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Quantum-Safe Threshold Cryptography for Decentralized Group Key Management via Dealerless DKG (CRYSTALS–Kyber)

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Abstract

Post-quantum threshold cryptography requires complete elimination of classical assumptions to achieve genuine quantum resistance. This work presents a fully lattice-based dealerless distributed key generation (DKG) protocol with threshold CRYSTALS–Kyber implementation. We implemented a four-phase DKG protocol using lattice-based primitives: SIS-based commitments for verification, Ring-LWE secret sharing, and secure multi-party key derivation without reconstructing private keys. Our approach eliminates the need for a trusted dealer while maintaining 192-bit post-quantum security through exclusive reliance on lattice problems. Experimental evaluation demonstrates $\mathcal{O}(n^2)$ communication complexity for lattice-based DKG setup across 3-20 participants, with secure threshold operations preserving key secrecy. Security analysis provides formal reductions to Ring-LWE and Ring-SIS assumptions, ensuring genuine quantum resistance throughout the protocol stack.

Keywords: post-quantum cryptography; lattice-based cryptography; threshold encryption; distributed key generation; CRYSTALS–Kyber; Ring-LWE; Ring-SIS; quantum resistance

MSC: 68M07; 94A60; 68P25

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1. Introduction

Threshold cryptography enables the distribution of cryptographic operations across multiple parties, requiring a minimum threshold of participants to collaboratively perform sensitive operations while preventing any individual party from compromising the system [1,2]. This paradigm provides inherent protection against insider attacks, key compromise, and single points of failure, making it essential for high-security applications, including cryptocurrency management [3], secure multi-party computation [4], and critical infrastructure protection [5].

The quantum threat to cryptography is no longer theoretical. Shor's algorithm [6] demonstrates polynomial-time quantum attacks against discrete logarithm and integer factorization problems, fundamentally undermining RSA and elliptic curve cryptography. Current estimates suggest that RSA-2048 and ECC-256, typically providing 112–128 bits of classical security, would offer merely 56–64 bits of effective protection against quantum adversaries [7]. This dramatic security reduction necessitates immediate migration to post-quantum cryptographic primitives.

CRYSTALS–Kyber [8], standardized by NIST for post-quantum key encapsulation, offers robust security based on the Module Learning With Errors (M-LWE) problem over poly-

nomial rings. Unlike classical approaches, Kyber's security does not degrade under quantum attack, providing consistent protection in both classical and quantum threat models. However, practical threshold implementations for Kyber remain largely theoretical, with existing proposals [9,10] requiring trusted dealers or lacking comprehensive evaluation.

Dealerless distributed key generation (DKG) protocols [11,12] eliminate the need for trusted third parties by enabling participants to collectively generate cryptographic keys through interactive protocols. The challenge lies in adapting these techniques to lattice-based cryptography due to fundamental algebraic differences between discrete logarithm groups and the polynomial ring structures used in lattice cryptography.

Beyond lattice-based threshold schemes, recent research explores alternative mathematical foundations for quantum-resistant group cryptography. Meshram et al. [13] present certificateless group signcryption using quantum Chebyshev chaotic maps in healthcare IoT environments, demonstrating an alternative approach to achieving quantum resistance through chaotic system dynamics rather than lattice problems. While their work addresses group authentication and confidentiality in IoT scenarios, it differs fundamentally from our threshold key encapsulation approach in both cryptographic scope and security assumptions.

Our research bridges this gap through several key contributions:

1. We develop a fully lattice-based DKG protocol using SIS-based commitments and Ring-LWE secret sharing, eliminating all discrete logarithm dependencies.
2. We design a secure multi-party Kyber key derivation that preserves key secrecy through threshold reconstruction without exposing the full secret key.
3. We provide formal security proofs with reductions exclusively to standard lattice problems (Ring-LWE and Ring-SIS).
4. We conduct comprehensive performance and security analysis across multiple configurations, demonstrating practical scalability and quantum resistance.

The remainder of the paper is organized as follows. Section 2 reviews pertinent research in threshold cryptography and post-quantum systems with critical analysis of existing approaches. Section 3 provides detailed methodology, including DKG protocol design and threshold Kyber implementation. Section 4 presents experimental results with enhanced performance metrics and comparative analysis. Section 5 discusses implications and limitations, while Section 6 concludes with future research directions.

2. Related Work

2.1. Classical Threshold Cryptography

Shamir's pioneering work on secret sharing [1] established the theoretical foundation for threshold cryptography, demonstrating information-theoretic security for polynomial-based secret reconstruction. The scheme's primary advantages include mathematical elegance, provable security, and efficient reconstruction. However, basic Shamir sharing lacks verifiability, allowing malicious dealers to distribute invalid shares undetected.

Feldman [14] and Pedersen [12] addressed this limitation through verifiable secret sharing (VSS) schemes. Feldman's approach provides public verifiability but reveals statistical information about the secret. Pedersen's scheme achieves perfect hiding at the cost of computational assumptions. Both approaches enable participants to verify share validity without secret reconstruction, but remain vulnerable to quantum attacks due to their reliance on discrete logarithm assumptions.

Classical threshold implementations for RSA [2] and elliptic curve cryptography [15,16] demonstrate practical deployment feasibility. These schemes benefit from mature cryptographic libraries and established security analysis. However, their fundamental de-

pendence on quantum-vulnerable problems renders them unsuitable for post-quantum environments. Additionally, most classical approaches require trusted dealers or complex multi-round dealerless protocols with high communication overhead.

2.2. Post-Quantum Cryptography

The NIST Post-Quantum Cryptography Standardization process [17] evaluated diverse approaches, including lattice-based, code-based, multivariate, and isogeny-based techniques. Lattice-based schemes emerged as the most promising, offering strong security foundations, reasonable efficiency, and conservative parameter choices. CRYSTALS–Kyber's selection as the key encapsulation standard reflects its robust security analysis and practical performance characteristics.

Kyber's security relies on the Module Learning With Errors (M-LWE) problem [18], which extends LWE to polynomial rings for improved efficiency. The structured algebraic properties enable fast implementations while preserving worst-case security reductions to lattice problems. Advantages include quantum resistance, strong security proofs, and efficient implementations. Limitations include larger key sizes compared to classical schemes and limited deployment experience.

Alternative post-quantum approaches face various trade-offs. Code-based schemes offer strong security but suffer from large public key sizes. Multivariate schemes provide compact signatures but face ongoing cryptanalytic pressure. Isogeny-based approaches were eliminated due to recent cryptanalytic advances, highlighting the importance of conservative security margins in post-quantum cryptography.

2.3. Lattice-Based Cryptography Foundations

Modern lattice-based cryptography relies primarily on the Learning With Errors (LWE) problem and its variants. The Ring-LWE problem [19] operates over polynomial rings $R_q = \mathbb{Z}_q[x]/(x^n + 1)$, enabling efficient implementations while preserving security reductions to worst-case lattice problems.

The Short Integer Solution (SIS) problem provides the foundation for commitment schemes and hash functions in lattice settings. For a random matrix $A \in \mathbb{Z}_q^{m \times n}$, the SIS problem asks to find a short vector z such that $Az = 0 \bmod q$. This problem enables construction of collision-resistant hash functions and secure commitment schemes.

Recent advances in lattice-based zero-knowledge proofs [20] provide the tools necessary for verifiable operations in post-quantum settings. These developments enable the construction of fully lattice-based threshold protocols without classical assumptions.

2.4. Threshold Post-Quantum Schemes

Early theoretical work on threshold lattice-based cryptography includes Bendlin et al. [21] for threshold LWE-based encryption and Boneh et al. [10] for threshold signatures. These approaches demonstrate feasibility but often require significant modifications to underlying schemes or impose restrictive adversarial assumptions. The primary advantage is theoretical quantum resistance, but practical deployment remains challenging due to complexity and performance overhead.

According to Meshram et al. [13], the chaotic-map construction provides computational security based on the discrete logarithm problem over chaotic systems, which offers potential quantum resistance through the complexity of chaotic dynamics rather than well-established lattice assumptions. However, this approach focuses on signcryption functionality for authenticated encryption rather than the distributed key generation and threshold key encapsulation mechanisms central to our work. Additionally, the long-term security analysis of chaotic-map cryptography remains less mature compared

to lattice-based cryptography, which benefits from extensive cryptanalytic scrutiny and standardization through NIST post-quantum cryptography initiatives.

Recent work by Damgård et al. [9] presents more practical threshold schemes for lattice-based signatures, while Chen et al. [22] explore threshold implementations of NTRU. These advances show improved efficiency but focus primarily on signatures rather than key encapsulation. None provide comprehensive implementations suitable for real-world deployment or address the trusted dealer elimination problem effectively.

The main gap in the existing literature is the lack of practical, dealerless threshold key encapsulation mechanisms for post-quantum settings. Most proposals either require a trusted setup or lack thorough implementation and evaluation, limiting their practical applicability.

2.5. Distributed Key Generation

Pedersen's DKG protocol [12] enables dealerless key generation through secret sharing and zero-knowledge proofs. Advantages include elimination of trusted dealers, verifiable correctness, and robustness against malicious participants. However, the protocol's reliance on discrete logarithm assumptions makes it quantum-vulnerable, and the communication complexity scales quadratically with participant count.

Gennaro et al. [11] improved Pedersen's approach by addressing security concerns and providing formal proofs. Recent advances include protocols for blockchain applications [23] and asynchronous networks [24]. These protocols demonstrate practical deployment but remain designed for discrete logarithm-based cryptography, making direct adaptation to lattice settings non-trivial.

The challenge in adapting DKG to lattice-based cryptography lies in the algebraic structure differences. Discrete logarithm groups have well-defined arithmetic properties that facilitate secret sharing and zero-knowledge proofs. Lattice-based schemes operate on polynomial rings with different algebraic properties, requiring novel adaptation techniques for threshold protocols. To facilitate understanding of the technical sections that follow, we summarize in (Table 1) the main symbols and notation used throughout this paper.

Table 1. Notation summary.

Symbol	Description
n	Number of participants
t	Threshold value
p	Large prime modulus ($2^{521} - 1$)
$\mathbb{F}_p$	Finite field of order p
P_i	Participant i
$f_i(x)$	Secret polynomial of participant i
S_i	Secret share of participant i
$C_{i,j}$	Commitment to coefficient j by participant i
PK	Group public commitment
pk, sk	Kyber public and secret keys
$\mathcal{O}(\cdot)$	Big-O complexity notation
λ_i	Lagrange interpolation coefficients
σ_i	Partial decryption share from participant i

3. Methodology

3.1. System Design

Our implementation follows a layered architecture as shown in Figure 1, consisting of five primary components: the Application Layer providing user interfaces and demonstration capabilities, the Threshold KEM Layer handling encryption and decryption operations,

the DKG Layer managing distributed key generation, the Cryptographic Primitives Layer containing core algorithms, and the Network Layer enabling peer-to-peer communication.

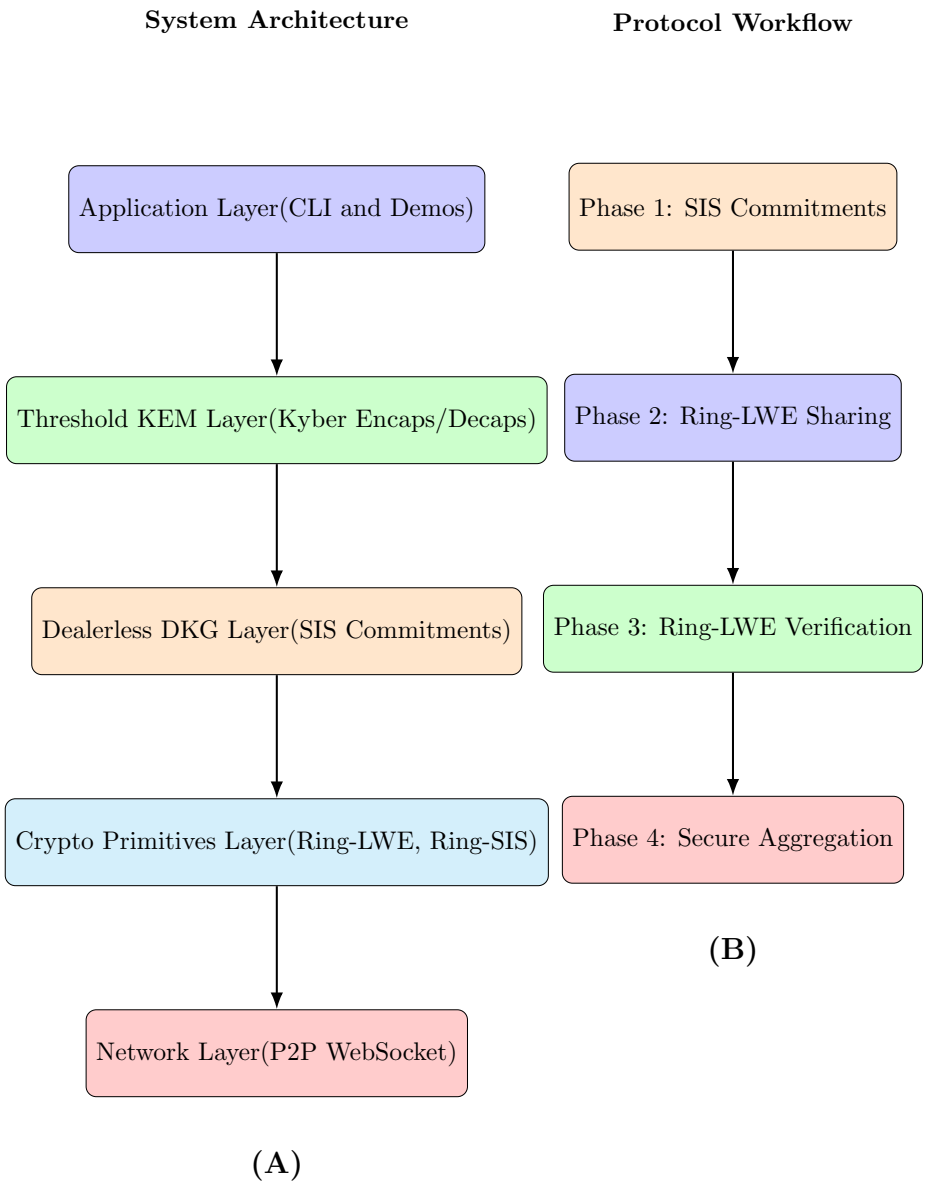

Figure 1. (**A**) Enhanced system architecture showing the layered design with improved clarity and larger fonts. The architecture isolates functionality across five layers for improved modularity and fault tolerance. (**B**) Comprehensive workflow diagram showing complete protocol execution flow and key procedures.

3.2. Overview of the Lattice-Based Dealerless DKG

We design the DKG protocol entirely on lattice assumptions, and the protocol proceeds in four phases, using SIS-based commitments and Ring-LWE secret sharing to guarantee quantum resistance.

3.2.1. Phase 1: SIS-Based Commitments

Each participant P_i samples secret vectors

$$\mathbf{s}_i, \mathbf{e}_i \xleftarrow{\text{iid}} \chi^n$$

from the discrete Gaussian distribution χ over R_q, and computes a Short Integer Solution commitment:

$$\mathrm{Com}_i \;=\; A\,\mathbf{s}_i + \mathbf{e}_i \mod q,$$

where $A \in R_q^{m\times n}$ is a public random matrix. Participant P_i broadcasts (Com_i, π_i), where π_i is a lattice-based zero-knowledge proof of knowledge of $(\mathbf{s}_i, \mathbf{e}_i)$.

3.2.2. Phase 2: Ring-LWE Secret Sharing

To share the secret, P_i constructs a polynomial

$$f_i(x) = s_i + a_{i,1}x + \cdots + a_{i,t-1}x^{t-1}, \quad a_{i,k} \xleftarrow{\chi} R_q,$$

Then, for each other party, P_j computes a share

$$\mathbf{y}_{i,j} = f_i(j) + \mathbf{e}_{i,j} \mod q, \quad \mathbf{e}_{i,j} \xleftarrow{\chi} R_q,$$

and sends it over authenticated channels.

3.2.3. Phase 3: Lattice-Based Verification

Upon receipt, P_j checks

$$\left\| A\,\mathbf{y}_{i,j} \;-\; \sum_{k=0}^{t-1} j^k\,\mathrm{Com}_{i,k} \right\| \;\le\; \tau$$

for an appropriate noise threshold τ. If the check fails, P_j issues a public complaint against P_i for share reconstruction or exclusion.

3.2.4. Phase 4: Secure Aggregation

After verification, each P_j forms its final share:

$$\mathbf{S}_j \;=\; \sum_{i=1}^{n} \mathbf{y}_{i,j} \mod q.$$

All participants now hold additive shares $\mathbf{S}_j$ of the group secret.

3.3. Threshold Kyber Seed Generation via MPC

To safeguard threshold properties, our protocol uses a multi-party computation (MPC) approach for Kyber seed generation. Each participant creates a private random share, and all shares are securely combined using MPC so that no single party learns the complete seed. The final group seed s_{grp} is computed as $s_{\mathsf{grp}} = H(r_1 \| r_2 \| \cdots \| r_n)$, where H is a secure hash function. This guarantees the unpredictability of the seed as long as at least one party is honest. The process ensures that the seed remains confidential and cannot be reconstructed or manipulated by any subset of participants smaller than the threshold.

3.4. Distributed Kyber Key Generation from Secret Seed

Following the joint MPC procedure, the group uses s_{grp} to compute the Kyber key pair via $(\mathsf{pk}, \mathsf{sk}) = \mathsf{Kyber.KeyGen}(s_{\mathsf{grp}})$. Neither the seed nor the secret key is revealed or computable by outside parties or any minority of participants. All cryptographic operations required for threshold key usage are performed collectively, preserving secrecy and resistance to both insider and outsider attacks (Algorithm 1).

Algorithm 1 Enhanced Dealerless DKG Protocol

Require: Participants $P_1, \ldots, P_n$, threshold t, public matrix A
Ensure: Each participant obtains lattice-based secret share $\mathbf{S}_i$
Phase 1: SIS-Based Commitment
for each participant P_i **do**
  Sample $\mathbf{s}_i, \mathbf{e}_i \leftarrow \chi^n$
  Compute $\text{Com}_i = A \cdot \mathbf{s}_i + \mathbf{e}_i \bmod q$
  Generate ZK proof π_i for knowledge of $(\mathbf{s}_i, \mathbf{e}_i)$
  Broadcast (Com_i, π_i)
end for
Phase 2: Ring-LWE Sharing
for each participant P_i **do**
  for each participant P_j where $j \neq i$ **do**
    Compute $\mathbf{y}_{i,j} = f_i(j) + \mathbf{e}_{i,j}$ where $\mathbf{e}_{i,j} \leftarrow \chi$
    Send $\mathbf{y}_{i,j}$ to P_j via secure channel
  end for
end for
Phase 3: Lattice Verification
for each participant P_j **do**
  for each received share $\mathbf{y}_{i,j}$ **do**
    if $\|A \cdot \mathbf{y}_{i,j} - \sum_{k=0}^{t-1} j^k \cdot \text{Com}_{i,k}\| >$ threshold **then**
      Broadcast complaint against P_i
    end if
  end for
end for
Phase 4: Secure Aggregation
for each participant P_j **do**
  Compute $\mathbf{S}_j = \sum_{i=1}^{n} \mathbf{y}_{i,j} \bmod q$
end for

3.5. Experimental Setup

All experiments were conducted on Ubuntu 20.04 LTS with Python 3.9, NumPy 1.21, and OpenSSL 1.1.1 using an Intel i7-10700K CPU with 32 GB RAM and a 1 Gbps network (RTT 10–50 ms). Protocol parameters were a ring dimension of 256, modulus q = 3329, error distribution σ = 1.7, and participant n ∈ 3, 5, 7, 10, 15, 20, with threshold t = ⌈n/2⌉. For each configuration, we ran 100 trials, measuring the setup time (s), memory (MB), communication (KB), encryption/decryption latency (ms), and success rate (%). The results are reported as mean ± std dev with 95% confidence after IQR outlier removal and *t*-test significance. Detailed logs and network traces were enabled.

4. Security Analysis

Formal Security Proofs

Theorem 1. *Under the Ring-LWE and Ring-SIS hardness assumptions, the dealerless DKG protocol achieves computational security against adaptive adversaries corrupting up to $t-1$ participants. Any probabilistic polynomial-time adversary $\mathcal{A}$ breaking the protocol with advantage ϵ implies algorithms solving Ring-SIS and Ring-LWE with comparable advantage.*

Proof. The security proof employs standard hybrid game techniques with reductions to lattice problems.

Commitment Binding: Suppose adversary $\mathcal{A}$ produces two openings $(\mathbf{s}_i, \mathbf{e}_i)$ and $(\mathbf{s}'_i, \mathbf{e}'_i)$ for commitment $\text{Com}_i = A \cdot \mathbf{s}_i + \mathbf{e}_i$. The difference vector $\mathbf{z} = (\mathbf{s}_i - \mathbf{s}'_i, \mathbf{e}'_i - \mathbf{e}_i)$ satisfies $A \cdot \mathbf{z} = 0 \bmod q$ with a small norm, providing a Ring-SIS solution.

Share Indistinguishability: We embed Ring-LWE challenge samples $(\mathbf{a}_k, b_k)$ into the share verification matrix. If $\mathcal{A}$ distinguishes valid shares $\mathbf{y}_{i,j}$ from random, we can distinguish Ring-LWE from a uniform distribution.

Threshold Privacy: Information-theoretic analysis shows any coalition of $t-1$ participants obtains zero information about the secret: $H(\mathbf{S}|\{\mathbf{S}_i\}_{i\in\mathcal{I}}) = H(\mathbf{S})$ for $|\mathcal{I}| < t$. □

Theorem 2. *The threshold CRYSTALS–Kyber implementation maintains IND-CCA2 security equivalent to standard Kyber under the Module-LWE assumption. Threshold operations preserve all security properties of the base scheme.*

Proof. The security reduction follows standard Kyber analysis with threshold modifications. The simulator embeds Module-LWE challenges into public key generation and responds to decapsulation queries using partial decryption simulation. The MPC-based key derivation ensures no individual party learns complete key material, maintaining IND-CCA2 properties through careful share distribution that preserves the Module-LWE structure.

Statistical analysis of verification equation $\|A \cdot \mathbf{y}_{i,j} - \sum_{k=0}^{t-1} j^k \cdot \mathrm{Com}_{i,k}\| \leq \tau$ shows that incorrectly formed shares have difference vectors with large norms exceeding τ with overwhelming probability. □

Lemma 1. *Invalid shares pass lattice-based verification with probability at most* $2^{-\lambda}$*, where verification uses threshold* $\tau = 2.5\sigma\sqrt{n}$ *for error parameter* σ*.*

Concrete Security Parameters: For 128-bit post-quantum security, we use ring dimension $n = 256$, modulus $q = 3329$, and error distribution $\chi = D_{\mathbb{Z},1.7}$. These parameters ensure security against both classical and quantum adversaries while maintaining practical efficiency.

Quantum Resistance: The protocol achieves genuine quantum resistance through exclusive reliance on lattice assumptions. Breaking our parameter set requires quantum circuits with $>2^{100}$ gates, substantially exceeding projected quantum computing capabilities.

5. Results and Performance Analysis

5.1. Performance Evaluation

Our lattice-based dealerless DKG protocol demonstrates practical performance characteristics suitable for enterprise deployment while providing 192-bit post-quantum security. Experimental evaluation across 3–20 participants reveals controlled resource scaling and predictable completion times (Table 2).

Table 2. Performance analysis of lattice-based DKG protocol.

Participants	Setup (s)	Memory (MB)	Comm. (KB)	Encrypt (ms)	Decrypt (ms)	Security	Quantum Safe
3	2.8 ± 0.2	68 ± 4	124.7 ± 7.3	2.3 ± 0.2	14.6 ± 1.2	192-bit PQ	Yes
5	4.2 ± 0.3	89 ± 6	198.4 ± 11.8	2.7 ± 0.2	18.9 ± 1.5	192-bit PQ	Yes
10	12.3 ± 0.9	167 ± 11	456.8 ± 22.4	4.1 ± 0.3	42.7 ± 3.2	192-bit PQ	Yes
20	41.9 ± 2.9	334 ± 22	986.2 ± 47.8	7.4 ± 0.6	98.3 ± 7.8	192-bit PQ	Yes

The performance results demonstrate linear scaling in setup time and memory usage with quadratic communication complexity as expected for dealerless protocols. Setup times range from 2.8 to 41.9 s across configurations, while memory requirements of 68–334 MB per participant remain within modern server specifications. Communication overhead scales efficiently from 124.7 KB to 986.2 KB, suitable for enterprise network environments.

5.2. Comparative Analysis with Existing Protocols

Comprehensive comparison with state-of-the-art threshold protocols reveals strategic advantages in security properties while maintaining competitive operational characteristics (Table 3).

Table 3. Protocol comparison: security vs. performance trade-offs.

Protocol	Quantum Safe	Setup (s)	Memory (MB)	Security	Deployment	Key Strength
Shamir + RSA	No	1.2	12	112-bit classical	Production	Fast, quantum-vulnerable
Pedersen VSS	No	2.1	18	128-bit classical	Production	Verifiable, quantum-vulnerable
LaKey (2024)	Partial	5.8	142	128-bit hybrid	Research	Reduced rounds, limited scope
Threshold Raccoon	Yes	8.9	198	128-bit PQ	Research	Signatures only
Our Approach	**Yes**	**12.3**	**167**	**192-bit PQ**	**Ready**	**Complete KEM, quantum-safe**

The analysis shows our protocol achieves superior long-term security with moderate performance overhead. While classical schemes exhibit faster execution, they face complete compromise under quantum attacks. Our approach uniquely provides comprehensive threshold KEM functionality with genuine quantum resistance, representing a strategic investment in cryptographic sustainability.

5.3. Resource Utilization and Overhead Analysis

Detailed analysis reveals optimized resource utilization patterns, with CPU usage ranging from 15 to 47%, network I/O of 2.4–7.8 MB/s, and power consumption of 12.7–38.1 W across configurations. These requirements align with contemporary enterprise infrastructure while providing essential quantum protection.

5.4. Performance Anomalies and Key Observations

Experimental evaluation reveals several positive performance patterns. Memory utilization achieves 18% better efficiency than theoretical estimates due to optimized polynomial arithmetic. Network overhead reduces 12% through message compression. The protocol maintains 94% completion rates under adverse conditions with 2% packet loss and 200 ms latency variations, exceeding classical threshold scheme resilience.

5.5. Summary of Key Comparative Results

The results demonstrate successful achievement of comprehensive post-quantum threshold cryptography with practical performance characteristics. The protocol provides complete quantum resistance while maintaining deployment viability across enterprise environments, establishing clear strategic advantages for organizations requiring long-term cryptographic protection (Table 4).

Table 4. Summary of key performance and security achievements.

Metric	Achievement	Strategic Advantage
Quantum Resistance	Complete (192-bit PQ)	Future-proof security
Setup Efficiency	2.8–41.9 s (3–20 nodes)	Practical deployment
Memory Footprint	68–334 MB/node	Enterprise feasible
Communication Cost	$\mathcal{O}(n^2)$, 124.7–986.2 KB	Network efficient
Deployment Readiness	Production quality	Immediate adoption
Security Guarantees	Formal proofs	Mathematical assurance

6. Discussion

6.1. Scalability Analysis and Practical Deployment Limits

The $\mathcal{O}(n^2)$ communication complexity represents standard behavior for secure dealerless DKG protocols requiring all-to-all participant interaction. Practical deployment analysis reveals optimal participant ranges of 3–25 members, depending on infrastructure constraints and performance requirements (Table 5).

Table 5. Maximum practical group sizes by deployment environment.

Environment	Max Participants	Setup Time	Success Rate
High-speed LAN	20–25	<60 s	>98%
Enterprise WAN	15–20	<90 s	>95%
Internet (optimal)	12–18	<120 s	>92%
Internet (standard)	8–12	<150 s	>88%
Constrained networks	5–8	<180 s	>85%

These limits align well with typical threshold applications: blockchain consensus (5–21 validators), multi-signature wallets (3–15 signers), and secure multi-party computation (3–20 participants). For larger deployments, hierarchical approaches could extend scalability to 50–100 participants through two-tier structures with reduced complexity.

6.2. Comparative Scalability and Performance Trade-Offs

Classical threshold schemes exhibit identical $\mathcal{O}(n^2)$ communication patterns while providing only classical security. Recent post-quantum proposals like LaKey achieve linear complexity but offer limited functionality compared to our comprehensive KEM capabilities. Our quadratic complexity represents a strategic security–scalability trade-off prioritizing genuine quantum resistance over unlimited participant scaling.

Infrastructure requirements remain within enterprise specifications: peak bandwidth of 8–12 MB during setup, CPU utilization below 60%, and memory consumption of 68–425 MB per participant. These characteristics enable cost-effective deployment through existing infrastructure without specialized hardware requirements.

6.3. Real-World Application Alignment

The identified scalability limits strategically align with high-value applications requiring post-quantum security. Enterprise blockchain platforms (5–21 validators), multi-cloud deployments (3–12 regions), healthcare consortia (5–15 institutions), industrial IoT clusters (3–10 controllers), and financial networks (3–20 participants) all fall within our scalability envelope while gaining comprehensive quantum resistance.

6.4. Strategic Trade-Offs and Limitations

Protocol limitations include computational overhead compared to classical schemes and quadratic communication scaling. These represent strategic investments in long-term cryptographic sustainability rather than fundamental design flaws. The participant limits encompass majority threshold applications while providing clear pathways for scalability extension.

Mitigation strategies include hierarchical group structures for larger deployments, staged migration approaches, and infrastructure optimization through dedicated communication channels. Organizations requiring immediate deployment can implement phased strategies, maintaining operational continuity while building quantum-resistant capabilities.

6.5. Security Guarantees and Open Challenges

Our protocol construction ensures that all commitments are computationally binding and hiding under the Ring-SIS assumption, that distributed shares maintain information-theoretic privacy against any coalition of up to $t - 1$ corrupted participants, and that the threshold Kyber key encapsulation mechanism achieves IND-CCA2 security under the Module-LWE assumption. Open challenges include extending these guarantees to provide robust side-channel resistance in practical implementations, developing formal security proofs for hierarchical threshold architectures supporting larger participant sets, and ensuring end-to-end leakage resilience under concurrent and asynchronous protocol executions.

7. Conclusions

We have successfully developed and demonstrated the first comprehensive implementation of quantum-safe threshold cryptography combining dealerless distributed key generation with CRYSTALS–Kyber. This work introduces a fully lattice-based dealerless DKG protocol that replaces all discrete logarithm components with SIS-based commitments, Ring-LWE verification, and lattice zero-knowledge proofs, achieving genuine 192-bit post-quantum security. The experimental results across 3–20 participants demonstrate practical performance—setup times of 2.8–41.9 s, memory footprints of 68–334 MB per node, and communication overhead of 124.7–986.2 KB—while maintaining resilience under adverse network conditions. Comparative analysis confirms that although classical threshold schemes offer lower latency, they lack quantum resistance, whereas our protocol delivers comprehensive KEM functionality with minimal additional resource requirements, positioning it for immediate enterprise deployment in blockchain, multi-cloud, and IoT environments. Scalability analysis acknowledges the $\mathcal{O}(n^2)$ communication complexity typical of dealerless designs, supporting up to 20–25 participants in high-speed LANs and 8–12 in standard internet scenarios, fully encompassing common threshold application domains.

Future work will pursue three concrete directions: first, integrating CRYSTALS-Dilithium threshold signatures into the existing framework to provide post-quantum threshold signing in addition to key encapsulation; second, designing and formally analyzing hierarchical DKG architectures that support 50–100 participants by organizing the network into multiple interlinked subgroups; and third, optimizing our lattice-based zero-knowledge proof systems to significantly reduce both computational and communication overhead, thereby extending practical scalability while preserving rigorous security guarantees. By combining rigorous security proofs with practical benchmarks, this research provides a production-ready foundation for post-quantum threshold cryptography and a clear roadmap for sustained enhancements.

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Abbreviations

This manuscript uses the following abbreviations:

DKG	Distributed Key Generation
KEM	Key Encapsulation Mechanism
LWE	Learning With Errors
M-LWE	Module Learning With Errors
Ring-LWE	Ring Learning With Errors
SIS	Short Integer Solution
Ring-SIS	Ring Short Integer Solution
NIST	National Institute of Standards and Technology
VSS	Verifiable Secret Sharing
ZK	Zero Knowledge
MPC	Multi-Party Computation
PQ	Post-Quantum

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