3D CHERN-SIMONS THEORIES AND THEIR RELATION TO QUANTUM GROUPS

CUPATITZIO RAMíREZ
Fac. de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Ap. Postal 1364, Puebla, Pue. México

and

LUIS URRUTIA
Departamento de Física
U.A.M.-I, México, D.F. México

(Received: February 24, 1994)

Abstract. The relation of 3D Chern-Simons theories to quantum groups is studied, it turns out that besides the already known quantum group realization for the quantized theory, a similar realization exists for the classical theory. The classical limit of the theory is considered in detail.

1. Introduction

In the last years 3D Chern-Simons theories have been studied due to their multiple applications [1, 2, 3, 4].

As topological field theories, Chern-Simons theorie do not depend on the metric of space-time manifold M.

If A is an algebra valued connexion of the group G on the manifold M, then the Chern-Simons action is given by:

$$I = \frac{k}{4\pi} \int_M d^3x \epsilon^{ijk} Tr\,(A_i \partial_j A_k + A_i A_j A_k) \tag{1}$$

where k is the coupling constant and Tr is the bilineal form of the algebra of the group G.

The action (1) is invariant under spacetime reparametrizations and under gauge transformations it is invariant up to an additive constant given by the winding number of the transforming group element.

The equations of motion following from (1) are gauge covariant:

$$F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j] = 0 \tag{2}$$

Thus, C-S theories will describe only "trivial" motions given by flat connexions.

If the group G is $ISO(2,1)$, then it was shown in [1] that the resulting theory is equivalent to 3D Einstein gravity.

For quantized theories the expectation value of Wilson lines along knotted closed curves will give the corresponding Jones polynomial. Moreover the quantum Hilbert space of a 2D spacelike section punctured by the intersection of the contained Wilson loops, describes the space of conformal blocks of 2D conformal field theories.

An interesting issue is the one of computing the commutator algebra of Wilson loops along spacelike curves [5]. As usual a foliation of $M = R\times\Sigma$ has been chosen. Thus, it is enough to study Wilson loops on Σ. In [5], $ISO(2,1)$ has been considerated in some detail; this study has been further pursued in [6] for the spinorial representations of $SO(3,1)$ and $SO(2,2)$ where after quantization the resulting algebra has been identified with $SL(2)_q$. Further, the Wilson loop algebra of Poincaré and conformal groups [7], and for de Sitter supergravity [8] have been calculated with similar results. Generalizations for $g > 1$ have been pursued in [9].

In this contribution $SU(2)$ C-S theory is considerated. In Sec. 2 it is shown that although the Poisson bracket algebra of integrated connexions is of braid type, the Jacobi identities are trivially satisfied. In Sec. 3 it is shown that the Poisson bracket algebra of traces, i.e. Wilson loops, has the structure of $SL(2)_q$. In Sec. 4 different quantization schemes are discussed. Conclusions are drawn in Sec. 5.

2. Quantum Symmetry of Classical Chern-Simons Theory

If $\gamma : R \to \Sigma$ is a noncontractible closed curve on Σ, then an integrated connexion

$$\Psi(\gamma) = Pe^{\int_\gamma A dx} \tag{3}$$

is a solution of the differential equation [5]:

$$\frac{d\Psi}{dt} = A_s\Psi \tag{4}$$

where A_s is the connexion tangent to γ at s. From the action (1) the canonical Poisson bracket relations can be derived:

$$\left[A_{a\alpha}(t,\mathbf{x}), A^b_\beta(t,\mathbf{x}')\right] = \frac{2\pi}{k}\epsilon_{\alpha\beta}\delta^a_b\delta^2(\mathbf{x}-\mathbf{x}') \tag{5}$$

where $\alpha,\beta = 1,2$ and a corresponds to the adjoint representation of G.

In order to compute the Poisson brackets of integrated connexions let us consider two crossing closed curves γ and σ [5, 6]. We take them as independent nontrivial homotopy classes, e.g., the cycles of a thorus. Both curves are decomposed into three pieces, the central one being in the neighborhood of the crossing point. (fig. 1).

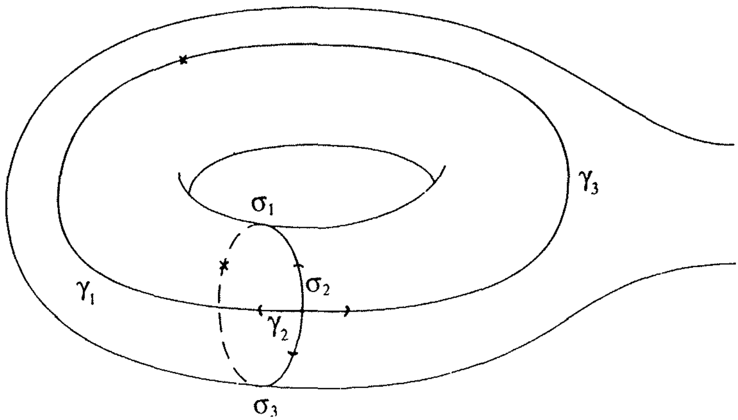

Figure 1

Thus $\Psi(\gamma) = \Psi(\gamma_3)\Psi(\gamma_2)\Psi(\gamma_1)$ and $\Psi(\sigma) = \Psi(\sigma_3)\Psi(\sigma_2)\Psi(\sigma_1)$. Taking (5) into account we obtain:

$$\begin{gathered}\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = \\ \Psi_1(\gamma_3)\Psi_2(\sigma_3)\,\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{PB}\,\Psi_1(\gamma_1)\Psi_2(\sigma_1)\end{gathered} \tag{6}$$

where, as usual, the notations are:

$$\Psi_1 = \Psi \otimes 1 \;\; and \;\; \Psi_2 = 1 \otimes \Psi \tag{7}$$

Further, we have:

$$\begin{aligned}\Psi(\gamma_2) &= 1 + \int_{s_o-\epsilon}^{s_o+\epsilon} ds\, A_\alpha\,[\mathbf{x}(s)]\, x'^\alpha(s) + \mathcal{O}(\epsilon^2) \\ \Psi(\sigma_2) &= 1 + \int_{u_o-\epsilon}^{u_o+\epsilon} du\, A_\alpha\,[\mathbf{x}(u)]\, x'^\alpha(u) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{8}$$

thus

$$\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{PB} = -\frac{2\pi}{k} s(\gamma, \sigma)\,(T^a \otimes T_a) \tag{9}$$

where $s(\gamma, \sigma) = \pm 1$ is the signature of the relative orientation of γ and σ.

Therefore, in the limit $\epsilon \to 0$:

$$\begin{gathered}\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = \\ -\tfrac{2\pi}{k} s(\gamma,\sigma)\Psi_1(\gamma_3)\Psi_2(\sigma_3)\,(T^a \otimes T_a)\,\Psi_1(\gamma_1)\Psi_2(\sigma_1)\end{gathered} \tag{10}$$

If we restrict ourselves to curves γ and σ with a common base point, the algebra (2.8) can be put in a closed form, so that for example in the limit $\epsilon \to 0$ we have $\Psi(\gamma_3) = \Psi(\sigma_3)$, then we can fix the gauge in such a way that we obtain the braid-like algebra:

$$\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = r_{12}(\gamma, \sigma)\Psi_1(\gamma)\Psi_2(\sigma) \tag{11}$$

where

$$r_{12}(\gamma,\sigma) = -\frac{2\pi}{k}s(\gamma,\sigma)(T^a \otimes T_a) \tag{12}$$

which obviously does not satisfy the classical Yang-Baxter equation. In fact, in order to satisfy the Jacobi identities of (11), we need three different, but equally based elements, say $\Psi(\gamma)$, $\Psi(\sigma)$ and $\Psi(\sigma')$ as in fig.1.2 (the fact that γ, σ and σ' are equally based is not explicitly shown). The point is that the gauge (11) cannot be implemented simultaneously for all possible brackets, for each of these brackets we must do separatedly a partition of the curves. fig. 1.1.

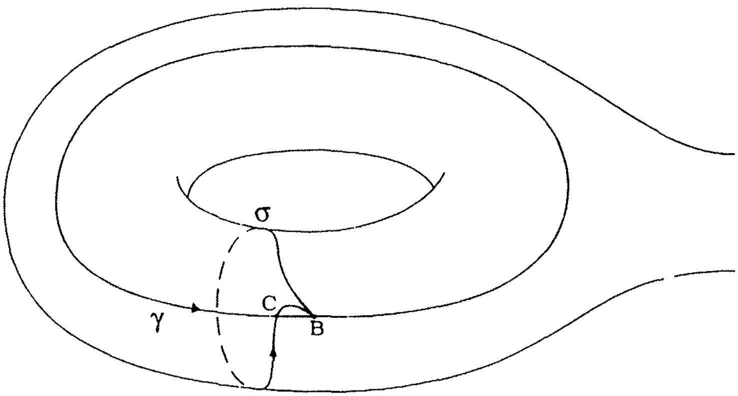

Figure 2

Taking that into account, it is easy to show that:

$$\begin{aligned}&\{\{\Psi_1(\gamma),\Psi_2(\sigma)\}_{PB},\Psi_3(\sigma')\}_{PB} + \\ &\{\{\Psi_1(\sigma),\Psi_2(\sigma')\}_{PB},\Psi_3(\gamma)\}_{PB} + \\ &\{\{\Psi_3(\sigma'),\Psi_1(\gamma)\}_{PB},\Psi_2(\sigma)\}_{PB} \equiv 0\end{aligned} \tag{13}$$

where the second term vanishes identically due to the fact that

$$\{\Psi_1(\sigma),\Psi_2(\sigma')\}_{PB} = 0 \tag{14}$$

Now we consider the Poisson bracket algebra of traces of integrated connexions (Wilson loops) for $SU(2)$:

$$C(\gamma) = Tr(\gamma) \tag{15}$$

$$\{C(\gamma),C(\sigma)\}_{PB} = -\frac{2\pi}{k}s(\gamma,\sigma)Tr\left[T^a\Psi(\gamma)\right]Tr\left[T_a\Psi(\sigma)\right] \tag{16}$$

where the Casimir element is given by:

$$T^a{}_{m'_1}{}^{m_1}T_{am'_2}{}^{m_2} = -\frac{1}{4}\delta_{m'_1}{}^{m_1}\delta_{m'_2}{}^{m_2} + \frac{1}{2}\delta_{m'_1}{}^{m_2}\delta_{m'_2}{}^{m_1} \tag{17}$$

resulting the algebra [5, 6]:

$$\{C(\gamma), C(\sigma)\}_{PB} = \frac{\pi}{k} s(\gamma,\sigma) \left[-\frac{1}{2} C(\gamma)C(\sigma) + C(\gamma\sigma) \right] \tag{18}$$

which closes due to the trace identities for 2x2 matrices [5, 6]:

$$Tr(AB) = Tr(A)Tr(B) - \det A\, Tr(A^{-1}B) \tag{19}$$

In our case the determinant is one and we have:

$$\begin{aligned} C(\gamma^2\sigma) &= -C(\sigma) + C(\gamma)C(\gamma\sigma) \\ C(\sigma^2\gamma) &= -C(\gamma) + C(\sigma)C(\gamma\sigma) \end{aligned} \tag{20}$$

and so on. Thus, the only independent generators are: $X_1 = C(\gamma)$, $X_2 = C(\sigma)$ and $X_3 = C(\gamma\sigma)$ with the resulting algebra [6]:

$$\{X_i, X_j\}_{PB} = \frac{\pi}{k} \left(\epsilon_{ij} X_i X_j + \epsilon_{ijk} X_k \right) \tag{21}$$

where $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$ and ϵ_{ijk} is the 3D Levi-Civita symbol. Relations similar to (21) arise for the monodromies of groups elements of $SU(2)$ WZW model in [10] where the resulting algebra has been interpreted as the semiclassical version of $SL(2)_q$. In the following we will show that in fact they constitute an exact representation of $SL(2)_q$.

Indeed, if we do the nonlinear reparametrization similar to the one used for the quantized theory in [6], see also e.g. [11]:

$$K^{\pm} = X_1 \pm i X_2 e^{\mp \frac{\pi}{k} H} \tag{22}$$

$$X_3 = \frac{i}{2} \left(e^{-\frac{\pi}{k} H} - e^{\frac{\pi}{k} H} \right) \tag{23}$$

We obtain:

$$\begin{aligned} \{K^+, K^-\}_{PB} &= \frac{\pi}{k} \left(e^{-\frac{\pi}{k} H} - e^{\frac{\pi}{k} H} \right) \\ \{H, K^{\pm}\}_{PB} &= \pm \frac{\pi}{k} K^{\pm} \end{aligned} \tag{24}$$

where the deformation parameter is given by:

$$q = e^{-\frac{\pi}{k}} \tag{25}$$

3. Quantization

In this section, we wish to obtain the algebra corresponding to (2.22) in the quantized theory. Due to the lack of regularization criteria like operator ordering, quantization of Chern-Simons theories imply a certain degree of arbitrariness.

In our case the indicated thing to do is canonical quantization. However, it would imply complicated operator manipulations to achieve (21). Instead of it we choose a way similar to [6]. We start from the following naive ansatz:

$$[X_i, X_j] = i\hbar\frac{\pi}{k}\left[\epsilon_{ij}\mathcal{O}(X_iX_j) + \epsilon_{ijk}X_k\right] \tag{26}$$

where

$$\mathcal{O}(X_iX_j) = \begin{cases} X_iX_j, & \epsilon_{ij} = 1 \\ X_jX_i, & \epsilon_{ij} = -1 \end{cases} \tag{27}$$

takes into account the nonconmutativity of X_i and X_j. In this case the nonlinear reparametrization is given by:

$$\begin{aligned} K^{\pm} &= \sqrt{1 - i\frac{\pi\hbar}{k}}X_1 \pm iX_2e^{\pm\mu} \\ X_3 &= i\frac{\sqrt{1 - i\frac{\pi\hbar}{k}}}{2 - i\frac{\pi\hbar}{k}}\left(e^{\mu} - e^{-\mu}\right) \end{aligned} \tag{28}$$

with the resulting algebra:

$$[K^+, K^-] = \frac{i\pi\hbar}{2k - i\pi\hbar}\left(e^{\mu} - e^{-\mu}\right) \tag{29}$$

$$[\mu, K^{\pm}] = \pm\ln\left(1 - i\frac{\pi\hbar}{k}\right)K^{\pm} \tag{30}$$

so that after some rescalings the canonical form turns out to be:

$$[K^+, K^-] = \frac{\left(q^{2H} - q^{-2H}\right)}{q^2 - q^{-2}} \tag{31}$$

$$[H, K^{\pm}] = \pm i\hbar K^{\pm} \tag{32}$$

where the quantized deformation parameter is given by:

$$q = \left(1 - i\frac{\pi\hbar}{k}\right)^{\frac{1}{i\hbar}} \tag{33}$$

such that the limit $\hbar \to 0$

$$\lim_{\hbar\to 0} q = \lim_{\hbar\to 0}\left[\left(1 - i\frac{\pi\hbar}{k}\right)^{-\frac{k}{i\pi\hbar}}\right]^{-\frac{\pi}{k}} = e^{-\frac{\pi}{k}} \tag{34}$$

gives the deformation parameter of the classical theory.

Now, we wish to make an ansatz of regularization for the operator product on the r.h.s. of (26) as follows:

$$X_i X_j \to \frac{1}{1+a}\left(X_i X_j + a X_j X_i\right) \tag{35}$$

Therefore

$$[X_1, X_2] = i\hbar u\left[\frac{1}{1+a}\left(X_1X_2 + aX_2X_1\right) + X_3\right] \tag{36}$$

hence

$$[X_1, X_2] = \frac{i\hbar u}{1 + ia\frac{\hbar u}{1+a}}\left(X_1X_2 + X_3\right) = i\hbar\tilde{u}\left(X_1X_2 + X_3\right) \tag{37}$$

which has the same form as (26).

Therefore the deformation parameter will be:

$$q = (1 - i\hbar\tilde{u})^{\frac{1}{i\hbar}} = \left(\frac{1 - i\frac{\pi\hbar/k}{1+a}}{1 + ia\frac{\pi\hbar/k}{1+a}}\right)^{\frac{1}{i\hbar}} \tag{38}$$

where we substituted $u \to \pi/k$.

It is interesting to expand (38) in power series on $\hbar$.

We obtain:

$$q = e^{-\frac{\pi}{k}} e^{-\sum_{n=2}^{\infty}\frac{1}{n}(i\hbar)^{n-1}\left(\frac{\pi/k}{1+a}\right)^n[1-(-a)^n]} \tag{39}$$

For example, if we take a symmetric ordering, i.e. $a = 1$, only even powers of $\hbar$ will survive and the deformation parameter will be real:

$$q_s = e^{-\frac{\pi}{k}} e^{-2\sum_{m=1}^{\infty}\frac{(-1)^m}{2m+1}\hbar^{2m}\left(\frac{\pi/k}{2a}\right)^{2m+1}} = e^{-\frac{\pi}{k}}\left[1 + \mathcal{O}\left(\hbar^2\right)\right] \tag{40}$$

Our results are based on a heuristic quantization of the trace algebra (11). Nevertheless, the resulting deformation parameter is consistent as far as the classical limit ($\hbar \to 0$) concerns.

It would be interesting to quantize (11) instead of (21). However in this case the noncommutativity of the operators leads to considerably complications, for example the trace identity (19) is not fulfiled anymore. Work is in progress in this direction.

Acknowledgements

This work was relized under partial support of CONACYT grants 1178 E9202. C. Ramírez would like to thank S. Theissen and P. Hess for useful discussıons. L. Urrutia would like to thank the High Energy Group of U.A.P. for his hospitality.

References

[1] Witten E., *Nucl. Phys.* **B311** (1988) 46.
[2] Witten E., *Nucl. Phys.* **B323** (1989) 113.
[3] Witten E., *Comm. Math. Phys.* **121** (1989) 351; S. Carlip, Nucl. Phys. **B324** (1989) 106.
[4] Verlinde H., *Nucl. Phys.* **B337** (1990) 652.
[5] Nelson J.E. and Regge T., *Nucl. Phys.* **B328** (1989) 190.
[6] Nelson J.E., Regge T. and Zertuche F., *Nucl. Phys.* **B339** (1990) 516.
[7] Zertuche F. and Urrutia L., SILARG VII.
[8] Urrutia L., Walbroeck H. and Zertuche F.,...
[9] Nelson J.E. and Regge T., *Comm. Math. Phys.* **155** (1993) 561.
[10] Gawędzki K., *Comm. Math. Phys.* **139** (1991) 201.
[11] Curtright T.L. and Zachos C.K., *Phys. Lett.* **B243** (1990) 237.
[12] Fadeev L.D., Reshetikin N.Yu. and Takhtajan L.A., *Algebra i Anal.* **1** (1989) 178 (in russian);
A. Alekseev, L. Fadeev and M. Semenov-Tian-Shansky, *Comm. Math. Phys.* **149** (1992) 335.