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UNDERSTANDING SEXTUPOLE*

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Abstract

In this study, we reassess the dynamics within a simple accelerator lattice featuring a single degree of freedom and incorporating a sextupole magnet. In the initial segment, we revisit the Hénon quadratic map, a representation of a general transformation with quadratic nonlinearity. In the subsequent section, we unveil that a conventional sextupole is essentially a composite structure, comprising an integrable McMillan sextupole and octupole, along with non-integrable corrections of higher orders. This fresh perspective sheds light on the fundamental nature of the sextupole magnet, providing a more nuanced understanding of its far-from-trivial chaotic dynamics. Importantly, it enables the description of driving terms of the second and third orders and introduces associated nonlinear Courant-Snyder invariant.

INTRODUCTION

In Ref. [1] we establish connections between canonical McMillan mappings and general chaotic maps in standard (McMillan-Hénon) form. Our investigation reveals that the McMillan sextupole and octupole serve as first-order approximations of dynamics around the fixed point, akin to the linear map and quadratic invariant (known as the Courant-Snyder invariant in accelerator physics), which represents zeroth-order approximations (referred to as linearization). Furthermore, we propose a novel formalism for nonlinear Twiss parameters, which accounts for the dependence of rotation number on amplitude. This stands in contrast to conventional betatron phase advance used in accelerator physics, which remains independent of amplitude. Notably, in the context of accelerator physics, this new formalism demonstrates its capability in predicting dynamical aperture around low-order resonances for flat beams, a critical aspect in beam injection/extraction scenarios.

STANDARD FORM OF THE MAP

Consider a simple accelerator lattice with one degree of freedom consisting of linear optics elements (drift spaces, dipoles, and quadrupoles) and a single thin nonlinear lens [2]:

$$\mathrm{F}: \quad \begin{bmatrix} x \\ \dot{x} \end{bmatrix}' = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ F(x) \end{bmatrix}.$$

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The effect on a test particle from all linear elements can be represented using a matrix with Courant-Snyder parametrization [3]:

$$\mathrm{M}: \quad \begin{bmatrix} x \\ \dot{x} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha\,\sin\Phi & \beta\,\sin\Phi \\ -\gamma\,\sin\Phi & \cos\Phi - \alpha\,\sin\Phi \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

where α, β and γ are *Twiss parameters* (also known as *Courant-Snyder parameters*) at the thin lens location, and Φ is the *betatron phase advance* over the linear optics insert

$$\Phi = \int \frac{\mathrm{d}s}{\beta(s)}.$$

Without the nonlinear lens, Twiss parameters are functions of longitudinal coordinate s with $\beta(s)$ referred to as the β-function, $\alpha(s) \equiv -\dot{\beta}(s)/2$, and $\gamma(s) \equiv [1+\alpha^2(s)]/\beta(s)$. At any location, the *Courant-Snyder invariant* is defined as:

$$\gamma(s)\,x^2(s) + 2\,\alpha(s)\,x(s)\,\dot{x}(s) + \beta(s)\,\dot{x}^2(s) = \mathrm{const}.$$

The rotation number (or *betatron tune* in accelerator physics) is independent of amplitude and given by:

$$\nu_0 = \frac{1}{2\,\pi} \oint \frac{\mathrm{d}s}{\beta(s)}.$$

When the nonlinear lens is introduced, the combined one-turn map $\mathrm{M}\circ\mathrm{F}$ can be rewritten in the McMillan-Hénon form

$$q' = p, \qquad p' = -q + f(p),$$

using a change of variables

$$\begin{cases} q = x, \\ p = x\,(\cos\Phi + \alpha\,\sin\Phi) + \dot{x}\,\beta\,\sin\Phi, \end{cases} \tag{1}$$

with the *force function* given by

$$f(q) = 2\,q\,\cos\Phi + \beta\,\sin\Phi\,F(q).$$

In this notations conventional thin (Th) sextupole and thin octupole lenses are given by

$$F_{\mathrm{sxt}}(x) = k_{\mathrm{sxt}}\,x^2, \qquad F_{\mathrm{oct}}(x) = \pm k_{\mathrm{oct}}\,x^3,$$

while McMillan sextupole and octupole provided by

$$f_{\mathrm{sxt}}(p) = p\,\frac{a-p}{1+p}, \qquad \text{and} \qquad f_{\mathrm{oct}}(p) = \frac{a\,p}{1\pm p^2}.$$

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PERTURBATION THEORY

For a general and even chaotic mappings in McMillan-Hénon form, with a differentiable but otherwise arbitrary force function $f(q)$, the symmetric McMillan maps approximate small amplitude dynamics. To demonstrate this, we introduce a small positive parameter ε representing the amplitude of oscillations. This is achieved through a change of variables $(q,p) \to \varepsilon\,(q,p)$:

$$q' = p$$
$$p' = -q + \frac{f(\varepsilon\, p)}{\varepsilon} = -q + a\,p + \varepsilon\, b\,p^2 + \varepsilon^2 c\,p^3 + \ldots$$

where f is expanded in a power series of $(\varepsilon\, p)$

$$a \equiv \partial_p f(0)/1!,\;\; b \equiv \partial_p^2 f(0)/2!,\;\; c \equiv \partial_p^3 f(0)/3!,\;\; \ldots,$$

and we assume the fixed point to be at the origin, necessitating $f(0) = 0$. Subsequently, we seek an *approximated invariant* of motion that is conserved with an accuracy of order $\mathcal{O}(\varepsilon^{n+1})$:

$$\mathcal{K}^{(n)}[p',q'] - \mathcal{K}^{(n)}[p,q] = \mathcal{O}(\varepsilon^{n+1}). \qquad (2)$$

The invariant is sought in the form of a polynomial:

$$\mathcal{K}^{(n)} = \mathcal{K}_0 + \varepsilon\,\mathcal{K}_1 + \varepsilon^2\,\mathcal{K}_2 + \ldots + \varepsilon^n\,\mathcal{K}_n,$$

where $\mathcal{K}_m$ consists of homogeneous polynomials in p and q of $(m+2)$ degree

$$\mathcal{K}_0 = C_{2,0}\,p^2 + C_{1,1}\,p\,q + C_{0,2}\,q^2,$$
$$\mathcal{K}_1 = C_{3,0}\,p^3 + C_{2,1}\,p^2 q + C_{1,2}\,p\,q^2 + C_{0,3}\,q^3,$$
$$\ldots,$$

and $C_{i,j}$ are coefficients to be determined to satisfy Eq. (2). The reader can check that, in the first two orders of this perturbation theory, a general result is provided

$$\mathcal{K}^{(2)}[p,q] = \mathcal{K}_0[p,q] - \varepsilon\,\frac{b}{a+1}\,(p^2 q + p\,q^2) + \\ + \varepsilon^2\left(\left[\frac{b^2}{a\,(a+1)} - \frac{c}{a}\right]p^2 q^2 + C\,\mathcal{K}_0^2[p,q]\right) \qquad (3)$$

where C is a coefficient such that Eq. (2) is satisfied for any value it takes. Setting $C = 0$, we define an approximated integral of motion (3) with

$$a = 2\,\cos\Phi + \beta\,\sin\Phi\,\partial_q F(0) = 2\,\cos[2\,\pi\,\nu_0],$$
$$b = \beta\,\sin\Phi\,\partial_{qq} F(0), \qquad (4)$$
$$c = \beta\,\sin\Phi\,\partial_{qqq} F(0).$$

This can be seen as a nonlinear analog of the Courant-Snyder invariant that includes higher-order terms and can be easily propagated through the linear part of the lattice, thus defined for any azimuth, $\mathcal{K}[p,q;s]$.

SEXTUPOLE MAGNET

Starting with the quadratic Hénon map $f_{\rm sxt}^{\rm (H)}(q) = a\,q + q^2$, we can go up to the second order of perturbation theory by first matching the quadratic term in the force function $(b = 1)$

$$f_{\text{SX-1}}^{\rm (MH)}(q) = \frac{a\,(a+1)+q}{(a+1)-q}\,q = a\,q + q^2 + \frac{q^3}{a+1} + \mathcal{O}(q^4),$$

and then removing the cubic term from the expansion $(c = 0)$

$$f_{\text{SX-2}}^{\rm (MH)}(q) = \frac{a\,(a+1)+q}{(a+1) - q + \frac{1}{a}\,q^2}\,q = a\,q + q^2 + \mathcal{O}(q^4).$$

While the first order is simply a rescaled McMillan sextupole (SX), the second order represents a specific mixture of both McMillan sextupole and focusing octupole (FO), as can be seen from the invariants:

$$\mathcal{K}_{\text{SX-1}}^{\rm (MH)}[p,q] = \mathcal{K}_0[p,q] - \frac{p^2 q + p\,q^2}{a+1},$$
$$\mathcal{K}_{\text{SX-2}}^{\rm (MH)}[p,q] = \mathcal{K}_0[p,q] - \frac{p^2 q + p\,q^2}{a+1} + \frac{p^2\,q^2}{a\,(a+1)}.$$

Using analytical results for detuning of McMillan multipoles [1], with the help of appropriate scaling

$$\left.\frac{\mathrm{d}\nu_{\text{SX-1}}^{\rm (MH)}}{\mathrm{d}J_{\text{SX-1}}^{\rm (MH)}}\right|_{J=0} = \left(\frac{-1}{1+a}\right)^2 \times \left.\frac{\mathrm{d}\nu_{\rm SX}}{\mathrm{d}J_{\rm SX}}\right|_{J=0}$$

and

$$\left.\frac{\mathrm{d}\nu_{\text{SX-2}}^{\rm (MH)}}{\mathrm{d}J_{\text{SX-2}}^{\rm (MH)}}\right|_{J=0} = \left.\frac{\mathrm{d}\nu_{\text{SX-1}}^{\rm (MH)}}{\mathrm{d}J_{\text{SX-1}}^{\rm (MH)}}\right|_{J=0} + \frac{1}{a\,(1+a)} \times \left.\frac{\mathrm{d}\nu_{\rm FO}}{\mathrm{d}J_{\rm FO}}\right|_{J=0}$$

we obtain

$$\left.\frac{\mathrm{d}\nu_{\text{SX-1}}^{\rm (MH)}}{\mathrm{d}J_{\text{SX-1}}^{\rm (MH)}}\right|_{J=0} = -\frac{1}{16\,\pi}\,\frac{9\,\cos(\pi\,\nu_0) + \cos(3\,\pi\,\nu_0)}{\sin^3(2\,\pi\,\nu_0)\,\sin(3\,\pi\,\nu_0)},$$
$$\left.\frac{\mathrm{d}\nu_{\text{SX-2}}^{\rm (MH)}}{\mathrm{d}J_{\text{SX-2}}^{\rm (MH)}}\right|_{J=0} = -\frac{1}{16\,\pi}\,\frac{3\,\cot(\pi\,\nu_0) + \cot(3\,\pi\,\nu_0)}{\sin^3(2\,\pi\,\nu_0)}. \qquad (5)$$

Transforming back to $(x,\dot{x})$, provides results in terms of physical variables:

$$\left.\frac{\mathrm{d}\nu_{\text{SX-1}}^{\rm (Th)}}{\mathrm{d}J_{\text{SX-1}}^{\rm (Th)}}\right|_{J=0} = -\frac{1}{16\,\pi}\;\frac{9\,\cos(\pi\,\nu_0) + \cos(3\,\pi\,\nu_0)}{\sin(3\,\pi\,\nu_0)}\;\beta^3\,k_{\rm sxt}^2,$$
$$\left.\frac{\mathrm{d}\nu_{\text{SX-2}}^{\rm (Th)}}{\mathrm{d}J_{\text{SX-2}}^{\rm (Th)}}\right|_{J=0} = -\frac{3}{16\,\pi}\,\left[3\,\cot(\pi\,\nu_0) + \cot(3\,\pi\,\nu_0)\right]\;\beta^3\,k_{\rm sxt}^2,$$

see Fig. 1. These results are consistent with other derivations using various perturbation theories including the Deprit perturbation theory [4] and the Lie algebra treatment [5, 6]. Notice that in addition to the scaling provided by Eqs. (4), an additional factor equal to the Jacobian of the transformation (1), $\mathbf{J} = \beta\,\sin\Phi$, must be taken into account to obtain the equations above from the McMillan-Hénon detunings (Eq. (5)).

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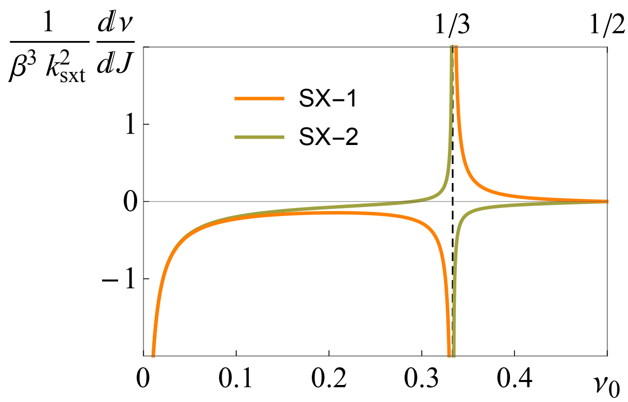

Figure 1: Detuning for the thin sextupole lens as a function of the betatron tune ν_0. The green curve corresponds to the value (SX-2) that matches numerical simulation, while the orange curve represents the first-order approximation.

Large Amplitudes

The applicability of any perturbation theory in a particular order greatly depends on the "leading" nonlinearity. While the area of stability exhibits a complex shape, yet McMillan multipoles offer a reliable estimate for the separatrix qsep and $\nu(q_0)$ near low-order (integer, half-integer, and third-integer) resonances. Although we do note a substantial alignment between our perturbation theory and other methods in addressing dynamics around the fixed point, it's important to highlight that the dependence of $\nu(q_0)$ on large amplitudes through elliptic functions inherently differs from the typical power series of q_0 often obtained in methods such as Lie algebra. In particular, analytical expressions for the rotation number of McMillan multipoles experience very rapid change around the limiting n-cycle, providing a more realistic description of behavior near the bounding separatrix.

As an example we consider the Hénon quadratic map $f^{(\mathrm{H})}_{\mathrm{sxt}}$ above the integer resonance ($a = 1.6$)

$$\delta\nu = \nu_0 - 0 \approx 0.1,$$

and then above the third-integer resonance ($a = -1.2$)

$$\delta\nu = \nu_0 - \frac{1}{3} \approx 0.02,$$

as shown in Fig. 2. The two rows in the middle show the first and second order approximated invariants, including corresponding dependencies $\nu(q_0)$ in the plot at the bottom. For the third-order resonance (plots d.), the approximation SX-1 fails to predict the proper sign of detuning, as expected, but in the next order SX-2 provides a quite accurate estimate of ν within the range of stable trajectories obtained by tracking. Despite the discrepancy, both orders provide useful information regarding the general shape and orientation of the phase space trajectories, which is valuable in practical applications such as resonant beam extraction, as demonstrated in Ref. [1].

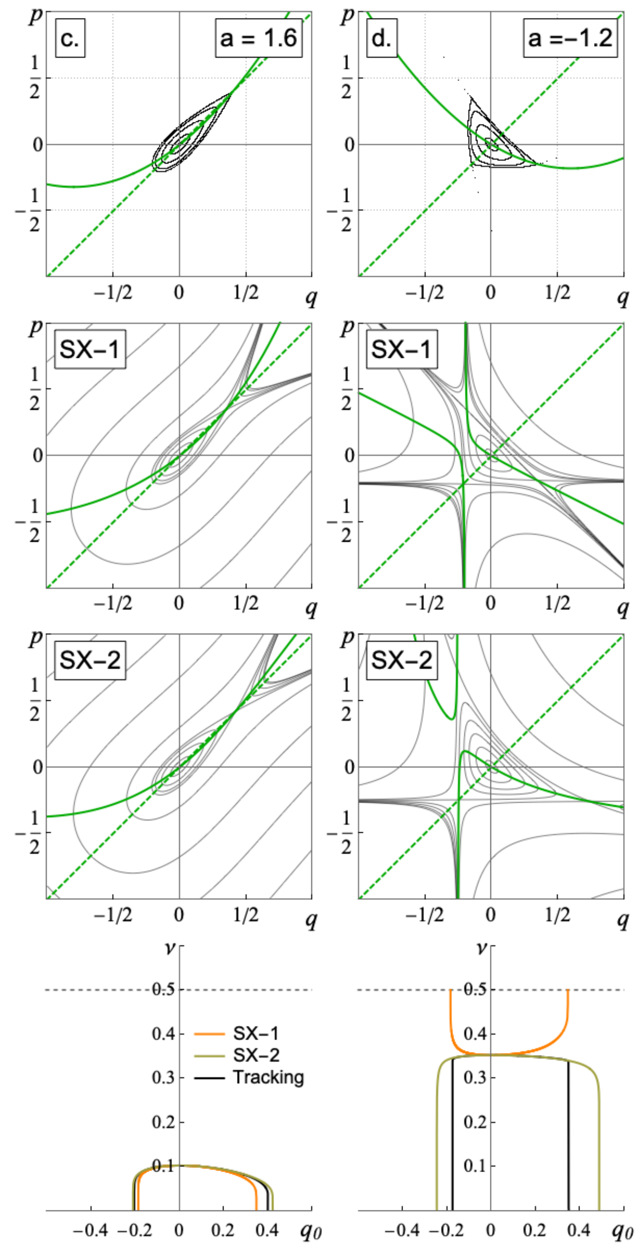

Figure 2: The top row illustrates phase space diagrams for the Hénon quadratic map $f(q) = a\,q + q^2$ obtained through tracking. The rows in the middle display level sets for the corresponding approximated McMillan-Hénon invariant of the first and second orders. Dashed and solid green curves are the first ($p = q$) and second ($p = f(q)/2$) symmetry lines, respectively. The bottom row presents a comparison of the rotation number as a function of the initial coordinate along the second symmetry line $\nu(q_0)$, evaluated from tracking (black curve) and the analytical approximations (shown in color).

CONCLUSION

For a comprehensive exploration of the solutions for McMillan multipoles and regular thin sextupole and octupole magnets readers are encouraged to refer to Ref. [1].

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