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RESEARCH ARTICLE

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Lemaitre–Tolman–Bondi model and accelerating expansion

Abstract I discuss the spherically symmetric but inhomogeneous Lemaitre–Tolman–Bondi (LTB) metric, which provides an exact toy model for an inhomogeneous universe. Since we observe light rays from the past light cone, not the expansion of the universe, spatial variation in matter density and Hubble rate can have the same effect on redshift as acceleration in a perfectly homogeneous universe. As a consequence, a simple spatial variation in the Hubble rate can account for the distant supernova data in a dust universe without any dark energy. I also review various attempts towards a semirealistic description of the universe based on the LTB model.

Keywords Dark energy, Supernovae, Cosmology, Gravitation

1 Introduction

The simplest homogeneous and isotropic cosmological models, based on the Friedmann–Robertson–Walker (FRW) metric, have proved to be remarkably successful ever since Edwin Hubble in 1929 rather cautiously suggested that the apparent linear correlation between the observed redshifts and distances of 24 galaxies could hint towards the possibility "that the velocity-distance relation may represent the de Sitter effect, and hence that numerical data may be introduced into discussions of the general curvature of space" [1]. Indeed, numerical data now guides the development of cosmology, which has become a precision science, albeit mostly within the framework of a perfectly homogenous background metric.

The FRW universe is characterized by two functions, the Hubble rate H and the density parameter Ω, or the average expansion rate and the average density of mass energy, respectively, which depend on time but are independent of the spatial

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location. However, one should keep in mind that their values cannot be extracted directly from the observations but must be deduced from the properties of light coming from the past light cone. In the context of the FRW model this is almost trivial, since the redshift z and scale factor $a(t)$ are everywhere related by $z = a(t_o)/a(t_e) - 1$, where the subscripts refer, respectively, to the observation and the emission of light. This theoretical simplicity should however not cloud the fact that all cosmological parameter determination requires an element of interpretation of the data. Of course, the FRW interpretation of the properties of the past light cone has served cosmology well, giving a good fit to observations and, until the late 90's, implying a matter dominated universe with $\Omega \approx \Omega_M$.

The situation changed dramatically with the WMAP [2] and distant supernova data [3; 4]. Considering the recent data from supernovae [5; 6], galaxy distributions [7] and anisotropies of the cosmic microwave background [8], the simplest FRW model would now lead to a highly contradictory picture of the universe, with the following best fit values for the average matter density:

- Cosmic microwave background: $\Omega_M \sim 1$
- Galaxy surveys: $\Omega_M \sim 0.3$
- Type Ia supernovae: $\Omega_M \sim 0$

As is well known, the glaring discrepancies between the different data sets have conventionally been remedied by introducing the cosmological constant Λ or vacuum energy Ω_Λ to the Einstein equations. This gives rise to an accelerated expansion of the universe. As a consequence, the apparent dimming of the luminosity of distant supernovae finds, in the context of perfectly homogeneous universe, a natural explanation.[1]

However, although the cosmological concordance ΛCDM-model [9] fits the observations well, there is no theoretical understanding of the origin of the cosmological constant or its magnitude. For particles physicists, who have spent a long time trying to prove that the cosmological constant must be zero, the tremendously small cosmological constant which just now happens to start to dominate the energy budget of the universe, is a theoretical nightmare. There exist a large number of different dark energy models (see e.g., [9; 10]) that attempt to provide a dynamical explanation for the cosmological constant, but none of them are compelling from particle physics point of view; moreover, very often they require fine-tuning. Modifications of the general theory of relativity on cosmological scales appear to suffer from analogous problems. For instance, $f(R)$ gravity theories [12] in the metric formalism are plagued by instabilities [13] while in the Palatini approach the cosmological constant seems to be essentially the only consistent modification that fits all the cosmological data [14].

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[1] Even if the primordial perturbation is not scale free, the combination of the CMB fluctuations and the shape of the correlation function up to $\sim 100h^{-1}$Mpc, seems to require dark energy for a homogeneous FRW model [11].

Facing such difficulties, one might be tempted to consider relinquishing the FRW assumption of the perfect homogeneity of the universe. After all, inhomogeneities are abundant in the universe: there are not only clusters of galaxies but also large voids. Because general relativity is a non-linear theory, even relatively small local inhomogeneities with a sufficiently large density contrast could in principle give rise to cosmological evolution that is not accessed by the usual cosmological perturbation theory in an FRW background. In fact, the potentially interesting consequences of the inhomogeneities were recognized already at the time when the homogeneous and isotropic models of the universe were first studied, but their impact on the global dynamics of the universe is still largely unknown (see e.g., [15]). Then the question arises: could the acceleration of the universe be just a trick of light, a misinterpretation that arises due to the oversimplification of the real, inhomogeneous universe inherent in the FRW model? Light, while traveling though inhomogeneities, does not see the average Hubble expansion but rather feels its variations, which could sum up to an important correction.[2] This effect is particularly important for the case of large scale inhomogeneities which will be the focus of the present paper. If the local Hubble expansion rate were to vary smoothly at scales of the order of, say, thousand megaparsecs, that would very much change our interpretation of the distant supernova redshifts. In such an inhomogeneous universe we could also just happen to be located in a special position. For instance, fate could have relegated us to an underdense region with a larger than average local Hubble parameter so that the discrepancy between nearby and distant supernovae luminosities could be resolved without dark energy.

Local inhomogeneities have recently been invoked as the culprit for the apparent acceleration of the expansion of the universe,[3] in particular by virtue of their so-called backreaction on the metric (for a discussion on the issues involved and a comprehensive list of references, see [18; 19]). One constructs an effective description of the universe by averaging out the inhomogeneities to obtain averaged, effective Einstein equations which, in addition to the terms found in the usual homogeneous case, include new terms that represent the effect of the inhomogeneities [20; 21; 22].

However, since we can only observe the redshift and energy flux of light arriving from a given source, not the expansion rate or the matter density of the universe nor their averages, one may wonder what are the actual observables related to the averaged equations. To wit, since we do not observe the average expansion of the universe directly, its average acceleration is also an indirect conclusion, arising from the fact that *in the perfectly homogeneous cosmological models* dark energy is required for a good fit. Consequently, there is no a priori reason to assume that an accelerated expansion is necessarily required to fit the data if one assumes a general inhomogeneous model of the universe. One may also add that the averaging procedure as such is not without problems: in general it is not correct to integrate out constrained degrees of freedom as if they were independent, and in cosmology the fact that we can make observations only along our past light cone makes the observable universe a constrained system. Hence it would be desirable

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[2] For a recent calculation of the small scale inhomogeneity-induced correction to the cosmological constant that one would infer from an analysis of the luminosities and redshifts of Type Ia supernovae, assuming a homogeneous universe, see [16].

[3] Inhomogeneities as an alternative to dark energy were first discussed in [17].

to study the effects of the the inhomogeneities on the directly observable light in an exact cosmological model. Unfortunately, in the presence of generic inhomogeneities this would be practically an impossible task. Instead, one must resort to toy models, the simplest of which is the spherically symmetric but inhomogeneous Lemaitre–Tolman–Bondi (LTB) model [23; 24; 25].

The great virtue of the LTB model is that it is exact. Because of its high degree of symmetry, it may not be realistic as such, but the LTB model is nevertheless interesting at least on two counts. First, it serves as a simple testing ground for the effects of inhomogeneities when fitting the cosmological data without dark energy. Second, since the fits can be performed unambiguously, the nature of the effective acceleration in the models where the spatial degrees of freedom have been averaged out, can be made transparent by comparing the averaged and "exact" models.

Of course, one can also take the LTB model more seriously. For instance, one may use the LTB metric to describe a local underdense bubble in FRW universe, for which there is some evidence both from supernova [26] and galaxy data [27]. First attempts along these directions [28; 29; 30] assumed an underdense region separated from the outside homogeneous FRW universe by a singular mass shell, followed by investigations of more realistic models with a continuous transition between the inner underdensity and the outer homogeneous universe (see e.g., [31; 32]). More complicated situations, including off-centered observers, can also be addressed, as will be discussed in Sect. 4.

2 The LTB metric

Let us consider a spherically symmetric dust universe with radial inhomogeneities as seen from our location at the center. Choosing spatial coordinates to comove $(dx^i/dt = 0)$ with the matter, the spatial origin $(x^i = 0)$ as the symmetry center, and the time coordinate $(x^0 \equiv t)$ to measure the proper time of the comoving fluid, the line element takes the general form [23; 24; 25]

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + A^2(r,t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (2.1)$$

where the functions $A(r,t)$ and $X(r,t)$ have both temporal and spatial dependence. The homogeneous FRW-metric is a special case and is obtained by letting

$$X(r,t) \rightarrow \frac{a(t)}{\sqrt{1 - kr^2}}, \quad A(r,t) \rightarrow a(t)r. \qquad (2.2)$$

The energy momentum tensor is given by

$$T^\mu_{\ \nu} = -\rho_M(r,t)\delta^\mu_0 \delta^0_\nu - \rho_\Lambda \delta^\mu_{\ \nu}, \qquad (2.3)$$

where $\rho_M(r,t)$ is the matter density, $u^\mu = \delta^\mu_0$ represent the components of the 4-velocity-field of the fluid, and we have kept the vacuum energy ρ_Λ for generality. Note that although the fluid is staying at fixed spatial coordinates, it can physically

move in the radial direction. Plugging Eq. (2.1) into the Einstein equation, $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$, one finds the set of equations

$$-2\frac{A''}{AX^2} + 2\frac{A'X'}{AX^3} + 2\frac{\dot{X}\dot{A}}{AX} + \frac{1}{A^2} + \left(\frac{\dot{A}}{A}\right)^2 - \left(\frac{A'}{AX}\right)^2 = 8\pi G(\rho_M + \rho_\Lambda), \qquad (2.4)$$

$$\dot{A}' = A'\frac{\dot{X}}{X}, \qquad (2.5)$$

$$2\frac{\ddot{A}}{A} + \frac{1}{A^2} + \left(\frac{\dot{A}}{A}\right)^2 - \left(\frac{A'}{AX}\right)^2 = 8\pi G\rho_\Lambda, \qquad (2.6)$$

and

$$-\frac{A''}{AX^2} + \frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\frac{\dot{X}}{X} + \frac{A'X'}{AX^3} + \frac{\ddot{X}}{X} = 8\pi G\rho_\Lambda. \qquad (2.7)$$

These contain only three independent differential equations, and we may solve $\dot{X}$ and $\ddot{X}$ from Eq. (3) and A'^2 and A'' from Eq. (**??**). Then one can substitute these into Eq. (2.7) and find that it yields an identity. Thus only two of Eqs. (3)–(6) are independent. One can easily solve Eq. (3) to obtain

$$X(r,t) = C(r)A'(r,t), \qquad (2.8)$$

where the function $C(r)$ depends only on the coordinate r. By redefining $C(r) \equiv 1/\sqrt{1-k(r)}$, where $k(r) < 1$, we can thus write the LTB metric Eq. (2.1) in its usual form:

$$ds^2 = -dt^2 + \frac{(A'(r,t))^2}{1-k(r)}dr^2 + A^2(r,t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (2.9)$$

where $k(r)$ is a function associated with the curvature of $t = $ const. hypersurfaces. The FRW metric is the limit $A(r,t) \to a(t)r$ and $k(r) \to kr^2$.

The two independent equations are given by

$$\frac{\dot{A}^2 + k(r)}{A^2} + \frac{2\dot{A}\dot{A}' + k'(r)}{AA'} = 8\pi G(\rho_M + \rho_\Lambda), \qquad (2.10)$$

$$\dot{A}^2 + 2A\ddot{A} + k(r) = 8\pi G\rho_\Lambda A^2. \qquad (2.11)$$

The first integral of Eq. (2.11) is

$$\frac{\dot{A}^2}{A^2} = \frac{F(r)}{A^3} + \frac{8\pi G}{3}\rho_\Lambda - \frac{k(r)}{A^2}, \qquad (2.12)$$

where $F(r)$ is a non-negative function that, like $k(r)$, is fixed by the boundary condition. Substituting Eq. (2.12) into Eq. (2.10) yields

$$\frac{F'}{A'A^2} = 8\pi G\rho_M. \qquad (2.13)$$

By combining Eqs. (2.10) and (2.11) we can construct the generalized acceleration equation

$$\frac{2}{3}\frac{\ddot{A}}{A} + \frac{1}{3}\frac{\ddot{A}'}{A'} = -\frac{4\pi G}{3}(\rho_M - 2\rho_\Lambda) \qquad (2.14)$$

which implies that the total acceleration, represented by the left hand side, is negative everywhere unless the vacuum energy is large enough: $\rho_\Lambda > \rho_M/2$. However, it does not exclude the possibility of having radial acceleration ($\ddot{A}'(r,t) > 0$), even in the pure dust universe, if the angular scale factor $A(r,t)$ is decelerating fast enough, and vice versa. This serves to demonstrate how the very notion of the acceleration becomes ambiguous in the presence of the inhomogeneities [33].

The boundary condition functions $F(r)$ and $k(r)$ are specified by the exact physical nature of the inhomogeneities. Their relation to the FRW model parameters can be recognized by comparing Eq. (2.12) with the Einstein equation for the homogeneous FRW-model

$$H^2(t) \equiv \frac{\dot{a}(t)}{a(t)} = \frac{8\pi G}{3}(\rho_M + \rho_\Lambda) - \frac{k}{a^2} \tag{2.15}$$

$$= H_0^2 \left[\Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + (1 - \Omega_\Lambda - \Omega_M)\left(\frac{a_0}{a}\right)^2 \right], \tag{2.16}$$

where $a_0 \equiv a(t_0)$ and $H_0 \equiv H(t_0)$. Thus, a comparison between Eqs. (2.12) and (2.15) motivates one to define the local Hubble rate as

$$H(r,t) \equiv \frac{\dot{A}(r,t)}{A(r,t)}. \tag{2.17}$$

The local matter density can be defined through

$$F(r) \equiv H_0^2(r)\Omega_M(r)A_0^3(r), \tag{2.18}$$

with

$$k(r) \equiv H_0^2(r)(\Omega_M(r) + \Omega_\Lambda(r) - 1)A_0^2(r), \tag{2.19}$$

where we have defined the boundary values at t_0 through $A_0(r) \equiv A(r,t_0)$, $H_0(r) \equiv H(r,t_0)$, and $\Omega_\Lambda(r) \equiv 8\pi G\rho_\Lambda/3H_0^2(r)$. With these definitions, the position-dependent Hubble rate, Eq. (2.12), takes a physically transparent form [34]:

$$H^2(r,t) = H_0^2(r) \left[\Omega_M(r)\left(\frac{A_0}{A}\right)^3 + \Omega_\Lambda(r) + \Omega_c(r)\left(\frac{A_0}{A}\right)^2 \right], \tag{2.20}$$

where $\Omega_c(r) \equiv 1 - \Omega_\Lambda(r) - \Omega_M(r)$.

The difference between the conventional Friedmann equation (2.15) and its LTB generalization, Eq. (2.20), is that all the quantities in the LTB case depend on the r-coordinate. Thus in the presence of inhomogeneities, the values of the Hubble rate and the matter density can vary at every spatial point so that the inhomogeneous dust models are defined by two functions of the spatial coordinates: $H_0(x^i)$ and $\Omega_M(x^i)$. As a consequence, the inhomogeneities are of two physically different kinds: inhomogeneities in the matter distribution, and inhomogeneities in the expansion rate. Although their dynamics are coupled via the Einstein equation, as boundary conditions they are independent. The universe could have an inhomogeneous big bang, where the universe came into being at different times at different points, and/or an inhomogeneous matter density. This opens up the possibility for an inhomogeneous universe that has a homogeneous present-day Ω_M; a model of this kind could potentially fit the supernova data as well as the galaxy

surveys without invoking dark energy. However, if $\Omega_M(r) = $ const., the physical matter distribution ρ_M itself has a spatial dependence provided $H_0(r) \neq$ const.. It can be made constant by choosing $\Omega_M(r)H_0^2(r) = $ const.

The spatial dependence holds true even for the gauge freedom of the scale function. In the FRW case the present value of the scale factor $a(t_0)$ can be chosen to be any positive number. Similarly, the corresponding present-day scale function $A(r,t_0)$ of the LTB model can be chosen to be any smooth and invertible positive function. In what follows we will choose the conventional gauge

$$A(r,t_0) = r. \tag{2.21}$$

Integrating Eq. (2.20) then gives the relation between the scale factor $A(r,t)$ and the coordinates r and t, which can also be used to find the age of the LTB universe. One finds

$$t_0 - t = \frac{1}{H_0(r)} \int_{\frac{A(r,t)}{A_0(r)}}^{1} \frac{dx}{\sqrt{\Omega_M(r)x^{-1} + \Omega_\Lambda(r)x^2 + \Omega_c(r)}}. \tag{2.22}$$

For any space-time point with coordinates (t, r, θ, φ), Eq. (2.22) determines the function $A(r,t)$ and all its derivatives. Thus the metric Eq. (2.9) is specified, and given the inhomogeneities, all the observable quantities can be computed. Equation (2.22) can be integrated in terms of elementary functions when $\Omega_\Lambda(r) = 0$ or $\Omega_\Lambda(r) + \Omega_M(r) = 1$; as an example, in the latter case one finds

$$(t - t_0)H_0 = \frac{2}{3\sqrt{1 - \Omega_M(r)}} \left[\text{arsinh}\sqrt{\omega(r)\left(\frac{A(r,t)}{A_0(r)}\right)^3} - \text{arsinh}\sqrt{\omega(r)} \right], \tag{2.23}$$

where

$$\omega(r) = \frac{1 - \Omega_M(r)}{\Omega_M(r)}. \tag{2.24}$$

In this particular case $A(r,t)$ can be found explicitly as

$$A(r,t) = A_0(r) \left[\cosh(\tau) + \sqrt{\frac{3}{8\pi G\rho_\Lambda}} H_0(r)\sinh(\tau) \right], \tag{2.25}$$

where $\tau = \sqrt{6\pi G\rho_\Lambda}(t - t_0)$.

3 Inhomogeneities and luminosity distance

To compare the inhomogeneous LTB model e.g., with the supernova observations, we need an equation that relates the redshift and energy flux of light with the exact nature of the inhomogeneities. For this, one must study propagation of light in the LTB universe.[4] Let us here derive the appropriate equations for notational clarity; a more general derivation for an off-center observer can be found in [36].

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[4] Luminosity distance in a perturbed FRW universe has been considered in [35].

From the symmetry of the situation, it is clear that light can travel radially, that is, there exist geodesics with $d\theta = d\varphi = 0$. Moreover, since light always travels along null geodesics, we have $ds^2 = 0$. Inserting these conditions into the equation for the line element, Eq. (2.9), we obtain the constraint equation for light rays

$$\frac{dt}{du} = -\frac{dr}{du}\frac{A'(r,t)}{\sqrt{1-k(r)}}, \tag{3.1}$$

where u is a curve parameter, and the minus sign indicates that we are studying radially *incoming* light rays.

Consider two light rays with solutions to Eq. (3.1) given by $t_1 = t(u)$ and $t_2 = t(u) + \lambda(u)$. Inserting these to Eq. (3.1) we obtain

$$\frac{d}{du}t_1 = \frac{dt(u)}{du} = -\frac{dr}{du}\frac{A'(r,t)}{\sqrt{1-k(r)}} \tag{3.2}$$

$$\frac{d}{du}t_2 = \frac{dt(u)}{du} + \frac{d\lambda(u)}{du} = -\frac{dr}{du}\frac{A'(r,t)}{\sqrt{1-k(r)}} + \frac{d\lambda(u)}{du} \tag{3.3}$$

$$\frac{d}{du}t_2 = -\frac{dr}{du}\frac{A'(r,t(u)+\lambda(u))}{\sqrt{1-k(r)}} = -\frac{dr}{du}\frac{A'(r,t)+\dot{A}'(r,t)\lambda(u)}{\sqrt{1-k(r)}}, \tag{3.4}$$

where Taylor expansion has been used in the last step and only terms linear in $\lambda(u)$ have been kept. Combining the right hand sides of Eqs. (3.3) and (3.4) gives the equality

$$\frac{d\lambda(u)}{du} = -\frac{dr}{du}\frac{\dot{A}'(r,t)\lambda(u)}{\sqrt{1-k(r)}}. \tag{3.5}$$

Differentiating the definition of the redshift, $z \equiv (\lambda(0) - \lambda(u))/\lambda(u)$, we obtain

$$\frac{dz}{du} = -\frac{d\lambda(u)}{du}\frac{\lambda(0)}{\lambda^2(u)} = \frac{dr}{du}\frac{(1+z)\dot{A}'(r,t)}{\sqrt{1-k(r)}}, \tag{3.6}$$

where in the last step we have used Eq. (3.5) and the definition of the redshift. Finally, we can combine Eqs. (2.19), (3.1) and (3.6) to obtain the pair of differential equations

$$\frac{dt}{dz} = \frac{-A'(r,t)}{(1+z)\dot{A}'(r,t)}, \tag{3.7}$$

$$\frac{dr}{dz} = \frac{\sqrt{1 + H_0^2(r)(1-\Omega_M(r)-\Omega_\Lambda(r))A_0^2(r)}}{(1+z)\dot{A}'(r,t)}, \tag{3.8}$$

which determine the relations between the coordinates and the observable redshift, i.e., $t(z)$ and $r(z)$.

Now that we have related the redshift to the inhomogeneities, we still need the relation between the redshift and the energy flux F, or the luminosity-distance, defined as $d_L \equiv \sqrt{L/4\pi F}$, where L is the total power radiated by the source. This is given by [37]

$$d_L(z) = (1+z)^2 A(r(z),t(z)). \tag{3.9}$$

Likewise, the angular distance diameter is given by

$$d_A(z) = A(r(z), t(z)). \tag{3.10}$$

As the z-dependence of t and r are determined by Eqs. (3.7) and (3.8) and the scale function $A(r,t)$ by Eq. (2.22), using Eq. (3.9) one can calculate d_L for a given z. All of these relations have a manifest dependence on the inhomogeneities (i.e., on the functions $H_0(r)$ and $\Omega_M(r)$). What remains is a comparison of Eq. (3.9) with the observed $d_L(z)$.

Because the boundary functions of the LTB model are arbitrary, it comes as no surprise that any isotropic set of observations can be explained by the appropriate inhomogeneities of the LTB model [38]. That the supernova data could be interpreted in terms of an inhomogeneous LTB model with no cosmological constant was first suggested by Célérier [39], who pointed out that the LTB model is degenerate with respect to any magnitude-redshift relation so that the accelerated expansion could be modeled by a very large number of inhomogeneity profiles. In this sense the LTB model is not predictive. The intriguing aspect here is rather the matter of principle which the LTB model can be used to demonstrate: that the supernova data does not necessarily imply accelerating expansion and hence the existence of dark energy is not an unavoidable consequence of the data but rather depends on the framework the data is interpreted in. Moreover, the inhomogeneities need not contradict the observed homogeneity in galaxy surveys [7], as is often claimed (see e.g., [40]), since the model admits solutions with constant Ω_M but with a position-dependent H.

To demonstrate this, let us consider the gold sample of 157 supernovae of Riess et al. [5] and disregard LSS and CMB data for the moment. In the FRW model the parameters that best describe our universe are found by maximizing the likelihood function $\exp(-\chi^2(H_0, \Omega_M, \Omega_\Lambda))$ constructed from the observations. However, to find the boundary conditions of the LTB universe that best describe our universe, we should in principle maximize the likelihood *functional* $\exp(-\chi^2(H_0(r), \Omega_M(r)))$. In practice, this is impossible. One can only consider some physically motivated types for the functions $H_0(r)$ and $\Omega_M(r)$ that contain free parameters; these are then fitted to the supernova observations by maximizing the leftover likelihood function. In the literature there exist several fits to the supernova data employing a simple LTB model with different authors having chosen different density profiles (and, unfortunately, often a different notation) [31; 34; 41; 42; 43; 44].

Since the expansion rate of the FRW universe has to accelerate in order to fit the supernova data, the second time derivative of the FRW scale function should be positive. In contrast, in the LTB universe the observations are affected by the variation of all the dynamical quantities along the past light cone, not just the time variation. Indeed, the directional derivative along the past light cone is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dt}{dr}\frac{\partial}{\partial r} = \frac{\partial}{\partial t} - \frac{A'(r,t)}{\sqrt{1-k(r)}}\frac{\partial}{\partial r} \approx \frac{\partial}{\partial t} - \frac{\partial}{\partial r}, \tag{3.11}$$

where the approximation in the last step is more accurate for small values of r, but is qualitatively correct even for larger r.

The main message of Eq. (3.11) is that from the observational point of view, the negative r-derivative roughly corresponds to the positive time derivative. This

Fig. 1 Confidence level contours in the LTB model with $\Omega_M(r) = \text{constant} = 0.45$ and $H_0(r) = 56.3 \text{ km/s/Mpc} + \Delta He^{-r/r_0}$. From [34]

is natural since by looking at a source, we simultaneously look into the past (i.e., along the *negative t*-axis) and into a spatial distance (i.e., along the *positive r*-axis). Hence, to mimic the acceleration, i.e., for the expansion rate to look as if it were to increase towards us along the past light cone, the expansion $H_0(r)$ must *decrease* as r grows: hence we should look for an LTB model with $H'_0(r) < 0$. Thus, keeping in mind the homogeneity of galaxy distributions, we could choose a simple four parameter LTB model like [34]

$$H_0(r) = H + \Delta He^{-r/r_0},$$
$$\Omega_M(r) = \Omega_0 = \text{constant}, \tag{3.12}$$

where H, ΔH, Ω_0 and r_0 are free parameters determined by the supernova observations. The best fit values are found to be [34]

$$H + \Delta H = 66.8 \text{ km/s/Mpc}, \ \Delta H = 10.5 \text{ km/s/Mpc}, \ r_0 = 500 \text{ Mpc}, \ \Omega_0 = 0.45. \tag{3.13}$$

The goodness of the fit is $\chi^2 = 172.6$ ($\chi^2/157 = 1.10$). The confidence level contours with fixed values of Ω_0 and H are shown in Fig. 1. For comparison with the homogeneous case, the best fit nonflat ΛCDM has $\Omega_M = 0.5$, $\Omega_\Lambda = 1.0$ with $\chi^2 = 175$ ($\chi^2/157 = 1.11$). What is perhaps surprising is the fact that the supernova fit is not only in qualitative agreement with the observed homogeneity in galaxy surveys but also automatically yields a value for the present-day matter density that is consistent with the observations. The smallness ($\sim 15\%$) of the spatial variation in the Hubble parameter is also somewhat surprising, considering that it is of the same order as the uncertainty of the model-independent[5] determination of the local Hubble rate by the Hubble space telescope [45]. The variation of the Hubble parameter found by Alnes et al. [31], who used a different model Ansatz, has also similar magnitude, but in contrast their model contains a large ($\sim 400\%$) variation in the matter density at scales larger than the current range of galaxy surveys.

One can also fit data with Eq. (3.12) together with a cosmological constant. Taking $\Omega_M(r) + \Omega_\Lambda(r) = 1$ one finds no improvement [34]. If instead of $\Omega_M = \text{const.}$ we assume a strictly uniform present-day matter distribution with $\rho_M(r, t_0) = \text{constant}$, which implies $H_0^2(r)\Omega_M(r) = \text{constant}$, we may choose the parametrization

$$H_0(r) = H + \Delta He^{-r/r_0},$$
$$\Omega_M(r) = \Omega_0(H + \Delta H)^2/(H + \Delta He^{-r/r_0})^2. \tag{3.14}$$

The best fit values in this case are

$$H + \Delta H = 67 \text{ km/s/Mpc}, \quad \Delta H = 10 \text{ km/s/Mpc}, \quad r_0 = 450 \text{ Mpc}, \quad \Omega_0 = 0.29. \tag{3.15}$$

Here the goodness of the fit is $\chi^2 = 172.6$ ($\chi^2/157 = 1.10$). The confidence level contours with Ω_0 and H fixed to their best fit values are displayed in Fig. 2.

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[5] Note that the smaller uncertainties found in the CMB data analysis cannot be used here as those fits assume that the entire universe is perturbatively close to the homogeneous FRW model.

All these models have an inhomogenous Big Bang. One could also have an inhomogenous expansion with a spatially constant age of the universe by choosing e.g.,

$$
H_0(r) = H \left[\frac{\sqrt{1 - \Omega_M(r)} - \Omega_M(r) \text{arsinh} \sqrt{\frac{1 - \Omega_M(r)}{\Omega_M(r)}}}{(1 - \Omega_M(r))^{3/2}} \right],
$$

$$
\Omega_M(r) = \frac{\Omega_0}{(1 + \delta e^{-r/r_0})^2}. \tag{3.16}
$$

The constraint of a simultaneous Big Bang leaves us with only one free function. The best fit values are [34]

$$
H = 76.5 \text{ km/s/Mpc}, \quad \delta = 1.21, \quad r_0 = 1000 \text{ Mpc}, \quad \Omega_0 = 0.29 \tag{3.17}
$$

with $\chi^2 = 175.5$ ($\chi^2/157 = 1.12$). Eq. (3.17) implies that the Hubble function $H_0(r)$ varies from the value $H_0(0) = 65$ km/s/Mpc near us to its asymptotic value $H_0(r \gg r_0) = 52$ km/s/Mpc. The age of the universe is then $t_{\text{age}} = 1/H = 12.8$ Gyr. Similar values have also been found in the model of ref. [31].

Thus simple and at least seemingly semirealistic LTB dust models can fit the supernova data. The point to note is that although the LTB equations of motion do not in general permit locally accelerated expansion, this does not exclude the possibility that there can be an effective, volume averaged acceleration, where a scale factor defined via the physical volume of some comoving region has a positive double time derivative [46]. However, it can be shown there is no effective average acceleration [34] for the models considered above.[6]

4 Towards more realistic LTB models

Whether the supernova data combined with the CMB and LSS data would nevertheless require an accelerating universe is an open question; cosmological perturbation theory in LTB background is still non-existent.

Some issues can be addressed, though. In particular, when the LTB metric models a local underdensity, one may assume that the evolution of perturbations is identical to that in a homogeneous universe until the time of last scattering. Adopting this approach, Alnes et al. [31] have considered in an approximation constraints arising from the position l_1 of first acoustic peak. They find a shift relative to the concordance ΛCDM model that is given by

$$
\mathscr{S} = \frac{l_1}{l_1^{\Lambda CDM}} = 0.01419(1 - \phi_1)\frac{d_A}{r_s}, \tag{4.1}
$$

---

[6] Although fitting the supernova data does not require accelerating expansion, for some profiles the LTB model may give rise to a suitably defined average acceleration [47]. For a discussion on backreaction in LTB models, see also [48].

Table 1 The best fit parameters of the locally underdense inhomogeneous model of [31]

Description	Symbol	Value
Density contrast parameter	$\Delta\alpha$	0.90
Transition point	r_0	1.35 Gpc
Transition width	$\Delta r/r_0$	0.40
Fit to supernovae	χ^2_{SN}	176.5
Position of first CMB peak	$\mathscr{S}$	1.006
Age of the universe	t_0	12.8 Gyr
Relative density inside underdensity	$\Omega_{m,in}$	0.20
Relative density outside underdensity	$\Omega_{m,out}$	1.00
Hubble parameter inside underdensity	h_{in}	0.65
Hubble parameter outside underdensity	h_{out}	0.51
Physical distance to last scattering surface	D_{LSS}	11.3 Gpc
Length scale of baryon oscillation from SDSS	$R_{0.35}$	107.1

where d_A is the angular diameter distance to the last scattering surface, given by Eq. (3.10); this is the part that depends on the local underdensity, whereas r_s, the sound horizon at recombination, and the (small) value of the parameter ϕ_1 can be obtained from the conventional homogeneous model. To be in agreement with the WMAP observations, the shift parameter should be within the range $\mathscr{S} = 1.00 \pm 0.01$. The locally underdense model depends on the density contrast parameter $\Delta\alpha$, functionally related to $A(r,t)$ and specifying the difference between the two region, the transition point r_0 from LTB to FRW, and the transition width $\Delta r/r_0$. A set of parameter values that yields a good fit both to the supernova data and the first CMB peak position can be found, as can be seen from Table 1. Generically, for the void picture to work, one should have a local underdense region that extends at least up to the nearby supernovae or about 300–400 Mpc/h.

These considerations hold if we occupy the exact center of the local LTB universe. For an observer that is located off-center, the universe appears to be anisotropic. Estimating the luminosity distance for an off-center observer is somewhat more complicated task than in the case of an observer at the center [36; 49]. One finds an anisotropic relation between the redshifts and the luminosity dis-

tances of supernovae, which however yields only a mild constraint as up to about 20% displacement from the center is consistent with the data [50]. In contrast, the constraint from the CMB dipole appears to be very stringent, allowing only a displacement of about 15 Mpc from the center of the underdense bubble [36]. This result is obtained by assuming that all of the observed dipole $a_{10} \sim 10^{-3}$ is due to the displacement. A cancelation of the dipole due to our local peculiar motion towards the center of the underdensity is a possibility that would allow for a larger displacement. Whether such a peculiar motion can arise naturally or only by an accident, remains to be seen.

For an off-center observer the direction towards the center of the bubble singles out a special axis. Therefore one could hope that a local LTB bubble could provide an explanation for the observed peculiar alignments of the CMB quadrupoles and octopoles [51; 52]. Because of the smallness of the displacement allowed by the dipole, the quadru- and octopoles appear not to have enough power to explain their observed alignment [36], although again the conclusion depends on the assumption that our local average motion has been accounted for correctly.

Instead of a single underdensity, one could also consider an "onion model" with a homogeneous background density, on top of which there are density fluctuations which are periodic as a function of the radial coordinate. The observer sits in some generic position and looks at sources along the radial direction, and the LTB dust solution incorporates the entire Universe. To study this set-up, Biswas et al. [49] have derived an expression for the luminosity distance in an LTB metric for an off-centre observer. The corrections due to underdensities to light propagation were found to have a tendency to cancel far away from the observer because a radial light ray unavoidably meets both underdense and overdense structures. However, in the real universe light encounters hardly any structure, so the cancellations might be an artifact of the onion model. Since in the real universe the photon is mostly traversing voids it should get redshifted faster as the nonlinearities increase with time and thereby effectively produce an apparent acceleration. In the onion model one can nevertheless mimic an accelerating ΛCDM cosmology under certain special conditions: the observer has to be located around a minimum of the density contrast that is required to be quite high [49].

Yet another approach is the "Swiss cheese" model of the inhomogeneous universe, where each spherical void is described by the LTB metric. At the boundary of these regions the LTB metric is matched with the FRW metric that describes the evolution between the inhomogeneities. One can then seek for the modifications of the luminosity distance as the light passes through the underdense regions [53]. In the extreme case where one assumes that light traverses the centers of all the inhomogeneities along its path, assuming that the locations of the source and the observer are random and inhomogeneities have sizes of order 10 Mpc, the relative increase of the luminosity distance is however just of the order of a few percent near $z \simeq 1$. A qualitatively similar conclusion has been reached in [54].

Structure formation and the smallness of CMB perturbations may in general pose a difficulty for LTB models. For instance, for a class of inhomogeneities a homogeneous universe is actually a late time attractor solution. This means that at earlier times matter density and/or Hubble rate tends to be even more inhomogeneous than today. Whether this presents an unsurmountable problem remains to be seen. Nevertheless, the LTB model serves as a reminder that the interpretation of

the cosmological data is not only quantitatively but even qualitatively very much model dependendent. Therefore, all options should be carefully examined before firm conclusions can be drawn. This is true in particular for dark energy, which is both an observational and theoretical enigma.

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