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Nijenhuis Operators and Abelian Extensions of Hom-δ-Jordan Lie Supertriple Systems

Qiang Li and Lili Ma *

School of Science, Qiqihar University, Qiqihar 161006, China
* Correspondence: limary@163.com

Abstract: Representations and cohomologies of Hom-δ-Jordan Lie supertriple systems are established. As an application, Nijenhuis operators and abelian extensions of Hom-δ-Jordan Lie supertriple systems are discussed. We obtain the infinitesimal deformation generated by virtue of a Nijenhuis operator. It is obtained that the sufficient and necessary condition for the equivalence of abelian extensions of Hom-δ-Jordan Lie supertriple systems.

Keywords: Hom-δ-Jordan Lie supertriple system; cohomology; deformation; Nijenhuis operator; abelian extension

MSC: 17B40; 17B56; 17B75

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1. Introduction

In this paper, we pay our main attention to Hom-δ-Jordan Lie supertriple systems over an arbitrary field. As is well-known, Lie triple systems have important applications in elementary particle theory and the theory of quantum mechanics [1]. Some results on Lie triple systems, including simple Lie triple systems over an algebraically closed field [2], the Yamaguti cohomology theory [3,4], infinitesimal deformations, abelian extensions [5], etc. were studied. Okubo reformulated the para-statistics as Lie supertriple systems and explained the relationship between Lie supertriple systems and ortho-symplectic supertriple systems [6]. In [7], the authors introduced δ-Jordan Lie supertriple systems, which are a generalization of Lie supertriple systems. Some examples of quasi-classical δ-Lie supertriple systems were given over a field of characteristic not two [8]. The Jordan superalgebra of F-type from Jordan Lie supertriple systems were discussed [9]. Later, Ma and Chen obtained the cohomology and deformations of δ-Jordan Lie triple systems in 2017 [10]. In [11], the authors determined derivations and deformations of δ-Jordan Lie supertriple systems. In 2022, we obtained 1-parameter formal deformations and abelian extensions of Lie color triple systems [12].

In recent years, structures and representations of many Hom-type algebras have been obtained [13–24]. In particular, the notion of Hom-Lie triple systems was introduced in [14]. Hom-Lie triple systems are ternary Hom–Nambu algebras whose triple product is left anti-symmetric. When the twisting maps of Hom–Lie triple systems are all equal to the identity map, one recovers Lie triple systems. In 2019, Chen and his students gave that cohomologies and 1-parameter formal deformations of Hom-δ-Jordan Lie triple systems [25]. In 2022, we obtained central extensions and Nijenhuis operators of Hom-δ-Jordan Lie triple systems [26].

The purpose of this paper is to consider the cohomology theory and deformations of Hom-δ-Jordan Lie supertriple systems based on some work in [3,4,10,12,13,25–27]. The paper is organized as follows. Section 2 is devoted to some basic definitions and the cohomology theory of multiplicative Hom-δ-Jordan Lie supertriple systems. In Section 3, we discuss Nijenhuis operators of Hom-δ-Jordan Lie supertriple systems and obtain the infinitesimal deformation generated using a Nijenhuis operator. Section 4 is dedicated to the abelian extension theory of multiplicative Hom-δ-Jordan Lie supertriple systems. We show the sufficient and necessary condition for the equivalence of abelian extensions.

Throughout this paper, $\mathbf{F}$ denotes an arbitrary field, and its characteristic is zero.

2. Preliminaries

We first recall some basic facts and definitions of Hom-Lie triple systems.

Definition 1 ([14]). *A Hom-Lie triple system $(T, [\cdot,\cdot,\cdot], \alpha = (\alpha_1, \alpha_2))$ consists of an $\mathbf{F}$-vector space T, a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, and linear maps $\alpha_i : T \to T$ for $i = 1, 2$, called twisted maps, such that for all $x, y, z, u, v \in T$,*

$$[x, x, z] = 0,$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0,$$

$$\begin{aligned}[\alpha_1(u), \alpha_2(v), [x, y, z]] =& [[u, v, x], \alpha_1(y), \alpha_2(z)] + [\alpha_1(x), [u, v, y], \alpha_2(z)] \\ &+ [\alpha_1(x), \alpha_2(y), [u, v, z]].\end{aligned}$$

If $|x|$ occurs in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of x.

Definition 2 ([28]). *A Hom-Lie supertriple system $(T, [\cdot,\cdot,\cdot], \alpha = (\alpha_1, \alpha_2))$ consists of a $\mathbb{Z}_2$-graded vector space $T = T_{\bar{0}} \oplus T_{\bar{1}}$ over $\mathbf{F}$ together with a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, and even linear maps $\alpha_i : T \to T$ for $i = 1, 2$, called twisted maps, such that for all $x, y, z, u, v \in T$,*

$$[x, y, z] = -(-1)^{|x||y|}[y, x, z], \tag{1}$$

$$(-1)^{|x||z|}[x, y, z] + (-1)^{|y||x|}[y, z, x] + (-1)^{|z||y|}[z, x, y] = 0, \tag{2}$$

$$\begin{aligned}[\alpha_1(u), \alpha_2(v), [x, y, z]] =& [[u, v, x], \alpha_1(y), \alpha_2(z)] + (-1)^{|x|(|u|+|v|)}[\alpha_1(x), [u, v, y], \alpha_2(z)] \\ &+ (-1)^{(|u|+|v|)(|x|+|y|)}[\alpha_1(x), \alpha_2(y), [u, v, z]].\end{aligned} \tag{3}$$

Definition 3. *A Hom-δ-Jordan Lie supertriple system $(T, [\cdot,\cdot,\cdot], \alpha = (\alpha_1, \alpha_2))$ consists of a $\mathbb{Z}_2$-graded vector space $T = T_{\bar{0}} \oplus T_{\bar{1}}$ over $\mathbf{F}$ together with a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, and even linear maps $\alpha_i : T \to T$ for $i = 1, 2$, called twisted maps, such that for all $x, y, z, u, v \in T$, $\delta = \pm 1$,*

$$[x, y, z] = -\delta(-1)^{|x||y|}[y, x, z], \tag{4}$$

$$(-1)^{|x||z|}[x, y, z] + (-1)^{|y||x|}[y, z, x] + (-1)^{|z||y|}[z, x, y] = 0, \tag{5}$$

$$\begin{aligned}[\alpha_1(u), \alpha_2(v), [x, y, z]] =& [[u, v, x], \alpha_1(y), \alpha_2(z)] + (-1)^{|x|(|u|+|v|)}[\alpha_1(x), [u, v, y], \alpha_2(z)] \\ &+ \delta(-1)^{(|u|+|v|)(|x|+|y|)}[\alpha_1(x), \alpha_2(y), [u, v, z]].\end{aligned} \tag{6}$$

Clearly , $T_{\bar{0}}$ is a Hom-δ-Jordan Lie triple system, and the case of $\alpha = \mathrm{id}, \delta = 1$ defines a Lie triple system, so any Hom-δ-Jordan Lie supertriple system is the generalization of a Lie triple system.

A Hom-δ-Jordan Lie supertriple system is said to be multiplicative if $\alpha_1 = \alpha_2 = \alpha$ and $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$, and denoted by $(T, [\cdot,\cdot,\cdot], \alpha)$.

A morphism $f : (T, [\cdot,\cdot,\cdot], \alpha = (\alpha_1, \alpha_2)) \to (T', [\cdot,\cdot,\cdot]', \alpha' = (\alpha'_1, \alpha'_2))$ of Hom-δ-Jordan Lie supertriple system is a linear map satisfying $f([x, y, z]) = [f(x), f(y), f(z)]'$ and $f \circ \alpha_i = \alpha'_i \circ f$ for $i = 1, 2$. An isomorphism is a bijective morphism.

Following the representation theory of Lie triple systems was studied by Yamaguti, we generalize the notion of the representation to Hom-δ-Jordan Lie supertriple systems.

Definition 4. *Let $(T, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative Hom-δ-Jordan Lie supertriple system, V a $\mathbb{Z}_2$-graded vector space over $\mathbf{F}$, and $A \in \mathrm{End}(V)$. V is called a $(T, [\cdot,\cdot,\cdot], \alpha)$-module with respect to A if there exists a bilinear map $\theta : T \times T \to \mathrm{End}(V)$, $(a, b) \mapsto \theta(a, b)$ such that for all $a, b, c, d \in T$,*

$$\theta(\alpha(a), \alpha(b)) \circ A = A \circ \theta(a, b), \tag{7}$$

$$\begin{aligned}(-1)^{(|a|+|b|)(|c|+|d|)}\theta(\alpha(c), \alpha(d))\theta(a, b) - \delta(-1)^{|a||b|+|d|(|a|+|c|)}\theta(\alpha(b), \alpha(d))\theta(a, c)\\ -\theta(\alpha(a), [b, c, d]) \circ A + (-1)^{|a|(|b|+|c|)}D(\alpha(b), \alpha(c))\theta(a, d) = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}\delta(-1)^{(|a|+|b|)(|c|+|d|)}\theta(\alpha(c), \alpha(d))D(a, b) - \delta D(\alpha(a), \alpha(b))\theta(c, d)\\ +\theta([a, b, c], \alpha(d))\circ A + \delta(-1)^{|c|(|a|+|b|)}\theta(\alpha(c), [a, b, d])\circ A = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}\delta(-1)^{(|a|+|b|)(|c|+|d|)}D(\alpha(c), \alpha(d))D(a, b) - D(\alpha(a), \alpha(b))D(c, d)\\ +\delta D([a, b, c], \alpha(d)) \circ A + \delta(-1)^{|c|(|a|+|b|)}D(\alpha(c), [a, b, d]) \circ A = 0,\end{aligned} \tag{10}$$

where $D(a, b) = (-1)^{|a||b|}\theta(b, a) - \delta\theta(a, b)$.

Then, θ is called the representation of $(T, [\cdot,\cdot,\cdot], \alpha)$ on V with respect to A. In the case $\theta = 0$, V is called the trivial $(T, [\cdot,\cdot,\cdot], \alpha)$-module with respect to A.

In particular, let $V = T$, $A = \alpha$, and $\theta(x, y)(z) = (-1)^{|z|(|x|+|y|)}[z, x, y]$. Then, $D(x, y)(z) = \delta[x, y, z]$ and (7)–(10) hold. In this case, T is said to be the adjoint $(T, [\cdot,\cdot,\cdot], \alpha)$-module and θ is called the adjoint representation of $(T, [\cdot,\cdot,\cdot], \alpha)$ on itself with respect to α.

As is the case of general algebras, we introduce the semidirect product of a multiplicative Hom-δ-Jordan Lie supertriple systems and its module.

Proposition 1. *Let θ be a representation of a multiplicative Hom-δ-Jordan Lie supertriple system $(T, [\cdot,\cdot,\cdot], \alpha)$ on V with respect to A. Define the operation $[\cdot,\cdot,\cdot]_V : (T \oplus V) \times (T \oplus V) \times (T \oplus V) \to T \oplus V$ by*

$$[(x, a), (y, b), (z, c)]_V = ([x, y, z], (-1)^{|x|(|y|+|z|)}\theta(y, z)(a) - \delta(-1)^{|y||z|}\theta(x, z)(b) + \delta D(x, y)(c));$$

and define the twisted map $\alpha + A : T \oplus V \to T \oplus V$ by

$$(\alpha + A)(x, a) = (\alpha(x), A(a)).$$

Then $T \oplus V$ is a multiplicative Hom-δ-Jordan Lie supertriple system.

Proof. Using $D(x, y) = (-1)^{|x||y|}\theta(y, x) - \delta\theta(x, y)$, we obtain

$$\begin{aligned}&[(x, a), (y, b), (z, c)]_V\\ =&([x, y, z], (-1)^{|x|(|y|+|z|)}\theta(y, z)(a) - \delta(-1)^{|y||z|}\theta(x, z)(b) + \delta D(x, y)(c))\\ =&-\delta(-1)^{|x||y|}([y, x, z], (-1)^{|y|(|x|+|z|)}\theta(x, z)(b) - \delta(-1)^{|x||z|}\theta(y, z)(a) - (-1)^{|x||y|}D(x, y)(c))\\ =&-\delta(-1)^{|x||y|}([y, x, z], (-1)^{|y|(|x|+|z|)}\theta(x, z)(b) - \delta(-1)^{|x||z|}\theta(y, z)(a) + \delta D(y, x)(c))\\ =&-\delta(-1)^{|x||y|}[(y, b), (x, a), (z, c)]_V,\end{aligned}$$

and

$$(-1)^{|x||z|}[(x, a), (y, b), (z, c)]_V + (-1)^{|y||x|}[(y, b), (z, c), (x, a)]_V + (-1)^{|z||y|}[(z, c), (x, a), (y, b)]_V$$

$$
\begin{aligned}
=&((-1)^{|x||z|}[x,y,z],(-1)^{|x||y|}\theta(y,z)(a)-\delta(-1)^{|z|(|x|+|y|)}\theta(x,z)(b)+\delta(-1)^{|x||z|}D(x,y)(c))\\
&+((-1)^{|y||x|}[y,z,x],(-1)^{|y||z|}\theta(z,x)(b)-\delta(-1)^{|x|(|y|+|z|)}\theta(y,x)(c)+\delta(-1)^{|y||x|}D(y,z)(a))\\
&+((-1)^{|z||y|}[z,x,y],(-1)^{|z||x|}\theta(x,y)(c)-\delta(-1)^{|y|(|x|+|z|)}\theta(z,y)(a)+\delta(-1)^{|z||y|}D(z,x)(b))\\
=&(0,(-1)^{|x||y|}\theta(y,z)(a)-\delta(-1)^{|y|(|x|+|z|)}\theta(z,y)(a)+\delta(-1)^{|y||x|}D(y,z)(a)\\
&+(-1)^{|y||z|}\theta(z,x)(b)-\delta(-1)^{|z|(|x|+|y|}\theta(x,z)(b)+\delta(-1)^{|z||y|}D(z,x)(b)\\
&+(-1)^{|z||x|}\theta(x,y)(c)-\delta(-1)^{|x|(|y|+|z|)}\theta(y,x)(c)+\delta(-1)^{|x||z|}D(x,y)(c))\\
=&(0,0).
\end{aligned}
$$

By (8)–(10), it follows that

$$
\begin{aligned}
&[[(x,a),(y,b),(u,c)]_V,(\alpha+A)(v,d),(\alpha+A)(w,e)]_V\\
=&[([x,y,u],(-1)^{|x|(|y|+|u|)}\theta(y,u)(a)-\delta(-1)^{|y||u|}\theta(x,u)(b)+\delta D(x,y)(c)),\\
&(\alpha(v),A(d)),(\alpha(w),A(e))]_V\\
=&([[x,y,u],\alpha(v),\alpha(w)],(-1)^{(|x|+|y|+|u|)(|v|+|w|)}\theta(\alpha(v),\alpha(w))((-1)^{|x|(|y|+|u|)}\theta(y,u)(a)\\
&-\delta(-1)^{|y||u|}\theta(x,u)(b)+\delta D(x,y)(c))-\delta(-1)^{|v||w|}\theta([x,y,u],\alpha(w))(A(d))\\
&+\delta D([x,y,u],\alpha(v))(A(e))),
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{|u|(|x|+|y|)}[(\alpha+A)(u,c),[(x,a),(y,b),(v,d)]_V,(\alpha+A)(w,e)]_V\\
=&(-1)^{|u|(|x|+|y|)}[(\alpha(u),A(c)),([x,y,v],(-1)^{|x|(|y|+|v|)}\theta(y,v)(a)-\delta(-1)^{|y||v|}\theta(x,v)(b)\\
&+\delta D(x,y)(d)),(\alpha(w),A(e))]_V\\
=&(-1)^{|u|(|x|+|y|)}([\alpha(u),[x,y,v],\alpha(w)],(-1)^{|u|(|x|+|y|+|v|+|w|)}\theta([x,y,v],\alpha(w))(A(c))\\
&-\delta(-1)^{(|x|+|y|+|v|)|w|}\theta(\alpha(u),\alpha(w))((-1)^{|x|(|y|+|v|)}\theta(y,v)(a)-\delta(-1)^{|y||v|}\theta(x,v)(b)\\
&+\delta D(x,y)(d))+\delta D(\alpha(u),[x,y,v])(A(e))),
\end{aligned}
$$

$$
\begin{aligned}
&\delta(-1)^{(|x|+|y|)(|u|+|v|)}[(\alpha+A)(u,c),(\alpha+A)(v,d),[(x,a),(y,b),(w,e)]_V]_V\\
=&\delta(-1)^{(|x|+|y|)(|u|+|v|)}[(\alpha(u),A(c)),(\alpha(v),A(d)),([x,y,w],(-1)^{|x|(|y|+|w|)}\theta(y,w)(a)\\
&-\delta(-1)^{|y||w|}\theta(x,w)(b)+\delta D(x,y)(e))]_V\\
=&\delta(-1)^{(|x|+|y|)(|u|+|v|)}([\alpha(u),\alpha(v),[x,y,w]],(-1)^{|u|(|v|+|x|+|y|+|w|)}\theta(\alpha(v),[x,y,w])(A(c))\\
&-\delta(-1)^{|v|(|x|+|y|+|w|)}\theta(\alpha(u),[x,y,w])(A(d))+\delta D(\alpha(u),\alpha(v))((-1)^{|x|(|y|+|w|)}\theta(y,w)(a)\\
&-\delta(-1)^{|y||w|}\theta(x,w)(b)+\delta D(x,y)(e))),
\end{aligned}
$$

$$
\begin{aligned}
&[(\alpha+A)(x,a),(\alpha+A)(y,b),[(u,c),(v,d),(w,e)]_V]_V\\
=&[(\alpha(x),A(a)),(\alpha(y),A(b)),([u,v,w],(-1)^{|u|(|v|+|w|)}\theta(v,w)(c)-\delta(-1)^{|v||w|}\theta(u,w)(d)\\
&+\delta D(u,v)(e))]_V\\
=&([\alpha(x),\alpha(y),[u,v,w]],(-1)^{|x|(|y|+|u|+|v|+|w|)}\theta(\alpha(y),[u,v,w])(A(a))\\
&-\delta(-1)^{|y|(|u|+|v|+|w|)}\theta(\alpha(x),[u,v,w])(A(b))+\delta D(\alpha(x),\alpha(y))((-1)^{|u|(|v|+|w|)}\theta(v,w)(c)\\
&-\delta(-1)^{|v||w|}\theta(u,w)(d)+\delta D(u,v)(e))).
\end{aligned}
$$

The calculation above shows that (1)–(3) hold.

Note that $\alpha+A$ is a linear map and, using (7), we have

$$
\begin{aligned}
&(\alpha+A)[(x,a),(y,b),(z,c)]_V\\
=&(\alpha+A)([x,y,z],(-1)^{|x|(|y|+|z|)}\theta(y,z)(a)-\delta(-1)^{|y||z|}\theta(x,z)(b)+\delta D(x,y)(c))\\
=&(\alpha([x,y,z]),A\circ((-1)^{|x|(|y|+|z|)}\theta(y,z)(a)-\delta(-1)^{|y||z|}\theta(x,z)(b)+\delta D(x,y)(c)))
\end{aligned}
$$

$$
\begin{aligned}
=&([\alpha(x),\alpha(y),\alpha(z)],(-1)^{|x|(|y|+|z|)}\theta(\alpha(y),\alpha(z))A(a)-\delta(-1)^{|y||z|}\theta(\alpha(x),\alpha(z))A(b)\\
&+\delta D(\alpha(x),\alpha(y))A(c))\\
=&[(\alpha(x),A(a)),(\alpha(y),A(b)),(\alpha(z),A(c))]_V\\
=&[(\alpha+A)(x,a),(\alpha+A)(y,b),(\alpha+A)(z,c)]_V.
\end{aligned}
$$

Thus , $(T\oplus V,[\cdot,\cdot,\cdot]_V,\alpha+A)$ is a multiplicative Hom-δ-Jordan Lie supertriple system. □

Let θ be a representation of $(T,[\cdot,\cdot,\cdot],\alpha)$ on V with respect to A. If an n-linear map $f:\underbrace{T\times\cdots\times T}_{n\text{ times}}\to V$ satisfies

$$A(f(x_1,\cdots,x_n))=f(\alpha(x_1),\cdots,\alpha(x_n)),$$

$$f(x_1,\cdots,x,y,x_{n-2})=-(-1)^{|x||y|}f(x_1,\cdots,y,x,x_{n-2}),$$

$$
\begin{aligned}
&(-1)^{|x||z|}f(x_1,\cdots,x_{n-3},x,y,z)+(-1)^{|y||x|}f(x_1,\cdots,x_{n-3},y,z,x)\\
&+(-1)^{|z||y|}f(x_1,\cdots,x_{n-3},z,x,y)=0,
\end{aligned}
$$

then f is called an n-Hom-cochain on T. Denote by $C^n_{\alpha,A}(T,V)$ the set of all n-Hom-cochains, $\forall n\geq 1$.

Definition 5. *For $n=1,2,3,4$, the coboundary operator $d^n_{hom}:C^n_{\alpha,A}(T,V)\to C^{n+2}_{\alpha,A}(T,V)$ is defined as follows.*

- *If $f\in C^1_\alpha(T,V)$, then*
$$
\begin{aligned}
&d^1_{hom}f(x_1,x_2,x_3)\\
=&(-1)^{(|f|+|x_1|)(|x_2|+|x_3|)}\theta(x_2,x_3)f(x_1)-\delta(-1)^{|x_2||x_3|+|f|(|x_1|+|x_3|)}\\
&\theta(x_1,x_3)f(x_2)+\delta(-1)^{|f|(|x_1|+|x_2|)}D(x_1,x_2)f(x_3)-f([x_1,x_2,x_3]).
\end{aligned}
$$
- *If $f\in C^2_\alpha(T,V)$, then*
$$
\begin{aligned}
&d^2_{hom}f(y,x_1,x_2,x_3)\\
=&(-1)^{(|f|+|y|+|x_1|)(|x_2|+|x_3|)}\theta(\alpha(x_2),\alpha(x_3))f(y,x_1)-\delta(-1)^{|x_2||x_3|+(|f|+|y_1|)(|x_1|+|x_3|)}\\
&\theta(\alpha(x_1),\alpha(x_3))f(y,x_2)+\delta(-1)^{(|f|+|y|)(|x_1|+|x_2|)}D(\alpha(x_1),\alpha(x_2))f(y,x_3)\\
&-f(\alpha(y),[x_1,x_2,x_3]).
\end{aligned}
$$
- *If $f\in C^3_\alpha(T,V)$, then*
$$
\begin{aligned}
&d^3_{hom}f(x_1,x_2,x_3,x_4,x_5)\\
=&(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\theta(\alpha(x_4),\alpha(x_5))f(x_1,x_2,x_3)\\
&-\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\theta(\alpha(x_3),\alpha(x_5))f(x_1,x_2,x_4)\\
&-\delta(-1)^{|f|(|x_1|+|x_2|)}D(\alpha(x_1),\alpha(x_2))f(x_3,x_4,x_5)\\
&+(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)}D(\alpha(x_3),\alpha(x_4))f(x_1,x_2,x_5)\\
&+f([x_1,x_2,x_3],\alpha(x_4),\alpha(x_5))+(-1)^{|x_3|(|x_1|+|x_2|)}f(\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5))\\
&+\delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}f(\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5])-f(\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]).
\end{aligned}
$$
- *If $f\in C^4_\alpha(T,V)$, then*
$$
\begin{aligned}
&d^4_{hom}f(y,x_1,x_2,x_3,x_4,x_5)\\
=&(-1)^{(|f|+|y|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\theta(\alpha^2(x_4),\alpha^2(x_5))f(y,x_1,x_2,x_3)\\
&-\delta(-1)^{(|f|+|y|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\theta(\alpha^2(x_3),\alpha^2(x_5))f(y,x_1,x_2,x_4)\\
&-\delta(-1)^{(|f|+|y|)(|x_1|+|x_2|)}D(\alpha^2(x_1),\alpha^2(x_2))f(y,x_3,x_4,x_5)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{(|f|+|y|+|x_1|+|x_2|)(|x_3|+|x_4|)}D(\alpha^2(x_3),\alpha^2(x_4))f(y,x_1,x_2,x_5)\\
&+f(\alpha(y),[x_1,x_2,x_3],\alpha(x_4),\alpha(x_5))\\
&+(-1)^{|x_3|(|x_1|+|x_2|)}f(\alpha(y),\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5))\\
&+\delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}f(\alpha(y),\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5])\\
&-f(\alpha(y),\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]).
\end{aligned}
$$

Theorem 1. *The coboundary operator d^n_{hom} defined above satisfies $d^{n+2}_{hom}d^n_{hom}=0$, $n=1,2$.*

Proof. From the definition of the coboundary operator it follows immediately that $d^3_{hom}d^1_{hom}=0$ implies $d^4_{hom}d^2_{hom}=0$. Then we only need to prove $d^3_{hom}d^1_{hom}=0$. In fact, by (7)–(10), we get

$$
\begin{aligned}
&d^3_{hom}(d^1_{hom}f)(x_1,x_2,x_3,x_4,x_5)\\
=&(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\theta(\alpha(x_4),\alpha(x_5))(d^1_{hom}f)(x_1,x_2,x_3)\\
&-\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\theta(\alpha(x_3),\alpha(x_5))(d^1_{hom}f)(x_1,x_2,x_4)\\
&-\delta(-1)^{|f|(|x_1|+|x_2|)}D(\alpha(x_1),\alpha(x_2))(d^1_{hom}f)(x_3,x_4,x_5)\\
&+(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)}D(\alpha(x_3),\alpha(x_4))(d^1_{hom}f)(x_1,x_2,x_5)\\
&+(d^1_{hom}f)([x_1,x_2,x_3],\alpha(x_4),\alpha(x_5))+(-1)^{|x_3|(|x_1|+|x_2|)}(d^1_{hom}f)(\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5))\\
&+\delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}(d^1_{hom}f)(\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5])\\
&-(d^1_{hom}f)(\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5])\\
=&(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\theta(\alpha(x_4),\alpha(x_5))((-1)^{(|f|+|x_1|)(|x_2|+|x_3|)}\theta(x_2,x_3)f(x_1)\\
&-\delta(-1)^{|x_2||x_3|+|f|(|x_1|+|x_3|)}\theta(x_1,x_3)f(x_2)+\delta(-1)^{|f|(|x_1|+|x_2|)}D(x_1,x_2)f(x_3)-f([x_1,x_2,x_3]))\\
&-\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\theta(\alpha(x_3),\alpha(x_5))((-1)^{(|f|+|x_1|)(|x_2|+|x_4|)}\theta(x_2,x_4)f(x_1)\\
&-\delta(-1)^{|x_2||x_4|+|f|(|x_1|+|x_4|)}\theta(x_1,x_4)f(x_2)+\delta(-1)^{|f|(|x_1|+|x_2|)}D(x_1,x_2)f(x_4)-f([x_1,x_2,x_4]))\\
&-\delta(-1)^{|f|(|x_1|+|x_2|)}D(\alpha(x_1),\alpha(x_2))((-1)^{(|f|+|x_3|)(|x_4|+|x_5|)}\theta(x_4,x_5)f(x_3)\\
&-\delta(-1)^{|x_4||x_5|+|f|(|x_3|+|x_5|)}\theta(x_3,x_5)f(x_4)+\delta(-1)^{|f|(|x_3|+|x_4|)}D(x_3,x_4)f(x_5)-f([x_3,x_4,x_5]))\\
&+(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)}D(\alpha(x_3),\alpha(x_4))((-1)^{(|f|+|x_1|)(|x_2|+|x_5|)}\theta(x_2,x_5)f(x_1)\\
&-\delta(-1)^{|x_2||x_5|+|f|(|x_1|+|x_5|)}\theta(x_1,x_5)f(x_2)+\delta(-1)^{|f|(|x_1|+|x_2|)}D(x_1,x_2)f(x_5)-f([x_1,x_2,x_5]))\\
&+(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}(\theta(\alpha(x_4),\alpha(x_5))f([x_1,x_2,x_3])\\
&-\delta(-1)^{|x_4||x_5|+|f|(|x_1|+|x_2|+|x_3|+|x_5|)}\theta([x_1,x_2,x_3],\alpha(x_5))f(\alpha(x_4))\\
&+\delta(-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)}D([x_1,x_2,x_3],\alpha(x_4))f(\alpha(x_5))-f([[x_1,x_2,x_3],\alpha(x_4),\alpha(x_5)]))\\
&+(-1)^{|x_3|(|x_1|+|x_2|)+(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)}(\theta([x_1,x_2,x_4],\alpha(x_5))f(\alpha(x_3))\\
&-\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\theta(\alpha(x_3),\alpha(x_5))f([x_1,x_2,x_4])\\
&+\delta(-1)^{|x_3|(|x_1|+|x_2|)+|f|(|x_1|+|x_2|+|x_3|+|x_4|)}D(\alpha(x_3),[x_1,x_2,x_4])f(\alpha(x_5))\\
&-(-1)^{|x_3|(|x_1|+|x_2|)}f([\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5)]))\\
&+\delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}((-1)^{(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)}\theta(\alpha(x_4),[x_1,x_2,x_5])f(\alpha(x_3))\\
&-\delta(-1)^{|x_4|(|x_1|+|x_2|+|x_5|)+|f|(|x_1|+|x_2|+|x_3|+|x_5|)}\theta(\alpha(x_3),[x_1,x_2,x_5])f(\alpha(x_4))\\
&+\delta(-1)^{|f|(|x_3|+|x_4|)}D(\alpha(x_3),\alpha(x_4))f([x_1,x_2,x_5])-f([\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5]]))\\
&-((-1)^{(|f|+|x_1|)(|x_2|+|x_3|+|x_4|+|x_5|)}\theta(\alpha(x_2),[x_3,x_4,x_5])f(\alpha(x_1))\\
&-\delta(-1)^{|f|(|x_1|+|x_3|+|x_4|+|x_5|)+|x_2|(|x_3|+|x_4|+|x_5|)}\theta(\alpha(x_1),[x_3,x_4,x_5])f(\alpha(x_2))\\
&+\delta(-1)^{|f|(|x_1|+|x_2|)}D(\alpha(x_1),\alpha(x_2))f([x_3,x_4,x_5])-f([\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]]))
\end{aligned}
$$

$$
\begin{aligned}
=& -f([[x_1,x_2,x_3],\alpha(x_4),\alpha(x_5)]) - (-1)^{|x_3|(|x_1|+|x_2|)} f([\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5)]) \\
& - \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f([\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5]]) + f([\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]]) \\
& + (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)+(|f|+|x_1|)(|x_2|+|x_3|)} \theta(\alpha(x_4),\alpha(x_5))\theta(x_2,x_3)f(x_1) \\
& - \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|+(|f|+|x_1|)(|x_2|+|x_4|)} \theta(\alpha(x_3),\alpha(x_5))\theta(x_2,x_4)f(x_1) \\
& + (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)+(|f|+|x_1|)(|x_2|+|x_5|)} D(\alpha(x_3),\alpha(x_4))\theta(x_2,x_5)f(x_1) \\
& - (-1)^{(|f|+|x_1|)(|x_2|+|x_3|+|x_4|+|x_5|)} \theta(\alpha(x_2),[x_3,x_4,x_5])f(\alpha(x_1)) \\
& - \delta(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)+|x_2||x_3|+|f|(|x_1|+|x_3|)} \theta(\alpha(x_4),\alpha(x_5))\theta(x_1,x_3)f(x_2) \\
& + (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4|(|x_2|+|x_5|)+|f|(|x_1|+|x_4|)} \theta(\alpha(x_3),\alpha(x_5))\theta(x_1,x_4)f(x_2) \\
& - \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)+|x_2||x_5|+|f|(|x_1|+|x_5|)} D(\alpha(x_3),\alpha(x_4))\theta(x_1,x_5)f(x_2) \\
& + \delta(-1)^{|f|(|x_1|+|x_3|+|x_4|+|x_5|)+|x_2|(|x_3|+|x_4|+|x_5|)} \theta(\alpha(x_1),[x_3,x_4,x_5])f(\alpha(x_2)) \\
& + \delta(-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)+|f|(|x_1|+|x_2|)} \theta(\alpha(x_4),\alpha(x_5))D(x_1,x_2)f(x_3) \\
& - \delta(-1)^{|f|(|x_1|+|x_2|)+(|f|+|x_3|)(|x_4|+|x_5|)} D(\alpha(x_1),\alpha(x_2))\theta(x_4,x_5)f(x_3) \\
& + (-1)^{|x_3|(|x_1|+|x_2|)+(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)} \theta([x_1,x_2,x_4],\alpha(x_5))f(\alpha(x_3)) \\
& + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)+(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)} \theta(\alpha(x_4),[x_1,x_2,x_5])f(\alpha(x_3)) \\
& - (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|+|f|(|x_1|+|x_2|)} \theta(\alpha(x_3),\alpha(x_5))D(x_1,x_2)f(x_4) \\
& + (-1)^{|f|(|x_1|+|x_2|)+|x_4||x_5|+|f|(|x_3|+|x_5|)} D(\alpha(x_1),\alpha(x_2))\theta(x_3,x_5)f(x_4) \\
& - \delta(-1)^{|x_4||x_5|+|f|(|x_1|+|x_2|+|x_3|+|x_5|)} \theta([x_1,x_2,x_3],\alpha(x_5))f(\alpha(x_4)) \\
& - (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)+|x_4|(|x_1|+|x_2|+|x_5|)+|f|(|x_1|+|x_2|+|x_3|+|x_5|)} \theta(\alpha(x_3),[x_1,x_2,x_5])f(\alpha(x_4)) \\
& - (-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)} D(\alpha(x_1),\alpha(x_2))D(x_3,x_4)f(x_5) \\
& + \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)+|f|(|x_1|+|x_2|)} D(\alpha(x_3),\alpha(x_4))D(x_1,x_2)f(x_5) \\
& + \delta(-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)} D([x_1,x_2,x_3],\alpha(x_4))f(\alpha(x_5)) \\
& + \delta(-1)^{|x_3|(|x_1|+|x_2|)+|f|(|x_1|+|x_2|+|x_3|+|x_4|)} D(\alpha(x_3),[x_1,x_2,x_4])f(\alpha(x_5)) \\
& - (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(\alpha(x_4),\alpha(x_5))f([x_1,x_2,x_3]) \\
& + \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(\alpha(x_3),\alpha(x_5))f([x_1,x_2,x_4]) \\
& + \delta(-1)^{|f|(|x_1|+|x_2|)} D(\alpha(x_1),\alpha(x_2))f([x_3,x_4,x_5]) \\
& - (-1)^{|f|(|x_3|+|x_4|)+(|x_1|+|x_2|)(|x_3|+|x_4|)} D(\alpha(x_3),\alpha(x_4))f([x_1,x_2,x_5]) \\
& + (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(\alpha(x_4),\alpha(x_5))f([x_1,x_2,x_3]) \\
& - \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(\alpha(x_3),\alpha(x_5))f([x_1,x_2,x_4]) \\
& + (-1)^{|f|(|x_3|+|x_4|)+(|x_1|+|x_2|)(|x_3|+|x_4|)} D(\alpha(x_3),\alpha(x_4))f([x_1,x_2,x_5]) \\
& - \delta(-1)^{|f|(|x_1|+|x_2|)} D(\alpha(x_1),\alpha(x_2))f([x_3,x_4,x_5]) \\
=&0.
\end{aligned}
$$

Therefore, the proof is complete. □

For $n = 1,2,3,4$, the map $f \in C^n_{\alpha,A}(T,V)$ is called an n-Hom-cocycle if $d^n_{hom}f = 0$. We denote by $Z^n_{\alpha,A}(T,V)$ the subspace spanned by n-Hom-cocycles and $B^n_{\alpha,A}(T,V) = d^{n-2}_{hom}C^{n-2}_{\alpha,A}(T,V)$.

Since $d^{n+2}_{hom}d^n_{hom} = 0$, $B^n_{\alpha,A}(T,V)$ is a subspace of $Z^n_{\alpha,A}(T,V)$. Hence, we can define a cohomology space $H^n_{\alpha,A}(T,V)$ of $(T,[\cdot,\cdot,\cdot],\alpha)$ as the factor space $Z^n_{\alpha,A}(T,V)/B^n_{\alpha,A}(T,V)$.

3. Nijenhuis Operators of Hom-δ-Jordan Lie Supertriple Systems

In this section, we study infinitesimal deformations of Hom-δ-Jordan Lie supertriple systems. We introduce the notion of Nijenhuis operators for Hom-δ-Jordan Lie supertriple systems, and obtain trivial deformations using this kind of Nijenhuis operators.

Let $(T, [\cdot,\cdot,\cdot], \alpha)$ be a Hom-δ-Jordan Lie supertriple system and $\psi : T \times T \times T \to T$ be an even trilinear map. Consider a λ-parametrized family of linear operations:

$$[x_1, x_2, x_3]_\lambda = [x_1, x_2, x_3] + \lambda\psi(x_1, x_2, x_3),$$

where λ is a formal variable, and $\lambda \neq 0$.

If $[\cdot,\cdot,\cdot]_\lambda$ endow T with Hom-δ-Jordan Lie supertriple system structure, which is denoted by T_λ, then we call that ψ generates a λ-parameter infinitesimal deformation of Hom-δ-Jordan Lie supertriple system.

Theorem 2. *ψ generates a λ-parameter infinitesimal deformation of Hom-δ-Jordan Lie supertriple system T is equivalent to (i) ψ itself defines a Hom-δ-Jordan Lie supertriple system structure on T and (ii) ψ is a 3-Hom-cocycle of T.*

Proof.

$$[x_1, x_2, x_3]_\lambda = [x_1, x_2, x_3] + \lambda\psi(x_1, x_2, x_3),$$

and

$$-\delta(-1)^{|x_1||x_2|}[x_2, x_1, x_3]_\lambda = -\delta(-1)^{|x_1||x_2|}[x_2, x_1, x_3] - \delta\lambda(-1)^{|x_1||x_2|}\psi(x_2, x_1, x_3),$$

we have

$$\psi(x_1, x_2, x_3) = -\delta(-1)^{|x_1||x_2|}\psi(x_2, x_1, x_3). \tag{11}$$

From the equality

$$\begin{aligned}
0 =&(-1)^{|x_1||x_3|}[x_1, x_2, x_3]_\lambda + (-1)^{|x_2||x_1|}[x_2, x_3, x_1]_\lambda + (-1)^{|x_3||x_2|}[x_3, x_1, x_2]_\lambda \\
=&(-1)^{|x_1||x_3|}[x_1, x_2, x_3] + (-1)^{|x_2||x_1|}[x_2, x_3, x_1] + (-1)^{|x_3||x_2|}[x_3, x_1, x_2] \\
&+ \lambda((-1)^{|x_1||x_3|}\psi(x_1, x_2, x_3) + (-1)^{|x_2||x_1|}\psi(x_2, x_3, x_1) + (-1)^{|x_3||x_2|}\psi(x_3, x_1, x_2)),
\end{aligned}$$

it follows that

$$(-1)^{|x_1||x_3|}\psi(x_1, x_2, x_3) + (-1)^{|x_2||x_1|}\psi(x_2, x_3, x_1) + (-1)^{|x_3||x_2|}\psi(x_3, x_1, x_2) = 0. \tag{12}$$

For the equality

$$\begin{aligned}
&[\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]_\lambda]_\lambda \\
&= [[x_1, x_2, y_1]_\lambda, \alpha(y_2), \alpha(y_3)]_\lambda + (-1)^{|y_1|(|x_1|+|x_2|)}[\alpha(y_1), [x_1, x_2, y_2]_\lambda, \alpha(y_3)]_\lambda \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]_\lambda]_\lambda,
\end{aligned}$$

the left hand side is equal to

$$\begin{aligned}
&[\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3] + \lambda\psi(y_1, y_2, y_3)]_\lambda \\
=&[\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]] + \lambda\psi(\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]) \\
&+ [\alpha(x_1), \alpha(x_2), \lambda\psi(y_1, y_2, y_3)] + \lambda\psi(\alpha(x_1), \alpha(x_2), \lambda\psi(y_1, y_2, y_3)) \\
=&[\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]] + \lambda(\psi(\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]) + [\alpha(x_1), \alpha(x_2), \psi(y_1, y_2, y_3)]) \\
&+ \lambda^2\psi(\alpha(x_1), \alpha(x_2), \psi(y_1, y_2, y_3)),
\end{aligned}$$

and the right hand side is equal to

$$
\begin{aligned}
&[[x_1, x_2, y_1] + \lambda\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3)]_\lambda + (-1)^{|y_1|(|x_1|+|x_2|)}[\alpha(y_1), [x_1, x_2, y_2] \\
&+ \lambda\psi(x_1, x_2, y_2), \alpha(y_3)]_\lambda + \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3] + \lambda\psi(x_1, x_2, y_3)]_\lambda \\
=&[[x_1, x_2, y_1], \alpha(y_2), \alpha(y_3)] + (-1)^{|y_1|(|x_1|+|x_2|)}[\alpha(y_1), [x_1, x_2, y_2], \alpha(y_3)] \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]] + \lambda(\psi([x_1, x_2, y_1], \alpha(y_2), \alpha(y_3)) \\
&+ [\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3)] + (-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), [x_1, x_2, y_2], \alpha(y_3)) \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\alpha(y_1), \psi(x_1, x_2, y_2), \alpha(y_3)] \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]) \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[\alpha(y_1), \alpha(y_2), \psi(x_1, x_2, y_3)]) + \lambda^2(\psi(\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3)) \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), \psi(x_1, x_2, y_2), \alpha(y_3)) \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1), \alpha(y_2), \psi(x_1, x_2, y_3))).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\psi(\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]) + \delta D(\alpha(x_1), \alpha(x_2))\psi(y_1, y_2, y_3) \\
=&\psi([x_1, x_2, y_1], \alpha(y_2), \alpha(y_3)) + (-1)^{(|x_1|+|x_2|+|y_1|)(|y_2|+|y_3|)}\theta(\alpha(y_2), \alpha(y_3))\psi(x_1, x_2, y_1) \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), [x_1, x_2, y_2], \alpha(y_3)) \\
&- \delta(-1)^{|y_1|(|x_1|+|x_2|)}\theta(\alpha(y_1), \alpha(y_3))\psi(x_1, x_2, y_2) \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]) \\
&+ (-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}D(\alpha(y_1), \alpha(y_2))\psi(x_1, x_2, y_3).
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\psi(\alpha(x_1), \alpha(x_2), \psi(y_1, y_2, y_3)) \\
=&\psi(\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3)) + (-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), \psi(x_1, x_2, y_2), \alpha(y_3)) \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1), \alpha(y_2), \psi(x_1, x_2, y_3)).
\end{aligned} \tag{14}
$$

Therefore, ψ defines a Hom-δ-Jordan Lie supertriple system structure on T by (11)–(14). Furthermore, by (13) ψ is a 3-Hom-cocycle. □

A deformation is said to be trivial if there exists an even linear map $N : T \to T$ such that for $\varphi_\lambda = \mathrm{id} + \lambda N : T_\lambda \to T$ we have

$$
\varphi_\lambda[x_1, x_2, x_3]_\lambda = [\varphi_\lambda x_1, \varphi_\lambda x_2, \varphi_\lambda x_3]. \tag{15}
$$

It is clear that

$$
\begin{aligned}
\varphi_\lambda[x_1, x_2, x_3]_\lambda =&[x_1, x_2, x_3] + \lambda\psi(x_1, x_2, x_3) + \lambda N([x_1, x_2, x_3] + \lambda\psi(x_1, x_2, x_3)) \\
=&[x_1, x_2, x_3] + \lambda(\psi(x_1, x_2, x_3) + N[x_1, x_2, x_3]) + \lambda^2 N\psi(x_1, x_2, x_3),
\end{aligned}
$$

and

$$
\begin{aligned}
[\varphi_\lambda x_1, \varphi_\lambda x_2, \varphi_\lambda x_3] =&[x_1 + \lambda N x_1, x_2 + \lambda N x_2, x_3 + \lambda N x_3] \\
=&[x_1, x_2, x_3] + \lambda([Nx_1, x_2, x_3] + [x_1, Nx_2, x_3] + [x_1, x_2, Nx_3]) \\
&+ \lambda^2([Nx_1, Nx_2, x_3] + [Nx_1, x_2, Nx_3] + [x_1, Nx_2, Nx_3]) \\
&+ \lambda^3[Nx_1, Nx_2, Nx_3].
\end{aligned}
$$

Thus, we have

$$\begin{aligned}\psi(x_1,x_2,x_3)=&[Nx_1,x_2,x_3]+[x_1,Nx_2,x_3]+[x_1,x_2,Nx_3]-N[x_1,x_2,x_3]\\=&(-1)^{|x_1|(|x_2|+|x_3|)}\theta(x_2,x_3)N(x_1)-\delta(-1)^{|x_2||x_3|}\theta(x_1,x_3)N(x_2)\\&+\delta D(x_1,x_2)N(x_3)-N[x_1,x_2,x_3]\end{aligned}\tag{16}$$

$$N\psi(x_1,x_2,x_3)=[Nx_1,Nx_2,x_3]+[Nx_1,x_2,Nx_3]+[x_1,Nx_2,Nx_3]\tag{17}$$

$$0=[Nx_1,Nx_2,Nx_3]\tag{18}$$

From the cohomology theory discussed in Section 2, (16) can be represented in terms of 1-coboundary as $\psi=d^1_{hom}N$. Moreover, it follows from (16) and (17) that N must satisfy the following condition

$$\begin{aligned}N^2[x_1,x_2,x_3]=&N[Nx_1,x_2,x_3]+N[x_1,Nx_2,x_3]+N[x_1,x_2,Nx_3]\\&-([Nx_1,Nx_2,x_3]+[Nx_1,x_2,Nx_3]+[x_1,Nx_2,Nx_3]).\end{aligned}\tag{19}$$

In the following, we denote by

$$\psi(x_1,x_2,x_3)=[x_1,x_2,x_3]_N,\tag{20}$$

then (17) is equivalent to

$$N[x_1,x_2,x_3]_N=[Nx_1,Nx_2,x_3]+[Nx_1,x_2,Nx_3]+[x_1,Nx_2,Nx_3].\tag{21}$$

Definition 6. *An even linear operator $N:T\to T$ is called a Nijenhuis operator if and only if $N\circ\alpha=\alpha\circ N$ and (18) and (19) hold.*

Theorem 3. *Let N be a Nijenhuis operator for T. Then, a deformation of T can be obtained by putting*

$$\begin{aligned}\psi(x_1,x_2,x_3)=&(-1)^{|x_1|(|x_2|+|x_3|)}\theta(x_2,x_3)N(x_1)-\delta(-1)^{|x_2||x_3|}\theta(x_1,x_3)N(x_2)\\&+\delta D(x_1,x_2)N(x_3)-N[x_1,x_2,x_3].\end{aligned}$$

Furthermore, this deformation is a trivial one.

Proof. It is obvious that $\psi=dN$ and $d\psi=d^2N=0$. Thus, ψ is a 3-Hom-cocycle of T. Now, we check that (3) holds for ψ. Considering (16), (20) and (21), it follows that

$$\begin{aligned}&\psi(\alpha(x_1),\alpha(x_2),\psi(y_1,y_2,y_3))\\=&[\alpha(x_1),\alpha(x_2),[Ny_1,y_2,y_3]+[y_1,Ny_2,y_3]+[y_1,y_2,Ny_3]-N[y_1,y_2,y_3]]_N\\=&[\alpha(x_1),\alpha(x_2),[Ny_1,y_2,y_3]]_N+[\alpha(x_1),\alpha(x_2),[y_1,Ny_2,y_3]]_N\\&+[\alpha(x_1),\alpha(x_2),[y_1,y_2,Ny_3]]_N-[\alpha(x_1),\alpha(x_2),N[y_1,y_2,y_3]]_N\\=&[N\alpha(x_1),\alpha(x_2),[Ny_1,y_2,y_3]]+[\alpha(x_1),N\alpha(x_2),[Ny_1,y_2,y_3]]\\&+[\alpha(x_1),\alpha(x_2),N[Ny_1,y_2,y_3]]-N[\alpha(x_1),\alpha(x_2),[Ny_1,y_2,y_3]]\\&+[N\alpha(x_1),\alpha(x_2),[y_1,Ny_2,y_3]]+[\alpha(x_1),N\alpha(x_2),[y_1,Ny_2,y_3]]\\&+[\alpha(x_1),\alpha(x_2),N[y_1,Ny_2,y_3]]-N[\alpha(x_1),\alpha(x_2),[y_1,Ny_2,y_3]]\\&+[N\alpha(x_1),\alpha(x_2),[y_1,y_2,Ny_3]]+[\alpha(x_1),N\alpha(x_2),[y_1,y_2,Ny_3]]\\&+[\alpha(x_1),\alpha(x_2),N[y_1,y_2,Ny_3]]-N[\alpha(x_1),\alpha(x_2),[y_1,y_2,Ny_3]]\\&-[N\alpha(x_1),\alpha(x_2),N[y_1,y_2,y_3]]-[\alpha(x_1),N\alpha(x_2),N[y_1,y_2,y_3]]\\&-[\alpha(x_1),\alpha(x_2),N^2[y_1,y_2,y_3]]+N[\alpha(x_1),\alpha(x_2),N[y_1,y_2,y_3]]\\=&[N\alpha(x_1),\alpha(x_2),[Ny_1,y_2,y_3]]+[\alpha(x_1),N\alpha(x_2),[Ny_1,y_2,y_3]]\end{aligned}$$

$$
\begin{aligned}
&- N[\alpha(x_1), \alpha(x_2), [Ny_1, y_2, y_3]] + [N\alpha(x_1), \alpha(x_2), [y_1, Ny_2, y_3]]\\
&+ [\alpha(x_1), N\alpha(x_2), [y_1, Ny_2, y_3]] - N[\alpha(x_1), \alpha(x_2), [y_1, Ny_2, y_3]]\\
&+ [N\alpha(x_1), \alpha(x_2), [y_1, y_2, Ny_3]] + [\alpha(x_1), N\alpha(x_2), [y_1, y_2, Ny_3]]\\
&- N[\alpha(x_1), \alpha(x_2), [y_1, y_2, Ny_3]] - [N\alpha(x_1), \alpha(x_2), N[y_1, y_2, y_3]]\\
&- [\alpha(x_1), N\alpha(x_2), N[y_1, y_2, y_3]] + N[\alpha(x_1), \alpha(x_2), N[y_1, y_2, y_3]]\\
&+ [\alpha(x_1), \alpha(x_2), [Ny_1, Ny_2, y_3] + [\alpha(x_1), \alpha(x_2), [Ny_1, y_2, Ny_3]\\
&+ [\alpha(x_1), \alpha(x_2), [y_1, Ny_2, Ny_3]].
\end{aligned}
$$

Similarly, a direct computation shows that

$$
\begin{aligned}
&\psi(\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3))\\
=&[[Nx_1, Nx_2, y_1], \alpha(y_2), \alpha(y_3)] + [[Nx_1, x_2, Ny_1], \alpha(y_2), \alpha(y_3)]\\
&+ [[x_1, Nx_2, Ny_1], \alpha(y_2), \alpha(y_3)]\\
&+ [[Nx_1, x_2, y_1], N\alpha(y_2), \alpha(y_3)] + [[Nx_1, x_2, y_1], \alpha(y_2), N\alpha(y_3)]\\
&- N[[Nx_1, x_2, y_1], \alpha(y_2), \alpha(y_3)] + [[x_1, Nx_2, y_1], N\alpha(y_2), \alpha(y_3)]\\
&+ [[x_1, Nx_2, y_1], \alpha(y_2), N\alpha(y_3)] - N[[x_1, Nx_2, y_1], \alpha(y_2), \alpha(y_3)]\\
&+ [[x_1, x_2, Ny_1], N\alpha(y_2), \alpha(y_3)] + [[x_1, x_2, Ny_1], \alpha(y_2), N\alpha(y_3)]\\
&- N[[x_1, x_2, Ny_1], \alpha(y_2), \alpha(y_3)] - [N[x_1, x_2, y_1], N\alpha(y_2), \alpha(y_3)]\\
&- [N[x_1, x_2, y_1], \alpha(y_2), N\alpha(y_3)] + N[N[x_1, x_2, y_1], \alpha(y_2), \alpha(y_3)],
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), \psi(x_1, x_2, y_2), \alpha(y_3))\\
=&(-1)^{|y_1|(|x_1|+|x_2|)}([N\alpha(y_1), [Nx_1, x_2, y_2], \alpha(y_3)] + [\alpha(y_1), [Nx_1, x_2, y_2], N\alpha(y_3)]\\
&- N[\alpha(y_1), [Nx_1, x_2, y_2], \alpha(y_3)] + [N\alpha(y_1), [x_1, Nx_2, y_2], \alpha(y_3)]\\
&+ [\alpha(y_1), [x_1, Nx_2, y_2], N\alpha(y_3)] - N[\alpha(y_1), [x_1, Nx_2, y_2], \alpha(y_3)]\\
&+ [N\alpha(y_1), [x_1, x_2, Ny_2], \alpha(y_3)] + [\alpha(y_1), [x_1, x_2, Ny_2], N\alpha(y_3)]\\
&- N[\alpha(y_1), [x_1, x_2, Ny_2], \alpha(y_3)] - [N\alpha(y_1), N[x_1, x_2, y_2], \alpha(y_3)]\\
&- [\alpha(y_1), N[x_1, x_2, y_2], N\alpha(y_3)] + N[\alpha(y_1), N[x_1, x_2, y_2], \alpha(y_3)]\\
&+ [\alpha(y_1), [Nx_1, Nx_2, y_2], \alpha(y_3)] + [\alpha(y_1), [Nx_1, x_2, Ny_2], \alpha(y_3)]\\
&+ [\alpha(y_1), [x_1, Nx_2, Ny_2], \alpha(y_3)]),
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1), \alpha(y_2), \psi(x_1, x_2, y_3))\\
=&\delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}([N\alpha(y_1), \alpha(y_2), [Nx_1, x_2, y_3]] + [\alpha(y_1), N\alpha(y_2), [Nx_1, x_2, y_3]]\\
&- N[\alpha(y_1), \alpha(y_2), [Nx_1, x_2, y_3]] + [N\alpha(y_1), \alpha(y_2), [x_1, Nx_2, y_3]]\\
&+ [\alpha(y_1), N\alpha(y_2), [x_1, Nx_2, y_3]] - N[\alpha(y_1), \alpha(y_2), [x_1, Nx_2, y_3]]\\
&+ [N\alpha(y_1), \alpha(y_2), [x_1, x_2, Ny_3]] + [\alpha(y_1), N\alpha(y_2), [x_1, x_2, Ny_3]]\\
&- N[\alpha(y_1), \alpha(y_2), [x_1, x_2, Ny_3]] - [N\alpha(y_1), \alpha(y_2), N[x_1, x_2, y_3]]\\
&- [\alpha(y_1), N\alpha(y_2), N[x_1, x_2, y_3]] + N[\alpha(y_1), \alpha(y_2), N[x_1, x_2, y_3]]\\
&+ [\alpha(y_1), \alpha(y_2), [Nx_1, Nx_2, y_3]] + [\alpha(y_1), \alpha(y_2), [Nx_1, x_2, Ny_3]]\\
&+ [\alpha(y_1), \alpha(y_2), [x_1, Nx_2, Ny_3]]).
\end{aligned}
$$

By the definition of N, we know that $N \circ \alpha = \alpha \circ N$. Using (3) and (21), and Theorem 2, it follows that

$$
\begin{aligned}
&\psi(\alpha(x_1), \alpha(x_2), \psi(y_1, y_2, y_3)) - \psi(\psi(x_1, x_2, y_1), \alpha(y_2), \alpha(y_3))\\
&- (-1)^{|y_1|(|x_1|+|x_2|)}\psi(\alpha(y_1), \psi(x_1, x_2, y_2), \alpha(y_3))
\end{aligned}
$$

$$
\begin{aligned}
&- \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\psi(\alpha(y_1),\alpha(y_2),\psi(x_1,x_2,y_3))\\
&= 0.
\end{aligned}
$$

The proof of the theorem is complete. □

4. Abelian Extensions of Hom-δ-Jordan Lie Supertriple Systems

In this section, we show that associated with any abelian extension, there is a representation and a 3-Hom-cocycle.

An ideal of a Hom-δ-Jordan Lie supertriple system T is a subspace I such that $[I,T,T] \subseteq I$. An ideal I of a Hom-δ-Jordan Lie supertriple system is called an abelian ideal if, moreover, $[T,I,I] = 0$. Notice that $[T,I,I] = 0$ implies that $[I,T,I] = 0$ and $[I,I,T] = 0$.

Definition 7. *Let $(T,[\cdot,\cdot,\cdot]_T,\delta)$, $(V,[\cdot,\cdot,\cdot]_V,\delta)$, and $(\hat{T},[\cdot,\cdot,\cdot]_{\hat{T}},\delta)$ be Hom-δ-Jordan Lie supertriple systems and $i: V \to \hat{T}$, $p: \hat{T} \to T$ be homomorphisms. The following sequence of Hom-δ-Jordan Lie supertriple systems is a short exact sequence if* $\mathrm{Im}(i) = \mathrm{Ker}(p)$, $\mathrm{Ker}(i) = 0$ *and* $\mathrm{Im}(p) = T$,

$$0 \longrightarrow V \xrightarrow{i} \hat{T} \xrightarrow{p} T \longrightarrow 0. \tag{22}$$

In this case, we call $\hat{T}$ an extension of T by V, and denote it by $E_{\hat{T}}$. It is called an abelian extension if V is an abelian ideal of $\hat{T}$, i.e., $[u,v,\cdot]_{\hat{T}} = [u,\cdot,v]_{\hat{T}} = [\cdot,u,v]_{\hat{T}} = 0$, for all $u,v \in V$.

A section $\sigma: T \to \hat{T}$ of $p: \hat{T} \to T$ consists of linear maps $\sigma: T \to \hat{T}$ such that $p \circ \sigma = \mathrm{id}_T$, $\widehat{\alpha} \circ \sigma = \sigma \circ \alpha$.

Definition 8. *Two extensions of Hom-δ-Jordan Lie supertriple systems $E_{\hat{T}}: 0 \longrightarrow V \xrightarrow{i} \hat{T} \xrightarrow{p} T \longrightarrow 0$ and $E_{\tilde{T}}: 0 \longrightarrow V \xrightarrow{j} \tilde{T} \xrightarrow{q} T \longrightarrow 0$ are equivalent. If there exists a Hom-δ-Jordan Lie supertriple system homomorphism $F: \hat{T} \to \tilde{T}$ such that the following diagram commutes*

$$
\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & \hat{T} \arrow[r, "p"] \arrow[d, "F"] & T \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & V \arrow[r, "j"] & \tilde{T} \arrow[r, "q"] & T \arrow[r] & 0
\end{tikzcd}
$$

Let $\hat{T}$ be an abelian extension of T by V, and a linear mapping $\sigma: T \to \hat{T}$ be a section. Define maps $T \otimes T \to End(V)$ by

$$D(x_1,x_2)(u) = \delta[\sigma(x_1),\sigma(x_2),u]_{\hat{T}}, \tag{23}$$

$$\theta(x_1,x_2)(u) = (-1)^{|u|(|x_1|+|x_2|)}[u,\sigma(x_1),\sigma(x_2)]_{\hat{T}} \tag{24}$$

Clearly, the following fact holds, i.e.,

$$D(x_1,x_2)(u) = (-1)^{|x_1||x_2|}\theta(x_2,x_1)(u) - \delta\theta(x_1,x_2)(u),$$

for all $(x_1,x_2) \in T \otimes T, u \in V$.

Let $\sigma: T \to \hat{T}$ be a section of the abelian extension. Define the following map $\omega: T \times T \times T \to \hat{T}$:

$$\omega(x_1,x_2,x_3) = [\sigma(x_1),\sigma(x_2),\sigma(x_3)]_{\hat{T}} - \sigma([x_1,x_2,x_3]_T), \tag{25}$$

for all $x_1,x_2,x_3 \in T$.

Theorem 4. *Let* $0 \longrightarrow V \longrightarrow \hat{T} \longrightarrow T \longrightarrow 0$ *be an abelian extension of* T *by* V*. Then,* ω *defined by* (25) *is a* 3*-Hom-cocycle of* T *with coefficients in* V*, where the representation* θ *is given by* (24)*.*

Proof. By the equality

$$\begin{aligned}
&[\hat{\alpha}(\sigma(x_1)), \hat{\alpha}(\sigma(x_2)), [\sigma(y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
=&[[\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \hat{\alpha}(\sigma(y_2)), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\hat{\alpha}(\sigma(y_1)), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}, \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[\hat{\alpha}(\sigma(y_1)), \hat{\alpha}(\sigma(y_2)), [\sigma(x_1), \sigma(x_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}}.
\end{aligned}$$

The left hand side shows that

$$\begin{aligned}
&[\hat{\alpha}(\sigma(x_1)), \hat{\alpha}(\sigma(x_2)), [\sigma(y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
=&[\hat{\alpha}(\sigma(x_1)), \hat{\alpha}(\sigma(x_2)), \omega(y_1, y_2, y_3) + \sigma([y_1, y_2, y_3]_T)]_{\hat{T}} \\
=&\delta D(\hat{\alpha}(x_1), \hat{\alpha}(x_2))\omega(y_1, y_2, y_3) + [\hat{\alpha}(\sigma(x_1)), \hat{\alpha}(\sigma(x_2)), \sigma([y_1, y_2, y_3]_T)]_{\hat{T}} \\
=&\delta D(\hat{\alpha}(x_1), \hat{\alpha}(x_2))\omega(y_1, y_2, y_3) + [\sigma(\alpha(x_1)), \sigma(\alpha(x_2)), \sigma([y_1, y_2, y_3])] \\
=&\delta D(\alpha(x_1), \alpha(x_2))\omega(y_1, y_2, y_3) + \omega(\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]_T) + \sigma([\alpha(x_1), \alpha(x_2), [y_1, y_2, y_3]_T]_T).
\end{aligned}$$

Similarly, the right side is equal to

$$\begin{aligned}
&[[\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \hat{\alpha}(\sigma(y_2)), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\hat{\alpha}(\sigma(y_1)), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}, \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\hat{\alpha}(\sigma(y_1)), \hat{\alpha}(\sigma(y_2)), [\sigma(x_1), \sigma(x_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
&= [\omega(x_1, x_2, y_1) + \sigma([x_1, x_2, y_1]_T), \hat{\alpha}(\sigma(y_2)), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}([\hat{\alpha}(\sigma(y_1)), \omega(x_1, x_2, y_2) + \sigma([x_1, x_2, y_2]_T), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\hat{\alpha}(\sigma(y_1)), \hat{\alpha}(\sigma(y_2)), \omega(x_1, x_2, y_3) + \sigma([x_1, x_2, y_3]_T)]_{\hat{T}} \\
&= [\omega(x_1, x_2, y_1), \hat{\alpha}(\sigma(y_2)), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} + [\sigma([x_1, x_2, y_1]_T), \hat{\alpha}(\sigma(y_2)), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\hat{\alpha}(\sigma(y_1)), \omega(x_1, x_2, y_2), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\hat{\alpha}(\sigma(y_1)), \sigma([x_1, x_2, y_2]_T), \hat{\alpha}(\sigma(y_3))]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\hat{\alpha}(\sigma(y_1)), \hat{\alpha}(\sigma(y_2)), \omega(x_1, x_2, y_3)]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\hat{\alpha}(\sigma(y_1)), \hat{\alpha}(\sigma(y_2)), \sigma([x_1, x_2, y_3]_T)]_{\hat{T}} \\
&= [\omega(x_1, x_2, y_1), \sigma(\alpha(y_2)), \sigma(\alpha(y_3))]_{\hat{T}} + [\sigma([x_1, x_2, y_1]_T), \sigma(\alpha(y_2)), \sigma(\alpha(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\sigma(\alpha(y_1)), \omega(x_1, x_2, y_2), \sigma(\alpha(y_3))]_{\hat{T}} \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}[\sigma(\alpha(y_1)), \sigma([x_1, x_2, y_2]_T), \sigma(\alpha(y_3))]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\sigma(\alpha(y_1)), \sigma(\alpha(y_2)), \omega(x_1, x_2, y_3)]_{\hat{T}} \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}[\sigma(\alpha(y_1)), \sigma(\alpha(y_2)), \sigma([x_1, x_2, y_3]_T)]_{\hat{T}} \\
&= (-1)^{(|x_1|+|x_2|+|y_1|)(|y_2|+|y_3|)}\theta(\alpha(y_2), \alpha(y_3))\omega(x_1, x_2, y_1) + \sigma([[x_1, x_2, y_1]_T, \alpha(y_2), \alpha(y_3)]_T) \\
&+ \omega([[x_1, x_2, y_1]_T, \alpha(y_2), \alpha(y_3)]_T) - \delta(-1)^{|y_1|(|x_1|+|x_2|)+|y_3|(|x_1|+|x_2|+|y_2|)}\theta(\alpha(y_1), \alpha(y_3))\omega(x_1, x_2, y_2) \\
&+ (-1)^{|y_1|(|x_1|+|x_2|)}\omega(\alpha(y_1), [x_1, x_2, y_2]_T, \alpha(y_3)) + (-1)^{|y_1|(|x_1|+|x_2|)}\sigma([\alpha(y_1), [x_1, x_2, y_2]_T, \alpha(y_3)]_T) \\
&+ (-1)^{(|x_1+x_2|)(|y_1+y_2|)}D(\alpha(y_1), \alpha(y_2))\omega(x_1, x_2, y_3) + \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}\omega(\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]_T) \\
&+ \delta(-1)^{(|x_1+x_2|)(|y_1+y_2|)}\sigma([\alpha(y_1), \alpha(y_2), [x_1, x_2, y_3]_T]_T).
\end{aligned}$$

Thus, it follows that

$$\omega([x_1, x_2, y_1]_T, \alpha(y_2), \alpha(y_3)) + (-1)^{|y_1|(|x_1|+|x_2|)}\omega(\alpha(y_1), [x_1, x_2, y_2]_T, \alpha(y_3))$$

$$
\begin{aligned}
&+\delta(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}\omega(\alpha(y_1),\alpha(y_2),[x_1,x_2,y_3]_T)\\
&+(-1)^{(|x_1|+|x_2|+|y_1|)(|y_2|+|y_3|)}\theta(\alpha(y_2),\alpha(y_3))\omega(x_1,x_2,y_1)\\
&-\delta(-1)^{(|x_1|+|x_2|)(|y_1+y_3|)+|y_2||y_3|}\theta(\alpha(y_1),\alpha(y_3))\omega(x_1,x_2,y_2)\\
&+(-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}D(\alpha(y_1),\alpha(y_2))\omega(x_1,x_2,y_3)\\
&-\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2,y_3]_T)-\delta D(\alpha(x_1),\alpha(x_2))\omega(y_1,y_2,y_3)\\
&=0.
\end{aligned}
$$

Therefore, ω is a 3-Hom-cocycle. □

Theorem 5. *Let T be a Hom-δ-Jordan Lie surpetriple system, (V,θ) be a T-module and ω be a 3-Hom-cocycle, then $T\oplus V$ is a Hom-δ-Jordan Lie surpetriple system under the following multiplication:*

$$
\begin{aligned}
&[x_1+u_1,x_2+u_2,x_3+u_3]_\omega\\
&=[x_1,x_2,x_3]+\omega(x_1,x_2,x_3)+\delta D(x_1,x_2)(u_3)\\
&-\delta(-1)^{|x_2||x_3|}\theta(x_1,x_3)(u_2)+(-1)^{|x_1|(|x_2|+|x_3|)}\theta(x_2,x_3)(u_1).
\end{aligned}
$$

and

$$(\alpha+A)(x+u)=\alpha(x)+A(u).$$

Proof. Similar to the proof of Proposition 1, we obtain (4)–(6) hold. □

Theorem 6. *Two abelian extensions of* Hom-δ-Jordan *Lie supertriple systems* $E_{\hat{T}}:0\longrightarrow V\xrightarrow{i}\hat{T}\xrightarrow{p}T\longrightarrow 0$ *and* $E_{\tilde{T}}:0\longrightarrow V\xrightarrow{j}\tilde{T}\xrightarrow{q}T\longrightarrow 0$ *are equivalent* $\Longleftrightarrow$ ω *and* ω' *are in the same cohomology class.*

Proof. $\Rightarrow$) Let $F:T\oplus_\omega V\to T\oplus_{\omega'}V$ be the corresponding homomorphism. Thus,

$$F[x_1,x_2,x_3]_\omega=[F(x_1),F(x_2),F(x_3)]_{\omega'}. \tag{26}$$

Note that F is an equivalence of extensions, so there is $\rho:T\to V$ such that

$$F(x_i+u_i)=x_i+\rho(x_i)+u_i,\quad i=1,2,3. \tag{27}$$

The left hand side of (26) is equal to

$$F([x_1,x_2,x_3]+\omega(x_1,x_2,x_3))=[x_1,x_2,x_3]+\omega(x_1,x_2,x_3)+\rho([x_1,x_2,x_3]),$$

and the right hand side of (26) is equal to

$$
\begin{aligned}
&[x_1+\rho(x_1),x_2+\rho(x_2),x_3+\rho(x_3)]_{\omega'}\\
=&[x_1,x_2,x_3]+\omega'(x_1,x_2,x_3)+\delta D(x_1,x_2)\rho(x_3)-\delta(-1)^{|x_2||x_3|}\theta(x_1,x_3)\rho(x_2)\\
&+(-1)^{|x_1|(|x_2|+|x_3|)}\theta(x_2,x_3)\rho(x_1).
\end{aligned}
$$

Then we obtain that

$$
\begin{aligned}
&(\omega-\omega')(x_1,x_2,x_3)\\
=&\delta D(x_1,x_2)\rho(x_3)-\delta(-1)^{|x_2||x_3|}\theta(x_1,x_3)\rho(x_2)\\
&+(-1)^{|x_1|(|x_2|+|x_3|)}\theta(x_2,x_3)\rho(x_1)-\rho([x_1,x_2,x_3]).
\end{aligned}
$$

Therefore, $\omega-\omega'=d^1\rho$, we get ω and ω' are in the same cohomology class.

$\Leftarrow$) If ω and ω' are in the same cohomology class, we assume that $\omega-\omega'=d^1\rho$, then F defined by (27) is an equivalence. □

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