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Dynamical System of Gauss-Bonnet Model with Vector Field

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Abstract. Modifying the Einsteins gravity is one of the interesting proposal to explain the dark energy that dominated our universe today. We investigated Gauss-bonnet model with vector field in the action function and we show the influence of the vector field to the density parameters of the model with a dynamical system approach. Parameter $\bar{\beta}$ that contributed by vector field give change to the evolution of density parameters. The model also can represent the condition of the present universe which dominated by dark energy.

Keywords: Modified gravity, Dynamical system, Gauss-Bonnet, Vector field.

1. Introduction

The cosmic acceleration of universe has been investigated by different observational data such as the supernovae type Ia [1-4], cosmic microwave background anisotropy [5], and the age of universe [6]. The interpretation of the observational data shows that this accelerated expansion is due to some kind of negative-pressure form of matter known as dark energy (DE). The analysis of cosmological observations also suggests that the universe is spatially flat, and consists of about 30% of dark matter, and 70% of homogeneously distributed dark energy with negative pressure. There are basically two approaches for construction of DE model. The first approach is called "modified matter models" where we modify the energy-momentum tensor $T_{\mu\nu}$ on the r.h.s. of the Einstein equations that contain matter source with a negative pressure. The second approach is called modified gravity models in which the Einstein tensor $G_{\mu\nu}$ on the l.h.s. of the Einstein equations is modified.

One of modification of gravity is add to the Einstein Lagrangian general functions of the Ricci and Riemann tensors, e.g., $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\bar{\beta}}R^{\mu\nu\alpha\bar{\beta}}, \dots)$ [6]. There is a way to modify gravity with a combination of Ricci and Riemann tensors that keeps the equations at second-order in the metric and does not necessarily give rise to instabilities, namely a GaussBonnet (GB) term coupled to scalar field [7-9]. In addition, we combined vector field to the model and identified the influence of it to the evolution of dark energy density in the model.

The organization of the paper is as follows. In section II, we construct Gauss-Bonnet Model with the vector field. In section III, we investigate the evolution of density parameters of the model with a dynamical system approach. The final section is devoted to the conclusion.

2. Gauss-Bonnet Model with Vector Field

In this section, we reconstruct Gauss-Bonnet model with add the action of the vector field. We assumed that the Lorentz symmetry is spontaneously broken by getting the expectation values of a vector field u^μ as $\langle 0 | u^\mu u_\mu | 0 \rangle = -1$ [10]. The action can be written as the sum of two distinct parts:

$$S = S_{GB} + S_u, \quad (1)$$

where the actions for the Gauss-Bonnet S_{GB} , is given by

$$S_{GB} = \int d^4x \sqrt{-g} \left[\frac{R}{2} - \frac{1}{2}(\nabla^2\phi) - V(\phi) - f(\phi)G \right] + S_m, \quad (2)$$

where $V(\phi)$ and $f(\phi)$ are functions of ϕ , and G is the GB term defined by

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad (3)$$

and the action of vector field S_u , is given by [11,12]

$$\begin{aligned} S_u = & \int d^4x \sqrt{-g} \left[\beta_1 \nabla^\mu u^\nu \nabla_\mu u_\nu - \beta_2 (\nabla_\mu u^\nu)^2 - \beta_3 \nabla^\mu u^\nu \nabla_\nu u_\mu \right. \\ & \left. + \lambda (u^\mu u_\mu + 1) \right]. \end{aligned} \quad (4)$$

In the above β_i ($i = 1, 2, 3, 4$) are arbitrary parameters and λ is a Lagrange multiplier. For the time-like vector field, we impose a constraint

$$u^\mu u_\mu = -1. \quad (5)$$

Vector u^μ is a dimensionless vector. For the background solutions, we use the homogeneity and isotropy of the universe spacetime

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 d\phi^2) \right], \quad (6)$$

The scale of universe determined by $a(t)$. We take the constraint [12],

$$u^\mu = (1, 0, 0, 0). \quad (7)$$

Varying the action (1) with respect to $g_{\mu\nu}$, we have field equation

$$\begin{aligned} T_{\mu\nu} = & 2R_{\mu\nu} - g_{\mu\nu}R + g_{\mu\nu}f(\phi)G - 4f(\phi)RR_{\mu\nu} + 8f(\phi)R_\mu^\rho R_{\nu\rho} \\ & - 4f(\phi)R_{\mu\nu\sigma\tau}R_\nu^{\rho\sigma\tau} + 8f(\phi)R_{\mu\rho\sigma\nu}R^{\rho\sigma} + 4f(\phi)(\nabla_\mu \nabla_\nu f(\phi))R \\ & - 4g_{\mu\nu}(\nabla^2 f(\phi))R - 8(\nabla^\rho \nabla_\mu f(\phi))R_{\nu\rho} - 8(\nabla^\rho \nabla_\nu f(\phi))R_{\mu\rho} \\ & + 8(\nabla^2 f(\phi))R_{\mu\nu} + 8g_{\mu\nu}(\nabla^\rho \nabla^\sigma f(\phi))R_{\rho\sigma} \\ & - 8(\nabla^\rho \nabla^\sigma f(\phi))R_{\mu\rho\nu\sigma}, \end{aligned} \quad (8)$$

where $T_{\mu\nu} = T_{\mu\nu}^{(u)} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}$ is the total energy-momentum tensor, $T_{\mu\nu}^{(u)}$, $T_{\mu\nu}^{(\phi)}$ dan $T_{\mu\nu}^{(m)}$ are the the energy-momentum tensors of vector, scalar fields and matter, respectively, defined by formula

$$\begin{aligned} T_{\mu\nu}^{(u)} &= 2\beta_1(\nabla_\mu u^\tau \nabla_\nu u_\tau - \nabla^\tau u_\mu \nabla_\tau u_\nu) - 2\nabla_\tau(u_{(\mu} J^\tau{}_{\nu)}) - 2\nabla_\tau(u^\tau J_{(\mu\nu)}) \\ &\quad + 2\nabla_\tau(u_{(\mu} J^\tau{}_{\nu)}) + 2u_\sigma \nabla_\tau J^{\tau\sigma} u_\mu u_\nu + g_{\mu\nu} \mathcal{L}_u, \end{aligned} \quad (9)$$

$$T_{\mu\nu}^{(\phi)} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\rho \phi \partial_\rho \phi + V(\phi) \right), \quad (10)$$

$$T_{\mu\nu}^{(m)} = (\rho_m + p_m) \nu_\mu \nu_\nu + p_m g_{\mu\nu}, \quad (11)$$

where

$$J^\mu{}_\nu = -\beta_1 \nabla^\mu u_\nu - \beta_2 \delta_\nu^\mu \nabla_\tau u^\tau - \beta_3 \nabla_\nu u^\mu. \quad (12)$$

We take the following exponential potential

$$V(\phi) = V_0 e^{-\lambda\phi}, \quad (13)$$

The GB coupling is generally given by the sum of the exponential terms

$$f(\phi) = (f_0/\mu) e^{\mu\phi}. \quad (14)$$

With assuming ϕ depends on time, the (tt) components of equation (8) give

$$3H^2 = \dot{\phi}^2/2 + V(\phi) + 24f_{,\phi}\dot{\phi}H^3 + \rho_m + \rho_r - 3\beta H^2, \quad (15)$$

where

$$\beta = \beta_1 + 3\beta_2 + \beta_3. \quad (16)$$

On the other hand, variation of equation (1) with ϕ give,

$$\ddot{\phi} + 3H\dot{\phi} - \lambda V + 24f_{,\phi}\dot{\phi}H^2(H^2 + \dot{H}) = 0, \quad (17)$$

3. Dynamical System of the Model

In order to study cosmological dynamics in the presence of a scalar field, Gauss-Bonnet term and a background fluid, from equation of motion (15), it is convenient to introduce the following dimensionless variables:

$$x_1 = \frac{\dot{\phi}}{\sqrt{6\bar{\beta}H}}, \quad x_2 = \frac{\sqrt{V(\phi)}}{\sqrt{3\bar{\beta}H}}, \quad x_3 = f_{,\phi}H^2, \quad x_4 = \frac{\sqrt{\rho_r}}{\sqrt{3\bar{\beta}H}}, \quad (18)$$

where $\bar{\beta} = \beta + 1$. Density parameters of scalar field, Gauss-Bonnet term, radiation and matter, we have, respectively

$$\Omega_\phi = x_1^2 + x_2^2, \quad \Omega_{GB} = 8\sqrt{6}x_1x_3, \quad \Omega_r = x_4^2, \quad \Omega_m = \frac{\rho_m}{3H^2}. \quad (19)$$

These satisfy the relation $\Omega_m + \Omega_r + \Omega_\phi + \Omega_{GB} = 1$ from equation (15). Differentiating the variables with respect to the number of e-foldings $N = \ln a$ and take $\bar{\beta}$ as constant, we obtain

the following autonomous equation

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda\sqrt{\bar{\beta}}x_2^2 - 4\sqrt{6}x_3 - (x_1 + 4\sqrt{6}x_3)\frac{\dot{H}}{H^2}, \quad (20)$$

$$\frac{dx_2}{dN} = -x_2 \left(\frac{\sqrt{6}}{2}\lambda x_1 + \frac{\dot{H}}{H^2} \right), \quad (21)$$

$$\frac{dx_3}{dN} = x_3 \left(\sqrt{6}\mu\bar{\beta}x_1 + 2\frac{\dot{H}}{H^2} \right), \quad (22)$$

$$\frac{dx_4}{dN} = -x_4 \left(2 + \frac{\dot{H}}{H^2} \right), \quad (23)$$

Taking the time derivative of Equation (15) and eliminating the term $\ddot{\phi}$ by using Equation (17), we find

$$\begin{aligned} & \left[6\bar{\beta} - 72\sqrt{6}\bar{\beta}x_1x_3 + \left(\sqrt{6}\bar{\beta}x_1 + 24\sqrt{\bar{\beta}}x_3 \right) 24\sqrt{\bar{\beta}}x_3 \right] \frac{\dot{H}}{H^2} = \\ & \left(\sqrt{6}\bar{\beta}x_1 + 24\sqrt{\bar{\beta}}x_3 \right) \left(-2\sqrt{6}\bar{\beta}x_1 + 3\lambda\bar{\beta}x_2^2 - 24\sqrt{\bar{\beta}}x_3 \right) \\ & -3\bar{\beta}\sqrt{6}\bar{\beta}\lambda x_1x_2^2 + 144\mu\bar{\beta}\sqrt{\bar{\beta}}x_1^2x_3 - 9 \left(1 - x_1^2 - x_2^2 - 8\sqrt{6}x_1x_3 \right. \\ & \left. - x_4^2 \right) - 12\bar{\beta}x_4^2. \end{aligned} \quad (24)$$

For the absence of radiation ($x_4 = 0$), with parameters $\beta = 0.1$, $\lambda = 2$, and $\mu = 7$, we got critical points like on the Table 1. The point (a) is matter dominated solution with density parameter $\Omega_m = 1$. The kinetic energy of scalar field is dominant for the points (b) and (c) because of the density parameter $\Omega_\phi = 1$ but it just was contributed by kinetic variable x_1 (variable $x_2 = 0$ as the variable of scalar field potential). Point (d) is the solution of scalar field dominated era with the most contributed variable is x_2 . We identify the stable solution with see the trajectory of critical points and see which one become attractor point. Fig. 1.a is the plot of trajectory of phase space (x_2, x_3) by set $x_1 = 0$. We are able to see that point (d) is the attractor solution. So we can conclude that point (d) is stable.

In Fig. 1.b we plot the evolution of Ω_m , Ω_r , Ω_ϕ , and Ω_{GB} versus N . The evolution of each

| Point | x_1 | x_2 | x_3 | Ω_ϕ | Ω_m |
|-------|-------|-------|-------|---------------|------------|
| a | 0 | 0 | 0 | 0 | 1 |
| b | -1 | 0 | 0 | 1 | 0 |
| c | +1 | 0 | 0 | 1 | 0 |
| d | 0 | 0.965 | 0.074 | $\simeq 1$ | $\simeq 0$ |

Table 1. Critical points of Gauss-Bonnet model with vector field for $\bar{\beta} = 0.1$, $\lambda = 2$, and $\mu = 7$.

density parameter starting from the epoch of matter-radiation equality ($\Omega_r = \Omega_m = 0.5$). As we know before equality epoch, the universe was dominated by radiation. But after that era, the density parameter of radiation go down significantly till reach the minimum value $\Omega_r = 0.003$ at $N = 21$. Otherwise, after equality epoch, the density parameter of matter rise up and reach the maximum value $\Omega_m = 0.899$ at $N = 14$. We call this epoch is matter-dominated era. At last, the density parameter of matter go down to $\Omega_m = 0.31$ at $N = 21$. The density parameter

of the scalar field started to rise up at $N = 9$ and dominated at $N = 21$ with $\Omega_\phi = 0.529$. The density parameter of Gauss-Bonnet term rise up at $N = 18$ and reach the maximum value $\Omega_{GB} = 0.157$ at $N = 21$. We assumed that the dark energy consist of scalar field and Gauss-Bonnet term, so we have shown the domination of dark energy in the present universe ($N = 21$).

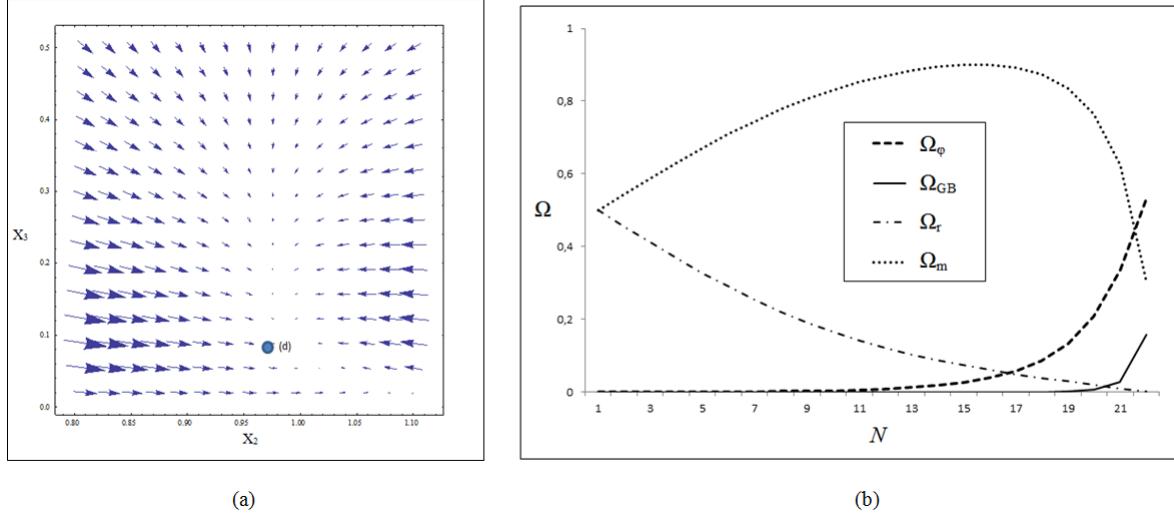


Figure 1. (a) The trajectory of the solution of Gauss-Bonnet model with vector field for $\bar{\beta} = 0.1$, $\mu = 7$, and $\lambda = 2$. (b) The evolution of variables Ω_ϕ , Ω_{GB} , Ω_r , and Ω_m for $\bar{\beta} = 11$, $\mu = 4$, and $\lambda = 23$. We choose initial condition $x_1 = 0.01$, $x_2 = 0.01$, $x_3 = 10^{-6}$, and $x_4 = 0.707$.

In Fig. 3 we show the evolution of each density parameters with variation of $\bar{\beta}$. At $N = 21$, the density parameter of each components for $\bar{\beta} = 12$ respectively $\Omega_r = 0.002$, $\Omega_m = 0.2273$, $\Omega_\phi = 0.447$, and $\Omega_{GB} = 0.276$. For $\bar{\beta} = 13$ at same N , the density parameter of each components are $\Omega_r = 0.001$, $\Omega_m = 0.196$, $\Omega_\phi = 0.346$, and $\Omega_{GB} = 0.456$. Evidently, the changed of $\bar{\beta}$ influence the value of each density parameters especially Ω_{GB} and Ω_m significantly (See Table 2). The result show that the increase of β will raise the density parameter value of Gauss-Bonnet term, but decrease the density parameter value of matter.

From the variations of $\bar{\beta}$ above, evolution of each density parameters for $\bar{\beta} = 12$ get near to the condition of present universe. The universe that showed at $N = 21$ is dominated by scalar field and Gauss-Bonnet term which represent the dark energy. The sum of density parameter both of them is $\Omega = 0.723$. The result is suitable with observations that show the universe today is dominated by 72% of dark energy [13,14].

| $\bar{\beta}$ | Ω_ϕ | Ω_{GB} | Ω_m | Ω_r |
|---------------|---------------|---------------|------------|------------|
| 11 | 0.529 | 0.157 | 0.310 | 0.003 |
| 12 | 0.447 | 0.276 | 0.273 | 0.002 |
| 13 | 0.346 | 0.456 | 0.196 | 0.001 |

Table 2. The values of each density parameter in Gauss-Bonnet model with vector field for variation of $\bar{\beta}$ at $N = 21$

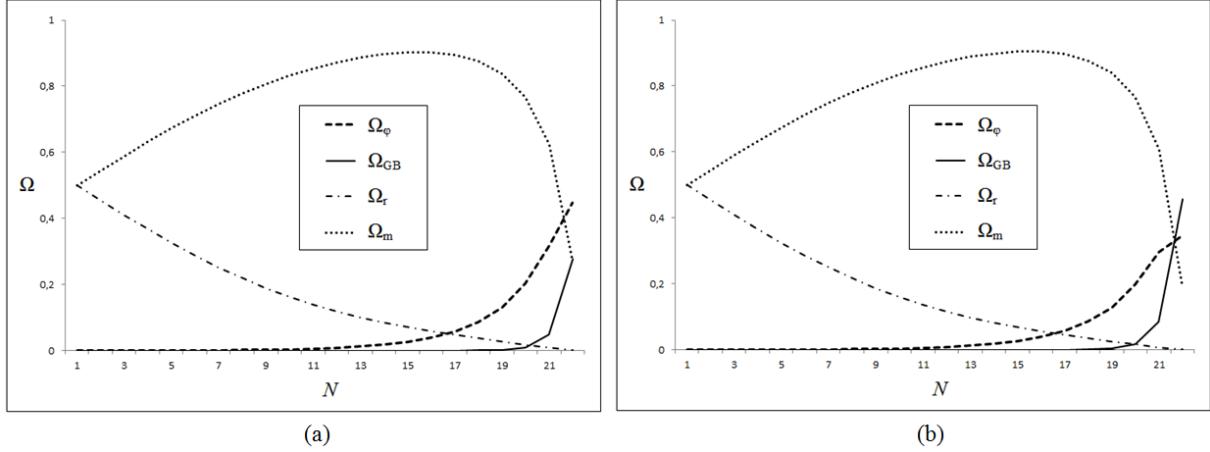


Figure 2. The evolution of variables Ω_ϕ , Ω_{GB} , Ω_r , and Ω_m with initial condition $x_1 = 0.01$, $x_2 = 0.01$, $x_3 = 10^{-6}$, and $x_4 = 0.707$ for : (a) $\bar{\beta} = 12$, $\mu = 4$, and $\lambda = 23$. (b) $\bar{\beta} = 13$, $\mu = 4$, and $\lambda = 23$

4. Conclusion

We studied The evolution of density parameter of matter, radiation, scalar field and Gauss-Bonnet term on Gauss-Bonnet Model with vector field. Parameter $\bar{\beta}$ which contributed by vector field give influence to the density parameters of the model. Variations of $\bar{\beta}$ give change of density parameters specially matter and Gauss-Bonnet term. The model that was analyzed represent the condition of the present universe today which dominated by dark energy. Scalar field and Gauss-Bonnet term in the model dominate the universe as dark energy in Gauss-Bonnet model with 72.3%.

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