General Relativity and Gravitation (2011)
DOI 10.1007/s10714-008-0690-9

RESEARCH ARTICLE

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Thermodynamics of regular black hole

Received: 4 July 2008 / Accepted: 27 August 2008

Abstract We investigate thermodynamics for a magnetically charged regular black hole (MCRBH), which comes from the action of general relativity and nonlinear electromagnetics, comparing with the Reissner–Norström (RN) black hole in both four and two dimensions after dimensional reduction. We find that there is no thermodynamic difference between the regular and RN black holes for a fixed charge Q in both dimensions. This means that the condition for either singularity or regularity at the origin of coordinate does not affect the thermodynamics of black hole. Furthermore, we describe the near-horizon AdS_2 thermodynamics of the MCRBH with the connection of the Jackiw–Teitelboim theory. We also identify the near-horizon entropy as the statistical entropy by using the AdS_2/CFT_1 correspondence.

Keywords Regular black holes, Thermodynamics, Jackiw–Teitelboim theory

1 Introduction

Hawking's semiclassical analysis of a black hole radiation suggests that most information about initial states is shielded behind event horizon and will not back to asymptotic region far from an evaporating black hole [1]. This means that the unitarity is violated by an evaporating black hole. However, this conclusion has been debated by many authors for three decades [2; 3; 4]. It is closely related to a long standing puzzle of the information loss paradox, which states the question of whether the formation and subsequent evaporation of a black hole is unitary. In order to determine the final state of evaporation process, a more precise treatment including quantum gravity effects and backreaction is generally

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required. In the semiclassical study of Schwarzschild black hole, the temperature ($T_H^{\rm Sch} \propto 1/m$) and the luminosity ($L_{\rm Sch} \propto 1/m^2$) diverge as the mass m of the black hole approaches zero. This means that the semiclassical approach breaks down for very light black holes. Furthermore, one has to take into account backreaction. It was shown that the effect of quantum gravity could cure this pathological short distance behavior [5; 6]. Also, if an extremal black hole is considered as the ground state of regular black hole (RBH), one may avoid the short distance behavior such as a terminal phase of evaporation and backreaction.

At present, two leading candidates for quantum gravity are the string theory and the loop quantum gravity. Interestingly, the semiclassical analysis of the loop quantum black hole provides a RBH without singularity in contrast to the classical one [7]. Its minimum size r_c is at Planck scale $\ell_{\rm Pl}$. On the other hand, in the continuing search for quantum gravity, the black hole thermodynamics may be related to a future experimental result at the LHC [8; 9; 10]. The causal structures of RBHs are similar to the Reissner–Nordström (RN) black hole with the singularity replaced by de Sitter space–time with curvature radius $r_0 = \sqrt{3/\Lambda}$ [11; 12; 13]. Recently, several authors have discussed the formation and evaporation process of a RBH with minimum size l [14; 15] induced from the string theory [16; 17]. The noncommutativity also provides another RBH with minimum scale $\sqrt{\theta}$ so called the noncommutative black hole [5; 6; 18; 19]. Very recently, we have investigated the thermodynamics and evaporation process of the noncommutative black hole [20]. It turned out that the final state of the evaporation process for all RBHs is a cold Planck-size remnant of extremal black holes with zero temperature. The connection between their minimum sizes is given by $r_c \sim r_0 \sim l \sim \sqrt{\theta} \sim Q \sim \ell_{\rm Pl}$, where Q is the charge of the RN black hole. We expect that the thermodynamics of RBHs is similar to the RN black hole [21], even though the latter has a timelike singularity [22].

In fact, RBHs have been considered, dating back to Bardeen [23], for avoiding the curvature singularity beyond event horizon in black hole physics [24]. Among various RBHs known to date, intriguing black holes are obtained from the action of Einstein gravity and nonlinear electrodynamics. The solutions to the coupled equations were found by Ayon-Beato and Garcia [25] and by Bronnikov [26]. The latter describes a magnetically charged regular black hole (MCRBH). Also its simplicity allows exact treatment such that the location of the horizons can be expressed in terms of the Lambert functions [27]. Moreover, Matyjasek investigated the extremal MCRBH with the near horizon geometry of $\mathrm{AdS}_2 \times \mathrm{S}^2$ [28; 29].

On the other hand, 2D dilaton gravity has been used in various situations as an effective description of 4D gravity after a black hole in string theory has appeared [30; 31]. Hawking radiation and thermodynamics of this black hole have been analyzed by several authors [32; 33; 34; 35; 36; 37]. Another 2D theories, which were originated from the Jackiw–Teitelboim (JT) theory [38; 39], have been also studied [40; 41; 42]. Although in this JT theory the curvature is constant and negative, it has a black hole solution, which implies the non-trivial thermodynamics [43; 44; 45; 46; 47; 48]. Moreover, Fabbri et al. [49] partially demonstrated the duality of the thermodynamics between a near-extremal RN black hole and the JT theory by considering temperature and entropy. Actually, 2D dilaton gravity approach is the s-wave approximation to 4D

gravity [50]. Recently, we have studied whether the entropy function approach [51] is suitable or not by obtaining the entropy of extremal MCRBH [52], and have investigated it in terms of the attractor mechanism [53]. The key ingredient is to find a 2D dilaton gravity with dilaton potential [54]. Note that several authors have recently mentioned how to derive the desired Bekenstein–Hawking entropy of extremal RBHs from the generalized entropy formula based on the Wald's Noether charge
formalism [55].

In this paper, we study thermodynamic properties of the MCRBH [28; 52; 53]. The motivation of studying this MCRBH is two folds: regularity and nonlinearity. The first issue is the regularity of the black hole solution. We exactly know the action for the MCRBH, in contrast to the noncommutative RBH whose action is unknown. The second one is the nonlinearity. We may introduce another nonlinear electromagnetics, Born–Infeld action. However, this action does not lead to a regularity of metric function in the limit of $r \to 0$ even though its presence softens the divergence of curvature scalar.

We observe that there exists an unstable point at $r_+ = r_m$ (known as Davies' point), where the temperature is maximum and the heat capacity changes from negative infinity to positive infinity. This Davies' point separates the whole thermodynamic process into the early stage with positive heat capacity and the late stage with negative heat capacity [56]. We also confirm this feature by using the effective 2D dilaton gravity.

2 Thermodynamic quantities of MCRBH

In order to analyze the thermodynamics of the MCRBH, let us start with the four-dimensional non-linear action [28; 29; 52]

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R - \mathscr{L}_M(B)], \tag{1}$$

where $\mathscr{L}_M(B)$ is a functional of $B = F_{\mu\nu}F^{\mu\nu}$ defined by

$$\mathscr{L}_M(B) = B\cosh^{-2}\left[a\left(\frac{B}{2}\right)^{1/4}\right]. \tag{2}$$

Here the free parameter a will be adjusted to guarantee regularity at the center. In the limit of $a \to 0$, this action reduces to the Einstein–Maxwell theory having the solution of the RN black hole. First, the tensor field $F_{\mu\nu}$ satisfies equations

$$\nabla_\mu \left(\frac{d\mathscr{L}(B)}{dB} F^{\mu\nu} \right) = 0, \tag{3}$$

$$\nabla_\mu {}^* F^{\mu\nu} = 0, \tag{4}$$

where the asterisk denotes the Hodge duality. Then, differentiating the action I with respect to the metric tensor $g_{\mu\nu}$ leads to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{5}$$

with the stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\frac{d\mathscr{L}(B)}{dB} F_{\rho\mu} F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} \mathscr{L}(B) \right). \tag{6}$$

For our purpose, we consider the spherically symmetric metric

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)} dr^2 + b^2(r) d\Omega_2^2, \tag{7}$$

where $b(r)$ plays a role of radius r of the two sphere S^2. To determine the metric function (7) defined by

$$U(r) = 1 - \frac{2m(r)}{r}, \tag{8}$$

we have to solve the Einstein equation. It leads to the mass distribution

$$m(r) = \frac{1}{4} \int^r \mathscr{L}[B(r')] r'^2 dr' + C, \tag{9}$$

where C is an integration constant. In order to determine $m(r)$, from Eq. (3) we choose the purely magnetic configuration by taking $F_{\mu\nu}$ to zero except for $F_{\theta\phi}$ as follows

$$F_{\theta\phi} = Q \sin\theta \to B = \frac{2Q^2}{r^4}, \tag{10}$$

where Q is an integration constant related to the magnetic charge of the solution. Hereafter we assume that $Q > 0$ for simplicity.

Considering the condition for the ADM mass at infinity as $m(\infty) = M = C$, the mass distribution takes the form

$$m(r) = M - \frac{Q^{3/2}}{2a} \tanh\left(\frac{aQ^{1/2}}{r} \right). \tag{11}$$

Moreover, setting $a = Q^{3/2}/2M$ determines the metric function (7) completely as

$$U(r) = 1 - \frac{2M}{r} \left(1 - \tanh \frac{Q^2}{2Mr} \right). \tag{12}$$

Fig. 1 Graph of the horizon mass M versus the horizon radius $r_\pm$ as the solution to $U(r_\pm)=0$ with a fixed $Q=Q_e$. The solid (*dashed*) curve describes the MCRBH (RN). For $M=M_e$, the degenerate event horizon is located at $r_e=0.872$, while at $r_e=1$ for the RN black hole. Three *horizontal lines* are for $M=1.5,1,0.8$

At this stage we note that $U(r)$ is regular ($U(r)\to 1$) as $r\to 0$ using $\lim_{r\to 0}\tanh[aQ^{1/2}/r]\sim 1$,[1] in contrast to the RN case ($a\to 0$ limit) whose metric function of $1-2M/r+Q^2/r^2$ diverges as r^{-2} in that limit. In order to find the horizon from $U(r)=0$, we use the Lambert functions $W_i(\xi)$ defined by the general formula $e^{W(\xi)}W(\xi)=\xi$ [27]. Here $W_0(\xi)$ and $W_{-1}(\xi)$ have real branches. Their values at branch point $\xi=-1/e$ are the same as $W_0(-1/e)=W_{-1}(-1/e)=-1$. Here, we set $W_0(1/e)\equiv w_0$ because the value of the principle branch of the Lambert function at $\xi=1/e=0.368$ plays a role in finding the location of degenerate horizon of the extremal MCRBH [28; 53].

Introducing a reduced radial coordinate $x=r/M$ and a charge-to-mass ratio $q=Q/M$, the condition for the event horizon is given by

$$U(x) = 1-\frac{2}{x}\left(1-\tanh\frac{q^2}{2x}\right)=0. \tag{13}$$

Here, one finds the outer x_+ and inner x_- horizons as

$$x_+(q)=-\frac{q^2}{W_0(-\frac{q^2e^{q^2/4}}{4})-q^2/4},\quad x_-(q)=-\frac{q^2}{W_{-1}(-\frac{q^2e^{q^2/4}}{4})-q^2/4}. \tag{14}$$

For $q=q_e=2\sqrt{w_0}$, the two horizons x_+ and x_- merge into a degenerate event horizon at

$$x_e=\frac{4q_e^2}{4+q_e^2}=\frac{4w_0}{1+w_0}, \tag{15}$$

where we have used the relation of $(q_e^2/4)e^{q_e^2/4}=1/e=w_0e^{w_0}$. That is, the degenerate horizon numerically appears at $(q_e=1.056, x_e=0.872)$ when $x_+=x_-=x_e$. Formally, Eq. (15) comes from the extremal condition of $U'(x)=0$. We have an ambiguity to determine the mass M_e of the extremal MCRBH. For simplicity, we choose $M_e=1$, and then $Q_e=M_eq_e=q_e$. On the other hand, for $q>q_e$ there is no horizon while two horizons appear for $q<q_e$. For comparison, we note the difference between $M_e=Q/1.056$ for the extremal MCRBH and $M_e=Q$ for the extremal RN black hole.

[1] Unless $a=Q^{3/2}/2M$, one could not recover a regularity at $r\to 0$. Hence, this choice of a is necessary and sufficient condition to obtain a regular black hole. One may consider three-parameter family of (a,Q,M). However, this is not the case, which could lead to a regular black hole. If this is the case, its solution of metric function has a singularity in the limit of $r\to 0$, like a Born–Infeld black hole.

From the condition $U(r_\pm) = 0$ for the horizons, one finds the mass as a function of horizon radius $r_\pm$ as

$$M(r_\pm) \equiv M_\pm = \frac{r_\pm}{2\left[1 - \tanh\left(\frac{Q^2}{2M_\pm r_\pm}\right)\right]}, \tag{16}$$

which is obviously a nonlinear relation between $M_\pm$ and $r_\pm$ due to preserving the regularity. Actually, the nonlinearity makes the thermodynamic analysis difficult. In order to see the relation, we plot the horizon mass M as a function of the horizon radius $r = r_\pm$ for a fixed $Q = Q_e$ numerically in Fig. 1. The degenerate event horizon locates at $r = r_e = 0.872$, where the minimum mass $M(r_e) = M_e$ appears from Eq. (16). Note that for $M(0.8) < M_e$, there is no horizon, which means that any solution to Eq. (16) does not exist, whereas for $M(1.5) > M_e$ one has two horizons: the inner $r = r_-$ and outer $r = r_+$ horizons. For a large $r > r_e$, we have the Schwarzschild relation $M = r_+/2$. This picture is similar to the case proposed by Hayward [14; 20] for a RBH.

Hereafter we consider the outer horizon $r = r_+$ only because we are interested in the thermodynamic analysis of the RBH. For our purpose, let us define the Bekenstein–Hawking entropy for the MCRBH as

$$S_{\rm BH} = \pi r_+^2. \tag{17}$$

The black hole temperature can be calculated to be

$$\begin{aligned} T(r_+) &= \frac{1}{4\pi}\left[\frac{dU}{dr}\right]_{r=r_+} \\ &= \frac{1}{4\pi}\left[\frac{1}{r_+} + \frac{Q^2}{4M_+^2 r_+}\left(1 - \frac{4M_+}{r_+}\right)\right]. \end{aligned} \tag{18}$$

Note that one recovers the Hawking temperature $T_H^{Sch} \propto r_+^{-1}$ of the Schwarzschild black hole for $r_+ > r_m$ with the Davies' point r_m, where the Hawking temperature reaches to the maximum value at $r_+ = r_m$ as shown in Fig. 2. It is important to investigate what happens as $r_+ \to 0$. In the Schwarzschild case, $T_H^{\rm Sch}$ diverges and this puts the limit on the validity of the evaporation process via the Hawking radiation. Against this scenario, the temperature T falls down to zero at $r_+ = r_e$[2] even where the extremal black hole appears as shown in Fig. 2a.

As is depicted in Fig. 2a, the temperature of the MCRBH grows until it reaches to the maximum value $T_m \simeq 0.03$ at $r_+ = r_m \simeq 1.689$ $(M = M_m = 1.166)$. As a result, the thermodynamics process is split into the right branch of $r_m < r_+ < \infty$ called the Schwarzschild phase and the left branch of $r_e \leq r_+ < r_m$ called the near-horizon thermal phase. In particular, one has the extremal black hole at $r_+ = r_e$

[2] The extremal black hole seems to be controversial because the entropy is non-zero ($S_e = \pi r_e^2$), while its temperature is zero. This is a long-standing problem for the extremal black hole. However, our guideline is that the first-law of thermodynamics should hold even for the extremal configuration and thus, it remains one of equilibria. In this case, we prove that the first-law is satisfied as $dM = TdS = 0$ at $M = M_e$. Hence the above case is compatible with the first-law of black hole thermodynamics.

Fig. 2 Three graphs for temperature, heat capacity, and free energy with a fixed $Q = Q_e$. The solid (*dashed*) curve denotes MCRBH (RN). The near-horizon thermal phase takes place for $r_e < r_+ < r_m$, the Schwarzschild phase is for $r_+ > r_m$. **a** Graph for the temperature T having the maximum value at $r_+ = r_m$. **b** Graph for the heat capacity C showing the blow-up at $r_+ = r_m$. The near-horizon thermodynamics takes the positive heat capacity $C > 0$, while the Schwarzschild phase has the negative heat capacity $C < 0$. **c** Plot of the free energy F

with $T(r_e) = 0$. In the region of $r < r_e$, there is no black hole for $M < M_e$ and thus the temperature cannot be defined. For $M > M_e$, we have the inner horizon at $r = r_-$ inside the outer horizon, but an observer at infinity does not recognize the presence of this horizon. Hence, we regard this region as the forbidden region in view of thermodynamic aspects.

In order to check the thermal stability of the MCRBH, we have to know the heat capacity [57]. Its heat capacity $C = \frac{dM(r_+)}{dT(r_+)}|_Q$ is calculated in appendix and given by

$$C(r_+) = \frac{16\pi M_+^3 r_+(4M_+^2 r_+ - 4M_+ Q^2 + Q^2 r_+)}{16M_+^2 Q^2 + 32M_+^3 Q^2 r_+ - r_+^2(4M_+^2 + Q^2)^2}, \quad (19)$$

where its variation is plotted in Fig. 2b. Here, we find a stable region of $C > 0$, which represents the near-horizon thermodynamics. We observe that a thermodynamically unstable region ($C < 0$) appears for $r_+ > r_m$ like the Schwarzschild black hole. We note that $C(r_e) = 0$ for the extremal black hole.

It is appropriate to comment on the value of $r_m = 1.689$ at which not only the Hawking temperature reaches to the maximum value, but also the specific heat blows up. In order to find the position $r_+ = r_m$ correctly, one has to include the variation of the mass function (16), as discussed in the appendix. Its value is shifted toward the inside of the black hole, when compared with the radius, $r_m^{\mathrm{RN}} = 1.732$, of the RN black hole. This means that the MCRBH could be thermodynamically stable in the more restricted region than the RN black hole's one. This is of course caused by the nonlinear mass function (16).

Finally, we may discuss a possible phase transition near $T = 0$ by introducing the Helmholtz free energy [58] as

$$F(r_+) = M(r_+) - M_e - T(r_+)S_{\mathrm{BH}}(r_+). \quad (20)$$

Its graph is shown in Fig. 2c. The Helmholtz free energy is zero ($F = 0$) at $r_+ = r_e$, as $F_{\mathrm{min}}^{\mathrm{RN}}(r_e = 1) = 0$ for the RN black hole. Both are monotonically decreasing functions of $r_e \leq r_+ < r_m$. For $r_+ > r_m$, one finds the Schwarzschild's free energy of $r_+/4$.

As is observed from Fig. 2, we split the whole thermal process into the near-horizon thermal and the Schwarzschild phase. The former is characterized by the increasing temperature and positive heat capacity, while the latter is determined by the decreasing temperature and negative heat capacity. We note that the near-horizon thermodynamics sharply contrasts to the conventional thermodynamics of the Schwarzschild black hole. Hence it is very important to explore thermodynamics of the MCRBH by using the other approach.

3 2D dilaton gravity approach of MCRBH

Various black holes in four dimensions have been widely studied through the dimensional reduction. Recently, its interest has increased as an example of AdS_2 arising as a near-horizon geometry. Very recently, we have shown that the 2D dilaton gravity approach provides all thermodynamic quantities of spherically symmetric RBHs in a simple way [54]. In this section, we shall explicitly show that the 4D MCRBH is equivalent to a 2D dilaton gravity.

After the dimensional reduction by integrating the action in Eq. (1) over S^2, the reduced effective action in two dimensions is obtained as [49]

$$I^{(2)} = \int d^2x\sqrt{-g}\left[\frac{1}{4}(b^2R_2 + 2g^{\mu\nu}\nabla_\mu b\nabla_\nu b + 2) - b^2\mathscr{L}_M\right]. \tag{21}$$

It is convenient to eliminate the kinetic term by using the conformal transformation

$$\bar{g}_{\mu\nu} = \sqrt{\phi}\, g_{\mu\nu}, \quad \phi = \frac{b^2(r)}{4}. \tag{22}$$

Fig. 3 Three graphs for $J(\phi)$, $V(\phi)$, and $V'(\phi)$ with $Q_e = 1.056$. The solid (*dashed*) curve describes the MCRBH (RN). $J(\phi)$ has a minimum at $\phi_e = 0.189$, $V(\phi_m)$ has a maximum value at $\phi_m = 0.714$, and $V'(\phi_m) = 0$, while for the extremal RN black holes those are at $\phi_e^{RN} = 0.25$ and $\phi_m^{\rm RN} = 0.75$

Then, we obtain the action of 2D dilation gravity with $G_2 = 1/2$ [38; 39]

$$\bar{I}_{\rm MCRBH} = \int d^2x\sqrt{-\bar{g}}\,[\phi\bar{R}_2 + V(\phi)]\,. \tag{23}$$

Here, the Ricci scalar and the dilaton potential are

$$\bar{R}_2 = -\frac{U''}{\sqrt{\phi}}, \tag{24}$$

$$V(\phi) = \frac{1}{2\sqrt{\phi}} - \frac{Q^2}{8\phi^{3/2}}\cosh^{-2}\left[\frac{Q^2}{4M\sqrt{\phi}}\right], \tag{25}$$

respectively. The two equations of motion are

$$\nabla^2\phi = V(\phi), \tag{26}$$

$$\bar{R}_2 = -V'(\phi), \tag{27}$$

where the derivative of $V'(\phi)$ takes the form

$$V'(\phi) = -\frac{1}{4\phi^{3/2}} + \frac{3Q^2}{16\phi^{5/2}}\cosh^{-2}\left[\frac{Q^2}{4M\sqrt{\phi}}\right] - \frac{Q^4}{32M\phi^3}\cosh^{-3}\left[\frac{Q^2}{4M\sqrt{\phi}}\right]\sinh\left[\frac{Q^2}{4M\sqrt{\phi}}\right]. \tag{28}$$

By choosing a conformal gauge of $\bar{g}_{tx} = 0$ [59; 60], we obtain the general solution to Eqs. (26) and (27) as

$$\frac{d\phi}{dx} = 2(J(\phi) - \mathscr{C}), \tag{29}$$

$$ds^2 = -(J(\phi) - \mathscr{C})dt^2 + \frac{dx^2}{J(\phi) - \mathscr{C}}, \tag{30}$$

where $J(\phi)$ is the integration of $V(\phi)$

$$J(\phi) = \int^{\phi} V(\tilde{\phi})d\tilde{\phi} = \sqrt{\phi} + M\tanh\left(\frac{Q^2}{4M\sqrt{\phi}}\right). \tag{31}$$

Here, $\mathscr{C}$ is a coordinate-invariant constant of the integration, which is identified with the mass M of the MCRBH $J(\phi), V(\phi)$, and $V'(\phi)$ are depicted in Fig. 3.

We note here the important connection between $J(\phi)$ and the metric function $U(r(\phi))$ with $r = 2\sqrt{\phi}$: $\sqrt{\phi}\ U(\phi) = J(\phi) - M$. A necessary condition that a 2D dilaton gravity admits an extremal MCRBH is that there exists at least one curve of $\phi = \phi_e = \text{const}$ such that $J(\phi_e) = M_e$. In addition, $J(\phi)$ is monotonic in a neighborhood of $\phi_e = r_e^2/4$ with $J'(\phi_e) = V(\phi_e) = 0$ and $J''(\phi_e) = V'(\phi_e) \neq 0$. The initial condition of the AdS_2-horizon $J(\phi_{\pm}) = M_{\pm}$ implies the outer (ϕ_+) and inner (ϕ_-) horizons, which satisfy

$$1 - \frac{M_{\pm}}{\sqrt{\phi_{\pm}}}\left[1 - \tanh\left(\frac{Q^2}{4M_{\pm}\sqrt{\phi_{\pm}}}\right)\right] = 0 \to U(\phi_{\pm}) = 0. \tag{32}$$

This is precisely the definition of the mass function $M_{\pm}$ in Eq. (16). Further conditions on the minimum value $J(\phi_e) = M_e$ in favor of its extremal configuration imply

$$U'(\phi_e) = 0, \quad U''(\phi_e) \neq 0, \tag{33}$$

which are the conditions for the degenerate horizon $r = r_e(Q_e = q_e)$. Hence, for $Q_e = q_e = 2\sqrt{w_0}$ and $M_e = 1$, we find the location of the degenerate horizon $r_e = x_e = w_0/(1+w_0)$. Here, we have an AdS_2 spacetime with negative constant curvature

$$\bar{R}_2|_{r=r_e} = -\frac{2h}{\sqrt{\phi_e}} = -\frac{1}{\sqrt{\phi_e}}U''(r_e) = -\frac{(1+\omega_0)^4}{32M_e^3\omega_0^3} = -V'(\phi_e). \tag{34}$$

There exists an unstable point of $\phi = \phi_m = 0.714$, which satisfies $J'(\phi_m) = V(\phi_m)$, $J''(\phi_m) = V'(\phi_m) = 0$.

Then, all thermodynamic quantities found in the previous section can be explicitly expressed in terms of the dilaton ϕ_+, the dilaton potential $\widetilde{V}(\phi_+)$, its integration $\widetilde{J}(\phi_+)$, and its derivative $\widetilde{V}'(\phi_+)$ as

$$S_{\rm BH}(\phi_+) = 4\pi\phi_+, \quad T_H(\phi_+) = \frac{\widetilde{V}(\phi_+)}{4\pi},$$
$$C(\phi_+) = 4\pi\frac{\widetilde{V}(\phi_+)}{\widetilde{V}'(\phi_+)}, \quad F(\phi_+) = \widetilde{J}(\phi_+) - J(\phi_e) - \phi_+\widetilde{V}(\phi_+), \tag{35}$$

where

$$\widetilde{V}(\phi_+) = \frac{1}{2\sqrt{\phi_+}} - \frac{Q^2}{8\phi_+^{3/2}}\cosh^{-2}\left[\frac{Q^2}{4M_+\sqrt{\phi_+}}\right],$$

Fig. 4 Graphs for the thermodynamic quantities as the functions of ϕ_+. Here, $4\pi\phi_+$ plays the role of the entropy. The solid (*dashed*) curve represents the MCRBH (RN) with $Q_e = 1.056$ ($Q_e = 1$). The regions in $\phi_e \le \phi_+ < \phi_m$ represent the JT phase corresponding to the near-horizon geometry of the MCRBH

$$\widetilde{J}(\phi_+) = \sqrt{\phi_+} + M_+ \tanh\left(\frac{Q^2}{4M_+\sqrt{\phi_+}}\right),$$
$$\widetilde{V}'(\phi_+) = \frac{16\pi M_+^3\phi_+(4M_+^2\sqrt{\phi_+} - 2M_+Q^2 + Q^2\sqrt{\phi_+})}{8M_+^2Q^2\sqrt{\phi_+} - 8M_+^3\phi_+ + 2M_+Q^4 - Q^4\sqrt{\phi_+} - 2M_+Q^2\phi_+}. \tag{36}$$

We note the difference between V, J, V' and $\widetilde{V}, \widetilde{J}, \widetilde{V}'$. The former quantities are obtained by considering the mass M as a constant, while the latter are obtained by considering the mass $M(r_+)$ as a function of r_+. Hence, for thermodynamic calculations we have to use the tilled variables $\widetilde{V}, \widetilde{J}$, and $\widetilde{V}'$.

In Fig. 4, we have the corresponding dual graphs, which are nearly the same as in Fig. 2. For $\phi_e < \phi < \phi_m$, we have the JT phase, whereas for $\phi > \phi_m$, we have the Schwarzschild phase. At the extremal point with $M_e = 1$ and $Q_e = 1.056$, we have $T_H = 0$, $C = 0$, and $F = 0$, which are determined by $V(\phi_e) = 0$. On the other hand, at the maximum point ($M = M_m$), one has $T_H = T_m,\ C = \pm\infty$, which are fixed by $V'(\phi_m) = 0$.

4 Near-horizon thermodynamics of extremal MCRBH

It is a nontrivial task to directly find the near-horizon thermodynamics from the full thermodynamic quantities because there exists a nonlinear dependence between the mass M and the horizon radius r_+ in the near-horizon geometry of the 4D extremal MCRBH. Instead, we use the 2D dilaton gravity because it was proved that the near-horizon thermodynamics could be effectively described by the corresponding JT theory for the RN black hole [38; 39]. In order to find the AdS_2 gravity of the JT theory, we consider perturbation around the degenerate event horizon as

$$J(\phi) = J(\phi_e) + J'(\phi_e)\varphi + \frac{J''(\phi_e)}{2}\varphi^2 = M_e + \frac{V'(\phi_e)}{2}\varphi^2, \tag{37}$$
$$M = M_e[1 + k\alpha^2] \equiv M_e + \Delta M \tag{38}$$

with $\varphi = \phi - \phi_e$. Although $\widetilde{V}, \widetilde{J}$, and $\widetilde{V}'$ should be used for thermodynamic calculation, here we use V, J, and V', respectively, for perturbation. This is because in the near-horizon one has $V \approx \widetilde{V}$, $J \approx \widetilde{J}$, and $V' \approx \widetilde{V}'$. That is, $\frac{dM_+}{dr_+} \approx 0$ near the degenerate horizon.

Introducing the new coordinates

$$\tilde{t} = \alpha t, \quad \tilde{x} = \frac{x - x_e}{\alpha}, \tag{39}$$

the perturbed dilaton and the metric are given by

$$\varphi = \alpha\tilde{x}, \tag{40}$$

$$ds^2_{AdS_2} = -\left[\frac{V'(\phi_e)}{2}\tilde{x}^2 - kM_e\right]d\tilde{t}^2 + \frac{d\tilde{x}^2}{\left[\frac{V'(\phi_e)}{2}\tilde{x}^2 - kM_e\right]}, \tag{41}$$

which show a locally AdS_2 spacetime. If $k = 0$, it is a global AdS_2 spacetime. Moreover, the mass deviation ΔM is the conserved parameter of the JT theory [60]

$$\Delta M = \frac{V'(\phi_0)}{2}\varphi^2 - |\nabla\varphi|^2. \tag{42}$$

Thus, the JT theory describes both the extremal ($\Delta M = 0$) and the near-extremal ($\Delta M \neq 0$) MCRBHs.

Now, we are in a position to derive the near-horizon AdS_2 thermodynamic quantities from the JT theory. From the null condition of the metric function in Eq. (41), we have the positive root

$$\tilde{x}_+ = \sqrt{\frac{2kM_e}{V'(\varphi_e)}}, \quad \varphi_+ = \sqrt{\frac{2\Delta M}{V'(\phi_e)}}. \tag{43}$$

Then, the JT entropy and temperature are given by

$$S_{\text{JT}} = 4\pi\varphi_+ = 4\pi\sqrt{\frac{2\Delta M}{V'(\phi_e)}}, \tag{44}$$

$$T_{\text{JT}} = \frac{V'(\phi_e)}{2\pi}\frac{\varphi_+}{2} = \frac{1}{4\pi}\sqrt{2V'(\phi_e)\Delta M}. \tag{45}$$

Furthermore, we may have the JT heat capacity and the free energy

$$C_{\text{JT}} = 4\pi\varphi_+ = 4\pi\sqrt{\frac{2\Delta M}{V'(\phi_e)}}, \tag{46}$$

$$F_{\text{JT}} = -\phi_e V'(\phi_e)\varphi_+ = -\frac{(M_e x_e)^2}{4}\sqrt{2V'(\phi_e)\Delta M}. \tag{47}$$

Note that $S_{\text{JT}} = C_{\text{JT}}$ as the case of the RN black hole as shown in Ref. [54]. Finally, all thermodynamic quantities take the following forms in the near-horizon region:

$$S^{\text{NH}}_{\text{BH}} = S_{\text{BH}}(M_e) + S_{\text{JT}} = \pi M_e^2 + 4\pi\sqrt{\frac{2\Delta M}{V'(\phi_e)}}, \tag{48}$$

$$T^{\text{NH}}_H = T_H(M_e) + T_{\text{JT}} = \frac{\sqrt{2V'(\phi_e)\Delta M}}{4\pi}, \tag{49}$$

$$C^{\text{NH}} = C(M_e) + C_{\text{JT}} = 4\pi\sqrt{\frac{2\Delta M}{V'(\phi_e)}}, \tag{50}$$

$$F^{\text{NH}} = F(M_e) + F_{\text{JT}} = -\frac{(M_e x_e)^2}{4}\sqrt{2V'(\phi_e)\Delta M}. \tag{51}$$

Fig. 5 Plot of the near-horizon (NH) entropy and temperature as functions of ΔM for $r_e \le r_+ < r_m$. Both are proportional to $\sqrt{\Delta M}$

Fig. 6 Plot of the near-horizon heat capacity and free energy as functions of ΔM for $r_e \le r_+ < r_m$

From Figs. 5 and 6, one finds that there is no thermodynamic difference between the MCRBH and RN black hole.

5 AdS_2/CFT_1 correspondence for entropy

In this section we interpret the JT entropy $S_{\rm JT}$ to be a statistical entropy by using AdS_2/CFT_1 correspondence according to the previous work [61]. This correspondence is available because of the near horizon isometery of SO(2,1) and an infinitely long throat of the AdS_2 spacetime near the extremal black hole. If the $\tilde{t}$ in the AdS_2 plays the role of a null coordinate, one may impose asymptotic symmetries on the boundary (mimicking the analysis of the 3D gravity) as

$$g_{\tilde{t}\tilde{t}} = -\frac{\bar{R}_e}{2}\tilde{x}^2 + \gamma_{\tilde{t}\tilde{t}} + \cdots, \tag{52}$$

$$g_{\tilde{t}\tilde{x}} = \frac{\gamma_{\tilde{t}\tilde{x}}}{\tilde{x}^3} + \cdots, \tag{53}$$

$$g_{\tilde{x}\tilde{x}} = \frac{2}{\bar{R}_e}\frac{1}{\tilde{x}^2} + \frac{\gamma_{\tilde{x}\tilde{x}}}{\tilde{x}^4} + \cdots \tag{54}$$

with $\bar{R}_e \equiv \bar{R}_2|_{r=r_e} = -V'(\phi_e)$. Choosing the boundary conformal gauge with $\gamma_{\tilde{t}\tilde{x}} = 0$, the charges can be derived easily. The infinitesimal diffeomorphisms $\zeta^a(\tilde{x},\tilde{t})$ preserving the above boundary conditions are $\zeta^{\tilde{t}} = \varepsilon(\tilde{t}),\ \zeta^{\tilde{x}} = -\tilde{x}\varepsilon'(\tilde{t})$. Its action on the 2D gravity in Eq. (21) induces the following transformation for the function $\Theta_{\tilde{t}\tilde{t}} = \kappa\left[\gamma_{\tilde{t}\tilde{t}} - (\bar{R}_e/2)^2\gamma_{\tilde{x}\tilde{x}}/2\right]$:

$$\delta_\varepsilon\Theta_{\tilde{t}\tilde{t}} = \varepsilon(\tilde{t})\Theta'_{\tilde{t}\tilde{t}} + 2\Theta_{\tilde{t}\tilde{t}}\varepsilon'(\tilde{t}) + \frac{2\kappa}{\bar{R}_e}\varepsilon'''(\tilde{t}). \tag{55}$$

$\Theta_{\tilde{t}\tilde{t}}$ behaves as the chiral component of the stress tensor of a boundary conformal field theory. To find its central charge, we have to know the coefficient κ in Eq. (55). For this purpose, we construct the full Hamiltonian $H = H_0 + K$, where K is a boundary term to have well-defined variational derivatives. This is determined as $K(\varepsilon) = \varepsilon(\tilde{t})2\alpha\left[\gamma_{\tilde{t}\tilde{t}} - (\bar{R}_e/2)^2\gamma_{\tilde{x}\tilde{x}}/2\right]$ with $\kappa = 2\alpha$. Assuming a periodicity of $2\pi\beta$ in $\tilde{t}$ [61], we find the central charge and its Virasoro generator

$$c = -\frac{48\alpha}{\bar{R}_e\beta}, \quad L_0^R = M_e k\alpha\beta. \tag{56}$$

Using the Cardy-formula for the right movers, one has the desired statistical entropy as follows

$$S_{st}^{\rm CFT_1} = 2\pi\sqrt{\frac{cL_0^R}{6}} = 2\pi\sqrt{\frac{8M_e k\alpha^2}{-\bar{R}_e}} = 4\pi\sqrt{\frac{2\Delta M}{V'(\phi_e)}} = S_{\rm JT}. \tag{57}$$

This statistical entropy accounts for the microscopic excitations around the extremal macroscopic state of the MCRBH.

6 Discussions

There are a few of approaches to understanding a magnetically charged regular black hole (MCRBH). However, it remains a nontrivial task to understand its full thermodynamic behaviors because this MCRBH was constructed from the combination of Einstein gravity and nonlinear electromagnetics. In this work, we have explored the thermodynamics of the MCRBH completely. Here, the extremal MCRBH is determined by zero temperature ($T = 0$), zero heat capacity ($C = 0$), and zero free energy ($F = 0$). We have also found an important point where the temperature is maximum, the heat capacity changes from positive infinity to negative infinity. This point separates the whole thermodynamic process into the near-horizon phase with positive heat capacity and the Schwarzschild phase with negative heat capacity. The former represents the near-horizon AdS_2 thermodynamics of the extremal MCRBH, which is characterized by the increasing temperature, positive heat capacity, and decreasing free energy. We have also reexamined the thermodynamics of the MCRBH by using the 2D dilaton gravity and its near-horizon thermodynamics by introducing the Jackiw-Teitelboim theory of AdS_2-gravity. All thermodynamic behaviors of the MCRBH are similar to those of the singular RN black hole. This means that an observer at infinity does unlikely distinguish between the regular and the singular black holes.

Concerning a possible phase transition, one expects that a phase transition occurs near $T = 0$, from the extremal MCRBH to the non-extremal MCRBH. However, in order to study the presumed phase transition, we have to introduce the negative cosmological constant because the free energy is positive for large r_+ [62]. Having the AdS-RBH, one may find the negative free energy for large r_+. Then, we may discuss the phase transition from the extremal MCRBH at $r_+ = r_e$ to a large MCRBH with $r_+ \gg r_e$ in AdS spacetime, similar to the Hawking–Page transition from the thermal AdS spacetime at $r_+ = 0$ to a large black hole [63; 64; 65].

In conclusion, we have shown that the thermodynamic behaviors of the MCRBH without singularity is the nearly same as those of the RN black hole with singularity. This is because the temperature in Fig. 2a, the heat capacity in Fig. 2b, and the free energy in Fig. 2c show the nearly same behaviors, regardless of singularity and regularity at the origin.

Acknowledgments Two of us (Y.S. Myung and Y.-J. Park) were supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021. Y.-W. Kim was supported by the Korea Research Foundation Grant funded by Korea Government (MOEHRD): KRF-2007-359-C00007.

Appendix: Proofs of Eqs. (19) and (36)

In this appendix, we will show how to get the concrete form of the specific heats for the two approaches. In the definition of the specific heat as

$$C = \left(\frac{dM}{dT}\right)_Q = \frac{dM(r_+)}{dr_+}\frac{dr_+}{dT(r_+)} \tag{58}$$

$$= \frac{dM(\phi_+)}{d\phi_+}\frac{d\phi_+}{dT(r_+)}, \tag{59}$$

the derivatives of the mass functions $M(r_+)$ $(M(\phi_+))$ with r_+ (ϕ_+) can be easily obtained from the metric function $U(r_+) = 0$ in Eq. (12) and $J(\phi) - M = 0$ in Eq. (30) as

$$\frac{dM(r_+)}{dr_+} = \frac{M_+}{r_+}\left(\frac{4M_+^2 r_+ - 4M_+ Q^2 + Q^2 r_+}{4M_+^2 r_+ + 4M_+ Q^2 - Q^2 r_+}\right), \tag{60}$$

$$\frac{dM(\phi_+)}{d\phi_+} = \frac{M_+}{2\phi_+}\left(\frac{4M_+^2\sqrt{\phi_+} - 4M_+ Q^2 + Q^2\sqrt{\phi_+}}{4M_+^2\sqrt{\phi_+} + 4M_+ Q^2 - Q^2\sqrt{\phi_+}}\right), \tag{61}$$

respectively. On the other hand, the derivatives of the temperature functions with r_+ (ϕ) can be also obtained as

$$\frac{dT(r_+)}{dr_+} = \frac{1}{4\pi}\left[-\frac{1}{r_+^2}\left(1 + \frac{Q^2}{4M_+^2}\right) + \frac{2Q^2}{M_+ r_+^3} + \left(\frac{Q^2}{M_+^2 r_+^2} - \frac{Q^2}{2M_+^3 r_+}\right)\frac{dM(r_+)}{dr_+}\right] \tag{62}$$

$$\frac{dT(\phi_+)}{d\phi_+} = -\frac{1}{4\phi_+^{3/2}} + \frac{Q^2}{4M_+\phi_+^2} - \frac{Q^2}{16M_+^2\phi_+^{3/2}} + \left(\frac{Q^2}{4M_+^2\phi_+} - \frac{Q^2}{4M_+^3\sqrt{\phi_+}}\right)\frac{dM(\phi_+)}{d\phi_+}. \tag{63}$$

Note in these calculations that one should be careful to differentiate the temperatures with r_+ (ϕ) because they also have the derivatives of the mass functions as shown in Eqs. (60) and (61). This contrasts to the usual calculations for the specific heats of the non-linear Born–Infeld and the RN black holes in which cases the mass functions can be explicitly separated with the horizon radius, while it is not for our non-linear MCRBH. Now, combining these Eqs. (62) and (63) with (60) and (61), respectively, we have the final expressions of the specific heat, Eqs.(19) and (36), which blow up at the radius r_m (ϕ_m) of giving the maximum Hawking temperature as expected.

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