M-Theory Solutions and Intersecting D-Brane Systems

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ABSTRACT

It is believed that fundamental M-theory in the low-energy limit can be described effectively by D=11 supergravity.

Extending our understanding of the different classical brane solutions in M-theory (or string theory) is important, and so there is a lot of interest in finding D=11 M-brane solutions such that after reduction to ten dimensions, they (or some combinations of them) reduce simply to the supersymmetric BPS saturated p-brane solutions.

In this thesis, we study and construct M2 and M5-branes solutions in D=11 supergravity. The M-brane solutions are constructed by lifting a D-brane to a four or higher dimensional geometry embedded in M-theory and then placing M-brane solutions in the background geometry.

We present new analytic M2 and M5-brane solutions in M-theory based on transverse Gibbons-Hawking and Bianchi spaces. These solutions provide realizations of fully localized type IIA D2/D6 and NS5/D6 brane intersections. One novel feature of these solutions is that the metric functions depend on more than two transverse coordinates, unlike all the other previous known solutions. Moreover since the metric functions in the Gibbons-Hawking geometries depends on more than one physical parameters, their embedding into M-theory yield new M-brane solutions with the M-brane metric functions depend on both compact and non-compact coordinates.

We show that all new solutions have eight preserved supersymmetries. Upon reduction to 10 dimensions, we find that the world-volume theories of the NS5-branes decouple from the bulk for these solutions.

ACKNOWLEDGEMENTS

I dedicate this thesis to my parents and to my friend Nasrin Sagai who tragically passed away on June 7, 2007.

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A_μ	Gauge potential
ALE	Asymptotically Locally Euclidean
$b_n^\mu, \tilde{b}_n^\mu$	Vibration modes in Neveu-Schwarz sector
$B^{\alpha\beta}$	Kalb-Ramond field
BPS	Bogomolnyi-Prasad-Sommerfield
$C_{(\mu\cdots\nu)}$	Gauge field (three-form field)
Cl_2	Clifford algebra
$d_n^\mu, \tilde{d}_n^\mu$	Vibration modes in Ramond sector
D	The dimension of spacetime
e	The determinant of the tetrad
e_α^2	Tetrad (vielbein)
e_x, e_y	Basis vectors in Clifford algebra
$F_{(\alpha_1\cdots\alpha_n)}$	Field strength
$G_{\mu\nu\rho\delta}$	Field strength (Four-form field)
GH	Gibbons-Hawking
$h_{\alpha\beta}$	Metric on worldsheet
$H_{(\alpha_1\cdots\alpha_n)}$	Field strength
H	Harmonic function in M2(M5)-brane
$\mathcal{H}_C$	Heun-C function
I_1	Modified Bessel function of the first kind
J_1	Bessel function of the first kind
K	Kaluza-Klein excitation number
KK	Kaluza-Klein theory
$K_{\mu ab}$	Contorsion
K_1	Modified Bessel function of the second kind
$\mathfrak{K}_{10}$	Gravitational constant
$\mathbf{K}$	Kretschmann scalar
$\mathcal{L}$	Lie derivative
l_s	Length scale
m	Mass
n	Winding mode
$\mathcal{N}$	The number of supersymmetries
N_L, N_R	Number operators
NS	Neveu-Schwarz sector
p^μ	Total momentum
Q_{M_2}, Q_{M_5}	The charge on the M2-brane or M5-brane
Q_α	Supercharges
R	Ramond sector
R	Ricci scalar
R	The radius of compactification
$R_{\alpha\beta}$	Ricci tensor

$R_{\alpha\beta\gamma\sigma}$	Riemann tensor
S	Action
$\vec{v}_T$	Transverse velocity
TOE	The theory of everything
$\mathcal{T}$	The tension of string
w	Winding
$\mathcal{W}_W$	Whittaker function type W
$\mathcal{W}_M$	Whittaker function type M
YM	Yang-Mills
Y_1	Bessel function of the second kind
x_A	M-brane coordinates
x^μ	Center of mass position in string theory
X^μ	Spacetime coordinates in target space
α'	Regge slope
$\alpha_m^\mu, \tilde{\alpha}_m^\mu$	Vibration modes
$\beta^{\mu\nu}$	Beta functions
$\Gamma^{a_1\cdots a_i}$	Antisymmetric products of Γ-matrices
ϵ^{ab}	Fully antisymmetric tensor
ϵ	Infinitesimal Susy transformation
$\epsilon_{\alpha_1\cdots\alpha_5}$	Levi-Civita symbol
λ	Dilatino
$\Lambda^\mu{}_\nu, \Lambda_\nu{}^\mu$	Lorentz transformation
ξ^α, ξ_α	Killing vectors
ρ	Linear mass density
ρ^0, ρ^1	Gamma matrices in 2D
ϕ	Scalar function
Φ	Dilaton
χ	Euler number
ψ_α^μ	Spinors
σ, τ	Spacetime coordinates on worldsheet
τ	Hirzebruch signature
$\omega_{\mu ab}$	Spin connection
$\omega_{\mu ab}^{(0)}$	Levi-Civita connection .+(Christoffel symbol)
$\oplus$	Direct sum
$\Box$	Laplace operator
$\star$	Hodge operator

CHAPTER 1

INTRODUCTION

One of the interesting and attractive problems in physics is that of understanding how to make sense of a quantum theory of gravity. According to our present knowledge, the best possible candidate for a quantum theory of gravity is superstring theory, which seems to exhibit good perturbative behavior (a brief history of string theory is provided in Appendix A).

In superstring theory, all the fundamental particles in particle physics and all known forces in nature are realizations of different states of the most fundamental object, that is, a one-dimensional string. However it has been known for some time that there are at least five distinct consistent superstring theories. It seemed that they simply exist with some dual relations between them although there was no known theory that explained why different superstring theories could have dual relations between them. In this regard, it is very difficult to determine which particular string theory describes our real world.

Recent developments in non-perturbative string theory starting with the discovery of various extended objects in superstring theories (D-branes) have begun to cast light on this question. It has become clear that these extended objects play important roles in the strong coupling dynamics of superstrings and in Membrane theory (M-theory). Superstring theories, when viewed in the strong coupling limit, are not just theories of strings but instead contain many extended objects (branes) as light degrees of freedom. The existence of these objects turns out to be the origin of the dual relations between different superstring theories. M-theory as it is called now is an 11-dimensional quantum theory of the many extended objects which produce all the superstring theories around their perturbative vacua. M-theory is an underlying theory in physics which tries to incorporate the five superstring theories. The theory was originally proposed by Edward Witten in 1995 and accoring to Witten's statment M can stand for magic, mystery, or membrane [1]. The compactification of the theory on an n-dimensional torus T^n results a matrix theory. The matrix theory in turn is an ordinary quantum field theory in $n + 1$ space-time dimensions.

Fundamental M-theory in the low-energy limit can be described effectively by 11D supergravity, underscoring the importance of understanding different classical brane solutions in M-theory. Extending our understanding of the different classical brane solutions in M-theory is very important because after reduction to 10D, these solutions yield supersymmetric solutions describing a large class of supersymmetric p-brane solutions and so there is a lot of interest in finding 11D M-brane solutions.

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Some supersymmetric solutions of two or three orthogonally intersecting 2-branes and 5-branes in M-theory were obtained some years ago and more such solutions have since been found.

Localized intersecting solutions, in which one brane's world-volume is completely inside another brane's world-volume, are very hard to find. These localized intersecting solutions have the important property that the solutions are not restricted to be in the near core regions of the branes in the system. Recently new localized intersecting solution was constructed by lifting a specific D-brane to self-dual geometries embedded in M-theory and then placing the different M-branes (M2 and M5) in the self-dual background geometries. Localized intersecting solutions are very interesting because the solutions are not restricted to be in the near core region of the branes.

In this work I have worked toward constructing new M-brane solutions and fully localized intersecting D-brane solutions in type IIA string theory. Specifically in the construction of new M-brane solutions, we start with the general Lagrangian from which the equations of 11D supergravity can be derived. In order to construct a solution to these equations that can be successfully reduced to 10D type IIA string theory, we assume a bosonic ground state, i.e. the vacuum expectation value of any fermion field should be zero. In this case, we have two sets of coupled equations of motion for the 11D metric tensor and four-form field strength. We use ansatze for the M-brane metrics which the metric functions depend on transverse coordinates to the brane. The eight (or five) dimensional transverse space that is not part of the M2 (M5) brane world-volume could be any combination of some low dimensional spaces. After finding the solutions for the metric functions, we use the well known Kaluza-Klein compactification method to get the different fields of the 10D type IIA supergravity: two Ramond-Ramond one-form and three-form fields, three Neveu-Schwarz/Neveu-Schwarz dilatons, an antisymmetric two- form and the 10D metric. Moreover, we explicitly check out that these fields satisfy properly the 10D supergravity equations.

Similar to M2 brane solutions, we can construct M5 brane solutions with self-dual geometries lifted to M-theory. Since in the 11D metric, the M5 brane itself only takes up five of the 10 spatial coordinates, we can embed a variety of different geometries. These include combinations of Bianchi space with itself, Taub-NUT and EH spaces. After compactification on a circle, we find the different fields of type IIA string theory which describe new completely localized intersecting NS5/D6 systems. Then I apply T-duality transformations on type IIA solutions and find type IIB NS5/D5 intersecting brane solutions. Finally, we consider the decoupling limit of new solutions and find evidence that in the limit of vanishing string coupling, the theory on the world-volume of the NS5-brane is a new little string theory. In fact the little string theory is a non-gravitational and non-local theory in six spacetime dimensions and similar to the string theories, the little string theories exhibit T-duality. The outline of this work is as follows.

Chapter 2 contains a summary of bosonic and superstring theories. This chapter is divided into three parts. In the first part, the actions of bosonic stings including closed and open strings, are introduced. The other features of bosonic string theory such as the equations of motion together with the solutions, the symmetries of the

action, and the bosonic string spectrum are reviewed. Moreover we introduce the effective action of bosonic strings while the strings are coupled to an antisymmetric tensor and a scalar field. In the second part, we consider superstring theory which includes fermions, bosons and supersymmetry transformations. In addition we review type IIA and IIB superstring theories in details. In the last part of chapter 2 we present the concept of compactification and we relate this concept to T-duality. Additionally by using the Lagrangian multiplier method, we show how the background fields in two different string theories are related under the T-duality transformation.

In **Chapter 3**, we recall gravitational instantons. Our main motivation in this chapter is to use the gravitational instantons in the transverse space of M2 and M5-branes solutions. These spaces also can be used to study intersecting brane configurations. In this chapter, first we briefly discuss instantons in Yang-Mills theory and then we introduce various gravitational instantons such as Bianchi models, Taub-NUT spaces and Gibbons-Hawking spaces (GH). These are four-dimensional Riemannian manifolds with Euclidean signatures which satisfy the vacuum Einstein equations and (anti) self-duality relation. Specially we give more details about Bianchi models and the related Lie algebra with three generators. Finally at the end of this chapter, we introduce two topological invariants known as the Euler characteristic and the Hirzebruch signature which are used in the classification of gravitational instantons.

We give an almost full description of D=11 supergravity in **Chapter 4**. In section 4.1 we start from the Poincare algebra and extend it to the super Poincare algebra, including central charges. The central charges commute with all elements in the algebra. In D=11 supergravity the central charges are related to two extended objects called M2 and M5-branes. Furthermore, we finish this section by reviewing the massless states in the super Poincare algebra, the field contents in D=11 and BPS (Bogomolnyi-Prasad-Sommerfield) states. These are states which have equal mass and charge.

To achieve the relation between the M-theory and superstring theory we should go from M-theory in D=11 to superstring theory in D=10 by cutting down one of spatial coordinates in D=11. In this method, the massless sector of lower dimension theory is considered to be independent of reduced coordinate. For the first time, this method was used to include both the D=4 gravity and electrodynamics in the content of pure D=5 gravity. In section 4.2 we discuss the dimensional reduction method and the Kaluza-Klein theory in details.

In section 4.3 we give the main highlights in D=11 supergravity. In this work no contribution comes from the gravitino therefore the spinor field is set to zero. We recall the eleven dimensional Lagrangian which contains the graviton, gravitino and gauge field. We skip the details of the calculation and give the components of the metric and other generated fields in D=10, in terms of the metric components in D=11, after dimensional reduction over a circle. We obtain the equations of motion for the metric and gauge field and also we give the equation of motion for the gravitino. We will use these equations later in chapter 5.

We give a brief introduction to membranes in section 4.3.4. Furthermore we start from the Killing equation and use the equation of motion for the gauge field to derive

the M2-brane solution. The solution satisfies Laplace equation in transverse space and preserve half of the original supersymmetries. The membranes are dynamical object carrying charge and mass. We use ADM (Richard Arnowitt, Stanley Deser and Charles W. Misner) formalism to derive mass and charge separately for M2 and M5-branes. In the rest of this section we discuss different configurations of intersecting M-branes and D-branes.

In **Chapter 5**, we present new solutions in D=11 supergravity. Gibbons-Hawking spaces and Bianchi space are two candidates to be in our embedded M-brane solutions. In section 5.1, we obtain new analytical solutions for the M2-brane functions in the Bianchi space. In this case the eight dimensional transverse space is divided into two parts. In the first part we use a four-dimensional flat metric and in the second part, the Bianchi space can be embedded.

In section 5.2, we consider the multi Gibbons-Hawking spaces (especially two-center and three-center) in the transverse space. As we mentioned earlier, satisfying the transverse Laplacian is the main requirement for embedding a four-dimensional metric in transverse space. In both M2 or M5-bare solutions, satisfying the Laplace equation leads to a lower dimensional Laplacian. Therefore in section 5.2, we solve the new Laplace equation and derive new metric functions.

In section 5.3 and 5.4, we study the embedding of four-dimensional multi (and explicitly double)-center Gibbons-Hawking spaces in M-theory and find analytical exact solutions for the M2-brane functions. These spaces are characterized with some NUT charges. We then discuss embedding products of Gibbons-Hawking metrics in M2-brane solutions. All of the solutions preserve some of the supersymmetry. In addition we give the D-brane solutions and the field contents in type IIA and IIB supergravities D=10. In section 5.5, similar to M2-brane solutions we present the M5-brane solutions.

We discuss briefly in section 5.6, the field equations of supergravity. There are three equations of motion in D=11 which can be extracted from the Lagrangian. We only use the equations of motion for the gauge field and metric (we ignore spinors). Any membrane solutions (M2 or M5) must satisfy the equations of motion, and the Laplacian operator in the transverse space. Thus in section 5.6, by assuming the new solutions fulfill the transverse Laplacian, we show that they also satisfy the equations of motion. Furthermore we check the preservation of supersymmetry for the new solutions by solving Killing equation.

In section 5.7, we consider the decoupling limit of our solutions and find evidence that in the limit of vanishing string coupling, the theory on the world-volume of the NS5-branes is a new little string theory. Moreover, we apply T-duality transformations on type IIA solutions and find type IIB NS5/D5 intersecting brane solutions and discuss the decoupling limit of the solutions.

In **Chapter 6**, we wrap up by some concluding remarks and future possible research directions. Finally we provide some technical details such as differential forms, Buschers rules for T-duality, Clifford algebra and the Heun-C functions related to the main text in the appendixes.

CHAPTER 2

STRING THEORY

String theory which is sometimes called the theory of everything (TOE) is one theoretical candidate for quantum theory of gravity, although so far, no experimental justification for the theory has been observed. The fundamental objects in this theory are not point objects. Instead, they are treated as one dimensional objects or strings (open or closed strings) (figure 2.1). The first part of this section is devoted to the bosonic string theory that includes only bosonic fields. In the second part we are going to introduce a realistic theory which contains both fermions and bosons, called superstring theory.

Figure 2.1: Fundamental objects in string theory assumed to be extended objects called open or closed strings.

2.1 Bosonic string theory

To start up, we consider the action of a free relativistic particle in D-dimensional target space.

$$S = -mc \int ds = -mc \int \sqrt{-g_{\mu\nu}dX^\mu dX^\nu} = -mc^2 \int \sqrt{1 - \frac{v^2}{c^2}} \, dt, \qquad (2.1)$$

where μ and $\nu = 0 \ldots D$ - 1. As we mentioned before, the strings are extended objects, therefore by modifying the free particle action in two steps, we can easily obtain an action which describes free bosonic strings. In the first step let us consider a small element of our string called $\delta m = \rho ds$ where ds, δm, and ρ are the differential length, the mass, and the linear density of element respectively. In the second step the velocity of the differential element must be replaced by the transverse velocity. By applying these two conditions the string action becomes [2]

$$S = -\rho c^2 \int \int_0^l \sqrt{1 - \frac{v_T^2}{c^2}} \, ds \, dt, \qquad (2.2)$$

where v_T is the transverse velocity and is defined as follows

$$\vec{v}_T = \frac{\partial \vec{x}}{\partial t} - (\frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial s})\frac{\partial \vec{x}}{\partial s}. \tag{2.3}$$

As it can be seen from (2.2) the action depends on two variables t and s. In general by assuming $t = t(\tau, \sigma)$, $\vec{x} = \vec{x}(\tau, \sigma)$ as functions of worldsheet coordinates τ and σ, and considering this as the representation of any point in target space by $X = (ct(\tau, \sigma), \vec{x}(t(\tau, \sigma), \sigma))$, the Lorentz invariant form of the action (known as Nambu-Goto action) can easily be derived. We adopt a flat metric $\eta_{\mu\nu}$ for the target space and set

$$X = X(c\tau, \vec{y}(\tau, \sigma)), \tag{2.4a}$$
$$\vec{y}(\tau, \sigma) = \vec{x}(t(\tau, \sigma), \sigma). \tag{2.4b}$$

Now we find the derivatives of $\vec{y}$ and X with respect to τ and σ, as shown below

$$\frac{\partial \vec{y}}{\partial \tau} = \frac{\partial \vec{x}}{\partial t}\frac{\partial t}{\partial \tau} \;\;\rightarrow\;\; \frac{\partial \vec{x}}{\partial t} = \frac{\frac{\partial \vec{y}}{\partial \tau}}{\frac{\partial t}{\partial \tau}}, \tag{2.5a}$$

$$\frac{\partial \vec{y}}{\partial \sigma} = \frac{\partial \vec{x}}{\partial t}\frac{\partial t}{\partial \sigma} + \frac{\partial \vec{x}}{\partial \sigma} \;\;\rightarrow\;\; \frac{\partial \vec{x}}{\partial \sigma} = \frac{\partial \vec{y}}{\partial \sigma} - \frac{\frac{\partial t}{\partial \sigma}}{\frac{\partial t}{\partial \tau}}\frac{\partial \vec{y}}{\partial \tau}. \tag{2.5b}$$

So $\frac{\partial X}{\partial \tau}$ and $\frac{\partial X}{\partial \sigma}$ are given by

$$\dot{X} = \frac{\partial X}{\partial \tau} = (c\frac{\partial t}{\partial \tau}, \frac{\partial \vec{y}}{\partial \tau}) \;\;\rightarrow\;\; \dot{X}^2 = -c^2(\frac{\partial t}{\partial \tau})^2 + (\frac{\partial \vec{y}}{\partial \tau})^2, \tag{2.6a}$$

$$X' = \frac{\partial X}{\partial \sigma} = (c\frac{\partial t}{\partial \sigma}, \frac{\partial \vec{y}}{\partial \sigma}) \;\;\rightarrow\;\; X'^2 = -c^2(\frac{\partial t}{\partial \sigma})^2 + (\frac{\partial \vec{y}}{\partial \sigma})^2. \tag{2.6b}$$

Writing the covariant form of the string action needs a manipulation of the transverse velocity (2.3). We write $(\vec{v}_T)^2$ as

$$(\vec{v}_T)^2 = (\frac{\partial \vec{x}}{\partial t})^2 - (\frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial s})^2, \tag{2.7a}$$

$$= (\frac{\partial \vec{x}}{\partial t})^2 - (\frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial \sigma})^2 (\frac{d\sigma}{ds})^2, \tag{2.7b}$$

where $\frac{ds}{d\sigma} = |\frac{\partial \vec{x}}{\partial \sigma}|$. We plug (2.5), (2.6) , and (2.7) into (2.2) to get the Nambu-Goto action. However the simplest way to derive the action is to use the static gauge (in this gauge we assume $t = \tau$) defined by

$$X = (ct, \vec{x}(t, \sigma)). \tag{2.8}$$

In this gauge the relative quantities are

$$\dot{X} = (c, \frac{\partial \vec{x}}{\partial t}) \;\;\rightarrow\;\; \dot{X}^2 = -c^2 + (\frac{\partial \vec{x}}{\partial t})^2, \tag{2.9a}$$

$$X' = (0, \frac{\partial \vec{x}}{\partial \sigma}) \;\;\rightarrow\;\; X'^2 = (\frac{\partial \vec{x}}{\partial \sigma})^2, \tag{2.9b}$$

$$X' \cdot \dot{X} = \frac{\partial \vec{x}}{\partial \sigma} \cdot \frac{\partial \vec{x}}{\partial t}. \tag{2.9c}$$

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Plugging (2.9a), (2.9b) and (2.9c) into (2.7) give

$$(\vec{v}_T)^2 = \dot{X}^2 + c^2 - (X' \cdot \dot{X})^2 (\frac{d\sigma}{ds})^2, \tag{2.10}$$

and

$$
\begin{aligned}
1 - \frac{\vec{v}_T^2}{c} &= \frac{-\dot{X}^2 + (X' \cdot \dot{X})^2 (\frac{d\sigma}{ds})^2}{c^2}, \\
&= (\frac{d\sigma}{ds})^2 \frac{(-\dot{X}^2 (\frac{ds}{d\sigma})^2 + (X' \cdot \dot{X})^2)}{c^2}, \\
&= (\frac{d\sigma}{ds})^2 \frac{(-\dot{X}^2 X'^2 + (X' \cdot \dot{X})^2)}{c^2}.
\end{aligned}
\tag{2.11}
$$

Using the string action (2.2) and (2.11), we get the Nambu-Goto action

$$S = -\frac{\mathcal{T}}{c} \int_0^\pi \int_{\tau_1(\sigma)}^{\tau_2(\sigma)} \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \ d^2\sigma, \tag{2.12}$$

where $d^2\sigma = d\sigma d\tau$, and $\mathcal{T} = \rho c^2$ is the string tension.

There is no convenient way to quantize the theory based on the Nambu-Goto action due to presence of square root. Hence we need to introduce an equivalent action called Polyakov action (or string sigma model action) that does not have the complicated square root. This action contains an intrinsic metric defined on the worldsheet, shown by $h_{\alpha\beta}(\tau, \sigma)$ and also the derivative of X^μ which are taken with respect to the coordinates on the worldsheet (τ or σ)[2, 3, 4]

$$S = -\frac{\mathcal{T}}{2} \int \sqrt{-h} h^{ab}(\tau, \sigma) \partial_a X^\gamma \partial_b X_\gamma \ d^2\sigma, \tag{2.13}$$

where $\mathcal{T}$ is the string tension, $h = det(h_{ab})$, X^μ are the coordinates of string in the target space, and $a, b = 0, 1$ (figure 2.2).
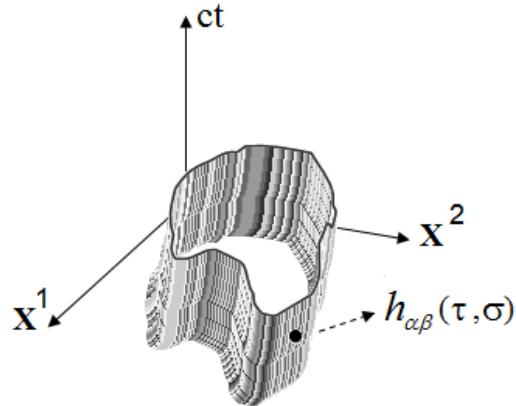
Figure 2.2: The evolution of a closed string in target space.

Three symmetry groups (or in other words gauge freedom) of the Polyakov action are

- Poincare transformation.
$$\tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$$

- Reparameterization of the worldsheet coordinates.
$$\tilde{\sigma} = \tilde{\sigma}(\sigma, \tau)$$
$$\tilde{\tau} = \tilde{\tau}(\sigma, \tau)$$

- Weyl transformation.
$$h_{ab} \longrightarrow e^{\phi(\sigma,\tau)} h_{ab}$$

Some properties of the Polyakov action are

- The trace of the energy-momentum tensor is zero.

  Using the Weyl transformation and $T^{ab} = -\frac{2}{\mathcal{T}} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}}$ one can show that the trace of the energy-momentum tensor $h^{ab}T_{ab}$ for a bosonic string is equal to zero. We notice this be true in general, in the quantum theory of the string. Considering the definition of T^{ab} and the Weyl transformation we have

  $$\delta S = \int d^2\sigma \frac{\delta S}{\delta h_{ab}} \delta h_{ab}, \tag{2.14a}$$

  $$\delta h_{ab} = h_{ab}\delta\phi, \tag{2.14b}$$

  $$\delta S = \int d^2\sigma (-\frac{\mathcal{T}}{2}\sqrt{-h}) T^{ab} h_{ab}\delta\phi \rightarrow T^a{}_a = 0. \tag{2.14c}$$

- The components of the energy-momentum tensor are zero.

  To show this we derive the equation of motion for h_{ab} by taking the variation of (2.13)

  $$\delta S = -\frac{\mathcal{T}}{2} \int \left[\delta(\sqrt{-h})h^{ab} + \sqrt{-h}\delta h^{ab} \right] \partial_a X^\gamma \partial_b X_\gamma \, d^2\sigma . \tag{2.15}$$

  One can easily show that
  $$\delta h = -h h_{ab}\delta h^{ab},$$

  which implies that

  $$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h} h_{ab}\delta h^{ab}. \tag{2.16}$$

  Substituting (2.16) in (2.15) gives the equation of motion for h_{ab} as follows

  $$\partial_a X^\gamma \partial_b X_\gamma = \frac{1}{2} h_{ab} h^{cd} \partial_c X^\gamma \partial_d X_\gamma. \tag{2.17}$$

  Now we are ready to obtain T_{ab}

  $$
  \begin{aligned}
  T_{ab} &= -\frac{2}{\mathcal{T}} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}}, \\
  &= \partial_a X^\gamma \partial_b X_\gamma - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\gamma \partial_d X_\gamma,
  \end{aligned}
  \tag{2.18}
  $$

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and using (2.17) results

$$T_{ab} = 0. \tag{2.19}$$

- Consistency with the Nambu-Goto action.

To see this we use (2.17) which gives us the desired result

$$\sqrt{-det(\partial_a X^\gamma \partial_b X_\gamma)} = \frac{1}{2}\sqrt{-h}h^{cd}\partial_c X^\gamma \partial_d X_\gamma. \tag{2.20}$$

The metric on worldsheet has three independent components, say h_{11}, h_{22}, and h_{12}, however the number of independent components can be reduced to one by using reparameterization invariance. By applying Weyl transformation, the remaining component can be gauged away and as a result the metric becomes a flat worldsheet metric shown by

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ h = |h_{ab}| = -1,$$

which simplifies the string action (2.13) to

$$S = \frac{\mathcal{T}}{2} \int (\dot{X}^2 - X'^2)d^2\sigma. \tag{2.21}$$

We write now the equation of motion derived from (2.12) and (2.21). Using Nambu-Goto action (2.12) the equation of motion becomes

$$\frac{\partial}{\partial \tau}\frac{(\dot{X}\cdot X')X'^\mu - X'^2\dot{X}^\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}} + \frac{\partial}{\partial \sigma}\frac{(\dot{X}\cdot X')\dot{X}^\mu - \dot{X}^2X'^\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}} = 0, \tag{2.22}$$

which does not have a simple form whereas by employing (2.21) this equation takes the following form

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0, \tag{2.23}$$

which is an ordinary wave equation. The string is either open with loose ends moving in spacetime or a closed string and so there are two different boundary conditions

- Closed string (periodic condition).

$$X^\mu(\tau, 0) = X^\mu(\tau, \pi) \tag{2.24}$$

- Open string.

$$\frac{\partial X^\mu(\tau, 0)}{\partial \sigma} = \frac{\partial X^\mu(\tau, \pi)}{\partial \sigma} = 0 \tag{2.25}$$

Remembering that the components of momentum can be written as

$$P_\sigma^\mu = -\mathcal{T}\frac{\partial X^\mu}{\partial \sigma}, P_\tau^\mu = \mathcal{T}\frac{\partial X^\mu}{\partial \tau}, \tag{2.26}$$

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(2.25) and P_σ^μ indicate that no momentum can flow off the ends of string.
The solution to(2.23) for a closed string is made of two parts called left and right movers which is given by

$$X^\mu = X_R^\mu(\text{right mover}) + X_L^\mu(\text{left mover}), \qquad (2.27)$$

where

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau - \sigma) + \frac{1}{2}l_s i \sum_{m \neq 0 \in Z} \frac{\alpha_m^\mu}{m} e^{-2im(\tau-\sigma)}, \qquad (2.28)$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau + \sigma) + \frac{1}{2}l_s i \sum_{m \neq 0 \in Z} \frac{\tilde{\alpha}_m^\mu}{m} e^{-2im(\tau+\sigma)}. \qquad (2.29)$$

The general solution to (2.23) for an open string is given by

$$X^\mu = x^\mu + l_s^2 p^\mu \tau + l_s i \sum_{m \neq 0 \in Z} \frac{\alpha_m^\mu}{m} e^{-im\tau} \cos(m\sigma), \qquad (2.30)$$

where

x^μ = center of mass position,
p^μ = total momentum,
l_s = characteristic length,
$(\mathcal{T}\pi l_s^2 = 1)$
α_m^μ , $\tilde{\alpha}_m^\mu$ = vibration modes.

We skip the quantization of bosonic string theory. However we mention that the quantization of bosonic string theory as well as consistency require the spacetime to be 26 dimensional (D=26). Moreover different kinds of particles with different mass and spin are introduced by various oscillation modes of string. The predicted fundamental-particles according to this theory are given in table (2.1)

Table 2.1: Bosonic string spectrum

String	Excited and Ground states	Particle
Open		
	ground - $|0; k>$	A tachyon
	$\alpha_{-1}^i |0; k>$	A massless vector boson
	$\alpha_{-2}^i |0; k>$ and $\alpha_{-1}^i \alpha_{-1}^j |0; k>$	A single massive spin-two particle
Closed		
	ground - $|0; k>$	A tachyon
	$|\zeta^{ij}> = \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0; k>$	A massless spin-two particle or graviton (the symmetric part of $|\zeta^{ij}>$) A massless scalar or dilaton (the trace of $\delta_{ij} |\zeta^{ij}>$) An antisymmetric tensor (the antisymmetric part of $|\zeta^{ij}>$)

2.2 Strings in curved target spaces

In a given string theory, for instance type IIA, the low energy dynamics of the theory is described by the effective action. In the low-energy limit the energies are lower than the energy scale m_s (string mass) and $\alpha' \to 0$. In other words the string length vanishes and the theory of particles is recovered. In this limit the massless modes are considered and their dynamics are described by the related massless fields. In practice, in order to obtain the effective action for the massless modes, we calculate string amplitudes. Then we find that the effective field theory which has an expansion in powers of α'. Since we assume $\alpha' \to 0$, the lower terms of the expansion are considered.

So far we assumed that the strings propagate in a flat-target space (Minkowski space) $\eta_{\alpha\beta}$. We continue our study about strings by assuming that the target space admit an arbitrary metric $G_{\alpha\beta}$. Our starting point is again the Polyakov action, however in order to generalize this action we need to introduce two background fields [5, 6, 7]. The first background field is an antisymmetric tensor field shown by $B_{\alpha\beta}(X)$ which can be coupled to the worldsheet and the second field is a scalar field (dilaton) $\Phi(X)$ coupled to the two dimensional Ricci scalar R, so the modified action becomes

$$S = -\frac{1}{4\pi\alpha'} \int \left\{ \left[h^{ab}(\tau,\sigma)G_{\alpha\beta}(X) + \epsilon^{ab}B_{\alpha\beta}(X) \right] \partial_a X^\alpha \partial_b X^\beta \right. $$
$$\left. + \alpha' \Phi(X) R \right\} \sqrt{-h} \ d^2\sigma, \tag{2.31}$$

where α' is the Regge slope which can be defined in terms of $\mathcal{T}$ by $\mathcal{T} = \frac{1}{2\pi\alpha'}$ and ϵ^{ab} is the fully antisymmetric tensor in two dimensions. The energy-momentum tensor becomes

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\alpha\beta}h^{ab}\partial_a X^\alpha \partial_b X^\beta - \frac{1}{2\alpha'}\beta^B_{\alpha\beta}\epsilon^{ab}\partial_a X^\alpha \partial_b X^\beta - \frac{1}{2}\beta^\Phi R, \tag{2.32}$$

where

$$\beta^G_{\alpha\beta} = \alpha'\left(R_{\alpha\beta} + 2\nabla_\alpha\nabla_\beta\Phi - \frac{1}{4}H_{\alpha\gamma\delta}H_\beta{}^{\gamma\delta}\right) + O(\alpha'^2), \tag{2.33a}$$

$$\beta^B_{\alpha\beta} = \alpha'\left(-\frac{1}{2}\nabla^\gamma H_{\gamma\alpha\beta} + \nabla^\gamma\Phi H_{\gamma\alpha\beta}\right) + O(\alpha'^2), \tag{2.33b}$$

$$\beta^\Phi = \alpha'\left(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\gamma\Phi\nabla^\gamma\Phi - \frac{1}{24}H_{\gamma\alpha\beta}H^{\gamma\alpha\beta}\right) + O(\alpha'^2), \tag{2.33c}$$

and

$$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta}, \tag{2.34}$$

is the rank three field strength tensor. The conformal invariance of (2.31) requires the vanishing of the beta functions (2.33) which in turns gives three equations. These

equations can be obtained by varying the following action in D-dimensional spacetime

$$S = \frac{1}{2} \int d^D X \sqrt{-G} \, e^{-2\Phi} \left(R + 4\nabla_\alpha \nabla^\alpha \Phi - \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - \frac{2(D-26)}{3\alpha'} \right. \\ \left. + O(\alpha') \right).$$

(2.35)

This action is known as the low-energy effective action in the target space. It is instructive to take a look at the last term in (2.31). By considering the constant mode of the dilaton ($\Phi = \Phi_0$) this term is proportional to the Euler number χ as

$$\sim \chi \Phi_0$$

(2.36)

where

$$\chi = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\sigma \sqrt{-h} R = 2(1-g).$$

(2.37)

g is the genus number of world sheet manifold $\mathcal{M}$ (figure 2.3), and R is the Ricci scalar of the world sheet.

Figure 2.3: Surfaces of genus 0, 1, and 2.

We can see that term (2.37) is related to the numbers of holes (loops) in the string scattering amplitudes. In other words a g-loop diagram in the path integral (the string S-matrix) gets weighted by a factor of $(e^{-<\Phi>})^\chi$ where the expectation value of Φ acts as the string coupling constant $g_s = e^{-<\Phi>}$ [8].

2.3   Superstring theory

As we know the real world contains two different types of particles, i.e. fermions and bosons. So just having bosons in the string theory (introduced in previous section) is not realistic and this theory must be modified to include both bosons and fermions simultaneously in the Polyakov action (2.13) [3, 4, 9, 10]. By including a two dimensional spinor in (2.13) and considering the gauge freedom, the total action becomes

$$S = -\frac{\mathcal{T}}{2} \int (\underbrace{\partial_a X_\mu \partial^a X^\mu}_{\mathfrak{L}_B} + \underbrace{\bar{\psi}^\mu \rho^a \partial_a \psi_\mu}_{\mathfrak{L}_F}) d^2\sigma.$$

(2.38)

where

$$\{\rho^a, \rho^b\} = 2\eta^{ab}, \ a, b = 0, 1 \ , \ \rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\{\psi^\mu, \psi^\nu\} = 0$, and $\bar\psi = \psi^\dagger i\rho^0$.

We should remind that the two dimensional spinor field of the string worldsheet

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}, \ \mu = 0 \ . \ . \ D - 1,$$

, is a D-dimensinal vector in target space.

Some highlights of (2.38) are

- The equation of motion for $\mathfrak{L}_F$ takes the form.

$$\rho^a \partial_a \psi = 0, \tag{2.39}$$

  which is the Dirac equation.

- The action (2.38) is invariant under the infinitesimal susy-transformations

$$\delta X^\mu = \bar\epsilon \psi^\mu \ , \ \delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon, \tag{2.40}$$

  where ϵ is an infinitesimal Majorana spinor satisfying anti-commuting Grassmann numbers $\{\epsilon_i, \epsilon_j\} = 0$, and $\bar\epsilon = \epsilon i\rho^0$.

- Two different boundary conditions for the fermionic part of the open string are available known as Ramond and Neveu-Schwarz boundary conditions and we often call them as R and NS sectors.

  The Ramond boundary conditions are given by (R sector)

$$\psi^\mu_+(0,\tau) = \psi^\mu_-(0,\tau) \ , \ \psi^\mu_+(\pi,\tau) = \psi^\mu_-(\pi,\tau), \tag{2.41}$$

  and the mode expansions for R sector are given by

$$\psi^\mu_-(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n \in Z} d^\mu_n e^{-in(\tau-\sigma)} \ , \ \psi^\mu_+(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n \in Z} d^\mu_n e^{-in(\tau+\sigma)}. \tag{2.42}$$

  The Neveu-Schwarz boundary conditions are as follow (NS sector)

$$\psi^\mu_+(0,\tau) = \psi^\mu_-(0,\tau) \ , \ \psi^\mu_+(\pi,\tau) = -\psi^\mu_-(\pi,\tau), \tag{2.43}$$

  and the mode expansions for NS sector are

$$\psi^\mu_-(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r \in Z+\frac{1}{2}} b^\mu_r e^{-ir(\tau-\sigma)} \ , \ \psi^\mu_+(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r \in Z+\frac{1}{2}} b^\mu_r e^{-ir(\tau+\sigma)}. \tag{2.44}$$

- Periodic or antiperiodic boundary condition for the fermionic closed string are given by

$$\psi_\pm(\sigma) = \pm\psi_\pm(\sigma + \pi), \tag{2.45}$$

which implies four distinct closed-string sectors. For the right movers, one can choose

$$\psi_-^\mu(\sigma, \tau) = \sum_{n \in Z} d_n^\mu e^{-2in(\tau - \sigma)} \text{ or } \psi_-^\mu(\sigma, \tau) = \sum_{r \in Z + \frac{1}{2}} b_r^\mu e^{-2ir(\tau - \sigma)}, \qquad (2.46)$$

while for the left-movers the mode expansions are

$$\psi_+^\mu(\sigma, \tau) = \sum_{n \in Z} \tilde{d}_n^\mu e^{-2in(\tau + \sigma)} \text{ or } \psi_+^\mu(\sigma, \tau) = \sum_{r \in Z + \frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau + \sigma)}. \qquad (2.47)$$

Considering different pairings of the left- and right- movers, four distinct closed-string sectors can be obtained as follows (table 2.2).

Table 2.2: Different sectors for the left- and right-movers

Left-movers (ψ_+^μ)	Right-movers (ψ_-^μ)	Closed-string sectors
R	R	R-R
R	NS	R-NS
NS	R	NS-R
NS	NS	NS-NS

The supersymmetry on the worldsheet induces supersymmetry transformations between the fermion and the boson fields in the spacetime. This in turn enable us to remove the non-physical states from the theory and also in comparison to the bosonic string theory, the dimension of target space in this theory is reduced to D = 10. In fact this is the only dimension which we can formulate a Lorentz invariant string theory. There are five different superstring theories based on supersymmetry [3, 4] which a brief explanation of each theory is given in the following table (2.3).

Table 2.3: Different types of string theories

Type	Oriented	$^*\mathcal{N}$	Open or Closed	Gauge symmetry
I	No	1	Both	$SO(32)$
IIA	Yes	2	Closed	$U(1)$
IIB	Yes	2	Closed	-
Heterotic $SO(32)$	Yes	1	Closed	$SO(32)$
Heterotic $E_8 \times E_8$	Yes	1	Closed	$E_8 \times E_8$

$^*\mathcal{N}$ is the number of supersymmetry

In the mid 1990s, it was realized that all five string theories are related to one another by a set of dualities which are known as T and S dualities [11]. A brief explanation about T and S dualities in string theories is provided in section 2.5.

2.4 Type IIA and IIB string theories

In D=10 there are two types of supergravities which can be formulated according to the number ($\mathcal{N} = 2$) of the transformation parameter ϵ. We will discuss each case separately as follows [12].

1. Type IIA supergravity $\mathcal{N} = (1,1)$. This theory is a non-chiral theory and can be obtained from D=11 supergravity by dimensional reduction. In fact the dimensional reduction on a circle yields type IIA theory. The field content is given by

Spinors	NS-NS Fields	R-R Fields	R-NS (NS-R)
Majorana - Weyl	e^a_μ, $B_{(2)}$, ϕ	$C_{(1)}$, $C_{(3)}$	ψ^+_μ, ψ^-_μ, λ^+, λ^-

where e^a_μ represents vielbein, $B_{\mu\nu}$ is an anti-symmetric tensor field (Kalb-Ramond field), ψ_μ stands for a $\frac{3}{2}$ spinor (gravitino) or a Rarita-Schwinger field with different chiralities ($\pm$), λ shows a $\frac{1}{2}$ spinor (dilatino) or a Dirac field with different chiralities ($\pm$), ϕ is dilaton and $C_{(1)}$, $C_{(3)}$ are gauge fields. The bosonic part of the Lagrangian for the massless fields takes the following form

$$\mathcal{S}_{\textbf{IIA}} = \frac{1}{2\mathfrak{K}_{10}} \int e^{-2\phi} \left(\star \mathbf{1} R + 4 d\phi \wedge \star d\phi \right) - \frac{1}{4\mathfrak{K}_{10}} \int e^{-2\phi} \mathbf{H}_{(\mathbf{3})} \wedge \star \mathbf{H}_{(\mathbf{3})} -$$
$$- \frac{1}{4\mathfrak{K}_{10}} \int \left(\mathbf{F}_{(\mathbf{2})} \wedge \star \mathbf{F}_{(\mathbf{2})} + \mathbf{F}_{(\mathbf{4})} \star \wedge \mathbf{F}_{(\mathbf{4})} + \mathbf{B}_{(\mathbf{2})} \wedge d\mathbf{C}_{(\mathbf{3})} \wedge d\mathbf{C}_{(\mathbf{3})} \right), \tag{2.48}$$

where the rank of each object is shown inside the parenthesis

$$\mathbf{H}_{(\mathbf{3})} = d\mathbf{B}_{(\mathbf{2})}, \tag{2.49a}$$
$$\mathbf{F}_{(\mathbf{2})} = d\mathbf{C}_{(\mathbf{1})}, \tag{2.49b}$$
$$\mathbf{F}_{(\mathbf{4})} = d\mathbf{C}_{(\mathbf{3})} + \mathbf{C}_{(\mathbf{1})} \wedge \mathbf{H}_{(\mathbf{3})}, \tag{2.49c}$$

and $\mathfrak{K}_{10}$ is gravitational constant (Appendix B).

2. Type IIB supergravity $\mathcal{N} = (2,0)$. This theory is a chiral theory and similar to type IIA can be obtained from D=11 supergravity by dimensional reduction on a circle. The field content is given by

Spinors	NS-NS Fields	R-R Fields	R-NS (NS-R)
Majorana - Weyl	e^a_μ, $B_{(2)}$, ϕ	$C_{(0)}$, $C_{(2)}$, $C_{(4)}$	ψ^+_μ, λ^-

where e^a_μ represents vielbein, $B_{\mu\nu}$ is an anti-symmetric tensor field (Kalb-Ramond field), ψ_μ stands for a $\frac{3}{2}$ spinor (gravitino) or a Rarita-Schwinger field with positive chirality, λ shows a $\frac{1}{2}$ spinor (dilatino) or a Dirac field with

negative chirality, ϕ is dilaton and $C_{(0)}$, $C_{(2)}$, and , $C_{(4)}$ are gauge fields. Moreover the field strength $\mathbf{F}_{(5)}$ satisfies a self-duality condition ($\mathbf{F}_{(5)} = \star\mathbf{F}_{(5)}$). We notice that the self-duality condition comes from the equation of motion for the gauge fields. Otherwise $\star\mathbf{F}_{(5)} \wedge \mathbf{F}_{(5)} = 0$, hence no contribution from $\mathbf{F}_{(5)}$ appears in the action (2.50). The bosonic part of the Lagrangian for the massless fields is given by

$$\mathcal{S}_{\mathbf{IIB}} = \frac{1}{2\mathfrak{K}_{10}} \int e^{-2\phi} \left(\star\mathbf{1}R + 4d\phi \wedge \star d\phi\right) - \frac{1}{4\mathfrak{K}_{10}} \int e^{-2\phi}\mathbf{H}_{(3)} \wedge \star\mathbf{H}_{(3)} -$$
$$-\frac{1}{4\mathfrak{K}_{10}} \int \left(\mathbf{F}_{(1)} \wedge \star\mathbf{F}_{(1)} + \mathbf{F}_{(3)} \wedge \star\mathbf{F}_{(3)} + \frac{1}{2}\mathbf{F}_{(5)} \wedge \star\mathbf{F}_{(5)} + \mathbf{C}_{(4)} \wedge \mathbf{H}_{(3)} \wedge d\mathbf{C}_{(2)}\right)$$
$$(2.50)$$

where

$$\mathbf{H}_{(3)} = d\mathbf{B}_{(2)}, \tag{2.51a}$$
$$\mathbf{F}_{(1)} = d\mathbf{C}_{(0)}, \tag{2.51b}$$
$$\mathbf{F}_{(3)} = d\mathbf{C}_{(2)} + \mathbf{C}_{(0)} \wedge \mathbf{H}_{(3)}, \tag{2.51c}$$
$$\mathbf{F}_{(5)} = d\mathbf{C}_{(4)} + \mathbf{C}_{(2)} \wedge \mathbf{H}_{(3)}. \tag{2.51d}$$

Finally as we show later, the type IIA and IIB superstrings can be transformed into each other by T-duality on a circle. In other words, compactifying a spatial direction in the type IIA leads to the type IIB theory.

2.5   Compactification

The word compactification in general means compactifying one of the spatial dimensions into a circle of radius R but sometimes the compactification can be performed over two or more spatial coordinates. We will study the procedure of compactification in a bosonic closed string as an example (for sake of simplicity and convenience we have considered the theory of the bosonic string) [2, 3, 4, 7]. As mentioned before the coordinates in target space were shown by $X^0....X^{25}$ where X^0 is the timelike coordinate while the rest forms the spatial coordinates. To see the effect of compactification on closed string let us assume that one of the spatial coordinates say X^{25} is curled up into a circle of radius R. The boundary condition for a closed string is

$$X^{25}(\tau, \sigma) = X^{25}(\tau, \sigma + 2\pi). \tag{2.52}$$

We remind that the boundary condition we use here is not similar to the one we mentioned before in (2.24). However the result we are looking for, does not depend on the way the boundary conditions are defined. As stated above,the 25th dimension is going to behave as a circle with radius R therefore the new form of the boundary condition (2.52) will be

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2n\pi R, \tag{2.53}$$

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where n is the winding number which shows the winding of the string around X^{25}. We note that the periodicity of X^{25} leads to the quantization of the total center of mass momentum, e.g. in the quantum regime the wave function contains the factor e^{ipx}, $\psi \sim e^{ipx}$ and applying the boundary condition $\psi(x) = \psi(x + 2\pi R)$ gives

$$
\begin{aligned}
e^{ip(x+2\pi R)} &= e^{ipx}, \\
e^{ip2\pi R} &= e^{i2\pi N}, \ N \in Z, \\
p &= \frac{N}{R}.
\end{aligned}
\tag{2.54}
$$

We will use this result soon in our case. Using the definition of winding number one can define the winding w as follows

$$
w = \frac{nR}{\alpha'},
\tag{2.55}
$$

where α' is Regge slope parameter ($\alpha' = \frac{1}{2}l_s{}^2$). The winding turns out to have units of momentum, or inverse length. In other words winding is a new kind of momentum. Using the mode expansion for 25th dimension and assuming the left center of mass momentum and the right center of mass momentum are different, X^{25} becomes

$$
X^{25}(\tau, \sigma) = x^{25} + \frac{\alpha'}{2}(p_L^{25} + p_R^{25})\tau + \frac{\alpha'}{2}(p_L^{25} - p_R^{25})\sigma + \text{modes}
\tag{2.56}
$$

Looking at equation (2.56) the first term corresponds to the total center of mass momentum $p^{25} = p_L^{25} + p_R^{25}$ and also the quantized value for the total momentum is given by

$$
p^{25} = \frac{K}{R},
\tag{2.57}
$$

where K is an integer known as Kaluza-Klein excitation number [3]. The second term is the winding mode of the string, satisfies the following equation

$$
nR = \frac{\alpha'}{2}(p_L^{25} - p_R^{25}).
\tag{2.58}
$$

Considering (2.55) and (2.58), the winding w becomes

$$
w = \frac{1}{2}(p_L^{25} - p_R^{25})
\tag{2.59}
$$

We skip the details of the calculation in this stage. For a compactified closed string, using the previous results and Virasoro operators lead to

$$
\alpha' m^2 = \left(\frac{nR}{\alpha'}\right)^2 + \left(\frac{K}{R}\right)^2 + 2(N_R + N_L) - 4,
\tag{2.60}
$$

where N_R and N_L are number operators.

Looking at the mass equation (2.60), one can see that the mass is invariant under the following duality transformation

$$
n \longleftrightarrow K , \ R' \longleftrightarrow \frac{\alpha'}{R}.
\tag{2.61}
$$

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In fact this symmetry which is called T-duality relates small distances in one theory to large distances in another. For instance, T-duality relates type IIA and type IIB theories. The second symmetry which exists between different string theories is S-duality. According to S-duality a strong interaction in one theory is the same as a weak interaction in another. The connection between distinct superstring theories and the related dualities are shown in figure 2.4.
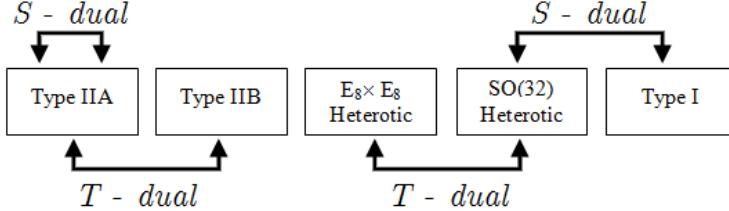
Figure 2.4: T and S dualities relate different superstring theories to each other.

According to table (2.4) two distinct superstring theories with different coupling constants can be connected by the S-duality transformation. The duality was known in 1995 [3, 13]. In this duality, a large coupling constant (g_s) or strong interaction is transformed to into a small one ($\frac{1}{g_s}$) or weak interaction and vice versa. For instance, a weak interaction in type I is the same as a strong interaction in heterotic SO(32), and vice versa.

2.6   T-dualization in curved background

As we pointed out earlier the various string theories are related to each other by different dualities e.g. by T-dualization of type IIA theory we can find type IIB theory. We should mention all discussions in the previous section (2.5) are based on two assumptions: 1) the target space is flat and 2) All other background fields vanish. In this section we remove these restrictions and we study the T-duality transformations along a circle when the target space is curved and the background fields take non-zero values. Again we just consider a closed-bosonic string theory and remind that under the T-duality transformation the right and left moving part of the X^{25} transform as

$$X_R^{25} \to -X_R^{25} \text{ and } X_L^{25} \to X_L^{25},$$
$$\text{or} \tag{2.62}$$
$$\tilde{X}^{25} = X_R^{25} - X_L^{25},$$

which we assumed that the T-dualization maps $X^{25} = X_R^{25} + X_L^{25}$ into $\tilde{X}^{25} = X_R^{25} - X_L^{25}$. We introduce a new action on the world sheet and claim that its equation of motion gives the same result as (2.62). First we start with a simple case and assume that all background fields are set to zero [4]. The new action is given by

$$\int (\frac{1}{2}V^c V_c - \epsilon^{ab} X^{25} \partial_b V_a) d^2\sigma, \tag{2.63}$$

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where V_c and x^{25} are dynamical fields in the action. The equation of motion for X^{25} reads

$$\epsilon^{ab}\partial_b V_a = 0. \tag{2.64}$$

One can easily obtain the solution to (2.64) by setting

$$V_a = \partial_a \tilde{X}^{25}, \tag{2.65}$$

where $\tilde{X}^{25}$ is an arbitrary function. Replacing V_a in (2.63) with $\partial_\alpha \tilde{X}^{25}$ gives

$$\int \frac{1}{2} \partial^c \tilde{X}^{25} \partial_c \tilde{X}^{25} d^2\sigma. \tag{2.66}$$

The equation of motion for V_a becomes

$$V_a = -\epsilon_a{}^b \partial_b X^{25}. \tag{2.67}$$

By substituting (2.67) into (2.63) we get

$$\frac{1}{2} \int \partial^a X^{25} \partial_a X^{25} d^2\sigma. \tag{2.68}$$

Comparing (2.67) and (2.65) results

$$\partial_a \tilde{X}^{25} = -\epsilon_a{}^b \partial_b X^{25}, \tag{2.69}$$

which satisfies (2.62). Now we consider the general case and rewrite (2.31) in the following form [7]

$$\begin{aligned}
\mathcal{S} = \frac{1}{4\pi\alpha'} \int \Big\{ & h^{ab}\Big[G_{25,25} V_a V_b + 2G_{25,\mu} V_a \partial_b X^\mu + G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \Big] \\
& + i\epsilon^{ab}\Big[2B_{25,\mu} V_a \partial_b X^\mu + B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\tilde{X}^{25}\partial_a V_b \Big] + \alpha' R\Phi \Big\} d^2\sigma\sqrt{-h}.
\end{aligned} \tag{2.70}$$

In (2.70) we have assumed that the T-dualization happens in X^{25}-direction which is the compactified coordinate of a circle of radius R and

$$\partial_{X^{25}}\Phi = \partial_{X^{25}}B_{\mu\nu} = \partial_{X^{25}}G_{\mu\nu} = 0. \tag{2.71}$$

On the other hand the dual coordinate, shown by $\tilde{X}^{25}$ is also a compactified coordinate of radius $\tilde{R}$ given by $\tilde{R} = \frac{\alpha'}{R}$. The equation of motion for $\tilde{X}^{25}$

$$\frac{\partial\mathcal{L}}{\partial\tilde{X}^{25}} = i\epsilon^{ab}\partial_a V_b = 0, \tag{2.72}$$

can be solved by setting $V_b = \partial_b X^{25}$. By plugging V_b in (2.70), we get

$$\begin{aligned}
\mathcal{S} = \frac{1}{4\pi\alpha'} \int \Big\{ & \Big[G_{25,25}\partial_a X^{25} h^{ab}\partial_b X^{25} + 2G_{25,\mu}\partial_a X^{25} h^{ab}\partial_b X^\mu + G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \Big] \\
& + i\epsilon^{ab}\Big[2B_{25,\mu}\partial_a X^{25}\partial_b X^\mu + B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \Big] + \alpha' R\Phi \Big\}\sqrt{-h}\, d^2\sigma,
\end{aligned} \tag{2.73}$$

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which reduces to the original action

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int \left\{ \left[G_{MN}h^{ab} + i\epsilon^{ab}B_{MN} \right] \partial_a X^M \partial_b X^N + \alpha' R\Phi \right\} \sqrt{-h} \; d^2\sigma, \qquad (2.74)$$

where $M, N = 0 \cdots 25$ and $a, b = 1, 2$. In a similar way one can obtain the dual action related to $\tilde{X}^{25}$. We start with the equation of motion V_a given by

$$\frac{\partial \mathcal{L}}{\partial V_a} - \partial_b \frac{\partial \mathcal{L}}{\partial(\partial_b V_a)} = 0$$

$$= h^{ab} \left[G_{25,25}V_b + G_{25,\mu}\partial_b X^\mu \right] + i\epsilon^{ab} \left[B_{25,\mu}\partial_b X^\mu + \partial_b \tilde{X}^{25} \right] = 0,$$

$$(2.75)$$

and from (2.75), V_c becomes

$$V_c = \frac{1}{G_{25,25}} \left[-G_{25\mu}\partial_c X^\mu - ih_{ac}\epsilon^{ab} \left(B_{25\mu}\partial_b X^\mu + \partial_b \tilde{X}^{25} \right) \right]. \qquad (2.76)$$

Substituting (2.76) into (2.70) gives an action of the form (2.74) with the new fields $\tilde{B}_{\mu\nu}$, $\tilde{G}_{\mu\nu}$ and $\tilde{\Phi}$ given by

$$\tilde{G}_{25,25} = \frac{1}{G_{25,25}}, \qquad (2.77\text{a})$$

$$\tilde{G}_{\mu,25} = \frac{B_{\mu,25}}{G_{25,25}}, \qquad (2.77\text{b})$$

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu,25}G_{\nu,25} - B_{\mu,25}B_{\nu,25}}{G_{25,25}}, \qquad (2.77\text{c})$$

$$\tilde{B}_{\mu,25} = \frac{G_{\mu,25}}{G_{25,25}}, \qquad (2.77\text{d})$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu,25}G_{\nu,25} - G_{\mu,25}B_{\nu,25}}{G_{25,25}}, \qquad (2.77\text{e})$$

$$e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{G_{25,25}}. \qquad (2.77\text{f})$$

We should mention that a different calculational method (using β-function equations) is needed to derive (2.77f).

CHAPTER 3

GRAVITATIONAL INSTANTONS

In this chapter we discuss various metrics called gravitational instantons. These are 4-dimensional Euclidean signature metrics which preserve some supersymmetries in D=11 supergravity and also satisfy [14, 15, 16, 17, 18]:

- Vacuum Einstein equation.

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0 \tag{3.1}$$

- (Anti) Self-duality-relation.

$$R^a{}_b = \varkappa(\star R^a{}_b), \tag{3.2}$$

where

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} \omega^c \wedge \omega^d. \tag{3.3}$$

and $\varkappa = 1$ for self-dual and $\varkappa = -1$ for anti self-dual solutions. If we apply the Hodge dual $(\star)$ to the $R^a{}_b$ we get

$$\begin{aligned}
\star R^a{}_b &= \tfrac{1}{2} R^a{}_{bcd} \star (\omega^c \wedge \omega^d), \\
&= \tfrac{1}{4} R^a{}_b{}^{cd} \epsilon_{cdef} (\omega^e \wedge \omega^f).
\end{aligned} \tag{3.4}$$

From (3.2) and (3.4), the (anti) self-duality-relation becomes

$$R^a{}_{bef} = \varkappa \frac{1}{2} R^a{}_b{}^{cd} \epsilon_{cdef}, \tag{3.5}$$

where ϵ_{cdef} is the totally antisymmetric Levi-Civita tensor [19]. Three different types of instantons used in embedding into the transverse direction of M-branes in this thesis are as follows:

- Bianchi Models.

- Taub-NUT space.

- Gibbons-Hawking space.

All these metrics have some applications in cosmology and quantum gravity. However the main motivation of finding gravitational instantons in gravity comes from nontrivial solutions to the Yang-Mills theory. To show this let us consider the Yang-Mills theory in a flat Euclidean space [20, 21], such that the line element becomes

$$ds^2 = dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2}, \tag{3.6}$$

According to the Yang-Mills theory, the Euclidean Lagrangian is given by

$$\mathcal{L}_{YM} = \frac{1}{2} \text{Tr}(F \wedge F), \tag{3.7}$$

where

$$\text{Tr}(T_\alpha T_\beta) = \frac{1}{2} \delta_{\alpha\beta}, \tag{3.8a}$$

$$F = dA + A \wedge A, \tag{3.8b}$$

$$A = A_a^\alpha dx^a T_\alpha, \tag{3.8c}$$

and the Lagrangian is invariant under gauge transformations. The gauge group is defined by its algebra

$$[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma, \tag{3.9}$$

where T_α is the generator and $f_{\alpha\beta}{}^\gamma$ is the structure constant of the gauge group, for instance the gauge group could be $U(1)$, $SU(2)$ or $SU(3)$ which corresponds to well known gauge symmetries of standard model of particle physics. The field strength F is given by

$$F = \frac{1}{2} F_{ac}^\alpha dx^a \wedge dx^c T_\alpha = d(A_c^\alpha) \wedge dx^c T_\alpha + (A_a^\beta dx^a T_\beta) \wedge (A_c^\gamma dx^c T_\gamma),$$

$$= \partial_a A_c^\alpha dx^a \wedge dx^c T_\alpha + \frac{1}{2} A_a^\beta A_c^\gamma [T_\beta, T_\gamma] dx^a \wedge dx^c, \tag{3.10}$$

$$= \frac{1}{2} (\partial_a A_c^\alpha - \partial_c A_a^\alpha + A_a^\beta A_c^\gamma f_{\beta\gamma}{}^\alpha) dx^a \wedge dx^c T_\alpha.$$

So, we can write F in terms of components

$$F_{ac}^\alpha = \partial_a A_c^\alpha - \partial_c A_a^\alpha + A_a^\beta A_c^\gamma f_{\beta\gamma}{}^\alpha, \tag{3.11}$$

or

$$F_{ac} = \partial_a A_c - \partial_c A_a + [A_a, A_c]. \tag{3.12}$$

By considering (3.11), the Lagrangian (3.7) takes a simple form as follows

$$\mathcal{L}_{YM} = \frac{1}{4} \left(F^\alpha \wedge F_\alpha \right), \tag{3.13}$$

where

$$F^\alpha = dA^\alpha + \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma. \tag{3.14}$$

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As we know, the field equations governing vacuum electrodynamics are

$$dF = 0, \tag{3.15a}$$

$$d * F = 0, \tag{3.15b}$$

where $A = A_a dx^a$ and $F = dA$. Similar equations in Yang-Mills are

$$D_A F = 0, \ D_A * F = 0, \tag{3.16}$$

where D_A is a gauge covariant derivative specified by

$$D_A F = dF + [A, F], \tag{3.17}$$

and $[A, B] = A \wedge B - B \wedge A$ [20].

Let us just focus on the Lagrangian (3.13). The action can be written as

$$\mathcal{S}_{YM} = \frac{1}{4} \int d^4x \ \mathrm{Tr} \left\{ (F_{\alpha\beta} \mp \star F_{\alpha\beta})(F_{\alpha\beta} \mp \star F_{\alpha\beta}) \pm 2 F_{\alpha\beta} \star F_{\alpha\beta} \right\}, \tag{3.18}$$

where

$$\star F_{\alpha\beta} = \frac{1}{2} F_{\mu\nu} \epsilon^{\mu\nu}{}_{\alpha\beta}, \tag{3.19}$$

and

$$\star F_{\alpha\beta} \star F^{\alpha\beta} = \star F_{\alpha\beta} \star F_{\alpha\beta} = F_{\alpha\beta} F^{\alpha\beta} = F_{\alpha\beta} F_{\alpha\beta}. \tag{3.20}$$

Using the identity $(F_{\alpha\beta} \mp \star F_{\alpha\beta})^2 \geq 0$, we get

$$\mathcal{S}_{YM} \geq \frac{1}{2} \left| \int d^4x \ \mathrm{Tr}(F_{\alpha\beta} \star F_{\alpha\beta}) \right|. \tag{3.21}$$

We work with the right hand side of (3.21) and show that this part is a total derivative

$$\mathrm{Tr}(F_{\mu\nu} \star F_{\mu\nu}) = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}), \tag{3.22a}$$

$$= 2\epsilon^{\mu\nu\alpha\beta} \mathrm{Tr} \left\{ (\partial_\mu A_\nu + A_\mu A_\nu)(\partial_\alpha A_\beta + A_\alpha A_\beta) \right\}, \tag{3.22b}$$

$$= 2\partial_\alpha \epsilon^{\alpha\beta\mu\nu} \mathrm{Tr} \left\{ A_\beta \partial_\mu A_\nu + \frac{2}{3} A_\beta A_\mu A_\nu \right\}, \tag{3.22c}$$

$$= 2\partial_\alpha Q_\alpha, \tag{3.22d}$$

where

$$Q_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \mathrm{Tr} \left\{ F_{\mu\nu} A_\beta - \frac{2}{3} A_\beta A_\mu A_\nu \right\}. \tag{3.23}$$

We have used $\mathrm{Tr}(ABCD) = \mathrm{Tr}(BCDA) = \mathrm{Tr}(CDAB) = \mathrm{Tr}(DABC)$ and note that $\mathrm{Tr}(A \wedge A \wedge A \wedge A) = 0$, hence no contribution comes from this term. Now we are ready to compare the action (3.21) to the topological charge q [22] defined by

$$q = \frac{1}{16\pi^2} \int d^4x \ \mathrm{Tr}(F_{\mu\nu} \star F^{\mu\nu}). \tag{3.24}$$

Plugging (3.22) and (3.24) into (3.21) leads

$$\mathcal{S}_{YM} \geq 8\pi^2 |q|. \tag{3.25}$$

As we can see from (3.25) the equality holds when the field strength is self-dual or anti-self-dual.

3.1 Bianchi models

In this section we consider the symmetry group of the spacetime and show that how the symmetry group and it's Lie algebra are related to the Killing vectors. There are two different ways to look at this problem:

- The metric of the spacetime is given and we are looking for the symmetry group (Lie algebra).

- The Lie algebra is known and we are interested in finding the metric of the spacetime manifold.

Let us just consider the first case and start with the cosmological principle. The cosmological principle states that on an adequately large scale the universe looks the same anywhere and in any direction. What the cosmological principle says, is that for two different observers at different locations the gravitation is similar or in other words all points of universe are equivalent. Mathematically this principle can be formulated by considering an infinitesimal variation of the coordinate system e.g.

$$x'^\mu = x^\mu + \varepsilon \xi^\mu(x^\alpha), \tag{3.26}$$

and studying the behavior of the metric under this transformation [23, 24, 25]. By employing the coordinate variation (3.26) and the cosmological principle which implies $g_{\mu\nu}(x^\alpha) = g'_{\mu\nu}(x^\alpha)$, we obtain

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{3.27}$$

where $\mathcal{L}$ stands for the Lie derivative. Linearly independent solutions to (3.27) are known as Killing vectors and they serve as generators for the Lie algebra of the symmetry group called isometry group. Since any linear combinations of Killing vectors satisfy (3.27), hence there is no unique way to determine the generators. It can be shown that the maximum number of independent Killing vectors on D-dimensional manifold is equal to $\frac{D(D+1)}{2}$ [26] e.g. D=4 gives 10 independent generators which generate the Poincare group. It is instructive to consider a simple situation and obtain the Killing vectors and the related isometry group. Let us start with the Euclidean plane $\mathcal{E}_2$. So it is feasible to write the metric as

$$ds^2 = dx^2 + dy^2, \tag{3.28}$$

and the Killing vectors can be obtained from

$$\partial_x \xi_x = 0, \ \partial_y \xi_y = 0, \ \partial_x \xi_y + \partial_y \xi_x = 0. \tag{3.29}$$

By solving (3.29) one can show that the generators are
$X_1 = \partial_x$ - translation in x-direction,
$X_2 = \partial_y$ - translation in y-direction,
$X_3 = x\partial_y - y\partial_x$ - rotation in the Euclidean plane,
and the commutation relations of the generators become

$$[X_1, X_3] = X_2 \ , \ [X_2, X_3] = -X_1 \ , \ [X_1, X_2] = 0, \tag{3.30}$$

which belong to the isometry group $\mathcal{E}_2$ or the 2-dimensional Euclidean group. Now we consider the second case to build a homogeneous space (the Lie algebra is given). For instance, the Lie algebra $[X_1, X_2] = X_1$ leads the following metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. In fact the Bianchi models [27] are based on the foliation of spacetime into slices of three-dimensional spatial spaces as $\mathcal{M}_4 = R \times \mathcal{M}_3$ where R shows the time variable. In this model the symmetry group G_3 has three generators shown by X_1, X_2 and X_3, satisfying the following commutation relations

$$[X_1, X_2] = n_3 X_3 - a X_2, \tag{3.31a}$$
$$[X_2, X_3] = n_1 X_1, \tag{3.31b}$$
$$[X_3, X_1] = n_2 X_2 + a X_3, \tag{3.31c}$$

where n_1, n_2, n_3, and a are the structure constants of the Lie algebra. In table (3.1) the list of possible homogeneous spaces is given [28].

Table 3.1: Bianchi models

Type	I	II	VII$_0$	VI$_0$	IX	VIII	V	IV	VII$_a$	III	VI$_a$
a	0	0	0	0	0	0	1	1	a	1	a
n_1	0	1	1	1	1	1	0	0	0	0	0
n_2	0	0	1	-1	1	1	0	0	1	1	1
n_3	0	0	0	0	1	-1	0	1	1	-1	-1

* a, n_1, n_2 and n_3 are the structure constants of the Lie algebra.

The only Bianchi model used in this work is the type IX which is a gravitational instanton [29].

$$ds^2 = \frac{dr^2}{\sqrt{F(r)}} + \frac{r^2}{4}\sqrt{F(r)}\left(\frac{\sigma_1{}^2}{1 - \frac{a_1{}^4}{r^4}} + \frac{\sigma_2{}^2}{1 - \frac{a_2{}^4}{r^4}} + \frac{\sigma_3{}^2}{1 - \frac{a_3{}^4}{r^4}}\right), \tag{3.32}$$

where

$$F(r) = \prod_{i=1}^{3}\left(1 - \frac{a_i{}^4}{r^4}\right), \tag{3.33}$$

and

$$\sigma_1 = d\psi + \cos\theta d\phi, \tag{3.34}$$
$$\sigma_2 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \tag{3.35}$$
$$\sigma_3 = \cos\psi d\theta + \sin\psi \sin\theta d\phi. \tag{3.36}$$

Numerical M-brane solutions in D=11 supergravity based on Bianchi type IX space can be found in [29]. In this thesis we will give an analytical solution for this model which is valid for $r \in (a, \infty)$. In fact this is the maximal range that the metric is well defined. In this case $a_1 = 0$ and $a_2 = a_3 = a$.

3.2 Taub-NUT metric

The Taub-NUT metric is the stationary and axisymmetric solutions of the Einstein field equation and admits two Killing vectors in $\phi \in [0, 2\pi]$ (∂_ϕ, around the symmetry axis z) and t (∂_t) directions [30, 31] (figure 3.1).
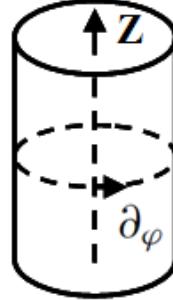
Figure 3.1: The spatial symmetry of the Taub-NUT space.

In order to introduce the Taub-NUT space, we follow the same treatment as given in reference [30]. The most general form of the line element having a cylindrical symmetry is

$$ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\phi}(r,\theta)dtd\phi + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\phi\phi}(r,\theta)d\phi^2. \quad (3.37)$$

Depending on how the $g_{t\phi}$ behaves at large distance, the solution to (3.37) falls into two categories:

- Kerr black hole.

$$g_{t\phi} \sim 2J\frac{\sin^2\theta}{r}, \qquad (3.38)$$
$$r \rightarrow \infty$$

  where J is the angular momentum in z direction. Of course this is not the case we are looking for. Hence we ignore this solution.

- Taub-NUT solution.

$$g_{t\phi} \sim 2N\cos\theta, \qquad (3.39)$$
$$r \rightarrow \infty$$

  where N stands for NUT charge.

The Euclidean section of the Taub-NUT metric has the following form

$$ds_\pm^2 = f(r)\left[d\tau^{(\pm)} \mp 2N(1 \mp \cos\theta)d\phi\right]^2 + f^{-1}(r)dr^2 + (r^2 - N^2)d\Omega^2, \qquad (3.40)$$

where

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 - N^2} \; , \; r_\pm = M \pm \sqrt{M^2 - N^2} \; , \; d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad (3.41)$$

and M, N stand for the mass and the NUT charge respectively. The Taub-NUT space contains two different singularities as follows

- Wire singularities (coordinate singularities).
  The singularities are located at $\theta = 0$ and $\theta = \pi$ therefore by defining two patches ds_-^2 and ds_+^2 (3.40), and also assuming τ is a compact coordinate with period of $8\pi N$ one can overcome this problem and regulate the metric.

- Curvature singularities.
  In general relativity the Kretschmann scalar $\mathbf{K}$ is used to study the singularities of spacetime. However the Kretschmann scalar is not the only quantity one can build from the Riemann tensor. One may use the Ricci scalar as alternative to the Kretschmann scalar. To compare these quantities, we start from the vacuum field equations in general relativity

$$g^{ab}(R_{ab} - \frac{R}{2}g_{ab}) = 0$$
$$R - \frac{R}{2}4 = 0 \qquad (3.42)$$
$$R = 0$$

as we can see from (3.42) for all vacuum solutions in general relativity the Ricci scalar vanishes whereas the Kretschmann scalar receives non-zero value. For instance, for the Schwarzschild metric $R = 0$ and $\mathbf{K} = \frac{48m^2}{r^6}$.
We use $\mathbf{K}$ to find the curvature singularity in the Taub-NUT metric. Let us start with calculating the Kretschmann scalar

$$\mathbf{K} = R^{abcd}R_{abcd}, \qquad (3.43\text{a})$$
$$= \frac{g(r, N, M)}{(r^2 - N^2)^6}, \qquad (3.43\text{b})$$

where $g(r, N, M)$ is a polynomial function and

$$g(r, N, N) = 96N^2(r - N)^6. \qquad (3.44)$$

As we can see from (3.43b) and (3.44) the metric is perfectly regular at $r = 0$ and only a curvature singularity arises at the metric where $r = N$ because $g(r = N, N, M) = 1536N^6(M - N)^2 \neq 0$. This singularity is removable if we set $M = N$. So $\mathbf{K}$ reduces to

$$\mathbf{K} = \frac{96N^2}{(r + N)^6}. \qquad (3.45)$$

A new form of the Taub-NUT metric is achievable if we just simply shift the radial coordinate r by M.

3.3 Gibbons-Hawking metric

The metric of Gibbons-Hawking [32] is given by

$$ds^2 = V^{-1}(d\Psi + \omega)^2 + V d\vec{x}.d\vec{x}, \tag{3.46}$$

where

$$\nabla V = \pm \nabla \times \vec{\omega}, \tag{3.47a}$$

$$V = \epsilon + \sum_{j=1}^{j=k} \frac{m_j}{|\vec{r} - \vec{r}_j|}, \tag{3.47b}$$

$$\omega = \sum_{j=1}^{j=k} m_j \frac{(z - z_j)}{|x - x_j|} \frac{(x - x_j)dy - (y - y_j)dx}{(x - x_j)^2 + (y - y_j)^2}, \tag{3.47c}$$

$$\tag{3.47d}$$

and (x_i, y_j, z_j) shows the location of NUTs (m_j) and $\epsilon \in \{0, 1\}$ is a constant. Self-duality implies $\nabla V = +\nabla \times \vec{\omega}$, while anti-self-duality implies $\nabla V = -\nabla \times \vec{\omega}$. Some special cases of this metric are given in table (3.2).

Table 3.2: Various metrics obtained from the Gibbons-Hawking metric

ϵ and k	Metric	
$\epsilon = 1$	Multi center Taub-NUT	Wire singularities can be removed if we set all $m_i = M$ and let the periodicity of Ψ be $\frac{8\pi M}{n}$
$\epsilon = 0$	*ALE	✓
$\epsilon = 0$ and $k = 1$	Flat	✓
$\epsilon = 0$ and $k = 2$	Eguchi-Hanson	✓

* Asymptotically Locally Euclidean

3.4 Topological invariants

We use topological invariants to characterize non-trivial solutions of Einstein's field equations. In four dimensions there are two independent topological invariants [14]. The topological invariants or in other words the possible quadratic terms that one can generate from the curvature of a compact manifold are

- Euler characteristic (number)

$$\chi \sim \int_{\mathcal{M}} R^{ab} \wedge \star R_{ab},$$

$$\chi = \frac{1}{128\pi^2} \int_{\mathcal{M}} d^4x \sqrt{g} \, \epsilon^{abef} \epsilon^{cdgh} R_{efgh} R_{abcd}, \tag{3.48}$$

and

- Hirzebruch signature (sometimes called index)

$$\tau \sim \int_{\mathcal{M}} R^{ab} \wedge R_{ab},$$

$$\tau = \frac{1}{96\pi^2} \int_{\mathcal{M}} d^4x \sqrt{g} \; \epsilon^{cdef} R^{ab}{}_{ef} R_{abcd}, \tag{3.49}$$

where $\epsilon^{abcd} = \frac{1}{\sqrt{g}} \varepsilon^{abcd}$ and ε^{abcd} is the Levi-Civita symbol. For non-compact manifolds some boundary terms relating to the extrinsic curvature (the second fundamental form) should be added to (3.48) and (3.49). More details about this case can be found somewhere else e.g. [15, 33]. The topological invariants for various spaces are given in table (3.3).

Table 3.3: χ and τ for various metrics

Metric	χ	τ
Schwarzschild and Kerr	2	0
Taub-NUT	1	0
Multi-Taub NUT	k	$k-1$
Eguchi-Hanson	2	1

CHAPTER 4

SUPERGRAVITY

In this chapter we study D=11 supergravity in details. In section 4.1 we introduce the super Poincare algebra and derive fermionic and bosonic states. We also show that there exist states whose the mass is equal to the charge. In section 4.2 we investigate the Kaluza-Klein theory in D=5 and show how pure gravity in higher-dimensional spacetime can generate a tower of massive and massless fields in a lower-dimensional spacetime. The same procedure can be generalized to the higher dimensions (e.g. D=11). In section 4.3 we review the D=11 dimensional Lagrangian and obtain the equations of motion for the metric $g_{\alpha\beta}$ and the gauge field $C_{\alpha\beta\gamma}$. Moreover we express the relation between fields and metric in D=11 and D=10 after doing dimensional reduction on a circle of radius R. In section 4.3.4 we use both the Killing spinor equation and the equation of motion for the gauge field to show how one-half of the supersymmetries is preserved in M2-brane solutions. In addition we obtain the general form of the metric for M2-brane solutions in D=11. Furthermore we calculate charge and mass of M2 and M5-branes and verify the equality of their mass and charge. At the end, we close this section by reviewing the intersecting M-branes in D=11 and D-branes solutions in D=10.

4.1 Supersymmetry algebra

The generators of the Poincare algebra which is based on spacetime symmetry (translation P_μ plus Lorentz transformation $M_{\mu\nu}$) are given by

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \tag{4.1a}$$

$$P_\mu = \partial_\mu, \tag{4.1b}$$

which satisfy the following algebra

$$[P_\mu, P_\nu] = 0, \tag{4.2a}$$

$$[P_\mu, M_{\alpha\beta}] = \eta_{\alpha\mu} P_\beta - \eta_{\beta\mu} P_\alpha, \tag{4.2b}$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = 2\eta_{\nu[\beta} M_{\alpha]\mu} + 2\eta_{\mu[\alpha} M_{\beta]\nu}. \tag{4.2c}$$

In eleven dimensional spacetimes i.e. $(\mathcal{N} = 1, D = 11)$, there is only one theory of supergravity which includes two bosonic fields and one fermionic field. By introducing new generators Q_α, known as supercharges, one can build the super Poincare algebra

[34, 35, 36, 37, 38, 39] for this theory. We should add the following anti-commutators of Q's to (4.2) as

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^N)_{\alpha\beta} P_N, \quad N = 0, \cdots 10, \tag{4.3}$$

where $C = \Gamma^0$ is the charge conjugation matrix and the supercharges Q_α are the 32 independent real supersymmetry generators in D=11. For massless states $P^2 = P^A P_A = 0$ and without losing generality, one can set $P^0 = P^1 = M$. Now by using (4.3) we get

$$
\begin{aligned}
P_M &= (-M, M, 0, \cdots, 0), \\
\{Q_\alpha, Q_\beta\} &= (-CM\Gamma^0 + MC\Gamma^1)_{\alpha\beta}, \\
&= M(-\Gamma^{0^2} + \Gamma^0\Gamma^1)_{\alpha\beta}, \\
&= M(1 + \Gamma^0\Gamma^1)_{\alpha\beta}.
\end{aligned}
\tag{4.4}
$$

Since $(\Gamma^0\Gamma^1)^2 = 1$ and $\Gamma^0\Gamma^1$ is traceless, the matrix $\Gamma^0\Gamma^1$ has 16 eigenvalues $+1$ and 16 eigenvalues -1. Hence (4.4) can take the following form

$$\{Q_\alpha, Q_\beta\} = 2M \left(\begin{smallmatrix} 0_{16} & 0_{16} \\ 0_{16} & I_{16} \end{smallmatrix} \right)_{\alpha\beta}, \tag{4.5}$$

which shows just half of 32 supersymmetries are unbroken and fulfill $SO(16)$ Clifford algebra

$$\left\{ \hat{Q}_\alpha, \hat{Q}_\beta \right\} = \delta_{\alpha\beta}, \tag{4.6}$$

where we set $\hat{Q} = \frac{Q}{\sqrt{2M}}$. By constructing a raising or lowering operator and introducing a vacuum state $|\Omega\rangle$, the massless states can easily be obtained as

State	Helicity	Degeneracy	Bosonic (B) / Fermionic (F)	e_μ^a	$A_{\alpha\beta\gamma}$	ψ_β^α
$|\Omega\rangle$	-2	1	B	1		
$\bar{Q}^1|\Omega\rangle$	$-\frac{3}{2}$	8	F			8
$\bar{Q}^2|\Omega\rangle$	-1	28	B	7	21	
$\bar{Q}^3|\Omega\rangle$	$-\frac{1}{2}$	56	F			56
$\bar{Q}^4|\Omega\rangle$	0	70	B	28	42	
$\bar{Q}^5|\Omega\rangle$	$\frac{1}{2}$	56	F			56
$\bar{Q}^6|\Omega\rangle$	1	28	B	7	21	
$\bar{Q}^7|\Omega\rangle$	$\frac{3}{2}$	8	F			8
$\bar{Q}^8|\Omega\rangle$	2	1	B	1		

where

$$\bar{Q}^i \equiv \underbrace{\bar{Q}^{m_1} \bar{Q}^{m_2} \cdots \bar{Q}^{m_i}}_{i-\text{times}}, \quad m_1, \cdots m_i = 1, \cdots 8. \tag{4.7}$$

As we can see from this table the field contents in D=11 supergravity contain states with of helicity 2 (graviton), $\frac{3}{2}$ (gravitino) and 1 (gauge field). If we count the degree

of freedom for fermions and bosons we get

$$\text{number of bosonic degrees of freedom} = \mathbf{1 + 1 + 7 + 7 + 28 + 21 + 21 + 42},$$
$$= \mathbf{128},$$
$$= \mathbf{8 + 8 + 56 + 56},$$
$$= \text{number of fermionic degrees of freedom}, \tag{4.8}$$

which comes from the supersymmetric properties of the D=11 supergravity.

Saturation of the BPS bound

In supersymmetry, BPS saturated states are states which preserve 1/2 (or 1/4, or 1/8) of the original supersymmetries and this occurs when the mass is equal to the central charge [34, 35, 36, 40, 41]. In order to show this we need to extend the Poincare algebra (4.2) as

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M + \frac{1}{p!}(C\Gamma^{M_1\cdots M_P})_{\alpha\beta} Z_{M_1\cdots M_P}, \tag{4.9}$$

where $\Gamma^{M_1\cdots M_P} = \Gamma^{[M_1}\Gamma^{M_2}\cdots\Gamma^{M_P]}$ and $Z_{M_1\cdots M_P}$ is an antisymmetric tensor or a charge carried by an extended object (e.g. M2 or M5-brane) which commutes with all the generators

$$[Z, P] = [Z, M] = [Z, Z] = [Z, Q] = 0. \tag{4.10}$$

Acting (4.9) on a physical state gives the Bogomolnyi bound (BPS) as

$$\langle\psi|\{Q_\alpha, Q_\beta\}|\psi\rangle = M - |Z| \geq 0. \tag{4.11}$$

We say that the BPS bound is saturated if there exist a state, say $|\phi\rangle$, that is annihilated by Q's. Under this condition the equality in, (4.11) holds. Later in this chapter we will show that two elementary solutions (M2 and M5) in D=11 supergravity fulfill saturated (4.11)

$$\mathcal{E}_2 = Q_2, \tag{4.12a}$$
$$\mathcal{E}_5 = Q_5, \tag{4.12b}$$

where $\mathcal{E}_2$ ($\mathcal{E}_5$) and Q_2 (Q_5) are the mass and charge of the M2 (M5)-brane solution respectively.

4.2 Kaluza-Klein theory

The main idea in Kaluza-Klein theory is to unify general relativity and classical electrodynamics by introducing a pure gravity in D=5. In the section we give more details about this theory and in particular we show how to connect geometrical objects in D=5 to gauge fields and geometrical objects in D=4.

4.2.1 Geometry

The main object in this section is to obtain an identity which relates the covariant derivative of an arbitrary vector to the Ricci tensor. We should note the procedure followed in this section can be used in the theory of hypersurfaces in both differential geometry and the dynamics of false vacuum bubbles in cosmology [42, 43, 44, 45]. Let us just start from the decomposition of a vector $\vec{A}$ in an arbitrary direction, say $\hat{n}$, by introducing an operator shown by $\mathcal{P}$ (figure 4.1). This operator will be used to construct new bases E_α from the old ones e_α. From the elementary geometry the vector $\vec{A}$ has two components given by

$$\vec{A} = A_\perp \hat{n} + \vec{A}_{||},$$
$$\vec{A}_{||} = \vec{A} - \xi(\vec{A}.\hat{n})\hat{n} = (1 - \xi\hat{n}\hat{n}).\vec{A}, \qquad (4.13)$$
$$= \mathcal{P}.\vec{A},$$

where $\mathcal{P} = 1 - \xi\hat{n}\hat{n}$ and

$$\xi = \hat{n}.\hat{n} = \left\{ \begin{array}{ll} 1, & \text{if } \hat{n} \text{ is a spacelike vector} \\ -1, & \text{if } \hat{n} \text{ is a timelike vector} \end{array} \right.$$

$$\vec{A} = \vec{A}_\perp + \vec{A}_{||}$$
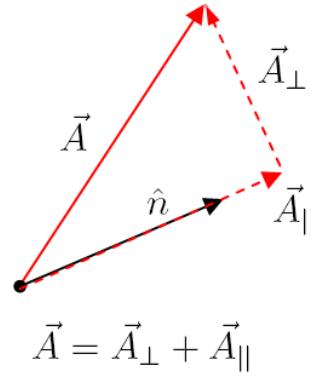
Figure 4.1: The vector $\vec{A}$ has two components ($\vec{A}_\perp$ and $\vec{A}_{||}$).

The new basis vector E_α can be obtained by acting $\mathcal{P}$ on the old basis vector e_α from left side as (figure 4.2)

$$E_\alpha = \mathcal{P}.e_\alpha,$$
$$= e_\alpha - \xi n_\alpha \hat{n}. \qquad (4.14)$$
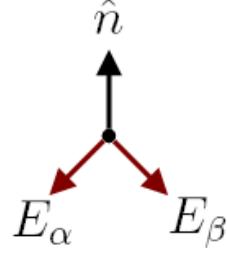
Figure 4.2: The new basis vector E_α is orthogonal to $\hat{n}$.

The new metric $h_{\alpha\beta}$, in terms of the old metric $g_{\alpha\beta}$ and $\hat{n}$ becomes

$$
\begin{aligned}
h_{\alpha\beta} &= E_\alpha.E_\beta = (e_\alpha - \xi n_\alpha \hat{n}).(e_\beta - \xi n_\beta \hat{n}), \\
&= g_{\alpha\beta} - \xi n_\alpha n_\beta.
\end{aligned}
\tag{4.15}
$$

So, the contravariant components of metric are given by

$$
h^{\alpha\beta} = g^{\alpha\beta} - \xi n^\alpha n^\beta.
\tag{4.16}
$$

Multiplying (4.16) from both sides by n_α and n_β gives

$$
\begin{aligned}
h^{\alpha\beta} n_\alpha n_\beta &= g^{\alpha\beta} n_\alpha n_\beta - \xi n^\alpha n^\beta n_\alpha n_\beta, \\
&= \hat{n}.\hat{n} - \xi(\hat{n}.\hat{n})^2, \\
&= \xi - \xi^2 \xi = 0.
\end{aligned}
\tag{4.17}
$$

which confirms that $h^{\alpha\beta}$ is tangent to the hypersurface. This tensor is known as the projection tensor. We will use $h^{\alpha\beta}$ and $h_{\alpha\beta}$ in the next section. We give the covariant derivative of basis vectors in terms of Christoffel symbols as

$$
\partial_\beta e_\alpha = e_{\alpha\beta} = \Gamma^\lambda{}_{\alpha\beta} e_\lambda.
\tag{4.18}
$$

Using (4.18) one can easily show that

$$
e_{\gamma\alpha\beta} - e_{\gamma\beta\alpha} = R^\lambda{}_{\gamma\beta\alpha} e_\lambda,
\tag{4.19}
$$

where $e_{\gamma\alpha\beta} \equiv \partial_\beta \partial_\alpha e_\gamma$ and $R^\lambda{}_{\gamma\beta\alpha}$ is the Riemann tensor defined by

$$
R^\lambda{}_{\gamma\beta\alpha} = 2\partial_{[\beta} \Gamma^\lambda{}_{\alpha]\gamma} + 2\Gamma^\lambda{}_{\delta[\beta} \Gamma^\delta{}_{\alpha]\gamma}.
\tag{4.20}
$$

Using $\hat{n} = n^\sigma e_\sigma$ and covariant differentiation of $\hat{n}$ with respect to coordinates get

$$
\begin{aligned}
\partial_\beta \partial_\alpha (n^\sigma e_\sigma) &= \\
&= (\nabla_\beta \nabla_\alpha n^\sigma) e_\sigma, \\
&= \partial^2_{\beta\alpha} \hat{n} + \partial_\alpha n^\sigma e_{\sigma\beta} + \partial_\beta n^\sigma e_{\sigma\alpha} + n^\sigma e_{\sigma\alpha\beta}.
\end{aligned}
\tag{4.21}
$$

Exchange between α and β implies that

$$
(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) n^\delta = n^\sigma R^\delta{}_{\sigma\beta\alpha},
\tag{4.22}
$$

and finally (4.22) and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ lead to

$$
\begin{aligned}
n^\sigma n^\alpha R_{\sigma\alpha} &= n^\alpha (\nabla_\gamma \nabla_\alpha - \nabla_\alpha \nabla_\gamma) n^\gamma, \\
&= \nabla_\gamma (n^\alpha \nabla_\alpha n^\gamma) - \nabla_\alpha (n^\alpha \nabla_\gamma n^\gamma) + \nabla_\alpha n^\alpha \nabla_\gamma n^\gamma - \nabla_\gamma n^\alpha \nabla_\alpha n^\gamma.
\end{aligned}
\tag{4.23}
$$

4.2.2 Dimensional reduction

Kaluza-Klein theory (KK) was developed by two physicists, Theodor Franz Eduard Kaluza and Oskar Klein, by introducing a small extra dimension(e.g. $x^4 = z$) to Einstein theory. The extra dimension is curled up into a circle of radius R so that we identify z with $z + 2\pi R$ (figure 4.3). In fact they realized that the 5D manifold could be used to unify two fundamental theories of gravitation and electromagnetism. However the prediction of the theory, the charge and the mass of the electron, never accomplished to match the real value of charge and mass. Therefore the Kaluza-Klein theory totally failed to fulfill Einstein's quest for a unified theory [13].
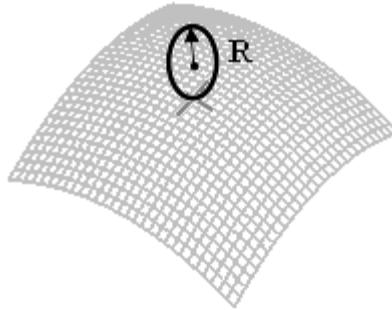
Figure 4.3: In KK-theory a small extra dimension is attached to any point of the spacetime.

In this section we discuss KK-theory in details [46, 47, 48, 49, 50, 51, 52] and then we show how a pure gravity in a higher dimensional spacetime (D=5) can generate many massless (or massive) fields after applying dimensional reduction method over one of its spatial coordinates. Our starting point is the Einstein-Hilbert action in D=5 but before that we need to make some assumptions as follows:

1. The coordinates in D=5 and D=4 are shown by x^μ and x^a respectively where $\mu = 0 \cdots 4$ and $a = 0 \cdots 3$.

2. There is a Killing vector for translations in z-direction.

3. Only massless fields are considered. In other words for an arbitrary field Φ, $\partial_z \Phi = 0$ or the zero-order term of Φ must be considered. To explain this we start with the Fourier series of $\Phi(z, x^a)$ given by

$$\Phi(z, x^a) = \sum_{n=-\infty}^{\infty} \Phi^{(n)}(x^a) e^{in\varphi}, \tag{4.24}$$

where

$$\varphi = \frac{z}{R}, \ \varphi \in [0, 2\pi), \tag{4.25}$$

and R is the compactification radius. As we can see from (4.24), in the massless case $\Phi(z, x^a) = \Phi^{(0)}(x^a)$ only $n = 0$ survives.

4. The spacetime admit a Riemannian metric with the following signatures $(-1, 1, 1, 1, 1)$.

Let us start with the five dimensional metric $g_{\mu\nu}$ which in general $g_{4a} \neq 0$ (or $g_{za} \neq 0$). We can always define new bases orthogonal to z-direction such that the metric takes a new form as follows

$$\gamma_{ab} = g_{ab} - \frac{g_{az}g_{bz}}{g_{zz}}. \tag{4.26}$$

Using $g_{\mu\nu}$, the line element in terms of γ_{ab} becomes

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$
$$= \gamma_{ab}dx^a dx^b + g_{zz}(dz + \frac{g_{za}dx^a}{g_{zz}})^2, \tag{4.27}$$

or

$$^{(5)}ds^2 = {}^{(4)} d\tilde{s}^2 + g_{zz}(dz + \underbrace{A_a dx^a}_{\mathcal{A}})^2, \tag{4.28}$$

where $\mathcal{A}$ is a one-form and $A_a = \frac{g_{za}}{g_{zz}}$. It is instructive to review the physical meaning of g_{zz} and A_a from a four-dimensional point of view. For observers living in D=4, g_{zz} is a scalar field and A_a looks like a vector field and also we can easily see that (4.28) is invariant under

$$z \rightarrow z - \Delta,$$
$$A_a \rightarrow A_a + \partial_a \Delta. \tag{4.29}$$

According to (4.29), it seems that the vector field A_a can be considered as a gauge field similar to the four-potential in the electromagnetism and in fact this might support the unification of gravity and electromagnetism in D=5.

Getting back to the original metric (4.28) and rewriting it in a new form for later convenience, we obtain

$$^{(5)}ds^2 = \gamma_{ab}dx^a dx^b + \phi^2(dz + \mathcal{A})^2, \tag{4.30}$$

where all the fields are independent of z and

$$\mathcal{A} = A_a dx^a. \tag{4.31}$$

4.2.3 5-Dimensional action

Similar to the Einstein-Hilbert action in D=4, the 5-Dimensional action is given by

$$I = \frac{1}{16\pi \mathbf{G}^{(5)}} \int d^5x \sqrt{-g^{(5)}} R^{(5)}, \tag{4.32}$$

where $\mathbf{G}^{(5)}$ is the 5-dimensional gravitational constant. As we mentioned before we deal only with massless fields, hence the volume element can be decomposed into $R^4 \times z$ as

$$I = \frac{1}{16\pi \mathbf{G}^{(5)}} \int_0^{2\pi R} dz \int d^4 x \sqrt{-g^{(5)}} R^{(5)}, \tag{4.33}$$

or

$$I = \frac{1}{16\pi \mathbf{G}^{(4)}} \int d^4 x \sqrt{-g^{(5)}} R^{(5)}, \tag{4.34}$$

where $\mathbf{G}^{(4)} = \frac{\mathbf{G}^{(5)}}{2\pi R}$ is the 4-dimensional gravitational constant. We use the relation between the determinant of the metrics in D=4 and D=5

$$\sqrt{-g^{(5)}} = \phi \sqrt{-\gamma}, \tag{4.35}$$

to derive the action in D=4. The action in D=4 reduces to

$$I = \frac{1}{16\pi \mathbf{G}^{(4)}} \int d^4 x \phi \sqrt{-\gamma} \ R^{(5)}. \tag{4.36}$$

In the tetrad formalism, the 5-dimensional metric becomes

$$^{(5)} ds^2 = \eta_{\hat{a}\hat{b}} \Omega^{\hat{a}} \otimes \Omega^{\hat{b}} + \Omega^{\hat{z}} \otimes \Omega^{\hat{z}}, \tag{4.37}$$

where

$$\Omega^{\hat{a}} = \omega^{\hat{a}},$$
$$\Omega^{\hat{z}} = \phi(dz + \mathcal{A}), \tag{4.38}$$

and $\mathcal{A}$ takes the following form

$$\mathcal{A} = A_a dx^a,$$
$$= A_a e_{\hat{a}}^a \omega^{\hat{a}}, \tag{4.39}$$
$$= A_{\hat{a}} \omega^{\hat{a}}.$$

Taking the exterior derivative of $\Omega^{\hat{z}}$ in (4.38) we have

$$d\Omega^{\hat{z}} = d(\phi(dz + \mathcal{A})),$$
$$= \partial_{\hat{a}}(\ln \phi)\Omega^{\hat{a}} \wedge \Omega^{\hat{z}} + \phi \frac{1}{2} F_{\hat{a}\hat{b}} \Omega^{\hat{a}} \wedge \Omega^{\hat{b}}, \tag{4.40}$$
$$= (\ln \phi)_{;\hat{a}} \Omega^{\hat{a}} \wedge \Omega^{\hat{z}} + \phi \frac{1}{2} F_{\hat{a}\hat{b}} \Omega^{\hat{a}} \wedge \Omega^{\hat{b}},$$

where

$$\mathcal{F} = d\mathcal{A},$$
$$= \frac{1}{2}(\partial_b A_a - \partial_a A_b) dx^b \wedge dx^b,$$
$$= \frac{1}{2}(A_{a;b} - A_{b;a}) dx^b \wedge dx^a, \tag{4.41}$$
$$= \frac{1}{2} F_{ab} dx^a \wedge dx^b,$$
$$= \frac{1}{2} F_{\hat{a}\hat{b}} \Omega^{\hat{a}} \wedge \Omega^{\hat{b}}.$$

37

Using Maurer-Cartan equation $(d\Omega^{\hat{z}} = -\Omega^{\hat{z}}{}_{\hat{\alpha}} \wedge \Omega^{\hat{\alpha}})$ and (4.40), the five-dimensional rotation forms become

$$\Omega^{\hat{z}}{}_{\hat{a}} = (\ln \phi)_{;\hat{a}}\Omega^{\hat{z}} + \frac{1}{2}\phi F_{\hat{a}\hat{b}}\Omega^{\hat{b}},$$

$$\Omega^{\hat{a}}{}_{\hat{b}} = \omega^{\hat{a}}{}_{\hat{b}} - \frac{1}{2}\phi F^{\hat{a}}{}_{\hat{b}}\Omega^{\hat{z}},$$

(4.42)

where $\omega^{\hat{a}}{}_{\hat{b}}$ is the four-dimensional rotation form. For later convenience we can easily obtain the Christoffel symbols from (4.42) as follows

$$\Gamma^{\hat{z}}{}_{\hat{a}\hat{b}} = -\Gamma^{\hat{a}}{}_{\hat{z}\hat{b}} = \frac{1}{2}\phi F_{\hat{a}\hat{b}}.$$

(4.43)

Moreover

$$
\begin{aligned}
\mathcal{R}^{\hat{a}}{}_{\hat{b}} &= \frac{1}{2}\mathcal{R}^{\hat{a}}{}_{\hat{b}\hat{\alpha}\hat{\beta}}\Omega^{\hat{\alpha}} \wedge \Omega^{\hat{\beta}} \\
&= d\Omega^{\hat{a}}{}_{\hat{b}} + \Omega^{\hat{a}}{}_{\hat{\alpha}} \wedge \Omega^{\hat{\alpha}}{}_{\hat{b}} \\
&= d(\omega^{\hat{a}}{}_{\hat{b}} - \frac{1}{2}\phi F^{\hat{a}}{}_{\hat{b}}\Omega^{\hat{z}}), \\
&+ (\omega^{\hat{a}}{}_{\hat{c}} - \frac{1}{2}\phi F^{\hat{a}}{}_{\hat{c}}\Omega^{\hat{z}}) \wedge (\omega^{\hat{c}}{}_{\hat{b}} - \frac{1}{2}\phi F^{\hat{c}}{}_{\hat{b}}\Omega^{\hat{z}}) \\
&- ((\ln \phi)^{;\hat{a}}\Omega^{\hat{z}} + \frac{1}{2}\phi F^{\hat{a}}{}_{\hat{l}}\Omega^{\hat{l}}) \wedge ((\ln \phi)_{;\hat{b}}\Omega^{\hat{z}} + \frac{1}{2}\phi F_{\hat{b}\hat{c}}\Omega^{\hat{c}}).
\end{aligned}
$$

(4.44)

After simplification of (4.44) and keeping terms that contain $(\Omega^{\hat{m}} \wedge \Omega^{\hat{l}})$, the second curvature forms become

$$\mathcal{R}^{\hat{a}}{}_{\hat{b}} = R^{\hat{a}}{}_{\hat{b}} - \frac{1}{8}\phi^2\left(2F^{\hat{a}}{}_{\hat{b}}F_{\hat{m}\hat{l}} + F^{\hat{a}}{}_{\hat{m}}F_{\hat{b}\hat{l}} - F^{\hat{a}}{}_{\hat{l}}F_{\hat{b}\hat{m}}\right)\Omega^{\hat{m}} \wedge \Omega^{\hat{l}}.$$

(4.45)

From (4.45) the components of the Riemann tensor are given by

$$\mathcal{R}^{\hat{a}}{}_{\hat{b}\hat{m}\hat{l}} = R^{\hat{a}}{}_{\hat{b}\hat{m}\hat{l}} - \frac{1}{4}\phi^2\left(2F^{\hat{a}}{}_{\hat{b}}F_{\hat{m}\hat{l}} + F^{\hat{a}}{}_{\hat{m}}F_{\hat{b}\hat{l}} - F^{\hat{a}}{}_{\hat{l}}F_{\hat{b}\hat{m}}\right).$$

(4.46)

We divide our calculation into two parts: first we consider the right hand side of (4.46) and obtain the Ricci scalar in D=4 and then we focus on the left hand side of (4.46) and apply the projection tensor of (4.16) to derive the Ricci scalar in D=5. From the right hand side we get

$$
\begin{aligned}
R^{\hat{a}\hat{b}}{}_{\hat{a}\hat{b}} &- \frac{1}{4}\phi^2\left(2F^2 + F^{\hat{a}}{}_{\hat{a}}F_{\hat{b}\hat{c}}\eta^{\hat{b}\hat{c}} + F^2\right), \\
&= R^{\hat{a}\hat{b}}{}_{\hat{a}\hat{b}} - \frac{3}{4}\phi^2 F^2, \\
&= R^{(4)} - \frac{3}{4}\phi^2 F^2,
\end{aligned}
$$

(4.47)

where $F^2 = F^{\hat{a}\hat{b}} F_{\hat{a}\hat{b}}$ and $R^{(4)}$ is the four dimensional Ricci scalar. The left hand side of (4.47) can be projected down to the hypersurface by using the projection tensor (4.16)

$$h_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta, \tag{4.48}$$

as

$$
\begin{aligned}
h^{\alpha\beta} h^{\mu\lambda} \mathcal{R}_{\lambda\alpha\mu\beta} &= \\
&= (g^{\alpha\beta} - n^\alpha n^\beta)(g^{\mu\lambda} - n^\mu n^\lambda) \mathcal{R}_{\lambda\alpha\mu\beta}, \\
&= \mathcal{R}^{(5)} - 2n^\alpha n^\beta \mathcal{R}_{\alpha\beta} + \underbrace{n^\alpha n^\beta n^\mu n^\lambda \mathcal{R}_{\lambda\alpha\mu\beta}}_{\text{zero}}, \\
&= \mathcal{R}^{(5)} - 2n^\alpha n^\beta \mathcal{R}_{\alpha\beta}.
\end{aligned}
\tag{4.49}
$$

We assume the normal vector is in $\hat{n} = e^z$ direction has the non-vanishing component $n_z = 1$. Using the identity (4.22 and 4.23) result

$$\nabla_\alpha n_\beta = -n_\gamma \Gamma^\gamma{}_{\alpha\beta} = -\Gamma^z{}_{\alpha\beta}. \tag{4.50}$$

In addition one can show that

$$\nabla^\alpha n_\alpha = 0, \tag{4.51a}$$
$$\nabla^\beta (n^\gamma \nabla_\gamma n_\beta) = -(\ln\phi)^{;a}{}_{;a}, \tag{4.51b}$$
$$(\nabla^\mu n^\nu)(\nabla_\nu n_\mu) = -\frac{1}{4}\phi^2 F_{\delta\sigma} F^{\delta\sigma} - (\ln\phi)_{;a}(\ln\phi)^{;a}, \tag{4.51c}$$

and

$$
\begin{aligned}
\mathcal{R}_{\alpha\beta} n^\alpha n^\beta &= \frac{1}{4}\phi^2 F^{ab} F_{ab} + (\ln\phi)_{;c}(\ln\phi)^{;c} - (\ln\phi)^{;c}{}_{;c}, \\
&= \frac{1}{4}\phi^2 F^{ab} F_{ab} - \frac{1}{\phi}\Box\phi,
\end{aligned}
\tag{4.52}
$$

and finally we get

$$\mathcal{R}^{(5)} = R^{(4)} - \frac{1}{4}\phi^2 F^{ab} F_{ab} - \frac{2}{\phi}\Box\phi, \tag{4.53}$$

where $\mathcal{R}$ and R are the Ricci scalars in D=5 and D=4 respectively. By plugging (4.53) into (4.34) we get

$$I = \frac{1}{16\pi \mathbf{G}^{(4)}} \int d^4x \phi\sqrt{-\gamma} \left(R^{(4)} - \frac{1}{4}\phi^2 F^{ab} F_{ab} - \frac{2}{\phi}\Box\phi \right). \tag{4.54}$$

As we can see from (4.54), at the end of the calculation we encounter with a pure gravity, coupling with the scalar field ϕ and the gauge field A_a. Furthermore all these fields live in D=4. In fact this is the main idea of the Kaluza-Klein theory.
In D=11 supergravity, the bosonic section of the Lagrangian contains the metric and gauge field. We have already studied the dimensional reduction of the metric while passing from D=5 to D=4. We shall now consider the dimensional reduction of an

arbitrary field strength which related to a specific gauge field. We start with a simple case, the Maxwell field strength $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (4.55)$$

We use the differential forms and assume that the reduction of the coordinate takes place in z direction, so the field strength becomes

$$
\begin{aligned}
\mathcal{F}_{(2)} &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \\
&= \frac{1}{2} F_{ab} dx^a \wedge dx^b + F_{az} dx^a \wedge dz, \\
&= \frac{1}{2} F_{ab} dx^a \wedge dx^b + \partial_a A_z dx^a \wedge dz, \\
&= dA_{(2)} + dA_{(1)} \wedge dz,
\end{aligned}
\qquad (4.56)
$$

where

$$
\begin{aligned}
dA_{(2)} &= \frac{1}{2} F_{ab} dx^a \wedge dx^b, \\
A_{(1)} &= A_z.
\end{aligned}
\qquad (4.57)
$$

Thus the general form of an antisymmetric field strength with an (n)-index can be written as follows

$$\mathcal{F}_{(n)} = dA_{(n-1)} + dA_{(n-2)} \wedge dz, \qquad (4.58)$$

or rewriting it in terms of the Kaluza-Klein potential gives

$$
\begin{aligned}
\mathcal{F}_{(n)} &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)}), \\
\mathcal{F}_{(n)} &= \underbrace{dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)}}_{F_{(n)}} + \underbrace{dA_{(n-2)}}_{F_{(n-1)}} \wedge (dz + \mathcal{A}_{(1)}), \\
\mathcal{F}_{(n)} &= F_{(n)} + F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)}),
\end{aligned}
\qquad (4.59)
$$

where $F_{(n)}$ and $F_{(n-1)}$ are the (D-1)-dimensional field strengths.

4.3   11-Dimensional supergravity

An extension of Einstein's general relativity in higher dimensions including supersymmetry (SUSY) or in other words the low effective action of M-theory is known as eleven-dimensional supergravity. This theory was constructed by Eugene Cremmer, Bernard Julia and Joel Scherk in 1978 [53]. The field content of eleven-dimensional supergravity is relatively simple and is made of both fermionic and bosonic massless particles. The bosonic field content is given by the metric $g_{\mu\nu}(g^{\mu\nu})$ and a totally antisymmetric three-form field $C_{\mu\nu\rho}$. The fermionic field content of the theory is a massless spin-$\frac{3}{2}$ field or gravitino (ψ_μ) thus the full eleven-dimensional supergravity

multiplet is given by $(g_{\mu\nu}, \psi_\mu, C_{\mu\nu\rho})$ [4, 54, 55]. The modified Lagrangian density of eleven-dimensional supergravity, is given by [56]

$$\mathfrak{L} = \frac{1}{4}eR + \frac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\frac{\omega+\hat{\omega}}{2})\psi_\rho - \frac{1}{4 \cdot 48}eG_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}$$

$$-\frac{1}{4 \cdot 48}e(\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\gamma\delta}\psi^\beta)(\frac{G_{\alpha\beta\gamma\delta} + \hat{G}_{\alpha\beta\gamma\delta}}{2}) \qquad (4.60)$$

$$+\frac{1}{4 \cdot 144^2}\epsilon^{\alpha_1...\alpha_4\beta_1...\beta_4\mu\nu\rho}G_{\alpha_1...\alpha_4}G_{\beta_1...\beta_4}C_{\mu\nu\rho},$$

where

$$D_\nu(\omega)\psi_\mu = \partial_\nu\psi_\mu - \frac{1}{4}\omega_{\nu ab}\Gamma^{ab}\psi_\mu,$$

$$G_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]},$$

$$\hat{G}_{\mu\nu\rho\sigma} = G_{\mu\nu\rho\sigma} + 6\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]},$$

$$K_{\mu ab} = \frac{1}{4}[\ \bar{\psi}_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta - 2(\bar{\psi}_\mu\Gamma_b\psi_a - \bar{\psi}_\mu\Gamma_a\psi_b + \bar{\psi}_b\Gamma_\mu\psi_a)\ ](\text{Contorsion}), \qquad (4.61)$$

$$\omega_{\mu ab} = \omega_{\mu ab}^{(0)} + K_{\mu ab},$$

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{1}{4}\bar{\psi}_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta.$$

The signature of the metric is $\eta_{ab} = (-1, 1....1)$ and a real representation of Γ-matrices satisfies the Clifford algebra (Appendix C and [56, 57])

$$\{\Gamma^a, \Gamma^a\} = 2\eta^{ab}\mathbf{1}_{32}. \qquad (4.62)$$

The equations of motion for eleven-dimensional supergravity fall into three groups by type of field content.

4.3.1   The equation of motion for $g_{\alpha\beta}$

Let us start with the equation of motion for $g_{\alpha\beta}$. The only terms having contribution in equation of motion are

$$\mathcal{L}_{g_{\alpha\beta}} = \frac{1}{4}\sqrt{-g}R - \frac{1}{4.48}\sqrt{-g}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}. \qquad (4.63)$$

The variation of $\mathcal{L}_{g_{\alpha\beta}}$ is given by

$$\delta\mathcal{L}_{g_{\alpha\beta}} = \frac{1}{4}\delta\left(\sqrt{-g}R\right) - \frac{1}{4.48}\delta\left(\sqrt{-g}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\right) \qquad (4.64)$$

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and the variation of different parts are given as follows

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}}, \tag{4.65a}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial g}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta}, \tag{4.65b}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{-g}}gg^{\alpha\beta}\delta g_{\alpha\beta}, \tag{4.65c}$$

$$= \frac{1}{2}\frac{1}{\sqrt{-g}}gg_{\alpha\beta}\delta g^{\alpha\beta}, \tag{4.65d}$$

$$= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{4.65e}$$

and

$$\delta R = \delta g^{\mu\nu}\, R_{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu}. \tag{4.66}$$

One can easily show that the contribution from $\delta R_{\alpha\beta}$ is equal to zero. To show this we use the definition of Ricci tensor

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}, \tag{4.67a}$$

$$= \Gamma^{\gamma}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\gamma}_{\lambda\beta}\Gamma^{\lambda}_{\gamma\alpha} + \partial_{\gamma}\Gamma^{\gamma}_{\beta\alpha} - \partial_{\beta}\Gamma^{\gamma}_{\gamma\alpha}, \tag{4.67b}$$

so the variation becomes

$$\delta R_{\alpha\beta} =,$$
$$= \delta\Gamma^{\gamma}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\alpha} + \Gamma^{\gamma}_{\lambda\gamma}\delta\Gamma^{\lambda}_{\beta\alpha} - \delta\Gamma^{\gamma}_{\lambda\beta}\Gamma^{\lambda}_{\gamma\alpha} - \Gamma^{\gamma}_{\lambda\beta}\delta\Gamma^{\lambda}_{\gamma\alpha} + \partial_{\gamma}\delta\Gamma^{\gamma}_{\beta\alpha} - \partial_{\beta}\delta\Gamma^{\gamma}_{\gamma\alpha}. \tag{4.68}$$

Since the $\delta\Gamma$ is a tensor this can be written as

$$\delta R_{\alpha\beta} = \nabla_{\gamma}(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_{\beta}(\delta\Gamma^{\gamma}_{\gamma\alpha}). \tag{4.69}$$

Multiplying both side by $g^{\alpha\beta}$ gives

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_{\gamma}g^{\alpha\beta}(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_{\beta}g^{\alpha\beta}(\delta\Gamma^{\gamma}_{\gamma\alpha}),$$
$$= \nabla_{\gamma}\, g^{\alpha\beta}(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_{\beta}\, g^{\alpha\beta}(\delta\Gamma^{\gamma}_{\gamma\alpha}), \tag{4.70}$$
$$= \nabla_{\sigma}S^{\sigma},$$

where $S^{\sigma} = g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\gamma\alpha})$. Now if we do the integration over the volume of the spacetime $\mathcal{M}$ we get

$$\int_{\mathcal{M}} \sqrt{-g}\nabla_{\sigma}S^{\sigma}d^4x = \int_{\mathcal{M}} \frac{\partial(\sqrt{-g}S^{\alpha})}{\partial x^{\alpha}}d^4x, \tag{4.71a}$$

$$= \int_{\Sigma} \sqrt{-g}S^{\alpha}d\Sigma_{\alpha}, \tag{4.71b}$$

$$= 0. \tag{4.71c}$$

We should notice that on the boundaries of the integration (4.71b), $\delta\Gamma$ vanishes. Finally the variation for the gauge field is

$$\delta(G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}) = \delta(G_{\mu\nu\rho\sigma}g^{\mu\mu_1}g^{\nu\nu_1}g^{\rho\rho_1}g^{\sigma\sigma_1}G_{\mu_1\nu_1\rho_1\sigma_1}), \tag{4.72a}$$

$$= 4G_{\alpha\nu\rho\sigma}G_\beta{}^{\nu\rho\sigma}\delta g^{\alpha\beta}. \tag{4.72b}$$

So, putting everything together gives the equation of motion

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\frac{1}{96}g_{\alpha\beta}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + \frac{1}{12}G_{\alpha\gamma_1\gamma_2\gamma_3}G_\beta{}^{\gamma_1\gamma_2\gamma_3}. \tag{4.73}$$

4.3.2   The equation of motion for $C_{\alpha\beta\gamma}$

For the gauge filed $C_{\alpha\beta\gamma}$

$$\mathcal{L}_C \equiv \mathcal{L}_{C_{ijk}} = \sqrt{-g}\frac{1}{48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + \frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4\gamma_1\gamma_2\gamma_3}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}C_{\gamma_1\gamma_2\gamma_3}. \tag{4.74}$$

First we notice two important identities that can be derived by using the definition of the field strength $G_{\mu\nu\rho\sigma}$. The identities are

$$\frac{\partial G_{\mu\nu\rho\sigma}}{\partial(\partial_\alpha C_{jkl})} = 4\delta^{\alpha jkl}_{\mu\nu\rho\sigma}, \tag{4.75a}$$

$$\delta^{\alpha_1\cdots\alpha_4}_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} = G^{\alpha_1\cdots\alpha_4}, \tag{4.75b}$$

where

$$\delta^{\alpha_1\cdots\alpha_4}_{\beta_1\cdots\beta_4} \equiv \delta^{\alpha_1\cdots\alpha_4}_{[\beta_1\cdots\beta_4]}. \tag{4.76}$$

Using the Euler-Lagrange equations for $C_{\alpha\beta\gamma}$

$$\partial_\alpha\left(\frac{\partial\mathcal{L}_C}{\partial(\partial_\alpha C_{ijk})}\right) - \frac{\partial\mathcal{L}_C}{\partial C_{ijk}} = 0, \tag{4.77}$$

the first term in the right hand side of (4.74) gives

$$\partial_\alpha\left(\frac{\partial\mathcal{L}_C}{\partial(\partial_\alpha C_{ijk})}\right) = \partial_\alpha\left(\frac{\partial}{\partial(\partial_\alpha C_{ijk})}\left(-\sqrt{-g}\frac{1}{48}G_{\mu\nu\rho\sigma}g^{\mu\alpha_1}g^{\nu\alpha_2}g^{\rho\alpha_3}g^{\sigma\alpha_4}G_{\alpha_1\alpha_2\alpha_3\alpha_4}\right)\right), \tag{4.78a}$$

$$= \partial_\alpha\left(-\frac{2.4}{48}\sqrt{-g}\,\delta^{\alpha jkl}_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\right), \tag{4.78b}$$

$$= \partial_\alpha\left(-\frac{2.4}{48}\sqrt{-g}\,G^{\alpha jkl}\right), \tag{4.78c}$$

and the second term yields

$$\partial_\alpha\frac{\partial}{\partial(\partial_\alpha C_{jkl})}\left(\frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4\gamma_1\gamma_2\gamma_3}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}C_{\gamma_1\gamma_2\gamma_3}\right), \tag{4.79a}$$

$$-\frac{\partial}{\partial C_{jkl}}\left(\frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4\gamma_1\gamma_2\gamma_3}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}C_{\gamma_1\gamma_2\gamma_3}\right), \tag{4.79b}$$

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and

$$= \frac{4.2}{144^2}\partial_\alpha \left(\epsilon^{\alpha_1\cdots\alpha_4\alpha ijk\gamma_1\gamma_2\gamma_3}G_{\alpha_1\cdots\alpha_4}C_{\gamma_1\gamma_2\gamma_3}\right) - \frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4},$$
(4.80a)

$$= \frac{4.2}{144^2}\left(\epsilon^{\alpha_1\cdots\alpha_4\alpha ijk\gamma_1\gamma_2\gamma_3}G_{\alpha_1\cdots\alpha_4}\partial_\alpha C_{\gamma_1\gamma_2\gamma_3}\right) - \frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4},$$
(4.80b)

$$= -\frac{2}{144^2}\left(\epsilon^{\alpha_1\cdots\alpha_4\alpha\gamma_1\gamma_2\gamma_3 ijk}G_{\alpha_1\cdots\alpha_4}4\partial_{[\alpha}C_{\gamma_1\gamma_2\gamma_3]}\right) - \frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4},$$
(4.80c)

$$= -\frac{2}{144^2}\left(\epsilon^{\alpha_1\cdots\alpha_4\alpha\gamma_1\gamma_2\gamma_3 ijk}G_{\alpha_1\cdots\alpha_4}G_{\alpha\gamma_1\gamma_2\gamma_3}\right) - \frac{1}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}.$$
(4.80d)

Finally the equation of motion for the gauge field by considering (4.80d) and (4.78c) becomes

$$\partial_\alpha(\sqrt{-g}\,G^{\alpha ijk}) + \frac{3.3!}{144^2}\left(\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}\right) = 0.$$
(4.81)

4.3.3 The equation of motion for ψ_α

Because we do not consider the spinors in this work thus we prefer to give directly the equation of motion for spinors without any details

$$\Gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho = 0,$$
(4.82)

where

$$\hat{D}_\nu\psi_\rho = D_\nu(\hat{\omega})\psi_\rho - \frac{1}{2.144}\left(\Gamma^{\alpha\beta\gamma\delta}{}_\nu - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\nu\right)\psi_\rho\hat{G}_{\rho\beta\gamma\delta}.$$
(4.83)

When the expectation values of the fermion fields are zero (we will show in chapter 5, setting the expectation values of the fermion fields equal to zero does not completely destroy the supersymmetry $\epsilon \neq 0$), the equations of motion are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{1}{48}(-\frac{1}{2}g_{\alpha\beta}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + 4G_{\alpha\nu\rho\sigma}G_\beta{}^{\nu\rho\sigma}),$$

$$\partial_\xi\left(\sqrt{-g}G^{\xi ijk}\right) = -\frac{1}{1152}(\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4 ijk}G_{\alpha_1\cdots\alpha_4}G_{\beta_1\cdots\beta_4}).$$
(4.84)

It is believed that there is an eleven-dimensional theory called M-theory which contains the eleven-dimensional supergravity as its low-energy limit and besides the compactification of M-theory on a circle with radius R in the low-energy limit turns out to be type IIA string theory [58, 59]. One can extract the action of type IIA string by considering the bosonic part of (4.60). As we stated before this part consists of a metric and a three-form potential as follows [56]

$$\mathcal{L}_{11} = \frac{1}{4}\sqrt{-g}R - \frac{1}{4\cdot 48}\sqrt{-g}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + \frac{1}{4\cdot 144^2}\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}.$$
(4.85)

Now we want to derive the Lagrangian in D=10 by compactifying one of the coordinates (e.g. x^{11}) on a circle. Assuming

$$\varphi = \frac{x^{11}}{R}, \ \varphi \in [0, 2\pi), \tag{4.86}$$

where R is the radius of the circle. The 11-dimensional metric tensor or any other fields can be written as

$$g_{\mu\nu}(x^1, \ldots, x^{11}) = \sum_{n=0}^{\infty} g_{\mu\nu}^{(n)}(x^1, \ldots, x^{10}) e^{in\varphi}. \tag{4.87}$$

For massless particles the term with $n = 0$ in (4.87) is required therefore $\partial_{x^{11}} g_{\mu\nu} = 0$. Introducing a gauge potential (A_μ) and a scalar field (ϕ) we can write the eleven-dimensional metric in terms of ten-dimensional fields $g_{\mu\nu}^{\text{String}}$, A_μ, and ϕ as follows

$$g_{\mu\nu}^{(11)} = e^{\frac{4}{3}\phi} \begin{pmatrix} e^{-2\phi} g_{\mu\nu}^{\text{String}} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}. \tag{4.88}$$

Using (4.85) and the new form of the metric, the Lagrangian density in D=10 or in other words the Lagrangian density of type IIA sting theory becomes [56]

$$\mathcal{L}_{(10)} = \frac{1}{4}\sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2 \cdot 3!}e^{-\phi}(H^{(3)})^2 \right]$$
$$- \frac{1}{4 \cdot 4}e^{\frac{3}{2}\phi}\sqrt{-g} \, (F_{\mu\nu}^{(2)})^2 - \frac{1}{4 \cdot 48}e^{\frac{\phi}{2}}\sqrt{-g} \left[F_{\alpha_1\ldots\alpha_4}^{(4)} + 8 \cdot A_{[a_1}^{(1)} H_{a_2 a_3 a_4]}^{(3)} \right]^2 \tag{4.89}$$
$$+ \frac{3}{2 \cdot 12^3}\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu} F_{\alpha_1\ldots\alpha_4}^{(4)} F_{\beta_1\ldots\beta_4}^{(4)} B_{\mu\nu}^{(2)}.$$

where

$$\begin{aligned} B_{ij}^{(2)} &= C_{ij11}, \\ F_{\mu\nu}^{(2)} &= 2\partial_{[\mu}A_{\nu]}, \\ H_{a_1\ldots a_3}^{(3)} &= G_{a_1\ldots a_3 11}, \\ F_{b_1\ldots b_4}^{(4)} &= G_{b_1\ldots b_4}. \end{aligned} \tag{4.90}$$

The connection between fields living in D=10 and D=11 after compactification is shown in (4.90). This suggests that solutions to D=11 supergravity after compactification over one of the coordinates, can be used in type IIA string theory.

4.3.4 M(D)-Brane solutions

Membranes are dynamical objects which play an important role in the theory of supergravities (e.g. D=10 and D=11) [60, 61, 62, 63]. They carry charge and mass and geometrically are defined by two groups of coordinate systems called tangential and transverse coordinates. The tangential coordinates are related to the word-volume of the membranes (x^1 and x^2 in figure 4.4). The transverse coordinates are used to show the location of membranes (x^3 in figure 4.4).
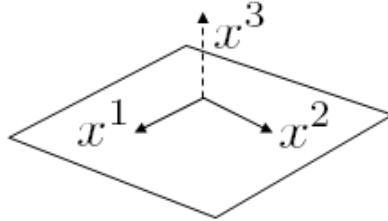
Figure 4.4: A D2-brane with two tangential coordinates x^1, x^2 and a transverse coordinate x^3. The location of the D2-brane is $x^3 = 0$.

We use word M-brane for the extended objects in D=11 whereas in D=10 the objects are called D-brane. A D(M)-brane is determined by spatial coordinates in it's world-volume e.g. M6-brane or D6-brane is a seven-dimensional object (1+6). In the first part of this section we will consider membrane solutions in D=11 and then we will discuss D-brane solutions in D=10. Let us start with D=11. It is known that D=11 supergravity and hence M-theory admit two basic solutions called M2-brane, and M5-brane, motivating interest in this subject. Both M2-brane and M5-brane solutions have mass and charge and also preserve 1/2 of the supersymmetry therefore they are BPS states (the states which preserve some supersymmetry). A brief summary of M2-brane and M5-brane solutions is provided in the subsequent sections.

M2-Branes

The membrane solutions which break one half of the spacetime supersymmetries and also saturate BPS bound, were discovered by Duff and Stelle in 1991 [64].
In constructing the membrane solutions, we will consider a two-dimensional object (M2-brane) with a world-volume oriented along $\{x^1, x^2, x^3, x^4 = 0, \cdots, x^{11} = 0\}$. The world volume of M2-brane respects Poincare invariance (P_3) and the whole configuration is invariant under rotation ($SO(8)$) in the transverse space, so we expect that the spacetime metric to be functions of distance r in the transverse space. Hence we start finding M2-brane solution by splitting the coordinates as

$$x^M = (x^a, x^i), \tag{4.91}$$

where $a, b, c \cdots = 1, 2, 3$ and $i, j, \cdots = 4, \cdots 11$. The coordinates on the tangent space carry a hat-sign e.g. $\hat{a}$ and $\hat{i}$. Similar to the coordinates splitting, the gamma

matrices also can be written in terms of longitudinal and transverse directions to the M2-brane as

$$
\begin{aligned}
\Gamma^A &= (\Gamma^a, \Gamma^i), \\
&= (\gamma^a \otimes \Sigma^9, \mathbb{1} \otimes \Sigma^i),
\end{aligned}
\tag{4.92}
$$

where γ^a are gamma matrices in Minkowski space (D=3), Σ^i are gamma matrices in the transverse Euclidean space (D=8) and Σ^9 satisfies the following properties

$$
\begin{aligned}
\Sigma^9 &= \Sigma^4 \cdots \Sigma^{11}, \\
(\Sigma^9)^2 &= 1_8.
\end{aligned}
\tag{4.93}
$$

Finally we remind that according to the decomposition of the gamma matrices the spinor field $\epsilon(x^a, x^i)$ can be decomposed as

$$
\epsilon = \eta_0 \otimes \xi(r),
\tag{4.94}
$$

where η_0 is a two-component spinor in D=3 which is constant and $\xi(r)$ is a spinor with 16 components lives in the transverse space (D=8). The D=11 line element which respect the symmetry group $P_3 \times SO(8)$ where P_3 stands for the 3-dimensional Poincare group, can be written in the following form

$$
ds_{11}^2 = e^{2A(r)} ds_3^2 + e^{2B(r)} ds_8^2,
\tag{4.95}
$$

where

$$
\begin{aligned}
ds_3^2 &= \eta_{ab} dx^a dx^b, \\
ds_8^2 &= \delta_{ij} dx^i dx^j,
\end{aligned}
\tag{4.96}
$$

and $r = \sqrt{\delta_{ij} x^i x^j}$. Since the membrane has three dimensions $(1+2)$, a 3-form gauge field A_{abc} as

$$
\begin{aligned}
A_{abc} &= \epsilon_{abc}\, e^{C(r)}, \\
\epsilon_{abc} &= +1,
\end{aligned}
\tag{4.97}
$$

is needed to be coupled to the world volume of the membrane. All other components of the gauge field A_{ijk} and all components of the gravitino ψ_M are set to zero. Since we require the fermions to vanish, from supersymmetry transformation it is clear that

$$
\delta_\epsilon \psi = D_\epsilon \epsilon = 0,
\tag{4.98}
$$

where (4.98) is known as the Killing spinor equation [65, 66, 67] in D=11 and D_ϵ is given by

$$
(D_\epsilon)_M = \partial_M - \frac{1}{4}\omega_{M\hat{A}\hat{B}}\Gamma^{\hat{A}\hat{B}} - \frac{1}{288}\left(\Gamma^{A_1 A_2 A_3 A_4}{}_M - 8\Gamma^{A_2 A_3 A_4}\delta_M^{A_1}\right) G_{A_1 A_2 A_3 A_4},
\tag{4.99}
$$

where the spin connection in terms of the Christoffel connection and the vielbein $(e_{\hat{B}}^{\hat{A}})$ is expressed by

$$
\omega_{M\hat{B}}{}^{\hat{A}} = \Gamma_{MF}{}^N e_N^{\hat{A}} e_{\hat{B}}^F - e_{\hat{B}}^F \partial_M e_F^{\hat{A}},
\tag{4.100}
$$

$$e^{\hat{A}}_B = \begin{cases} e^{A(r)} & \text{if } \hat{A} = \hat{a} \\ e^{B(r)} & \text{if } \hat{A} = \hat{i} \end{cases}, \tag{4.101}$$

and the Christoffel connection in terms of vielbein becomes

$$\Gamma_{ABC} = g_{CF}\Gamma_{AB}{}^F$$
$$= \frac{1}{2}\left(e_{\hat{I}C}\partial_A e^{\hat{I}}_B + e^{\hat{I}}_B\partial_A e_{\hat{I}C} + e_{\hat{I}A}\partial_B e^{\hat{I}}_C + e^{\hat{I}}_C\partial_B e_{\hat{I}A} - e_{\hat{I}A}\partial_C e^{\hat{I}}_B - \partial_C e_{\hat{I}A}e^{\hat{I}}_B\right). \tag{4.102}$$

According to the value of $M = (a, i)$, we split the calculation of (4.99) into two parts

1. $M = a$.

   In this case the Killing equation (4.99) reads as

$$\underbrace{\partial_a}_{0} - \frac{1}{4}\underbrace{\omega_{a\hat{A}\hat{B}}\Gamma^{\hat{A}\hat{B}}}_{I} - \frac{1}{288}\left(\underbrace{\Gamma^{A_1A_2A_3A_4}{}_a}_{II} - \underbrace{8\Gamma^{A_2A_3A_4}\delta^{A_1}_a}_{III}\right)G_{A_1A_2A_3A_4} = 0. \tag{4.103}$$

   Simplifying I gives

$$I = 2e^{-B(r)}e^{A(r)}\frac{\partial A(r)}{\partial x^i}\Gamma^{\hat{i}\hat{a}},$$
$$= 2e^{-B(r)}e^{A(r)}\partial_i A(r)\Gamma^{\hat{i}\hat{a}}, \tag{4.104}$$

   where a summation over i is implied. Using the gamma matrices we get

$$I = 2\partial_{\hat{i}}A(r)e_{a\hat{b}}\Gamma^{\hat{i}\hat{b}},$$
$$= 2\partial_{\hat{i}}A(r)e_{a\hat{b}}\Sigma^{\hat{i}}\gamma^{\hat{b}}\Sigma^{(9)}. \tag{4.105}$$

   One can easily show that the contribution from term II vanishes,

$$II = \Gamma^{A_1A_2A_3A_4}{}_a G_{A_1A_2A_3A_4} = 4!\Gamma^{123i}{}_a G_{123i} = 0, \tag{4.106}$$

   and finally term III reduces to

$$III = \Gamma^{A_2A_3A_4}\delta^{A_1}_a G_{A_1A_2A_3A_4},$$
$$= 3\Gamma^{bci}\epsilon_{abc}\partial_i e^{C(r)},$$
$$= 3\gamma^{bc}\Sigma^i\epsilon_{abc}\partial_i e^{C(r)}, \tag{4.107}$$
$$= 6e^{-3A(r)}\gamma_a\Sigma^i\partial_i e^{C(r)},$$

   where $G_{abci} = -\epsilon_{abc}\partial_i e^{C(r)}$. Plugging I, II, and III into (4.103), we obtain

$$(D_\epsilon)_a = -\frac{1}{6}e^{-3A(r)}\partial_{\hat{i}}e^{3A(r)}e_{a\hat{b}}\Sigma^{\hat{i}}\gamma^{\hat{b}}\Sigma^{(9)} + \frac{1}{6}e^{-3A(r)}\gamma_a\Sigma^{\hat{i}}\partial_{\hat{i}}e^{C(r)},$$
$$= -\frac{1}{6}e^{-3A(r)}\Sigma^{\hat{i}}\gamma_a\partial_{\hat{i}}\left(e^{3A(r)}\Sigma^{(9)} - e^{C(r)}\right). \tag{4.108}$$

   A projection can be built by taking $e^{3A(r)} = e^{C(r)}$ which implies that $A = \frac{1}{3}C$ and so (4.108) becomes

$$(D_\epsilon)_a = \frac{1}{6}\Sigma^{\hat{i}}\gamma_a\partial_{\hat{i}}C\left(1 - \Sigma^{(9)}\right). \tag{4.109}$$

One can solve this equation by making the following ansatz

$$\epsilon = f(x^4, \cdots, x^{11})(1 + \Sigma^{(9)})\epsilon_0, \tag{4.110}$$

where ϵ_0 is a constant spinor. We will use this solution after calculating the second part of the Killing equation in the transverse space.

2. $M = i$.
   The Killing equation (4.99) becomes

$$(D_\epsilon)_i = \partial_i - \frac{1}{4}\underbrace{\omega_{i\hat{A}\hat{B}}\Gamma^{\hat{A}\hat{B}}}_{\mathrm{I}} - \frac{1}{288}\left(\underbrace{\Gamma^{A_1 A_2 A_3 A_4}{}_i}_{\mathrm{II}} - \underbrace{8\Gamma^{A_2 A_3 A_4}\delta_i^{A_1}}_{\mathrm{III}}\right)G_{A_1 A_2 A_3 A_4} = 0. \tag{4.111}$$

The first part (I) can be simplified as

$$\mathrm{I} = 2\partial_k B\Sigma^k{}_i. \tag{4.112}$$

The second part (II) reduces to

$$\begin{aligned}
\mathrm{II} &= \Gamma^{A_1 A_2 A_3 A_4}{}_i G_{A_1 A_2 A_3 A_4}, \\
&= 4\Gamma^{A_1 A_2 A_3 m}{}_i G_{A_1 A_2 A_3 m}, \\
&= -24e^{-3A(r)}\gamma^{123}\Sigma^{(9)}\Sigma^m{}_i\partial_m e^{C(r)},
\end{aligned} \tag{4.113}$$

and finally the simplification of the last term (III) gives

$$\begin{aligned}
\mathrm{III} &= \Gamma^{A_2 A_3 A_4}\delta_i^{A_1}G_{A_1 A_2 A_3 A_4}, \\
&= \Gamma^{A_2 A_3 A_4}G_{iA_2 A_3 A_4}, \\
&= -6e^{-3A(r)}\gamma^{123}\Sigma^{(9)}\partial_i e^{C(r)}.
\end{aligned} \tag{4.114}$$

Substituting the above expressions in (4.111) we find

$$\begin{aligned}
(D_\epsilon)_i &= \partial_i - \frac{1}{2}\partial_k B\Sigma^k{}_i - \frac{1}{12}e^{-3A(r)}\gamma^{123}\Sigma^m{}_i\Sigma^{(9)}\partial_m e^{C(r)} - \frac{1}{6}e^{-3A(r)}\gamma^{123}\Sigma^{(9)}\partial_i e^{C(r)}, \\
&= \partial_i - \frac{1}{6}e^{-3A(r)}\gamma^{123}\Sigma^{(9)}\partial_i e^{C(r)} - \frac{1}{2}\Sigma^k{}_i\left(\partial_k B + \frac{1}{6}e^{-3A(r)}\gamma^{123}\Sigma^{(9)}\partial_k e^{C(r)}\right), \\
&= \partial_i - \frac{1}{6}e^{-3A(r)}\Sigma^{(9)}\partial_i e^{C(r)} - \frac{1}{2}\Sigma^k{}_i\left(\partial_k B + \frac{1}{6}\Sigma^{(9)}\partial_k e^{C(r)}\right).
\end{aligned} \tag{4.115}$$

Again a projection can be obtained if we set

$$A = \frac{1}{3}C, \tag{4.116a}$$

$$B = -\frac{1}{6}C, \tag{4.116b}$$

and (4.115) reads as

$$(D_\epsilon)_i = \partial_i - \frac{1}{6}\Sigma^{(9)}\partial_i C - \frac{1}{2}\Sigma^k{}_i\partial_k B\left(1 - \Sigma^{(9)}\right). \tag{4.117}$$

Plugging (4.110) into (4.117) gives

$$f(x^4, \cdots, x^{11}) = e^{\frac{1}{6}C},$$
$$\epsilon = e^{\frac{1}{6}C}(1 + \Sigma^{(9)})\epsilon_0, \tag{4.118}$$

or in general form

$$\epsilon = e^{\frac{1}{6}C}\eta_0 \otimes \xi_0, \tag{4.119}$$

where ξ_0 is a constant spinor satisfying

$$(1 - \Sigma^{(9)})\xi_0 = 0. \tag{4.120}$$

We note that the Killing equation does not imply the equations of motion, hence in order to determine $C = C(r)$ we use the equation of motion for the gauge field (4.81). Inserting the field strength $G_{A_1 B_1 C_1 M}$ into (4.81) gives

$$\partial_M(\sqrt{-g}G^{A_1 B_1 C_1 M}) + 0 = 0,$$
$$\partial_M(e^{3A+8B}g^{A_1 A_2}g^{B_1 B_2}g^{C_1 C_2}g^{MM_1}G_{A_2 B_2 C_2 M_1}) = 0,$$
$$\partial_i(e^{3A+8B}g^{aa_1}g^{bb_1}g^{cc_1}e^{-2B}\delta^{ij}\epsilon_{a_1 b_1 c_1}\partial_j e^C) = 0,$$
$$\epsilon_{abc}\partial_i(e^{3A+8B}e^{-6A}e^{-2B}\delta^{ij}\partial_j e^C) = 0,$$
$$\partial_i(e^{-3A+6B}\delta^{ij}\partial_j e^C) = 0,$$
$$\partial_i(e^{-2C}\delta^{ij}\partial_j e^C) = 0,$$
$$\delta^{ij}\partial_i(\partial_j e^{-C}) = 0. \tag{4.121}$$

which shows e^{-C} satisfies the Laplace equation (4.121) in the transverse space and has the following solution

$$e^{-C} \equiv H(r) = 1 + \frac{\mathcal{Q}_{M_2}}{r^6}, \tag{4.122}$$

where $r = \sqrt{x_4^2 + \cdots x_{11}^2}$ and $\mathcal{Q}_2$ is a constant. Knowing e^C, one can easily show that

$$e^{2A} = H^{-\frac{2}{3}}, \tag{4.123}$$
$$e^{2B} = H^{\frac{1}{3}}, \tag{4.124}$$

and the metric becomes

$$ds_{11}^2 = H^{-\frac{2}{3}}ds_3^2 + H^{\frac{1}{3}}ds_8^2, \tag{4.125}$$

which subsequently fulfills the requirement of the metric at infinity

$$\text{Minkowski space} = \lim_{r \to +\infty} ds_{11}^2 = ds_3^2 + ds_8^2. \tag{4.126}$$

In the following paragraphs we summarize the results obtained from the previous discussion and we complete the M2-brane solution by describing the mass and charge of the membrane. In particular we use ADM (Arnowitt, Deser and Misner) formalism

to derive the mass of the membranes. As we obtained earlier the M2-brane solution takes the following form

$$ds^2 = \underbrace{H^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2)}_{\text{M2-brane}} + H^{\frac{1}{3}}(dx_3^2 + \cdots + dx_{10}^2), \qquad (4.127)$$

or in the matrix form, the eleven-dimensional metric becomes

$$g_{\mu\nu} = H^{-\frac{2}{3}} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & H.\mathbf{1}_8 \end{pmatrix},$$

$$G_{012i} = c\frac{\partial_i H}{H^2}, \; c = \pm 1, \qquad (4.128)$$

where H is a harmonic function satisfying Laplace equation $\nabla^2 H = 0$, and depends on $x_3 \cdots x_{10}$. G_{012i} is a four-form field, and $c = \pm 1$ correspond to M2-brane $(c = +1)$ and anti M2-brane $(c = -1)$. The functional form of H in terms of transverse coordinates is given by

$$H = 1 + \frac{\mathcal{Q}_{M_2}}{\bar{r}^6},$$
$$\bar{r}^2 = x_3^2 + \cdots + x_{10}^2, \qquad (4.129)$$

where $\mathcal{Q}_{M_2}$ is related to the charge of the M2-brane. The solution specifies a membrane locating at the point $\bar{r} = 0$ in the transversal space and the world-volume of the membrane is oriented along the t, x_1, and x_2 directions. M2-brane solution preserves $\frac{1}{2}$ of the initial supersymmetry [4, 55, 56, 68]. The Killing spinors in this case takes the following form

$$\varepsilon = H^{-\frac{1}{6}}\eta, \qquad (4.130a)$$
$$\Gamma^{tx_1 x_2}\eta = c\eta, \qquad (4.130b)$$

where η is a constant spinor, $c = \pm 1$ and $\Gamma^{tx_1 x_2} = \Gamma^t \Gamma^{x_1} \Gamma^{x_2}$.

As we can see from (4.129), the solution contains a singularity at $\bar{r} = 0$. The nature of this singularity can be studied by analyzing the behavior of the Kretschmann scalar $(\mathbf{K} = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta})$ at $\bar{r} = 0$. In order to simplify the calculation we assume that the 8-dimensional metric in the transverse space takes the following form

$$ds_8^2 = dy^2 + y^2 d\alpha^2 + y^2 \sin^2(\alpha) \left(d\beta^2 + \sin^2(\beta)d\eta^2\right) +$$
$$+ dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) \left(d\phi^2 + \sin^2(\phi)d\psi^2\right). \qquad (4.131)$$

So the M2-brane metric becomes

$$ds_{11}^2 = H(r,y)^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2) + H(r,y)^{\frac{1}{3}}ds_8^2, \qquad (4.132)$$

where $H(r,y) = 1 + \frac{\mathcal{Q}_{M_2}}{(y^2+r^2)^3}$. Calculating the Kretschmann scalar, we find

$$\mathbf{K} = \frac{\hat{F}(r,y,\mathcal{Q}_{M_2})}{\left(r^6 + 3\,r^4 y^2 + 3\,r^2 y^4 + y^6 + \mathcal{Q}_{M_2}\right)^{14/3}}, \qquad (4.133)$$

where

$$\hat{F}(0, 0, \mathcal{Q}_{M_2}) = 468\, \mathcal{Q}_{M_2}{}^4. \tag{4.134}$$

This reveals that the value of the **K** is finite in the limit as $r, y \to 0$, hence $\bar{r} = \sqrt{r^2 + y^2} = 0$ is simply a coordinate singularity. We rewrite the metric [69] in terms of an isotropic coordinate system $(\tilde{r})$ as

$$ds_{11}^2 = H(\tilde{r})^{-\frac{2}{3}}(-dt^2 + dx_1{}^2 + dx_2{}^2) + H(\tilde{r})^{\frac{1}{3}}(d\tilde{r}^2 + \tilde{r}^2 d\Omega_7^2), \tag{4.135}$$

where $d\Omega_7^2$ is the angular part of 8-dimensional flat space in the hyperspherical coordinate system. Now we introduce the variable $\tilde{R}$ by

$$\tilde{r} = \mathcal{Q}_{M_2}{}^{\frac{1}{6}} \frac{\sqrt{\tilde{R}}}{(1 - \tilde{R}^3)^{\frac{1}{6}}}, \tag{4.136}$$

$$1 > \tilde{R} > 0,$$

and

$$H(\tilde{R}) = 1 + \frac{\mathcal{Q}_{M_2}}{\tilde{r}^6} = \frac{1}{\tilde{R}^3}. \tag{4.137}$$

The metric (4.135) in terms of the new variable $\tilde{R}$ reads as

$$ds_{11}^2 = \tilde{R}^2(-dt^2 + dx_1{}^2 + dx_2{}^2) + \frac{1}{4}\mathcal{Q}_{M_2}{}^{\frac{1}{3}}\tilde{R}^{-2}d\tilde{R}^2 + \mathcal{Q}_{M_2}{}^{\frac{1}{3}}d\Omega_7^2 +$$
$$+ \frac{1}{4}\mathcal{Q}_{M_2}{}^{\frac{1}{3}}\left((1 - \tilde{R}^3)^{-\frac{7}{3}} - 1\right)\tilde{R}^{-2}d\tilde{R}^2 + \mathcal{Q}_{M_2}{}^{\frac{1}{3}}\left((1 - \tilde{R}^3)^{-\frac{1}{3}} - 1\right)d\Omega_7^2. \tag{4.138}$$

Two interesting cases in (4.138) are

1. $\tilde{R} \to 1$.
   the metric becomes asymptotically flat.

2. $\tilde{R} \to 0$.
   the metric is given by

$$ds_{11}^2 = \tilde{R}^2(-dt^2 + dx_1{}^2 + dx_2{}^2) + \frac{1}{4}\mathcal{Q}_{M_2}{}^{\frac{1}{3}}\tilde{R}^{-2}d\tilde{R}^2 + \mathcal{Q}_{M_2}{}^{\frac{1}{3}}d\Omega_7^2, \tag{4.139}$$

which is the metric of $AdS_4 \times S^7$ (figure 4.5). By setting $\Lambda = \frac{1}{4}R = -\frac{12}{\mathcal{Q}_{M_2}{}^{\frac{1}{3}}}$ where R is the Ricci scalar the AdS_4 part satisfies

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 0. \tag{4.140}$$
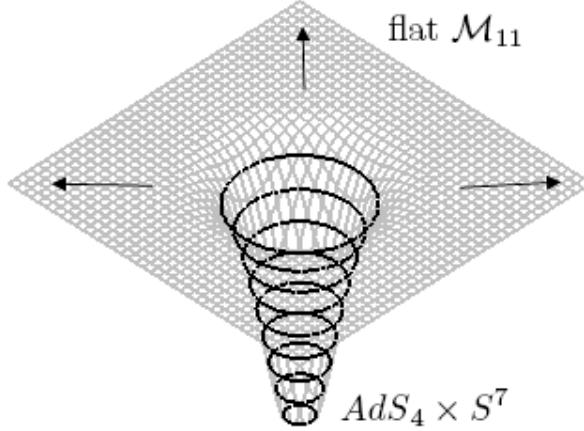
Figure 4.5: The topology of M2-brane solution at $R \to 0$ and $R \to 1$.

Charge and mass

M2-branes and M5-branes are dynamical objects which carry charge and mass. M2-branes are electrically and M5-branes are magnetically charged under a 3-form gauge field shown by $A_{\alpha_1 \alpha_2 \alpha_3}$. As a matter of fact we say electrically charged branes and the magnetically charged branes are dual to each other. The charge of a single M2-brane is given by

$$Q_2 = \int_{\Sigma_7} \star G^{(4)}, \tag{4.141}$$

where Σ_7 is a 7-sphere (S^7) enclosing the M2-brane, $G_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 4\partial_{[\alpha_1} A_{\alpha_2 \alpha_3 \alpha_4]}$ and

$$G = \frac{1}{4!} G_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge dx^{\alpha_3} \wedge dx^{\alpha_4}. \tag{4.142}$$

For the sake of simplicity we use the spherical coordinates to obtain the metric in the transverse space. So, the metric for the 8-dimensional Euclidean space in terms of spherical coordinates is given by

$$ds_8^2 = H(y)^{\frac{1}{3}} y^2 \begin{bmatrix} \frac{1}{y^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{(\theta_2 \theta_2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{(\theta_3 \theta_3)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{(\theta_4 \theta_4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{(\theta_5 \theta_5)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{(\theta_6 \theta_6)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{(\theta_7 \theta_7)} \end{bmatrix}$$

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where

$$g_{(\theta_2\theta_2)} = \sin^2(\theta_1), \tag{4.143a}$$

$$g_{(\theta_3\theta_3)} = \sin^2(\theta_1)\sin^2(\theta_2), \tag{4.143b}$$

$$g_{(\theta_4\theta_4)} = \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3), \tag{4.143c}$$

$$g_{(\theta_5\theta_5)} = \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3)\sin^2(\theta_4), \tag{4.143d}$$

$$g_{(\theta_6\theta_6)} = \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3)\sin^2(\theta_4)\sin^2(\theta_5), \tag{4.143e}$$

$$g_{(\theta_7\theta_7)} = \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3)\sin^2(\theta_4)\sin^2(\theta_5)\sin^2(\theta_6), \tag{4.143f}$$

where $0 < \theta_7 \leq 2\pi$ and $0 < \theta_i \leq \pi$, $i = 1 \cdots 6$. The volume element can be obtain from the metric as

$$\sqrt{-g}\, d\theta_1 \cdots d\theta_7 = y^7 H(y)^{\frac{1}{3}} d\Omega_7, \tag{4.144}$$

which yields

$$\Omega_7 = \int_0^\pi \cdots \int_0^\pi \sin^6(\theta_1)\sin^5(\theta_2)\cdots\sin(\theta_6)\, d\theta_1 \cdots d\theta_6 \int_0^{2\pi} d\theta_7 = \frac{\pi^4}{3}, \tag{4.145}$$

where Ω_7 is the volume of 7-sphere (S^7). The metric function H and the gauge field $(A_{tx_1x_2})$ are given by

$$H(y) = 1 + \frac{\mathcal{Q}_{M_2}}{y^6},$$

$$A_{tx_1x_2} = \frac{1}{H(y)}, \tag{4.146}$$

where $dA_{tx_1x_2}$ gives $G^{(4)}$. Inserting (4.144) and (4.146), into (4.141) we find

$$Q_2 = 2\pi^4 \mathcal{Q}_{M_2}. \tag{4.147}$$

As we can see from the M2-brane solution

$$\lim_{y \to +\infty} H(y) = 1, \tag{4.148}$$

which shows the asymptotically flatness of space, hence this allows us to use the ADM formalism to obtain the mass of the M2-brane $(\mathcal{E}_2)$ [56, 69]. In this formalism

$$\mathcal{E}_2 = \lim_{y \to +\infty} \int_{\Sigma_7} [\nabla^\alpha g_{\beta\alpha} - \nabla_\beta(\eta^{\mu\nu} g_{\mu\nu})]\, n^\beta d\Omega_7, \tag{4.149}$$

where $\eta^{\mu\nu}$ stands for the flat metric, n^β is the components of unit normal vector to S^7, $d\Omega_7$ is the volume element (S^7) and ∇ is the covariant derivative associated with the metric $\eta^{\mu\nu}$. The metric $\eta_{\mu\nu}$ is given by

$$ds_{10}^2 = dx_1^2 + dx_2^2 + dy^2 + y^2 d\Omega_7^2, \tag{4.150}$$

and $g_{\mu\nu}$ takes the following form

$$ds_{10}^2 = H(y)^{-\frac{2}{3}}(dx_1^2 + dx_2^2) + H(y)^{\frac{1}{3}}(dy^2 + y^2 d\Omega_7^2), \tag{4.151}$$

where $d\Omega_7^2$ is the differential element of a 7-sphere (S^7). Employing (4.149), (4.150) and (4.151) we find the ADM mass as

$$\mathcal{E}_2 = \lim_{y \to +\infty} \frac{\mathcal{Q}_{M_2}(3 + \frac{7}{y^6}\mathcal{Q}_{M_2})2\pi^4}{3(1 + \frac{\mathcal{Q}_{M_2}}{y^6})\left(1 + \frac{\mathcal{Q}_{M_2}}{y^6}\right)^{\frac{2}{3}}} = 2\pi^4 \mathcal{Q}_{M_2}. \tag{4.152}$$

Comparing (4.147) and (4.152) we can conclude that

$$\mathcal{E}_2 = Q_2, \tag{4.153}$$

which means that the M2-brane solution saturates the BPS bound

$$\mathcal{E}_2 \geq Q_2. \tag{4.154}$$

M5-Branes

Similar to the M2-brane solution, one can define another important solution in eleven-dimensional supergravity called M5-brane. This solution is given by [56]

$$ds^2 = \underbrace{H^{-\frac{1}{3}}(-dt^2 + dx_1^2 + \cdots + dx_5^2)}_{\text{M5-brane}} + H^{\frac{2}{3}}(dx_6^2 + \cdots + dx_{10}^2), \tag{4.155}$$

or

$$g_{\mu\nu} = \begin{pmatrix} -H^{-\frac{1}{3}} & & & & & \\ & \ddots & & & & \\ & & H^{-\frac{1}{3}} & & & \\ & & & H^{\frac{2}{3}} & & \\ & & & & \ddots & \\ & & & & & H^{\frac{2}{3}}(x_6 \cdots x_{10}) \end{pmatrix},$$

$$G_{\alpha_1 \cdots \alpha_4} = cH^{\frac{2}{3}}\epsilon_{\alpha_1 \cdots \alpha_5}\partial^{\alpha_5} H(x_6, \cdots, x_{10}), \ \ c = \pm 1, \tag{4.156}$$

where H is a harmonic function $\nabla^2 H = 0$, $\epsilon_{\alpha_1 \cdots \alpha_5}$ is Levi-Civita symbol, and $c = \pm 1$ stands for M5-brane and anti M5-brane respectively. H takes the functional form

$$H = 1 + \frac{\mathcal{Q}_{M_5}}{r^3}, \tag{4.157}$$
$$r^2 = x_6^2 + \cdots + x_{10}^2,$$

where $\mathcal{Q}_{M_5}$ is related to the charge of the M5-brane. The Killing spinors for M5-brane have the following form

$$\varepsilon = H^{-\frac{1}{12}}\eta, \tag{4.158}$$

where η is a constant spinor satisfying the projection

$$\Gamma^{tx_1 x_2 x_3 x_4 x_5}\eta = c\eta, \tag{4.159}$$

where $\Gamma^{tx_1 x_2 x_3 x_4 x_5} = \Gamma^t \Gamma^{x_1} \Gamma^{x_2} \Gamma^{x_3} \Gamma^{x_4} \Gamma^{x_5}$. The M5-brane shows an event horizon without any singularity and the metric interpolates between Minkowski space-time at infinity and $AdS_7 \times S^4$ space-time near the horizon [69].

Charge and mass

Similar to M2-brane the magnetic charge of a M5-brane [56, 69] is defined as

$$Q_5 = \int_{\Sigma_5} G^{(4)} = 8\pi^2 \mathcal{Q}_{M_5}, \tag{4.160}$$

and the ADM mass $\mathcal{E}_5$ is

$$\mathcal{E}_5 = \lim_{y \to +\infty} \frac{\mathcal{Q}_{M_5}(3 + \frac{8}{y^3}\mathcal{Q}_{M_5})8\pi^2}{3(1 + \frac{\mathcal{Q}_{M_5}}{y^3})\left(1 + \frac{\mathcal{Q}_{M_5}}{y^3}\right)^{\frac{1}{3}}} = 8\pi^2 \mathcal{Q}_{M_5}, \tag{4.161}$$

and again we see that the M5-brane solution saturates the BPS bound.

$$\mathcal{E}_5 \geq Q_5. \tag{4.162}$$

In the following, we briefly introduce intersecting membranes in D=11 supergravity [70, 71, 72, 73, 74, 75, 76, 77, 78]. We just consider only three combinations of M2 and M5-branes which give the following configurations of intersecting branes in D=11 (table 4.1).

Table 4.1: Intersecting branes with different configurations.

	t	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
M2	×	×	×	−	−	−	−	−	−	−	−
M2	×	−	−	×	×	−	−	−	−	−	−
M2	×	×	−	−	−	−	×	−	−	−	−
M5	×	×	×	×	×	×	−	−	−	−	−
M5	×	×	×	×	×	×	−	−	−	−	−
M5	×	×	×	×	−	−	×	×	−	−	−

* × and − stand for the world-volume and the transverse coordinates respectively.

The general form of the metric in terms of the world-volume and the transverse coordinates become

$$ds^2 = \overbrace{-- \cdots -}^{W}\underbrace{\times \times \cdots \times}_{T}, \tag{4.163}$$

where W and T stand for the world-volume and the transverse space respectively.

M2-M2 Intersecting branes

We start with two intersecting M2-branes [71, 77, 78, 79] which overlaps in a point and are oriented along (x_1, x_2) and (x_3, x_4) directions respectively. They may be

located at different locations. The metric for two-M2-branes is given by

$$ds^2 = (H_1 H_2)^{\frac{1}{3}} \left(-\frac{dt^2}{H_1 H_2} + \frac{1}{H_1}(dx_1^2 + dx_2^2) + \frac{1}{H_2}(dx_3^2 + dx_4^2) + (dx_5^2 + \cdots + dx_{10}^2) \right),$$

(4.164)

where

$$G_{t12\alpha} = \frac{c_1}{2} \frac{1}{H_1^2} \frac{\partial H_1}{\partial x^\alpha}, \quad G_{t34\alpha} = \frac{c_2}{2} \frac{1}{H_2^2} \frac{\partial H_2}{\partial x^\alpha}, \quad \alpha = 5 \cdots 10,$$

(4.165)

and

$$H_i = H_i(x_5, \cdots, x_{10}), \quad \nabla^2 H_i = 0, \quad c_i = \pm 1, \quad i = 1, 2.$$

(4.166)

The metric functions H_i become

$$H_i = 1 + \frac{Q_i}{r_i^4}, \quad \vec{x}_i = (x_{i5} \cdots x_{i10}), \quad \vec{x} = (x_5 \cdots x_{10}), \quad i = 1, 2,$$

(4.167)

where $r_i = |\vec{x} - \vec{x}_i|$ is the relative distance between the location of the each brane $(\vec{x}_i)$ and the position vector $(\vec{x})$ in the transverse space. The Killing spinors become

$$\epsilon = (H_1 H_2)^{-\frac{1}{6}} \eta,$$

(4.168)

where η is a constant spinor satisfying

$$\Gamma_{012} \eta = c_1 \eta,$$

(4.169a)

$$\Gamma_{034} \eta = c_2 \eta,$$

(4.169b)

and

$$[\Gamma_{012}, \Gamma_{034}] = 0,$$

(4.170a)

$$\mathrm{Tr}(\Gamma_{012} \Gamma_{034}) = 0.$$

(4.170b)

Each condition in (4.170) preserve half of the spinors thus the solution preserve $\frac{1}{4}$ of the original supersymmetry.

M2-M5 Intersecting branes

In this configuration the M2-brane is oriented in (x_1, x_6) directions and the M5-brane is smeared out in $(x_1, x_2, x_3, x_4, x_5)$ directions and the overlapping coordinate is (x_1). For this solution the metric is given by

$$ds^2 = H_1^{-\frac{1}{3}} H_2^{-\frac{2}{3}}(-dt^2 + dx_1^2) + H_1^{-\frac{1}{3}} H_2^{\frac{1}{3}}(dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2)$$
$$+ H_1^{\frac{2}{3}} H_2^{-\frac{2}{3}}(dx_6^2) + H_1^{\frac{2}{3}} H_2^{\frac{1}{3}}(dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2),$$

(4.171)

where $H_i = H_i(x_7, x_8, x_9, x_{10})$, $i = 1, 2$ and the non vanishing components of the field strength are

$$G_{6\alpha\beta\gamma} = \frac{c_1}{2} \epsilon_{\alpha\beta\gamma\delta} \partial_\delta H_1,$$

(4.172a)

$$G_{t16\alpha} = \frac{c_2}{2} \frac{\partial_\alpha H_2}{H_2^2}.$$

(4.172b)

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The Killing spinors in this configuration are given by

$$\epsilon = (H_1)^{-\frac{1}{12}}(H_2)^{-\frac{1}{6}}\eta, \tag{4.173}$$

where η is a constant spinor which satisfy the constrains

$$\Gamma_{016}\eta = c_1\eta, \tag{4.174a}$$
$$\Gamma_{012345}\eta = c_2\eta. \tag{4.174b}$$

M5-M5 Intersecting branes

Two M5-intersecting branes are oriented in $(x_1, x_2, x_3, x_4, x_5)$ and $(x_1, x_2, x_3, x_6, x_7)$ directions and the overlapping coordinates (x_1, x_2, x_3). The metric for this solution has the form

$$\begin{aligned} ds^2 &= (H_1 H_2)^{-\frac{1}{3}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H_1^{-\frac{1}{3}}H_2^{\frac{2}{3}}(dx_4^2 + dx_5^2) \\ &\quad + H_1^{\frac{2}{3}}H_2^{-\frac{1}{3}}(dx_6^2 + dx_7^2) + (H_1 H_2)^{\frac{2}{3}}(dx_8^2 + dx_9^2 + dx_{10}^2), \end{aligned} \tag{4.175}$$

where $H_i = H_i(x_8, x_9, x_{10})$ and $i = 1, 2$. The field strengths reduce to

$$G_{67\alpha\beta} = \frac{c_1}{2}\epsilon_{\alpha\beta\gamma}\partial_\gamma H_1, \tag{4.176a}$$
$$G_{45\alpha\beta} = \frac{c_2}{2}\epsilon_{\alpha\beta\gamma}\partial_\gamma H_2. \tag{4.176b}$$

The Killing spinors in this configuration are given by

$$\epsilon = (H_1 H_2)^{-\frac{1}{12}}\eta, \tag{4.177}$$

where again η fulfills

$$\Gamma_{012345}\eta = c_1\eta, \tag{4.178a}$$
$$\Gamma_{012367}\eta = c_2\eta. \tag{4.178b}$$

This configuration preserve $\frac{1}{4}$ of the original supersymmetry.

D_p-branes

We start from the gauge fields in type IIA theory (2.48) and predict the existence of D-brane solutions in D=10. There are three gauge fields and each gauge field can couple to a D_p-brane as

Gauge Field	Electrically	Magnetically
$\mathbf{C}_{(1)}$	D_0	D_6
$\mathbf{B}_{(2)}$	F_1	NS_5
$\mathbf{C}_{(3)}$	D_2	D_4

where F_1 and NS_5 are the fundamental branes. In a similar way one can obtain some possible D-branes in type IIB theory as

Gauge Field	Electrically	Magnetically
$\mathbf{C}_{(2)}$	D_1	D_5
$\mathbf{C}_{(4)}$	D_3	D_3
$\mathbf{B}_{(2)}$	F_1	NS_5

where again F_1 and NS_5 are the fundamental branes. We briefly present D_p-branes solutions [71, 77] in D=10 in terms of metrics and the scalar field (dilaton). We start with the fundamental branes F_1 and NS_5. The metric for the simplest brane F_1 in type IIA and IIB takes the following form

$$ds^2 = e^{2\phi}(-dt^2 + dx_1^2) + dx_2^2 + \cdots + dx_9^2, \tag{4.179}$$

where

$$e^{2\phi} = H(x_2, \cdots, x_9)^{-1}, \ \nabla^2 H = 0. \tag{4.180}$$

The metric for NS_5 is given by

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + e^{2\phi}(dx_6^2 + \cdots + dx_9^2), \tag{4.181}$$

where

$$e^{2\phi} = H(x_6, \cdots, x_9), \ \nabla^2 H = 0. \tag{4.182}$$

Finally we give the spacetime metric for D_p-brane as

$$ds^2 = H^{-\frac{1}{2}}(-dt^2 + dx_1^2 + \cdots + dx_p^2) + H^{\frac{1}{2}}(dx_{p+1}^2 + \cdots + dx_9^2), \tag{4.183}$$

where

$$e^{2\phi} = H^{-\frac{p-3}{2}}, \ \nabla^2 H = 0. \tag{4.184}$$

Intersecting D-branes

We should note that M-branes solutions in D=11 provide D2/D6 and NS5/D6-brane intersections in type IIA supergravity [70]. Here we just consider D2-branes inside D6-branes [71]. The spacetime metric takes the following form

$$ds^2 = \frac{1}{\sqrt{H_1 H_2}}(-dt^2 + dx_1^2 + dx_2^2) + \frac{\sqrt{H_1}}{\sqrt{H_2}}\underbrace{(dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2)}_{\vec{V}} +$$
$$+ \sqrt{H_1 H_2}\underbrace{(dx_7^2 + dx_8^2 + dx_9^2)}_{\vec{W}}, \tag{4.185}$$

where H_1 and H_2 are the harmonic functions for D2 and D6-branes respectively. The harmonic functions satisfy

$$\partial_{\vec{W}}^2 H_1 + H_2 \partial_{\vec{V}}^2 H_1 = 0, \tag{4.186a}$$
$$\partial_{\vec{W}}^2 H_2 = 0. \tag{4.186b}$$

One can show that the solutions to (4.186) are

$$H_1 = 1 + \sum_k \frac{\mathcal{Q}_{2k}}{\left[\left|\vec{V} - \vec{V}_{0k}\right|^2 + 4\mathcal{Q}_6|\vec{W} - \vec{W}_0|\right]^3}, \qquad (4.187a)$$

$$H_2 = \frac{\mathcal{Q}_6}{|\vec{W} - \vec{W}_0|}. \qquad (4.187b)$$

There is a particular interest in finding new M2 and M5-brane solutions in M-theory. These solutions can be obtained by embedding various spaces in the transverse space to M-branes. For M2-brane solutions the metric (4.127) can be written in the following form

$$ds^2 = \underbrace{H^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2)}_{\text{M2-brane}} + \underbrace{H^{\frac{1}{3}}(ds_1^2 + ds_2^2)}_{\text{eight-dimensional metric}}, \qquad (4.188)$$

which indicates that the eight-dimensional space in (4.188) is labeled by two distinct metrics called ds_1^2 and ds_2^2. In table (4.2) some combinations of metrics, which can be embedded in ds_1^2 and ds_2^2, are given.

Table 4.2: Different combinations of metrics for ds_8^2

$ds_8^2 = ds_1^2 + ds_2^2$	ds_1^2	ds_2^2
ds_8^2	Flat	2-center
ds_8^2	2-center	2-center
ds_8^2	Flat	3-center
ds_8^2	3-center	3-center
ds_8^2	Flat	k-center
ds_8^2	k-center	k-center
ds_8^2	Flat	Bianchi IX

* 1 and 2 are four dimensional spaces

For the M5-brane solutions the metric takes the following form

$$ds^2 = \underbrace{H^{-\frac{1}{3}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2)}_{\text{M5-brane}} + \underbrace{H^{\frac{2}{3}}(dy^2 + dw^2)}_{\text{five-dimensional metric}}, \qquad (4.189)$$

where dw^2 is a four-dimensional space equal to Gibbons-Hawking or Bianchi IX spaces. Some of these combinations have already been studied, for example, M2 and M5-brane solutions based on transverse Taub-NUT, Atiyah-Hitchin, and Bianchi type IX spaces have been discussed in [29, 80, 81]. In chapter 5 we present new M2 and M5-brane solutions by lifting Gibbons-Hawking and Bianchi spaces to D=11 supergravity.

CHAPTER 5

NEW M-BRANE SOLUTIONS

Fundamental M-theory in the low-energy limit is generally believed to be effectively described by D=11 supergravity [58, 59, 82]. This suggests that brane solutions in the latter theory furnish classical soliton states of M-theory, motivating considerable interest in this subject. There is particular interest in finding D=11 M-brane solutions that reduce to supersymmetric p-brane solutions (that saturate the BPS bound) upon reduction to 10 dimensions. Some supersymmetric BPS solutions of two or three orthogonally intersecting 2-branes and 5-branes in D=11 supergravity were obtained some years ago [79], and more such solutions have since been found [83].

Recently interesting new supergravity solutions for localized D2/D6, D2/D4, NS5/D6 and NS5/D5 intersecting brane systems were obtained [29, 70, 80, 81, 84]. By lifting a D6 (D5 or D4)-brane to four-dimensional self-dual geometries embedded in M-theory, these solutions were constructed by placing M2- and M5-branes in different self-dual geometries. A special feature of this construction is that the solution is not restricted to be in the near core region of the D6 (or D5) brane, a feature quite distinct from the previously known solutions [85, 86]. For all of the different BPS solutions, 1/4 of the supersymmetry is preserved as a result of the self-duality of the transverse metric. Moreover, in [87], partially localized D-brane systems involving D3, D4 and D5 branes were constructed. By assuming a simple ansatz for the eleven dimensional metric, the problem reduces to a partial differential equation that is separable and admits proper boundary conditions.

The aim of this chapter is to construct the fully localized supergravity solutions of D2 (and NS5) intersecting D6 branes without restricting to the near core region of the D6 by reduction of ALE geometries lifted to M-theory. In fully localized solutions the world volume of the lower dimensional brane is entirely inside the world volume of the higher dimensional brane. Our main motivation for considering ALE geometries (and specially multi-center Gibbons-Hawking spaces) is that in all previously constructed M-brane solutions [29, 70, 80, 81, 84], we have at most one parameter in each solution. For example, NUT/Bolt parameter n for embedded transverse Taub-NUT/Bolt spaces, Eguchi-Hanson parameter a in the case of embedded transverse Eguchi-Hanson geometry and a constant number with unit of length that is related to the NUT charge of metric at infinity obtained from Atiyah-Hitchin metric in the case of embedded transverse Atiyah-Hitchin geometry. Moreover, in all the above mentioned solutions, the metric functions depend (at most) only on two non-compact

coordinates. The metric functions in the multi-center Gibbons-Hawking geometries depend (in general) on more physical parameters, hence their embeddings into M-theory yield new results for the metric functions with both non-compact and compact coordinates.

We have obtained several different supersymmetric BPS solutions of interest [88, 89]. We should mention the condition of preserved supersymmetry is distinct from that of BPS which is defined in the bosonic theory. However as we will show in this chapter all solutions preserve some supersymmetry ($\epsilon \neq 0$) hence they are BPS states. Due to the general M2 and M5 ansatze that we consider in this chapter, the metric functions for all M2 solutions, as well as M5 solutions are harmonic. Hence all our brane configurations are determined by solutions of Laplace equations and they obey the BPS property. Specifically, since in the 11 dimensional metric for an M2-brane, the M2-brane itself only takes up two of the 10 spatial coordinates, we can embed a variety of different geometries. These include the double Taub-NUT metric, two-center Eguchi-Hanson metric and products of these 4-dimensional metrics. After compactification on a circle, we find the different fields of type IIA string theory.

In our procedure we begin with a general ansatz for the metric function of an M2-brane in 11-dimensional M-theory. After compactification on a circle (T^1), we find a solution to type IIA theory for which the highest degree of the field strengths is four. Hence the non-compact global symmetry for massless modes is given by the maximal symmetry group $E_{1(1)} = \mathbb{R}$, without any need to dualize the field strengths [90]. For the full type IIA theory, only the discrete subgroup $E_{1(1)}(\mathbb{Z}) = \mathbb{Z}$ survives, in particular by its action on the BPS spectrum and as a discrete set of identifications on the supergravity moduli space.

In the following sections we present in details our new solutions and show that they are indeed satisfy the equations of motion. After that, we use the Killing spinor equations to obtain the number of supersymmetries.
Moreover, we consider the decoupling limits of our new solutions.

As a guidance for the reader, here we present the summary of all possible embedded metrics in M-theory, given in table (5.1).

In the case of embedded Bianchi space, we use a special map as

$$
\begin{aligned}
r = a &\to +\infty, \\
r = +\infty &\to 0,
\end{aligned}
\tag{5.1}
$$

to find the analytical solutions. In the case of embedded 2-center GH space the solutions are exact solutions and we give both series solutions and closed-form solutions. In the case of embedded 3-center GH space, the partial differential equations are hard to be solved exactly, so we use different approximations to find their solutions (as shown in table 5.1). Similar to 3-center case, solutions in k-center case require some approximations.

Table 5.1: Possible metrics, achieved from Gibbons-Hawking (Multi Taub-NUT) and Bianchi spaces

Metrics	$V(r,\theta)$ or other approximations	Range of r	Page
Bianchi space			63
IX	$a_1 = 0,\, a_2 = a_3 = a$	$r \in (a, \infty)$	
2-center			68
$N_1 = 1,\, N_2 = 0$	$V = \epsilon + \frac{n}{r} + \frac{n}{\sqrt{a^2 + r^2 + 2ar\cos\theta}}$	$r \in (0, \infty)$	
k-center			75
$N_1 \neq N_2$	$V \sim \epsilon + \breve{C}_0\,\frac{1}{r} + \breve{C}_1\,\frac{1}{r^2}\cos\theta$	$r > \mathcal{N}a$	
$N_1 \neq N_2$	$V \sim \breve{B}_0 + \breve{B}_1\,\frac{1}{r} + \breve{B}_2\, r\cos\theta$	$r \in (0, a)$	
$N_1 = N_2$	$V \sim \breve{A}_0 + \breve{A}_1\frac{1}{r} + \breve{A}_2(3\cos^2\theta - 1)r^2$	$r \in (0, a)$	
3-center			84
$N_1 = N_2 = 1$	Similar to $(N_1 = N_2)$	$r \in (0, a)$	
-	$\frac{R_1 R_2}{r} \sim \mu$	$r \in (a, \infty)$	

---

[*] ϵ, a, n, a_1, a_2, a_3, $\breve{A}_0$, $\breve{A}_1$, $\breve{A}_2$, $\breve{B}_0$, $\breve{B}_1$, $\breve{B}_2$, $\breve{C}_0$, $\breve{C}_1$ and $\mathcal{N}$ are constants.
[**] k, N_1 and N_2 are the number of NUT charges.

5.1 Bianchi space

The D=11 M2-brane with an embedded transverse Bianchi type IX metric is given by the following metric

$$ds_{11}^2 = H(y,r)^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H(y,r)^{1/3}\left(dy^2 + y^2 d\Omega_3^2 + ds_{\text{(Bianchi IX)}}^2\right),$$
(5.2)

where the Bianchi metric type IX is given by (3.32). The metric (5.2) is a solution to the eleven dimensional supergravity if $H(y,r)$ satisfies the following partial differential equation

$$y\sqrt{A_1 A_2 A_3}\left[2r\frac{\partial^2 H}{\partial r^2} + \left\{6 + r\left(\frac{1}{A_1}\frac{dA_1}{dr} + \frac{1}{A_2}\frac{dA_2}{dr} + \frac{1}{A_3}\frac{dA_3}{dr}\right)\right\}\frac{\partial H}{\partial r}\right] + \left\{2yr\frac{\partial^2 H}{\partial y^2} + 6r\frac{\partial H}{\partial y}\right\} = 0,$$
(5.3)

where $A_i = 1 - \frac{a_i^4}{r^4}$. One can easily solve (5.3) by defining as

$$H(y,r) = 1 + \mathcal{Q}_{M_2}Y(y)R_c(r),$$
(5.4)

which leads to

$$Y(y) = \frac{J_1(\frac{c}{\sqrt{2}}y)}{y},$$
(5.5)

and

$$2rA_1A_2A_3\frac{d^2R_c(r)}{dr^2} + \{6A_1A_2A_3 + r(A_2A_3\frac{dA_1}{dr} + A_3A_1\frac{dA_2}{dr} + A_1A_2\frac{dA_3}{dr})\}\frac{dR_c(r)}{dr}$$
$$- c^2r\sqrt{A_1A_2A_3}R_c(r) = 0,$$

(5.6)

where c is the separation constant. It is unlikely to find exact solutions to (5.6) however we can simplify the problem by considering $a_1 = 0$ and $a_2 = a_3 = a$ which results the Eguchi-Hanson type II metric. Using the later assumption the differential equation (5.6) becomes

$$r(1 - \frac{a^4}{r^4})\frac{d^2R_c(r)}{dr^2} + \left\{3\left(1 - \frac{a^4}{r^4}\right) + 4\frac{a^4}{r^4}\right\}\frac{dR_c(r)}{dr} - \frac{1}{2}c^2rR_c(r) = 0.$$ (5.7)

We introduce the new variable t as

$$r = \frac{a}{\sqrt{\tanh t}},$$ (5.8)

the differential equation (5.7) in terms of the new variable t is given by

$$\frac{d^2R_c(t)}{dt^2} = \frac{a^2c^2}{8}\frac{\cosh(t)}{\sinh^3(t)}R_c(t).$$ (5.9)

For small t

$$\frac{\cosh t}{\sinh^3 t} \sim \frac{1}{t^3} - \frac{1}{15}t + \frac{4}{189}t^3 + O\left(t^5\right),$$ (5.10)

and (5.9) reduces to

$$\frac{d^2R_c(t)}{dt^2} \approx \frac{a^2c^2}{8}\frac{1}{t^3}R_c(t).$$ (5.11)

We find the most general solutions to (5.11) as

$$R_c(r) = \sqrt{\tanh^{-1}\left(\frac{a^2}{r^2}\right)}I_1(1, \frac{ac}{\sqrt{2\tanh^{-1}\left(\frac{a^2}{r^2}\right)}})\Big\{C_1+$$
$$+ C_2\int\frac{rdr}{(r^4 - a^4)\tanh^{-1}\left(\frac{a^2}{r^2}\right)I_1(1, \frac{ac}{\sqrt{2\tanh^{-1}\left(\frac{a^2}{r^2}\right)}})^2}\Big\},$$

(5.12)

where I_1 is the modified Bessel function of the first kind, and C_1, C_2 are two constants. We found that the first term in (5.12) does not meet the boundary conditions at infinity (figure 5.1) and therefore the only acceptable solution vanishing at infinity comes from the second term.
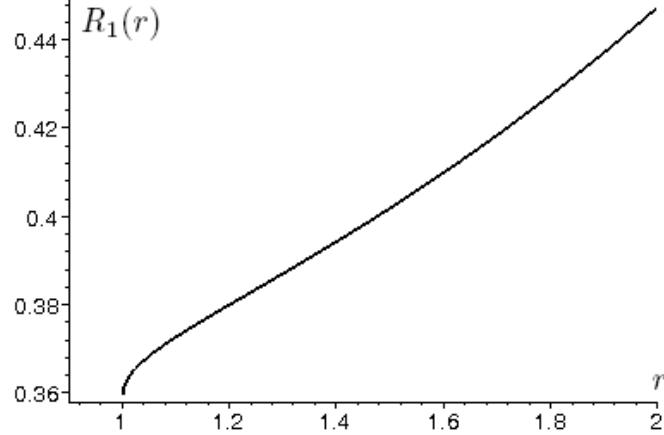
**Figure 5.1:** The first term in (5.12) is divergent as r tends to infinity. We set $a = c = 1$.

Analytically continuing $c \to ic$ yields new solutions for $R_c(r)$ (figure 5.2) and $Y(y)$ as

$$R_c(r) = \sqrt{\tanh^{-1}\left(\frac{a^2}{r^2}\right)} \left[C_1 J_1 \left(\frac{ac}{\sqrt{2\tanh^{-1}\left(\frac{a^2}{r^2}\right)}} \right) + C_2\, Y_1 \left(\frac{ac}{\sqrt{2\tanh^{-1}\left(\frac{a^2}{r^2}\right)}} \right) \right],$$

(5.13a)

and

(5.13b)

$$Y(y) = \frac{K_1\left(\frac{c}{\sqrt{2}}y\right)}{y},$$

(5.13c)

where J_1, Y_1 and K_1 are Bessel functions.
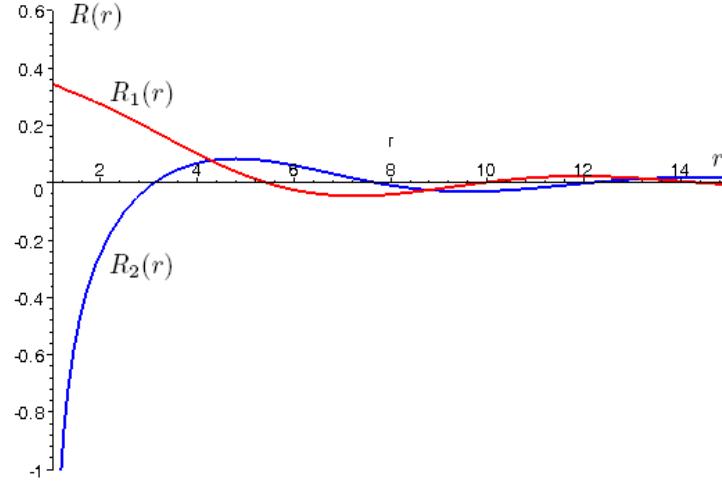
Figure 5.2: Both acceptable solutions in (5.13a) vanish at $r = \infty$.

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In order to give a comparison between the numerical and analytical solutions we set $a = 1$ and $c = 2$. In figures 5.3 and 5.4 we compare the numerical and analytical solutions. Similar comparison can be done by using different boundary conditions at different locations e.g. $r = 1.5$.
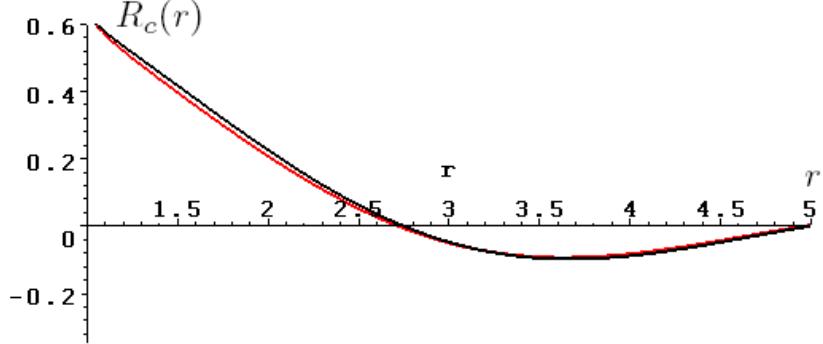
Figure 5.3: Numerical and analytical (J_1) solutions are compared at $r = 1.01$. The black curve shows the solution to differential equation (5.7) while the red curve shows the numerical solutions.
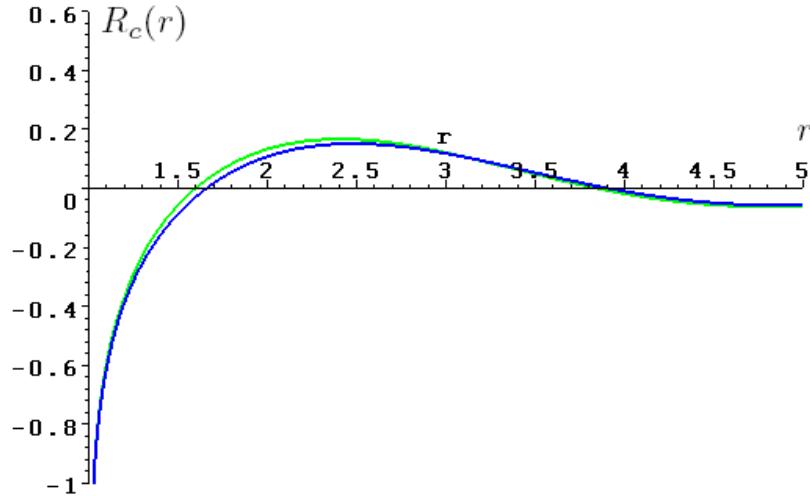
Figure 5.4: Numerical and analytical (Y_1) solutions are compared at $r = 1.01$. The blue curve shows the solution to differential equation (5.7) while the green curve shows the numerical solutions.

By knowing $R_c(r)$ and $Y(y)$, the most general solution for the metric function is a superposition of all possible solutions which takes the following form

$$H(y, r) = 1 + \mathcal{Q}_{M_2} \int_0^\infty R_c(r) Y(y) dc. \tag{5.14}$$

As we will see later in next section both M2-brane and M5-brane solutions with embedded Gibbons-Hawking space, the 11-dimensional equation of motion is separable

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if we set $H = 1 + QY(y)R(r,\theta)$, where Q is a constant related to the charge of the M-branes. In addition we will find that $R(r,\theta)$ satisfies the same partial differential equation in both M2 and M5 brane cases.

## 5.2 Gibbons-Hawking space and solutions for $R(r,\theta)$

We start with the equation of motion for $R(r,\theta)$

$$\frac{2}{r}\frac{\partial R(r,\theta)}{\partial r} + \frac{\partial^2 R(r,\theta)}{\partial r^2} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial R(r,\theta)}{\partial\theta} + \frac{\partial^2 R(r,\theta)}{\partial\theta^2}\right) = c^2 V R(r,\theta), \quad (5.15)$$

where

$$V = \epsilon + \frac{n}{r} + \sum_{k=1}^{k=N_1}\frac{n}{\sqrt{r^2 + (ka)^2 + 2kar\cos\theta}} + \sum_{k=1}^{k=N_2}\frac{n}{\sqrt{r^2 + (ka)^2 - 2kar\cos\theta}}. \quad (5.16)$$

and N_1 and N_2 are the number of NUT-charges along z-direction (figure 5.5).
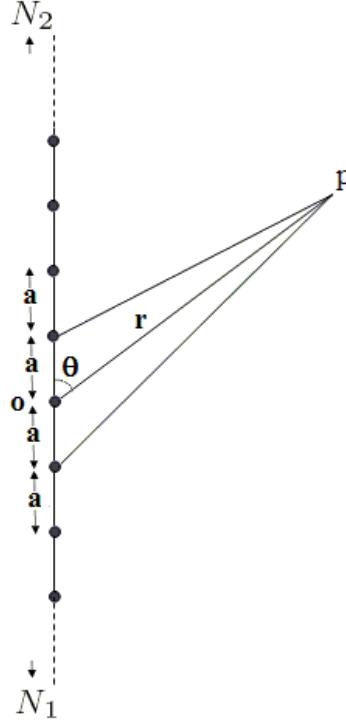
Figure 5.5: The geometry of charges in $k = N_1 + N_2 + 1$-center instanton.

It is easy to see that the differential equation (5.15) strongly depend on V which in turn is determined by N_1 and N_2. Thus choosing different values for N_1 and N_2 lead various solutions. We study the solutions to (5.15) according to the values of N_1 and N_2 as follows:

$N_1 = 1$ **and** $N_2 = 0$ **(2-center)**

In this case V reduces to

$$V = \epsilon + \frac{n_1}{r} + \frac{n_2}{\sqrt{r^2 + a^2 + 2ar\cos\theta}}. \tag{5.17}$$

We change the coordinates (r, θ) to a new pair of coordinates (μ, λ) defined by

$$\mu = r' + r \ , \ \lambda = r' - r, \tag{5.18}$$

where $r' = \sqrt{r^2 + a^2 + 2ar\cos\theta}$ and $\mu \geq a$ and $-a \leq \lambda \leq a$. We notice that the coordinate transformations (5.18) is well defined everywhere except along the z-axis.
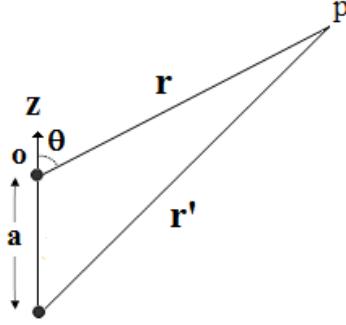
Figure 5.6: The geometry of charges in 2-center instanton.

The differential equation (5.15) in terms of new coordinates becomes

$$-2\lambda\frac{\partial R}{\partial\lambda} + (a^2 - \lambda^2)\frac{\partial^2 R}{\partial\lambda^2} + 2\mu\frac{\partial R}{\partial\mu} + (\mu^2 - a^2)\frac{\partial^2 R}{\partial\mu^2} = c^2\Big[\frac{1}{4}\epsilon(\mu^2 - \lambda^2) + \\ + \frac{1}{2}\mu(n_1 + n_2) + \frac{1}{2}\lambda(n_1 - n_2)\Big]R. \tag{5.19}$$

This equation is separable and yields

$$2\lambda\frac{1}{G}\frac{\partial G}{\partial\lambda} + (\lambda^2 - a^2)\frac{1}{G}\frac{\partial^2 G}{\partial\lambda^2} - \frac{1}{2}c^2(n_1 - n_2)\lambda - \frac{1}{4}\epsilon c^2\lambda^2 - M^2c^2 = 0, \tag{5.20a}$$

$$2\mu\frac{1}{F}\frac{\partial F}{\partial\mu} + (\mu^2 - a^2)\frac{1}{F}\frac{\partial^2 F}{\partial\mu^2} - \frac{1}{2}c^2(n_1 + n_2)\mu - \frac{1}{4}\epsilon c^2\mu^2 - M^2c^2 = 0, \tag{5.20b}$$

upon substituting in $R(\mu, \lambda) = F(\mu)G(\lambda)$ where M is the separation constant. The solution to equation (5.20a) is given by

$$G(\lambda) = \tilde{\mathcal{H}}_C(\lambda)\Big\{\hat{g}_{c,M} + \hat{g}'_{c,M}\int\frac{1}{(a-\lambda)(a+\lambda)\tilde{\mathcal{H}}_C^2(\lambda)}d\lambda\Big\}, \tag{5.21}$$

where $\tilde{\mathcal{H}}_C(\lambda)$ (Appendix D) stands for

$$\tilde{\mathcal{H}}_C(\lambda) =$$
$$e^{\frac{c}{2}\sqrt{\epsilon}(a-\lambda)}\mathcal{H}_C\Big(2ca\sqrt{\epsilon}, 0, 0, ac^2N_-, -\frac{1}{4}(\epsilon a^2 + 2aN_- + 4M^2)c^2, \frac{1}{2}(1 - \frac{\lambda}{a})\Big). \tag{5.22}$$

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In equations (5.21) and (5.22), $N_- = n_2 - n_1$ and $\tilde{g}_{c,M}, \tilde{g}'_{c,M}$ are two constants in λ. The power series expansion of $\tilde{\mathcal{H}}_C(\lambda)$ is

$$\tilde{\mathcal{H}}_C(\lambda) = 1 - (\frac{aN_- c^2}{4} + \frac{M^2 c^2}{2} + \frac{\epsilon a^2 c^2}{8})(1 - \frac{\lambda}{a}) +$$
$$+ (\frac{\epsilon a^2 c^2}{32} - \frac{M^2 c^2}{8} + \frac{\epsilon^2 a^4 c^4}{256} + \frac{\epsilon a^3 c^4 N_-}{64} + \frac{c^4 M^4}{16} + \frac{\epsilon a^2 c^4 M^2}{32} + \frac{ac^4 N_- M^2}{16} +$$
$$+ \frac{a^2 c^4 N_-^2}{64})(1 - \frac{\lambda}{a})^2 + O(\lambda^3). \tag{5.23}$$

Hence we obtain

$$G(\lambda) = \tilde{\mathcal{H}}_C(\lambda)\{g_{c,M} + g'_{c,M} \ln \left|1 - \frac{\lambda}{a}\right|\} + g'_{c,M} \sum_{n=1}^{\infty} d_n (1 - \frac{\lambda}{a})^n, \tag{5.24}$$

where $g_{c,M}, g'_{c,M}$ and d_n's are constants in λ. The first few d_n's are

$$d_1 = \frac{1}{2} + M^2 c^2 + \frac{\epsilon a^2 c^2}{4} + \frac{aN_- c^2}{2}$$
$$d_2 = \frac{M^2 c^2}{8} - \frac{\epsilon a^2 c^2}{32} + \frac{1}{8} - \frac{3\epsilon^2 a^4 c^4}{256} - \frac{3\epsilon a^3 c^4 N_-}{64} -$$
$$- \frac{3c^4 M^4}{16} - \frac{3\epsilon a^2 c^4 M^2}{32} - \frac{3ac^4 N_- M^2}{16} - \frac{3a^2 c^4 N_-^2}{64}. \tag{5.25}$$

The same approach can be used to find the solution to equation (5.20b). We find

$$F(\mu) = \tilde{\mathcal{H}}_C(\mu)\{\hat{f}_{c,M} + \hat{f}'_{c,M} \int \frac{1}{(\mu - a)(a + \mu)\tilde{\mathcal{H}}_C^2(\mu)} d\mu\} \tag{5.26}$$

where $\tilde{\mathcal{H}}_C(\mu)$ stands for

$$\tilde{\mathcal{H}}_C(\mu) =$$
$$e^{\frac{c}{2}\sqrt{\epsilon}(a-\mu)}\mathcal{H}_C(2ca\sqrt{\epsilon}, 0, 0, ac^2 N_+, -\frac{1}{4}(\epsilon a^2 + 2aN_+ + 4M^2)c^2, \frac{1}{2}(1 - \frac{\mu}{a})). \tag{5.27}$$

In equation (5.27), $N_+ = n_1 + n_2$ which yields the power series expansion as

$$\tilde{\mathcal{H}}_C(\mu) = 1 - (\frac{aN_+ c^2}{4} + \frac{M^2 c^2}{2} + \frac{\epsilon a^2 c^2}{8})(1 - \frac{\mu}{a}) +$$
$$+ (\frac{\epsilon a^2 c^2}{32} - \frac{M^2 c^2}{8} + \frac{\epsilon^2 a^4 c^4}{256} + \frac{\epsilon a^3 c^4 N_+}{64} + \frac{c^4 M^4}{16} + \frac{\epsilon a^2 c^4 M^2}{32} + \frac{ac^4 N_+ M^2}{16} +$$
$$+ \frac{a^2 c^4 N_+^2}{64})(1 - \frac{\mu}{a})^2 + O(\mu^3). \tag{5.28}$$

So, we obtain

$$F(\mu) = \tilde{\mathcal{H}}_C(\mu)\{f_{c,M} + f'_{c,M} \ln \left|1 - \frac{\mu}{a}\right|\} + f'_{c,M} \sum_{n=1}^{\infty} b_n (1 - \frac{\mu}{a})^n \tag{5.29}$$

where b_n's are given by (5.25) upon replacing N_- by N_+. We find the most general solution to equation (5.19) (or equivalently to equation (5.15) after coordinate transformations (5.18)) given by

$$R(r,\theta) = \left\{\tilde{\mathcal{H}}_C(\mu)\{f_{c,M} + f'_{c,M}\ln\left|1 - \tfrac{\mu}{a}\right|\}\delta_{a,\mu_0} + f'_{c,M}\sum_{n=0}^{\infty} b_{n,\mu_0}(1 - \tfrac{\mu}{\mu_0})^n\right\} \times$$

$$\times \left\{\tilde{\mathcal{H}}_C(\lambda)\{g_{c,M} + g'_{c,M}\ln\left|1 - \tfrac{\lambda}{a}\right|\}\delta_{a,\lambda_0} + g'_{c,M}\sum_{n=0}^{\infty} d_{n,\lambda_0}(1 - \tfrac{\lambda}{\lambda_0})^n\right\}. \quad (5.30)$$

where $\mu_0 \geq a, |\lambda_0| \leq a$. In (5.30), $d_{0,a} = 0$ and $d_{n>0,a}$ are given by (5.25). The other coefficients are given by

$$\begin{aligned}
b_{0,\mu_0>a} &= 1, \\
b_{1,\mu_0>a} &= -\mu_0, \\
b_{2,\mu_0>a} &= \{-\frac{\mu_0}{(\mu_0^2 - a^2)} + \frac{c^2(\epsilon\mu_0^2 + 4M^2 + 2N_+\mu_0)}{8(\mu_0^2 - a^2)}\}\mu_0^2, \\
b_{3,\mu_0>a} &= \{\frac{c^2(\epsilon\mu_0^3 + 8\mu_0 M^2 + 3N_+\mu_0^2 + N_+a^2 + \epsilon\mu_0 a^2)}{12(\mu_0^2 - a^2)^2} \\
&\quad + \frac{\tilde{A}_1}{24(\mu_0^2 - a^2)^2}\}\mu_0^3,
\end{aligned} \quad (5.31)$$

where

$$\begin{aligned}
\tilde{A}_1 = &- 24\mu_0^2 - 8a^2 - c^2\epsilon\mu_0^4 + c^2\epsilon\mu_0^2 a^2 - 4c^2 M^2\mu_0^2 + 4c^2 M^2 a^2 - 2c^2 N_+\mu_0^3 \\
&+ 2c^2 N_+\mu_0 a^2,
\end{aligned}$$

and

$$\begin{aligned}
d_{0,|\lambda_0|<a} &= 1, \\
d_{1,|\lambda_0|<a} &= -\lambda_0, \\
d_{2,|\lambda_0|<a} &= \{-\frac{c^2(\epsilon\lambda_0^2 + 4M^2 + 2N_-\lambda_0)}{8(a^2 - \lambda_0^2)} + \frac{\lambda_0}{(a^2 - \lambda_0^2)}\}\lambda_0^2, \\
d_{3,|\lambda_0|<a} &= \{\frac{c^2(\epsilon\lambda_0^3 + 8\lambda_0 M^2 + 3N_-\lambda_0^2 + N_-a^2 + \epsilon\lambda_0 a^2)}{12(\lambda_0^2 - a^2)^2} + \\
&\quad + \frac{\tilde{A}_2}{24(a^2 - \lambda_0^2)^2}\}\lambda_0^3,
\end{aligned} \quad (5.32)$$

where

$$\begin{aligned}
\tilde{A}_2 = &-24\lambda_0^2 - 8a^2 - c^2\epsilon\lambda_0^4 + c^2\epsilon\lambda_0^2 a^2 - 4c^2 M^2\lambda_0^2 + 4c^2 M^2 a^2 - 2c^2 N_-\lambda_0^3 + \\
&+ 2c^2 N_-\lambda_0 a^2.
\end{aligned}$$

The recursion relations that we have used to derive the coefficients (5.31) and (5.32), both are in the form of

$$Q_n = \mathcal{Q}_1 Q_{n-1} + \mathcal{Q}_2 Q_{n-2} + \mathcal{Q}_3 Q_{n-3} + \mathcal{Q}_4 Q_{n-4}, \quad (5.33)$$

where $n \geq 2$ and $Q_0 = Q_1 = 1$. Moreover $Q_{n<0} = 0$. The coefficients (5.31) are related to Q's by

$$b_{n,\mu_0>a} = (-\mu_0)^n Q_n, \qquad (5.34)$$

and the functions $\mathcal{Q}$ depend on $\epsilon, \mu_0, n, c, a, M, N_+$. For (5.32), the relation to Q's is

$$d_{n,|\lambda_0|<a} = (-\lambda_0)^n Q_n, \qquad (5.35)$$

where the functions $\mathcal{Q}$ depend on $\epsilon, \mu_0, n, c, a, M, N_-$. In both cases, the radius of convergence is large enough to find the membrane function (5.105) at many intermediate-zone points. As an example, for the choice of $a = \epsilon = M = 1, c = N_+ = 2$ and $\mu_0 = 10.75$, the series is divergent for $0.9906 < \mu < 20.5093$.

In figures 5.8 and 5.7, we plot the slices of the most general solution (5.30) at $\lambda =$const. and $\mu =$const. respectively, for different values of separation constant c.
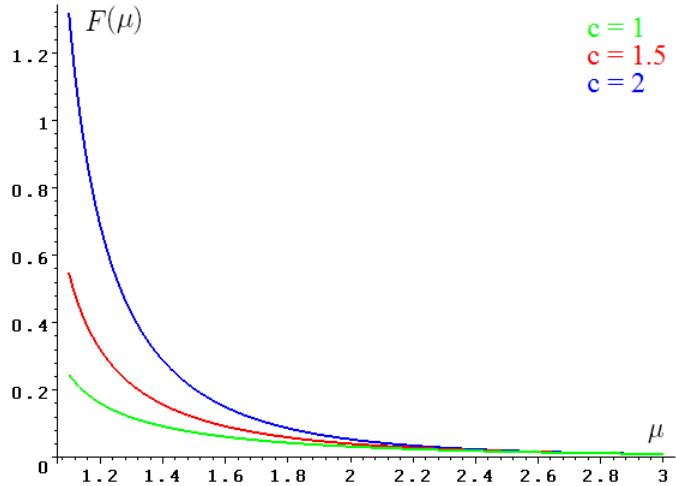
Figure 5.7: The first bracket in (5.30) as a function of $\mu - a = \frac{1}{z}$.
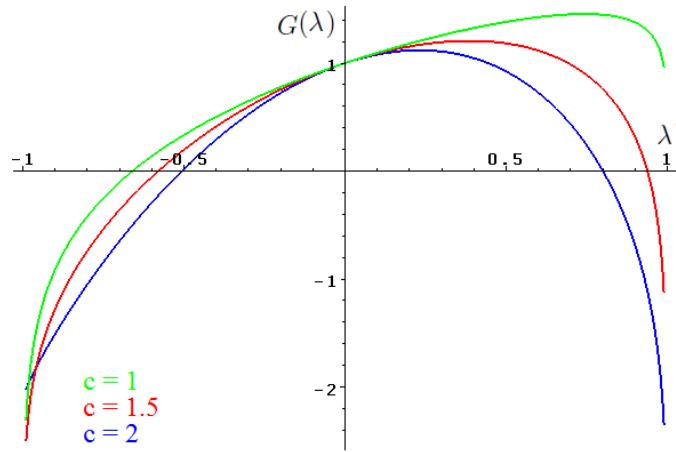
**Figure 5.8:** The second bracket in (5.30) as a function of λ.

Before discussing the closed form solutions of (5.20b), we should note that the second series of solutions (5.20a) and (5.20b) can be obtained upon replacing c (or M) by ic (or iM).

Closed form solutions

By defining a new variable η as

$$\mu = \frac{a}{\tanh(\eta)}, \tag{5.36}$$

one can map μ to η as shown in figure 5.11
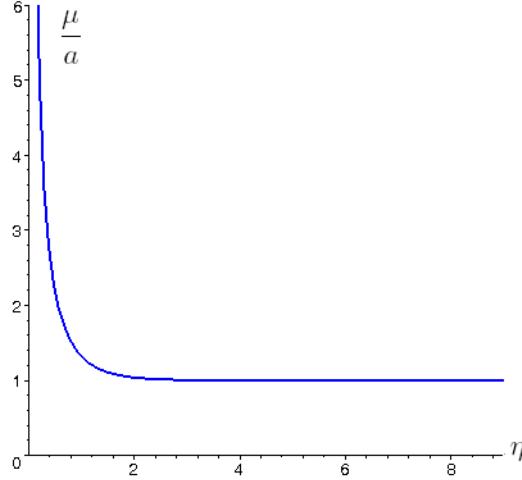
Figure 5.9: According to (5.36), $\eta = +\infty$ and $\eta = 0$ are mapped to $\mu = a$ and $\mu = +\infty$ respectively.

The differential equation (5.20b) in terms of the new variable η turns out to be

$$\frac{d^2 F(\eta)}{d\eta^2} =$$
$$\frac{1}{4} \frac{(\epsilon a^2 + 4 M^2) c^2 \cosh^2(\eta) + 2a(n_1 + n_2) c^2 \sinh(\eta)\cosh(\eta) - 4 M^2 c^2}{\sinh^4(\eta)} F(\eta). \tag{5.37}$$

By considering the series expansion of the right hand side in (5.37) around $\eta = 0$ the new differential equation becomes

$$\frac{d^2 F(\eta)}{d\eta^2} \approx \left\{ \frac{\epsilon a^2 c^2}{4\eta^4} + \frac{c^2 a^2 (n_1 + n_2)}{2\eta^3} + \frac{\epsilon a^2 c^2 + 12 M^2 c^2}{12\eta^2} \right\} F(\eta), \tag{5.38}$$

which has a solution as

$$F(\eta) \sim \eta \mathcal{W}_W \left(-\frac{1}{2} \frac{c(n_1 + n_2)}{\sqrt{\epsilon}}, \frac{\sqrt{1 + (\frac{1}{3}\epsilon a^2 + 4 M^2) c^2}}{2}, \frac{ca\sqrt{\epsilon}}{\eta} \right), \tag{5.39}$$

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where $\mathcal{W}_W$ is the Whittaker function and $\eta = \tanh^{-1}\left(\frac{a}{\mu}\right)$.

In figure 5.10 the analytical solutions to (5.39) are compared to the numerical solutions to (5.20b). For both solutions we have used the same initial conditions. It can be seen from figure 5.10 when $\mu \to a$, we get a tiny difference between numerical and analytical solutions. To remove this difference between numerical and analytical solutions, we consider two solutions, one for $a < \mu < 2a$ and the other for $\mu \geq 2a$ and match them at $\mu = 2a$. So, we get

$$F(\mu) = \Theta(2a - \mu)F_1(\mu) + \Theta(\mu - 2a)F_2(\mu), \tag{5.40}$$

where

$$F_2(\mu) =$$

$$\tanh^{-1}\left(\frac{a}{\mu}\right)\mathcal{W}_W\left(-\frac{1}{2}\frac{c(n_1 + n_2)}{\sqrt{\epsilon}}, \frac{\sqrt{1 + (\frac{1}{3}\epsilon a^2 + 4M^2)c^2}}{2}, \frac{ca\sqrt{\epsilon}}{\tanh^{-1}\left(\frac{a}{\mu}\right)}\right), \tag{5.41}$$

$F_1(\mu)$ given by (5.26) and Θ stands for the Heaviside step function.
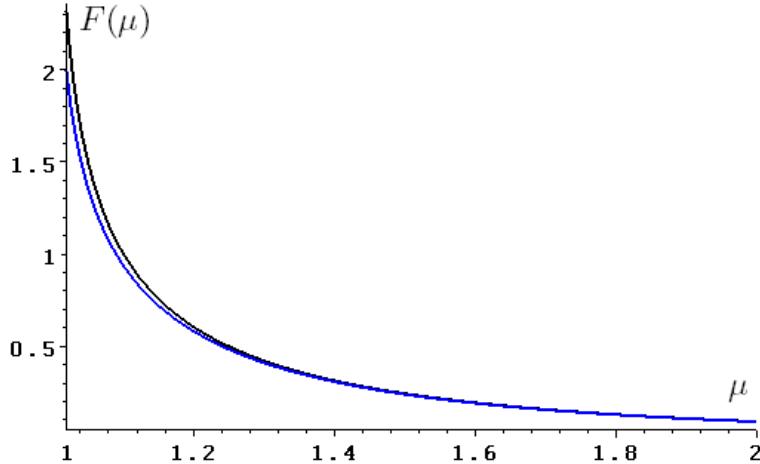
Figure 5.10: A comparison between the numerical solution to (5.20b) and analytical solutions to (5.39). The black curve shows the analytical solution.

We should note by choosing proper values for $\hat{f}_{c,M}$ and $\hat{f}'_{c,M}$ in (5.26) two solutions $F_1(\mu)$ and $F_2(\mu)$ are C^∞ continuous at $\mu = 2a$ (figure 5.11).
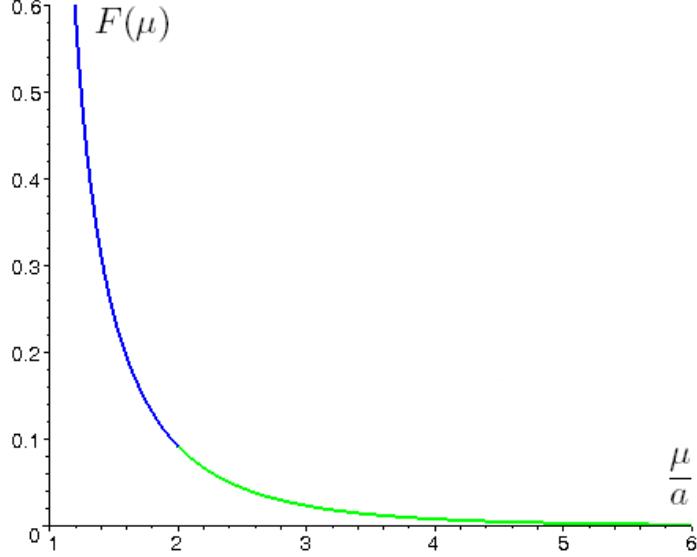
Figure 5.11: The full solution for $F(\mu)$ is made of $F_1(\mu)$ (blue) and $F_2(\mu)$ (green).

The closed form solution for $G(\lambda)$ in general is given by (5.21) however by setting $n_1 = n_2$, an interesting solution can be obtained as follows

$$F(\lambda) = \bar{f}_{cM}\mathcal{H}_C(0, -\frac{1}{2}, 0, -\frac{a^2c^2\epsilon}{16}, \frac{1}{4} - \frac{M^2c^2}{4}, \frac{\lambda^2}{a^2}) + \\ + \bar{f}'_{cM}\mathcal{H}_C(0, \frac{1}{2}, 0, -\frac{a^2c^2\epsilon}{16}, \frac{1}{4} - \frac{M^2c^2}{4}, \frac{\lambda^2}{a^2})\lambda, \tag{5.42}$$

where $\bar{f}_{cM}$ and $\bar{f}'_{cM}$ are constants and the first leading terms in the power series expansion of $\mathcal{H}_C(0, \bar{Q}, 0, -\frac{a^2c^2\epsilon}{16}, \frac{1}{4} - \frac{M^2c^2}{4}, \frac{\lambda^2}{a^2})$ are given by

$$1 + \frac{2\bar{Q} + 1 - M^2c^2}{4a^2(\bar{Q} + 1)}\lambda^2 + \\ + \frac{24\,\bar{Q} + 9 - 10\,M^2c^2 + 12\,\bar{Q}^2 - 8\,\bar{Q}M^2c^2 + M^4c^4 - a^2c^2\epsilon\,\bar{Q} - a^2c^2\epsilon}{32(\bar{Q} + 1)(\bar{Q} + 2)a^4}\lambda^4 + O(\lambda^5), \tag{5.43}$$

where $\bar{Q} = \pm\frac{1}{2}$. We notice that the analytical solution (5.39) in the limit of large r

$$\begin{aligned} r &\gg a, \\ a^2 &\approx 0, \\ \mu &\approx 2r, \end{aligned} \tag{5.44}$$

and using

$$\tanh^{-1}(x) \approx x + O\left(x^3\right), \tag{5.45}$$

becomes

$$F(r) \sim \frac{1}{r}\mathcal{W}_W(-\frac{1}{2}\frac{c\tilde{n}}{\sqrt{\epsilon}}, \frac{\sqrt{1 + 4M^2c^2}}{2}, 2c\sqrt{\epsilon}r), \tag{5.46}$$

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where $\tilde{n} = n_1 + n_2$. This result exactly resembles the solution for a single NUT-charge $\tilde{n}$ embedded in M2-brane [80].

k-center instantons

We try to find solutions to (5.15) in the presence of $k = N_1 + N_2 + 1$ charges (figure 5.5) where the functional form of $V = V(r, \theta)$ is given by (5.16). In general it is unlikely to find exact analytic solutions to (5.15), hence we need to make some approximations. In this section, we find the solutions of (5.15) in region $r > \mathcal{N}a$ where $\mathcal{N} = \max(N_1, N_2)$ and region $r < a$, respectively.

Let us just start with the case $r > \mathcal{N}a$. In region $r > \mathcal{N}a$, $V(r, \theta)$ reduces to

$$V(r, \theta) \approx \epsilon + \frac{n(1 + N_1 + N_2)}{r} + \left[\frac{N_2(N_2 + 1) - N_1(N_1 + 1)}{2} \right] \frac{an \cos \theta}{r^2}, \quad (5.47)$$

where we keep the terms up to the second-order in $1/r$. The separated differential equations after applying (5.47) are

$$r^2 \frac{d^2 f(r)}{dr^2} + 2r \frac{df(r)}{dr} - c^2 (\epsilon r^2 + n(N_1 + N_2 + 1)r + M^2) f(r) = 0, \quad (5.48)$$

$$\frac{d^2 g(\theta)}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{dg(\theta)}{d\theta} + c^2 (M^2 + \tilde{m} \cos \theta) g(\theta) = 0, \quad (5.49)$$

where

$$\tilde{m} = \frac{(N_1(N_1 + 1) - N_2(N_2 + 1))}{2} na, \quad (5.50)$$

the constants c and M are considered as real positive numbers that we call this case as case 1. The solution to equation (5.48) is given by

$$f(r) \sim \frac{1}{r} \mathcal{W}_W \left(-\frac{cn(N_1 + N_2 + 1)}{2\sqrt{\epsilon}}, \frac{\sqrt{1 + 4M^2 c^2}}{2}, 2c\sqrt{\epsilon}r \right), \quad (5.51)$$

where $\mathcal{W}_W$ is the Whittaker function and the solution to equation (5.49) is given by

$$g(\xi) = \mathcal{H}_C \left(0, 0, 0, 2\tilde{m}c^2, -(M^2 + \tilde{m})c^2, \frac{\xi}{2} \right) \left\{ C_{c,M} + \right.$$
$$\left. + C'_{c,M} \int \frac{d\xi}{\xi(\xi - 2) \mathcal{H}_C \left(0, 0, 0, 2\tilde{m}c^2, -(M^2 + \tilde{m})c^2, \frac{\xi}{2} \right)^2} \right\}, \quad (5.52)$$

where $\mathcal{H}_C$ is the Heun-C function (see Appendix D), $\xi = 1 - \cos \theta$ and $C_{c,M}, C'_{c,M}$ are constants. Figure 5.12 shows the behavior of the first and second lines of (5.52) where the constants are set to $a = 1$, $n = 1$, $\tilde{m} = 12$ ($N_1 = 5$ and $N_2 = 2$), $M = 1$, and $c = 1$.
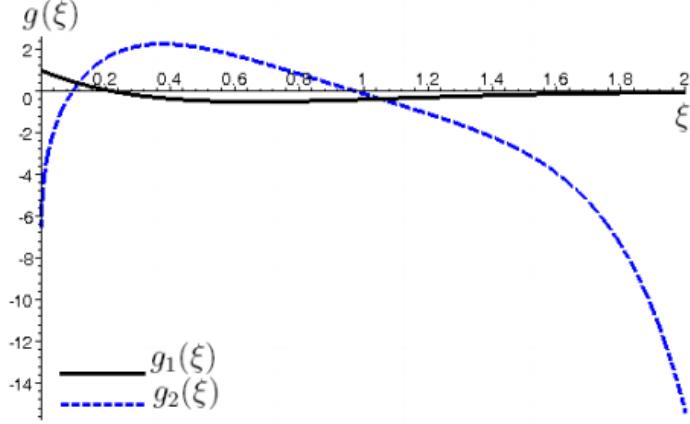
Figure 5.12: The first and second lines of solution (5.52) represented by $g_1(\xi)$ and $g_2(\xi)$, respectively.

As it is shown below, the second line of (5.52) has a logarithmic divergence at $\xi = 1$. In fact the angular function (5.52) has a series expansion around $\xi = 0$, given by

$$
\begin{aligned}
g\left(\xi\right) \equiv \chi(y, M, c) = C_{c,M}&\left[1 - \frac{1}{2}c^2(M^2 + \tilde{m})\xi + \cdots\right] + \\
&+ C'_{c,M}\left[(1 - \frac{1}{2}c^2(M^2 + \tilde{m})\xi + \cdots)\ln(\xi) + (\frac{1}{2} + c^2(M^2 + \tilde{m}))\xi + \cdots\right].
\end{aligned}
\tag{5.53}
$$

The other divergent behavior of $g_2(\xi)$ at $\xi = 2$ (in figure 5.12) could be obtained easily by expansion of (5.52) around $\xi = 2$. Finally, the general solution to (5.15) is given by $R(r, \theta) = f(r)g(\xi)$. For later convenience, we define function $\Im(r, c, M)$ as

$$
\begin{aligned}
\Im(r, c, M) = &f_1 \; r^{-\frac{1}{2} - \frac{\sqrt{1+4M^2c^2}}{2}}\left(1 - \frac{nc^2(N_1 + N_2 + 1)}{-1 + \sqrt{1 + 4M^2c^2}}r + \cdots\right) + \\
&f_2 \; r^{-\frac{1}{2} + \frac{\sqrt{1+4M^2c^2}}{2}}\left(1 + \frac{nc^2(N_1 + N_2 + 1)}{1 + \sqrt{1 + 4M^2c^2}}r + \cdots\right).
\end{aligned}
\tag{5.54}
$$

Case 2:
We can analytically continue $c \to ic$ that yields a new solution. In this case the solutions for $g(y)$ is $\chi(y, M, ic)$ and the radial part has two-vanishing solutions at infinity as follows
1) if $4M^2c^2 \neq 1$

$$
f(r) = \Im(r, ic, M).
\tag{5.55}
$$

2) if $4M^2c^2 = 1$

$$
\begin{aligned}
f(r) = &C_1 \; \frac{1}{\sqrt{r}}\left(1 - nc^2(N_1 + N_2 + 1)r + \cdots\right) + \\
&C_2 \; \frac{1}{\sqrt{r}}\left(ln(r)(1 - nc^2(N_1 + N_2 + 1)r + \cdots) + 2nc^2(N_1 + N_2 + 1)r + \cdots\right).
\end{aligned}
\tag{5.56}
$$

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where C_1 and C_2 are constants. We find the numerical solutions to (5.48) and compare with series solutions given by $\Im(r, ic, M)$ as

$$f_1(r) = \frac{1}{r^{0.9841229182}} \Big(1.0 - 283.4274006\, r + 1235.685153\, r^2 - 1778.059435\, r^3 +$$

$$+ \cdots - 0.1586133408 \times 10^{-9}\, r^{17} + 4.162117735 \times 10^{-12}\, r^{18} \Big),$$

$$(5.57)$$

and

$$f_2(r) = \frac{1}{r^{0.0158770818}} \Big(1.0 - 4.572599538\, r + 6.763825856\, r^2 - 4.729363676\, r^3 +$$

$$+ \cdots - 1.873282837 \times 10^{-14}\, r^{17} - 1.420219368 \times 10^{-15}\, r^{18} \Big).$$

$$(5.58)$$

The constants were set to: $N_1 = 5$, $N_2 = 5$, $M = \frac{1}{8}$, $n = 1$, $c = 1$, $\epsilon = 1$, $a = 1$ and the initial conditions were calculated at $r = 3$. The results, given in figures 5.13 and 5.14 reveal that in order to achieve the exact numerical solution we need to keep higher order terms in (5.57) and (5.58).
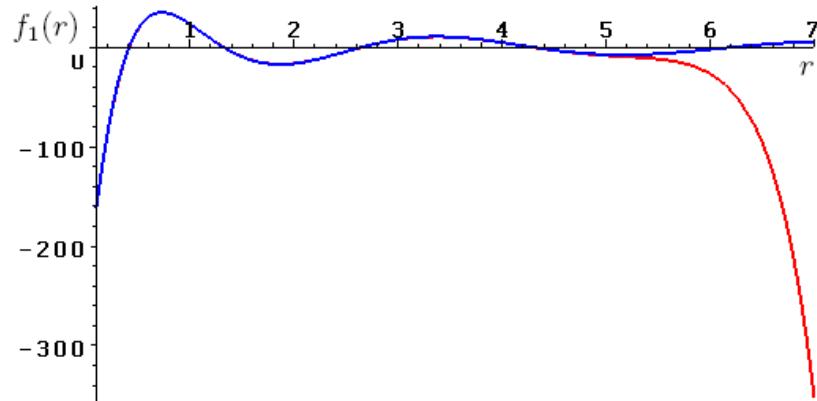
Figure 5.13: The series solution $f_1(r)$ (red) is compared with the numerical solution.
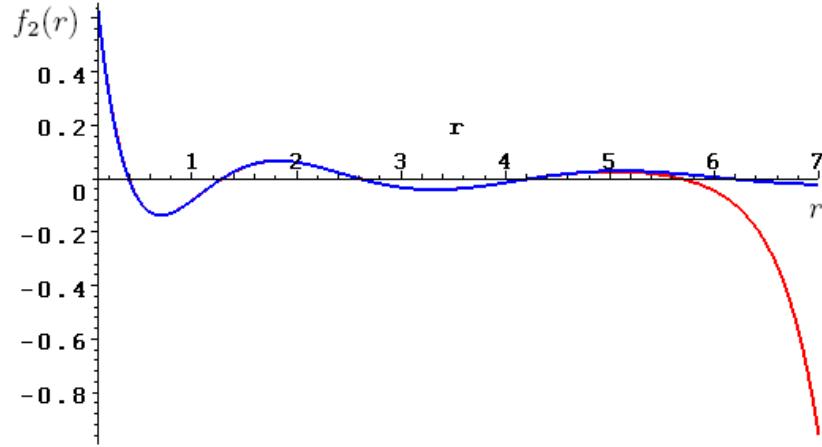
Figure 5.14: The series solution $f_2(r)$ (red) is compared with the numerical solution.

Case 3:
We analytically continue both $c \to ic$ and $M \to iM$. The solutions are

$$f(r) = \Im(r, ic, iM), \tag{5.59}$$
$$g(y) = \chi(r, ic, iM). \tag{5.60}$$

As an example, in figures 5.15 and 5.16 we plot the numerical solution to $g(y)$ where the constants are $n = 1$, $c = 1$, $a = 1$, $M = 0.5$, $N_1 = 5$, and $N_2 = 3$.
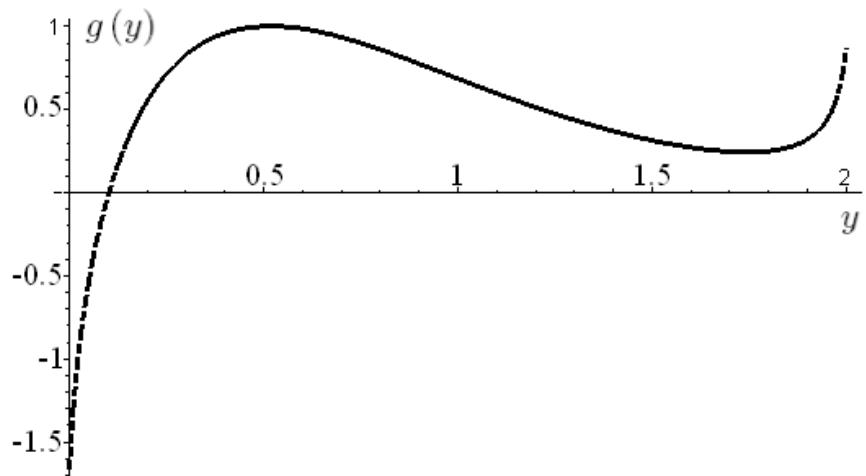
Figure 5.15: The logarithmically divergent part of $g(y)$ at $y = 0$.
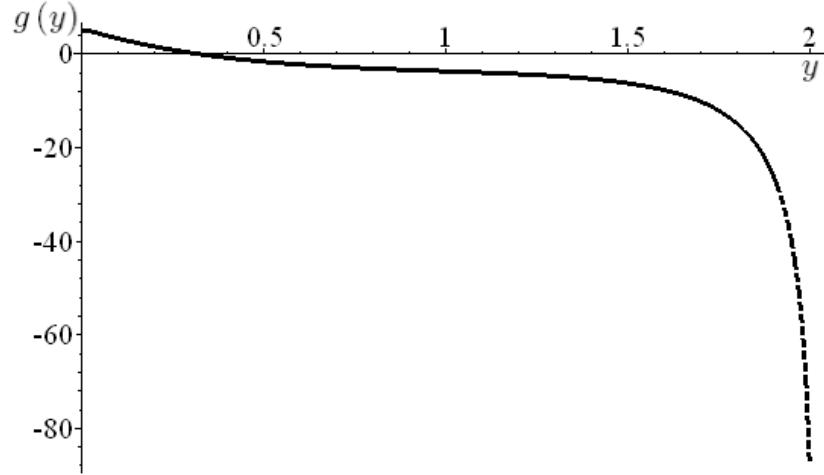
Figure 5.16: The regular part of $g(y)$ at $y = 0$.

Case 4:

Finally, we analytically continue only $M \rightarrow iM$. Similar to case 2, we have $g(y) = \chi(r, c, iM)$ and the radial solution becomes

$$f(r) = \frac{1}{r} \mathcal{W}_W \left(-\frac{cn(N_1 + N_2 + 1)}{2\sqrt{\epsilon}}, \frac{\sqrt{1 - 4M^2 c^2}}{2}, 2c\sqrt{\epsilon}r \right). \qquad (5.61)$$

Finally the total solution $R(r, \theta)$ for each case (1, 2, 3, or 4) is given by

$$R(r, \theta) = f(r)g(y) \mid_{y=1-\cos\theta}. \qquad (5.62)$$

If we let the number of charges be finite and geometrically, they have been located inside a region (figure 5.17) such that the maximum dimension of the region is very smaller than r ($L << r$), then the solution to equation (5.48) becomes

$$R(r, \theta) = \frac{1}{r} \mathcal{W}_W \left(-\frac{cn(N_1 + N_2 + 1)}{2\sqrt{\epsilon}}, \frac{1}{2}, 2c\sqrt{\epsilon}r \right), \qquad (5.63)$$

where unlike the solution (5.62), there is no dependence to angular coordinate θ since $M = 0$. It is straightforward to derive (5.63) from (5.46) by setting $M = 0$.

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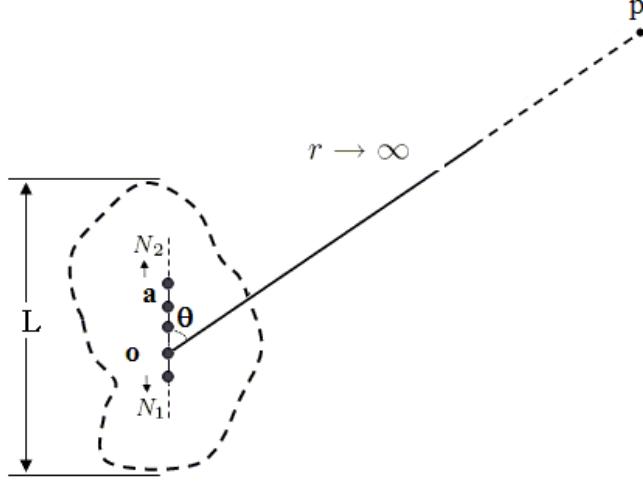
Figure 5.17: At $r \to \infty$, $V \approx \epsilon + \frac{1+N_1+N_2}{r}$.

Now, we present the solutions for M-brane metric functions in near region where $r < a$ (figure 5.18).
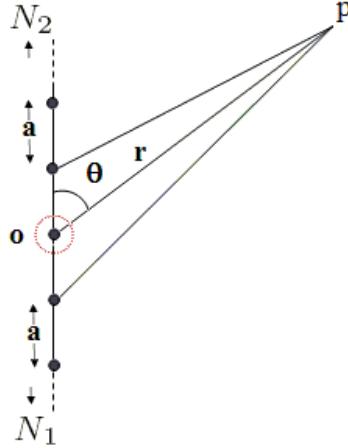
Figure 5.18: The geometry of charges in $k = N_1 + N_2 + 1$-center instanton.

In this region, we notice

$$V \approx \epsilon + \frac{n}{r} + \underbrace{\sum_{k=1}^{N_1} \frac{n}{ka} + \sum_{k=1}^{N_2} \frac{n}{ka}}_{A} + \frac{nr\cos\theta}{a^2} \underbrace{\left[\sum_{k=1}^{N_2} \frac{1}{k^2} - \sum_{k=1}^{N_1} \frac{1}{k^2} \right]}_{B}, \qquad (5.64)$$

and the equation of motion (5.15) becomes

$$\frac{2}{r} \frac{\partial R(r,\theta)}{\partial r} + \frac{\partial^2 R(r,\theta)}{\partial r^2} + \frac{1}{r^2} \left(\frac{\cos\theta}{\sin\theta} \frac{\partial R(r,\theta)}{\partial \theta} + \frac{\partial^2 R(r,\theta)}{\partial \theta^2} \right) =$$
$$c^2 \left(\epsilon + A + \frac{n}{r} + \frac{nBr\cos\theta}{a^2} \right) R(r,\theta), \qquad (5.65)$$

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where we assume $B \neq 0$ ($N_1 \neq N_2$). If $B = 0$, we should consider higher order terms in (5.64) which we will consider it later in this section. We redefine $R(r, \theta)$ as follows

$$R(r, \theta) = e^{\beta \cos \theta} \Psi(r, \theta), \tag{5.66}$$

where $\beta = \frac{naBc^2}{2}$. As we know ($\frac{r}{a} < 1$), so the partial differential equation in terms of $\Psi(r, \theta)$ approximates to be

$$2\,r\frac{\partial \Psi\,(r, \theta)}{\partial r} + r^2\frac{\partial^2 \Psi\,(r, \theta)}{\partial r^2} + \left(\frac{\cos \theta}{\sin \theta} - 2\,\beta\,\sin \theta\right)\frac{\partial \Psi\,(r, \theta)}{\partial \theta} +$$
$$+ \frac{\partial^2 \Psi(r, \theta)}{\partial \theta^2} + \beta^2 \sin^2 \theta \Psi(r, \theta) - 2\,\beta\,\cos \theta \Psi(r, \theta) - c^2 \left((\epsilon + A)r^2 + nr\right) \Psi(r, \theta) = 0. \tag{5.67}$$

The partial differential equation (5.67) separates into

$$r^2\frac{d^2 f\,(r)}{dr^2} + 2\,r\frac{df\,(r)}{dr} - c^2 \left((\epsilon + A)\,r^2 + nr + M^2\right) f\,(r) = 0, \tag{5.68a}$$

$$\frac{d^2 g\,(\theta)}{d\theta^2} + \left(\frac{\cos \theta}{\sin \theta} - 2\,\beta\,\sin \theta\right)\frac{dg\,(\theta)}{d\theta} + \left(M^2 c^2 + (\beta \sin \theta)^2 - 2\,\beta\,\cos \theta\right) g\,(\theta) = 0. \tag{5.68b}$$

Solution to (5.68a) is a Whittaker M function

$$f(r) = \frac{f_0}{r}\mathcal{W}_M\left(-\frac{cn}{2\sqrt{\epsilon + A}}, \frac{1\sqrt{1 + 4M^2 c^2}}{2}, c\sqrt{\epsilon + A}\,\,r\right). \tag{5.69}$$

The solutions to (5.68b), in terms of coordinate $\zeta = \cos \theta$, are given by

$$g(\zeta) = e^{-\beta\zeta}\mathcal{F}(\nu, 1 - \nu, \frac{1}{2}(1 - \zeta))\left\{g_1 + g_2 \int \frac{d\zeta}{(\zeta^2 - 1)\mathcal{F}(\nu, 1 - \nu, 1, \frac{1}{2}(1 - \zeta))^2}\right\}, \tag{5.70}$$

where $\mathcal{F}$ is the hypergeometric function and $\nu = \frac{1}{2} + \frac{\sqrt{1 + 4M^2 c^2}}{2}$. The solution can be expressed in the series forms as

$$g\,(\xi) = C1\left(1 + \frac{2\beta - M^2 c^2}{2}\xi + O(\xi^2)\right) +$$
$$C2\left(\ln(\xi)\left\{1 + \frac{2\beta - M^2 c^2}{2}\xi + O(\xi^2)\right\} + \left\{(\frac{1}{2} + M^2 c^2)\xi + O(\xi^2)\right\}\right), \tag{5.71}$$

where $\xi = 1 - \zeta$.
As we mentioned before, if $N_1 = N_2 = N_0$, we should keep higher order terms in (5.64). Starting from (5.67) and changing the coordinates to

$$x = \cos \theta, \,\, z = \frac{r}{a}, \tag{5.72}$$

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We get

$$z^2\frac{\partial^2 R(z,x)}{\partial z^2} + 2z\frac{\partial R(z,x)}{\partial z} + (1-x^2)\frac{\partial^2 R(z,x)}{\partial x^2} - 2x\frac{\partial R(z,x)}{\partial x}$$
$$-\left[c^2(a^2\epsilon + 2naA_0)z^2 + nac^2z + naB_0c^2z^4(3x^2-1)\right]R(z,x) = 0, \tag{5.73}$$

where $A_0 = \sum_{k=1}^{N_0}\frac{1}{k}$ and $B_0 = \sum_{k=1}^{N_0}\frac{1}{k^2}$. To solve (5.73), we introduce the function $\Omega(x,z)$ as follows

$$R(z,x) = e^{\beta x}\Omega(z,x), \tag{5.74}$$

where $\beta = \sqrt{3naB_0}c$. Hence the differential equation (5.73) in terms of $\Omega(z,x)$ becomes

$$(2\beta - 2x - 2x^2\beta)\frac{\partial\Omega(z,x)}{\partial x} + 2z\frac{\partial\Omega(z,x)}{\partial z} + (1-x^2)\frac{\partial^2\Omega(z,x)}{\partial x^2} + (z^2)\frac{\partial^2\Omega(z,x)}{\partial z^2}$$
$$+ (\beta^2 - 2\beta x - x^2\beta^2)\Omega(z,x) +$$
$$+ \left[nac^2B_0z^4 - nac^2z - (c^2a^2\epsilon + 2c^2naA_0)z^2\right]\Omega(z,x) = 0. \tag{5.75}$$

Separating the variables in $\Omega(z,x)$ by $\Omega(z,x) = \Upsilon(z)\Theta(x)$ and substituting into (5.75), we find two separated second order differential equations for $\Theta(x)$ and $\Upsilon(z)$, as follows

$$(1-x^2)\frac{d^2\Theta(x)}{dx^2} + 2\left[(1-x^2)\beta - x\right]\frac{d\Theta(x)}{dx} - (2x\beta + \beta^2x^2 - M^2c^2 - \beta^2)\Theta(x) = 0, \tag{5.76}$$

$$z^2\frac{d^2\Upsilon(z)}{dz^2} + 2z\frac{d\Upsilon(z)}{dz} + \left[-M^2c^2 + nac^2B_0z^4 - nac^2z - (c^2a^2\epsilon + 2c^2naA_0)z^2\right]\Upsilon(z) = 0. \tag{5.77}$$

The solutions to (5.76) are given by (5.70) as $\Theta(x) = g(\zeta)|_{\zeta=x}$ while the solutions to (5.77) can be written as

$$\Upsilon(z) = z^{-\frac{1+\sqrt{1+4M^2c^2}}{2}}\Upsilon_1(z) + z^{\frac{-1+\sqrt{1+4M^2c^2}}{2}}, \Upsilon_2(z) \tag{5.78}$$

where

$$\Upsilon_1(z) = 1 + \frac{nac^2}{1 - \sqrt{1+4M^2c^2}}z + O(z^2),$$
$$\Upsilon_2(z) = 1 + \frac{nac^2}{1 + \sqrt{1+4M^2c^2}}z + O(z^2), \tag{5.79}$$

and $\Upsilon_i(z)$, $i = 1,2$ are two independent polynomials of z. In figures 5.19 and 5.20 we obtained the numerical solutions to (5.77) and compared with

$$\bar{\Upsilon}_1(z) = \frac{1 - 0.8090169927z - 6.758610452z^2 - 4.361067969z^3 - 4.626809522z^4}{z^{1.618033988}}, \tag{5.80}$$

and

$$\bar{\Upsilon}_2(z) = z^{0.6180339880}\Bigg(1 + 0.3090169945z + 0.5086104635z^2 + $$

$$+0.1110679776z^3 + 0.03590041978z^4\Bigg),$$

(5.81)

where we set $A = \frac{3}{2}$, $B = \frac{5}{4}$, $\epsilon = M = c = a = n = 1$ and 5th-order and higher order terms of z were omitted.
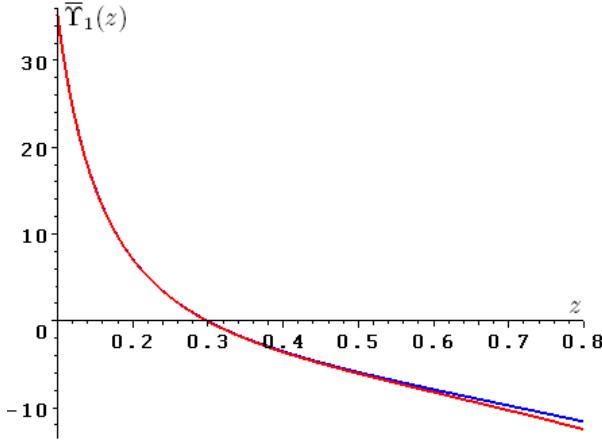
Figure 5.19: The $\bar{\Upsilon}_1(z)$ is compared with the numerical solution (red). The difference between curves is a result of omitting 5th and higher order terms in the series solutions.
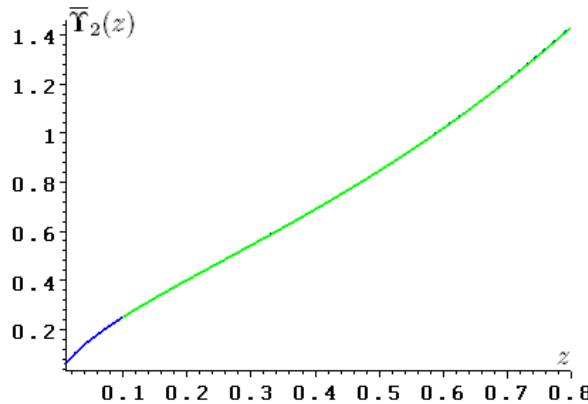
Figure 5.20: The $\bar{\Upsilon}_2(z)$ is compared with the numerical solution. Both solutions are in perfect agreement.

We can analytically continue the near region solutions (as we did for far region) and get new solutions, however we do not consider them here.

$N_1 = 1$ and $N_2 = 1$ (3-center instantons)

The solutions in this case can be divided into two cases where $r < a$ and $r > a$. For $r < a$ we can use the results in the previous section (k-center) by setting $N_1 = N_2 = 1$. To find the solutions to (5.15) over region $r > a$, we define a pair of new independent coordinates μ, λ given by

$$\mu = \frac{R_2 + R_1}{2} = \frac{\sqrt{r^2 + a^2 + 2ar\cos\theta} + \sqrt{r^2 + a^2 - 2ar\cos\theta}}{2}, \quad (5.82)$$

$$\lambda = \frac{R_2 - R_1}{2} = \frac{\sqrt{r^2 + a^2 + 2ar\cos\theta} - \sqrt{r^2 + a^2 - 2ar\cos\theta}}{2}. \quad (5.83)$$

A geometrical interpretation of μ and λ can be obtained using figure 5.21. According to figure 5.21 we can easily show that $|R_2 - R_1| < 2r < (R_1 + R_2)$ and $|R_2 - R_1| < 2a < (R_1 + R_2)$ or in other words $\lambda < r < \mu$ and $\lambda < a < \mu$.
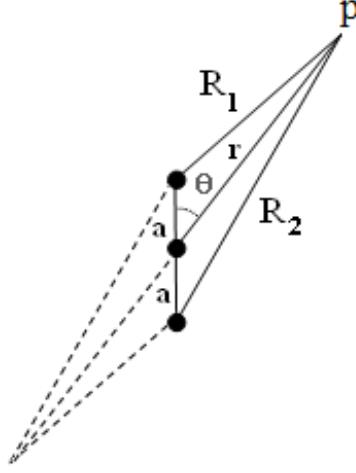
Figure 5.21: The relation between μ, λ and r.

The equation (5.15) turns into

$$(\mu^2 - a^2)\frac{\partial^2 R(\mu,\lambda)}{\partial\mu^2} + 2\mu\frac{\partial R(\mu,\lambda)}{\partial\mu} + (a^2 - \lambda^2)\frac{\partial^2 R(\mu,\lambda)}{\partial\lambda^2} - 2\lambda\frac{\partial R(\mu,\lambda)}{\partial\lambda} =$$
$$c^2 \left[\epsilon(\mu^2 - \lambda^2) + 2\mu n + \frac{n R_2 R_1}{r} \right] R(\mu,\lambda). \quad (5.84)$$

In the absence of cross-term $\frac{n R_2 R_1}{r}$, the equation (5.84) could be easily solved by the method of separation of variables. In region $r > a$, one can show $R_1 \approx r - a\cos\theta$ and $R_2 \approx r + a\cos\theta$. hence we get

$$\frac{R_2 R_1}{r} \approx \mu - \frac{a^2}{r}\cos^2\theta \approx \mu \quad (5.85)$$

So, in terms of new coordinates μ and λ, the equation (5.84) turns into

$$(\mu^2 - a^2)\frac{\partial^2 R(\mu,\lambda)}{\partial\mu^2} + 2\mu\frac{\partial R(\mu,\lambda)}{\partial\mu} + (a^2 - \lambda^2)\frac{\partial^2 R(\mu,\lambda)}{\partial\lambda^2} - 2\lambda\frac{\partial R(\mu,\lambda)}{\partial\lambda} =$$
$$c^2 \left[\epsilon(\mu^2 - \lambda^2) + 3\mu n \right] R(\mu,\lambda). \quad (5.86)$$

This differential equation (5.86) separates into two ordinary second-order differential equations, given by

$$(\mu^2 - a^2)\frac{d^2 G(\mu)}{d\mu^2} + 2\mu\frac{dG(\mu)}{d\mu} - c^2(\epsilon\mu^2 + 3\mu n + M^2)G(\mu) = 0, \quad (5.87)$$

$$(a^2 - \lambda^2)\frac{d^2 F(\lambda)}{d\lambda^2} - 2\lambda\frac{dF(\lambda)}{d\lambda} + c^2(\epsilon\lambda^2 + M^2)F(\lambda) = 0. \quad (5.88)$$

For $\mu \geq 2a$, introducing the new coordinate $0 \leq q \leq \tanh^{-1}(\frac{1}{2})$ related to μ by $\mu = \frac{a}{\tanh(q)}$, the equation (5.87) changes to

$$\frac{d^2 G(q)}{dq^2} - \left(\frac{M^2 c^2}{\sinh^2(q)} + \frac{\beta^2 \cosh(q)}{\sinh^3(q)} + \frac{\alpha^2 \cosh^2(q)}{\sinh^4(q)}\right)G(q) = 0, \quad (5.89)$$

where $\beta^2 = 3nc^2 a$, $\alpha^2 = \epsilon c^2 a^2$. The solutions to (5.89) can be obtained as

$$G_1(q) = g_1 q \mathcal{W}_W\left(-1/2\frac{\beta^2}{\alpha}, \ 1/2\sqrt{1 + 4\gamma^2}, \ 2\frac{\alpha}{q}\right), \quad (5.90)$$

where $\gamma^2 = M^2 c^2 + 1/3\,\alpha^2$ and g_1 is a constant. For $a < \mu \leq 2a$, the solutions to (5.87) become

$$G_2(z) = e^{-ca\sqrt{\epsilon}z}\mathcal{H}_C\left(4\,ca\sqrt{\epsilon}, 0, 0, 6\,c^2 an, -c^2\left(3\,na + M^2 + \epsilon\,a^2\right), -\frac{z}{2}\right) \times$$

$$(1 + g_2 \int \frac{e^{2\,ca\sqrt{\epsilon}z}}{z\,(z+2)\,\mathcal{H}_C\left(4\,ca\sqrt{\epsilon}, 0, 0, 6\,c^2 an, -c^2\left(3\,na + M^2 + \epsilon\,a^2\right), -\frac{z}{2}\right)^2}dz),$$

$$(5.91)$$

where $z = \frac{\mu}{a} - 1$ and g_2 is a constant. We should note that by choosing proper values for g_1 and g_2, two solutions (5.90) and (5.91) are C^∞ continuous at $\mu = 2a$.

For the second differential equation (5.88), the solutions are given by

$$F(\lambda) = f_{cM}\mathcal{H}_C(0, -\frac{1}{2}, 0, -\frac{a^2 c^2 \epsilon}{4}, \frac{1}{4} - \frac{M^2 c^2}{4}, \frac{\lambda^2}{a^2}) +$$

$$+ f'_{cM}\mathcal{H}_C(0, \frac{1}{2}, 0, -\frac{a^2 c^2 \epsilon}{4}, \frac{1}{4} - \frac{M^2 c^2}{4}, \frac{\lambda^2}{a^2})\lambda,$$

$$(5.92)$$

where f_{cM}, and f'_{cM} are constants.

For completeness, we also numerically solve the equation (5.88) and the results are illustrated in figure 5.22.
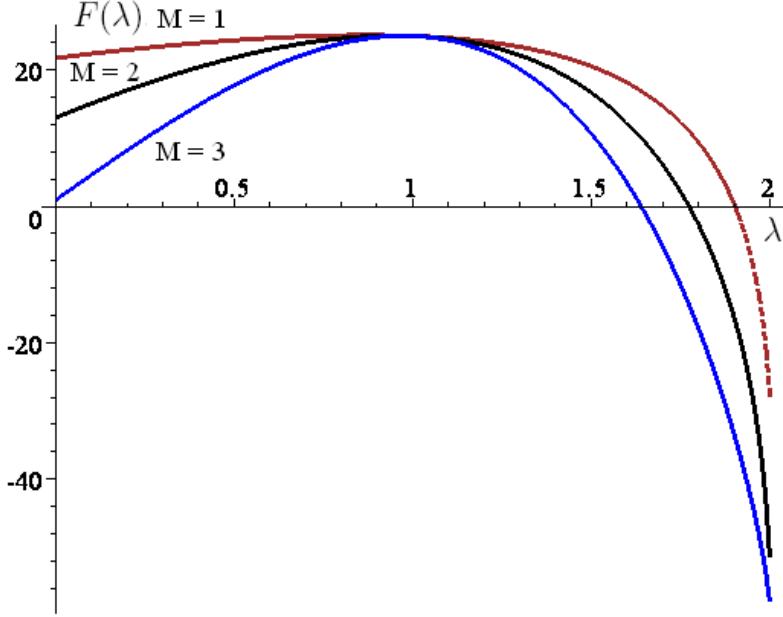
Figure 5.22: Numerical solutions to equation (5.88).

As the final result, the most general solution for the $R(r, \theta)$ in region $r > a$, is given by:

$$R(r, \theta) = G_t(\mu)F(\lambda), \tag{5.93}$$

where $G_t(\mu) = G_1(\tanh^{-1}(\frac{a}{\mu}))\Theta(\frac{\mu}{a} - 2) + G_2(\frac{\mu}{a} - 1)\Theta(2 - \frac{\mu}{a})$ and Θ stands for the Heaviside step function.

In the following sections we discuss the general aspects of M2-brane and M5-brane solutions. Since these solutions depend on three transverse coordinates y, r, and θ and also satisfy various 11-dimensional Laplace equations, we consider each case separately.

5.3 M2-branes with one transverse Gibbons-Hawking space

The 11-dimensional M2-brane metric with an embedded Gibbons-Hawking space is given by

$$ds_{11}^2 = H(y, r, \theta)^{-\frac{2}{3}}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H(y, r, \theta)^{\frac{1}{3}}\left(ds_4^2 + ds_{\text{GH}}^2\right), \tag{5.94}$$

where ds_4^2 a three-sphere (flat space) given by

$$ds_4^2 = dy^2 + y^2\left(d\alpha_1^2 + \sin^2(\alpha_1)(d\alpha_2^2 + \sin^2(\alpha_1)d\alpha_3^2)\right), \tag{5.95}$$

and

$$ds^2_{\text{GH}} = V(r,\theta)\left\{dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right\} + \frac{(d\psi + \omega(r,\theta)d\phi)^2}{V(r,\theta)}. \tag{5.96}$$

The non-vanishing components of the field strength

$$F_{tx_1x_2y} = -\frac{1}{2H^2}\frac{\partial H}{\partial y}, \tag{5.97a}$$

$$F_{tx_1x_2r} = -\frac{1}{2H^2}\frac{\partial H}{\partial r}, \tag{5.97b}$$

$$F_{tx_1x_2\theta} = -\frac{1}{2H^2}\frac{\partial H}{\partial \theta}. \tag{5.97c}$$

The metric (5.94) is a solution to the equations of motions, provided $H(y,r,\theta)$ is a solution to the partial differential equation

$$2ry\sin\theta\frac{\partial H}{\partial r} + y\cos\theta\frac{\partial H}{\partial\theta} + r^2y\sin\theta\frac{\partial^2 H}{\partial r^2} + y\sin\theta\frac{\partial^2 H}{\partial\theta^2} +$$
$$+ \left(r^2y\sin\theta\frac{\partial^2 H}{\partial y^2} + 3r^2\sin\theta\frac{\partial H}{\partial y}\right)V = 0, \tag{5.98}$$

where $V = V(r,\theta)$ and $H = H(y,r,\theta)$. We notice that solutions to harmonic equation (5.98) determine metric function everywhere except at the location of the brane source. We consider the M2-brane is placed at the point $r = 0$, $y = 0$. Substituting

$$H(y,r,\theta) = 1 + \mathcal{Q}_{M_2}Y(y)R(r,\theta), \tag{5.99}$$

where $\mathcal{Q}_{M_2}$ is the charge M2-brane, we obtain two differential equations as follows

$$\frac{d^2Y(y)}{dy^2} + \frac{3}{y}\frac{dY(y)}{dy} + c^2Y(y) = 0, \tag{5.100a}$$

$$\frac{2}{r}\frac{\partial R(r,\theta)}{\partial r} + \frac{\partial^2 R(r,\theta)}{\partial r^2} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial R(r,\theta)}{\partial\theta} + \frac{\partial^2 R(r,\theta)}{\partial\theta^2}\right) = c^2VR(r,\theta). \tag{5.100b}$$

The solution to (5.100a) is

$$Y(y) \sim \frac{J_1(y)}{y}, \tag{5.101}$$

where $J_1(y)$ is the Bessel functions of the first and if we convert c to ic, the solution becomes

$$Y(y) \sim \frac{K_1(y)}{y}, \tag{5.102}$$

where $K_1(y)$ is the modified Bessel function of the first kind.
We note that the general solution of the metric function could be written as a superposition of the solutions with separation constants c and M

$$H(y,r,\theta) = 1 + \mathcal{Q}_{M_2}\int_0^{c_{\max}}\int_0^{M_{\max}}Y(y)R(r,\theta)dcdM, \tag{5.103}$$

where the integration is calculated over two separation constants c, M and $R(r, \theta)$, $Y(y)$ are well known functions. We remind that some solutions of $R(r, \theta)$ contain terms which constrain the value of c and M e.g. having term like $\sqrt{1 - 4M^2 c^2}$ in solutions, implies that $\frac{1}{2M} \gg c$. In decoupling limit as we will see later the coupling can happen if there are no restrictions on the values of c and M or in other words $c, M \in [0, \infty)$ and hence the only acceptable solutions for $H(y, r, \theta)$ are

$$H(y, r, \theta) = 1 + \mathcal{Q}_{M_2} \int_0^\infty \int_0^\infty Y(y) R(r, \theta) dc dM. \tag{5.104}$$

For instance, the general first set of solution (corresponding to embedded Gibbons-Hawking space with $k = 2$ and $\epsilon \neq 0$) is

$$H(y, r, \theta) = 1 + \mathcal{Q}_{M2} \int_0^\infty dc \int_0^\infty dM \frac{J_1(cy)}{y} \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\mu) \{ f_{c,M} + f'_{c,M} \ln \left| 1 - \frac{\mu}{a} \right| \} \delta_{a, \mu_0} + f'_{c,M} \sum_{n=0}^\infty b_{n, \mu_0} (1 - \frac{\mu}{\mu_0})^n \right\} \times \tag{5.105}$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\lambda) \{ g_{c,M} + g'_{c,M} \ln \left| 1 - \frac{\lambda}{a} \right| \} \delta_{a, \lambda_0} + g'_{c,M} \sum_{n=0}^\infty d_{n, \lambda_0} (1 - \frac{\lambda}{\lambda_0})^n \right\}.$$

As we notice, the solution (5.105) depends on four combinations of constants $f_{c,M}, f'_{c,M}$ and $g_{c,M}, g'_{c,M}$ in form of $fg, f'g, fg'$ and $f'g'$ which each combination has dimension of inverse charge (or inverse length to six). Hence, the functional form of each constant could be considered as an expansion of the form $c^{3+2\beta} M^\beta$ where $\beta \in \mathbb{Z}_+$. Moreover we should mention the meaning of μ_0 and λ_0 in equation (5.105) that have dimensions of length. We recall that the near-zone solutions (5.24) and (5.29) are given partly by series expansions around $r \simeq a$. The intermediate-zone solutions are given by similar power series expansions (with substitutions $a \to \lambda_0$ and $d_n \to d_{n, \lambda_0}$ in (5.24) and $a \to \mu_0$ and $b_n \to b_{n, \mu_0}$ in (5.29) around some fixed points, denoted by μ_0 and λ_0. To calculate numerically the membrane metric function (5.105) at any μ, λ (or equivalently any r and θ), we consider some fixed values for μ_0 and λ_0 (5.31 and 5.32).

From D=11 to D=10

The 11D metric and four-form field strength can be easily reduced down to ten dimensions using the following equations

$$g_{mn} = \begin{bmatrix} e^{-2\Phi/3} \left(\bar{g}_{\alpha\beta} + e^{2\Phi} C_\alpha C_\beta \right) & e^{4\Phi/3} C_\alpha \\ e^{4\Phi/3} C_\beta & e^{4\Phi/3} \end{bmatrix}, \tag{5.106}$$

$$F_{(4)} = \mathcal{F}_{(4)} + \mathcal{H}_{(3)} \wedge d\psi, \tag{5.107}$$

where ψ is the eleventh dimension, on which we compactify. The indices $\alpha, \beta, \cdots$ refer to ten-dimensional space-time components after compactification. Reducing

the metric to ten dimensions (5.107) gives the following NSNS fields

$$e^{\frac{4\Phi}{3}} = g_{\psi\psi} \;\;\rightarrow\;\; \Phi = \frac{3}{4} \ln\left\{ \frac{H^{\frac{1}{3}}(y,r,\theta)}{V(r,\theta)} \right\}, \tag{5.108}$$

$$B_{\alpha\beta} = 0, \tag{5.109}$$

and Ramond-Ramond (RR) fields

$$e^{\frac{4\Phi}{3}} C_\alpha = g_{\alpha\psi},$$

$$e^{\frac{4\Phi}{3}} C_\phi = g_{\phi\psi} = e^{\frac{4\Phi}{3}} \omega(r,\theta),$$

and C_ϕ becomes

$$C_\phi = \omega(r,\theta). \tag{5.110}$$

Using $\mathcal{F}_{(4)} = dA$ the non-vanishing components of the three-from gauge field read as

$$A_{tx_1x_2} = \frac{1}{2H(y,r,\theta)}. \tag{5.111}$$

After compactification along ψ direction type IIA supergravity metric can be obtained from

$$\bar{g}_{\alpha\beta} = e^{\frac{2\Phi}{3}} \left(g_{\alpha\beta} - \frac{g_{\alpha\psi}g_{\beta\psi}}{g_{\psi\psi}} \right), \tag{5.112}$$

which gives the following line element

$$\begin{aligned}
ds_{10}^2 \;=\; & H^{-\frac{1}{2}}(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)\left(-dt^2 + dx_1^2 + dx_2^2\right) + \\
& + H^{\frac{1}{2}}(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)\left(dy^2 + y^2 d\Omega_3^2\right) + \\
& + H^{\frac{1}{2}}(y,r,\theta)V^{\frac{1}{2}}(r,\theta)(dr^2 + r^2 d\Omega_2^2),
\end{aligned} \tag{5.113}$$

which describes a localized D2-brane at $y = r = 0$ along the world-volume of D6-brane.

From type IIA to type IIB

Applying the T-duality transformations (Appendix E) on x_1 direction yield the field contents and the metric in the type IIB superstring theory. The line element takes the following form

$$\begin{aligned}
ds_{IIB}^2 \;=\; & H^{-\frac{1}{2}}(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)\left(-dt^2 + H(y,r,\theta)V(r,\theta)dx_1^2 + dx_2^2\right) + \\
& + H^{\frac{1}{2}}(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)\left(dy^2 + y^2 d\Omega_3^2\right) + \\
& + H^{\frac{1}{2}}(y,r,\theta)V^{\frac{1}{2}}(r,\theta)(dr^2 + r^2 d\Omega_2^2).
\end{aligned} \tag{5.114}$$

The background fields are given by

$$\tilde{\Phi} = \frac{1}{2} \ln\left\{ \frac{H(r,y,\theta)}{V(r,\theta)} \right\}, \tag{5.115a}$$

$$\tilde{C}^{(2)}_{\phi x_1} = \omega(r,\theta), \tag{5.115b}$$

$$\tilde{C}^{(4)}_{\alpha\beta\gamma\eta} = 0. \tag{5.115c}$$

5.4 M2-branes with two transverse Gibbons-Hawking spaces

We can also embed two four dimensional Gibbons-Hawking spaces into the eleven dimensional membrane metric. For the sake of simplicity, here we consider the embedding of two double-NUT (or two double-center Eguchi-Hanson) metrics of the form (5.96) with $\epsilon \neq 0$ (or $\epsilon = 0$). The M-brane metric is

$$
\begin{aligned}
ds_{11}^2 = &\, H(y, \alpha, r, \theta)^{-2/3} \left(-dt^2 + dx_1^2 + dx_2^2 \right) + \\
&+ H(y, \alpha, r, \theta)^{1/3} \left(ds_{\mathrm{GH}(1)}^2 + ds_{\mathrm{GH}(2)}^2 \right),
\end{aligned} \tag{5.116}
$$

where $ds_{\mathrm{GH}(i)}$, $i = 1, 2$ are two copies of the metric (5.96) with coordinates (r, θ, ϕ, ψ) and $(y, \alpha, \beta, \gamma)$. The non-vanishing components of four-form field are

$$
F_{tx_1 x_2 x} = -\frac{1}{2H^2} \frac{\partial H(y, \alpha, r, \theta)}{\partial x}, \tag{5.117}
$$

where $x = r, \theta, y, \alpha$. The metric (5.116) and four-form field (5.117) satisfy the eleven dimensional equations of motion if

$$
\begin{aligned}
&2ry \sin(\alpha) \sin(\theta) \{ V(r, \theta) y \frac{\partial H}{\partial r} + V(y, \alpha) r \frac{\partial H}{\partial y} \} + \\
+\ &\sin(\alpha) y^2 \cos(\theta) V(r, \theta) \frac{\partial H}{\partial \theta} + r^2 \sin(\theta) \cos(\alpha) V(y, \alpha) \frac{\partial H}{\partial \alpha} + \\
+\ &r^2 \sin(\alpha) y^2 \sin(\theta) \{ V(r, \theta) \frac{\partial^2 H}{\partial r^2} + V(y, \alpha) \frac{\partial^2 H}{\partial y^2} \} + \\
+\ &\sin(\theta) \sin(\alpha) \{ r^2 V(y, \alpha) \frac{\partial^2 H}{\partial \alpha^2} + y^2 V(r, \theta) \frac{\partial^2 H}{\partial \theta^2} \} = 0, \tag{5.118}
\end{aligned}
$$

where $V(y, \alpha) = \epsilon + \frac{n_3}{y} + \frac{n_4}{\sqrt{y^2 + b^2 + 2by \cos(\alpha)}}$. The equation (5.118) is separable if we set $H(y, \alpha, r, \theta) = 1 + \mathcal{Q}_{M_2} R_1(y, \alpha) R_2(r, \theta)$. This gives two equations

$$
2x_i \frac{\partial R_i}{\partial x_i} + x_i^2 \frac{\partial^2 R_i}{\partial x_i^2} + \frac{\cos y_i}{\sin y_i} \frac{\partial R_i}{\partial y_i} + \frac{\partial^2 R_i}{\partial^2 y_i} = u_i c^2 x_i^2 V(x_i, y_i) R_i, \tag{5.119}
$$

where $(x_1, y_1) = (y, \alpha)$ and $(x_2, y_2) = (r, \theta)$. There is no summation on index i and $u_1 = +1$, $u_2 = -1$, in equation (5.119). We already know the solutions to the two differential equations (5.119) as given in section 5.2, hence the most general solution to (5.118) is

$$
H(y, \alpha, r, \theta) = 1 + \mathcal{Q}_{M_2} \int_0^\infty dc \int_0^\infty dM \int_0^\infty d\tilde{M} R(y, \alpha) \tilde{R}(r, \theta). \tag{5.120}
$$

From D=11 to D=10

We can choose to compactify down to ten dimensions by compactifying on either ψ or γ coordinates. In the first case, we find the type IIA string theory with the NSNS

fields

$$\Phi = \frac{3}{4} \ln \left(\frac{H^{1/3}}{V(r,\theta)} \right), \tag{5.121}$$

$$B_{\mu\nu} = 0, \tag{5.122}$$

and RR fields

$$C_\phi = \omega(r,\theta), \tag{5.123}$$

$$A_{tx_1x_2} = \frac{1}{2H(y,\alpha,r,\theta)}. \tag{5.124}$$

The metric is given by

$$\begin{aligned} ds_{10}^2 &= H(y,\alpha,r,\theta)^{-1/2}V(r,\theta)^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \\ &+ H(y,\alpha,r,\theta)^{1/2}V(r,\theta)^{-1/2}\left(ds_{\text{GH}(1)}^2\right) + \\ &+ H(y,\alpha,r,\theta)^{1/2}V(r,\theta)^{1/2}\left(dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right). \end{aligned} \tag{5.125}$$

In the latter case, the type IIA fields are in the same form as (5.121), (5.122), (5.123), (5.124) and (5.125), just by replacements $(r,\theta,\phi,\psi) \Leftrightarrow (y,\alpha,\beta,\gamma)$. In either cases, we get a fully localized D2/D6 brane system. We can further reduce the metric (5.125) along the γ direction of the first Gibbons-Hawking space. However the result of this compactification is not the same as the reduction of the M-theory solution (5.116) over a torus, which is compactified type IIB theory. The reason is that to get the compactified type IIB theory, we should compactify the T-dual of the IIA metric (5.125) over a circle, and not directly compactify the 10D IIA metric (5.125) along the γ direction. We note also an interesting result in reducing the 11D metric (5.116) along the ψ (or γ) direction of the GH(1) (or GH(2)) in large radial coordinates. As y (or r) $\to \infty$ the transverse geometry in (5.116) locally approaches $\mathbb{R}^3 \otimes S^1 \otimes$ GH(2) (or GH(1)$\otimes \mathbb{R}^3 \otimes S^1$). Hence the reduced theory, obtained by compactification over the circle of the Gibbons-Hawking, is IIA. Then by T-dualization of this theory (on the remaining S^1 of the transverse geometry), we find a type IIB theory.

5.5   M5-brane solutions

The 11-dimensional M5-brane metric with an embedded Gibbons-Hawking metric has the following form

$$\begin{aligned} ds_{11}^2 = H(y,r,\theta)^{-\frac{1}{3}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + \\ +H(y,r,\theta)^{\frac{2}{3}}\left(dy^2 + ds_{\text{GH}}^2\right), \end{aligned} \tag{5.126}$$

with field strength components

$$F_{\psi\phi ry} = \frac{\alpha}{2}\sin(\theta)\frac{\partial H}{\partial\theta}, \tag{5.127a}$$

$$F_{\psi\phi\theta y} = -\frac{\alpha}{2}r^2\sin(\theta)\frac{\partial H}{\partial r}, \tag{5.127b}$$

$$F_{\psi\phi\theta r} = \frac{\alpha}{2}r^2\sin(\theta)V(r,\theta)\frac{\partial H}{\partial y}. \tag{5.127c}$$

$$\tag{5.127d}$$

where $\alpha = +1$ and $\alpha = -1$ correspond to the M5-brane and the anti M5-brane respectively. The metric (5.126) is a solution to the equations of motions, provided $H(y, r, \theta)$ is a solution to the partial differential equation

$$2r\frac{\sin\theta}{V}\frac{\partial H}{\partial r} + \frac{\cos\theta}{V}\frac{\partial H}{\partial\theta} + r^2\sin\theta\frac{\partial^2 H}{\partial y^2} + \frac{\sin\theta}{V}\left\{\frac{\partial^2 H}{\partial\theta^2} + r^2\frac{\partial^2 H}{\partial r^2}\right\} = 0, \tag{5.128}$$

where $V = V(r, \theta)$ and $H = H(y, r, \theta)$. This equation is separable upon substituting

$$H(y, r, \theta) = 1 + \mathcal{Q}_{M_5}Y(y)R(r, \theta), \tag{5.129}$$

where $\mathcal{Q}_{M_5}$ is the charge M5-brane. The separated differential equations become

$$\frac{\partial^2 Y(y)}{\partial y^2} + c^2 Y(y) = 0, \tag{5.130a}$$

$$\frac{2}{r}\frac{\partial R(r,\theta)}{\partial r} + \frac{\partial^2 R(r,\theta)}{\partial r^2} + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial R(r,\theta)}{\partial\theta} + \frac{\partial^2 R(r,\theta)}{\partial\theta^2}\right) = c^2 V R(r,\theta). \tag{5.130b}$$

The solution to (5.130a) is

$$Y(y) \sim \cos(cy + \zeta), \tag{5.131}$$

and converting c to ic gives

$$Y(y) \sim e^{-cy}. \tag{5.132}$$

The same argument for the decoupling limit as we discussed in the M2-brane solutions is valid here. Hence the most general M5-brane function is given by

$$H(y, r, \theta) = 1 + \mathcal{Q}_{M_5}\int_0^\infty\int_0^\infty Y(y)R(r,\theta)dcdM. \tag{5.133}$$

For example corresponding to embedded Gibbons-Hawking space with $k = 2$ and $\epsilon \neq 0$ is given by

$$H(y, r, \theta) = 1 + \mathcal{Q}_{M_5}\int_0^\infty dc\int_0^\infty dM\cos(cy + \varsigma)\times$$

$$\times\left\{\tilde{\mathcal{H}}_C(\mu)\{f_{c,M} + f'_{c,M}\ln\left|1 - \frac{\mu}{a}\right|\}\delta_{a,\mu_0} + f'_{c,M}\sum_{n=0}^\infty b_{n,\mu_0}(1 - \frac{\mu}{\mu_0})^n\right\}\times$$

$$\times\left\{\tilde{\mathcal{H}}_C(\lambda)\{g_{c,M} + g'_{c,M}\ln\left|1 - \frac{\lambda}{a}\right|\}\delta_{a,\lambda_0} + g'_{c,M}\sum_{n=0}^\infty d_{n,\lambda_0}(1 - \frac{\lambda}{\lambda_0})^n\right\}. \tag{5.134}$$

Similar result holds for embedded Gibbons-Hawking space with $k = 2$ and $\epsilon = 0$. The solution (5.134) depends on four combinations of constants in form of $fg, f'g, fg'$ and $f'g'$ which each combination should have dimension of inverse length. Hence, the functional form of each constant could be considered as an expansion of the form $c^{1/2+2\beta}M^{\beta}$ where $\beta \in \mathbb{Z}_{+}$.

From D=11 to D=10

As with M2-brane case, reducing the metric to ten dimensions gives the following NSNS dilaton

$$\Phi = \frac{3}{4}\ln\left\{\frac{H^{2/3}(y,r,\theta)}{V(r,\theta)}\right\}. \tag{5.135}$$

The NSNS field strength of the two-form associated with the NS5-brane, is given by

$$\mathcal{H}_{(3)} = F_{\phi yr\psi}d\phi \wedge dy \wedge dr + F_{\phi y\theta\psi}d\phi \wedge dy \wedge d\theta + F_{\phi r\theta\psi}d\phi \wedge dr \wedge d\theta, \tag{5.136}$$

where the different components of 4-form F, are given by (5.127). The RR fields are

$$C_{\phi} = \omega(r,\theta), \tag{5.137}$$
$$A_{\alpha\beta\gamma} = 0, \tag{5.138}$$

where C_{α} is the field associated with the D6-brane, and the metric in ten dimensions is given by:

$$\begin{aligned}ds_{10}^2 &= V^{-\frac{1}{2}}(r,\theta)\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + H(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)dy^2 + \\ &+ H(y,r,\theta)V^{\frac{1}{2}}(r,\theta)\left(dr^2 + r^2 d\Omega_2^2\right).\end{aligned} \tag{5.139}$$

We can see the above ten dimensional metric is an NS5$\perp$D6 brane solution.

From type IIA to type IIB

Similar to M2-brane and considering the compactification on x_1 direction, the metric and background fields for the type IIB have the following forms, ,

$$\begin{aligned}ds_{10}^2 &= V^{-\frac{1}{2}}(r,\theta)\left(-dt^2 + V(r,\theta)dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + \\ &+ H(y,r,\theta)V^{-\frac{1}{2}}(r,\theta)dy^2 + H(y,r,\theta)V^{\frac{1}{2}}(r,\theta)\left(dr^2 + r^2 d\Omega_2^2\right), \end{aligned} \tag{5.140}$$

and

$$\tilde{\Phi} = \frac{1}{2}\ln\left\{\frac{H(r,y,\theta)}{V(r,\theta)}\right\}, \tag{5.141a}$$
$$\tilde{C}_{\phi x_1}^{(2)} = \omega(r,\theta), \tag{5.141b}$$
$$\tilde{C}_{\alpha\beta\gamma\eta}^{(4)} = 0. \tag{5.141c}$$

5.6 Equations of motion and Killing spinor equation

Both Gibbons-Hawking and Bianchi spaces embedded in M2 and M5-brane solutions fulfill the equations of motion for the gauge field (4.81) and the metric (4.73). They also preserve some supersymmetries which can be obtained from the Killing equation (4.99). In this part we just consider M5-brane solutions and show these solutions meet all requirements in D=11 supergravity and maintain some supersymmetries.

5.6.1 Equations of motion

For a M5-brane solution, the eleven dimensional metric admits the following form

$$ds^2 = H(y, r, \theta)^{-\frac{1}{3}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2) + H(y, r, \theta)^{\frac{2}{3}}(dy^2 + ds_4^2(r, \theta)), \tag{5.142}$$

where ds_4^2 is a four dimensional manifold with the Euclidean signature. We assume ds_4^2 is a multi center Taub-NUT, for example two-center space with two NUT charges (n_1 and n_2) is given by

$$ds_4^2 = V(r, \theta)\left\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right\} + \frac{1}{V(r, \theta)}(d\psi + \omega(r, \theta)d\phi)^2, \tag{5.143}$$

where $V(r, \theta)$ and $\omega(r, \theta)$ are

$$V(r, \theta) = \epsilon + \frac{n_1}{r} + \frac{n_2}{\sqrt{r^2 + a^2 + 2ar\cos\theta}}, \tag{5.144a}$$

$$\omega(r, \theta) = n_1\cos\theta + \frac{n_2(a + r\cos\theta)}{\sqrt{r^2 + a^2 + 2ar\cos\theta}}. \tag{5.144b}$$

The equation of motions according to [91] are given by

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{1}{3}\left[F_{\alpha\gamma_1\gamma_2\gamma_3}F_\beta{}^{\gamma_1\gamma_2\gamma_3} - \frac{1}{8}g_{\alpha\beta}F_{\delta_1\delta_2\delta_3\delta_4}F^{\delta_1\delta_2\delta_3\delta_4}\right], \tag{5.145a}$$

$$\nabla_{\alpha_1}F^{\alpha_1\alpha_2\alpha_3\alpha_4} = -\frac{1}{576}\epsilon^{\delta_1\cdots\delta_4\delta_5\cdots\delta_8\alpha_2\alpha_3\alpha_4}F_{\delta_1\cdots\delta_4}F_{\delta_5\cdots\delta_8}, \tag{5.145b}$$

where $\alpha_1, \gamma_1, \delta_1, \alpha, \beta\cdots$ are 11-dimensional world space indices, $F_{\alpha_1\cdots\alpha_4}$ is the field strength defined by

$$F_{\delta_1\cdots\delta_4} = \frac{\alpha}{2}\epsilon_{\delta_1\cdots\delta_4\delta_5}\partial^{\delta_5}H(y, r, \theta), \tag{5.146}$$

where $\alpha = 1$ corresponds to M5-brane and $\alpha = -1$ corresponds to an anti-M5 brane. So the non-vanishing components of the field strength become

$$F_{\psi\phi ry} = \frac{\alpha}{2}\sin\theta\frac{\partial H(y, r, \theta)}{\partial\theta}, \tag{5.147a}$$

$$F_{\psi\phi\theta y} = -\frac{\alpha}{2}r^2\sin\theta\frac{\partial H(y, r, \theta)}{\partial r}, \tag{5.147b}$$

$$F_{\psi\phi\theta r} = V(r, \theta)\frac{\alpha}{2}r^2\sin\theta\frac{\partial H(y, r, \theta)}{\partial y}. \tag{5.147c}$$

which satisfy the equation of motion for the gauge field (5.145b) as

$$\nabla_{\alpha_1} F^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = -\frac{1}{576} \epsilon^{\delta_1 \cdots \delta_4 \delta_5 \cdots \delta_8 \alpha_2 \alpha_3 \alpha_4} F_{\delta_1 \cdots \delta_4} F_{\delta_5 \cdots \delta_8} = 0. \tag{5.148}$$

As an example $\nabla_{\alpha_1} F^{\alpha_1 \phi r \psi}$ is given by

$$\nabla_{\alpha_1} F^{\alpha_1 \phi r \psi} =$$

$$-\frac{\alpha}{2V r^2 \sin \theta H^{\frac{8}{3}}} \left(\frac{1}{V} \frac{\partial H}{\partial y} \frac{\partial V}{\partial \theta} + \frac{\cos \theta}{\sin \theta} \frac{\partial H}{\partial y} \right) \tag{5.149}$$

$$+\frac{1}{3V H r \sin \theta} \left(\frac{3\alpha}{2r V H^{\frac{5}{3}}} \frac{\partial H}{\partial y} \frac{\partial V}{\partial \theta} + \frac{3\alpha \cos \theta}{2r \sin \theta H^{\frac{5}{3}}} \frac{\partial H}{\partial y} \right) = 0.$$

Now we consider the first equation of motion (5.145a). We define a new-rank two tensor $\hat{G}_{\alpha\beta}$ as follows

$$\hat{G}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - \frac{1}{3} \left[F_{\alpha\gamma\lambda\delta} F_\beta{}^{\gamma\lambda\delta} - \frac{1}{8} g_{\alpha\beta} F_{\delta_1 \delta_2 \delta_3 \delta_4} F^{\delta_1 \delta_2 \delta_3 \delta_4} \right]. \tag{5.150}$$

and for simplicity we show $V(r,\theta) = V$, $\omega(r,\theta) = \omega$ and $H(y,r,\theta) = H$ and obtain $\hat{G}_{\alpha\beta}$. The components of $\hat{G}_{\alpha\beta}$ fall into two categories: components which are equal to zero, for example $\hat{G}_{yr}$ and $\hat{G}_{\theta y}$

$$\hat{G}_{yr} = \frac{1}{2} \frac{1}{H^2} \left(\alpha^2 - 1 \right) \frac{\partial H}{\partial y} \frac{\partial H}{\partial r} \bigg|_{\alpha = \pm 1} = 0, \tag{5.151a}$$

$$\hat{G}_{\theta y} = \frac{1}{2} \frac{1}{H^2} \left(\alpha^2 - 1 \right) \frac{\partial H}{\partial \theta} \frac{\partial H}{\partial y} \bigg|_{\alpha = \pm 1} = 0. \tag{5.151b}$$

The components which are not equal to zero, such as $\hat{G}_{tt}$

$$\hat{G}_{tt} =$$

$$\frac{1}{4H^3 V^3 r^4 \sin^2 \theta} \Bigg\{ -2 \sin^2 \theta \ r^4 V^2 H \frac{\partial^2 H}{\partial r^2} - 2 \sin^2 \theta \ r^4 V^3 H \frac{\partial^2 H}{\partial y^2}$$

$$- 4 \frac{\partial H}{\partial r} r^3 H V^2 \sin^2 \theta + V^2 \left(\frac{\partial H}{\partial \theta} \right)^2 r^2 \sin^2 \theta + V^2 \left(\frac{\partial H}{\partial r} \right)^2 r^4 \sin^2 \theta + V^3 \left(\frac{\partial H}{\partial y} \right)^2 r^4 \sin^2 \theta$$

$$- 2 \sin^2 \theta \ r^2 V^2 H \frac{\partial^2 H}{\partial \theta^2} - 2 \sin \theta \cos \theta \ r^2 V^2 H \frac{\partial H}{\partial \theta} - 2 \sin \theta \cos \theta \ r^2 V H^2 \frac{\partial V}{\partial \theta} - \left(\frac{\partial \omega}{\partial \theta} \right)^2 H^2$$

$$- 2 \sin^2 \theta \ r^2 V H^2 \frac{\partial^2 V}{\partial \theta^2} - 4 \sin^2 \theta \ r^3 V H^2 \frac{\partial V}{\partial r} + \sin^2 \theta \ r^4 H^2 \left(\frac{\partial V}{\partial r} \right)^2 + \sin^2 \theta \ r^2 H^2 \left(\frac{\partial V}{\partial \theta} \right)^2$$

$$- r^2 H^2 \left(\frac{\partial \omega}{\partial r} \right)^2 - 2 \sin^2 \theta \ r^4 V H^2 \frac{\partial^2 V}{\partial r^2} \Bigg\} - \frac{\alpha^2}{4 r^2 V H^3} \left\{ \left(\frac{\partial H}{\partial \theta} \right)^2 + r^2 \left(\frac{\partial H}{\partial r} \right)^2 + r^2 V \left(\frac{\partial H}{\partial y} \right)^2 \right\}. \tag{5.152}$$

Here we claim that this component becomes zero as well if the Laplacian of H vanishes

$$\nabla^2 H = 0 \rightarrow \hat{G}_{tt} = 0. \tag{5.153}$$

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To show this, we need to obtain $\nabla^2 H$. The Laplacian takes the following form

$$\nabla^2 H = 2r \frac{\sin\theta}{V(r,\theta)} \frac{\partial H}{\partial r} + \frac{\cos\theta}{V(r,\theta)} \frac{\partial H}{\partial\theta} + r^2 \sin\theta \frac{\partial^2 H}{\partial y^2} + \frac{\sin\theta}{V(r,\theta)} \left(\frac{\partial^2 H}{\partial\theta^2} + r^2 \frac{\partial^2 H}{\partial r^2} \right) = 0. \tag{5.154}$$

From (5.154) one can simply derive $\frac{\partial^2 H}{\partial y^2}$ which becomes

$$\frac{\partial^2 H}{\partial y^2} = -\frac{2}{Vr} \frac{\partial H}{\partial r} - \frac{\cos\theta}{Vr^2\sin\theta} \frac{\partial H}{\partial\theta} - \frac{1}{V} \frac{\partial^2 H}{\partial r^2} - \frac{1}{Vr^2} \frac{\partial^2 H}{\partial\theta^2}. \tag{5.155}$$

We plug (5.155) into (5.152) and use the functional form of $\omega(r,\theta)$ (5.144b) and $V(r,\theta)$ (5.144a). After simplifying the final result and setting $\alpha = \pm 1$, we get $\hat{G}_{\alpha\beta} = 0$. All non-zero components of $\hat{G}_{tt}$ upon substitution of $\frac{\partial^2 H}{\partial y^2}$ by (5.155) and functional form of $\omega(r,\theta)$ and $V(r,\theta)$ turn out to be zero.

5.6.2 Killing spinor equation

In this section, we explicitly show all our BPS solutions presented in the previous sections preserve $1/4$ of the supersymmetry. This means setting expectation values of all fermions in the theory equal to zero does not destroy completely the supersymmetry. Generically a configuration of n intersecting branes preserves $\frac{1}{2^n}$ of the supersymmetry. In general, the Killing spinors are projected out by product of Gamma matrices with indices tangent to each brane. If all the projections are independent, then $\frac{1}{2^n}$-rule can give the right number of preserved supersymmetries. On the other hand, if the projections are not independent then $\frac{1}{2^n}$-rule can't be trusted. There are some important brane configurations when the number of preserved supersymmetries is more than that by $\frac{1}{2^n}$-rule [78, 92].

As we briefly mentioned in the introduction, the number of non-trivial solutions to the Killing spinor equation

$$\partial_M \varepsilon + \frac{1}{4} \omega_{abM} \Gamma^{ab} \varepsilon + \frac{1}{144} \Gamma_M{}^{npqr} F_{Mpqr} \varepsilon - \frac{1}{18} \Gamma^{pqr} F_{Mpqr} \varepsilon = 0, \tag{5.156}$$

determine the amount of supersymmetry of the solution where the indices $M, N, P, ...$ are eleven dimensional world indices and $a, b, ...$ are eleven dimensional non-coordinate tangent space indices. The connection one-form is given by $\omega^a_b = \Gamma^a_{bc} \hat{\theta}^b$, in terms of Ricci rotation coefficients Γ_{abc} and non-coordinate basis $\hat{\theta}^a = e^a_M dx^M$ where e^M_a are vielbeins. The eleven dimensional M-brane metrics (5.94) and (5.126) are $ds^2 = \eta_{ab} \hat{\theta}^a \otimes \hat{\theta}^b$ in non-coordinate basis. The connection one-form ω^a_b satisfies torsion- and curvature-free Cartan's structure equations

$$d\hat{\theta}^a + \omega^a_b \wedge \hat{\theta}^b = 0, \tag{5.157}$$
$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = 0. \tag{5.158}$$

In (5.156), Γ^a matrices make the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = -2\eta^{ab}. \tag{5.159}$$

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and $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$. Moreover, $\Gamma^{M_1 \ldots M_k} = \Gamma^{[M_1} \ldots \Gamma^{M_n]}$. A representation of the algebra is given in Appendix C.

For our purposes, we use the thirty two dimensional representation of the Clifford algebra (5.159), given by [93]

$$\Gamma_i = \begin{bmatrix} 0 & -\widetilde{\Gamma}_i \\ \widetilde{\Gamma}_i & 0 \end{bmatrix}, \quad (i = 1 \ldots 8), \tag{5.160}$$

$$\Gamma_9 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{5.161}$$

$$\Gamma_\star = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.162}$$

$$\Gamma_0 = -\Gamma_{123456789\star}. \tag{5.163}$$

We note $\Gamma_{0123456789\star} = \epsilon_{0123456789\star} = 1$. For a given Majorana spinor ϵ, its conjugate is given by $\bar{\epsilon} = \epsilon^T \Gamma_0$. Moreover we notice that $\Gamma_0 \Gamma_{a_1 a_2 \cdots a_n}$ is symmetric for $n = 1, 2, 5$ and antisymmetric for $n = 0, 3, 4$. The $\widetilde{\Gamma}_i$'s in (5.160), the sixteen dimensional representation of the Clifford algebra in eight dimensions, are given by [94]

$$\widetilde{\Gamma}_i = \begin{bmatrix} 0 & L_i \\ L_i & 0 \end{bmatrix}, \quad (i = 1 \ldots 7), \tag{5.164}$$

$$\widetilde{\Gamma}_8 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{5.165}$$

in terms of L_i, the left multiplication by the imaginary octonions on the octonions. The imaginary unit octonions satisfy the following relationship

$$o_i \cdot o_j = -\delta_{ij} + c_{ijk} o_k, \tag{5.166}$$

where c_{ijk} is totally skew symmetric and its non-vanishing components are given by

$$c_{124} = c_{137} = c_{156} = c_{235} = c_{267} = c_{346} = c_{457} = 1. \tag{5.167}$$

We take the L_i to be the matrices such that the relation (5.166) holds. In other words, given a vector $v = (v_0, v_i)$ in $\mathbb{R}^8$, we write $\hat{v} = v_0 + v_j o_j$, where the effect of left multiplication is $o_i(\hat{v}) = v_0 o_i - v_i + c_{ijk} v_j o_k$, we then construct the 8×8 matrix $(L_i)_{\xi\zeta}$ by requiring $o_i(\hat{v}) = (L_i)_{\xi\zeta} o_\xi v_\zeta$, where $\xi, \zeta = 0, 1, \ldots 7$. We consider first the M2-brane solutions, for example (5.105). Substituting $\varepsilon = H^{-1/6}\epsilon$ in the Killing spinor equations (5.156) yields solutions that (in what follows in this section, we show the non-coordinate tangent space indices of Γ's by $t, x_1, x_2, \cdots, \phi, \psi$, to simplify the notation)

$$\Gamma^{tx_1 x_2}\epsilon = -\epsilon, \tag{5.168}$$

and so at most half the supersymmetry is preserved due to the presence of the brane. We note that if we multiply all the components of four-form field strength, given in (5.97a),(5.97b) and (5.97c), by -1, then the projection equation (5.168) changes to

$\Gamma^{tx_1x_2}\epsilon = +\epsilon$. The other remaining equations in (5.156), arising from the left-over terms from $\partial_M\epsilon + \frac{1}{4}\omega_{Mab}\Gamma^{ab}\epsilon$ portion, are

$$\partial_{\alpha_1}\epsilon \; - \; \frac{1}{2}\Gamma^{y\alpha_1}\epsilon = 0, \tag{5.169}$$

$$\partial_{\alpha_2}\epsilon \; - \; \frac{1}{2}\left[\sin(\alpha_1)\Gamma^{y\alpha_2} + \cos(\alpha_1)\Gamma^{\alpha_1\alpha_2}\right]\epsilon = 0, \tag{5.170}$$

$$\partial_{\alpha_3}\epsilon \; - \; \frac{1}{2}\left[\sin(\alpha_2)(\sin(\alpha_1)\Gamma^{y\alpha_3} + \cos(\alpha_1)\Gamma^{\alpha_1\alpha_3}) + \cos(\alpha_2)\Gamma^{\alpha_2\alpha_3}\right]\epsilon = 0, \tag{5.171}$$

$$\partial_{\psi}\epsilon \; + \; \frac{1}{4r^2\sin\theta}\left[-V^2(\frac{\partial\omega}{\partial\theta}\Gamma^{\theta\phi} + r\frac{\partial\omega}{\partial r}\Gamma^{r\phi}) + r\sin\theta(\frac{\partial V}{\partial\theta}\Gamma^{\psi\phi} + r\frac{\partial V}{\partial r}\Gamma^{\psi r})\right]\epsilon = 0, \tag{5.172}$$

$$\partial_{\theta}\epsilon \; + \; \frac{1}{4r\sin\theta}\left[-V\frac{\partial\omega}{\partial\theta}\Gamma^{\psi\phi} + \frac{r\sin\theta}{V}(r\frac{\partial V}{\partial r} - 2V)\Gamma^{r\theta}\right]\epsilon = 0, \tag{5.173}$$

$$\partial_{\phi}\epsilon \; + \; \frac{1}{4}\left[\frac{\partial(V\omega)}{\partial r}\Gamma^{\psi r} - \frac{1}{rV\sin\theta}(V^3\omega\frac{\partial\omega}{\partial r} - r^2\sin^2\theta\frac{\partial V}{\partial r} + 2rV\sin^2\theta)\Gamma^{r\phi}\right.$$
$$\left. - \frac{1}{r^2V\sin\theta}(V^3\omega\frac{\partial\omega}{\partial\theta} - r^2\sin^2\theta\frac{\partial V}{\partial\theta} + 2r^2V\sin\theta\cos\theta)\Gamma^{\theta\phi} + \frac{1}{4r}\frac{\partial(V\omega)}{\partial\theta}\Gamma^{\psi\theta}\right]\epsilon = 0. \tag{5.174}$$

We can solve the first three equations, (5.169), (5.170) and (5.171) by using the Lorentz transformation

$$\epsilon = \exp\left\{\frac{\alpha_1}{2}\Gamma^{y\alpha_1}\right\}\exp\left\{\frac{\alpha_2}{2}\Gamma^{\alpha_1\alpha_2}\right\}\exp\left\{\frac{\alpha_3}{2}\Gamma^{\alpha_2\alpha_3}\right\}\eta, \tag{5.175}$$

where η is independent of α_1, α_2 and α_3. To solve equation (5.172), we note that the equation can be written as

$$\partial_{\psi}\eta + \left[f(r,\theta)(\Gamma^{\theta\phi} + \Gamma^{\psi r}) + g(r,\theta)(\Gamma^{r\phi} - \Gamma^{\psi\theta})\right]\eta = 0, \tag{5.176}$$

where

$$f(r,\theta) \;=\; \frac{(r^2 + a^2 + 2ar\cos\theta)^{3/2}n_1 + an_2r^2\cos\theta + n_2r^3}{4(r^2 + a^2 + 2ar\cos\theta)^{1/2}\{(r^2 + a^2 + 2ar\cos\theta)^{1/2}(r + n_1) + n_2r\}^2}, \tag{5.177}$$

$$g(r,\theta) \;=\; \frac{an_2r^2\sin\theta}{4(r^2 + a^2 + 2ar\cos\theta)^{1/2}\{(r^2 + a^2 + 2ar\cos\theta)^{1/2}(r + n_1) + n_2r\}^2}. \tag{5.178}$$

So, the solution to equation (5.176) satisfies

$$\Gamma^{\psi r\theta\phi}\eta = \eta. \tag{5.179}$$

This equation eliminates another half of the supersymmetry provided η is independent of ψ, too. With this projection operator, (5.173) and (5.174) can be solved to give

$$\eta = \exp\left\{-\frac{\theta}{2}\Gamma^{\hat\psi\hat\phi}\right\}\exp\left\{\frac{\phi}{2}\Gamma^{\hat\theta\hat\phi}\right\}\lambda, \tag{5.180}$$

where λ is independent of θ and ϕ. Finally, we conclude due to two projections (5.168) and (5.179), embedding Gibbons-Hawking space in M2 metric preserves 1/4 of supersymmetry.

Next, we consider the M5-brane solutions considered in section 5.5, given by (5.134). Substituting $\varepsilon = H^{-1/12}\epsilon$ in the Killing spinor equations (5.156) yields

$$\Gamma^{tx_1x_2x_3x_4x_5}\epsilon = \epsilon. \tag{5.181}$$

We note that for the anti-M5-brane $\alpha = -1$ in (5.127), the projection equation (5.181) changes to $\Gamma^{tx_1x_2x_3x_4x_5}\epsilon = -\epsilon$. Moreover, we get three equations for ϵ that are given exactly by equations (5.172), (5.173) and (5.174). The solutions to these three equations imply

$$\Gamma^{\psi r\theta\phi}\epsilon = \epsilon, \tag{5.182}$$

and

$$\epsilon = \exp\left\{-\frac{\theta}{2}\Gamma^{\hat\psi\hat\phi}\right\}\exp\left\{\frac{\phi}{2}\Gamma^{\hat\theta\hat\phi}\right\}\xi, \tag{5.183}$$

where ξ is independent of θ and ϕ. So, the two projection operators given by (5.181) and (5.182) show M5-brane solutions preserve 1/4 of supersymmetry.

Finally we consider how much supersymmetry could be preserved by the solutions (5.116) with metric function (5.120), given in section 5.4. As in the case of M2-brane, we get the projection equation

$$\Gamma^{tx_1x_2}\epsilon = -\epsilon, \tag{5.184}$$

that remove half the supersymmetry, after substituting $\varepsilon = H^{-1/6}\epsilon$ into the Killing spinor equations (5.156). The remaining equations could be solved by considering

$$\Gamma^{\psi r\theta\phi}\epsilon = \epsilon, \tag{5.185}$$
$$\Gamma^{\alpha_3 y\alpha_1\alpha_2}\epsilon = \epsilon. \tag{5.186}$$

However, the three projection operators in (5.184),(5.185) and (5.186) are not independent, since their indices altogether cover all the non-coordinate tangent space. Hence, we have only two independent projection operators, meaning 1/4 of the supersymmetry is preserved.

5.7 Decoupling limits of solutions

In this section we consider the decoupling limits for the various solutions we have presented above. The specifics of calculating the decoupling limit are shown in detail elsewhere (see for example [95]), so we will only provide a brief outline here. The process is the same for all cases, so we will also only provide specific examples of a few of the solutions above.

At low energies, the dynamics of the D2-brane decouple from the bulk, with the region close to the D6-brane corresponding to a range of energy scales governed by the IR fixed point [96]. For D2-branes localized on D6-branes, this corresponds in

the field theory to a vanishing mass for the fundamental hyper-multiplets. Near the D2-brane horizon ($H \gg 1$), the field theory limit is given by

$$g_{YM2}^2 = g_s \ell_s^{-1} = \text{fixed}. \tag{5.187}$$

In this limit the gauge couplings in the bulk go to zero, so the dynamics decouple there. In each of our cases above, we scale the coordinates y and r such that

$$Y = \frac{y}{\ell_s^2}, \quad U = \frac{r}{\ell_s^2}, \tag{5.188}$$

are fixed (where Y and U, are used where appropriate). As an example we note that this will change the harmonic function of the D6-brane in the Gibbons-Hawking case to the following (recall that to avoid any conical singularity, we should have $n_1 = n_2 = n$, hence the asymptotic radius of the 11th dimension is $R_\infty = n = g_s \ell_s$)

$$V(U, \theta) = \epsilon + g_{YM2}^2 N_6 \{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU \cos\theta}} \}, \tag{5.189}$$

where we rescale a to $a = A\ell_s^2$ and generalize to the case of N_6 D6-branes. We notice that the metric function $H(y, r, \theta)$ scales as $H(Y, U, \theta) = \ell_s^{-4} h(Y, U, \theta)$ if the coefficients $f_{c,M}, f'_{c,M}, \cdots$ obey some specific scaling. The scaling behavior of $H(Y, U, \theta)$ causes then the D2-brane to warp the ALE region and the asymptotically flat region of the D6-brane geometry. As an example, we calculate $h(Y, U, \theta)$ that corresponds to (5.105). It is given by

$$h(Y, U, \theta) = 32\pi^2 N_2 g_{YM}^2 \int_0^\infty dC \int_0^\infty d\mathcal{M} \frac{J_1(CY)}{Y} \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\Omega, g_{YM}) \{ F_{C,\mathcal{M}} + F'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Omega}{A} \right| \} \delta_{A,\Omega_0} + F'_{C,\mathcal{M}} \sum_{n=0}^\infty b_{n,\Omega_0} (1 - \frac{\Omega}{\Omega_0})^n \right\} \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\Lambda, g_{YM}) \{ G_{C,\mathcal{M}} + G'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Lambda}{A} \right| \} \delta_{A,\Lambda_0} + G'_{C,\mathcal{M}} \sum_{n=0}^\infty d_{n,\Lambda_0} (1 - \frac{\Lambda}{\Lambda_0})^n \right\}. \tag{5.190}$$

where we rescale $c = C/\ell_s^2$ and $M = \mathcal{M}\ell_s^4$. We notice that decoupling demands rescaling of the coefficients $f_{c,M}, f'_{c,M}, \cdots$ in (5.105) by $f_{c,M} = F_{C,\mathcal{M}}/\ell_s^6, f'_{c,M} = F'_{C,\mathcal{M}}/\ell_s^6, \cdots$. In (5.190), $\Omega = \sqrt{U^2 + A^2 + 2AU \cos\theta} + U$ and $\Lambda = \sqrt{U^2 + A^2 + 2AU \cos\theta} - U$ and we use $\ell_p = g_s^{1/3} \ell_s$ to rewrite $\mathcal{Q}_{M2} = 32\pi^2 N_2 \ell_p^6$ in terms of ℓ_s given by $\mathcal{Q}_{M2} = 32\pi^2 N_2 g_{YM2}^4 \ell_s^8$.

The respective ten-dimensional supersymmetric metric (5.113) scales as

$$\begin{aligned} ds_{10}^2 &= \ell_s^2 \{ h^{-1/2}(Y, U, \theta) V^{-1/2}(U, \theta) \left(-dt^2 + dx_1^2 + dx_2^2 \right) + \\ &+ h^{1/2}(Y, U, \theta) V^{-1/2}(U, \theta) \left(dY^2 + Y^2 d\Omega_3^2 \right) + \\ &+ h^{1/2}(Y, U, \theta) V^{1/2}(U, \theta) (dU^2 + U^2 d\Omega_2^2) \}, \end{aligned} \tag{5.191}$$

and so there is only one overall normalization factor of ℓ_s^2 in the metric (5.191). This is the expected result for a solution that is a supergravity dual of a QFT. The other M2-brane and supersymmetric ten-dimensional solutions, given by (5.104), (5.105), (5.120) and (5.125) have qualitatively the same behaviors in decoupling limit.

We now consider an analysis of the decoupling limits of M5-brane solution given by metric function (5.134). At low energies, the dynamics of IIA NS5-branes will decouple from the bulk [96]. Near the NS5-brane horizon ($H >> 1$), we are interested in the behavior of the NS5-branes in the limit where string coupling vanishes

$$g_s \to 0, \tag{5.192}$$

and

$$\ell_s = \text{ fixed.} \tag{5.193}$$

In these limits, we rescale the radial coordinates such that they can be kept fixed

$$Y = \frac{y}{g_s \ell_s^2}, \ U = \frac{r}{g_s \ell_s^2}. \tag{5.194}$$

This causes the harmonic function of the D6-brane for the Gibbons-Hawking solution (5.139), change to

$$V(r,\theta) = \epsilon + \frac{N_6}{\ell_s} \{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU \cos\theta}} \} \equiv V(U,\theta), \tag{5.195}$$

where we generalize to N_6 D6-branes and rescale $a = A\ell_s^2 g_s$.

We can show the harmonic function for the NS5-branes (5.134) rescales according to $H(Y, U, \theta) = g_s^{-2} h(Y, U, \theta)$. In fact, we have

$$H(Y,U,\theta) = \frac{\pi N_5 \ell_s^5}{g_s^2} \int_0^\infty dC \int_0^\infty d\mathcal{M} \cos(CY + \varsigma) \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\Omega, \ell_s) \{ F_{C,\mathcal{M}} + F'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Omega}{A} \right| \} \delta_{A,\Omega_0} + F'_{C,\mathcal{M}} \sum_{n=0}^\infty b_{n,\Omega_0} (1 - \frac{\Omega}{\Omega_0})^n \right\} \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\Lambda, \ell_s) \{ G_{C,\mathcal{M}} + G'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Lambda}{A} \right| \} \delta_{A,\Lambda_0} + G'_{C,\mathcal{M}} \sum_{n=0}^\infty d_{n,\Lambda_0} (1 - \frac{\Lambda}{\Lambda_0})^n \right\}. \tag{5.196}$$

where we use $\ell_p = g_s^{1/3} \ell_s$ to rewrite $\mathcal{Q}_{M5} = \pi N_5 \ell_p^3$ as $\pi N_5 g_s \ell_s^3$. To get (5.196), we rescale $c = C/(g_s \ell_s^2)$, $M = \mathcal{M} g_s^2 \ell_s^4$ and $a = A g_s \ell_s^2$ such that $h(Y, U, \theta)$ doesn't have any g_s dependence. In decoupling limit, the ten-dimensional metric (5.139) becomes,

$$ds_{10}^2 = V^{-1/2}(U,\theta) \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right)$$
$$+ \ell_s^4 \{ h(Y,U,\theta) V^{-1/2}(U,\theta) dY^2 + h(Y,U,\theta) V^{1/2}(U,\theta) \left(dU^2 + U^2 d\Omega_2^2 \right) \}. \tag{5.197}$$

In the limit of vanishing g_s with fixed l_s (as we did in (5.192) and (5.193)), the decoupled free theory on NS5-branes should be a little string theory [97] (i.e.

a 6-dimensional non-gravitational theory in which modes on the 5-brane interact amongst themselves, decoupled from the bulk). We note that our NS5/D6 system is obtained from M5-branes by compactification on a circle of self-dual transverse geometry. Hence the IIA solution has T-duality with respect to this circle. The little string theory inherits the same T-duality from IIA string theory, since taking the limit of vanishing string coupling commutes with T-duality. Moreover T-duality exists even for toroidally compactified little string theory. In this case, the duality is given by an $O(d, d, \mathbb{Z})$ symmetry where d is the dimension of the compactified toroid. These are indications that the little string theory is non-local at the energy scale l_s^{-1} and in particular in the compactified theory, the energy-momentum tensor can't be defined uniquely [98].

As the last case, we consider the analysis of the decoupling limits of the IIB solution that can be obtained by T-dualizing the compactified M5-brane solution (5.126). The type IIA NS5$\perp$ D6(5) configuration is given by the metric (5.139) and fields (5.135), (5.136), (5.137) and (5.138). The metric (5.140) describes a IIB NS5$\perp$D5(4) brane configuration (along with the dualized dilaton, NSNS and RR fields). At low energies, the dynamics of IIB NS5-branes will decouple from the bulk. Near the NS5-brane horizon ($H >> 1$), the field theory limit is given by

$$g_{YM5} = \ell_s = \text{ fixed,} \tag{5.198}$$

We rescale the radial coordinates y and r as in (5.194), such that their corresponding rescaled coordinates Y and U are kept fixed. The harmonic function of the D5-brane is

$$V(r, \theta) = \epsilon + \frac{N_5}{g_{YM5}} \{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU\cos\theta}} \}, \tag{5.199}$$

where N_5 is the number of D5-branes. The harmonic function of the NS5$\perp$D5 system (5.140), rescales according to $H(Y, U, \theta) = g_s^{-2} h(Y, U, \theta)$, where

$$h(Y, U, \theta) = \pi N_5 g_{YM5}^5 \int_0^\infty dC \int_0^\infty d\mathcal{M} \cos(CY + \zeta) \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\mu, g_{YM5}) \{ F_{C,\mathcal{M}} + F'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Omega}{A} \right| \} \delta_{A, \Omega_0} + F'_{C,\mathcal{M}} \sum_{n=0}^\infty b_{n, \Omega_0} (1 - \frac{\Omega}{\Omega_0})^n \right\} \times$$

$$\times \left\{ \tilde{\mathcal{H}}_C(\lambda, g_{YM5}) \{ G_{C,\mathcal{M}} G'_{C,\mathcal{M}} \ln \left| 1 - \frac{\Lambda}{A} \right| \} \delta_{A, \Lambda_0} + G'_{C,\mathcal{M}} \sum_{n=0}^\infty d_{n, \Lambda_0} (1 - \frac{\Lambda}{\Lambda_0})^n \right\}. \tag{5.200}$$

In this case, the ten-dimensional metric (5.140), in the decoupling limit, becomes

$$\widetilde{ds}_{10}^2 = V^{-1/2}(U, \theta) \left(-dt^2 + V(U, \theta)dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) +$$

$$+ g_{YM5}^2 h(Y, U, \theta) \{ V^{-1/2}(U, \theta)dY^2 + + V^{1/2}(U, \theta) \left(dU^2 + U^2 d\Omega_2^2 \right) \}. \tag{5.201}$$

The decoupling limit illustrates that the decoupled theory in the low energy limit is super Yang-Mills theory with $g_{YM} = \ell_s$. In the limit of vanishing g_s with fixed l_s, the

decoupled free theory on IIB NS5-branes (which is equivalent to the limit $g_s \to \infty$ of decoupled S-dual of the IIB D5-branes) reduces to a IIB (1,1) little string theory with eight supersymmetries.

Chapter 6

Summary

The central thrust of this thesis is the explicit and exact construction of supergravity solutions for fully localized D2/D6 and NS5/D6 brane intersections without restricting to the near core region of the D6-branes [88, 89]. Unlike all the other known solutions, the novel feature of these solutions is the dependence of the metric function to three (and four) transverse coordinates. These solutions are new M2 and M5 brane metrics that are presented in chapter 5 which are the main results of this work. The common feature of all of these solutions is that the brane function is a convolution of an decaying function with a damped oscillating one. The metric functions vanish far from the M2 and M5-branes and diverge near the brane cores. Which means the field strength diverges near the brane cores. This is an expected result due to the brane electric or magnetic charge.

Dimensional reduction of the M2 solutions to ten dimensions gives us intersecting IIA D2/D6 configurations that preserve 1/4 of the supersymmetry. For the M5 solutions, dimensional reduction yields IIA NS5/D6 brane systems overlapping in five directions. The latter solutions also preserve 1/4 of the supersymmetry and in both cases the reduction yields metrics with acceptable asymptotic behaviors.

We considered the decoupling limit of our solutions and found that D2 and NS5-branes can decouple from the bulk, upon imposing proper scaling on some of the coefficients in the integrands.

In the case of M2-brane solutions; when the D2-brane decouples from the bulk, the theory on the brane is 3 dimensional $\mathcal{N} = 4$ $SU(N_2)$ super Yang-Mills (with eight supersymmetries) coupled to N_6 massless hypermultiplets [99]. This point is obtained from dual field theory and since our solutions preserve the same amount of supersymmetry, a similar dual field description should be attainable.

In the case of M5-brane solutions; the resulting theory on the NS5-brane in the limit of vanishing string coupling with fixed string length is a little string theory. In the standard case, the system of N_5 NS5-branes located at N_6 D6-branes can be obtained by dimensional reduction of $N_5 N_6$ coinciding images of M5-branes in the flat transverse geometry. In this case, the world-volume theory (the little string theory) of the IIA NS5-branes, in the absence of D6-branes, is a non-local non-gravitational six dimensional theory [100]. This theory has (2,0) supersymmetry (four supercharges in the **4** representation of Lorentz symmetry $Spin(5,1)$) and an R-symmetry $Spin(4)$ remnant of the original ten dimensional Lorentz symmetry. The presence of the D6-branes breaks the supersymmetry down to (1,0), with eight supersymmetries. Since

we found that some of our solutions preserve 1/4 of supersymmetry, we expect that the theory on NS5-branes is a new little string theory. By T-dualization of the 10D IIA theory along a direction parallel to the world-volume of the IIA NS5, we find a IIB NS5⊥D5(4) system, overlapping in four directions. The world-volume theory of the IIB NS5-branes, in the absence of the D5-branes, is a little string theory with (1,1) supersymmetry. The presence of the D5-brane, which has one transverse direction relative to NS5 world-volume, breaks the supersymmetry down to eight supersymmetries. This is in good agreement with the number of supersymmetries in 10D IIB theory: T-duality preserves the number of original IIA supersymmetries, which is eight. Moreover we conclude that the new IIA and IIB little string theories are T-dual: the actual six dimensional T-duality is the remnant of the original 10D T-duality after toroidal compactification.

A useful application of the exact M-brane solutions in this thesis is to employ them as supergravity duals of the NS5 world-volume theories with matter coming from the extra branes. More specifically, these solutions can be used to compute some correlation functions and spectrum of fields of our new little string theories.

In the standard case of A_{k-1} (2,0) little string theory, there is an eleven dimensional holographic dual space obtained by taking appropriate small g_s limit of an M-theory background corresponding to M5-branes with a transverse circle and k units of 4-form flux on $S^3 \otimes S^1$. In this case, the supergravity approximation is valid for the (2,0) little string theories at large k and at energies well below the string scale. The two point function of the energy-momentum tensor of the little string theory can be computed from classical action of the supergravity evaluated on the classical field solutions [97].

Near the boundary of the above mentioned M-theory background, the string coupling goes to zero and the curvatures are small. Hence it is possible to compute the spectrum of fields exactly. In [98], the full spectrum of chiral fields in the little string theories was computed and the results are exactly the same as the spectrum of the chiral fields in the low energy limit of the little string theories. Moreover, the holographic dual theories can be used for computation of some of the states in our little string theories.

We conclude with a few comments about possible directions for future work. Investigation of the different regions of the metric (5.126) or alternatively the 10D string frame metric (5.140) with a dilaton for small and large Higgs expectation value U would be interesting, as it could provide a means for finding a holographical dual relation to the new little string theory we obtained. Moreover, the Penrose limit of the near-horizon geometry may be useful for extracting information about the high energy spectrum of the dual little string theory [101]. The other open issue is the possibility of the construction of a pp-wave spacetime which interpolates between the different regions of the our new IIA NS5-branes.

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Appendix A

The Development of string theory

Table A.1: Historical development of string theory

1921	Kaluza-Klein (unification of gravitation and electromagnetism)
1970	String theory (the official birth of string theory)
1971	Supersymmetry
1974	Gravitons
1976	Supergravity (supersymmetry was added to gravity, making supergravity)
1980	Superstrings (string theory plus supersymmetry)
1991-1995	Duality Revolution (using a set of dualities to relate various superstring theories)
1996	Black Hole Entropy

A brief history about the development of string theory can be found in [102].

APPENDIX B

DIFFERENTIAL FORMS

In this appendix we provide a very brief summary of differential forms. Let us just start with the definition of a p-form. A p-form C is given by

$$C_{(p)} = \frac{1}{p!} C_{a_1 \cdots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}, \tag{B.1}$$

and the Hodge dual of the basis in D dimensions is defined as follows

$$\star dx^{a_1} \wedge \cdots \wedge dx^{a_p} = \frac{1}{(D-p)!} \epsilon_{b_1 \cdots b_{D-p}}{}^{a_1 \cdots a_p} dx^{b_1} \wedge \cdots \wedge dx^{b_{D-p}}, \tag{B.2}$$

where

$$\begin{aligned} \epsilon_{a_1 \cdots a_D} &= \sqrt{-g}\, \varepsilon_{a_1 \cdots a_D} \\ \epsilon^{a_1 \cdots a_D} &= \frac{1}{\sqrt{-g}} \varepsilon^{a_1 \cdots a_D} \end{aligned} \tag{B.3}$$

and

$$\varepsilon^{a_1 \cdots a_D} = -\varepsilon_{a_1 \cdots a_D}, \ \varepsilon_{a_1 \cdots a_D} = \begin{cases} 1 & \text{even permutation,} \\ -1 & \text{odd permutation,} \\ 0 & \text{equal indices.} \end{cases}$$

If we apply the Hodge dual to $C_{(p)}$ we get

$$\star C_{(p)} = \frac{1}{p!} C_{a_1 \cdots a_p} \star dx^{a_1} \wedge \cdots \wedge dx^{a_p}. \tag{B.4}$$

For instance by acting the Hodgge dual to 1 we obtain

$$\star 1 = \sqrt{-g}\, dx^1 dx^2 \cdots dx^D. \tag{B.5}$$

If we assume A and B are any two p-forms, then $\star A \wedge B = \star B \wedge A$ is given by

$$\star A \wedge B = \frac{1}{p!} A_{a_1 \cdots a_p} B^{a_1 \cdots a_p} \star 1. \tag{B.6}$$

APPENDIX C

INTRODUCTION TO CLIFFORD ALGEBRA

This section provides a summary of Clifford algebra and also explains how this algebra is graded over $Z_2 = \{0, 1\}$ [104].

C.0.1 Z_2 - Graded algebra

In order to define Clifford algerba[1] in R^2 first we start with ordinary vectors. Let $\vec{a} \in R^2$ then the scalar product of $\vec{a}$ with itself gives the magnitude of the vector which is a non-negative real number and is equal to

$$\vec{a}.\vec{a} = (a_x\hat{e}_x + a_y\hat{e}_y).(a_x\hat{e}_x + a_y\hat{e}_y) = a_x^2 + a_y^2 \tag{C.1}$$

where $\hat{e}_i.\hat{e}_j = \delta_{ij}{}^2$.

Taking the same vector and using a new rule for multiplication of $\vec{a}$ with itself gives

$$\vec{a}\vec{a} = (a_x\hat{e}_x + a_y\hat{e}_y).(a_x\hat{e}_x + a_y\hat{e}_y) = a_x^2\hat{e}_x^2 + a_y^2\hat{e}_y^2 + a_xa_y\hat{e}_x\hat{e}_y + a_ya_x\hat{e}_y\hat{e}_x \tag{C.2}$$

Employing the main assumption in Cl_2 which says that $\vec{a}.\vec{a} = \vec{a}\vec{a}$ and considering (C.2) results

$$\hat{e}_y^2 = \hat{e}_x^2 = 1 \ , \ \hat{e}_x\hat{e}_y + \hat{e}_y\hat{e}_x = 0 \tag{C.3}$$

or

$$\{\hat{e}_i, \hat{e}_j\} = 2\delta_{ij} \ , \ i,j = 1,2 \tag{C.4}$$

Considering $\vec{a}, \vec{b} \in Cl_2$, and using (table C.1) $\vec{a}\vec{b}$ becomes [3]

$$\vec{a}\vec{b} = (a_x\hat{e}_x + a_y\hat{e}_y)(b_x\hat{e}_x + b_y\hat{e}_y) = a_xb_x + a_yb_y + (a_xb_y - a_yb_x)\hat{e}_x\hat{e}_y \tag{C.5}$$

$$\vec{a}\vec{b} = \vec{a}.\vec{b} + (a_xb_y - a_yb_x)\hat{e}_x\hat{e}_y \tag{C.6}$$

where $\hat{e}_x\hat{e}_y = -\hat{e}_y\hat{e}_x$.

Table C.1: Multiplication table for Cl_2

	e_1	e_2	e_{12}
e_1	1	e_{12}	e_2
e_2	$-e_{12}$	1	$-e_1$
e_{12}	$-e_2$	e_1	-1

In general any object belongs to Cl_2 (say q)[4] is usually made of three parts known

---

[1]Shown by Cl_2

[2]$x \equiv 1$ and $y \equiv 2$

[3]Keeping in mind that Cl_2 is a direct sum of R, R^2, and $\wedge^2 R$ ($Cl_2 = R \oplus R^2 \oplus \wedge^2 R$)

[4]No arrow is used

as, Scalars ($\in R$), Vectors ($\in R^2$), and Bivectors ($\in \wedge^2 R$) hence, q shall take the following form

$$q = \underbrace{a_0}_{\text{Scalar}} + \underbrace{\vec{p}}_{\text{Vector}} + \underbrace{\mathbf{B}}_{\text{Bivector}} \tag{C.7}$$

where $\mathbf{B} = c_0 \hat{e}_{12}$.

As it can be seen from (C.7) the Clifford algebra Cl_2 contains R, R^2, and $\wedge^2 R$ and also the basis vectors fall into two categories

- The even part $Cl_2{}^+$ $\{1, e_{12}\}$.

- The odd part $Cl_2{}^-$ $\{e_1, e_2\}$.

One can verify that $Cl_2 = Cl_2{}^+ \oplus Cl_2{}^-$ and subsequently show that

$$Cl_2{}^+ Cl_2{}^+ \subset Cl_2{}^+$$
$$Cl_2{}^- Cl_2{}^+ \subset Cl_2{}^-$$
$$Cl_2{}^+ Cl_2{}^- \subset Cl_2{}^-$$
$$Cl_2{}^- Cl_2{}^- \subset Cl_2{}^+$$

By writing $Cl_2{}^+ = (Cl_2)_0$ and $Cl_2{}^- = (Cl_2)_1$, we are able to reduce all above equations to a simple term given by

$$(Cl_2)_i (Cl_2)_j \subset (Cl_2)_{i+j} \tag{C.8}$$

Letting $i, j \in Z_2$ it can be shown that (C.8) follows the $Z2$-multiplication table (C.2)

Table C.2: Z_2-multiplication table

$\oplus$	0	1
0	0	1
1	1	0

C.0.2 Clifford algebra in D(1,10)

In the previous section the Clifford algebra was introduced in R^2. Starting from (C.4) and changing $\delta_{ij}(\delta^{ij})$ to $\eta_{ij}(\eta^{ij})$, and $\hat{e}$ to Γ a representation of the Clifford algebra in higher dimensions is given by

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbf{1} \tag{C.9}$$

where $a, b = 0, 1...n$ and $\eta^{ab} = (-1, 1, ..., 1)$.

Taking a proper combination of Pauli matrices, a representation of eleven dimensional Clifford algebra by real matrices Γ^a can be obtained as follows [56]

$$\Gamma^0 = -i\sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \qquad \Gamma^1 = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \otimes \sigma_1$$
$$\Gamma^2 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \qquad \Gamma^3 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_3 \otimes i\sigma_2 \otimes \sigma_1$$
$$\Gamma^4 = i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1 \qquad \Gamma^5 = i\sigma_2 \otimes \sigma_3 \otimes i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1$$
$$\Gamma^6 = i\sigma_2 \otimes i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1 \otimes \sigma_1 \qquad \Gamma^7 = i\sigma_2 \otimes i\sigma_2 \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_1$$
$$\Gamma^8 = i\sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes i\sigma_2 \qquad \Gamma^9 = \sigma_1 \otimes \mathbf{1}_{16}$$
$$\Gamma^{10} = \sigma_3 \otimes \mathbf{1}_{16}$$

where $\mathbf{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

APPENDIX D

THE HEUN-C FUNCTIONS

The Heun-C function $\mathcal{H}_C(\alpha, \beta, \gamma, \delta, \lambda, z)$ is the solution to the confluent Heun's differential equation [105]

$$\mathcal{H}_C'' + (\alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1})\mathcal{H}_C' + (\frac{\mu}{z} + \frac{\nu}{z - 1})\mathcal{H}_C = 0, \qquad (\text{D.1})$$

where $\mu = \frac{\alpha - \beta - \gamma + \alpha\beta - \beta\gamma}{2} - \lambda$ and $\nu = \frac{\alpha + \beta + \gamma + \alpha\beta + \beta\gamma}{2} + \delta + \lambda$. The equation (D.1) has two regular singular points at $z = 0$ and $z = 1$ and one irregular singularity at $z = \infty$. The $\mathcal{H}_C$ function is regular around the regular singular point $z = 0$ and is given by $\mathcal{H}_C = \Sigma_{n=0}^{\infty} h_n(\alpha, \beta, \gamma, \delta, \lambda)z^n$, where $h_0 = 1$. The series is convergent on the unit disk $|z| < 1$ and the coefficients h_n are determined by the recurrence relation

$$h_n = \Theta_n h_{n-1} + \Phi_n h_{n-2}, \qquad (\text{D.2})$$

where we set $h_{-1} = 0$ and

$$\Theta_n = \frac{2n(n-1) + (1-2n)(\alpha - \beta - \gamma) + 2\lambda - \alpha\beta + \beta\gamma}{2n(n + \beta)}, \qquad (\text{D.3})$$

$$\Phi_n = \frac{\alpha(\beta + \gamma + 2(n-1)) + 2\delta}{2n(n + \beta)}. \qquad (\text{D.4})$$

Appendix E

From type IIA to IIB

Buscher's rules for T-duality

T-duality along the compact direction z relates IIA and IIB theories. The relations between background fields and merics are given by [103]

$$
\tilde{G}_{zz} = \frac{1}{G_{zz}}
$$

$$
\tilde{G}_{\alpha\beta} = G_{\alpha\beta} - \frac{G_{\alpha z}G_{\beta z} - B_{\alpha z}B_{\beta z}}{G_{zz}} \tag{E.1}
$$

$$
\tilde{G}_{\alpha z} = \frac{B_{\alpha z}}{G_{zz}}
$$

,

$$
\tilde{\Phi} = \Phi - \frac{1}{2}\ln(G_{zz})
$$

$$
\tilde{B}_{\alpha z} = \frac{G_{\alpha z}}{G_{zz}} \tag{E.2}
$$

$$
\tilde{B}_{\alpha\beta} = B_{\alpha\beta} - \frac{B_{\alpha z}G_{\beta z} - G_{\alpha z}B_{\beta z}}{G_{zz}}
$$

and for the gauge fields

$$
\tilde{C}^{(2n)}_{\alpha\cdots\beta\gamma z} = C^{(2n-1)}_{\alpha\cdots\beta\gamma} - (2n-1)\frac{C^{(2n-1)}_{[\alpha\cdots\beta|z}G_{|\gamma]z}}{G_{zz}}
$$

$$
\tilde{C}^{(2n)}_{\alpha\cdots\beta\gamma\eta} = C^{(2n+1)}_{\alpha\cdots\beta\gamma\eta} + 2nC^{(2n-1)}_{[\alpha\cdots\beta\gamma}\ B_{|\eta]z} + 2n(2n-1)\frac{C^{(2n-1)}_{[\alpha\cdots\beta|z}B_{|\gamma|z}G_{|\eta]z}}{G_{zz}}. \tag{E.3}
$$

where the fields in type IIB are shown by $\tilde{g}_{\alpha\beta}, \tilde{B}_{\alpha\beta}\cdots$.