produce associated 'shadow' sequences when they were extended to the dual numbers. Here we consider the most general version of the Somos-5 recurrence defined over the ring of dual numbers $\mathbb{D}$ with complex coefficients, that is the ring $\mathbb{C}[\varepsilon]$ modulo the relation $\varepsilon^2 = 0$. We present three different ways to present the general solution of the initial value problem for Somos-5 and its shadow part: in analytic form, using the Weierstrass sigma function with arguments in $\mathbb{D}$; in terms of the solution of a linear difference equation; and using Hankel determinants constructed from $\mathbb{D}$-valued moments, via a connection with a Quispel–Roberts–Thompson map over the dual numbers.

Keywords: somos sequence, dual number, shadow sequence, discrete integrability, elliptic function

* Author to whom any correspondence should be addressed.

1. Introduction

Supersymmetry is a proposed physical framework in which bosons and fermions can be treated on the same footing. In terms of algebra, this means that one should work with a $\mathbb{Z}_2$-graded ring that is a direct sum of even and odd components, and one can then define geometric structures (supergeometry) by working over such a ring. One of the simplest examples, corresponding to the minimal case of $N = 1$ supersymmetry in physics [3], is to extend the real numbers $\mathbb{R}$ by two Grassmann variables ξ_1, ξ_2, satisfying

$$\xi_j\xi_k + \xi_k\xi_j = 0, \qquad j,k = 1,2, \tag{1.1}$$

which produces the ring

$$(\mathbb{R} \oplus \mathbb{R}\xi_1\xi_2) \oplus (\mathbb{R}\xi_1 \oplus \mathbb{R}\xi_2) = R_0 \oplus R_1, \tag{1.2}$$

whose even part $R_0 = \mathbb{R} \oplus \mathbb{R}\xi_1\xi_2$ contains nilpotent elements, namely the multiples of $\varepsilon = \xi_1\xi_2$. The ring (or $\mathbb{R}$-algebra) corresponding to the even part $D = R_0$, that is

$$D = \mathbb{R} \oplus \mathbb{R}\varepsilon \qquad \text{with} \quad \varepsilon^2 = 0,$$

is known as the set of dual numbers, and was first introduced by Clifford. Replacing $\mathbb{R}$ by $\mathbb{C}$ or another field (or a ring) gives analogues of the dual numbers, which are useful in computer algebra (for automatic differentiation) and in algebraic geometry (for defining the tangent space of an algebraic variety). For the purposes of this paper, it will be convenient to take the complex numbers as the ambient field, and work with the commutative $\mathbb{C}$-algebra of dual numbers $\mathbb{D} = D \otimes \mathbb{C}$ given by

$$\mathbb{D} = \left\{ x + y\varepsilon \,|\, x,y \in \mathbb{C}, \varepsilon^2 = 0 \right\},$$

which is isomorphic to the quotient $\mathbb{C}[\varepsilon] / \langle \varepsilon^2 \rangle$.

Although the algebraic and geometric aspects of supergeometry have been developed for some time, it seems that certain arithmetic aspects of superalgebras have only begun to be explored very recently. There are two especially noteworthy examples: the 'shadows' of integer sequences [11, 25, 27, 28]; and the notion of supersymmetric continued fractions associated with the supermodular group $\mathrm{OSp}(1|2,\mathbb{Z})$ [5], namely the supergroup $\mathrm{OSp}(1|2)$ with coefficients in the ring $\mathbb{Z}[\xi_1, \xi_2]$ with two Grassmann variables satisfying (1.1). Both of the latter examples have come about as a byproduct of the search, starting with [24], for an appropriate notion of a cluster superalgebra, including mutations of both even and odd cluster variables, together with supersymmetric analogues of related objects like frieze patterns and snake graphs [19–21, 26].

The general philosophy of shadow sequences is explained in the paper [27], where Ovsienko considers integer sequences that are obtained by replacing variables x_n in nonlinear recurrence relations by dual variables $X_n = x_n + y_n\varepsilon \in \mathbb{D}$, as well as taking $\mathbb{D}$-valued coefficients in any such recurrence. Working in $\mathbb{D}$ (where we can regard ε on its own as a single Grassmann variable), the resulting recurrence relation for X_n can be split into its odd and even parts in terms of powers of ε, with the original recurrence relation is recovered in the even (ε^0) part. The remaining odd (ε^1) part defines the new associated sequence of values y_n, satisfying an inhomogeneous linear relation in terms of the x_n and coefficients of the original equation and their new dual parts. If we restrict the coefficients to be $\mathbb{C}$-valued, then the sequence (y_n) is referred to as a 'shadow' of the original sequence (x_n): in that case, y_n is a solution of a homogeneous linear reurrence, which is just the linearization of the original recurrence around x_n. Hence a single original sequence (x_n) has multiple shadows, belonging to a vector space with the same dimension as the original nonlinear recurrence relations that it satisfies; but in the case of integer sequences $x_n \in \mathbb{Z}$ generated by nonlinear recurrence, the problem of characterizing its integer shadows $y_n \in \mathbb{Z}$ appears to be an interesting arithmetical question. As an example, Ovsienko demonstrates that the tetrahedral numbers may be obtained as a shadow of the natural numbers when they are viewed as solutions of the nonlinear relation

$$x_{n+1}x_{n-1} = x_n^2 - 1. \tag{1.3}$$

The latter is one of the simplest examples of a recurrence which exhibits the Laurent phenomenon [8], meaning that if the two initial values x_0, x_1 are viewed as variables then all of the iterates are Laurent polynomials in these variables with integer coefficients: $x_n \in \mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}]$ for all n.

Another example of the Laurent phenomenon is provided by the Somos-5 recurrence,

$$x_n x_{n+5} = \alpha x_{n+1}x_{n+4} + \beta x_{n+2}x_{n+3} \tag{1.4}$$

with coefficients α, β. In this case, the Laurent property for (1.4) means all iterates of the recurrence are Laurent polynomials in the initial 5 entries x_0, x_1, x_2, x_3, x_4 with coefficients belonging to the polynomial ring $\mathbb{Z}[\alpha, \beta]$, i.e.

$$x_n \in \mathbb{Z}\left[\alpha, \beta, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}\right], \quad \forall n \in \mathbb{Z}. \tag{1.5}$$

It follows that setting all five initial conditions $x_j = 1$ for $j = 0, 1, 2, 3, 4$ and taking $\alpha, \beta \in \mathbb{Z}$ ensures that the whole sequence (x_n) consists of integers for all n. The original example considered is when $\alpha = \beta = 1$, commonly referred to as *the* Somos-5 sequence, in which case first few terms are given by

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, \ldots, \tag{1.6}$$

this being the OEIS sequence A006721 [22]. However, there are considerably more choices of initial conditions and coeffcients that give rise to integer sequences, due to the way that the recurrence is connected with elliptic curves (see below).

Shadow sequences obtained from an analogous recurrence of fourth order, namely Somos-4 given by

$$x_n x_{n+4} = \alpha x_{n+1}x_{n+3} + \beta x_{n+2}^2, \tag{1.7}$$

were presented for general initial conditions by one of us in [11]. Sequences of Somos type, or Gale-Robinson sequences, have many structural similarities with Somos-4 and Somos-5 sequences: they inherit the Laurent property via reduction from partial difference equations, specifically the octahedron/cube recurrences (a.k.a. the discrete Hirota/Miwa equations) [8, 18], which means they can also be interpreted as discrete integrable systems [10].

The purpose of this article is to explore how Somos-5 sequences extend to sequences of dual numbers, and to determine explicit expressions for the shadow sequences defined by this change. In the next section we describe basic properties of Somos-5 and its extension to $\mathbb{D}$. Section 3 presents analytic formulae in terms of Weierstrass functions. The fourth section provides a derivation of a complete set of shadow Somos-5 sequences, given by a combination of analytic and algebraic expressions, based on variation of parameters for a linear difference equation, while the fifth section presents Hankel determinant formulae for dual Somos-5 sequences, making use of recent results from [13]. Section 6 is concerned with the Quispel–Roberts–Thompson (QRT) map associated with Somos-5, and the interpretation of its dual number version as a discrete integrable system, via the construction of a compatible pencil of Poisson brackets (a bi-Hamiltonian structure). We end with some brief comments and conclusions.

2. Somos-5 recurrence and dual version

We begin by briefly reviewing some properties of the Somos-5 recurrence, which are summarized in [12]. The recurrence (1.4) has a 2-invariant given by

$$K_n = \frac{x_n x_{n+4} + \alpha x_{n+2}^2}{x_{n+1}x_{n+3}}; \tag{2.1}$$

it is straightforward to verify directly from the recurrence that this satisfies the period 2 condition $K_n = K_{n+2}$. This leads immediately to two independent invariant quantities I, J for the Somos-5 recurrence, which are obtained from the product and sum, respectively, of the odd and even values of this 2-invariant, according to

$$I = K_n + K_{n+1}, \tag{2.2}$$

$$J = \frac{K_n K_{n+1} - \beta}{\alpha} \tag{2.3}$$

(where it is assumed that $\alpha \neq 0$). These invariants can both be given explicitly in terms of the coefficients α, β and any 5 adjacent iterates, by solving the following pair of homogeneous relations of degree 5 in x_j:

$$\begin{aligned} x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4} I = {} & x_n^2 x_{n+2} x_{n+4}^2 + \alpha\left(x_{n+1}^3 x_{n+3} x_{n+4}\right. \\ & \left. + x_n x_{n+2}^3 x_{n+4} + x_n x_{n+1} x_{n+3}^3\right) + \beta x_{n+1}^2 x_{n+2} x_{n+3}^2, \end{aligned} \tag{2.4}$$

$$\begin{aligned} x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4} J = {} & x_n^2 x_{n+3}^2 x_{n+4} + x_n x_{n+1}^2 x_{n+4}^2 \\ & + \alpha\left(x_{n+1}^2 x_{n+2}^2 x_{n+4} + x_n x_{n+2}^2 x_{n+3}^2\right) \\ & + \beta x_{n+1} x_{n+2}^3 x_{n+3}. \end{aligned} \tag{2.5}$$

These relations allow I, J to be simply computed from any set of 5 initial conditions, and then they are both constant along each orbit of (1.4). Both relations also extend directly to the case of dual numbers.

To extend the recurrence relation (1.4) to the dual numbers $\mathbb{D}$ in the most general way possible, we write the dual Somos-5 recurrence as

$$X_n X_{n+5} = \alpha X_{n+1} X_{n+4} + \beta X_{n+2} X_{n+3}, \tag{2.6}$$

with not only dual number variables $X_n = x_n + y_n \varepsilon$, but also coefficients $\alpha, \beta \in \mathbb{D}$, which can be expanded as

$$\alpha = \alpha^{(0)} + \alpha^{(1)}\varepsilon, \qquad \beta = \beta^{(0)} + \beta^{(1)}\varepsilon.$$

In their paper [25], Ovsienko and Tabachnikov consider the specific cases

$$X_n X_{n+5} = \left(1 + \alpha^{(1)}\varepsilon\right) X_{n+1} X_{n+4} + X_{n+2} X_{n+3}, \tag{2.7}$$

$$X_nX_{n+5} = X_{n+1}X_{n+4} + \left(1+\beta^{(1)}\varepsilon\right)X_{n+2}X_{n+3}. \tag{2.8}$$

In the context of more general Gale-Robinson sequences, showing these produce integer sequences for $x_0 = x_1 = x_2 = x_3 = x_4 = 1$ and an arbitrary set of 5 integer initial values $y_j \in \mathbb{Z}$ for $0 \leqslant j \leqslant 4$.

We can immediately expand the general case into its odd and even parts, giving the system:

$$x_nx_{n+5} = \alpha^{(0)}x_{n+1}x_{n+4} + \beta^{(0)}x_{n+2}x_{n+3}, \tag{2.9}$$

in the even (ε^0) part and

$$\begin{aligned} &x_ny_{n+5} + y_nx_{n+5} - \alpha^{(0)}\left(x_{n+1}y_{n+4} + y_{n+1}x_{n+4}\right) - \beta^{(0)}\left(x_{n+2}y_{n+3} + y_{n+2}x_{n+3}\right) \\ &\quad = \alpha^{(1)}x_{n+1}x_{n+4} + \beta^{(1)}x_{n+2}x_{n+3}, \end{aligned} \tag{2.10}$$

in the odd (ε^1) part. Clearly the even part recovers the original Somos-5 recurrence (1.4) for x_n, with coefficients $\alpha^{(0)}, \beta^{(0)}$, as expected. The odd part gives the inhomogeneous linear relation for the new sequence (y_n). The homogeneous version of (2.10), corresponding to $\alpha^{(1)} = \beta^{(1)} = 0$, gives the shadow sequence y_n in the sense defined by Ovsienko [27], and is the linearization of the even part. Five linearly independent shadow sequences can then be specified implicitly in terms of 5 independent sets of initial conditions y_0, y_1, y_2, y_3, y_4, but in what follows we will derive 5 specific shadow sequences related to the base sequence x_n, using the explicit solution to the Somos-5 recurrence.

Fomin and Zelevinsky's original proofs of the Laurent property for Somos-5 and other sequences including Somos-4 [8] are based on treating the initial data as formal variables. The same method carries over directly to (2.6) with dual numbers, because the dual version has the same form as (1.4) and $\mathbb{D}$ is a commutative ring. Thus we see that X_n has the Laurent property in its coefficients and initial data, that is

$$X_n = \mathbb{Z}\left[\alpha, \beta, X_0^{\pm 1}, X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}, X_4^{\pm 1}\right]. \tag{2.11}$$

Upon splitting the relation (2.6) into its even and odd parts, we can state a more precise version of the Laurent property for each part, by making use of the standard reciprocal formula for dual numbers:

$$(x + y\varepsilon)^{-1} = x^{-1}\left(1 - x^{-1}y\varepsilon\right). \tag{2.12}$$

Thus we see that the two parts of the system, (2.9) and (2.10), together have the Laurent property in the sense that

$$x_n \in \mathbb{Z}\left[\alpha^{(0)}, \beta^{(0)}, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}\right], \quad \forall n \in \mathbb{Z}. \tag{2.13}$$

$$y_n \in \mathbb{Z}\left[\alpha^{(0)}, \beta^{(0)}, \alpha^{(1)}, \beta^{(1)}, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, y_0, y_1, y_2, y_3, y_4\right], \quad \forall n \in \mathbb{Z}. \tag{2.14}$$

Also note that to iterate the dual Somos-5 for numerical values at any particular step n, we require $x_n \neq 0$ in order to determine x_{n+5}. This means that if we take the 5 dual initial data to be units, i.e. $X_0, X_1, X_2, X_3, X_4 \in \mathbb{D}^*$ where $\mathbb{D}^* = \{x + y\varepsilon \in \mathbb{D} \,|\, x \neq 0\}$, then the Laurent property ensures that the whole orbit $(X_n)_{n\in\mathbb{Z}}$ can be defined by evaluating suitable Laurent polynomials in these 5 initial values.

The conserved quantities likewise carry over to the dual system, as an analogous 2-invariant $K_n \in \mathbb{D}$ can be defined with exactly the same formula (2.1) but replacing each $x_j \to X_j$ and with $\alpha \in \mathbb{D}$. Thus dual invariants (first integrals) $I = I^{(0)} + I^{(1)}\varepsilon$ and $J = J^{(0)} + J^{(1)}\varepsilon$ can be found. To find these explicitly it is best to simply take the original homogeneous relations (2.4) and (2.5), and change all terms to their dual counterparts, $x_0 \to X_0$ etc.

In what follows, the first integral $J \in \mathbb{D}$ will play a central role, so we consider it first. For J, the even part of the dual version of (2.5) yields precisely the same formula as (2.5) does for $J^{(0)}$, as expected, but with the parameters replaced by $\alpha^{(0)}, \beta^{(0)}$:

$$J^{(0)} = \frac{x_n^2 x_{n+3}^2 x_{n+4} + x_n x_{n+1}^2 x_{n+4}^2 + \alpha^{(0)}\left(x_{n+1}^2 x_{n+2}^2 x_{n+4} + x_n x_{n+2}^2 x_{n+3}^2\right) + \beta^{(0)} x_{n+1} x_{n+2}^3 x_{n+3}}{x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}}. \tag{2.15}$$

As it stands, dualizing (2.5) and taking the odd part gives a relation for $J^{(1)}$ which includes $J^{(0)}$ as well. After substituting for $J^{(0)}$, we can get an expression for $J^{(1)}$ in terms of the initial data and coefficients alone:

$$J^{(1)} = \frac{D_n - \sum_{j=0}^{4} C_n^{(j)} x_{n+j}^{-1} y_{n+j}}{x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}}, \tag{2.16}$$

where

$$C_n^{(0)} = \alpha^{(0)} x_{n+1}^2 x_{n+2}^2 x_{n+4} + \beta^{(0)} x_{n+1} x_{n+2}^3 x_{n+3} - x_n^2 x_{n+3}^2 x_{n+4}, \tag{2.17}$$

$$C_n^{(1)} = \alpha^{(0)}\left(x_n x_{n+2}^2 x_{n+3}^2 - x_{n+1}^2 x_{n+2}^2 x_{n+4}\right) + x_n^2 x_{n+3}^2 x_{n+4} - x_n x_{n+1}^2 x_{n+4}^2, \tag{2.18}$$

$$\begin{aligned} C_n^{(2)} = {} & x_n^2 x_{n+3}^2 x_{n+4} + x_n x_{n+1}^2 x_{n+4}^2 - \alpha^{(0)}\left(x_{n+1}^2 x_{n+2}^2 x_{n+4} + x_n x_{n+2}^2 x_{n+3}^2\right) \\ & - 2\beta^{(0)} x_{n+1} x_{n+2}^3 x_{n+3}, \end{aligned} \tag{2.19}$$

$$C_n^{(3)} = \alpha^{(0)}\left(x_{n+1}^2 x_{n+2}^2 x_{n+4} - x_n x_{n+2}^2 x_{n+3}^2\right) + x_n x_{n+1}^2 x_{n+4}^2 - x_n^2 x_{n+3}^2 x_{n+4}, \tag{2.20}$$

$$C_n^{(4)} = \alpha^{(0)} x_n x_{n+2}^2 x_{n+3}^2 + \beta^{(0)} x_{n+1} x_{n+2}^3 x_{n+3} - x_n x_{n+1}^2 x_{n+4}^2, \tag{2.21}$$

$$\begin{aligned} D_n = {} & \alpha^{(1)}\left(x_{n+1}^2 x_{n+2}^2 x_{n+4} + x_n x_{n+2}^2 x_{n+4} + x_n x_{n+2}^2 x_{n+3}^2\right) \\ & + \beta^{(1)} x_{n+1} x_{n+2}^3 x_{n+3}. \end{aligned} \tag{2.22}$$

As we might expect, there are many symmetries in this expression. This would be even more apparent if we shifted n down by two to see how the $n-j$ and $n+j$ terms for $j = 0, 1, 2$ balance each other in many expressions. We will use this expression later when examining one of the linearly independent shadow sequences of Somos-5 and note some of its other properties.

For completeness, we also include the form of the second dual invariant, $I \in \mathbb{D}$. Similarly the even part gives $I^{(0)}$ by the same expression (2.4) for the original I. The odd part of I can then be found from the odd component, to yield

$$I^{(1)} = \frac{\bar{B}_n - \sum_{j=0}^{4} \bar{A}_n^{(j)} x_{n+j}^{-1} y_{n+j}}{x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}}, \tag{2.23}$$

where

$$\bar{A}_n^{(0)} = \alpha^{(0)} x_{n+1}^3 x_{n+3} x_{n+4} + \beta^{(0)} x_{n+1}^2 x_{n+2} x_{n+3}^2 - x_n^2 x_{n+2} x_{n+4}^2, \tag{2.24}$$

$$\bar{A}_n^{(1)} = x_n^2 x_{n+2} x_{n+4}^2 + \alpha^{(0)} \left[x_n x_{n+2}^3 x_{n+4} - 2x_{n+1}^3 x_{n+3} x_{n+4} \right] - \beta^{(0)} x_{n+1}^2 x_{n+2} x_{n+3}^2, \tag{2.25}$$

$$\bar{A}_n^{(2)} = \alpha^{(0)} \left[x_{n+1}^3 x_{n+3} x_{n+4} - 2x_n x_{n+2}^3 x_{n+4} + x_n x_{n+1} x_{n+3}^3 \right], \tag{2.26}$$

$$\bar{A}_n^{(3)} = x_n^2 x_{n+2} x_{n+4}^2 + \alpha^{(0)} \left[x_n x_{n+2}^3 x_{n+4} - 2x_n x_{n+1} x_{n+3}^3 \right] - \beta^{(0)} x_{n+1}^2 x_{n+2} x_{n+3}^2, \tag{2.27}$$

$$\bar{A}_n^{(4)} = \alpha^{(0)} x_n x_{n+1} x_{n+3}^3 + \beta^{(0)} x_{n+1}^2 x_{n+2} x_{n+3}^2 - x_n^2 x_{n+2} x_{n+4}^2, \tag{2.28}$$

$$\bar{B}_n = \alpha^{(1)} \left[x_{n+1}^3 x_{n+3} x_{n+4} + x_n x_{n+2}^3 x_{n+4} + x_n x_{n+1} x_{n+3}^3 \right] + \beta^{(1)} x_{n+1}^2 x_{n+2} x_{n+3}^2. \tag{2.29}$$

The quantity I is considerably less useful in studying the Somos-5 shadow solutions, essentially because the quantity J leads to a connection with elliptic curves, which leads to explicit analytic solutions of the recurrence in terms of Weierstrass functions, as described in the next section.

3. Analytic solution of dual Somos-5

An explicit analytic solution for the Somos-5 recurrence was given in [9], via the relation with elliptic curves and associated functions. We will state one version of this solution here, following the notation used in [12], and subsequently show how to extend it to dual Somos-5 sequences.

Theorem 3.1. *The general solution of the initial value problem for the Somos-5 recurrence (1.4) over $\mathbb{C}$ is*

$$x_n = A_\pm B_\pm^{\lfloor \frac{n}{2} \rfloor} \mu^{\lfloor \frac{n}{2} \rfloor^2} \sigma(n\kappa + z_0) \tag{3.1}$$

(where the subscripts $+/-$ apply for even and odd n respectively), given in terms of the Weierstrass sigma function $\sigma(z) = \sigma(z; g_2, g_3)$ associated with the elliptic curve

$$\mathrm{y}^2 = 4\mathrm{x}^3 - g_2 \mathrm{x} - g_3 \tag{3.2}$$

with the parameters g_2, g_3, μ and κ appearing in (3.1) being explicitly determined from the coefficients α, β of (1.4) and the conserved quantity J defined by (2.5), via *the formulae*

$$g_2 = 12\tilde{\lambda} - 2J, \qquad g_3 = 4\tilde{\lambda}^3 - g_2\tilde{\lambda} - \tilde{\mu}^2. \tag{3.3}$$

$$\tilde{\mu} = (\beta + \alpha J)^{\frac{1}{4}}, \quad \tilde{\lambda} = \frac{1}{3\tilde{\mu}^2}\left(\frac{J^2}{4} + \alpha\right), \qquad \mu = \frac{\tilde{\mu}}{\sigma(2\kappa)} = -\sigma(\kappa)^{-4}. \tag{3.4}$$

The arbitrary parameter z_0 is determined from the initial data, while the remaining constants $A_\pm, B_\pm$ (also related to the initial data) are arbitrary up to the constraint

$$B_+ = -\mu^{-1} B_- = \sigma(\kappa)^4 B_-. \tag{3.5}$$

The solution (3.1) corresponds to a sequence of points $P_0 + nP$ along the curve (3.2), for an arbitrary initial point $P_0 = (\wp(z_0), \wp'(z_0))$ translated by $P = (\wp(\kappa), \wp'(\kappa)) = (\tilde{\lambda}, \tilde{\mu})$. The proof in [9] makes use of the fact that if x_n is a solution of (1.4) then the sequence of ratios

$$w_n = \frac{x_{n+2} x_{n-1}}{x_{n+1} x_n}. \tag{3.6}$$

Satisfies a nonlinear recurrence relation of second order, namely

$$w_{n-1}w_nw_{n+1} = \alpha w_n + \beta, \tag{3.7}$$

which is a particular example of a QRT map in the plane. The conserved quantity J for Somos-5 can be rewritten in terms of two adjacent ratios (3.6), as the expression

$$J = w_{n-1} + w_n + \alpha\left(\frac{1}{w_{n-1}} + \frac{1}{w_n}\right) + \frac{\beta}{w_{n-1}w_n}. \tag{3.8}$$

The explicit solution is found by associating the curve (3.2) with the recurrence, where the coefficients are given in terms of the sigma function by

$$\alpha = \frac{\sigma(3\kappa)}{\sigma(\kappa)^9}, \qquad \beta = -\frac{\sigma(4\kappa)}{\sigma(2\kappa)\sigma(\kappa)^{12}}. \tag{3.9}$$

For fixed initial conditions, the parameters z_0, κ can be computed from the elliptic integrals

$$z_0 = \pm\int_\infty^{x_0}\frac{\mathrm{d}x}{y}, \qquad \kappa = \pm\int_\infty^{\tilde{\lambda}}\frac{\mathrm{d}x}{y}, \tag{3.10}$$

where x_0 and a consistent relative choice of signs above must be determined from

$$x_0 = \tilde{\lambda} + \frac{\tilde{\mu}^2}{w_{-1} + w_0 - J}, \qquad \tilde{\mu} = \wp'(\kappa), \qquad \wp'(\kappa)\wp'(z_0) = (x_0 - \lambda)(w_{-1} - w_0),$$

while the coefficients $A_\pm, B_\pm$ can be found from the initial data. To see that the number of parameters in the analytic solution match the initial value problem, note that the recurrence (1.4) requires 5 pieces of initial data to be specified, x_0, x_1, x_2, x_3, x_4, say, together with 2 coefficients α, β, making a total of 7 parameters, while the analytic solution (3.1) is completely specified by choosing the 7 quantities $A_+, A_-, B_-, z_0, \kappa, g_2, g_3$. (Note that B_+ is fixed by the other data by the constraint (3.5), while μ is determined from (3.4).)

To extend this to the dual system, we note the standard identity

$$\Phi(X) = \Phi(x) + \Phi'(x)y\varepsilon, \tag{3.11}$$

for any differentiable function Φ and $X = x + y\varepsilon \in \mathbb{D}$. This can be used to give solutions to the dual system (2.6) in terms of the analytic solution (3.1) and derivatives.

Proposition 3.2. *Given dual number parameters* $A_\pm = A_\pm^{(0)} + A_\pm^{(1)}\varepsilon$, $B_- = B_-^{(0)} + B_-^{(1)}\varepsilon$, $Z_0 = z^{(0)} + z^{(1)}\varepsilon$, $K = \kappa^{(0)} + \kappa^{(1)}\varepsilon$, $g_2 = g_2^{(0)} + g_2^{(1)}\varepsilon$, $g_3 = g_3^{(0)} + g_3^{(1)}\varepsilon$, *with* $A_\pm, B_-, \sigma(K) \in \mathbb{D}^*$, *and letting* $\mu = -\sigma(K)^{-4}$, $B_+ = -\mu^{-1}B_-$, *the sequence*

$$X_n = A_\pm B_\pm^{\lfloor \frac{n}{2}\rfloor}\mu^{\lfloor \frac{n}{2}\rfloor^2}\sigma(nK + Z_0), \qquad n \in \mathbb{Z} \tag{3.12}$$

satisfies the dual Somos-5 recurrence (2.6) with coefficients

$$\alpha = \alpha^{(0)} + \alpha^{(1)}\varepsilon = \frac{\sigma(3K)}{\sigma(K)^9}, \qquad \beta = \beta^{(0)} + \beta^{(1)}\varepsilon = -\frac{\sigma(4K)}{\sigma(2K)\sigma(K)^{12}}. \tag{3.13}$$

In terms of even/odd components, this may be written as

$$X_n = x_n + x_n \left(\frac{A_{\pm}^{(1)}}{A_{\pm}^{(0)}} + \left\lfloor \frac{n}{2} \right\rfloor \frac{B_{\pm}^{(1)}}{B_{\pm}^{(0)}} + z_0^{(1)} \zeta\left(z_0^{(0)} + n\kappa^{(0)}\right) \right.$$
$$\left. + \left(\kappa^{(1)} \partial_{\kappa^{(0)}} + g_2^{(1)} \partial_{g_2^{(0)}} + g_3^{(1)} \partial_{g_3^{(0)}}\right) \log x_n \right) \varepsilon \tag{3.14}$$

where ∂ denotes a partial derivative, $\zeta(z) = \zeta(z; g_2^{(0)}, g_3^{(0)})$ is the Weierstrass zeta function, and x_n is the right-hand side of (3.12) with all parameters replaced by their even components, i.e. $A_+ \to A_+^{(0)}$, etc.

Proof. This is analogous to the proof of the analytic solution of the dual Somos-4 recurrence in [11]. We can make use of part of the original proof of theorem 3.1 for the regular Somos-5 recurrence in [9]: in one direction, the fact that the analytic expression satisfies the recurrence relies only on the three-term relation for the sigma function. The sigma function $\sigma(z; g_2, g_3)$ is an analytic function of the argument $z \in \mathbb{C}$, and of the parameters g_2, g_3, and so the three-term relation holds as an identity of formal series, which is still valid when z, g_2, g_3 are replaced by elements of the commutative ring $\mathbb{D}$. As for the even and odd parts, we can expand the right-hand side of (3.12) into its even and odd parts, $X_n = x_n + y_n\varepsilon$, by writing

$$A\pm = A_{\pm}^{(0)} \left(1 + \frac{A_{\pm}^{(1)}}{A_{\pm}^{(0)}}\varepsilon\right), \quad B_{\pm}^{\lfloor \frac{n}{2} \rfloor} = \left(B_{\pm}^{(0)}\right)^{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{B_{\pm}^{(1)}}{B_{\pm}^{(0)}}\varepsilon\right)^{\lfloor \frac{n}{2} \rfloor} = \left(B_{\pm}^{(0)}\right)^{\lfloor \frac{n}{2} \rfloor} \left(1 + \left\lfloor \frac{n}{2} \right\rfloor \frac{B_{\pm}^{(1)}}{B_{\pm}^{(0)}}\varepsilon\right),$$

where we used the binomial theorem, and also

$$\sigma(Z_0 + nK; g_2, g_3) = \sigma\left(z_0^{(0)} + nK; g_2, g_3\right)\left(1 + z_0^{(1)} \zeta(z_0 + nK; g_2, g_3)\varepsilon\right)$$
$$= \sigma\left(z_0^{(0)} + nK; g_2, g_3\right)\left(1 + z_0^{(1)} \zeta\left(z_0 + n\kappa^{(0)}; g_2^{(0)}, g_3^{(0)}\right)\varepsilon\right)$$

by (3.11). Upon multiplying out and keeping only terms of order zero and one in ε, this yields

$$X_n = A_{\pm}^{(0)} \left(B_{\pm}^{(0)}\right)^{\lfloor \frac{n}{2} \rfloor} \mu^{\lfloor \frac{n}{2} \rfloor^2} \sigma\left(z_0^{(0)} + nK; g_2, g_3\right) \left(1 + \left(\frac{A_{\pm}^{(1)}}{A_{\pm}^{(0)}} + \left\lfloor \frac{n}{2} \right\rfloor \frac{B_{\pm}^{(1)}}{B_{\pm}^{(0)}} \right.\right.$$
$$\left.\left. + z_0^{(1)} \zeta\left(z_0 + n\kappa^{(0)}; g_2^{(0)}, g_3^{(0)}\right) \right) \varepsilon \right),$$

whose even (ε^0) coefficient x_n just corresponds to the solution of the regular Somos-5 recurrence with all parameters being the even parts of the dual ones, while at order ε^1 above one can see the first three terms that make up the expression for y_n in (3.14). The remaining terms at order ε^1, involving a linear combination of the partial derivatives $\partial_{\kappa^{(0)}}, \partial_{g_2^{(0)}}, \partial_{g_3^{(0)}}$ applied to $\log x_n$, follow by considering the analytic dependence of the factor $\mu^{\lfloor \frac{n}{2} \rfloor^2} \sigma(z_0^{(0)} + nK; g_2, g_3)$ on the parameters $K, g_2, g_3 \in \mathbb{D}$ and applying (3.11) to each of these variables in turn. □

As it stands, the preceding result is not a complete analogue of theorem 3.1: while (3.12) provides a solution of the dual Somos-5 recurrence, depending on 7 dual parameters, we have not shown that it solves the general initial value problem for (2.6). To do this, we would need to have an analogue of the elliptic integrals (3.10) in order to reconstruct the parameters $Z_0, K \in \mathbb{D}$

from α, β and the initial data. This would seem to require a proper theory of elliptic curves and integrals over the dual numbers, which seems to be lacking. Nevertheless, the analytic formulae appearing in proposition 3.2 will be useful in what follows, for the construction of the shadow Somos-5 sequences.

4. Somos-5 shadows

Returning to the recurrence (2.10) for y_n, we noted previously that the case $\alpha^{(1)} = 0 = \beta^{(1)}$, that is

$$x_n y_{n+5} + y_n x_{n+5} - \alpha^{(0)} (x_{n+1} y_{n+4} + y_{n+1} x_{n+4}) - \beta^{(0)} (x_{n+2} y_{n+3} + y_{n+2} x_{n+3}) = 0, \tag{4.1}$$

corresponds to the linearization of (2.9), whose solutions are the shadow sequences in the sense of [27]. The above homogeneous linear equation for y_n is of order five, the same as the order of (2.9), hence the shadow Somos-5 sequences form a vector space of dimension 5. In particular, we can consider differentiating the explicit solution (3.1) with respect to suitable parameters, in order to obtain 5 linearly independent solutions of the linearized equation (4.1).

Lemma 4.1. *For fixed coefficients $\alpha^{(0)}, \beta^{(0)}$, the Somos-5 shadow equation (4.1) has 5 linearly independent solutions, which can be chosen as follows:*

$$y_n^{(i)} = \begin{cases} 0 & \text{for } n \text{ odd}, \\ x_n & \text{for } n \text{ even}, \end{cases} \tag{4.2}$$

$$y_n^{(ii)} = \begin{cases} x_n & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases} \tag{4.3}$$

$$y_n^{(iii)} = n x_n \tag{4.4}$$

$$y_n^{(iv)} = \zeta\left(z_0^{(0)} + n\kappa^{(0)}\right) x_n \tag{4.5}$$

$$y_n^{(v)} = x_n \partial_{J^{(0)}} \log(x_n) = x_n \left(\frac{\mathrm{d}\kappa^{(0)}}{\mathrm{d}J^{(0)}} \partial_{\kappa^{(0)}} + \frac{\mathrm{d}g_2^{(0)}}{\mathrm{d}J^{(0)}} \partial_{g_2^{(0)}} + \frac{\mathrm{d}g_3^{(0)}}{\mathrm{d}J^{(0)}} \partial_{g_3^{(0)}} \right) \log x_n \tag{4.6}$$

Proof. For fixed $\alpha, \beta \in \mathbb{C}$, the solution (3.1) depends on the 5 complex parameters A_+, A_-, B_-, z_0, J, which can be chosen arbitrarily, so we can obtain independent solutions of the linearized equation by differentiating with respect to each of these in turn. In the context of the dual equation, we wish to consider each of these parameters as the even part of a corresponding dual number, so that they should acquire a superscript (0), but for the purposes of this proof the superscripts are omitted. The parameters $A_\pm$ appear alternately in even/odd terms, and the derivatives with respect to these scaling parameters are proportional to x_n for each choice of parity of n, hence produce the alternating forms of (4.2) and (4.3) above. Using the constraint of (3.5) to rewrite B_+ in terms of B_-, B_- appears in both odd and even terms, and the derivative with respect to B_- up to scaling brings down a $\lfloor \frac{n}{2} \rfloor$ factor on each term, giving $\lfloor \frac{n}{2} \rfloor x_n$. Then we can set $y_n^{(\mathrm{iii})} = y_n^{(\mathrm{ii})} + 2\lfloor \frac{n}{2} \rfloor x_n$ to obtain the simpler form of (4.4). The fourth solution, as in (4.5), contains the Weierstrass zeta function $\zeta(z) = \zeta(z; g_2, g_3)$ for the same g_2, g_3 as in the solution, which comes from taking the derivative of $\sigma(z_0 + n\kappa)$ and using the definition

$$\zeta(z; g_2, g_3) = \frac{\sigma'(z; g_2, g_3)}{\sigma(z; g_2, g_3)}.$$

The final linearly independent solution, $y_n^{(\mathrm{v})}$, is found using the derivative with respect to the quantity J, the conserved quantity of the Somos-5 sequence, defined by (2.5), since the parameters κ, g_2, g_3 appearing in the formula (3.1) all depend on this quantity. The dependence on J is not straightforward, since it is determined by variations of the sigma function both with respect to its argument as a quasiperiodic function on the torus $\mathbb{C}/\Lambda$, and with respect to the modular parameters g_2, g_3 which determine the period lattice Λ of the elliptic curve (3.2). Hence (4.6) is best left in the form of a sum of logarithmic derivatives of x_n with respect to the quantities κ, g_2, g_3. $\square$

There are several different ways to construct these explicit shadow solutions. On the one hand, notice that the result of lemma 4.1 can be viewed as a corollary of proposition 3.12, since when $\alpha^{(1)} = \beta^{(1)} = 0$ the odd part of the solution (3.14) satisfies the homogeneous equation (4.1): the freedom of choice in the pair of parameters $A_{\pm}^{(1)}$ gives linear combinations of $y_n^{(\mathrm{i})}$ and $y_n^{(\mathrm{ii})}$; terms involving the parameters $B_{\pm}^{(1)}$ (of which only one can be chosen freely) results in including multiples of $y_n^{(\mathrm{iii})}$; the parameter $z_{\pm}^{(1)}$ produces linear combinations of $y_n^{(\mathrm{iv})}$; and the last three logarithmic derivatives correspond to the inclusion of the fifth linearly independent solution $y_n^{(\mathrm{v})}$. On the other hand, it is also easy to verify that $y_n^{(\mathrm{i})}, y_n^{(\mathrm{ii})}, y_n^{(\mathrm{iii})}$ are shadow solutions by direct substitution.

Example 4.2. The solution $y_n^{(\mathrm{iii})} = nx_n$ can be shown to satisfy the linear homogeneous recurrence as follows: Substituting $y_n = nx_n$ into the left-hand side of (4.1) produces

$$(n+5)\,x_n x_{n+5} + n x_n x_{n+5} - \alpha^{(0)}\left((n+4)\,x_{n+1}x_{n+4} + (n+1)\,x_{n+1}x_{n+4}\right)$$
$$- \beta^{(0)}\left((n+3)\,x_{n+2}x_{n+3} + (n+3)\,x_{n+2}x_{n+3}\right),$$

which is equal to

$$(2n+5)\left(x_n x_{n+5} - \alpha^{(0)}\,x_{n+1}x_{n+4} - \beta^{(0)}\,x_{n+2}x_{n+3}\right) = 0,$$

since x_n is a solution of the original Somos-5 recurrence (1.4) with $\alpha \to \alpha^{(0)}$, $\beta \to \beta^{(0)}$.

Similar direct substitutions verify that $y_n^{(\mathrm{i})}$ and $y_n^{(\mathrm{ii})}$ are shadow solutions, and even more straightforward to check is their sum $y_n = y_n^{(\mathrm{i})} + y_n^{(\mathrm{ii})} = x_n$. Compared with the analogous results for Somos-4 in [11], where the original sequence is also a shadow, the fact that the splitting of the original solution x_n into its odd and even index components are separate shadow solutions for Somos-5 is not unexpected: it is a consequence of the fact that the nonlinear recurrence admits a symmetry whereby the odd and even terms of the sequence can be scaled independently.

The solution $y_n^{(\mathrm{iv})}$ is more complicated, being expressed in terms of the Weierstrass ζ function associated with the elliptic curve (3.2). To obtain algebraic relations for this shadow, we note that from the solution (3.1), after shifting the index, the ratios w_n in (3.6) are given by

$$w_n = \frac{\sigma\left(z_0 + (n-1)\,\kappa\right)\sigma\left(z_0 + (n+2)\,\kappa\right)}{\sigma\left(\kappa\right)^4\sigma\left(z_0 + n\kappa\right)\sigma\left(z_0 + (n+1)\,\kappa\right)} = C\left(\zeta\left(z_0 + (n+1)\,\kappa\right) - \zeta\left(z_0 + n\kappa\right) + \tilde{c}\right), \tag{4.7}$$

for $C = \frac{\sigma(2\kappa)}{\sigma(\kappa)^4}$ and $\tilde{c} = \zeta(\kappa) - \zeta(2\kappa)$, where the second equality (used in [13] in connection with solutions of the Volterra lattice) follows from addition formulae for Weierstrass functions. We can compare this formula directly with the shadow solution $y_n^{(\mathrm{iv})}$.

Lemma 4.3. *Up to subtracting multiples of $y_n^{(i)}, y_n^{(ii)}, y_n^{(iii)}$ and overall scale, a fourth linearly independent shadow sequence for Somos-5 is $\bar{y}_n^{(iv)}$ given by*

$$\bar{y}_n^{(iv)} = x_n \sum_{j=0}^{n-1} w_j = C\Big(y_n^{(iv)} + \tilde{c}y_n^{(iii)} - \zeta(z_0^{(0)})\big(y_n^{(i)} + y_n^{(ii)}\big)\Big) \tag{4.8}$$

for $w_n = \frac{x_n x_{n+3}}{x_{n+1}x_{n+2}}$ which satisfies

$$w_{n+1}w_{n-1} = \alpha^{(0)} + \frac{\beta^{(0)}}{w_n}, \tag{4.9}$$

Proof. Using the formula (4.7) but with parameters $z_0 \to z_0^{(0)}$, $\kappa \to \kappa^{(0)}$, and so forth, we have the telescopic sum

$$\begin{aligned}\sum_{j=0}^{n-1} w_j &= C\left(\zeta\left(z_0^{(0)} + n\kappa^{(0)}\right) - \zeta\left(z_0^{(0)} + (n-1)\kappa^{(0)}\right) + \tilde{c} + \cdots + \zeta\left(z_0^{(0)} + \kappa^{(0)}\right) - \zeta\left(z_0^{(0)}\right) + \tilde{c}\right) \\ &= C\left(\zeta\left(z_0^{(0)} + n\kappa^{(0)}\right) + n\tilde{c} - \zeta\left(z_0^{(0)}\right)\right),\end{aligned}$$

and upon multiplying by x_n and comparing with the result of lemma 4.1, we obtain (4.8). □

Upon computing another telescopic sum, namely $x_n\bar{y}_{n+1}^{(\mathrm{iv})} - x_{n+1}\bar{y}_n^{(\mathrm{iv})}$, we find a first order inhomogeneous linear relation for this alternative fourth shadow solution.

Corollary 4.4. *The terms of the shadow Somos-5 sequence $(\bar{y}_n^{(iv)})$ satisfy the relation*

$$x_n\bar{y}_{n+1}^{(iv)} = x_{n+1}\bar{y}_n^{(iv)} + x_{n-1}x_{n+2}. \tag{4.10}$$

To look into the final sequence $y_n^{(\mathrm{v})}$ we return to the conserved quantity J as defined by the relation (2.5). We noted previously that extending to dual numbers changes this to a dual quantity given by $J = J^{(0)} + J^{(1)}\varepsilon$ where $J^{(0)}$ fulfills the same role as the original conserved quantity and $J^{(1)}$ relates to the y_n sequence and is given by (2.16). We can rewrite this equation in terms of a linear operator acting on y_n by defining

$$L_n = \sum_{j=0}^{4} C_n^{(j)} x_{n+j}^{-1} \mathcal{S}^j, \tag{4.11}$$

$$F_n = D_n - J^{(1)} x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4}, \tag{4.12}$$

where $\mathcal{S}$ is the shift operator that sends $n \to n+1$. Then (2.16) can be rewritten as

$$L_n(y_n) = F_n, \tag{4.13}$$

which, for fixed $J^{(1)}$, is a linear inhomogeneous difference equation of order 4 for y_n, reducing the order by 1 from the original recurrence (2.10) for y_n.

The 4th order homogeneous equation that arises when $\alpha^{(1)} = \beta^{(1)} = 0$ and $J^{(1)} = 0$, namely

$$L_n(y_n) = 0, \tag{4.14}$$

is the linearization of the equation defining $J^{(0)}$, whose solutions form a vector space of dimension 4, spanned by the 4 linearly independent shadow solutions $y_n^{(\mathrm{i})}, y_n^{(\mathrm{ii})}, y_n^{(\mathrm{iii})}, y_n^{(\mathrm{iv})}$. A fifth linearly independent shadow solution for Somos-5, such as $y_n^{(\mathrm{v})}$, then arises from an inhomogenous solution of (4.13), by keeping $\alpha^{(1)} = \beta^{(1)} = 0$, but taking a non-zero value $J^{(1)} \neq 0$. The other dual invariant I has even/odd components $I^{(0)}$ and $I^{(1)}$ which are functionally independent of the components of J, so the values taken by I depend on the orbits of dual Somos-5; but here we will not pursue the analogous linear equation obtained from $I^{(1)}$ any further.

Examining the other sequences $y_n^{(\mathrm{i})}, y_n^{(\mathrm{ii})}, y_n^{(\mathrm{iii})}, y_n^{(\mathrm{iv})}$ as solutions to this homogeneous equation, we can note some identities for the coefficients of the operator L_n. For $y_n^{(\mathrm{i})}$, $L_n(y_n^{(\mathrm{i})}) = 0$ gives

$$0 = \sum_{j=0}^{4} C_n^{(j)} x_{n+j}^{-1} y_{n+j} = \begin{cases} C_n^{(1)} + C_n^{(3)} & \text{for } n \text{ odd,} \\ C_n^{(0)} + C_n^{(2)} + C_n^{(4)} & \text{for } n \text{ even.} \end{cases} \tag{4.15}$$

The corresponding equations obtained from $L_n(y_n^{(\mathrm{ii})}) = 0$ are the same but with n odd/even switched, so hence we must have $C_n^{(0)} + C_n^{(2)} + C_n^{(4)} = 0$ and $C_n^{(1)} + C_n^{(3)} = 0$ for all n; and from the original definitions of the coefficients $C_n^{(j)}$ in (2.17), we can see that these are satisfied identically. By taking the sum $y_n^{(\mathrm{i})} + y_n^{(\mathrm{ii})}$ of these two shadow solutions, from (4.2) and (4.3) we get the equation $L_n(x_n) = 0$, which corresponds to the identity $\sum_{j=0}^{4} C_n^{(j)} = 0$. Upon examining the equation $L_n(y_n^{(\mathrm{iii})}) = 0$, we obtain the identity

$$0 = \sum_{j=0}^{4} C_n^{(j)} (n+j) = n \sum_{j=0}^{4} C_n^{(j)} + C_n^{(1)} + 2C_n^{(2)} + 3C_n^{(3)} + 4C_n^{(4)}. \tag{4.16}$$

Hence, by making use of $\sum_{j=0}^{4} C_n^{(j)} = 0$ and the other relations for the coefficients $C_n^{(j)}$, we find that $C_n^{(2)}$, $C_n^{(3)}$ and $C_n^{(4)}$ can be written in terms of $C_n^{(0)}$ and $C_n^{(1)}$, thus:

$$C_n^{(2)} = -2C_n^{(0)} - C_n^{(1)}, \qquad C_n^{(3)} = -C_n^{(1)}, \qquad C_n^{(4)} = C_n^{(0)} + C_n^{(1)}. \tag{4.17}$$

The relations (4.17) allow us to rewrite the homogeneous equation (4.14) in a much simpler way, which makes the first three independent shadow solutions even more obvious. Indeed, if we define

$$y_n = x_n Y_n,$$

then the homogeneous equation becomes

$$L_n(y_n) \equiv x_n x_{n+1} x_{n+2} x_{n+3} x_{n+4} \tilde{L}_n(Y_n) = 0, \tag{4.18}$$

where a short calculation with the above relation yields the 4th order difference operator

$$\tilde{L}_n = \left[\tilde{C}_n^{(0)} (\mathcal{S}+1) + \tilde{C}_n^{(1)} \mathcal{S}\right] (\mathcal{S}+1)(\mathcal{S}-1)^2, \tag{4.19}$$

where

$$\tilde{C}_n^{(0)} = \alpha^{(0)} \frac{x_{n+1} x_{n+2}}{x_n x_{n+3}} + \beta^{(0)} \frac{x_{n+2}^2}{x_n x_{n+4}} - \frac{x_n x_{n+3}}{x_{n+1} x_{n+2}}, \tag{4.20}$$

$$\tilde{C}_n^{(1)} = \alpha^{(0)} \left(\frac{x_{n+2}x_{n+3}}{x_{n+1}x_{x+4}} - \frac{x_{n+1}x_{n+2}}{x_n x_{n+3}} \right) + \frac{x_n x_{n+3}}{x_{n+1}x_{n+2}} - \frac{x_{n+1}x_{n+4}}{x_{n+2}x_{n+3}}. \tag{4.21}$$

Due to the factorized cubic part of $\tilde{L}_n$ with constant coefficients, namely $(\mathcal{S}+1)(\mathcal{S}-1)^2$, it is clear that the kernel of this operator has a 3-dimensional subspace spanned by $Y_n = 1$, $Y_n = n$ and $Y_n = (-1)^n$, and multiplying by x_n we immediately find the subspace of $\ker L_n$ spanned by $y_n^{(\mathrm{i})}, y_n^{(\mathrm{ii})}, y_n^{(\mathrm{iii})}$.

We now present an algebraic formula for the general solution of the 5th order recurrence (2.10) for y_n, which is based on applying the method of variation of parameters in the discrete setting (see [6]) to the 4th order equation (4.13). The main observation to make initially is that every solution of (2.10) is also a solution of (4.13), for some value of the first integral $J^{(1)}$. Hence, to solve the original problem, it is sufficient to find the general solution of the 4th order equation with an arbitrary parameter $J^{(1)}$.

For variation of parameters, we start with any 4 linear independent solutions of the homogeneous equation (4.14), $y_n^{(\mathrm{i})}, y_n^{(\mathrm{ii})}, y_n^{(\mathrm{iii})}, y_n^{(\mathrm{iv})}$ say, and assume that the solution of the full inhomogeneous problem (4.13) takes the form

$$y_n = \sum_j f_n^{(j)} y_n^{(j)}, \tag{4.22}$$

where $f_n^{(j)}$ are some coefficient functions, as yet undetermined, and the sum runs over $j =$ i, ii, iii, iv. Before we substitute this into the equation, we first set the constraints

$$\sum_j \left(f_{n+1}^{(j)} - f_n^{(j)} \right) y_{n+k}^{(j)} = 0 \qquad \text{for} \quad k = 1,2,3, \tag{4.23}$$

which also implies that

$$y_{n+k} = \sum_j f_n^{(j)} y_{n+k}^{(j)}, \qquad k = 0,1,2,3.$$

Then substituting this form of y_n into (4.13) and applying the constraints (4.23) gives

$$\begin{aligned} L_n(y_n) &= C_n^{(4)} x_{n+4}^{-1} \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)} \right) y_{n+4}^{(j)} + \sum_j f_n^j L_n \left(y_n^{(j)} \right) \\ &= C_n^{(4)} x_{n+4}^{-1} \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)} \right) y_{n+4}^{(j)}, \end{aligned}$$

as the shadow sequences $y_n^{(j)} \in \ker L_n$ for $j =$ i, ii, iii, iv. Hence, from (4.13), we have

$$C_n^{(4)} x_{n+4}^{-1} \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)} \right) y_{n+4}^{(j)} = F_n,$$

which, together with the 3 constraints (4.23), gives a system of 4 simultaneous equations for the differences

$$\delta_n^{(j)} := f_{n+1}^{(j)} - f_n^{(j)},$$

and solving this linear system then determines the functions $f_n^{(j)}$, up to a set of arbitrary constants $f_0^{(j)}$, corresponding to the choice of initial conditions.

Theorem 4.5. *A general solution of (2.10) can be given in the form*

$$y_n = \sum_j f_n^{(j)} y_n^{(j)}, \tag{4.24}$$

where the sum runs over $j = i, ii, iii, iv$ for the 4 shadow sequences $y_n^{(i)}, y_n^{(ii)}, y_n^{(iii)}, y_n^{(iv)}$. The coefficients $f_n^{(j)}$ are given by

$$f_n^{(j)} = f_0^{(j)} + \sum_{k=0}^{n-1} \delta_k^{(j)} \tag{4.25}$$

for arbitrary constants $f_0^{(j)}$, and

$$\begin{pmatrix} \delta_n^{(i)} \\ \delta_n^{(ii)} \\ \delta_n^{(iii)} \\ \delta_n^{(iv)} \end{pmatrix} = \frac{x_{n+4}F_n}{C_n^{(4)}} \begin{vmatrix} y_{n+1}^{(i)} & y_{n+1}^{(ii)} & y_{n+1}^{(iii)} & y_{n+1}^{(iv)} \\ y_{n+2}^{(i)} & y_{n+2}^{(ii)} & y_{n+2}^{(iii)} & y_{n+2}^{(iv)} \\ y_{n+3}^{(i)} & y_{n+3}^{(ii)} & y_{n+3}^{(iii)} & y_{n+3}^{(iv)} \\ y_{n+4}^{(i)} & y_{n+4}^{(ii)} & y_{n+4}^{(iii)} & y_{n+4}^{(iv)} \end{vmatrix}^{-1} \begin{pmatrix} d(ii,iii,iv) \\ d(i,iii,iv) \\ d(i,ii,iv) \\ d(i,ii,iii) \end{pmatrix}, \tag{4.26}$$

where

$$d(a,b,c) = \begin{vmatrix} y_{n+1}^{(a)} & y_{n+1}^{(b)} & y_{n+1}^{(c)} \\ y_{n+2}^{(a)} & y_{n+2}^{(b)} & y_{n+2}^{(c)} \\ y_{n+3}^{(a)} & y_{n+3}^{(b)} & y_{n+3}^{(c)} \end{vmatrix}. \tag{4.27}$$

Proof. The variation of parameters method reduces the problem to solving the linear system

$$\begin{pmatrix} y_{n+1}^{(\mathrm{i})} & y_{n+1}^{(\mathrm{ii})} & y_{n+1}^{(\mathrm{iii})} & y_{n+1}^{(\mathrm{iv})} \\ y_{n+2}^{(\mathrm{i})} & y_{n+2}^{(\mathrm{ii})} & y_{n+2}^{(\mathrm{iii})} & y_{n+2}^{(\mathrm{iv})} \\ y_{n+3}^{(\mathrm{i})} & y_{n+3}^{(\mathrm{ii})} & y_{n+3}^{(\mathrm{iii})} & y_{n+3}^{(\mathrm{iv})} \\ y_{n+4}^{(\mathrm{i})} & y_{n+4}^{(\mathrm{ii})} & y_{n+4}^{(\mathrm{iii})} & y_{n+4}^{(\mathrm{iv})} \end{pmatrix} \begin{pmatrix} \delta_n^{(\mathrm{i})} \\ \delta_n^{(\mathrm{ii})} \\ \delta_n^{(\mathrm{iii})} \\ \delta_n^{(\mathrm{iv})} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(C_n^{(4)}\right)^{-1} x_{n+4}F_n \end{pmatrix}.$$

The four independent shadow sequences $y_n^{(j)} \in \ker L_n$ for $j = \mathrm{i, ii, iii, iv}$ can be chosen arbitrarily. □

We end this section by presenting some explicit examples of shadow sequences.

Example 4.6. Taking the sequence x_n to be the original well-known Somos-5 sequence (1.6), i.e. with $\alpha = \beta = x_0 = x_1 = x_2 = x_3 = x_4 = 1$, we immediately find the first three shadow sequences in terms of these x_n from (4.2)–(4.4). For the fourth shadow sequence $y_n^{(\mathrm{iv})}$, we can take $\bar{y}_n^{(\mathrm{iv})}$ given by (4.8) in terms of x_n and the quantities w_n, which can either be calculated directly from the x_n using (3.6), or found recursively using (4.9) with $w_1 = w_2 = 1$. The $n = 0$ term is fixed to be $\bar{y}_0^{(\mathrm{iv})} = 0$ (an empty sum) which is consistent with the relation (4.10). For $y_n^{(\mathrm{v})}$ we must solve (4.13) for $\alpha^{(1)} = \beta^{(1)} = 0$ and $J^{(1)} \neq 0$. For simplicity we start with $y_n^{(\mathrm{v})} = 0$ for $n = 0, 1, 2, 3$ and set $J^{(1)} = -1$ to cancel the $-$ sign in F_n, which gives $y_4^{(\mathrm{v})} = 1$, and the rest of the sequence is found by continuing with the definition of $J^{(1)}$ or by iterating the base recurrence (4.1). Numerical values up to $n = 10$ are given in table 1.

The 5 sequences are linearly independent, as can be seen clearly for these values by considering the initial conditions as y_0, y_1, y_2, y_3, y_4 and the corresponding vectors $\mathbf{y}^{(j)} = (y_0^{(j)}, y_1^{(j)}, y_2^{(j)}, y_3^{(j)}, y_4^{(j)})$ for $j = \mathrm{i,ii,iii,iv,v}$. Then we can explicitly expand the standard basis

Table 1. The original Somos-5 sequence (1.6) and 5 linearly independent Shadow sequences associated with it.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	1	1	1	1	1	2	3	5	11	37	83
$y_n^{(\mathrm{i})}$	1	0	1	0	1	0	3	0	11	0	83
$y_n^{(\mathrm{ii})}$	0	1	0	1	0	2	0	5	0	37	0
$y_n^{(\mathrm{iii})}$	0	1	2	3	4	10	18	35	88	333	830
$y_n^{(\mathrm{iv})}$	0	2	3	4	6	15	25	49	130	475	1147
$y_n^{(\mathrm{v})}$	0	0	0	0	1	1	2	5	17	23	118

for $\mathbb{C}^5$ in terms of these vectors as

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{y}^{(\mathrm{i})} + \mathbf{y}^{(\mathrm{v})} - \mathbf{y}^{(\mathrm{iv})} + \mathbf{y}^{(\mathrm{ii})} + \mathbf{y}^{(\mathrm{iii})},\\
\mathbf{e}_2 &= \mathbf{y}^{(\mathrm{iv})} - \frac{3}{2}\mathbf{y}^{(\mathrm{iii})} + \frac{1}{2}\mathbf{y}^{(\mathrm{ii})},\\
\mathbf{e}_3 &= \mathbf{y}^{(\mathrm{iv})} - 2\mathbf{y}^{(\mathrm{v})} - \mathbf{y}^{(\mathrm{ii})} - \mathbf{y}^{(\mathrm{iii})},\\
\mathbf{e}_4 &= \frac{1}{2}\mathbf{y}^{(\mathrm{ii})} - \mathbf{y}^{(\mathrm{iv})} + \frac{3}{2}\mathbf{y}^{(\mathrm{iii})},\\
\mathbf{e}_5 &= \mathbf{y}^{(\mathrm{v})},
\end{aligned}$$

which shows that $\{\mathbf{y}^(\mathrm{i}), \mathbf{y}^(\mathrm{ii}), \mathbf{y}^(\mathrm{iii}), \mathbf{y}^(\mathrm{iv}), \mathbf{y}^(\mathrm{v})\}$ span the space of initial conditions (although not as a $\mathbb{Z}$-module).

5. Hankel determinant formulae

Using a combinatorial approach and determinant identities, Hankel determinant formulae for Somos-5 sequences were derived in [4], while in recent work on discrete integrable systems related to the Volterra lattice, a different set of Hankel determinant formulae were found by one of us via the connection with Stieltjes continued fractions [13]. In the notation used in the latter work, the sequence of Hankel determinants is specified by

$$\Delta_{2k-1} = \det\left(s_{i+j-1}\right)_{i,j=1,2,\ldots,k}, \qquad \Delta_{2k} = \det\left(s_{i+j}\right)_{i,j=1,2,\ldots,k} \tag{5.1}$$

for $k \geqslant 1$, with $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$, where the entries s_j (the moments) are defined by the recursion relation

$$s_j = \gamma_1 s_{j-1} + \sum_{i=1}^{j-1} s_i s_{j-i} + \gamma_2 \sum_{i=1}^{j-2} s_i s_{j-i-1} \tag{5.2}$$

for $j \geqslant 3$, with initial values $s_1 = w_1, s_2 = w_1 w_2$, and

$$\gamma_1 = -2w_1 - \frac{c_1}{2}, \qquad \gamma_2 = \frac{1}{4w_1}\left(\frac{c_1^2}{4} - c_2\right) - w_2. \tag{5.3}$$

Here, as usual, the w_n are given by the recurrence (3.7), which is the QRT map associated with the Somos-5 relation (1.4) via the substitution

$$w_n = \frac{x_n x_{n+3}}{x_{n+1}x_{n+2}} \tag{5.4}$$

(and the reader should note that in this section we have made an overall shift of index compared with (3.6), in order to be consistent with the conventions used in [13]). The remaining constants c_1, c_2, which appear in the coefficients γ_1, γ_2 defined by (5.3), can be fixed from a related elliptic curve, or by a variety of relations to the other parameters. They can be written

$$c_1 = -2\left(\frac{\alpha}{w_0} + w_1 + w_0 + w_{-1}\right) = -2J,$$
$$c_2 = 4\alpha + \frac{c_1^2}{4} = 4\alpha + J^2 = 12\tilde{\mu}\tilde{\lambda},$$

where the substitution (5.4) has been used to obtain the extra equalities in each line above, rewriting c_1, c_2 in terms of other constants used earlier, namely the first integral J defined in (2.5), and $\tilde{\mu}$, $\tilde{\lambda}$ as in (3.4). Equivalently, the sequence of moments s_j, and the corresponding Hankel determinants, are completely determined by fixing the values of s_1, s_2 and the coefficients γ_1, γ_2 in (5.2). The w_n can then be given in terms of these determinants by

$$w_n = \frac{\Delta_{n-3}\Delta_n}{\Delta_{n-1}\Delta_{n-2}} = \frac{x_n x_{n+3}}{x_{n+1}x_{n+2}}.$$

So the Δ_n generate a Somos-5 sequence with $x_1 = x_2 = x_3 = 1, x_4 = s_1, x_5 = s_2$, if we identify

$$x_n = \Delta_{n-3}.$$

Example 5.1. For the original Somos-5 sequence we have $x_0 = x_1 = x_2 = x_3 = x_4 = \alpha = \beta = 1$ and $J = 5$, thus $s_1 = w_1 = 1$ and $s_2 = w_1 w_2 = 2$. From the above, the entries of the Hankel matrices are found recursively using

$$s_j = 3s_{j-1} + \sum_{i=1}^{j-1} s_i s_{j-i} - 3\sum_{i=1}^{j-2} s_i s_{j-i-1}, \tag{5.5}$$

which recovers example 3.9 in [13], and the Hankel determinants $\Delta_n = x_{n+3}$ can be seen to reproduce the original Somos-5 sequence (1.6).

In [13], integrable maps and Hankel determinant solutions are constructed from continued fraction expansions on hyperelliptic curves of arbitrary genus g, but for what follows it will be convenient to paraphrase the main results about Hankel determinants in the case $g = 1$, which correspond to Somos-5 sequences in the following way.

Proposition 5.2. *Suppose that a sequence $(\Delta_n)_{n\geqslant -2}$ is specified by $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$, and Hankel determinants Δ_{2k-1}, Δ_{2k} given by (5.1) for $k \geqslant 1$, with entries given by the moment sequence (s_j) obtained from the recursion (5.2) for $j \geqslant 3$, for a fixed choice of coefficients γ_1, γ_2 and initial values s_1, s_2. Then Δ_n is a solution of the Somos-5 recurrence*

$$\Delta_{n+3}\Delta_{n-2} = \alpha\Delta_{n+2}\Delta_{n-1} + \beta\Delta_{n+1}\Delta_n, \qquad n \geqslant 0,$$

where the coefficients α *and* β *are given by*

$$\alpha=-\gamma_2 s_1-s_2, \qquad \beta=(\gamma_1+\gamma_2)\,s_2+\gamma_2 s_1^2+2s_1s_2, \tag{5.6}$$

and the resulting value of the first integral J is

$$J=\gamma_1+2s_1. \tag{5.7}$$

Note that fixing initial values $\Delta_{-2}=\Delta_{-1}=\Delta_0=1$ means that there are just 4 free parameters $s_1,s_2,\gamma_1,\gamma_2$, compared with the 7 parameters required to specify the general initial value problem for Somos-5. However, other initial values for the sequences can be obtained, and the missing degrees of freedom can be restored, by applying the 3-parameter group of scaling symmetries

$$x_{2k-1}\to A_-x_{2k-1}, \qquad x_{2k}\to A_+x_{2k}, \qquad x_n\to B^n x_n \tag{5.8}$$

for arbitrary non-zero constants $A_-,A_+,B\in\mathbb{C}^*$.

Hankel determinant formulae for negative values of the index n are also presented in [13], given by a similar formula and recursion for the entries. The scaling symmetries above can be applied in order to ensure the positive and negative index formulae line up to form a full sequence for all $n\in\mathbb{Z}$.

As with the explicit form of the Somos-5 solution previously, because $\mathbb{D}$ is a commutative ring, the above Hankel determinant formula extends directly to the dual numbers, simply by taking all variables and constants to be in $\mathbb{D}$.

Proposition 5.3. *Suppose that constants* $\gamma_1,\gamma_2\in\mathbb{D}$ *and* $s_1,s_2\in\mathbb{D}^*$ *are given, and* $\Delta_{-2}=\Delta_{-1}=\Delta_0=1$. *Then the sequence* (Δ_n) *defined from the Hankel determinants*

$$\Delta_{2k-1}=\det\left(s_{i+j-1}\right)_{i,j=1,2,\dots,k} \qquad \Delta_{2k}=\det\left(s_{i+j}\right)_{i,j=1,2,\dots,k}, \qquad k\geqslant 1, \tag{5.9}$$

with entries given by the sequence of moments $s_j=s_j^{(0)}+s_j^{(1)}\varepsilon\in\mathbb{D}$ *generated recursively from*

$$s_j=\gamma_1 s_{j-1}+\sum_{i=1}^{j-1}s_is_{j-i}+\gamma_2\sum_{i=1}^{j-2}s_is_{j-i-1}, \qquad j\geqslant 3, \tag{5.10}$$

is a solution of the dual Somos-5 recurrence

$$\Delta_{n+3}\Delta_{n-2}=\alpha\Delta_{n+2}\Delta_{n-1}+\beta\Delta_{n+1}\Delta_n, \qquad n\geqslant 0, \tag{5.11}$$

where the coefficients $\alpha=\alpha^{(0)}+\alpha^{(1)}\varepsilon$ *and* $\beta=\beta^{(0)}+\beta^{(1)}\varepsilon$ *are given by*

$$\alpha^{(0)}=-\gamma_2^{(0)}s_1^{(0)}-s_2^{(0)}, \quad \beta^{(0)}=\gamma_2^{(0)}s_2^{(0)}+\gamma_1^{(0)}s_2^{(0)}+\gamma_2^{(0)}\left(s_1^{(0)}\right)^2+2s_1^{(0)}s_2^{(0)}, \tag{5.12}$$

and

$$\alpha^{(1)}=-\gamma_2^{(0)}s_1^{(1)}-\gamma_2^{(1)}s_1^{(0)}-s_2^{(1)}, \tag{5.13}$$

$$\beta^{(1)}=\gamma_2^{(0)}s_2^{(1)}+\gamma_2^{(1)}s_2^{(0)}+\gamma_1^{(1)}s_2^{(0)}+\gamma_1^{(0)}s_2^{(1)}+\gamma_2^{(1)}\left(s_1^{(0)}\right)^2 + 2\gamma_2^{(0)}s_1^{(1)}s_1^{(0)}+2s_1^{(1)}s_2^{(0)}+2s_1^{(0)}s_2^{(1)}. \tag{5.14}$$

and the components of the resulting dual first integral $J = J^{(0)} + J^{(1)}\varepsilon$ *take the values*

$$J^{(i)} = \gamma_1^{(i)} + 2s_1^{(i)} \quad \textit{for } i = 0,1. \tag{5.15}$$

The constants γ_1, γ_2 can also be given explicitly in terms of an initial set of values of x_n, y_n, via the formulae

$$\gamma_1^{(0)} = J^{(0)} - 2\frac{x_1x_4}{x_2x_3}, \qquad \gamma_2^{(0)} = \frac{x_1x_4}{x_2x_3} + \frac{x_0x_3}{x_1x_2} - J^{(0)}, \tag{5.16}$$

and

$$\gamma_1^{(1)} = J^{(1)} - 2\frac{y_1x_4}{x_2x_3} - 2\frac{x_1y_4}{x_2x_3} + 2\frac{x_1y_2x_4}{x_2^2x_3} + 2\frac{x_1y_3x_4}{x_2x_3^2}, \tag{5.17}$$

$$\gamma_2^{(1)} = \frac{y_1x_4}{x_2x_3} + \frac{x_1y_4}{x_2x_3} - \frac{x_1y_2x_4}{x_2^2x_3} - \frac{x_1y_3x_4}{x_2x_3^2} + \frac{y_0x_3}{x_1x_2} + \frac{x_0y_3}{x_1x_2} - \frac{x_0y_1x_3}{x_1^2x_2} - \frac{x_0y_2x_3}{x_1x_2^2} - J^{(1)}, \tag{5.18}$$

which can be obtained via the formulae for the coefficients γ_1, γ_2 in terms of the w_n and J from [13], and using (2.12) to expand into even/odd components after changing all variables into dual numbers. Assuming $x_1 = x_2 = x_3 = 1$ and $y_1 = y_2 = y_3 = 0$ to match up with the initial terms of the sequence (Δ_n), with $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$, this simplifies considerably, to give

$$\gamma_1^{(0)} = J^{(0)} - 2x_4, \qquad \gamma_2^{(0)} = x_4 + x_0 - J^{(0)}, \tag{5.19}$$

and

$$\gamma_1^{(1)} = J^{(1)} - 2y_4, \qquad \gamma_2^{(1)} = y_4 + y_0 - J^{(1)}. \tag{5.20}$$

The Hankel determinant formulae can now be demonstrated to recover some of the dual Somos-5 sequences that were found previously in the literature.

Example 5.4. To obtain the Shadow sequence $y_n^{(\mathrm{v})}$ as in example 4.6, we have $\alpha^{(0)} = \beta^{(0)} = x_0 = x_1 = x_2 = x_3 = x_4 = 1$ and $\alpha^{(1)} = \beta^{(1)} = y_0 = y_1 = y_2 = y_3 = 0$ with $J^{(1)} = -1$. Hence $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$ from $\Delta_n = x_{n+3} + y_{n+3}\varepsilon$ as required. Then $J^{(0)} = 5$ can be found from (2.5) using the initial values of x_n. Hence $s_1^{(0)} = x_4 = 1$ via $w_1 = \frac{X_1X_4}{X_2X_3}$ and the reciprocal formula (2.12). Similarly, $s_2^{(0)} = x_5 = 2$ which can be found using the recurrence for the x_n. Then from the y_n values, we find $s_1^{(1)} = 1, s_2^{(1)} = 1$. We can make use of the relations for the components of α and β to get the coefficients γ_1, γ_2. In fact, without using (5.12), we already know from example 5.1 that we have $\gamma_1^{(0)} = 3$ and $\gamma_2^{(0)} = -3$, because the even parts must agree with the parameters for the ordinary Somos-5 over $\mathbb{C}$. From the $n = 1$ case of (5.15) we find $\gamma_1^{(1)} = -3$ and via (5.13) we get $\gamma_2^{(1)} = 2$. Hence

$$s_j = (3 - 3\varepsilon)s_{j-1} + \sum_{i=1}^{j-1} s_is_{j-i} - (3 - 2\varepsilon)\sum_{i=1}^{j-2} s_is_{j-i-1}, \qquad j \geqslant 3, \tag{5.21}$$

with initial moments $s_1 = 1 + \varepsilon, s_2 = 2 + \varepsilon$. From this we can find the first few dual moments: $s_3 = 7 - \varepsilon, s_4 = 27 - 18\varepsilon, s_5 = 109 - 119\varepsilon$, and verify that the determinants recover the sequence. We have $\Delta_1 = s_1 = 1 + \varepsilon$, $\Delta_2 = 2 + \varepsilon$ and

$$\Delta_3 = \begin{vmatrix} 1+\varepsilon & 2+\varepsilon \\ 2+\varepsilon & 7-\varepsilon \end{vmatrix} = 3 + 2\varepsilon, \tag{5.22}$$

$$\Delta_4 = \begin{vmatrix} 2+\varepsilon & 7-\varepsilon \\ 7-\varepsilon & 27-18\varepsilon \end{vmatrix} = 5+5\varepsilon, \tag{5.23}$$

$$\Delta_5 = \begin{vmatrix} 1+\varepsilon & 2+\varepsilon & 7-\varepsilon \\ 2+\varepsilon & 7-\varepsilon & 27-18\varepsilon \\ 7-\varepsilon & 27-18\varepsilon & 109-119\varepsilon \end{vmatrix} = 11+17\varepsilon. \tag{5.24}$$

So from the even parts we can see the expected sequence x_n: $1,1,1,1,1,2,3,5,11,\ldots$, while the odd parts give the shadow sequence $y_n^{(\mathrm{v})}$ as y_n: $0,0,0,0,1,1,2,5,17,\ldots$.

We can also demonstrate the use of the scalings (5.8) to get other dual sequences, when we consider an example of one of Ovsienko and Tabachnikov's specific cases, namely (2.7).

Example 5.5. From (2.7) we have $\alpha = 1+\alpha^{(1)}\varepsilon$, $\beta = 1$. Letting $\alpha^{(1)} = 2$ for this example, with the usual initial conditions $x_0 = x_1 = x_2 = x_3 = x_4 = 1$ and $y_0 = y_1 = y_2 = y_3 = y_4 = 0$ so that $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$, we have from (5.19) that $\gamma_1^{(0)} = 3, \gamma_2^{(0)} = -3$. Similarly we can find $J^{(1)} = 6$ from (2.16) and thus, via (5.20), $\gamma_1^{(1)} = 4, \gamma_2^{(1)} = -4$. Also have $s_1 = X_4 = 1$, and using the full recurrence for the y_n, (2.10), with the above values we find $y_5 = 2$ so $s_2 = X_5 = 2+2\varepsilon$. Hence the entries can be generated by

$$s_j = (3+4\varepsilon)\, s_{j-1} + \sum_{i=1}^{j-1} s_i s_{j-i} - (3+4\varepsilon) \sum_{i=1}^{j-2} s_i s_{j-i-1}, \tag{5.25}$$

which gives $s_3 = 7+14\varepsilon, s_4 = 27+78\varepsilon, s_5 = 109+402\varepsilon$. Hence

$$\Delta_3 = \begin{vmatrix} 1 & 2+2\varepsilon \\ 2+2\varepsilon & 7+14\varepsilon \end{vmatrix} = 3+6\varepsilon, \tag{5.26}$$

$$\Delta_4 = \begin{vmatrix} 2+2\varepsilon & 7+14\varepsilon \\ 7+14\varepsilon & 27+78\varepsilon \end{vmatrix} = 5+14\varepsilon, \tag{5.27}$$

$$\Delta_5 = \begin{vmatrix} 1 & 2+2\varepsilon & 7+14\varepsilon \\ 2+2\varepsilon & 7+14\varepsilon & 27+78\varepsilon \\ 7+14\varepsilon & 27+78\varepsilon & 109+402\varepsilon \end{vmatrix} = 11+42\varepsilon. \tag{5.28}$$

Hence we can see the expected sequence x_n: $1,1,1,1,1,2,3,5,11,\ldots$, and we have the sequence of odd parts y_n: $0,0,0,0,0,2,6,14,42,\ldots$ which can be checked via the recurrence (2.10). We can also demonstrate rescaling using (5.8) to get sequences with initial values other than $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$. For example, to obtain the initial conditions $y_n = n$ for $n = 0,\ldots,4$ we can use $A_- = A_+ = 1$ and $B = 1+\varepsilon$. Then by scaling the $X_n = x_n + y_n\varepsilon$ for the sequences above to $X_n \to B^n X_n$, we obtain the same sequence x_n: $1,1,1,1,1,2,3,5,11,\ldots$ but the alternative odd component sequence y_n: $0,1,2,3,4,12,24,49,130,\ldots$ which can again be verified by (2.10). Hence sequences with other initial conditions can still be obtained from the original Somos-5 sequence, by using the Hankel determinant formulae together with appropriate scalings.

6. Bi-Hamiltonian structure for dual QRT map

Iterating the recurrence (3.7) is equivalent to making iterations of the birational map

$$\varphi: \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \longmapsto \begin{pmatrix} w_2 \\ w_1^{-1}\left(\alpha+\beta w_2^{-1}\right) \end{pmatrix}, \tag{6.1}$$

in the affine plane with coordinates (w_1, w_2), which preserves the log-canonical symplectic form

$$\omega = \frac{\mathrm{d}w_1 \wedge \mathrm{d}w_2}{w_1 w_2},$$

in the sense that $\varphi^*(\omega) = \omega$. The first integral J for Somos-5 can be rewritten in terms of a pair of coordinates (w_1, w_2) in the plane, by fixing $n = 1$ in (3.8), to obtain

$$J = w_1 + w_2 + \alpha\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + \frac{\beta}{w_1 w_2},$$

and the family of level sets of J defines a pencil of biquadratic plane curves. The map (6.1) is a particular example of a symmetric QRT map, and is associated with elliptic solutions of the Volterra lattice (see [13] and references).

As already mentioned above, the substitution (3.6) relates solutions of the Somos-5 recurrence to iterates of the map (6.1), and we can extend this to the dual numbers by setting

$$W_n = \frac{X_{n+2}X_{n-1}}{X_{n+1}X_n}, \tag{6.2}$$

where X_n is a solution of (2.6). Upon expanding into even/odd components we find

$$W_n = w_n + \varepsilon v_n,$$

where the even component w_n corresponds to iterates of φ as before, but with the replacement $\alpha \to \alpha^{(0)}$, $\beta \to \beta^{(0)}$ in (6.1), and the odd components are given in terms of x_n and y_n by

$$v_n = \frac{x_{n-1}y_{n+2}}{x_{n+1}x_n} - \frac{x_{n-1}x_{n+2}y_{n+1}}{x_{n+1}^2 x_n} - \frac{x_{n-1}x_{n+2}y_n}{x_{n+1}x_n^2} + \frac{x_{n+2}y_{n-1}}{x_{n+1}x_n}. \tag{6.3}$$

Moreover, the odd components satisfy the recurrence

$$w_{n-1}w_n v_{n+1} + w_{n-1}w_{n+1}v_n + w_n w_{n+1} v_{n-1} = \alpha^{(0)}\, v_n + \alpha^{(1)}\, w_n + \beta^{(1)}, \tag{6.4}$$

which comes from the odd part of the recurrence for W_n, that is

$$W_{n+1}W_n W_{n-1} = \alpha\, W_n + \beta, \qquad \alpha, \beta \in \mathbb{D}. \tag{6.5}$$

The above recurrence corresponds to the dual QRT map

$$\varphi_{\mathrm{dual}}: \quad \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \longmapsto \begin{pmatrix} W_2 \\ W_1^{-1}\left(\alpha + \beta W_2^{-1}\right) \end{pmatrix}, \qquad (W_1, W_2) \in \mathbb{D}^*, \tag{6.6}$$

which can be written in components as the 4D map

$$\begin{pmatrix} w_1 \\ w_2 \\ v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} w_2 \\ w_3 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \text{where} \quad w_3 = w_1^{-1}w_2^{-1}\left(\alpha^{(0)}w_2 + \beta^{(0)}\right), \tag{6.7}$$

and

$$v_3 = -w_2^{-1}w_3 v_2 - w_1^{-1}w_3 v_1 + w_1^{-1}w_2^{-1}\left(\alpha^{(0)}v_2 + \alpha^{(1)}w_2 + \beta^{(1)}\right).$$

Remark 6.1. In the general setting of noncommutative variables, where W_1, W_2 are considered as units in an associative algebra, with parameters α, β both set to 1, Duzhin and Kontsevich discovered the map (6.6) in the form

$$(W_1, W_2) \mapsto \left(W_1 W_2 W_1^{-1}, \left(1 + W_2^{-1}\right) W_1^{-1}\right),$$

which they found was a discrete symmetry of the ODE system

$$\dot{W}_1 = W_1 W_2 - W_1 W_2^{-1} - W_2^{-1}, \quad \dot{W}_2 = -W_2 W_1 + W_2 W_1^{-1} + W_1^{-1}. \tag{6.8}$$

(See [14] and [29] for more details.)

We now derive a Poisson structure that is compatible with the dual QRT map $\varphi_{\rm dual}$, which is obtained simply by noticing that the dual analogue of the symplectic form ω, namely

$$\Omega = \frac{\mathrm{d}W_1 \wedge \mathrm{d}W_2}{W_1 W_2}, \tag{6.9}$$

is preserved by (6.6), so that $\varphi^*_{\rm dual}(\Omega) = \Omega$. Then we can expand Ω into its even and odd components,

$$\Omega = \omega^{(0)} + \varepsilon \omega^{(1)},$$

where

$$\omega^{(0)} = \omega$$

is the same as the original log-canonical 2-form on the (w_1, w_2) plane, while

$$\omega^{(1)} = -\left(\frac{v_1}{w_1} + \frac{v_2}{w_2}\right)\omega + \frac{1}{w_1 w_2}\left(\mathrm{d}v_1 \wedge \mathrm{d}w_2 + \mathrm{d}w_1 \wedge \mathrm{d}v_2\right),$$

and note that each component is separately invariant under the map in the extended 4D phase space:

$$\varphi^*_{\rm dual}\left(\omega^{\,(j)}\right) = \omega^{\,(j)}, \quad j = 0, 1.$$

Theorem 6.2. *The dual QRT map φ_{dual} defined by (6.7) is bi-Hamiltonian, in the sense that it preserves the pencil of Poisson brackets $\{\,,\}$ with parameter ζ defined by*

$$\begin{aligned} &\{w_1, w_2\} = \{w_1, v_1\} = \{w_2, v_2\} = 0, \\ &\{w_2, v_1\} = -\{w_1, v_2\} = w_1 w_2, \\ &\{v_1, v_2\} = w_1 w_2 \zeta - w_1 v_2 - w_2 v_1. \end{aligned} \tag{6.10}$$

Furthermore, the even and odd components of the first integral J are in involution with respect to this pencil, hence φ_{dual} is a Liouville integrable map in 4D.

Proof. The form (6.9) can be written as $\mathrm{d}\log W_1 \wedge \mathrm{d}\log W_2$, so $\mathrm{d}\Omega = 0$ which implies that the even and odd parts are both closed. Hence $\omega^{(0)}$ and $\omega^{(1)}$ are both symplectic forms that are preserved by the map $\varphi_{\rm dual}$, as is the linear combination

$$\omega_\zeta = \zeta \omega^{(0)} + \omega^{(1)}.$$

The 2-form ω_ζ is nondegenerate for any $\zeta \in \mathbb{C}$. Hence its inverse defines a nondegenerate Poisson tensor, which corresponds to the pencil of brackets $\{\,,\}$ given in (6.10), and $\varphi^*_{\mathrm{dual}}(\omega_\zeta) = \omega_\zeta$ implies that this bracket is preserved by the dual QRT map for all ζ. In the form (6.6), it is clear that the map preserves the invariant $J \in \mathbb{D}$ given by the dual analogue of (3.8), that is

$$J = W_1 + W_2 + \alpha\left(\frac{1}{W_1} + \frac{1}{W_2}\right) + \frac{\beta}{W_1 W_2} = J^{(0)} + \varepsilon J^{(1)},$$

where

$$\begin{aligned} J^{(0)} &= w_1 + w_2 + \alpha^{(0)}\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + \frac{\beta^{(0)}}{w_1 w_2}, \\ J^{(1)} &= \left(1 - \frac{\alpha^{(0)}}{w_1^2} - \frac{\beta^{(0)}}{w_1^2 w_2}\right) v_1 + \left(1 - \frac{\alpha^{(0)}}{w_2^2} - \frac{\beta^{(0)}}{w_1 w_2^2}\right) v_2 \\ &\quad + \alpha^{(1)}\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + \frac{\beta^{(1)}}{w_1 w_2}, \end{aligned} \tag{6.11}$$

which correspond to rewriting the conserved quantities (2.15) and (2.16) for the dual Somos-5 recurrence in terms of the coordinates for the 4D phase space. The latter define two independent invariants for the map in 4D, and $\{J^{(0)}, J^{(1)}\} = 0$ for all $\zeta \in \mathbb{C}$, hence the map φ_{dual} is integrable in the Liouville sense. □

Remark 6.3. Because they are in involution, the conserved quantities (6.11) generate a pair of commuting vector fields from any member of the Poisson pencil. Fixing $\zeta = 0$ to obtain the bracket corresponding to the symplectic form $\omega^{(1)}$, the first vector field is $\{\cdot, J^{(0)}\}|_{\zeta=0}$ given by

$$\begin{aligned} w_1' &= 0, \\ w_2' &= 0, \\ v_1' &= -w_1 w_2 + \left(\alpha^{(0)} w_1 + \beta^{(0)}\right) w_2^{-1}, \\ v_2' &= w_1 w_2 - \left(\alpha^{(0)} w_2 + \beta^{(0)}\right) w_1^{-1} \end{aligned}$$

(with the derivative denoted by prime), while the second vector field is $\{\cdot, J^{(1)}\}|_{\zeta=0}$ given by

$$\begin{aligned} \dot{w}_1 &= -w_1 w_2 + \left(\alpha^{(0)} w_1 + \beta^{(0)}\right) w_2^{-1}, \\ \dot{w}_2 &= w_1 w_2 - \left(\alpha^{(0)} w_2 + \beta^{(0)}\right) w_1^{-1}, \\ \dot{v}_1 &= \left(-w_2 + \alpha^{(0)} w_2^{-1}\right) v_1 - \left(w_1 + \alpha^{(0)} w_1 w_2^{-2} + \beta^{(0)} w_2^{-2}\right) v_2 + \left(\alpha^{(1)} w_1 + \beta^{(1)}\right) w_2^{-1}, \\ \dot{v}_2 &= \left(w_2 + \alpha^{(0)} w_2 w_1^{-2} + \beta^{(0)} w_1^{-2}\right) v_1 + \left(w_1 - \alpha^{(0)} w_1^{-1}\right) v_2 - \left(\alpha^{(1)} w_2 + \beta^{(1)}\right) w_1^{-1}. \end{aligned}$$

In terms of the original dual coordinates, the latter is

$$\dot{W}_1 = -W_1 W_2 + (\alpha W_1 + \beta) W_2^{-1}, \quad \dot{W}_2 = W_1 W_2 - (\alpha W_2 + \beta) W_1^{-1},$$

which, when $\alpha = \beta = 1$, corresponds to (6.8) (up to an overall minus sign). Observe that the first two components $\dot{w}_1$, $\dot{w}_2$ above are the Hamiltonian vector field produced by $J^{(0)}$ with the bracket $\{w_1, w_2\} = w_1 w_2$, corresponding to the form ω in the (w_1, w_2) plane, but this form becomes degenerate when it is extended to the 4D phase space.

The map φ_{dual} provides a geometric interpretation of the shadow Somos-5 sequences. If we take the additional parameters $\alpha^{(1)} = \beta^{(1)} = 0$, then (6.7) is just the original QRT map together with its linearization, and the points (v_1, v_2) are the shadow in the plane. We can obtain a sequence of such points by using the formula (6.3), where (x_n) is an ordinary Somos-5 sequence and (y_n) is any one of the shadow sequences $(y_n^{(j)})$ for $j = \text{i, ii, iii, iv, v}$, or any linear combination of the latter. Now the first three shadows (that is, $j = \text{i, ii, iii}$) just correspond to the infinitesimal action of the three-dimensional group of scaling symmetries (gauge transformations) for Somos-5, namely (5.8) for arbitrary $A_+, A_-, B \in \mathbb{C}^*$. These scaling symmetries leave w_n invariant, and by extending them to $A_+, A_-, B \in \mathbb{D}^*$, at the level of the linearization in the plane this means that they do not appear: substituting $y_n = y_n^{(\text{i})}, y_n^{(\text{ii})}, y_n^{(\text{iii})}$ or any linear combination thereof into (6.3) gives a trivial shadow sequence in the plane, namely $(v_n, v_{n+1}) = (0, 0)$ for all n. The fourth shadow sequence (or multiples of it) corresponds to a sequence of points on tangent lines of the level curve of the first integral $J^{(0)}$ in the (w_1, w_2) plane: substituting $y_n = y_n^{(\text{iv})}$ into (6.3) gives a sequence of points (v_n, v_{n+1}) that satisfy

$$\left(1 - \frac{\alpha^{(0)}}{w_n^2} - \frac{\beta^{(0)}}{w_n^2 w_{n+1}}\right) v_n + \left(1 - \frac{\alpha^{(0)}}{w_{n+1}^2} - \frac{\beta^{(0)}}{w_n w_{n+1}^2}\right) v_{n+1} = 0, \tag{6.12}$$

which corresponds to $J^{(1)} = 0$ (with $\alpha^{(1)} = 0$, $\beta^{(1)} = 0$ as before). Each generic level curve of $J^{(0)}$ has genus 1, being isomorphic to a Weierstrass elliptic curve (3.2) as in theorem 3.1, and the vector (v_n, v_{n+1}) satisfying (6.12) is tangent to the level curve at the point (w_n, w_{n+1}).

As an example, let us consider the original Somos-5 sequence, for which the points (w_n, w_{n+1}) lie on the curve

$$w_n + w_{n+1} + \frac{1}{w_n} + \frac{1}{w_{n+1}} + \frac{1}{w_n w_{n+1}} = 5 \tag{6.13}$$

(corresponding to $J^{(0)} = 5$), and densely fill a real compact connected component of this curve (an oval). Taking the shadow sequence $y_n^{(\text{iv})}$, as given in table 1, one can obtain a sequence of vectors (v_n, v_{n+1}) satisfying the equation

$$\left(1 - \frac{1}{w_n^2} - \frac{1}{w_n^2 w_{n+1}}\right) v_n + \left(1 - \frac{1}{w_{n+1}^2} - \frac{1}{w_n w_{n+1}^2}\right) v_{n+1} = 0$$

for all n, which provide a tangent vector to the oval at the point (w_n, w_{n+1}), for each n. We have plotted the same orbit of such tangent vectors twice, in figures 1 and 2 (points shown in red), where they can be seen to densely fill a closed curve starting from the point $(v_1, v_2) = (-1, 1)$, which is tangent to the curve (6.13) at $(w_1, w_2) = (1, 1)$.

The fifth shadow sequence corresponds to linear perturbations of the QRT map that are transverse to the level curves of the first integral $J^{(0)}$, involving modular deformations of the underlying elliptic curve: substituting $y_n = y_n^{(\text{v})}$ into (6.3), or taking this shadow sequence with a linear combination of any of the other four, gives points (v_n, v_{n+1}) that produce a non-zero value of the second invariant $J^{(1)}$ in (6.11). For the original Somos-5 sequence, the fifth shadow

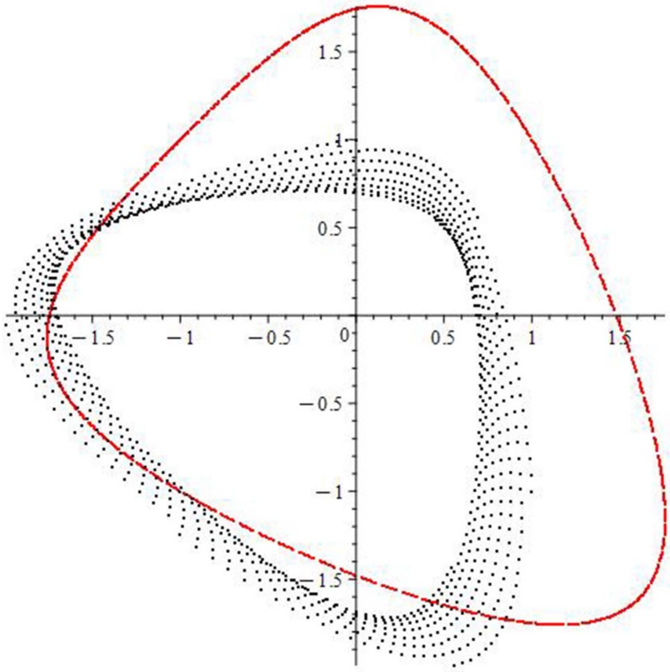

Figure 1. Plot in (v_1, v_2) plane of 1000 points on two different shadow orbits of the map φ_{dual} for $\alpha = \beta = 1$ and $(w_1, w_2) = (1,1)$ with initial values $(v_1, v_2) = (-1,1)$ and $(v_1, v_2) = (0,1)$, respectively. The red points, starting from $(-1,1)$, densely fill a closed curve, while the black points, starting from (0, 1), appear to spiral outwards.

sequence $y_n^{(\mathrm{v})}$ is given in the last column of table 1, and we can use this to generate vectors (v_n, v_{n+1}) that are transverse to the level curve $J_0 = 5$, given by (6.13) above, at each point (w_n, w_{n+1}). Figure 1 shows 1000 points on an orbit starting from $(v_1, v_2) = (0,1)$, which gives $J^{(1)} = -1$, and figure 2 shows the same number of points on an orbit starting from $(v_1, v_2) = (1,1)$, which gives $J^{(1)} = -2$. Both of these orbits including transverse perturbations (with points shown in black) appear to spiral gradually out from the origin.

7. Conclusion

We have shown that the main results on the dual Somos-4 sequences from [11] have exact analogues for dual Somos-5 sequences: analytic solutions in terms of Weierstrass functions with arguments in $\mathbb{D}$; an algebraic form of the general solution for the odd parts using variation of parameters; and exact algebraic expressions in terms of $\mathbb{D}$-valued Hankel determinants. We have also constructed explicit formulae for a complete basis of shadow Somos-5 sequences, in the sense of [27], and showed how to reproduce other examples of dual sequences studied by Ovsienko and Tabachnikov.

The fact that we are able to obtain so many explicit results is due to integrability lurking in the background, which persists when we move from the shadows to dual sequences. From a naive viewpoint, integrability just means that the original system has a certain number of conserved quantities, and these extend in the obvious way when all variables and parameters are replaced by dual numbers. However, there appears to be a much stronger set of properties

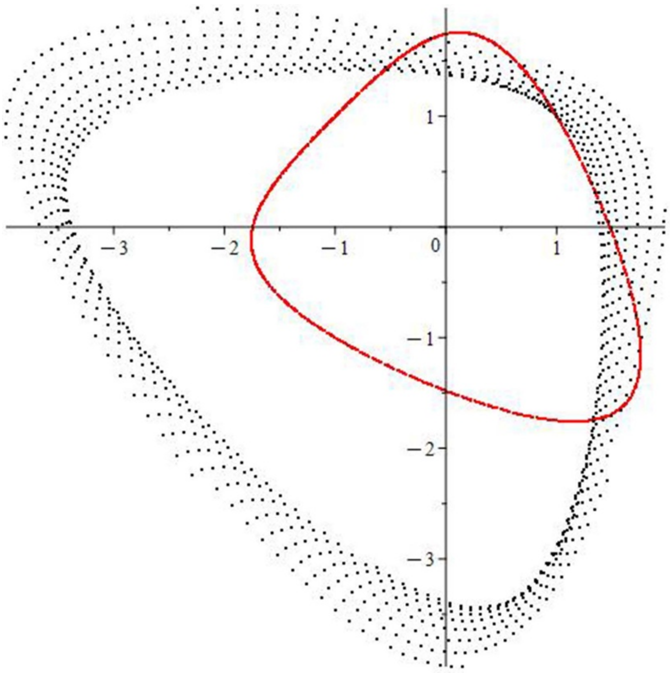

Figure 2. Another plot in the (v_1, v_2) plane, as in figure 1, but now comparing orbits of φ_{dual} with initial values $(v_1, v_2) = (-1, 1)$ (the same red points as before) and $(v_1, v_2) = (1, 1)$ (black points that spiral outwards), respectively.

that is inherited in the context of dual numbers, since we have shown that the symplectic structure for the QRT map associated with Somos-5 extends to give a bi-Hamiltonian structure in the dual setting, and this leads to a proof that the dual QRT map is integrable in the Liouville sense.

Supersymmetric analogues of integrable PDEs and the construction of their associated bi-Hamiltonian structures have been considered for several decades [7, 16, 23], and supersymmetric soliton equations continue to be a subject of current interest [17]. However, some analogues of discrete integrable systems over Grassmann algebras have only been obtained quite recently [1, 2, 15], and there does not appear to be a fully developed theory of Liouville integrability in this context.

Data availability statement

No new data were created or analysed in this study.

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ORCID iD

A N W Hone https://orcid.org/0000-0001-9780-7369

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