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Entanglement in High Dimensional Quantum Systems

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Résumé

La détection de l'intrication est une étape indispensable dans le contexte de l'information et du calcul quantique. Cette tâche importante est difficile pour les systèmes quantiques de dimensions supérieures à 2×3. En effet, contrairement aux cas 2×3 et 2×2 où il existe des conditions nécessaires et suffisantes bien établies comme le critère de Peres-Horodecki, pour le cas de dimensions plus grandes, il n'existe que des conditions suffisantes à l'intrication. Par grandes dimensions, on entend soit un système bi-partite où chacune des parties a une dimension supérieure à 3, ou un système multipartite composé d'un grand nombre de particules.

On s'intéresse à développer de nouveaux critères pour détecter l'intrication dans de tels systèmes en utilisant des grandeurs mesurables sans avoir besoin de faire une tomographie complète de l'état quantique considéré. Notre approche consiste à réduire la dimension du problème, pour avoir un système plus simple à étudier. Pour ce faire, on transforme localement chaque sous-système en un qubit sans créer de l'intrication.
Nous montrons que cette transformation est caractérisée par les valeurs moyennes de trois opérateurs arbitraires, prises dans l'état quantique du système. Nous donnons, dans le second chapitre, des conditions nécessaires et suffisantes pour que cette transformation soit valide du point de vue physique, fournissant ainsi un outil simple et puissant pour le problème de la réduction des dimensions.

Nous exploitons ce formalisme pour dériver des critères d'intrication pour des systèmes bipartites ou multipartites qui découlent des critères existants pour les qubits. En transformant localement chaque sous-système dans le paradigme des LOCC sans communications classiques, tel que la séparabilité est conservée, l'application de critères d'intrication connus pour les qubits induit automatiquement des critères d'intrication en fonction des 3 opérateurs utilisés pour réaliser la transformation. Nous prenons l'exemple où les observables choisis sont les 3 composantes du moment angulaire et on obtient

ainsi un critère simple dont on étudie la performance en le comparant au critère de Peres-Horodecki pour une famille d'états de qudits.

Dans le troisième chapitre, on s'intéresse aux critères d'intrication, pour des systèmes multipartites, basés sur des mesures d'observables collectives. Après avoir transformé l'état du système en un état multipartite de qubits, on applique les inégalités de compression de spin pour des qubits. Cependant, lorsqu'on applique notre formalisme à ce cas, il est possible d'obtenir une superposition cohérente d'états avec des nombres de particules différentes. Par conséquent, nous avons dû prendre en compte les fluctuations quantiques et/ou classiques que l'opérateur du nombre de particules peut présenter. Nous avons obtenu une forme généralisée des inégalités de compression de spin pour un nombre de particules fluctuant et pour des observables collectifs arbitraires. Nous avons appliqué nos résultats à un système d'atomes de chrome ultrafroids piégés dans un réseau optique, en collaboration avec l'équipe Gazes Dipolaires Quantiques du laboratoire LPL de l'université Paris 13. Nous avons montré, à l'aide d'une simulation numérique, que nos inégalités généralisées sont capables de détecter l'intrication à l'aide d'opérateurs collectifs mesurables en utilisant des techniques accessibles dans ce type de dispositif.

Dans le quatrième chapitre, Nous généralisons le formalisme du second chapitre pour transformer un système bi-partite en deux qubits. Nous donnons une caractérisation mathématique pour les opérations LOCC avec un tour de communication classique en fonction des observables locaux des deux parties. Nous donnons aussi une stratégie pour mesurer les quantités nécessaires pour les inégalités de compression de spin généralisé dans des expériences semblables à celles mentionné dans le troisième chapitre. Finalement, on généralise un critère de compression de spin capable de détecter la profondeur d'intrication d'un ensemble de spins dont le nombre fluctue, pour des observables arbitraires.

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General introduction and layout

When reading almost any paper on the subject of quantum information, quantum communication and many other areas involving quantum theory, one can skip the first couple of lines because it would be something like, "[...] entanglement lies at the heart of quantum [...]". This is not lack of creativity but rather people acknowledging the importance of quantum entanglement.
Indeed, entanglement and coherence are the defining features of quantum mechanics. They open up a plethora of applications not allowed for in classical mechanics. From the early days these weird features, allowing for things like Schrodinger's cat or EPR paradox, were debated but nowadays have become an experimental reality and a basis for numerous applications.
Since entanglement is an important resource, it needs to be measured and quantified. A lot of effort has been spent in order to develop criteria able to certify its presence in a given state. However, beyond bipartite systems of dimensions 2×3, there are no known operational necessary and sufficient conditions for separability, rendering entanglement characterization quite hard. It is, therefore, a necessity to develop criteria for high dimensional systems that do not require full tomography nor excessive number of measurements which is the main focus of this thesis.

In chapter 1, I start by a small introduction to motivate the importance of entanglement in different areas and the need to characterize it, or better quantify it, for fundamental and experimental applications. Then, I move on to recall very briefly some of the most important tools to infer entanglement in the bipartite case, like reduction and PPT criteria, to end with entanglement witnesses and entanglement measures. For the multipartite case, a special attention is drawn to spin squeezing inequalities which are of big importance for systems with a very large number of particles. I first recall the concept of squeezing in bosonic systems and then introduce the different known squeezing

parameters for spins including spin squeezing inequalities.

Chapter 2 is concerned with the problem of entanglement detection in high dimensional bipartite systems. Our main idea is to perform a dimensionality reduction of the bipartite system by mapping it into a two-qubit system such that separability is conserved. Since separability is conserved, entanglement presence in the resulting two-qubit state implies entanglement in the original state to which the mapping is applied.
The chapter starts with introducing the physical idea behind the mapping for a single system which consists in letting the original system interact with a qubit, initialized in some reference state, then discarding the original system and keeping the qubit in its new state acquired due to the interaction. Next, we give a complete characterization of all such maps in terms of expectation values of three operators acting on the original Hilbert space. We also draw an explicit connection between different representations of a quantum channel that maps a state of an arbitrary system into a qubit state.
Once the mapping for a single particle is introduced, we carry on to describe a mapping of a two-particle system into two qubits. We restrict our analysis to mappings implemented by Local operations with no classical communications. In other words we apply the introduced formalism to each subsystem independently where no entanglement can be created such that entanglement of the two qubit system witnesses entanglement of the original one. Then we study in detail a straight forward example of a mapping using the operators of angular momentum and compare the resulting criterion for a family of two qudit states. Finally, we apply the same mapping for N spin-j system to obtain a system of N spin-$\frac{1}{2}$ where one can show a connection between spin squeezing inequalities for spin-j particles and those for spin-$\frac{1}{2}$ ones.

Chapter 3 is concerned with entanglement detection in many-particle systems using spin squeezing inequalities. More precisely, it concerned with spin squeezing inequalities for systems where the particle number may present classical or quantum fluctuations. The former occurs commonly in experiments where one has to repeat measurement many times in order to get a meaningful quantity and account for systematic errors. The latter, however, is more exotic and occurs in systems where a super-selection rule on particle number does not apply. This situation appears naturally in our formalism when we map each particle into a fictitious spin-$\frac{1}{2}$ particle. Depending on the state of the N-particle system, we can end up with a coherent superposition of qubits' state with different particle number.

The chapter starts by recalling the spin squeezing inequalities and the basic steps to prove them. We next introduce our main result to generalize these inequalities to arbitrary operators and adequate particle number operators. We then study the case for a special class of operators that allow for inequalities analogous to spin squeezing inequalities that are independent of choice of the coordinate system. We also generalize the Sørensen-Mølmer criterion for extreme spin squeezing to the mentioned general frame work for arbitrary operators and fluctuating particle number.
Finally, we study an experimental case that inspired our work thanks to the collaboration with the experimental group "Quantum Dipolar Gazes" at "LPL" lab in Villetaneuse, France. We first show, using a numerical simulation, that spin squeezing inequalities are unlikely to detect entanglement in their experiment. We also show that our generalized inequalities are very good candidates to detect entanglement in their particular setup using their available measurement scheme, a point we elaborate more in the next chapter.

In the last chapter, chapter 4, we present some partial results and possible directions to improve and build upon the results presented in the previous chapters.
In the first part, using the formalism introduced in chapter 2, we extend the mapping for bipartite systems beyond local operations without any communication. We present an operational form and characterize all maps to two qubits implemented via local operations and one way classical communications.
In the second part, we visit the problem of measurability of the collective operators needed for our inequalities defined in chapter 3. We provide a possible scheme based on measurements commonly accessible in cold atoms experiments. Finally, we discuss generalizing squeezing parameters, in the way done in chapter 3, to detect entanglement depth which is of greater interest than simple entanglement detection for quantum information and quantum metrology applications.

Chapters 2, 3 and 4 constitute the original work of this thesis. Part of the results presented in Chapter 2 were published in [1]. The main result of chapter 3 was published in [2].

1

Introduction: Entanglement criteria and spin squeezing

In this chapter, I will recall some useful notions about entanglement theory, and some of the tools to detect it in different systems. However, I will assume the reader to be familiar with the basic notions of quantum mechanics like Hilbert spaces, observables, density matrices, Heisenberg uncertainty principle,$\cdots$*etc* that can be found in numerous standard textbooks about the subject. A very brief reminder is presented here as an introduction to entanglement, the main focus of this thesis and this chapter. Hence, the first half of this chapter is dedicated to recalling some of the basic results in entanglement theory and its different detection criteria. For the next half, special attention is drawn to entanglement detection in systems consisting of large number of spins using spin squeezing inequalities that I will make use of for the remainder of this thesis.

1.1 Introduction

After quantum superposition, quantum entanglement is arguably one of the weirdest and headache provoking features of quantum mechanics. So much so, that it spooked Erwin Schrödinger[1], one of the fathers of quantum mechanics and Albert Einstein himself along with Podolsky and Prosen. The later three went on and claimed in their famous paper [4] that quantum mechanics theory is incomplete. They inferred the existence of some Local Hidden Variables (LHV), unaccounted for by quantum theory, to solve what has become

[1] "It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it." [3]

to be known as the EPR paradox.
At the heart of this paradox was entanglement, named by Schrödinger, which describes the inability to describe subsystems individually despite the ability to describe the system as a whole for some states [5]. There is no classical analogy to such type of correlations and it represented a theoretical contradictory aspect of quantum mechanics.
This was the case until the EPR paradox was reformulated by Bohm and Aharanov [6] in terms of spin variables which led to John Bell's famous inequality [7]. These inequalities opened the way to experimental tests and verification of non-locality of entanglement, long after being a theoretical and philosophical controversy, "simply" by looking at coincidence statistics of measurements performed by two parties.
The most famous test proposal was due to John Clauser, Michael Horne, Abner Shimony, and Richard Holt, who derived in their famous paper [8] the CHSH inequality. Both mentioned inequalities will hold true assuming the existence of some local hidden variables model but are expected to be violated for some quantum states. However, the most convincing experimental proof that closed the locality loophole was performed by Alain Aspect and his colleagues in [9, 10]. this experiment is considered a big triumph of quantum theory and has inspired later efforts [11–14] to tackle the remaining loopholes [15–17] and put local realism[2] to rest.

Entanglement is now accepted as a reality of quantum mechanics, but still ,like quantum mechanics [3], not fully understood at a fundamental level. It is quite a rare occurrence in science where a theory is very successful in predicting the outcome of experiments and has a lot of applications yet, there is still an active research and so much debate going on about its foundations [19]. Since there are many interpretations for quantum mechanics [20], each of which describes the same mathematical formulation, entanglement is far from understood from a pure physics' angle. All we can do, for now, is try to understand it as a mathematical property of multi-partite systems.

From a mathematical point of view, it is easy to define separable states, or states that are not entangled. A separable state of a multi-partite system is simply a convex combination of some product states;i.e.

$$\rho = \sum_k \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)} \quad : \lambda_k \geq 0, \sum_k \lambda_k = 1.$$

[2] The ability to predict with certainty the value of a physical quantity without disturbing the system[4].
[3] "I think I can safely say that nobody understands quantum mechanics", Richard Feynman [18].

This convex sum represents a statistical superposition of product states $\rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)}$, where each one of them occurs with probability λ_k. The product state $\rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)}$ represents the simplest separable state and describes a state in which the different parties are completely independent. The above convex combination of product states gives rise to classical correlations among the different parties and is rather difficult to characterize for an unknown state before hand.

However, from a geometric perspective, one can see that separable states are a convex subset of the set of all states. Inside this subset lie all separable states while entangled states reside outside it, see figure 1.4 for a pictorial representation. It is therefore quite important to understand the geometry of quantum states [21–36] in order to gain more insight into the distinction between separable and entangled states and understand their evolution under transformations.

While a qubit; i.e. a two-level system, enjoys the simplest geometric representation [37], Fig. 1.1, a two qubit system is the simplest setup for which entanglement appears. Algebraically, a 2-qubit state is represented by a 4×4 Hermitian matrice. Then the set of states is a submanifold embedded in a 16 dimensional linear space, which makes it extremely difficult to visualize geometrically. In order to circumvent this difficulty we define classes of equivalent states with respect to some property. As far as entanglement is concerned, one should consider local operations that does not create entanglement, preferably operations that does not destroy it either. The paradigm of operations that do not create entanglement is known as LOCC (Local Operations and Classical communications)where each party can apply, locally to its share of the state, the most general transformation allowed by the laws of quantum mechanics (a trace-preserving completely positive map) which can be conditioned on the other parties' outcomes through successive rounds of classical communication [38–40]. LOCC and SLOCC(Stochastic LOCC) [41] do not lead to equivalence relation but rather to partial ordering as one can define classes of states where a state can only be transformed to another such that entanglement is not increased [21, 42–44].

A smaller class of Local Operations LO$\subset$ LOCC [45] with invertible matrices [30] can be used to define equivalence classes of states. If state normalization is ignored, one is able to describe two-qubit states in three dimensional space [30, 27]. The same representation in Fig. 1.2 was first obtained by Horodecki et al. [22] when considering 2-qubit states with maximally mixed subsystems.

In analogy to the 2-qubit case, one can construct a "magic simplex" of 2-qutrit states with maximally mixed subsystems [34, 33, 35] and later generalized to 2-qudit states [46, 47].

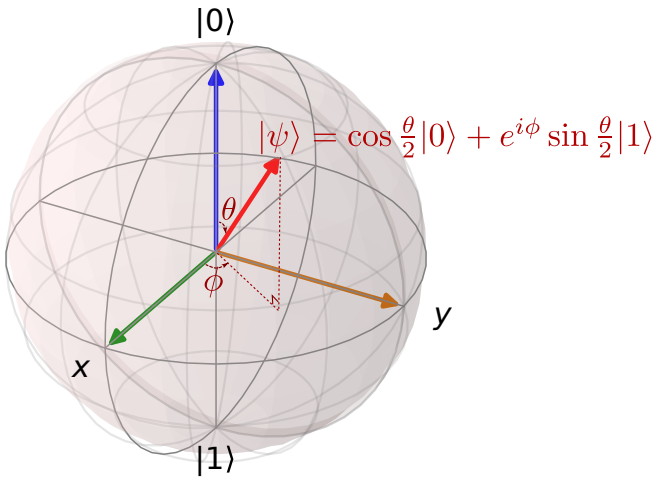

Figure 1.1: The Bloch sphere with antipodal points corresponding to a pair of mutually orthogonal qubit states. The north and south poles of the Bloch sphere are chosen to be $|0\rangle$ and $|1\rangle$. The points on the surface of the sphere correspond to the pure states of the qubit, whereas the interior points correspond to the mixed states.

Despite it not containing all states with maximally mixed reduced density matrices, the magic simplex has high symmetry and offers a simple geometric representation of an interesting class of states. This in turn, allows to define and parametrize families of the illusive bound entangled states, not distillable by SLOCC operations [48–52], and construct geometrical entanglement witnesses [53, 47, 54, 35].

Another fundamental reason to study entanglement than geometry of quantum states, is its role as one of the main tools to determine quantumness or non-classicality of a multi-partite state. This notion of non-classicality differs from context to another and from system to another. For example, in optics, such states are squeezed single mode states [55, 56] and superposition of coherent states[4] like cat states [57, 58]. Also, while Fock states are considered to be non-classical states in optics, since they give a non-positive P-representation [59] (Glauber-Sudarshan P representation [60, 61]), they are considered as classical states in the context of ressource theories of coherence, where coherent superpostion of Fock states[5] are non-classical [62–64]. In the multi-partite scenario, entangled states represent a subset of a bigger family of non-classical states

[4] Classical states analoguous to those of a classical harmonic oscillator as we will see later in 1.5.1.
[5] like coherent states

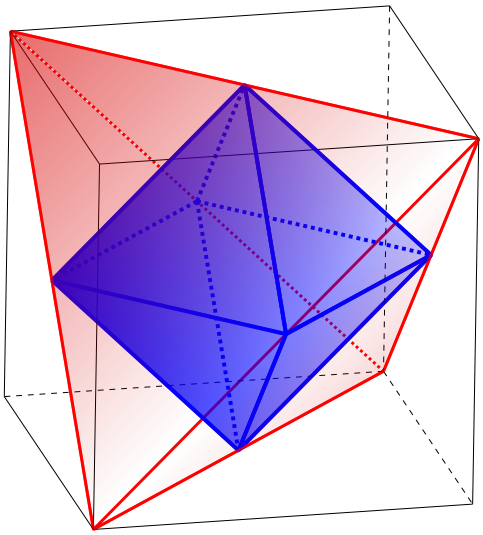

Figure 1.2: The octahedron (in blue) represents the equivalence class of separable states. The set of points that lie outside the octahedron but inside the tetrahedron (in red) are entangled states. The vertices of the tetrahedron represent the four Bell states [22].

which present correlations with no classical counterpart. These correlations are called quantum discord and are sometimes present even in separable states [65–67].
Despite this variety of non-classicality concepts, entanglement remains the corner stone and the most important among them. To begin with, it was shown that entanglement is a necessary feature for any physical theory that has a classical limit like decoherence [68]. Moreover, non-classicality can be interchangeably transformed into entanglement even for single-partite systems using ancillary systems [69]. In [70], a non-classicality measure for quantum states of a single-mode electromagnetic field was proposed and called entanglement potential (EP). This measure is based upon the amount of two-mode entanglement that can be generated from the state using linear optics and auxilliary clasical states. It was shown that indeed the notion of non-clasicality for single mode optics and two-mode entanglement are closely related and can be transformed into one another [71, 62] and later was generalized to different contexts other than optics [69].

The most non-classical feature about entangled states is the possibility to have non-local correlations[16], i.e. correlations that cannot be explained by local models. In other words, one is unable to construct local theories with some hidden random variable shared among the different parties, or shared randomness, to explain the joint measurements' statistics as mentioned earlier in the introduction.
While it is known that any state exhibiting non-locality must be entangled, the relation-

ship between entanglement and non-locality is rather rich and far more complex. To begin with, all pure entangled states, bi-partite or multi-partite, violate some form of Bell inequalities[6] [74, 75, 73], and are, hence, non-local. The same cannot be said about mixed entangled states as some of them admit a local model for all local measurements and, therefore, cannot violate any Bell inequality [76–80].
However, mixed entangled states may have what is called hidden locality that cannot be revealed by direct measurements, involved in Bell inequalities, but may be revealed with some pre-processing and local filtering of the state [81–85]. Moreover, it was shown in [86, 87] that all bi-partite states have hidden non-locality that can be activated when local filtering them jointly with other states that do not show non-locality neither. Another phenomenon is the super activation of non-locality, where a state not showing non-locality before hand, becomes non-local when performing joint measurements on multiple copies of the same state [88–90]. Finally, it was shown that using semi-quantum games [91], a variant of the Bell tests where the input states are quantum, all entangled states can be witnessed and reveal non-locality by a class of non-local games. This shows how intricate the relationship between entanglement and non-locality is and how non-locality might be a generic feature of entanglement.
The mentioned non-classical features exhibited by entangled states inspired and lead to a large variety of applications that have been demonstrated experimentally. Many of these applications take advantage of non-locality of entangled states like quantum cryptography [92–98] ,quantum teleportation [99–104], and entanglement swapping [105–107]. More importantly, entanglement is a valuable resource for quantum information and quantum computation. This resource [108] cannot be created for free by the means of LOCC operations [39] but can be broadcasted [109, 110] and controlled [111]. Moreover, entanglement is an essential ingredient in many quantum algorithms that allow a big advantage over the existent classical ones [112–114]. At last, but not least, entanglement is essential to enhance precision of measurements in linear metrology applications. In such applications, non-classical correlations in certain entangled states can help improve sensitivity and beat the standard quantum limit of classical states [115–118].

The above mentioned applications of entanglement are nothing but a tiny sample of a huge list beyond the scope of this small introduction [52]. It is no wonder that a big effort has been spent to find methods and criteria to quantify and detect this precious

[6] More precisely, a CHSH inequality [72] or an extension of it in the multi-partite case [73].

and fragile resource. The aim of this chapter is to recall only a few of these criteria that will be useful for the following chapters.

1.2 States in quantum mechanics

For the sake of completeness, I will try to recall the bare minimum necessary to introduce the notion of entangled states. For that, I will introduce what is a state in quantum mechanics from a mathematical point of view then the notion of purity which lead us to the introduction of density matrices. Once these basics are recalled, we can move on to describe the state of a multi-partite system and distinguish between separable and entangled states.

The mathematical foundations of quantum mechanics are due to Von Neumann [119]. As a starting point, any physical system may be represented via a Hilbert space representing all the possible states the system may be in for some degree of freedom. These degrees of freedom might be quantized; as for spin systems or energy levels for a particle in a box, or continuous like position or moment of a particle. This vectorial representation allows for one of the most interesting features of quantum theory, quantum superposition.

In what follows, I will only consider the case of finite dimensional Hilbert spaces. This case captures the basic mathematical and more importantly physical concepts of quantum mechanics. The generalization to infinite dimensions can be done with extra care taken to account for the various mathematical subtleties related to infinite Hilbert spaces.

1.2.1 Pure states

Let us start by describing a system in pure state in the statistical sense. Statistical physics deals with ensembles of systems, where the system is replicated a big number of times. Hence, a system is in a pure state iff any replica is in the exact same state.

Let $\mathcal{S}$ be a system and let $\mathcal{H}^{(d)}$ be a Hilbert space spanned by the orthonormal basis $\{|e_i\rangle\}_{i=1}^{d}$ where the vector is written in Dirac's notation. This basis is an abstract representation of the different distinguishable outcomes of a measurement of the considered degrees of freedom. Upon performing a measurement, the system will be in one of those outcomes and never in a superposition. However, repeating the measurement might give

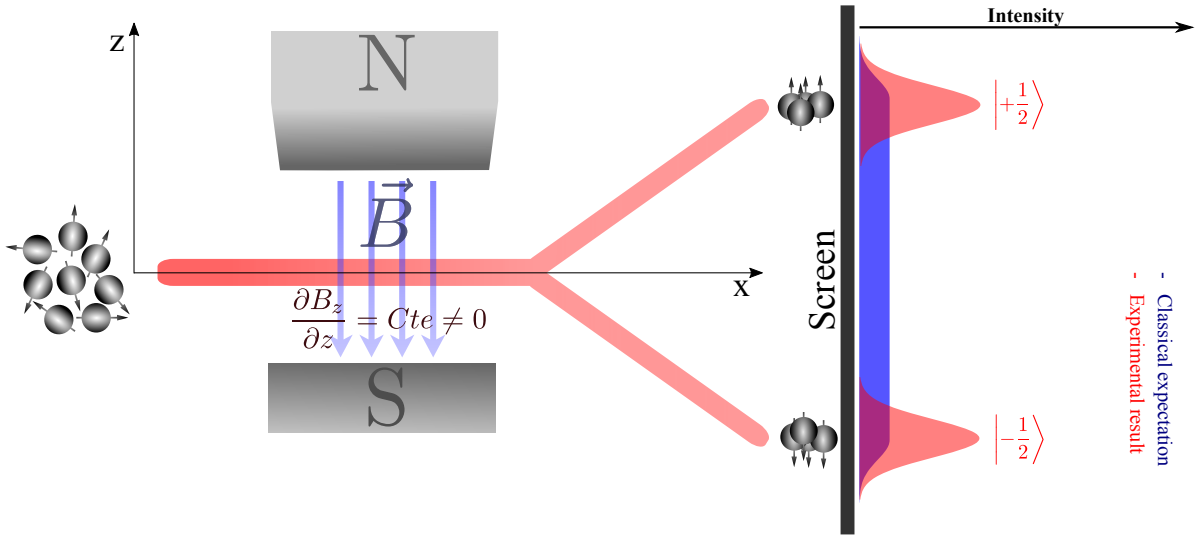

Figure 1.3: Sketch of Stern-Gerlach experiment: A beam of silver atoms is sent through an inhomogeneous magnetic field with a constant gradiant along the z axis. The silver atoms come out of the magnetic field and hit a screen perpendicular to their initial path. On the right, the intensity distribution is observed on the screen; in blue classical expectation versus quantum result in red. An atom after the measurement can only be in one of the two spin states $|+\frac{1}{2}\rangle$ or $|-\frac{1}{2}\rangle$.

different outcome to the previous measurements with a probability depending on the state of the system beforehand.

As one of the most famous and important examples is the experiment of Stern-Gerlach [120–123]. This experiment, first performed in 1922, has long been considered as the quintessential experiment to illustrate the electronic intrinsic angular momentum; spin. Another important aspect of it, is the simplicity of the results of the experiment, only two outcomes, making it the perfect tool to illustrate and gain direct access to the priciples of quantum mechanics, as was done in [120].

The experiment is summarized in Fig.1.3 where a collimated beam of silver atoms is sent, in the x direction for example, through a magnetic field before hitting a screen to measure the spatial distribution of silver atoms. The magnetic field pointing in the z direction has a gradient which, for simplicity sake, is assumed to be a constant; i.e. $\frac{\partial B_z}{\partial z} = cte \neq 0$. Without the magnetic field gradient, the silver atoms with magnetic moment $\vec{\mu}$ each will precess around the z-axis. This precession will have no influence on the spatial distribution of silver atoms, which will be a Gaussian centered around $(0, 0, 0)$ with a spread depending on how good beam collimation is and time of flight of silver atoms. On the other hand, with the presence of magnetic field gradient, each atom will

feel a force in the z-direction: $F_z = -\dfrac{\partial U}{\partial z}$, where $U = -\vec{\mu} \cdot \vec{B} = -\mu_z B_z$ the potential energy of a silver atom in a magnetic field,thus:

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

Since the orientation of the magnetic moment of a silver atom $\vec{\mu}$ at the exit of the oven, source of the beam, is random, a classical mindset would expect its projection on the z-axis to be a uniform distribution $\mu_z \in [-|\vec{\mu}|, |\vec{\mu}|]$. Hence, the original spot at the origin of the screen with no magnetic field gradient will be stretched in the z-direction with borders corresponding to atoms having $\mu_z = \pm|\vec{\mu}|$ before entering the magnetic field region. Instead, two spots were observed corresponding to two values of $\mu_z = \pm\mu_B = \pm|\vec{\mu}|$, where $\mu_B = \dfrac{e\hbar}{2m_e}$ is Bohr magneton and m_e is the mass of an electron. Since, for silver atoms, $\vec{\mu} = \dfrac{g\mu_B}{\hbar}\vec{S}$, where g[7] is Landé g-factor and $\vec{S}$ is the spin vector, this experiment offers a direct measurement of spin component along the z-axis. In other words, what this experiment showed clearly is that there are only two possible values for S_z:$+\dfrac{\hbar}{2}$ and $-\dfrac{\hbar}{2}$ as predicted by quantum mechanics.

After the measurement, all the atoms are in a pure state being either $|+\frac{1}{2}\rangle$ or $|-\frac{1}{2}\rangle$; taking $\hbar = 1$. Since states live in a Hilbert space, linear superposition can arise and a general pure state of a spin-$\frac{1}{2}$ can be written as a complex combination of the form:

$$|\psi\rangle = \alpha|+\frac{1}{2}\rangle + \beta|-\frac{1}{2}\rangle \quad : |\alpha|^2 + |\beta|^2 = 1$$

where $|\alpha|^2$ $\left(|\beta|^2\right)$ represent the probability to find the spin-$\frac{1}{2}$ in the pure state $|+\frac{1}{2}\rangle$ $\left(|-\frac{1}{2}\rangle\right)$ after performing the measurement. Despite having two different outcomes for measurement in the $\{|+\frac{1}{2}\rangle, |-\frac{1}{2}\rangle\}$, the state $|\psi\rangle$ is pure since there exists a projective measurement, in the basis $\{|\psi\rangle, |\psi^\perp\rangle\}$[8], that leaves it unchanged. It is called a projective measurement since the states after the measurement $\{|\psi\rangle, |\psi^\perp\rangle\}$ are orthogonal, *i.e.* $\langle\psi|\psi^\perp\rangle = 0$. Note that the atoms before the measurement in the experiment above were not in a pure state, but rather we have an ensemble of states $\{|+\frac{1}{2}\rangle, |-\frac{1}{2}\rangle\}$ with probabilities $\{\frac{1}{2}, \frac{1}{2}\}$. We say that each atom is in a mixed state. To describe such states,

[7] $g = 2$ for free electrons

[8] $|\psi^\perp\rangle = \beta|+\frac{1}{2}\rangle - \alpha|-\frac{1}{2}\rangle$

we use a density matrix formulation which is a very powerful and useful tool to describe the state of a quantum system in general whether pure or mixed.

1.2.2 mixed states

Mixed states describe an ensemble of states $\{|\psi_i\rangle\}$ with probabilities $\{p_i\}$ where in a large number of copies of the considered system, each copy will be found in the state $|\psi_i\rangle$ with probability p_i. A more compact way of describing such situations is to use a density matrix to describe the state.

A density matrix ρ is a bound operator acting on the Hilbert space of the considered system with the following conditions:

- *ρ is hermitian, $\rho^\dagger = \rho$*
- *ρ is positive $\rho \geq 0$ and normalized $Tr[\rho] = 1$.*

Being a hermitian operator, ρ will always be diagonalizable $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$. Positivity and normalization will ensure that $\{\lambda_i\}_i$ is a probability distribution. Hence ρ is a representation of the ensemble $\{\lambda_i, |\psi_i\rangle\}$.
A special case is when $\rho = |\psi\rangle\langle\psi|$ which represents a pure state since $\rho^2 = \rho$. In other words, ρ is a projector onto the state $|\psi\rangle$. Conversely, a mixture of states $\{p_i, |\phi_i\rangle\}$, with $p_i \geq 0$ and $\sum_i p_i = 1$, can be represented by the density matrix $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$.

In fact, the density matrix formalism is the most general way of describing a quantum system as was proven by Gleason in Ref. [124]. It is also worth noting that a given density matrix does not represent a unique ensemble of states. The simplest example is to take an ensemble of non orthogonal states $\{p_i, |\phi_i\rangle\}$. Clearly, $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ has a spectral decomposition $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, where the states $\{|\psi_i\rangle\}$ are orthogonal.

In general [37], *two ensembles $\{\lambda_i, |\psi_i\rangle\}$, $\{p_i, |\phi_i\rangle\}$ will give rise to the same density matrix;i.e. $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ iff there exists a unitary matrix U such that:*

$$\sqrt{\lambda_i}|\psi_i\rangle = \sum_j U_{ij}\sqrt{p_i}|\phi_i\rangle.$$

1.2.3 Many-body states

So far, we have the mathematical representation of a single particle state. While, one can always treat a many-body system as a whole, it is more interesting to have a more detailed description with respect to its one-body components.

Let us consider a system S composed of n subsystems $\{S^{(i)}\}_{i\in\mathbb{N}^n}$. Each subsystem $S^{(i)}$ is described by a Hilbert space $\mathcal{H}^{(i)}$. Then we can describe the system S by the tensor product $\bigotimes_i \mathcal{H}^{(i)} = \mathcal{H}$. Moreover, any pure state $|\Psi\rangle \in \mathcal{H}$ can be written as:

$$|\Psi\rangle = \sum_{\vec{k}} \alpha_{\vec{k}} \bigotimes_i |\psi_{k_i}^{(i)}\rangle \tag{1.1}$$

where $\vec{k} = (k_1, \cdots, k_n)$ is a vector of indices, and the index k_i labels the elements of the basis $\{|\psi_{k_i}^{(i)}\rangle\}_{k_i}$ of $\mathcal{H}^{(i)}$. An interesting point here is that despite S is in a pure state $|\Psi\rangle$, the subsystem $S^{(i)}$ is not necessarily in a pure state and that depends on the coefficients $\alpha_{\vec{k}}$ (1.1). The purity of the state of $S^{(i)}$ will mean that it is separable of the rest of the subsystems of S as we will see later when we study entanglement. Once pure many-body states are defined, their extension to mixed states is straightforward.

We also can extend an operator $A^{(i)}$ acting on $\mathcal{H}^{(i)}$ into an operator acting on the whole Hilbert space $\mathcal{H}$, simply, by taking its tensor product with the identity over the other Hilbert space, i.e. $A^{(i)} \otimes \bigotimes_{j\neq i} \mathbb{1}^{(j)}$. These operators are called local operators since they only act on a local part of the system. By abuse of notation, we will drop the identity over other subsystems and we will, henceforth, use $A^{(i)}$ to represent an operator acting on $\mathcal{H}^{(i)}$ or $\mathcal{H}$ depending on the context. We will also be referring to operators of the form

$$A = \sum_{i=1}^{n} A^{(i)} \tag{1.2}$$

as collective operators. For example the operator:

$$J_x = \sum_{i=1}^{n} J_x^{(i)}$$

is the x-component of the collective spin operator of an ensemble of n j-spins.

Partial trace

Given the state ρ of the many-body system S, we can obtain the state ρ_A of its subsystem A as:

$$\rho^{(A)} = \mathrm{Tr}_B\left[\rho\right], \tag{1.3}$$

where $B = S\backslash A$ and $\mathrm{Tr}_B\left[.\right]$ represents the partial trace over B, the remaining part of the system.
This can be justified by the fact that $\rho^{(A)}$ gives the correct expectation value of any local operator $\mathcal{D}^{(A)}$:

$$\langle \mathcal{D}^{(A)} \rangle = \mathrm{Tr}\left[\rho \mathbb{1}^{(B)} \otimes \mathcal{D}^{(A)}\right] = \mathrm{Tr}_A\left[\rho^{(A)} \mathcal{D}^{(A)}\right] \tag{1.4}$$

Now we are in position to discuss one of the most interesting features of quantum mechanics related to multi-partite systems, entanglement.

1.3 Bipartite entanglement

In this section, we recall the definition of bipartite entanglement and some of the most important criteria to detect it. Let $\rho^{(AB)}$ be a state of bipartite system AB. $\rho^{(AB)}$ is said to be *a product state* iff it can be written as a product $\rho^{(A)} \otimes \rho^{(B)}$, where $\rho^{(A)} = \mathrm{Tr}_B\left[\rho^{(AB)}\right]$ and $\rho^{(B)} = \mathrm{Tr}_A\left[\rho^{(AB)}\right]$. More generally, $\rho^{(AB)}$ is said to be a *separable state* iff it can be written as a convex sum of product states; i.e.

$$\rho^{(AB)} = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)} \quad : p_k \geq 0, \sum_k p_k = 1. \tag{1.5}$$

Once separable states are defined, we are able to define entangled states as the states that cannot be written in the above form. In other words, $\rho^{(AB)}$ is said to be entangled iff it cannot be written as a convex sum of product states. This definition, albeit clear, is useless in practice. As we have seen, a mixed state represents many ensembles as long as they verify the condition introduced in 1.2.2. Testing for all possible ensembles is quite complex and prohibitive especially when the dimension of the Hilbert space gets bigger. That is why a lot of effort has been and still being spent to find efficient entanglement criteria [52, 125]. Next, we will start by criteria for pure bi-partite state and then we will move on to some of the criteria for mixed states. I will only mention criteria relevant to this thesis for finite dimensional systems and skip over criteria for continuous variable systems [126].

1.3.1 Entanglement of pure bipartite states

Characterizing entanglement for bipartite pure state can be achieved rather easily. We say that a pure bipartite state $|\Psi\rangle \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$ is entangled iff it cannot be written as a product state:

$$|\Psi\rangle = \left|\phi^A\right\rangle \otimes \left|\phi^B\right\rangle : \left(\left|\phi^A\right\rangle, \left|\phi^B\right\rangle\right) \in \mathcal{H}^{(A)} \times \mathcal{H}^{(B)}$$

In general, we can write any state $|\Psi\rangle$ using the orthonormal basis $\left\{\left|\psi_i^A\right\rangle \otimes \left|\psi_j^B\right\rangle\right\}_{i,j=1}^{d_A,d_B}$, where $dim(\mathcal{H}^{(A)}) = d_A, dim(\mathcal{H}^{(B)}) = d_B$ as:

$$|\Psi\rangle = \sum_{i,j}^{d_A,d_B} \alpha_{ij} \left|\psi_i^A\right\rangle \otimes \left|\psi_j^B\right\rangle \tag{1.6}$$

Using Schmidt decomposition[37], we can find an orthonormal basis $\{\left|e_i^A\right\rangle \otimes \left|e_i^B\right\rangle\}_{i=1}^{k}$ and coefficients $\{\beta_i \geq 0\}_{i=1}^{k} : k \leq \min\{d_A, d_B\}$ such that:

$$|\Psi\rangle = \sum_{i=1}^{k} \beta_i \left|e_i^A\right\rangle \otimes \left|e_i^B\right\rangle, \tag{1.7}$$

where $\beta_i = \sqrt{s_i}$, and s_i are the singular values of the matrix α whose elements are α_{ij} defined in (1.6). k is called the Schmidt rank. Hence, according to the previous separability condition for pure states, a state is separable iff it has a Schmidt rank equal to 1. It is worth noting that the Schmidt rank k is equal to the rank of either of the reduced density matrices $\rho^{(A)} = \mathrm{Tr}_B\left[|\Psi\rangle\langle\Psi|\right]$ or $\rho^{(B)} = \mathrm{Tr}_A\left[|\Psi\rangle\langle\Psi|\right]$. In fact, one can easily see that they share the same eigenvalues s_i. Thus, it is quite simple to fully characterize entanglement for pure states by calculating the spectrum of the reduced density matrix.

From experimental point of view, having a pure state and conserving its purity is almost an impossibility due to systematic errors (small errors in the system preparation, noise ...). One has to deal with mixed states for which separability is quite hard to characterize. Next, I will recall some of the most important criteria that I will make use of through this work.

1.3.2 Entanglement of mixed bipartite states

We have seen that it is quite easy to distinguish pure separable states. Mixed states, on the other hand, are harder to be judged entangled or not. It is still an open question to

find necessary and sufficient conditions for separability for arbitrary bipartite state. All the criteria that we are going to cite here, are necessary conditions for separability but not sufficient.

Majorization criterion and entropic inequalities

Let x, y be two vectors in $\mathbb{R}^n$ and let $x^{\downarrow}, y^{\downarrow}$ be the two vectors obtained from x, y by ordering their components in descending order. We say that x is majorized by y [127] and we write $x \prec y$ if:

$$\forall k \in \{1, \cdots, n\}; \sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \qquad \sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow} \tag{1.8}$$

If the last equaltiy does not hold, then we say that x is weakly majorized by y and we write $x \prec^w y$. Once this notion is introduced, we can look at an important separability criterion introduced bu Nielsen and Kempe [128] for bipartite state of arbitrary dimension $\rho^{(AB)}$:
If $\rho^{(AB)}$ is separable then

$$\Lambda\left(\rho^{(AB)}\right) \prec \Lambda\left(\rho^{(A)}\right), \qquad \Lambda\left(\rho^{(AB)}\right) \prec \Lambda\left(\rho^{(B)}\right) \tag{1.9}$$

where $\rho^{(A)}, \rho^{(B)}$ are the reduced density matrices and $\Lambda(\rho)$ is the $d_A \times d_B$ eigenvalues vector of ρ in descending order - padded with zeros for the eigenvalues' vector of the reduced density matrices.

From the above inequality, follows a weaker inequality in terms of Von Neumann entropy defined for a given density matrix ρ as:

$$\mathcal{S}(\rho) = -\mathrm{Tr}\left[\rho \log(\rho)\right]. \tag{1.10}$$

Since $S(\rho)$ is Schur concave [129] and from the previous criterion, one can obtain the following separability criterion:

$$\mathcal{S}\left(\rho^{(AB)}\right) \geq \mathcal{S}\left(\rho^{(A)}\right), \qquad \mathcal{S}\left(\rho^{(AB)}\right) \geq \mathcal{S}\left(\rho^{(B)}\right) \tag{1.11}$$

$S(\rho)$ can be thought of as a measure of the lack of information about the state of a

system. If the system is in a pure state, hence, as we discussed above, is in a particular state with certainty, then the von Neumann entropy is 0. On the other hand, if the system is in a mixed state, then the corresponding von Neumann entropy is strictly positive. The maximally mixed state of a, say, d-level system, $\rho = \frac{1}{d}\mathbb{1}$ represents the complete absence of information about the state of the system and has the maximal value of the von Neumann entropy $S(\rho) = \log(d)$. In other words, the higher the entropy is, the less we know about the system.
Then we see that the above criterion (1.11) tells us that, for separable states, the global system is more disordered than either of its subsystems. In other words, for some entangled states, we have more information about the global system than about its components. This is never the case for classical variables, where the Shannon entropy [130] for a single random variable is always smaller than the joint entropy [131] for two random variables:

$$H(X,Y) \geq H(X) \quad , \quad H(X,Y) \geq H(Y) \tag{1.12}$$

where $H(X) = -\sum_x P(X=x)\log\left(P(X=x)\right)$ is the classical counter part for the von Neumann entropy.

The Von Neumann entropy is a limit case for a whole family of entropy called Rényi entropy defined [132] for a parameter $\alpha \geq 0$ as:

$$S_\alpha(\rho) = \frac{\log\left(\mathrm{Tr}\left[\rho^\alpha\right]\right)}{1-\alpha} \tag{1.13}$$

where Von Neumann entropy is obtained as $\lim_{\alpha\to 1^+} S_\alpha(\rho)$. Then [22, 133, 134], For any separable state $\rho^{(AB)}$ the following inequalities are verified

$$\mathcal{S}_\alpha\left(\rho^{(AB)}\right) \geq \mathcal{S}_\alpha\left(\rho^{(A)}\right), \qquad \mathcal{S}_\alpha\left(\rho^{(AB)}\right) \geq \mathcal{S}_\alpha\left(\rho^{(B)}\right) \tag{1.14}$$

for all $\alpha \geq 0$.

Finally, one can write an equivalent separability criterion to (1.14) in terms of Tsallis entropy [135–137] as:

$$T_\alpha\left(\rho^{(AB)}\right) \geq T_\alpha\left(\rho^{(A)}\right), \qquad T_\alpha\left(\rho^{(AB)}\right) \geq T_\alpha\left(\rho^{(B)}\right) \tag{1.15}$$

where $T_\alpha(\rho)$ is Tsallis entropy defined for $\alpha \geq 0$ as

$$T_\alpha(\rho) = \frac{\mathrm{Tr}\left[\rho^\alpha\right] - 1}{1-\alpha}. \tag{1.16}$$

Inseparability with positive maps

A linear map $\mathcal{M}$ from $\mathcal{B}\left(\mathcal{H}^{(1)}\right)$ to $\mathcal{B}\left(\mathcal{H}^{(2)}\right)$, with $\mathcal{B}(\mathcal{H})$ being the bounded operator space acting on $\mathcal{H}$, is said to be positive iff for all positive $\rho \in \mathcal{B}\left(\mathcal{H}^{(1)}\right)$, $\mathcal{M}(\rho)$ is positive. On the other hand, $\mathcal{M}$ is said to be completely positive [138] iff the linear map $\mathbb{1}^{(n)} \otimes \mathcal{M}$ is positive for all n, where $\mathbb{1}^{(n)}$ is the identity map from $\mathcal{B}\left(\mathcal{H}^{(n)}\right)$,where $\mathcal{H}^{(n)}$ is any Hilbert space of dimension n, to itself.

From the above two definitions, one can see that a complete positive map is not useful to detect entanglement since it maps any density matrix into a positive one. However, a positive but not completely positive map can map some bipartite density matrices into non positive ones. In other words, for a positive but not completely positive $\mathcal{M}$, there exists n or a Hilbert space $\mathcal{H}^{(A)}$, such that $\mathbb{1}^{(A)} \otimes \mathcal{M}$ is not positive. Hence the following separability criterion [139]:

A state $\rho^{(AB)}$ is separable iff for all positive but not completely positive maps $\mathcal{M}$ from $\mathcal{B}\left(\mathcal{H}^{(B)}\right)$ to $\mathcal{B}\left(\mathcal{H}^{(A)}\right)$, $\mathbb{1}^{(A)} \otimes \mathcal{M}\left(\rho^{(AB)}\right)$ is positive.

It is easy to see why the above requirement is a necessary condition for separability. Say $\rho^{(AB)}$ is a separable state and $\mathcal{M}$ is a positive map. Then, we can write $\rho^{(AB)}$ in the following form:

$$\rho^{(AB)} = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)} \quad : p_k \geq 0, \sum_k p_k = 1. \tag{1.17}$$

From which follows that

$$\mathbb{1}^{(A)} \otimes \mathcal{M}\left(\rho^{(AB)}\right) = \sum_k p_k \rho_k^{(A)} \otimes \mathcal{M}\left(\rho_k^{(B)}\right) \tag{1.18}$$

is positive since $\mathcal{M}$ is positive. As for the sufficiency part see [139].

While testing for all positive maps is quite prohibitive, one can use certain maps as a sufficient condition for entanglement. The most important ones among these are the reduction criterion and PPT criterion.

Reduction criterion

The positive map to consider for the reduction criterion is defined as:

$$\mathcal{M}(\rho) = \mathrm{Tr}[\rho]\,\mathbb{1} - \rho \tag{1.19}$$

which gives the following criterion [140, 141]:
For all separable states $\rho^{(AB)}$, the following inequalities hold

$$\rho^{(A)} \otimes \mathbb{1} - \rho^{(AB)} \geq 0 \quad , \quad \mathbb{1} \otimes \rho^{(B)} - \rho^{(AB)} \geq 0. \tag{1.20}$$

One corollary of the above inequalities are the entropic inequalities (1.14) seen in the previous section [141]. Another corollary is the majorization criterion [128]. This follows from the implication [142],

$$A \leq B \Rightarrow \lambda(A) \prec^{w} \lambda(B)$$

for any Hermitian operators A and B. Hence, the reduction criterion is stronger than both the mojorization and entropic criteria [143].
It is also equivalent to the PPT criterion for the case of two qubits, hence it constitutes a necessary and sufficient condition for entanglement, as we will see in the next section. It is however, weaker than the PPT criterion for higher dimensions. Moreover, reduction criterion is closely related to entanglement distillability, defined hereafter, where it was shown in [141] that any state that violates (1.20) is distillable.

Entanglemnt distillation The notion of entanglement distillation was introduced in [39] as a way to concentrate and purify entanglement to allow for quantum communication protocols in the presence of noise [39, 144].
Entanglement distillation, when two parties share n copy of some entangled bipartite state, consists in performing some LOCC operations to obtain the maximal number, $k(n) \leq n$, possible of pairs of maximally entangled states [145][9]. The asymptotic ratio $\frac{k}{n}$ in the limit of large number of copies n is called entanglement of distillation [144] which quantifies how much pure state entanglement we can extract from a given state ρ.
The task of deciding whether a bipartite state ρ is distillable or not was significantly reduced in [49] to the following condition,

[9] We call ρ a maximally entangled state of an $M \times N$ dimensional bipartite system iff the reduced density matrix for any subsytem is of the form $P^{(k)}/k$ with $k = \min(M, N)$ and $P^{(k)}$ is a projector of rank k.

A bipartite state $\rho \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$ *is distillable iff there exists a number* n *and rank-2 projectors* P *and* Q *such that the unnormalized state* $\rho' = P \otimes Q \rho^{\otimes n} P \otimes Q$ *is entangled.* In other words, ρ' is a state acting on a 2×2 subspace of $\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$. Hence, distillable entanglement can be seen as a form of two-qubit entanglement.
As I have just mentioned earlier, any state violating the reduction criterion is distillable. Since, reduction criterion is a necessary and sufficient condition for entanglement in the 2-qubit case, it follows that all 2-qubit states are distillable. However, not all states can be distilled and the question of distillability of a given state has been linked [49] to the PPT criterion mentioned herafter.

PPT criterion

Positive Partial Transpose (PPT) criterion or Peres-Horodecki criterion[146, 139] is one of the most important separability criterion out there for bipartite systems. Its importance is specially appreciated for providing a necessary and sufficient condition for entanglement of bipartite systems with dimensions 2×2 and 2×3.
As can be inferred from the name Partial Transpose, it is the transpose map to be considered $\mathcal{M} = T$, and it was shown that [146, 139]:

For all separable states $\rho^{(AB)}$, $\rho^{(AB)^{T_B}} = \mathbb{1} \otimes T\left(\rho^{(AB)}\right)$ *is a density matrix.*

where the symbol $.^{T_B}$ means that the partial transpose is taken with respect to the subsystem B. One can see that taking the partial transpose does not matter whether it is with respect to A or to B. that is because

$$\rho^{T_A} = \left(\rho^{T_B}\right)^T .$$

and the transpose is positive but not completely positive map.
Despite it being only a sufficient criterion for entanglement, the PPT was shown [146, 139] to be a necessary and sufficient for the case of 2×2 and 2×3 systems. Hence a full characterization of the separable states for these cases. This is because there exists a decomposition of positive maps in terms of completely positive maps as shown by Størmer and Woronowicz [147, 148] for systems of dimensions 2×2 and 2×3. For these systems, the positivity of $\mathbb{1} \otimes \mathcal{M}(\rho)$ for any positive map $\mathcal{M}$ becomes equivalent to the positivity of $\mathbb{1} \otimes \mathcal{M}_1^{CP}(\rho) + \mathbb{1} \otimes \mathcal{M}_2^{CP}\left(\rho^{T_B}\right)$, where $\mathcal{M}_1^{CP}$ and $\mathcal{M}_2^{CP}$ are completely positive, which, in turn, is related to the positivity of ρ^{T_B}.

This decomposition, however, does not exist for higher dimensional systems, hence the existence of states that are entangled with positive partial transpose [48]. These states are called PPT entangled states and they are bound entangled since they cannot be distilled by means of LOCC operations [49]. In contrast, the states that are not positive under partial transpose are called NPT entangled states. It is still an open question whether all NPT states are distillable or not. As mentioned earlier, all NPT states that violate the reduction criterion are distillable. It is also known that all NPT $2 \times N$ states are distillable [149]. However, the existence of NPT bound entanglement does not have a definite answer yet [150–153].

Finally, let us give an example of states that violates PPT criterion but satisfies the reduction criterion. One family of such $d \times d$ states was introduced by Werner [76]:

$$\rho_\beta = \frac{1}{d^2 + d\beta} (\mathbb{1} + \beta V) \quad , -1 \leq \beta \leq 1 \tag{1.21}$$

where $V = \sum_{i,j} |i,j\rangle\langle j,i|$ is the flip operator. These states were introduced as states that do admit a hidden variable model but exhibit non-classical correlations. Another property of these states that they are the only states that are invariant under local unitary transformations of the form

$$\rho \longrightarrow U \otimes U \rho U^\dagger \otimes U^\dagger. \tag{1.22}$$

Consequently, any state ρ can be transformed into a Werner state via a twirl operation

$$\int dU U \otimes U \rho U^\dagger \otimes U^\dagger \tag{1.23}$$

where the integral is carried out with respect to Haar measure on the unitary group $U(d)$. Werner showed that ρ_β is separable iff $\mathrm{Tr}\,[\rho_\beta V] \geq 0$. The condition $\mathrm{Tr}\,[\rho_\beta V] \geq 0$ is equivalent to $\beta \geq -\frac{1}{d}$ or having a positive partial transpose. Hence, PPT criterion detetcts all entangled Werner states whereas the reduction criterion fails to detect any state for $d \geq 3$ [141].

These states (1.21) are highly symmetric and are of great importance in the context of entanglement distillation. As we mentioned earlier, the existence of NPT bound entanglement is still an open question but it can be reduced to the study of entanglement distillation of Werner states. Indeed, it was shown in [141] that distillability of all NPT states is equivalent to distillability of all NPT Werner states.

Range criterion

This criterion was introduced by Horodecki [48] as a way to construct and detect the entangled states that are undetectable via the PPT criterion. It is called the range[10] criterion and it is stated as follows:

Let us consider a bipartite state $\rho \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$. *If* $\rho^{(AB)}$ *is separable, then there exists a set of product states* $\{|\psi_i\rangle \otimes |\phi_k\rangle\}$, $|\psi_i\rangle \in \mathcal{H}^{(A)}$, $|\phi_k\rangle \in \mathcal{H}^{(B)}$, *and* $(i,k) \in I$ *where* I *is a finite set of indices with cardinal number* $m \leq d^{(A)} \times d^{(B)}$, *such that:*

- *the set* $\{|\psi_i\rangle \otimes |\phi_k\rangle\}$ *span the range of* ρ
- *the set* $\{|\psi_i\rangle \otimes |\phi_k^*\rangle\}$ *span the range of* ρ^{T_B},

where $|\phi_i^*\rangle$ *is obtained by taking the complex conjugate of* $|\phi_i\rangle$ *in the basis in which the partial transpose was performed.*

As mentioned in [48], this criterion is useful when the range of ρ does not span the entire Hilbert space. Another point to mention, is that this criterion is not stronger than PPT in general. For example, it fails to detect 2×2 entangled Werner states which violate the PPT criterion. Finally, an example of states satisfying both the PPT and range criteria was provided in [48].

CCNR criterion

Another simple and powerful criterion is the computable cross norm or realignment (CCNR) criterion. The name CCNR comes from the fact that this criterion has been discovered in two different forms, namely, by cross norms [154, 155] and by realignment of density matrices [156].

Let us start with the CCN norm that is defined for an operator $C \in \mathcal{L}\left(\mathcal{H}^{(A)}\right) \otimes \mathcal{L}\left(\mathcal{H}^{(B)}\right)$as [155, 157]:

$$\|C\|_{CCN} = \inf\{\sum_{i=1}^{n} \|A_i\|_2 \|B_i\|_2 : C = \sum_{i=1}^{n} A_i \otimes B_i\} \tag{1.24}$$

where the infimum runs over all finite decompositions of C into elementary tensor product [154] and $\|.\|_2$ is Hilbert-Schmidt norm. While this does not seem so computable,

[10]The range of an operator A acting on a Hilbert space $\mathcal{H}$ is given by $\mathcal{R}(A) = \{A|\psi\rangle : |\psi\rangle \in \mathcal{H}\}$.

it can be shown for a density matrix $\rho^{(AB)}$ to be:

$$\left\|\rho^{(AB)}\right\|_{CCN} = \sum_i \Gamma_i \tag{1.25}$$

where Γ_i are the Schmidt coefficients obtained from the Schmidt decomposition of $\rho^{(AB)}$. Like we have seen for pure states (1.7), we can carry out Schmidt decomposition in the operator linear space and find orthonormal basis $\{E_k^{(A)}\}$ and $\{E_k^{(B)}\}$ such that

$$\rho^{(AB)} = \sum_k \Gamma_k E_k^{(A)} \otimes E_k^{(B)}. \tag{1.26}$$

Moreover, it can be shown that [158]:

$$\left\|\rho^{(AB)}\right\|_{CCN} = \|\tilde{\rho}\|_1 \tag{1.27}$$

where $\|.\|_1$ is the trace norm and $\tilde{\rho}$ is the realigned matrix [156] defined as:

$$\langle m|\langle n|\tilde{\rho}|m'\rangle|n'\rangle = \langle m|\langle m'|\rho^{(AB)}|n'\rangle|n\rangle. \tag{1.28}$$

Then, we can state the CCNR criterion as follows:
For a separable state $\rho \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$*, the following inequality holds*

$$\|\rho\|_{CCN} = \|\tilde{\rho}\|_1 \leq 1. \tag{1.29}$$

Finally, one can generalize the realignment criterion to any contraction linear mapping Λ that does not increase the trace norm for product states [159]. That is, if $\left\|\Lambda\left(|\psi^{(A)}\rangle\langle\psi^{(A)}| \otimes |\phi^{(B)}\rangle\langle\phi^{(B)}|\right)\right\|_1 \leq 1$, then
for any separable state $\rho \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$ *we have*

$$\|\Gamma(\rho)\|_1 \leq 1. \tag{1.30}$$

The CCNR criterion as well is complementary to the PPT criterion where it was shown to detect some PPT entangled states and fail to detect some entangled 2×2 states [160].

Entanglement witness

All the above mentioned criteria require full knowledge of the state, which is hard to do in practice for high dimensional systems, hence the idea of entanglement witness [161]. As depicted in Fig 1.4, an entanglement witness is an observable whose expectation

value is positive for all separable states, but negative for some entangled states. This considerably simplifies the task if the objective is to only testify entanglement rather than full knowledge of the state.

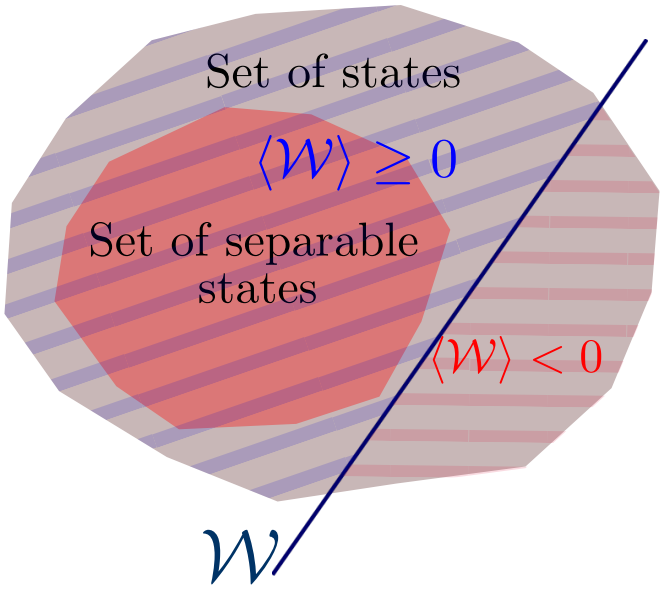

Figure 1.4: Pictorial view of the set of all bipartite states. Separable states form a convex subset depicted in red. The blue line represents an entanglement witness $\mathcal{W}$. The states, that lie in the area with blue stripes, give rise to positive expectation value of $\mathcal{W}$. Thus, only the entangled states in the area with red stripes, for which $\langle W \rangle < 0$, are detected by $\mathcal{W}$.

The origin of the idea is geometric [139]. Since the set of separable states is convex and compact, one can separate any entangled state from the set of separable states by a hyper-plane characterized by a Hermitian W[11] orthogonal to it. All the points of this hyper-plane correspond to states ρ such that $\mathrm{Tr}\,[\rho W] = 0$. Thus, one can choose W such that its expectation value is positive for all separable states and negative for the considered entangled state. This point was introduced in [139] as follows:

For every entangled state $\rho \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$, *there exists a Hermitian operator* $W \in \mathcal{B}\left(\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}\right)$ *such that*

- *For any separable state* σ; $Tr\,[\sigma W] \geq 0$
- $Tr\,[\rho W] < 0$.

[11] W is a vector in the operator vector space.

As stated by the above theorem, every entangled state can be detected via a measurable observable. However, constructing such an observable for every state is equivalent to solving the separability problem which has been shown to be an NP-hard problem [162–164]. Constructing entanglement witnesses in general, and finding the minimal set of them that allows for the detection of all entangled states is one of the most challenging open questions [165].

Entanglement witnesses $\mathcal{W}$ can be constructed from positive but not completely positive maps $\mathcal{M} : \mathcal{B}\left(\mathcal{H}^{(A)}\right) \to \mathcal{B}\left(\mathcal{H}^{(B)}\right)$ via Choi-Jamiolkowski isomorphism [166, 167]:

$$\mathcal{W} = (\mathbb{1} \otimes \mathcal{M})\left(|\Psi_+\rangle\langle\Psi_+|\right) \quad : |\Psi_+\rangle = \frac{1}{\sqrt{d_A}} \sum_{k=1}^{d_A} |k,k\rangle, \tag{1.31}$$

where $|\Psi_+\rangle$ is a maximally entangled state in $\mathcal{H}^{(A)} \otimes \mathcal{H}^{(A)}$. This follows from the properties of the above isomorphism where a map $\mathcal{M}$ is positive iff the corresponding operator $\mathcal{W}$ via (1.31) has positive expectation value for all product states. Moreover, a map $\mathcal{M}$ is completely positive iff the corresponding operator $\mathcal{W}$ via (1.31) is positive, see lemma in the appendix of [168]. Hence, a map $\mathcal{M}$ is positive but not completely positive iff the corresponding operator $\mathcal{W}$ is an entanglement witness. Despite this one to one correspondence, the separability criterion based on a positive but not completely positive map $\mathcal{M}$, mentioned previously 1.3.2, is stronger than the corresponding witness constructed via (1.31) [169]. However, the point of entanglement witnesses is its measurability in experiments, where, ideally, one is able to infer entanglement from a minimum number of measurements.

Often, in experiments, the target entangled state $|\Psi\rangle$ is known beforehand. In this case, on can construct the following entanglement witness

$$\mathcal{W} = \alpha\mathbb{1} - |\Psi\rangle\langle\Psi| \tag{1.32}$$

which depicts how "similar" a state is to the entangled state $|\Psi\rangle$. α is taken to be the maximum overlap of product states with $|\Psi\rangle$ [170]:

$$\alpha = \max_{|\phi\rangle product} |\langle\phi|\Psi\rangle|^2 \tag{1.33}$$

Hence, any state ρ with fidelity[12] $\langle\Psi|\rho|\Psi\rangle$ exceeding α is entangled. This kind of witnesses, despite being unable to detect PPT entangled states, has the advantage of being easy to generalize for multipartite cases.
Another important way of constructing witnesses that can be generalized for the multipartite case was provided in [172] as follows: Given an observable H, one can construct the following entanglement witness

$$\mathcal{W}_H = H - \inf_{|\phi\rangle product} \langle\phi|H|\phi\rangle \tag{1.34}$$

where the infimum is taken for all pure product states. The above witness can be useful for a spin system whose hamiltonian has an entangled ground state [172].

Finally, we mention Bell inequalities which was firstly considered as entanglement witness by Terhal [161]. The simplest and most famous form is the CHSH inequality [72] which can be put into witness form as:

$$\mathcal{W}_{CHSH} = 2\mathbb{1} - \mathcal{A}_{CHSH} \tag{1.35}$$

where A_{CHSH} is the CHSH operator defined as:

$$\mathcal{A}_{CHSH} = A_1 \otimes B_1 + A_2 \otimes B_1 + A_1 \otimes B_2 - A_2 \otimes B_2 \tag{1.36}$$

and A_i and B_i are the dichotomic measurements performed by the first and second party with eigenvalues $\{-1, +1\}$. For the 2-qubit state, the maximum negative value for the above witness is $2 - 2\sqrt{2}$ that can be achieved with the singlet state [13]

$$|\Psi_-\rangle = \frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}} \tag{1.37}$$

and the following operators:

$$A_1 = \sigma_x\,,\; A_2 = \sigma_y\,,\; B_1 = -\frac{\sigma_x + \sigma_y}{\sqrt{2}}\,,\; B_2 = \frac{\sigma_y - \sigma_x}{\sqrt{2}} \tag{1.38}$$

[12] Fidelity of two quantum states ρ and σ is defined as $F(\sigma,\rho) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right]\right)^2$ [171]. If $\sigma = |\Psi\rangle$, we get $F(|\Psi\rangle\langle\Psi|,\rho) = \langle\Psi|\rho|\Psi\rangle$.

[13] or any maximally entangled state with the application of proper local unitaries to CHSH operator.

$2\sqrt{2}$ represents the maximum achievable value for $\mathcal{A}_{CHSH}$ in quantum mechanics and is called the Tsirelson bound [173] which can be exceeded when considering some non-local models other than quantum mechanics [174].
It is important to emphasize that the above witness rather witnesses non-locality which implies entanglement. In other words, $\mathcal{W}_{CHSH}$ is a non optimal witness that detects only entangled states that do not admit LHVM models. For example, Werner states (1.21) for the 2-qubit case, which can be written in this equivalent form:

$$\rho_W = (1-p)\,\frac{\mathbb{1}}{4} + p|\Psi_-\rangle\langle\Psi_-| \tag{1.39}$$

is entangled for $p > \frac{1}{3}$ and have been shown to admit a local hidden variable model for $p \leq 0.66$ [78]. The entangled Werner states are only detected by the CHSH witness for $p > \frac{1}{\sqrt{2}}$.

Entanglement measures

So far we have presented some of the criteria to certify the presence of entanglement in a state. However, if someone is interested in quantifying entanglement, which is useful, for example, in quantum communication and distillation protocols [175, 144, 176], one may use what is called an entanglement measure.
An entanglement measure is a function that maps any bipartite state ρ into a positive number $E(\rho) \in \mathbb{R}^+$ with the following axiomatic properties [177, 176]:

- For any separable state σ, $E(\sigma) = 0$ meaning that the state contains no entanglement.
- $E(\rho)$ should remain invariant under local unitary transformations; i.e. $E(\rho) = E\left(U \otimes V \rho U^\dagger \otimes V^\dagger\right)$, where U and V are unitaries on the first and second subsystem respectively. That is because the mentioned transformation represents a simple change of basis that is incapable of changing the present correlations in a state.
- Since manipulating entanglement in quantum information and communication protocols is usually done within the paradigm of LOCC operations, where entanglement is a resource and LOCC do not create entanglement, $E(.)$ should not increase on average under such transformations.

Any function verifying the above requirements is called an entanglement monotone. One may require additional properties [125, 176] like convexity and additivity, however these

may be relaxed since they are not verified by some entanglement measures [178, 179]. As an example of an entanglement measure for pure states is the Von-Neumann entropy of any of the two subsystems. We can check easily that it verifies the above three conditions. To begin with, If the state is separable, i.e. of the form $|\Psi\rangle = |\psi^{(A)}\rangle \otimes |\psi^{(B)}\rangle$, then $S\left(\rho^{(A)}\right) = 0$. A local unitary would not change the eigenvalues of $\rho^{(A)}$, thus $S\left(\rho^{(A)}\right)$ is unchanged. Finally, it was shown by Nielsen [42] that a state $|\Psi\rangle$ can be transformed into another $|\Phi\rangle$ via LOCC operations iff $\lambda_\Psi \prec \lambda_\Phi$ where λ_Ψ and λ_Φ are the eigenvalues of the reduced density matrices of $|\Psi\rangle$ and $|\Phi\rangle$ respectively. This implies that $S\left(\rho_\Psi^{(A)}\right) \geq S\left(\rho_\Phi^{(A)}\right)$ since $S(.)$ is Schur concave. However, this measure is only defined for pure states and it fails for mixed states, which can be checked taking the completely mixed state as a counterexample. It was extended in [39] to mixed states by taking the convex roof of the Von neumann entropy:

$$E_F(\rho) = \inf_{p_k, |\Psi_k\rangle} \sum_k p_k S\left(\rho_k^{(A)}\right) \tag{1.40}$$

where $\rho_k^{(A)}$ is the reduced density matrix of the pure state $|\Psi_k\rangle$ and the infimum is taken over all pure states such that $\rho = \sum_k |\Psi_k\rangle\langle\Psi_k|$. This measure is called entanglement of formation and can be efficiently computed for 2×2 states [180]. It is, however, quite challenging to compute for higher dimensions because of the optimizations over all possible decomposition of a given density matrix.

Another famous entanglement measure is concurrence [181–183] defined for pure states as:

$$\mathcal{C}(\rho) = \sqrt{2\left(1 - \mathrm{Tr}\left[\rho^{(A)^2}\right]\right)} \tag{1.41}$$

Then the above measure can be extended to mixed states by taking the convex roof of the concurrence for mixed states. For the 2×2 case, concurrence has a simple form:

$$\mathcal{C}(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{1.42}$$

where $\{\lambda_i\}_{i=1}^4$ are the eigenvalues of

$$M = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \quad : \tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y \tag{1.43}$$

in descending order.

Finally, a very simple entanglement measure that can be calculated easily for mixed states is negativity [184] ,which is connected to the violation of the PPT criterion via

the relation:

$$\mathcal{N}(\rho) = \frac{\left\|\rho^{T_B}\right\|_1 - 1}{2} \tag{1.44}$$

Another equivalent measure is the logarithmic negativity [184]

$$E_{\mathcal{N}}(\rho) = \log_2 \left\|\rho^{T_B}\right\|_1 \tag{1.45}$$

where both of these measures is useful for NPT states only.
There are other measures based on geometrical considerations, distance, entanglement distillability and entanglement cost that I will skip here and leave it to the interested reader to look it up in these reviews for example [125, 176, 52].

1.4 Multipartite entanglement

If dealing with two systems introduced a very complex and rich features to the states, one should expect the same as one adds more systems where the dimension grows exponentially larger.
The notion of entanglement, in which we are interested, can be generalized relatively easily. However, we can distinguish different degrees of separability that are not possible with two systems only.

Full separability: We say that a state $\rho \in \mathcal{B}\left(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \cdots \otimes \mathcal{H}^{(N)}\right)$ to be fully separable iff it can be written in the form:

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)} \quad : p_k \geq 0\,,\ \sum_k p_k = 1 \tag{1.46}$$

On the other hand, one may divide the N-partite systems into different partitions,i.e. grouping some of the subsystems into one subsystem for example, which leads to the following notion of separability with respect to the new partition despite the presence or not of entanglement in the newly formed subsystems.

k-producibility [185]: We say that a state $\rho \in \mathcal{B}\left(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \cdots \otimes \mathcal{H}^{(N)}\right)$ is k-producible iff it can be written in the form:

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(n)} \quad : p_k \geq 0\,,\ \sum_k p_k = 1 \tag{1.47}$$

where the number of parties is $n \geq \frac{N}{k}$ and $\rho^{(i)}$ is a state of k_i particles such that $k_i \leq k$. If we denote the convex set of k-producible states by $\mathcal{S}_{\|}$, then one can have the following hierarchy $S_1 \subset S_2 \subset \cdots \subset S_{N-1} \subset S_N$ where S_1 is the set of fully separable states and S_N corresponds to the set of all states of the N-partite system, see Fig. 1.5.

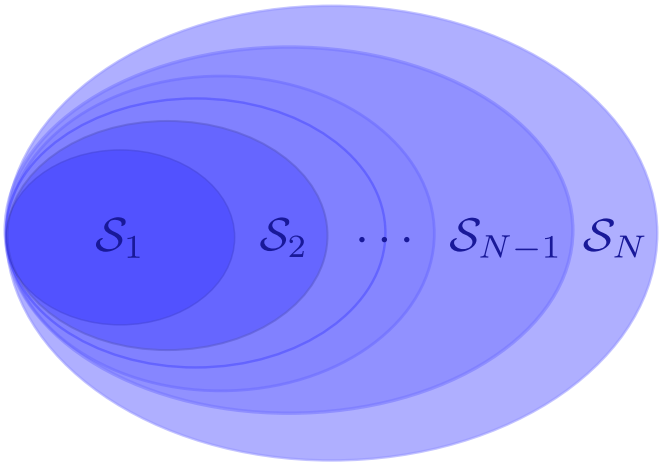

Figure 1.5: Schematic view of the hierarchy of the convex set S_k containing k-producible states for $1 \leq k \leq N$.

Entanglement depth: We say that a state $\rho \in \mathcal{B}\left(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \cdots \otimes \mathcal{H}^{(N)}\right)$ has entanglement depth k iff $\rho \in S_k \setminus S_{k-1}$. This means that one is unable to find a decomposition of the form (1.47) with parties containing less than k particles. Hence, states with entanglement depth k are called genuine k-entangled states [185]. In particular, states in the set $S_N \setminus S_{N-1}$ are called genuine multipartite entangled states.
At this point, mentioning the Greenberger-Horne-Zeilinger (GHZ) states is a must [186, 187]. This state can be defined for N-qubits as:

$$|GHZ\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}} \tag{1.48}$$

and can be generalized for N-qudits:as:

$$|GHZ_d\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle^{\otimes N} \tag{1.49}$$

These states can be considered as maximally entangled states for N-partite systems analogous to those for 2-qudits. These states were initially proposed to refute Einstein-Podolsky-Rosen (EPR) ideas on the relation between locality and elements of reality with

quantum mechanics, but they proved to be extremely useful for many applications like metrology [188] and quantum computation [189]. However, despite it being maximally entangled, it is extremely sensitive to particle loss where the loss of one particle will give a completely separable state.

Another class of genuine multipartite entangled states are the W states defined for 3-qubits as [149]:

$$|W_3\rangle = \frac{1}{3}\left(|0,0,1\rangle + |0,1,0\rangle + |1,0,0\rangle\right) \tag{1.50}$$

They can be written for N-qubits in the form [149]:

$$|W_N\rangle = \frac{1}{\sqrt{N}}|N-1,1\rangle \tag{1.51}$$

where $|N-1,1\rangle$ is the completely symmetric state with $N-1$ zeros and 1 ones. Unlike, the GHZ states, the W-states have entangled reduced density matrices for any pair of particles [149].

Entanglement detection in multi-partite systems

Since entanglement structure of multipartite systems is richer than the bi-partite case, entanglement criteria are not always a simple generalization of the bi-partite case even in the pure case. For instance, Schmidt decomposition exists for all pure bi-partite states, but not all pure entangled states admit a generalized Schmidt decomposition [190]. An example of states admitting a generalized Schmidt decomposition would be the GHZ states (1.49) and one that does not admit one is the W state (1.50).

However, many of the bi-partite entanglement criteria may prove useful for the multi-partite case. If one splits the total system into two subsystems, one can apply the bipartite criteria. For example, for pure states of spin systems, the Von-neumann entropy of a bi-partition is studied to understand how entanglement is spreading for large systems [191]. Other bi-partite criteria can be extended as well, like the range criterion [52] and the realignment criterion [192]. Moreover, there are many extensions of Bell inequalities to the multi-partite case like Mermin inequalities [193] ans Svetlichny inequality [194]. For more details on the different multi-partite entanglement criteria, the reader might check these reviews [52, 125] and the references there in. As for the rest of this chapter, we will only focus on entanglement detection using collective operators, especially with spin squeezing inequalities.

1.5 Spin squeezing inequalities

In this section, I will try to recall, very briefly, the theory of squeezd spin states. The most famous definition of spin squeezing is due to Kitagawa and Ueda [195] which naturally attracted a lot of attention later on. The notion of squeezed light had already been introduced as a handy way of reducing quantum fluctuations of a light quadrature below the Heisenberg limit. The definition of squeezing for spin systems, like in optical systems, is therefore an essential tool for achieving measurement precision beyond the standard quantum limit SQL in many metrology protocols [196]. SQL or shot-noise limit, in atomic interferometry experiments, is the limit of precision when we use uncorrelated atoms and is given by $\frac{1}{\sqrt{N}}$ for N uncorrelated particles. However, the ultimate limit allowed by quantum mechanics, for linear metrology, is the Heisenberg limit with scaling $\frac{1}{N}$ [197].

Moreover, spin squeezing in multipartite systems can be a sign of entanglement. In fact, for a system of N spin-$\frac{1}{2}$, no spin squeezing can be achieved without entanglement. In that regard, a spin squeezing criteria can be used as entanglement witness [198–201]. This is not surprising, taking the previous point into consideration. It is well known that entanglement is required to beat the shot noise limit in linear metrology ignoring, for the moment, the discussion on the required type and scaling of entanglement [202–204]. Hence, squeezed states, that give a better scaling than the shot noise limit, are entangled.

1.5.1 The concept of squeezing

Coherent states were introduced as early as 1926 by Schrödinger [205], where he introduced them as quantum states of a harmonic oscillator that behave classically; i.e. they give rise to expectation values that follow equations of classical mechanics. Then, it was Glauber [60] who coined the name for these semi-classical states for quantum optics.
The concept of squeezing was firstly introduced in bosonic systems [59, 206, 207, 55]. In such systems we can define a position operator X and a momentum operator P in terms of the bosonic mode creation and annihilation operators, $a^\dagger$ and a:

$$X = \frac{a + a^\dagger}{\sqrt{2}} \quad , \quad P = \frac{a - a^\dagger}{i\sqrt{2}}. \tag{1.52}$$

a and $a^\dagger$ verify the bosonic commutation relation:

$$\left[a, a^\dagger\right] = \mathbb{1} \tag{1.53}$$

where $\mathbb{1}$ is the identity over the infinte dimensional Hilbert space spanned by Fock states [208] $|n\rangle$ which are defined as the eigenstates of the number operator $N = a^\dagger a$ and verify the following relations:

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad , \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad , \quad N|n\rangle = n|n\rangle \tag{1.54}$$

From Eq. (1.53), we find that X and P (1.52) verify the following commutation relation:

$$[X, P] = i \,^{14} \tag{1.55}$$

Putting the above relation into the Heisenberg uncertainty inequality [209–211] for two operators A and B:

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{|[A, B]|^2}{4}, \tag{1.56}$$

where $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$ is the variance of A, we get:

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4}. \tag{1.57}$$

There are no states that violate the above inequality. However, there are states that do saturate it and these states are coherent states and squeezed states.

Coherent states

Coherent states can be defined as the states that saturate the Heisenberg uncertainty relation (1.57) with the additional property:

$$\Delta X = \Delta P = \frac{1}{\sqrt{2}} \tag{1.58}$$

These states can also be defined as the eigenstates of the annihilation operator a as:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{1.59}$$

When written in Fock basis, it gives the following expression:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1.60}$$

[14] i here stands for $i\mathbb{1}$.

From the above relation, one can easily see that coherent states are not orthogonal, since for any coherent states, say $|\alpha\rangle$ and $|\beta\rangle$, we have:

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{1.61}$$

Hence, the set of coherent states is over-complete since they also verify

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = \mathbb{1}. \tag{1.62}$$

A special case of these coherent states is the vacuum state $|0\rangle$ which allows to obtain all other coherent states via the application of the displacement operator $\mathcal{D}(\alpha)$ [60]:

$$|\alpha\rangle = \mathcal{D}(\alpha)|0\rangle \quad , \mathcal{D}(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{1.63}$$

which is a transformation that belongs to the group generated by the Heisenberg-Weyl algebra given by the three commutation relation [212]:

$$\left[a, a^\dagger\right] = \mathbb{1}, \quad [a, \mathbb{1}] = 0, \quad \left[a^\dagger, \mathbb{1}\right] = 0 \tag{1.64}$$

This method of applying a displacement operator on a reference state can be generalized to construct generalized coherent states for different algebras and can be used to construct coherent spin states [212], which will be introduced later on.

Squeezed states

Squeezed states, like coherent states, saturate the Heisenberg uncertainty relation (1.57) with the exception that

$$\Delta X \neq \Delta P. \tag{1.65}$$

Squeezed states can be obtained by applying the squeezing operator [213, 214]

$$S(\zeta) = e^{\frac{1}{2}\left(\zeta a^2 - \zeta^* a^{\dagger 2}\right)} \quad ; \zeta = re^{i\theta} \tag{1.66}$$

to a coherent state $|\alpha\rangle$ to obtain the squeezed state

$$|\zeta;\alpha\rangle = S(\zeta)|\alpha\rangle. \tag{1.67}$$

If we define a rotated quadratures X_θ and P_θ in the X-P plane:

$$X_\theta = \quad e^{i\frac{\theta}{2}a^\dagger a} X e^{-i\frac{\theta}{2}a^\dagger a} = \frac{a e^{-i\frac{\theta}{2}} + a^\dagger e^{i\frac{\theta}{2}}}{\sqrt{2}} \tag{1.68}$$

$$P_\theta = \quad e^{i\frac{\theta}{2}a^\dagger a} P e^{-i\frac{\theta}{2}a^\dagger a} = \frac{a e^{-i\frac{\theta}{2}} - a^\dagger e^{i\frac{\theta}{2}}}{i\sqrt{2}} \tag{1.69}$$

Then the variances of X_θ and P_θ in the squeezed state $|\zeta;\alpha\rangle$, where $\zeta = re^{i\theta}$, is found to be[59]:

$$(\Delta X_\theta)^2 = \frac{e^{-2r}}{2}, \quad , \quad (\Delta P_\theta)^2 = \frac{e^{2r}}{2}, \tag{1.70}$$

where we can see the effect of the squeezing operator on the coherent state is to reduce the variance along the quadrature X_θ by a factor $\frac{e^{-2r}}{2}$. This will result in increasing the orthogonal quadrature P_θ by a factor $\frac{e^{2r}}{2}$ so that the quantity $(\Delta X_\theta)^2 (\Delta P_\theta)^2 = \frac{1}{4}$ remains unchanged. r, hence, determines the squeezing and is called squeezing parameter. A nice illustration for coherent and squeezd states can be found at [215].

1.5.2 Coherent spin states

In a similar way to bosonic coherent states, one can define coherent spin states for [216, 212, 217] based on the action of of $SU(2)$ rotations on a finite dimensional Hilbert space, namely the space of a spin-j particle. In this case the algebra that generates the group is 3-dimensional, the basis (J_z, J_+, J_-) ,where $J_\pm = J_x \pm iJy$, satisfying the commutation relations:

$$[J_z, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = 2J_z \tag{1.71}$$

Noticing that $J_-|j,-j\rangle = 0$ [15], one can define coherent spin states by applying the Wigner rotation matrix $\mathcal{D}(\theta,\phi)$ to the reference state $|j,-j\rangle$;

$$|\theta,\phi\rangle = \mathcal{D}(\theta,\phi)|j,-j\rangle \tag{1.72}$$

where

$$\mathcal{D}(\theta,\phi) = e^{\xi J_+ - \xi^* J_-} = e^{i\theta(\sin\phi J_x - \cos\phi J_y)} \tag{1.73}$$

[15] $|j,m\rangle$ with $m \in \{-j,\cdots,j\}$ are simultaneous eigen-states of J_z and the Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ with eigenvalues m and $j(j+1)$ respectively.

and $\xi = \frac{\theta}{2}e^{-i\phi}$. In other words, $\mathcal{D}(\theta,\phi)$ represents a rotation by an angle θ around the axis y', obtained by a rotation of the axis y by an angle ϕ around the z-axis. The analogy with bosonic coherent states is quite remarkable, mainly if we replace the annihilation and creation operators, a and $a^\dagger$, with the ladder operators J_- and J_+. Also the state $|j,-j\rangle$ plays the role of the vacuum state $|0\rangle$ of the bosonic systems.
The analogy becomes even clearer in the high spin number limit $j \to \infty$. First we consider an equivalent form of the Wigner matrix [217]:

$$\mathcal{D}(\theta,\phi) = e^{\xi J_+ - \xi^* J_-} = e^{\tau J_+} e^{\ln\left(1+|\tau|^2\right) J_z} e^{-\tau^* J_-} \tag{1.74}$$

where as mentioned above $\xi = \frac{\theta}{2}e^{-i\phi}$ and $\tau = \tan\frac{\theta}{2}e^{-i\phi}$. From this, we can get an equivalent definition of coherent spin states (1.72)

$$|\theta,\phi\rangle = \frac{1}{\left(1+|\tau|^2\right)^j} e^{\tau J_+}|j,-j\rangle \tag{1.75}$$

Now performing the following substitutions

$$J_+ = \sqrt{2j}a^\dagger \quad , \quad \tau = \frac{\alpha}{\sqrt{2j}}, \tag{1.76}$$

we can see that the spin coherent state approaches a bosonic coherent state $|\alpha\rangle$ for $j \to \infty$.
Coherent spin states defined in Eq. (1.72) are completely polarized along the direction $\vec{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$; i.e. if we define the spin component along the direction $\vec{n}$, $J_{\vec{n}} = \mathcal{D}(\theta,\phi)J_z\mathcal{D}(\theta,\phi)^\dagger$, we have:

$$J_{\vec{n}}|\theta,\phi\rangle = -j|\theta,\phi\rangle \tag{1.77}$$

Like bosonic coherent states, coherent spin states form an over complete basis with the following resolution of the identity:

$$\frac{2j+1}{4\pi}\int \sin\theta d\theta d\phi |\theta,\phi\rangle\langle\theta,\phi| = \mathbb{1} \tag{1.78}$$

And they can be expanded in terms of the orthogonal basis $\{|j,m\rangle\}_m$:

$$|\theta,\phi\rangle = \frac{1}{\left(1+|\tau|^2\right)^j}\sum_{m=-j}^{j}\sqrt{C_{2j}^{j+m}}\tau^{j+m}|j,m\rangle \tag{1.79}$$

More importantly, coherent spin states saturate the Heisenberg uncertainty inequality (1.56):

$$(\Delta J'_x)^2 \left(\Delta J'_y\right)^2 \geq \frac{\langle J'_z\rangle^2}{4} \tag{1.80}$$

where J'_k, $k = x, y, z$, are rotated J_k in certain directions, like $J'_k = \mathcal{D}(\theta,\phi)J_k\mathcal{D}(\theta,\phi)^\dagger$ for example. However, an important distinction with the bosonic case (1.57) is that the right hand side of the above inequality is not a constant and does change with the state in question. This is why there is no unique definition of squeezed spin states like it is the case for bosonic squeezed states, as will be explained in the next section.

1.5.3 Squeezed spin states and spin squeezing parameters

In similar way to squeezed bosonic states, and since coherent spin states verify:

$$(\Delta J'_x)^2 = \left(\Delta J'_y\right)^2 = \frac{|\langle J'_z\rangle|}{2} \tag{1.81}$$

where $J'_k = \mathcal{D}(\theta,\phi)J_k\mathcal{D}(\theta,\phi)^\dagger$ and $k = x, y, z$ for some θ and ϕ, we are tempted to define squeezed states to be the states that verify [218]:

$$(\Delta J'_x)^2 < \frac{|\langle J'_z\rangle|}{2}. \tag{1.82}$$

Or equivalently, define a squeezing parameter

$$\zeta^2 = \frac{2\,(\Delta J'_k)^2}{|\langle J'_l\rangle|}, \tag{1.83}$$

where $k \neq l$ stand for two directions among x, y, z, such that whenever we have $\zeta < 1$, we conclude that the state is squeezed.

However, the above definition fails as it classifies some of the coherent spin states as squeezed states. To see this, when considering coherent spin states (1.72), one should keep in mind that the parameter (1.83) has two coordinate dependencies: firstly, the coherent state (1.72) depends on (θ,ϕ). Secondly, the rotated operator $J'_k = \mathcal{D}(\theta',\phi')J_k\mathcal{D}(\theta',\phi')^\dagger$, with $k = x, y, z$, depends on the coordinates (θ',ϕ'). In the case where (θ,ϕ) and (θ',ϕ') are the same, we get $\zeta = 1$ as we have seen before. Yet, when for some different directions; i.e. $(\theta,\phi) \neq (\theta',\phi')$, we have $\zeta < 1$ inferring squeezing of these coherent states[195, 216]. In other words, a simple coordinate change can provide squeezing! This is useless squeezing since coordinate change in the sense defined above provides

no quantum correlations among particles, in the multi-partite case, and , hence, cannot provide better precision in metrology applications, as expected from squeezed states.
The definition (1.83) needs modification to obtain a meaningful squeezing parameter. It was first done by Kitagawa and Ueda [195] where they defined the following squeezing parameter:

$$\zeta^2 = \frac{2\min_{\vec{n}_1}\left(\Delta J_{\vec{n}_1}\right)^2}{\left|\langle \vec{J} \rangle\right|} = \frac{2\min_{\vec{n}_1}\left(\Delta J_{\vec{n}_1}\right)^2}{j}. \tag{1.84}$$

The minimum is taken over all directions $\vec{n}_1$ in the plane orthogonal to $\vec{n}_0$, the mean spin direction for which $\langle J_{\vec{n}_0} \rangle = \left|\langle \vec{J} \rangle\right|$.
Comparing (1.83) with (1.84), we can see two differences between them. The first one is that we are taking the minimum variance of the spin component in the plane orthogonal to $\vec{n}_0$. Secondly, and more importantly, $\vec{n}_0$ is taken to be along the mean spin direction. Since coherent spin states are completely polarized (1.77), we have $|\langle J_{\vec{n}_0} \rangle| = j$, hence, the second equality in (1.84). With these modifications, all coherent states yield $\zeta = 1$, and simple coordinate change cannot produce spin squeezing. Kitagawa and Ueda studied the case where the spin j is composed of $2j$ spin $\frac{1}{2}$ particles and argued that any spin squeezing is due to inter-particle quantum correlations [195].

Later on, Wineland et al. [196] defined a slightly modified version of the squeezing parameter (1.84):

$$\zeta_R^2 = \frac{2j\min_{\vec{n}_1}\left(\Delta J_{\vec{n}_1}\right)^2}{\langle J_{\vec{n}_0} \rangle^2}, \tag{1.85}$$

where $\vec{n}_0$ and $\vec{n}_1$ are the same as explained above. They showed that this squeezing parameter is connected to the sensitivity of spin to rotation. In fact, they have defined the squeezing parameter (1.85) as

$$\zeta_R^2 = \frac{(\Delta\theta)^2}{(\Delta\theta_{CSS})^2} \tag{1.86}$$

where θ is the angle of rotation that we would like to measure when rotating J_y, for example [196], around J_x;

$$J_y' = e^{i\theta J_x} J_y e^{-i\theta J_x} = \cos\theta J_y - \sin\theta J_z. \tag{1.87}$$

$(\Delta\theta_{CSS})^2$ represents the best precision that can be achieved in measuring θ using coherent spin states. For the multi-partite case, for $N = 2j$ spin-$\frac{1}{2}$ particles, that Wineland et al.

considered in [196], $(\Delta\theta_{CSS})^2$ is the best precision achievable using uncorrelated particles. In this case, it can be easily shown that

$$(\Delta\theta_{CSS})^2 = \frac{1}{N} = \frac{1}{2j} \tag{1.88}$$

which accounts for the factor $2j$ in the nominator in (1.85) and is known to be the SQL for N particles.

To see where the rest comes from, we may assume, without loss of generality, the state to be polarized along the z-direction;i.e.

$$\langle J_{\vec{n}_0}\rangle = \langle J_z\rangle \text{ and } \langle J_x\rangle = \langle J_y\rangle = 0.$$

Then, using the relation, verified by a function $f(\theta)$,

$$\Delta f(\theta) = \left|\frac{\partial f(\theta)}{\partial\theta}\right| \Delta\theta \tag{1.89}$$

we find that:

$$(\Delta\theta)^2 = \frac{\left(\Delta J'_y\right)^2}{\left|\frac{\partial\langle J'_y\rangle}{\partial\theta}\right|^2} = \frac{\left(\Delta J'_y\right)^2}{\cos^2\theta\,|\langle J_z\rangle|^2}. \tag{1.90}$$

For small angles of rotation, the above equation is minimized for $\theta = 0$ which results in (1.85).

This clearly shows the connection between spin squeezing and the improved sensitivity to rotations beyond the standard quantum limit. Such improvement cannot be achieved, for most metrology protocols, without inter-particle correlations.

Even for the case of single spin j, with $j > \frac{1}{2}$, we can see it as if it was a collection of spin-$\frac{1}{2}$ particles, where any state of the spin-j can be represented as a symmetric Dicke state [219] of $2j$ spin-$\frac{1}{2}$ particles. Hence, spin squeezing in the single particle case can be seen as entanglement of $2j$ fictitious particles.

One might be tempted to use this argument to understand why there are no spin squeezed states for a spin-$\frac{1}{2}$. However, this is not rigorous since entanglement and spin squeezing are not equivalent. For example, the state $|\frac{1}{2},+\frac{1}{2}\rangle$ can be seen as an entangled state of two spins, say j and $j+\frac{1}{2}$, with j being an integer or half-integer:

$$|\frac{1}{2},+\frac{1}{2}\rangle = \sum_{m_1,m_2} |j,m_1;j+\frac{1}{2},m_2\rangle\langle j,m_1;j+\frac{1}{2},m_2|\frac{1}{2},+\frac{1}{2}\rangle \tag{1.91}$$

where $\langle j_1, m_1; j_2, m_2 | J, M\rangle$ is the Clebsch-Gordan coefficient. This state, despite being entangled, is not squeezed as all spin-$\frac{1}{2}$ states are obtained by a rotation from the state $|\frac{1}{2}, -\frac{1}{2}\rangle$ and, thus, they are coherent states according to the definition (1.72).

1.5.4 Spin squeezing and entanglement

So far, we have seen the connection of the squeezing parameter (1.85) to improved precision in metrology when using squeezed states. This improvement, for multi-parite case is attributed to entanglement. Nonetheless, the way (1.85) is defined does not distinguish single particle from multi-particle case. As we discussed above, squeezing in the single particle case can be attributed to entanglement among fictitious spin-$\frac{1}{2}$ particles. In other words, Definition (1.85) does not distinguish between physical and fictitious entanglement. The former being the most interesting for most applications, it is important to define a squeezing parameter, that we want to use as entanglement witness, for the multi-particle case explicitly.

As we have discussed above, a spin-j can be seen as a collection of N spin-$\frac{1}{2}$ particles such that $N = 2j$. Hence, we may simply replace j by $\dfrac{N}{2}$ in (1.85) to get the following squeezing parameter for N spin-$\frac{1}{2}$ particles:

$$\zeta^2 = \frac{N \min_{\vec{n}_1} \left(\Delta J_{\vec{n}_1}\right)^2}{\langle J_{\vec{n}_0}\rangle^2}. \tag{1.92}$$

Sørensen et al. showed in [220] that $\zeta \geq 1$ for all separable states. In other words, squeezed states, for which $\zeta < 1$, are entangled.
The appealing feature about the above entanglement witness is that it uses only the expectation values and variance of global or collective operators. This is very appealing from experimental point of view where measuring individual particles might be a daunting task. This might be even more difficult when considering two-particle correlations, essential for witnessing entanglement. Whereas the variance in (1.92) encompasses an average over all two-particle correlations since:

$$\left\langle J_k^2 \right\rangle = \left\langle \left(\frac{1}{2} \sum_{n=1}^{N} \sigma_k^{(n)} \right)^2 \right\rangle = \frac{N}{4} + \frac{1}{4} \sum_{n \neq m}^{N} \left\langle \sigma_k^{(n)} \sigma_k^{(m)} \right\rangle$$

where k is x, y or z and σ_k are Pauli operators[16].

At this point, multiple questions come to mind. In no particular order, the first questions is how small ζ^2 can get and what is the ultimate squeezing limit achievable for spin systems. The second question is whether it is possible to generalize and elaborate other squeezing parameters to use as entanglement witness. If there are, how many are needed to detect as many entangled states as possible. Last question, but not the least, is what kind of entanglement can be detected using these squeezing parameters which uses only the variance of collective spin operators to capture two-particle correlations. While it is obvious that no entanglement can be detected using one-particle moments, the same cannot be said about the entanglement depth of states detected using two particle correlations. Next, we will go over the first two questions in detail and briefly over the last one.

Extreme spin squeezing

Let us begin with the first question, which was answered by Sørensen and Mølmer in their famous paper [220]. They have looked at the squeezing parameter Eq. 1.85:

$$\zeta_R^2 = \frac{2j\left(\Delta J_x\right)^2}{\langle J_z\rangle} \tag{1.93}$$

where the x-axis is the direction that minimizes the variance, $\vec{n}_1$ for a state completely polarized along the z-direction. They have set out to find out the states that the smallest ζ_R^2 possible for a given $\langle J_z\rangle$ which , in turn, will have less noise for a given value of signal in spectroscopy. They have shown that, for integer values of j, these states are eigenstates corresponding to the minimum eigenvalue of the Hamiltonian:

$$H = J_x^2 + \mu J_z \tag{1.94}$$

where the parameter μ is a Lagrange multiplier to ensure the value of $\langle J_z\rangle$. By varying μ over a wide range, one is able to numerically determine the minimum eigenvalue and calculate the corresponding $\left(\Delta J_x\right)^2$ and $\langle J_z\rangle$. Then, one is able to find the state that verifies $\left(\Delta J_x\right)^2 = \langle J_x^2\rangle$ [17] for the given value of $\langle J_z\rangle$. However, for half-integer values of j, states that minimize $\langle J_x^2\rangle$ do not minimize $\left(\Delta J_x\right)^2$ for a given value of $\langle J_z\rangle$. This

[16] We have used the following relation for Pauli operators: $\{\sigma_i, \sigma_j\} = 2\delta_{i,j}\mathbb{1}$, with $\{\,.\,,\,.\,\}$ being the anti-commutator.

[17] $\langle J_x\rangle = 0$ since the state is polrized along the z- direction.

is because the minimum eigenvalue of J_z^2 is $\frac{1}{4}$ and states with $\langle J_x\rangle = \langle J_y\rangle = 0$ do not minimize the variance $(\Delta J_x)^2$ for a given $\langle J_z\rangle$, hence, not the most squeezed according to (1.93). The above method can be extended, for half-integer j, by considering the Hamiltonian $J_x^2+\mu J_z+\nu J_x$ with two Lagrange multipliers μ and ν, see ref [221] for details.

Consequently, with the above numerical procedure in mind, one is able to calculate the function [220]:

$$F_j(X) = \frac{1}{j}\min_{\langle J_z\rangle/j=X}(\Delta J_x)^2 \tag{1.95}$$

which is the minimum variance of J_x divided by j for a given value of $\langle J_z\rangle$. As shown in [220], $F_j(X)$ is convex and monotonically increasing with values in the interval $[0, 0.5]$, where $F_j(0) = 0$ and $F_j(1) = 0.5$. Finally, for $j_1 \geq j_2$, we have $F_{j_1}(X) \leq F_{j_2}(X)$.
Once $F_j(X)$ is defined, Sørensen and Mølmer [220] showed that for all separable N j-spin states, the following inequality is verified:

$$(\Delta J_x)^2 \geq NjF_j\left(\frac{\langle J_z\rangle}{Nj}\right) \tag{1.96}$$

As a result, any state violating the above inequality is entangled. They even showed the amount of violation of the above inequality can be used to detect entanglement depth where we may replace $F_j(.)$ by $F_{kj}(.)$ in the above inequality. Hence, we can write a set of inequalities as follows:

Extreme spin squeezing inequalities: All k-producible states of N spin-j particles verify the following inequality:

$$(\Delta J_x)^2 \geq NjF_{kj}\left(\frac{\langle J_z\rangle}{Nj}\right) \tag{1.97}$$

Hence, any state that violates the above inequality is entangled and has $k+1$ entanglement depth. In what follows, I will recall the proof for these inequalities, presented in [220, 222, 223], which will be used later in chapter 3. We start by proving the inequalities for product k-producible states of the form

$$\rho = \rho^{(1)}\otimes\cdots\otimes\rho^{(m)} \quad : m \geq \frac{N}{k}$$

where each state $\rho^{(i)}$ is a state of $k_i \leq k$ particles such that $\sum_{i=1}^m k_i = N$. If we define the operators $J_\alpha^{(i)}$, with $\alpha = x, y, z$, to be the operators of a spin-$(k_i j)$ particle such that

$J_\alpha = \sum_{i=1}^m J_\alpha^{(i)}$, we can write

$$(\Delta J_x)^2 = \sum_{i=1}^m \left(\Delta J_x^{(i)}\right)^2 \geq \sum_{i=1}^m k_i j F_{k_i j}\left(\frac{\langle J_z^{(i)}\rangle}{k_i j}\right) \geq \sum_{i=1}^m N j \frac{k_i}{N} F_{kj}\left(\frac{\langle J_z^{(i)}\rangle}{k_i j}\right)$$

where the first equality follows from the fact we have a product state ρ, and the last inequality follows from the fact that $k_i \leq k$ and the properties of the function $F_j(.)$. Finally, from the convexity of the function $F_j(.)$, we get:

$$(\Delta J_x)^2 \geq N j F_{kj}\left(\sum_{i=1}^m \frac{k_i}{N}\frac{\langle J_z^{(i)}\rangle}{k_i j}\right) = N j F_{kj}\left(\frac{\langle J_z\rangle}{N j}\right) \tag{1.98}$$

This finishes the proof for product states. For mixed states, inequalities (1.97) follow from the concavity of the variance and the convexity of the function $F_{kj}(.)$.

Optimal spin squeezing inequalities

Now, we move on to the next question about generalizing spin squeezing parameters which is discussed in great detail in [223]. As a starter point, we see that that the squeezing parameter (1.92) can be put into an equivalent inequality for separable states as:

$$N \min_{\vec{n}_1} (\Delta J_{\vec{n}_1})^2 \geq \langle J_{\vec{n}_0}\rangle^2$$

Other squeezing parameters were proposed to detect entanglement in N spin-$\frac{1}{2}$ particles. Raghavan et al. proposed the following squeezing parameter

$$\zeta_D^2 = \frac{N(\Delta J_{\vec{n}})^2}{N^2/4 - \langle J_{\vec{n}}\rangle^2} \tag{1.99}$$

to detect entanglement in Dicke states [224], which can be turned into the following inequality

$$N(\Delta J_{\vec{n}})^2 \geq N^2/4 - \langle J_{\vec{n}}\rangle^2 \tag{1.100}$$

for separable states. Another spin squeezing inequality

$$(\Delta J_{\vec{n}_1})^2 + (\Delta J_{\vec{n}_2})^2 \geq N C_j \tag{1.101}$$

was proposed by He et al. [225] to detect entanglement of planar squeezed states; i.e. states where the direction of minimum variance and the mean spin direction lie in the same plane. The above inequality is defined for N spin-j particles where $\vec{n}_1$ and $\vec{n}_2$ define

the plane of squeezing and C_j represent the lower bound of the sum of two variances $\left(\Delta J^{(i)}_{\vec{n}_1}\right)^2 + \left(\Delta J J^{(i)}_{\vec{n}_2}\right)^2$ for a single spin-j.

All the different spin squeezing inequalities mentioned above are linear combinations with respect to the quantities $\{\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle\}$ and $\{\langle J_x\rangle^2, \langle J_y\rangle^2, \langle J_z\rangle^2\}$. While certain combinations detects certain entangled states better than others , it is interesting to ask what is the maximum amount of entangled states detected using these quantities only. Tóth et al. [201, 226] introduced a complete set of generalized spin squeezing inequalities for N spin-$\frac{1}{2}$ particles and later were generalized to the following inequalities for arbitrary spin [227, 228]:

$$\tilde{\Delta}^2 J_x + \tilde{\Delta}^2 J_y + \tilde{\Delta}^2 J_z \geq -Nj^2 \tag{1.102a}$$

$$(N-1)\left[\tilde{\Delta}^2 J_k + \tilde{\Delta}^2 J_l\right] \geq \langle \tilde{J}_m^2\rangle - N(N-1)j^2 \tag{1.102b}$$

$$\langle \tilde{J}_l^2 + \tilde{J}_m^2\rangle - N(N-1)j^2 \leq (N-1)\tilde{\Delta}^2 J_k \tag{1.102c}$$

$$\langle \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2\rangle \leq N(N-1)j^2, \tag{1.102d}$$

where $\langle \tilde{J}_k^2\rangle = \langle J_k^2\rangle - \sum_{i=1}^N \langle \left(J_k^{(i)}\right)\rangle$, and $\tilde{\Delta}^2 J_k = \langle \tilde{J}_k^2\rangle - \langle J_k\rangle^2$ are the modified second moment and the modified variance of the spin component $k \in \{x, y, z\}$.
It was shown that these inequalities are complete if one considers the quantities $\{\langle \tilde{J}_x^2\rangle, \langle \tilde{J}_y^2\rangle, \langle \tilde{J}_z^2\rangle\}$ and $\{\langle J_x\rangle^2, \langle J_y\rangle^2, \langle J_z\rangle^2\}$. For fixed values $\{\langle J_x\rangle^2, \langle J_y\rangle^2, \langle J_z\rangle^2\}$, the above inequalities (1.102) define a polytope in the space $\{\langle \tilde{J}_x^2\rangle, \langle \tilde{J}_y^2\rangle, \langle \tilde{J}_z^2\rangle\}$. For the limit of large N, all states corresponding to points inside the polytope are separable [228]. Hence, the optimality of these inequalities in the limit of very large N where any entangled state, detected by some combination of the first and second spin moments, is going to be detected by one of the inequalities (1.102a)-(1.102c). The last inequality (1.102d) is verified for all states.

In order to prove the above inequalities, one may start by rewriting the four inequalities into one compact inequality [227, 228]:

$$(N-1)\sum_{k\notin\mathcal{I}} \tilde{\Delta}^2 J_k - \sum_{k\in\mathcal{I}} \langle \tilde{J}_k^2\rangle \geq -N(N-1)j^2, \tag{1.103}$$

where $\mathcal{I}$ can be any subset of $\{x, y, z\}$ (including the empty set). Each inequality in Eqs. (1.102) is obtained by increasing the number of elements in $\mathcal{I}$ by one, starting from the empty set. Next, we may proceed to prove this inequality for product states like we

did before. for product states, we have the following facts:

$$\langle \tilde{J}_k^2 \rangle := \sum_{i\neq j} \langle J_k^{(i)} J_k^{(j)} \rangle = \sum_{i\neq j} \langle J_k^{(i)} \rangle \langle J_k^{(j)} \rangle = \langle J_k \rangle^2 - \sum_i \langle J_k^{(i)} \rangle^2 \tag{1.104}$$

$$\tilde{\Delta}^2 J_k := \langle \tilde{J}_k^2 \rangle - \langle J_k \rangle^2 = -\sum_i \langle J_k^{(i)} \rangle^2 \tag{1.105}$$

Now we make use of the following two inequalities: Cauchy-Schwartz inequality

$$\langle J_k \rangle^2 \leq N \sum_{i=1}^{N} \langle J_k^{(i)} \rangle^2 \tag{1.106}$$

and the inequality of angular momentum j:

$$\langle J_x^{(i)} \rangle^2 + \langle J_y^{(i)} \rangle^2 + \langle J_z^{(i)} \rangle^2 \leq j^2 \tag{1.107}$$

Putting (1.106) and (1.104) into the left hand side of inequality (1.103), we find:

$$(N-1) \sum_{k\notin\mathcal{I}} \tilde{\Delta}^2 J_k - \sum_{k\in\mathcal{I}} \langle \tilde{J}_k^2 \rangle \geq -(N-1) \sum_i \sum_{k\in\{x,y,z\}} \langle J_k^{(i)} \rangle^2 \tag{1.108}$$

Finally, plugging (1.107) into the above inequality completes the proof of (1.103) for product states. Finally, using the concavity of the variance, one is able to prove (1.103) for all separable states.

It is important to note that Cauchy-Schwartz inequality is the main ingredient to proving the above inequality along side with the inequality (1.107). Both of these are not restrictive to spin operators. As it has been shown in Ref. [227], a set of M collective observables $A_k = \sum_{i=1}^{N} A_k^{(i)}$, where $k = 1, \cdots, M$, can be used instead. To be able to derive inequalities as Eq. (1.103), it is only required that there exists a positive number α such that

$$\sum_{k=1}^{M} \langle A_k^{(i)} \rangle^2 \leq \alpha^2; \forall i = 1, 2, \cdots, N, \tag{1.109}$$

which is satisfied by the spin operators $J_k^{(i)}$ with $\alpha = j$. Then, using Cauchy-Schwartz inequality, one is able to get the inequality

$$(N-1) \sum_{k\notin\mathcal{I}} \tilde{\Delta}^2 A_k - \sum_{k\in\mathcal{I}} \langle \tilde{A}_k^2 \rangle \geq -N(N-1)\alpha^2, \tag{1.110}$$

which is the same as concavity of the variance, we obtain the inequalities (1.103) with J_k replaced by A_k, j replaced by α and $\mathcal{I}$ being any subset of $\{1, \cdots, M\}$, including the empty set. This gives a huge freedom and opens the door for many choices to construct

new entanglement witnesses which we will use in chapter 3. One such choice was given in [227] using the $Su(d)$ generators for a system of N d-dimensional systems.

The above inequalities can be rearranged and put into the form of a squeezing parameter with some extra care taken to avoid negative values [228]. For example, eq. (1.102c) can be put into the following generalized spin squeezing parameter

$$\zeta_{os}^2 = (N-1)\frac{N\jmath^2 + \tilde{\Delta}^2 J_k}{\langle \tilde{J}_l^2 + \tilde{J}_m^2 \rangle} \tag{1.111}$$

defined for $\langle \tilde{J}_l^2 + \tilde{J}_m^2 \rangle > 0$, where if not verified, the corresponding state is separable. Squeezing in the sense of the above inequalities implies entanglement, hence it represents a generalized form of squeezing that ignores squeezing of individual spins. It was shown [226] that, for spin-$\frac{1}{2}$ systems, ζ_{os}^2 detects a wider class of entangled states than the squeezing parameter (1.92). The same observation holds for higher spin systems [228]. Moreover, Spin squeezing inequalities (1.102) can detect some PPT entangled states, that cannot be distilled with respect to any bi-partition of the multipartite spin system [226].

Finally, many generalizations to the spin squeezing inequalities [223] have been made such as coordinate system independent spin squeezing inequalities [226–228], which we will recall in the third chapter. Another generalization was made to the extreme spin squeezing inequalities (1.97) in [221]. Furthermore , the above results were extended to a classical fluctuations of particles' number N [229, 221]. Our main contribution was to generalize spin squeezing inequalities for particles presenting general fluctuations, quantum and/or classical [2], as we will see in chapter 3.

2

Mapping to two-dimensional systems: formalism and applications

In many situations where non classicality of a physical system is under investigation, the first attempt it to dichotomize the measurement outcomes, to finally obtain a yes/no or 0/1 answer. In most of the experiments, the physical system under consideration posses several degrees of freedoms, which measurements can give many outputs even continuous ones. Classifying the outcomes of the measurements into two categories such that standard test of non-classicality (like entanglement, non-locality, non contextuality or Legget-Garg) is a common practice in quantum information science. In each specific experimental situation a recipe is developed to perform this dichotomization.

The main objective of this chapter is to set-up a theoretical framework that allows a general description of this dichotomization procedure of a quantum system. To this end, I will first introduce how to map any system of arbitrary dimensions to a two level system. A general 2-level system is fully characterized by the 3 components of the Bloch vectors that are equal to the expectation value of the 3 Pauli operators. This is why the mapping introduced in this chapter is characterized by the expectation values of three observables, which are mapped to the expectation of the 3 Pauli operators of the 2-level system.

In a second step I will address bi-partite systems, by using the same mapping locally in each of the parts. In this way 2-qudit systems can be mapped to 2-qubit. Because the mapping acts locally it can not create entanglement, and map two separable qudits to two separable qubits. Hence, entanglement of the mapped 2-qubit state reveal entanglement of the original 2-qudits state.

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2.1 General framework

Let consider a state ρ in a d dimensional Hilbert space $\mathcal{H}^{(d)}$, we would like to define a mapping $\rho \to \mathcal{M}(\rho)$, such that $\mathcal{M}(\rho)$ represent a qubit state in $\mathcal{H}^{(2)}$. To realize this mapping, we consider the combined system formed by the original system in state ρ and an ancillary qubit initially in an arbitrary reference state $|0\rangle\langle 0|$. The mapping is obtained by switching on an interaction between both subsystems and then by discarding the original system and keeping the ancillary qubit. This very general procedure ensure that the final operator describing the qubit state is a physical state, that is a positive, hermitian, normalized 2×2 matrix.

The interaction between both subsystems can represented by an Hamiltonian H acting on the global system, original one plus the ancilla, for a specific amount of time Δt necessary for the change of the qubit state to take place. The resulting state of the global system at Δt will be $e^{-i\Delta t H}\rho \otimes |0\rangle\langle 0| e^{i\Delta t H}$, hence the state of the qubit will be:

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}^{(d)},}\left[e^{-i\Delta t H}\rho \otimes |0\rangle\langle 0| e^{i\Delta t H}\right]$$

where $\mathrm{Tr}_{\mathcal{H}^{(d)}}[\cdot]$ denote the partial trace with respect to the original system $\mathcal{H}^{(d)}$.

The above procedure can be summarized as

$$U : \mathcal{H}^{(d)} \otimes \mathcal{H}^{(2)} \to \mathcal{H}^{(d)} \otimes \mathcal{H}^{(2)}; \quad \text{where } U = e^{-i\Delta t H}$$

that entangles the original system with the ancillary qubit, then tracing out the original system. Applying only unitary operations to the global system is rather restrictive, and can be generalized by applying an *isometry* to a larger system composed of the original system and the ancillary qubit plus an arbitrary ancillary qudit. Equivalently, one may apply an isometry:

$$U : \mathcal{H}^{(d)} \otimes \mathcal{H}^{(2)} \to \mathcal{H}^{(D)} \otimes \mathcal{H}^{(2)}; U^\dagger U = \mathbb{1}_{2d}$$

Where $D > d$ and where the restriction $UU^\dagger = \mathbb{1}_{2d}$ for unitary operator U is relaxed. Thanks to the property $U^\dagger U = \mathbb{1}_{2d}$, the desired mapping

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}^{(D)}}\left[U\rho \otimes |0\rangle\langle 0| U^\dagger\right]$$

is positive and trace preserving. By changing the isometry, one might get a different dichotomization scheme. At this stage, the relationship between the chosen isometry and the dichotomisation form is unclear and complicated. In the next section I give an operational form of the mapping where one chooses the mapping form through a more convenient choice of three operators which expectations are the expectation if the σ_x,σ_y and σ_z components of the resulting qubit state.

2.1.1 General formalism: mapping to qubit

We start by noticing that the isometry U, from $\mathcal{H}^{(d)} \otimes \mathcal{H}^{(2)}$ to $\mathcal{H}^{(D)} \otimes \mathcal{H}^{(2)}$, which characterizes the mapping $\mathcal{M}_U$ can be parametrized by four linear operators A_i $(i = 0, 1, 2, 3)$ from $\mathcal{H}^{(d)}$ to $\mathcal{H}^{(D)}$ as follows:

$$U = \sum_{i=0}^{3} A_i \otimes \sigma_i, \tag{2.1}$$

where σ_i's, for $i = 1, 2, 3$, are the Pauli matrices and $\sigma_0 = \mathbb{1}$ is the identity operator in $\mathcal{H}^{(2)}$ and where A_i $(i = 1, \cdots, 4)$ are operators from $\mathcal{H}^{(d)}$ to $\mathcal{H}^{(D)}$. The isometric property $U^\dagger U = \mathbb{1}$ implies that:

$$\begin{aligned} &\vec{A}^\dagger \cdot \vec{A} + {A_0}^\dagger A_0 = \mathbb{1}, \\ &\vec{A}^\dagger A_0 + {A_0}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A} = \vec{0}, \end{aligned} \tag{2.2}$$

where $\vec{A} = (A_1, A_2, A_3)^T$. The vectorial form is simply a compact way of writing three equations, and unlike outer product of real vectors, the order of operators should be respected, e.g., $\left(\vec{A}^\dagger \wedge \vec{A}\right)_1 = {A_2}^\dagger A_3 - {A_3}^\dagger A_2$. In addition, in the special case where the operators A_i are Hermitian, one has $\vec{A}^\dagger = \vec{A}$ but not $\vec{A}^\dagger \wedge \vec{A} = \vec{0}$ necessarily. Actually this is only true when the different operators A_i for $i = 1, 2, 3$ mutually commute.

With the parametrization Eq. (2.1) the mapping defined in the previous section can be written as:

$$\begin{aligned}\mathcal{M}:\mathcal{B}\left(\mathcal{H}^{(d)}\right) &\rightarrow \mathcal{B}\left(\mathcal{H}^{(2)}\right):\\ \rho &\rightarrow \mathcal{M}(\rho)=\mathrm{Tr}_{\mathcal{H}^{(D)}}\left[U\rho\otimes|0\rangle\langle 0|U^{\dagger}\right]=\sum_{i,j=0}^{3}\left\langle {A_j}^{\dagger}A_i\right\rangle_{\rho}\sigma_i|0\rangle\langle 0|\sigma_j.\end{aligned} \tag{2.3}$$

It is worth noting that the explicit reference to $\mathcal{H}^{(D)}$ in the first line of (2.3) has become implicit in the product ${A_j}^{\dagger}A_i$, which is an operator in $\mathcal{B}\left(\mathcal{H}^{(d)}\right)$[1]. Inserting the relations fulfilled by the Pauli matrices:

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k+\delta_{ij}\mathbb{1} \tag{2.4}$$

$$\sigma_i\sigma_j\sigma_k = i\epsilon_{ijk}\mathbb{1}+\delta_{jk}\sigma_i-\delta_{ik}\sigma_j+\delta_{ij}\sigma_k \tag{2.5}$$

where ϵ_{ijk} is the Levi-Civita tensor, and $|0\rangle\langle 0|=\dfrac{\mathbb{1}+\sigma_3}{2}$ into (2.3), we get:

$$\begin{aligned}\mathcal{M}(\rho)=&\frac{1}{2}\sum_{i,j=1}^{3}\left\langle {A_j}^{\dagger}A_i\right\rangle_{\rho}\left(i\epsilon_{ijk}\sigma_k+(\delta_{ij}+i\epsilon_{i3j})\,\mathbb{1}-\delta_{ij}\sigma_3+\delta_{i3}\sigma_j+\delta_{j3}\sigma_i\right)\\ &+\left\langle {A_0}^{\dagger}A_0\right\rangle_{\rho}\frac{\mathbb{1}+\sigma_3}{2}+\left\langle {A_3}^{\dagger}A_0+{A_0}^{\dagger}A_3\right\rangle_{\rho}\frac{\mathbb{1}}{2}+\frac{1}{2}\left\langle \vec{A}^{\dagger}A_0+{A_0}^{\dagger}\vec{A}\right\rangle_{\rho}.\vec{\sigma}\\ &+\frac{i}{2}\left\langle {A_0}^{\dagger}A_2-{A_2}^{\dagger}A_0\right\rangle_{\rho}\sigma_1-\frac{i}{2}\left\langle {A_0}^{\dagger}A_1-{A_1}^{\dagger}A_0\right\rangle_{\rho}\sigma_2.\end{aligned}$$

We can simplify the above equation further by using Eq. (2.2) into the following compact form:

$$\begin{aligned}\mathcal{M}(\rho)=&\frac{\mathbb{1}}{2}+\left\langle -i\vec{A}^{\dagger}\wedge\vec{A}+\frac{1}{2}\vec{A}^{\dagger}A_3+\frac{1}{2}{A_3}^{\dagger}\vec{A}\right\rangle_{\rho}.\vec{\sigma}\\ &+\left\langle {A_0}^{\dagger}A_0-\frac{\mathbb{1}}{2}\right\rangle_{\rho}\sigma_3+\frac{i}{2}\left[\left\langle \vec{A}^{\dagger}A_0-{A_0}^{\dagger}\vec{A}\right\rangle_{\rho}\wedge\vec{\sigma}\right]_3\end{aligned} \tag{2.6}$$

which can be written as:

$$\mathcal{M}(\rho)=\frac{\mathbb{1}}{2}+\left\langle\vec{B}\right\rangle_{\rho}.\frac{\vec{\sigma}}{2} \tag{2.7}$$

[1]Where we have used the identity $\mathrm{Tr}[AB]=\mathrm{Tr}[BA]$, which holds for compact operators of trace class [230].

where we have defined

$$\frac{B_x}{2} = -iA_2{}^\dagger A_3 + iA_3{}^\dagger A_2 + \frac{A_1{}^\dagger A_3 + A_3{}^\dagger A_1}{2} + \frac{iA_0{}^\dagger A_2 - iA_2{}^\dagger A_0}{2}, \tag{2.8}$$

$$\frac{B_y}{2} = iA_1{}^\dagger A_3 - iA_3{}^\dagger A_1 + \frac{A_2{}^\dagger A_3 + A_3{}^\dagger A_2}{2} + \frac{-iA_0{}^\dagger A_1 + iA_1{}^\dagger A_0}{2}, \tag{2.9}$$

$$\frac{B_z}{2} = -iA_1{}^\dagger A_2 + iA_2{}^\dagger A_1 + A_3{}^\dagger A_3 + A_0{}^\dagger A_0 - \frac{\mathbb{1}}{2}. \tag{2.10}$$

The form (2.7) is the operational form we desire and equations (2.8, 2.9, 2.10) set the connection between this form and the isometry U used to implement it.

In order for this mapping to be operational, one has to ask the inverse question: given a set of three operators B_x, B_y, B_z, what are the conditions on these operators such that there exists an isometry U that implement the mapping of the form (2.7)? In the next section I will give a first answer this question.

2.1.2 Existence of the mapping

In order for the previous mapping (2.7) to exist, we clearly need to find proper operators A_is that verify Eqs. (2.2) and (2.8, 2.9, 2.10). Next we simplify these equations by writing the set of conditions Eqs. (2.2) explicitly:

$$A_1{}^\dagger A_0 + A_0{}^\dagger A_1 + iA_2{}^\dagger A_3 - iA_3{}^\dagger A_2 = 0 \tag{2.11}$$

$$A_2{}^\dagger A_0 + A_0{}^\dagger A_2 - iA_1{}^\dagger A_3 + iA_3{}^\dagger A_1 = 0 \tag{2.12}$$

$$A_3{}^\dagger A_0 + A_0{}^\dagger A_3 + iA_1{}^\dagger A_2 - iA_2{}^\dagger A_1 = 0 \tag{2.13}$$

From (2.10) and (2.13), we can eliminate A_1, A_2 and get:

$$\left(A_3{}^\dagger + A_0{}^\dagger\right)(A_3 + A_0) = \frac{\mathbb{1} + B_z}{2}$$

Which motivates the definition of $K_0 = A_3 + A_0$, so that:

$$K_0{}^\dagger K_0 = \frac{\mathbb{1} + B_z}{2} \tag{2.14}$$

Doing the same thing for equations (2.8), (2.9), and using (2.11) (2.12), we get:

$$A_1{}^\dagger K_0 + K_0{}^\dagger A_1 - iA_2{}^\dagger K_0 + iK_0{}^\dagger A_2 = B_x$$

$$A_2{}^\dagger K_0 + K_0{}^\dagger A_2 + iA_1{}^\dagger K_0 - iK_0{}^\dagger A_1 = B_y$$

We also define $K_1 = A_1 + iA_2$, then the above equations would simplify into:

$$\begin{aligned} {K_1}^\dagger K_0 + {K_0}^\dagger K_1 &= B_x \\ i{K_1}^\dagger K_0 - i{K_0}^\dagger K_1 &= B_y \end{aligned}$$

We can regroup the equations verified by K_0 ,K_1 as:

$$\begin{aligned} {K_0}^\dagger K_0 = \frac{\mathbb{1} + B_z}{2}, \qquad & {K_1}^\dagger K_1 = \frac{\mathbb{1} - B_z}{2} \\ {K_0}^\dagger K_0 + {K_1}^\dagger K_1 = \mathbb{1}, \qquad & {K_0}^\dagger K_1 = \frac{B_x + iB_y}{2} \end{aligned} \tag{2.15}$$

Hence, in order for the mapping Eq. (2.7) to exist, we should be able to find two operators K_0 and K_1 such that the set of equations Eqs. (2.15) is verified.

To show that the two operators K_0 and K_1 that verifies the conditions (2.15) are sufficient to find an isometry U such that the mapping (2.7) exists, we can use the fact that U is an isometry and define its image to be in a larger Hilbert space. Once the operators K_0 and K_1 are found, such that conditions (2.15) are verified, one can also define the following operators

$$\tilde{K}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} K_0 \\ K_0 \end{pmatrix} \quad , \quad \tilde{K}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} K_1 \\ K_1 \end{pmatrix} \tag{2.16}$$

acting on the hilbert space $\mathcal{H}^{(d)}$ with codmain $\mathcal{H}^{(D)} \oplus \mathcal{H}^{(D)}$, where $\mathcal{H}^{(D)}$ is the codomain of K_0 and K_1. These operators also verify the conditions (2.15), i.e. we have:

$$\tilde{K}_0^\dagger \tilde{K}_0 = {K_0}^\dagger K_0 = \frac{\mathbb{1} + B_z}{2} \, , \, \tilde{K}_1^\dagger \tilde{K}_1 = {K_1}^\dagger K_1 = \frac{\mathbb{1} - B_z}{2} \, , \, \tilde{K}_0^\dagger \tilde{K}_1 = {K_0}^\dagger K_1 = \frac{B_+}{2} \tag{2.17}$$

where we have defined $B_+ = B_x + iB_y$. Then we can simply define the operators:

$$\begin{aligned} \tilde{A}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} K_0 \\ 0 \end{pmatrix} \, , \, \tilde{A}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ K_0 \end{pmatrix} \\ \tilde{A}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} K_1 \\ 0 \end{pmatrix} \, , \, \tilde{A}_2 = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 \\ K_1 \end{pmatrix} \end{aligned} \tag{2.18}$$

which verify

$$\tilde{A}_0 + \tilde{A}_3 = \tilde{K}_0 \quad , \quad \tilde{A}_1 + i\tilde{A}_2 = \tilde{K}_1 \tag{2.19}$$

and they also verify the conditions (2.2). Hence, there exists an isometry

$$\tilde{U} = \sum_{i=0}^{3} \tilde{A}_i \otimes \sigma_i \tag{2.20}$$

that implements the mapping (2.7)

$$\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \left\langle \vec{B} \right\rangle_\rho . \frac{\vec{\sigma}}{2}$$

Next, I give an example of a mapping where these conditions (2.15) can be explicitly calculated. For a specific choice of B_is. namely the angular momentum components, we can write explicitly the form of the isometry needed to implement the mapping.

2.1.3 Dichotomization using angular momentum operators

An important example, that will become very handy for spin squeezing inequalities and dichotomic observables, is the following mapping:

$$\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \frac{\left\langle \vec{J} \right\rangle_\rho}{j} . \frac{\vec{\sigma}}{2} \tag{2.21}$$

Where we have made the choice $B_i = J_i/j$, $i = x, y, z$. The three operators J_i are the components of the angular momentum operator that has the advantage of being easily accessible from experimental point of view. We have also introduced the normalization constant j, where the eigenvalue of the total angular J^2 is given by $j(j+1)$[2]. What this mapping does, is to consider a spin-j as a spin-1/2 such that

$$\frac{\left\langle \vec{J} \right\rangle_\rho}{j} = \frac{\langle \vec{\sigma} \rangle_{\mathcal{M}(\rho)}}{2}$$

The above equality, motivates our normalization factor j of the angular momentum operators.

All left is to show that the mapping (2.21) can be implemented via an isometry as in (2.3). In other words, we will try to find the operators K_0, K_1 such that the conditions (2.15) are verified.

We note that the equations (2.15) can be expressed in a simple form with the help of the two bosonic annihilation operators a and b corresponding to the Schwinger

[2]We set $\hbar$=1.

representation of spin-j [231][3]:

$$J_3 = \frac{1}{2}(a^\dagger a - b^\dagger b)\,,\; J_- = b^\dagger a\,,\; J_+ = a^\dagger b \tag{2.22}$$

where $J_\pm = J_1 \pm iJ_2$ with the restriction $a^\dagger a + b^\dagger b = 2j$. Rewriting conditions (2.15) in terms of bosonic operators $a(b)$ and $a^\dagger(b^\dagger)$, one gets:

$$K_0{}^\dagger K_0 = \frac{a^\dagger a}{2j}\,,\; K_1{}^\dagger K_1 = \frac{b^\dagger b}{2j}\,,\; K_0{}^\dagger K_1 = \frac{a^\dagger b}{2j} \tag{2.23}$$

Which suggests the choice

$$K_0 = \frac{a}{\sqrt{2j}} \quad , \quad K_1 = \frac{b}{\sqrt{2j}} \tag{2.24}$$

Now, one should find operators A_i, $i = 0, 1, 2, 3$ such that:

$$A_0 + A_3 = K_0 \quad , \quad A_1 + iA_2 = K_1$$

and such that the conditions (2.2) are verified. One possible choice of such operators [1] A_i realizing the isometry U through Eqs. (2.1) and Eqs. (2.2) can be shown to be (see Appendix A.3)

$$\begin{aligned} A_0 &= \frac{1}{2\sqrt{2}}\left(\frac{a}{\sqrt{j}} + \frac{a^\dagger}{\sqrt{j+1}}\right), A_3 = \frac{1}{2\sqrt{2}}\left(\frac{a}{\sqrt{j}} - \frac{a^\dagger}{\sqrt{j+1}}\right), \\ A_1 &= \frac{1}{2\sqrt{2}}\left(\frac{b}{\sqrt{j}} - \frac{b^\dagger}{\sqrt{j+1}}\right), iA_2 = \frac{1}{2\sqrt{2}}\left(\frac{b}{\sqrt{j}} + \frac{b^\dagger}{\sqrt{j+1}}\right). \end{aligned} \tag{2.25}$$

Please note that despite involving bosonic operators a and b in (2.25) acting on infinite dimensional Hilbert space, they are acting on a restricted sub-space $a^\dagger a + b^\dagger b = 2j$.

Although the set of equations (2.15) minimizes the task of verifying the validity of the mapping (2.7), it is still a hard problem to solve and it is not always as straightforward as we have seen from this example. We can actually simplify the problem by using the property of completely positive maps (CP), which will give us a very simple condition as we will see next.

[3]Each spin-j can be seen as $2j$ spin-1/2 particles. Each level can be seen as a bosonic mode, say a for spin up and b for spin down where a, b are the annihilation operators for each mode respectively. They verify the commutation relation $[a^\dagger, a] = 1$, $[b^\dagger, b] = 1$ and $[b^\dagger, a] = 0$. Since the number of spin-1/2 particles representing a spin-j is $2j$, the total number of particles in the two modes is $a^\dagger a + b^\dagger b = 2j$.

2.1.4 Necessary and sufficient conditions

The form (2.3) is known as the Stinespring representation of the trace preserving channel $\mathcal{M}$ [232, 233]. Stinespring representation in the open system representation states that [234]:

Stinespring representation *For a completely positive and trace preserving (CPTP) linear mapping* $\mathcal{M} : \mathcal{B}\left(\mathcal{H}^{(d)}\right) \to \mathcal{B}\left(\mathcal{H}^{(2)}\right)$*, there exists an isometry* V *from* $\mathcal{H}^{(d)}$ *to* $\mathcal{H}^{(D)} \otimes \mathcal{H}^{(2)}$ *such that* $\mathcal{M}(\rho) = Tr_E\left[V\rho V^\dagger\right]$ *where* E *is the environment represented by the Hilbert space* $\mathcal{H}^{(D)}$*.*

A comparison with the form of our mapping (2.3) gives that $V : \mathcal{H}^{(d)} \to \mathcal{H}^{(D)} \otimes \mathcal{H}^{(2)}$ can be seen as a simple restriction of the isometry $U : \mathcal{H}^{(d)} \otimes \mathcal{H}^{(2)} \to \mathcal{H}^{(D)} \otimes \mathcal{H}^{(2)}$.

Kraus representation The mapping defined in (2.3) is trace preserving completely positive map and one can easily write its Kraus representation [235, 21] as (see appendix A.1):

$$\mathcal{M}(\rho) = \sum_{i=1}^{D} \mathcal{K}_i \rho \mathcal{K}_i^\dagger \tag{2.26}$$

where

$$\mathcal{K}_i = |0\rangle\langle e_i|K_0 + |1\rangle\langle e_i|K_1 \quad : 1 \leq i \leq D, \tag{2.27}$$

K_0 and K_1 verify the conditions (2.15) and $\{|e_i\rangle : 1 \leq i \leq D\}$ is an orthonormal basis of $\mathcal{H}^{(D)}$ the Hilbert space over which the trace is taken in (2.3). The above property holds even for infinite Hilbert spaces where one finds a countable set of Kraus operators [236]. This will become helpful to extend Theorem 1, we will introduce hereafter, for bound operators of trace class.

In the finite dimensional case, we can make use of the property of CP maps proved in [168]:

Lemma I *A linear mapping* $\mathcal{M} : \mathcal{B}\left(\mathcal{H}^{(d)}\right) \to \mathcal{B}\left(\mathcal{H}^{(2)}\right)$ *is completely positive iff the operator* $\mathcal{D} = I \otimes \mathcal{M}(P_+)$ *is positive, where* I *is the identity map from* $\mathcal{B}\left(\mathcal{H}^{(d)}\right)$ *to itself and* $P_+ = \frac{1}{d}\sum_{i,j=1}^{d} |i,i\rangle\langle j,j|$ *a maximally entangled state in* $\mathcal{B}\left(\mathcal{H}^{(d)} \otimes \mathcal{H}^{(d)}\right)$*.*

Now let us calculate the operator $\mathcal{D}$ explicitly for the mapping (2.7):

$$\mathcal{D} = \frac{1}{d}\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \mathcal{M}\left(|i\rangle\langle j|\right) = \frac{1}{2d}\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \left(\delta_{i,j}\sigma_0 + \langle j|\vec{B}|i\rangle.\vec{\sigma}\right)$$

Finally, we get a very simple form of $\mathcal{D}$ as:

$$\mathcal{D} = \frac{\mathbb{1}^{(d)}}{2} \otimes \sigma_0 + \frac{B_x^T}{2} \otimes \sigma_1 + \frac{B_y^T}{2} \otimes \sigma_2 + \frac{B_z^T}{2} \otimes \sigma_3 \tag{2.28}$$

Since the positivity of $\mathcal{D}$ is equivalent to the positivity of its transpose $\mathcal{D}^T$ we finally get our necessary and sufficient condition for the mapping (2.7) to be completely positive that we state in the following theorem:

Theorem 1 *The mapping* $\mathcal{M} : (\rho) = \frac{\mathbb{1}}{2} + \left\langle \vec{B} \right\rangle_\rho . \frac{\vec{\sigma}}{2}$ *is CP iff*

$$\mathcal{D} = \begin{pmatrix} \mathbb{1}^{(d)} + B_z & B_x + iB_y \\ B_x - iB_y & \mathbb{1}^{(d)} - B_z \end{pmatrix} \succeq 0 \tag{2.29}$$

For the proof of this theorem, see appendix A.2, the intermediate step [168] which helped find the condition, is discarded, thus, the above theorem is not restricted to finite dimensions and can be generalized to compact operators of trace class [230].

The appealing feature about the above criterion, is that any three operators, B_x, B_y and B_z can be used for the mapping with some normalization factor β, such that the mapping

$$\mathcal{M}\left(\rho\right) = \frac{\mathbb{1}}{2} + \frac{\left\langle \vec{B} \right\rangle_\rho}{\beta} . \frac{\vec{\sigma}}{2} \tag{2.30}$$

is completely positive. Where β can be chosen to be the minimum value for which the matrix

$$\mathcal{D} = \begin{pmatrix} \beta\mathbb{1}^{(d)} + B_z & B_x + iB_y \\ B_x - iB_y & \beta\mathbb{1}^{(d)} - B_z \end{pmatrix} \tag{2.31}$$

is positive semi-definite. This can be determined with a simple diagonalization of the matrix

$$\mathcal{B} = \begin{pmatrix} B_z & B_x + iB_y \\ B_x - iB_y & -B_z \end{pmatrix} \tag{2.32}$$

as

$$\beta = -\lambda_{min}\left(B\right) \tag{2.33}$$

where $\lambda_{min}(B)$ is the smallest eigenvalue of $\mathcal{B}$. Please note that β cannot be zero except for the trivial choice $B_i = 0$, $i = x, y, z$. That is because one should have

$$\beta^2 \geq \sup\{\langle B_x\rangle_\rho^2 + \langle B_y\rangle_\rho^2 + \langle B_z\rangle_\rho^2 \mid \rho \in \mathcal{B}(\mathcal{H})\}$$

at least to ensure positivity of the qubit state $\mathcal{M}(\rho)$. Usually, one would like to choose the minimum value of β, such that the condition (2.31) is verified, to approach a pure state as much as possible.

Hence complete positivity of the mapping (2.30) is finally determined via the condition (2.31), that depends only on the chosen observables and avoids any reference to the isometry U used to perform this mapping (2.3).

Next, we will study two examples of operators B_i for which one can determine β explicitly. These choices will appear frequently throughout the remainder of the thesis. The first example will be the angular momentum operators, whereas the next example is concerned with Pauli like operators that we will define here after.

2.1.5 Dichotomization using angular momentum operators: revisited

In the light of this preceding result, we revisit the case where the mapping is defined through angular momentum operators that we have already discussed in section 2.1.3 (see page 55). That is the mapping:

$$\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \frac{\left\langle \vec{J} \right\rangle_\rho}{\beta}.\frac{\vec{\sigma}}{2}, \tag{2.34}$$

where J_i, $i = x, y, z$ are the components of angular momentum.

Now, we set out to determine the best value of β such that the mapping (2.34) is completely positive. Previously (see section 2.1.3), we have set $\beta = j$. Here we will show, using our simplified criterion (2.31) that the corresponding mapping is indeed CP.

To show that, we borrow a theorem for positivity of block matrices [237, 238]

Theorem 2 *A block matrix*

$$M = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$$

is positive semi-definite iff:

$$A \succeq 0 \, and \, \left(\mathbb{1} - AA^{-1}\right) B = 0 \, and C - B^{\dagger} A^{-1} B \succeq 0$$

where A^{-1} *is the pseudo inverse of the matrix* A *[239–242].*

Since $\mathcal{D}$ should be positive semi-definite, theorem 2 [237, 238] tells us that $A = j\mathbb{1} + J_z \succeq 0$. This is indeed the case since:

$$j\mathbb{1} + J_z = \sum_{m=-j}^{j} (j+m)|m\rangle\langle m| = \sum_{m=0}^{2j} m|m-j\rangle\langle m-j| \tag{2.35}$$

The pseudo inverse of A can be calculated easily, since it is diagonal, to be:

$$A^{-1} = (j\mathbb{1} + J_z)^{-1} = \sum_{m=1}^{2j} \frac{1}{m}|m-j\rangle\langle m-j| \tag{2.36}$$

Thus, we have the following relation:

$$\mathbb{1} - AA^{-1} = \sum_{m=0}^{2j} |m-j\rangle\langle m-j| - \sum_{m=1}^{2j} |m-j\rangle\langle m-j| = |-j\rangle\langle -j| \tag{2.37}$$

Since $\langle -j|B = \langle -j|J_+ = 0$, we find that the condition $(\mathbb{1} - AA^{-1})\, B = 0$ is verified. All left is to check the last condition

$$C - B^{\dagger} A^{-1} B = j\mathbb{1} - J_z - J_- \left(j\mathbb{1} + J_z\right)^{-1} J_+ \succeq 0$$

We have:

$$J_- \left(j\mathbb{1} + J_z\right)^{-1} J_+ = \sum_{m=1}^{2j} \frac{1}{m} J_-|m-j\rangle\langle m-j|J_+$$

We also have [120]:

$$J_+|m\rangle = \sqrt{j\,(j+1) - m\,(m+1)}|m+1\rangle$$

$$J_-|m\rangle = \sqrt{j\,(j+1) - m\,(m-1)}|m-1\rangle$$

Then we get:

$$J_- \left(j\mathbb{1} + J_z\right)^{-1} J_+ = \sum_{m=1}^{2j} \frac{j(j+1) - (m-j)(m-j-1)}{m} |m-j-1\rangle\langle m-j-1|$$
$$= \sum_{m=1}^{2j} \frac{m(2j+1) - m^2}{m} |m-j-1\rangle\langle m-j-1| = \sum_{m=0}^{2j-1} (2j-m)|m-j\rangle\langle m-j|$$
$$= \sum_{m=-j}^{j} (j-m)|m\rangle\langle m| = j\mathbb{1} - J_z \tag{2.38}$$

Hence, we finally get that $C - B^\dagger A^{-1} B = 0 \succeq 0$ is verified as well. As a result, the matrix $\mathcal{D}$ is positive semi-definite and we can conclude that the mapping (2.34) is CP.

We emphasize that this mapping is practical, as the 3 expectation values $\langle J_i \rangle_\rho$ can be easily measured. In addition, it conserves the rotational invariance. Indeed, suppose that we perform a rotation $\mathcal{R}_{\vec{n}}(\alpha)$ by an angle α around a given vector $\vec{n}$. Then, ρ is transformed as $\rho' = e^{-i\alpha \vec{J}\cdot\vec{n}} \rho e^{i\alpha \vec{J}\cdot\vec{n}}$. It is not difficult to show that ρ' is mapped to the rotated qubit :

$$\mathcal{M}_U(e^{-i\alpha \vec{J}\cdot\vec{n}} \rho e^{i\alpha \vec{J}\cdot\vec{n}}) = e^{-i\alpha \vec{\sigma}\cdot\vec{n}/2} \mathcal{M}_U(\rho) e^{i\alpha \vec{\sigma}\cdot\vec{n}/2}. \tag{2.39}$$

The invariance displayed in Eq. (2.39) is a consequence of the simple vectorial relations:

$$\left\langle \mathcal{R}_{\vec{n}}^{-1}(\alpha) \left[\vec{J}\right] \right\rangle_\rho \cdot \vec{\sigma} = \left\langle \vec{J} \right\rangle_{\rho'} \cdot \sigma = \left\langle \vec{J} \right\rangle_\rho \cdot \mathcal{R}_{\vec{n}}(\alpha) \left[\vec{\sigma}\right], \tag{2.40}$$

where $\vec{J}$ is the vector whose 3 components are the 3 operators J_i, for $i = 1, 2, 3$, and $\mathcal{R}_{\vec{n}}^{-1}(\alpha) \left[\vec{J}\right]$ is the corresponding vector in the rotated frame. Since SO(3) and SU(2) are isomorphic, we can apply all possible unitaries to the mapped qubit by rotating the original angular momentum correspondingly.

2.1.6 Dichotomization using Pauli like operators

A more general and popular choice is that of dichotomic Pauli operators. In many situations, it is very convenient to consider only a bi-dimensional subspace of the total Hilbert space on which we define Pauli operators and study the system as a two-level system. Here we will introduce a very general class of Pauli like operators for any system and we will discuss some applications later on. I will give examples of such operators for discrete and continuous variable systems as well.

Introduction

Let us start by defining the Pauli-like operators. First let us recall the definition of Pauli operators for spin-1/2 where the Hilbert space is spanned by the two vectors $\{|0\rangle, |1\rangle\}$:

$$\begin{aligned} \sigma_x &= \sigma_+ + \sigma_- \,,\, \sigma_y = -i\,(\sigma_+ - \sigma_-)\,,\, \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \\ \sigma_- &= |1\rangle\langle 0|\,,\, \sigma_+ = {\sigma_-}^\dagger \end{aligned} \tag{2.41}$$

One can write σ_z as $P_0 - P_1$ where P_0, P_1 are projectors in $\mathcal{H}_0$ spanned by $\{|0\rangle\}$ and $\mathcal{H}_1$ spanned by $\{|1\rangle\}$ respectively.

Inspired from the above observations, we can generalize and extend the definition of Pauli operators for any system of arbitrary dimensions. First, we define two Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ such that:

$$\mathcal{H}_0 \oplus \mathcal{H}_1 \subseteq \mathcal{H} \tag{2.42}$$

Where $\mathcal{H}$ is the Hilbert space of the considered system. Next, we define projectors $P_0(P_1)$ onto $\mathcal{H}_0(\mathcal{H}_1)$ which will be used to define the generalized S_z operator as:

$$S_z = P_0 - P_1 \tag{2.43}$$

The generalized S_z, like the qubit σ_z, can be seen as the difference of population between the 0 and 1 subspaces, i.e. $P_0 - P_1$.

Now, let us take a look at a generalization of the σ_+ operator. It will be denoted as S_+, and it can be defined as:

$$S_+ = P_0 \Lambda P_1,$$

where we have introduced Λ a linear mapping from $\mathcal{H}_1$ to $\mathcal{H}_0$ with singular value 1. The generalization of σ_-, denoted S_-, can be simply defined from S_+ as:

$$S_- = {S_+}^\dagger = P_1 \Lambda^\dagger P_0$$

Mapping with Pauli like operators

We thus have the ingredients to map any state ρ in the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ to a qubit $\mathcal{M}(\rho)$, using these Pauli like operators. Indeed, we define the mapping as:

$$\rho \to \mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \frac{\left\langle \vec{S} \right\rangle_\rho . \vec{\sigma}}{2} \tag{2.44}$$

where the x, y, z components of the vector $\vec{S}$ are defined as:

$$S_z = P_0 - P_1, \quad S_x = P_0 \Lambda P_1 + P_1 \Lambda^\dagger P_0, \quad S_y = \frac{P_0 \Lambda P_1 - P_1 \Lambda^\dagger P_0}{i}. \tag{2.45}$$

We impose the requirement that $\|\Lambda\|_2 \equiv \sqrt{\lambda_{max}\left(\Lambda^\dagger \Lambda\right)} = 1$ to ensure the positivity of the mapping. Indeed, we show in appendix A.4 that this requirement implies:

$$\mathrm{Tr}\left[\mathcal{M}(\rho)^2\right] = \frac{1 + \langle S_x \rangle_\rho^2 + \langle S_y \rangle_\rho^2 + \langle S_z \rangle_\rho^2}{2} \leq 1$$

ensuring that the mapped state $\mathcal{M}(\rho)$ is positive. We defer to appendix A.5 the proof that the above mapping is completely positive.

Commutation relations of the Pauli-like operators

At this stage, it is interesting to check out the commutation relations of the Pauli like operators defined in Eq. (2.45). Since the projectors P_0 and P_1 are orthogonal, it is trivial to check that:

$$[S_z, S_x] = 2iS_y \quad , \quad [S_y, S_z] = 2iS_x \tag{2.46}$$

However $[S_x, S_y] = 2iS_z$ is not true in general. In fact, we find that:

$$[S_x, S_y] = 2i\left(P_0 \Lambda P_1 \Lambda^\dagger P_0 - P_1 \Lambda^\dagger P_0 \Lambda P_1\right)$$

In order for the right hand side of the above equality to be equal to $S_z = P_0 - P_1$, we should have:

$$\Lambda P_1 \Lambda^\dagger = P_0 \quad , \quad \Lambda^\dagger P_0 \Lambda = P_1 \tag{2.47}$$

Now let us recall the singular value decomposition of Λ

$$\Lambda = \sum_n \sqrt{\lambda_n} |e_n\rangle\langle f_n|$$

$\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers such that

$$\max\{\lambda_n\}_{n\in\mathbb{N}} = 1$$

because of the condition $\|\Lambda\|_2 = 1$. $\{|e_n\rangle\}_{n\in\mathbb{N}}$ and $\{|f_n\rangle\}_{n\in\mathbb{N}}$ are orthonormal basis of $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. From the SVD decomposition of Λ it immediately follows:

$$\Lambda P_1 \Lambda^\dagger = \sum_n \lambda_n |e_n\rangle\langle e_n| \quad , \quad \Lambda^\dagger P_0 \Lambda = \sum_n \lambda_n |f_n\rangle\langle f_n|$$

Comparing with (2.47), we see that in order for the commutation relation $[S_x, S_y] = 2iS_z$ to be fulfilled, we need to set all the singular values of Λ to 1. In other words, Λ should be of the following form:

$$\Lambda = \sum_n |e_n\rangle\langle f_n|$$

which in turn implies that there is one to one correspondence between $\{|e_n\rangle\}_n$ and $\{|f_n\rangle\}_n$. For finite dimensional Hilbert spaces, this means $\mathcal{H}_0$ and $\mathcal{H}_1$ should have the same dimension. In general, Λ should be an isomorphism from $\mathcal{H}_1$ to $\mathcal{H}_0$ of the above form.

The commutation relations $[S_i, S_j] = 2i\epsilon_{ijk}S_k$, ensure that the Pauli like operators are the generators of SU(2) in $\mathcal{H}_0 \oplus \mathcal{H}_1$, and as a consequence, the components of $\vec{S}$ transform as a vector under rotations. This is very useful to generate all the rotations in the mapped state (2.44) via rotations on the original state as we did see in the last example with angular momentum operators. This feature will become very useful, as we will see in the next chapter, to derive a direction independent form of spin squeezing inequalities.

However, when the commutation relations are not needed, and the Pauli like operators are only needed to perform the mapping (2.44), the above requirements for the commutation relation $[S_x, S_y] = 2iS_z$ can be relaxed. The choice depends on what are we looking for when performing the mapping (2.44).

Examples

One of the major interests of the Pauli-like operators is to test for non-locality [4, 7] in high dimensional systems as we will see in the next section. More precisely, define spin-1/2 like operators to test Clauser-Horne-Shimony-Holt (CHSH) inequalities [8] for discrete variable systems [243] and continuous variable systems [244]. Here, however, I will only give examples that are known and discussed in the literature to get a taste of such Pauli-like operators that will be useful for later discussion.

Finite dimensional systems

Let us consider the simplest choice possible considered in [243] of an even dimensional Hilbert space $\mathcal{H}^{(2n)}$, where the case $n = 1$ corresponds to the case of a 2-level system.

Now let us define $\mathcal{H}_0$ to be the even Hilbert space spanned by the vectors $\{|2i\rangle : 0 \leq i < n\}$, and $\mathcal{H}_1$ to be the odd Hilbert space spanned by the vectors $\{|2i+1\rangle : 0 \leq i < n\}$. Then, we define $P_0 \Lambda P_1$ to be

$$P_0 \Lambda P_1 = \sum_{i=0}^{n-1} |2i\rangle\langle 2i+1|.$$

Then recalling the definition of Pauli-like operators (2.45), we get:

$$\begin{aligned} S_z &= \textstyle\sum_{i=0}^{n-1} |2i\rangle\langle 2i| - |2i+1\rangle\langle 2i+1| \\ S_x &= \textstyle\sum_{i=0}^{n-1} |2i\rangle\langle 2i+1| + |2i+1\rangle\langle 2i| \\ S_y &= -i \textstyle\sum_{i=0}^{n-1} |2i\rangle\langle 2i+1| - |2i+1\rangle\langle 2i| \end{aligned} \tag{2.48}$$

At this point, it is instructive to see that any pure even state $|\psi_0\rangle \in \mathcal{H}_0$ would be mapped via the mapping (2.44) into the pure qubit state $|0\rangle$, since

$$\langle\psi_0|S_z|\psi_0\rangle = 1 \quad , \quad \langle\psi_0|S_x|\psi_0\rangle = \langle\psi_0|S_y|\psi_0\rangle = 0$$

with the same reasoning, any pure odd state $|\psi_1\rangle \in \mathcal{H}_1$ would be mapped into the qubit state $|1\rangle$. Thus, the role of the mapping (2.44) is to replace any state from $\mathcal{H}_0\,(\mathcal{H}_1)$ by the qubit state $|0\rangle\,(|1\rangle)$.

Since our formalism and definition of Pauli-like operators is versatile and allows for two unitary degrees of freedom in $\mathcal{H}_0$ and $\mathcal{H}_1$ separately, we can choose the desired final qubit state, by choosing $\mathcal{H}_0$ and $\mathcal{H}_1$ properly. For example, let us look at the state in $\mathcal{H}^{(d)}$ with $d > 2$, for the sake of illustration:

$$|\Psi_+\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$$

If we take $\mathcal{H}_0$ to be the Hilbert space spanned by $\{|2\rangle\}$ and $\mathcal{H}_1$ to be the Hilbert space spanned by $\{|1\rangle\}$, and use the same Pauli-like operators defined above, the above state will be mapped via (2.44) into the following qubit state

$$|\psi\rangle = \frac{|1\rangle + |0\rangle}{\sqrt{2}}$$

However, if we choose $\mathcal{H}_0$ to be the Hilbert space spanned by $\{|\Psi_+\rangle\}$ and $\mathcal{H}_1$ to be the Hilbert space spanned by $\{|\Psi_-\rangle\}$, where

$$|\Psi_-\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}},$$

and if we define $P_0 \Lambda P_1 = |\Psi_+\rangle\langle\Psi_-|$ in the Pauli-like operator definition (2.45), the state of interest $|\Psi_+\rangle$ will be mapped into the qubit state $|0\rangle$.
Finally, for any of the previous choices, the qudit state $|0\rangle$ will be mapped into the qubit mixed state $\frac{\mathbb{1}}{2}$.

The above discussion can be seen as an easy recipe to map any system into a qubit state, at least for the cases where the studied state is known or can be predicted. All the mathematical tools that we provided boil down, for this special choice of operators, to labelling some states 0 and others 1 depending on one's needs.

Continuous variable systems

Here, we will consider the example introduced by Chen et al. [244] for a single-mode light field. They have introduced the following Pauli like operators "pseudospin" for photons:

$$\begin{aligned} S_z &= \textstyle\sum_{i=0}^{\infty} |2i\rangle\langle 2i| - |2i+1\rangle\langle 2i+1| \\ S_x &= \textstyle\sum_{i=0}^{\infty} |2i\rangle\langle 2i+1| + |2i+1\rangle\langle 2i| \\ S_y &= -i \textstyle\sum_{i=0}^{\infty} |2i\rangle\langle 2i+1| - |2i+1\rangle\langle 2i| \end{aligned} \tag{2.49}$$

where $|i\rangle$ represents the Fock state with i photons. Like the previous case, S_z represents the parity of photon number and S_x and S_y are parity flip operators.

We can see like the previous case, that any state with even photon number will be mapped via (2.44) into the qubit state $|0\rangle$, whereas, states with odd photon number will be mapped to $|1\rangle$. Perhaps, a more famous example, is the even(odd) cat state:

$$|\text{cat}_\pm\rangle \propto |\alpha\rangle \pm |-\alpha\rangle \tag{2.50}$$

where $|\alpha\rangle$ is the coherent state defined for any complex α as:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{2.51}$$

We can easily see that the even (odd) cat state is written as a superposition of even(odd) Fock states, thus will be mapped via (2.44) into the qubit state $|0\rangle$ ($|1\rangle$), for all values of α. Since coherent states are a superposition of even and odd cat states, more precisely:

$$|\alpha\rangle = \sqrt{\frac{1+e^{-2|\alpha|^2}}{2}}|\text{cat}\rangle_+ + \sqrt{\frac{1-e^{-2|\alpha|^2}}{2}}|\text{cat}\rangle_- \tag{2.52}$$

it will be mapped into the qubit state

$$\sqrt{\frac{1+e^{-2|\alpha|^2}}{2}}|0\rangle + \sqrt{\frac{1-e^{-2|\alpha|^2}}{2}}|1\rangle$$

We have thus far studied the mapping of any system into a qubit, the main step for the next section. The next step naturally, will be the mapping of a bipartite system into two qubits to, as mentioned in the introduction, study the entanglement properties of the system based on the entanglement properties of the resulting qubit.

2.2 Mapping bipartite system to two qubits

As we have already mentioned, the main objective of this mapping is to simplify the study of entanglement by reducing the dimensions. Indeed, entanglement detection can be a daunting task in systems of dimensions larger than 2×3, since there are no known necessary and sufficient conditions for entanglement in such systems. Whereas for systems with dimensions 2×2 and 2×3, there exists necessary and sufficient conditions, especially, the Peres-Horodecki criterion or PPT criterion [83, 139]. Hence, by mapping each of the subsystems into a qubit as we have introduced in the previous section, we get a 2×2 system for which PPT criterion can be easily applied. If the mapping is done properly, such that no entanglement is created in the process, entanglement in the resulting 2×2 system is a sufficient condition for entanglement in the original system. Such mappings that does not create entanglement can be performed using what is known as Local Operations and Classical Communication (LOCC) operations [45, 40, 38].

Description of LOCC operations is hard and there is no known mathematical form to describe all possible operations. However, one can define different subclasses of such operations. It is worth mentioning that there exits a class called the separable operations class that has an elegant simple mathematical form. However, it was shown that separable

operations are different from LOCC ones [45], and that the name separable is somewhat deceptive.

For this work, we will restrict our attention to a specific class of Local Operations with no classical communication that can be described quite easily using Kraus operators. I will give the description in the next section for this class, but we can have an intuitive picture of it. The two parties, usually called Alice and Bob, apply local operations to their part of the shared state with no possibility to communicate or receive any information from the other party. In other words, they blindly apply their operation irrespective of what the other party does. One can naturally expand this class by adding one way communication where only Alice can tell Bob, classically, what operation to apply depending on the results of her local operations. This class, in turn, can be expanded by having two rounds of classical communication and so on.

2.3 Mapping bipartite system to two qubits: Local mapping with no communication

In this paradigm, as introduced earlier, Alice and Bob apply their mapping locally to their part of the state they share with no knowledge or information conveyed from one party to another. This is the most straightforward way to map a bipartite system into 2-qubit system. Let us call $\mathcal{M}_a$ and $\mathcal{M}_b$, the mapping Alice and Bob apply to their share of the state respectively, then the overall mapping applied to the bipartite state ρ_{AB} will simply be $\mathcal{M}_a \otimes \mathcal{M}_b$ and we will end up with a 2×2 state $\mathcal{M}_a \otimes \mathcal{M}_b(\rho_{AB})$.

As we have seen before, any trace preserving CP map can be implemented with an isometry U and an ancillary qubit (2.3). Hence, if we note by U_a the isometry Alice uses, $|0\rangle_a$ the state of the ancilla and the same for Bob but with the subscript b instead, then the total bipartite mapping $\mathcal{M}_a \otimes \mathcal{M}_b$:

$$\begin{aligned}\mathcal{M}_a \otimes \mathcal{M}_b : \mathcal{B}\left(\mathcal{H}^{(d_a)} \otimes \mathcal{H}^{(d_b)}\right) &\to \mathcal{B}\left(\mathcal{H}^{(2_a)} \otimes \mathcal{H}^{(2_b)}\right) : \\ \rho &\to \mathcal{M}_a \otimes \mathcal{M}_b(\rho)\end{aligned}$$

with

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \mathrm{Tr}_{\mathcal{H}^{(D_a)} \otimes \mathcal{H}^{(D_b)}}\left[U_a \otimes U_b \rho \otimes |0\rangle_a\langle 0| \otimes |0\rangle_b\langle 0| U_a^\dagger \otimes U_b^\dagger\right]. \tag{2.53}$$

is trace preserving CP mapping implemented via the isometry $U_a \otimes U_b$ and the two ancillary qubits of Alice and Bob. The tensor product reflects the obvious fact that there is no communication between Alice and Bob.

Next, our goal is to write the previous mapping as a function of local operators acting on each party. Similar to the steps followed in Sec. 2.1.1, we write the isometries U_a and U_b in the form (2.1):

$$U_a = \sum_{i=0}^{3} A_i^{(d_1)} \otimes \sigma_i^{(1)} \quad , \quad U_b = \sum_{i=0}^{3} A_i^{(d_2)} \otimes \sigma_i^{(2)} \tag{2.54}$$

where $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are Pauli operators acting on Alice's qubit and Bob's respectively.[4] Substituting U_a and U_b in (2.53) by their expressions (2.54) and simplifying using the same steps in Sec.2.1.1 for each party, we can write the mapping in the desired form:

$$\begin{aligned} \mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\Big[& \mathbb{1}^{(4)} + \textstyle\sum_{i=1}^{3} \left\langle B_i^{(d_1)} \right\rangle_\rho \sigma_i^{(1)} + \sum_{i=1}^{3} \left\langle B_i^{(d_2)} \right\rangle_\rho \sigma_i^{(2)} \\ & + \textstyle\sum_{i,j=1}^{3} \left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho \sigma_i^{(1)} \otimes \sigma_j^{(2)} \Big] \end{aligned} \tag{2.55}$$

Where the operators $B_i^{(d_k)}$, $k = 1, 2$ and $i = 1, 2, 3$, are defined as a function of the operators $A_j^{(d_k)}$ as in Sec.2.1.1, Eqs.(2.8). The above equation is the desired form of the mapping where we have written the mapping as a function of 6 mean values and 9 correlations of local operators calculated in the original state. We can rewrite the last equation in a more compact form by defining

$$B_0^{(d_k)} = \mathbb{1}^{(d_k)} \quad : k = 1, 2$$

to finally get:

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4} \sum_{i,j=0}^{3} \left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho \sigma_i^{(1)} \otimes \sigma_j^{(2)} \tag{2.56}$$

This form, like in the case of mapping to a single qubit, expresses the 2-qubit state as a function of correlations of local operators, three for each party that can be measured easily. More importantly, it does not contain any indication to the isometries used to implement the mapping.

[4] From this point on, we will refer to operators acting on Alice(Bob) Hilbert space $\mathcal{H}^{(d_1)}(\mathcal{H}^{(d_2)})$ with the superscript $^{(d_1)}(^{(d_2)})$ instead of the subscript $_{a(b)}$, to point out that the Hilbert spaces are of different dimensions in general and will be dropped for the infinite dimensional case.

As we have seen before, we pick six operators, three for each party $B_i^{(d_k)}$, where to ensure complete positivity of the mapping, we should normalize them by a factor $\beta^{(d_k)}$: $k = 1, 2$ determined by theorem (1), so we can write explicitly the mapping (2.56):

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\Bigg[\mathbb{1}^{(4)} + \sum_{i=1}^{3} \frac{\left\langle B_i^{(d_1)} \right\rangle_\rho}{\beta^{(d_1)}} \sigma_i^{(1)} + \sum_{i=1}^{3} \frac{\left\langle B_i^{(d_2)} \right\rangle_\rho}{\beta^{(d_2)}} \sigma_i^{(2)} + \sum_{i,j=1}^{3} \frac{\left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho}{\beta^{(d_1)}\beta^{(d_2)}} \sigma_i^{(1)} \otimes \sigma_j^{(2)}\Bigg] \tag{2.57}$$

Which represents the practical form of the mapping. Complete positivity of the mapping, ensures the physicality, ability to implement, of the mapping. In other words, applying this mapping to any state of the system will give always a state (positive, hermitian and trace one) no matter how large the environment the system might be interacting with. However, for this work, we do not care how the mapping is implemented and that is why we expressed the mapping as a function of correlations and mean values of local observables and some normalization factor that can be calculated as a function of the chosen observables. Here, we are rather interested to test if a given bipartite state ρ is entangled or not. Complete positivity, on the other hand, is not needed as we will see in later discussion where we will relax the condition of complete positivity to positivity only.

Next, I will show two important properties of the above mapping that are useful from entanglement detection perspective. Then I will show how this mapping can provide entanglement criteria with focus on criteria based on angular momentum and Pauli like operators that we introduced previously.

2.3.1 Properties of Mapping (2.57)

Preserves separability

Preserving separability is an obvious property of this mapping since LOCC operations do not increase entanglement. One can directly prove it using the form (2.57). For a product state $\rho = \rho_1 \otimes \rho_2$ we have

$$\left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho = \left\langle B_i^{(d_1)} \right\rangle_{\rho_1} \left\langle B_i^{(d_2)} \right\rangle_{\rho_2}$$

which simplifies the expression of Eq.(2.57) to:

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\left[\mathbb{1}^{(4)} + \sum_{i=1}^{3}\frac{\left\langle B_i^{(d_1)}\right\rangle_\rho}{\beta^{(d_1)}}\sigma_i^{(1)} + \sum_{i=1}^{3}\frac{\left\langle B_i^{(d_2)}\right\rangle_\rho}{\beta^{(d_2)}}\sigma_i^{(2)} + \sum_{i,j=1}^{3}\frac{\left\langle B_i^{(d_1)}\right\rangle_{\rho_1}\left\langle B_i^{(d_2)}\right\rangle_{\rho_2}}{\beta^{(d_1)}\beta^{(d_2)}}\sigma_i^{(1)}\otimes\sigma_j^{(2)}\right]$$
$$= \left[\frac{\mathbb{1}}{2} + \frac{1}{2}\sum_{i=1}^{3}\frac{\left\langle B_i^{(d_1)}\right\rangle_\rho}{\beta^{(d_1)}}\sigma_i^{(1)}\right]\otimes\left[\frac{\mathbb{1}}{2} + \frac{1}{2}\sum_{i=1}^{3}\frac{\left\langle B_i^{(d_2)}\right\rangle_\rho}{\beta^{(d_2)}}\sigma_i^{(2)}\right] = \mathcal{M}_a(\rho_1)\otimes\mathcal{M}_b(\rho_2) \tag{2.58}$$

Hence, any product state is mapped into a product state. By linearity of the mapping, any convex combination of product states will be mapped via (2.57) into the same convex combination of product 2-qubit states.

Separability conservation is a key point to witness entanglement and it is an essential requirement for any entanglement measure. Since the mapping preserves separability, any entanglement in the resulting 2-qubit state, for which entanglement is easily characterized, implies entanglement in the original bipartite state which is usually hard to detect.

Maps PPT states into PPT states

PPT states stand for states with positive partial transpose of the density matrix. A state with a non positive partial transpose or NPT is entangled. This is the so called Peres-Horodecki criterion or PPT criterion [83, 139]. For systems of dimensions larger than 2×3, this criterion is sufficient, i.e. there exists entangled states that are PPT [49]. Which leads to an interesting question: can we, via our mapping (2.54), do better than the PPT criterion? In other words, can we map a PPT state into an NPT 2-qubit state?

The answer to this question is no for the case of mapping (2.54), i.e. a mapping via LO operations with no communication. For the general case with a mapping implemented by LOCC operations, the above questions is related to entanglement distillabilty [245, 145] of the state[5] to which the answer is also no.

The answer to the question in the special case can be shown explicitly within the framework introduced in this chapter. From equations (2.15), we can see that the

[5] ability to obtain a maximally entangled 2-qubit state using LOCC operations

operators that define the mapping can be written in the form:

$$\begin{aligned}\frac{B_1^{(d_k)}}{\beta^{(d_k)}} &= {K_1^{(d_k)}}^\dagger K_0^{(d_k)} + {K_0^{(d_k)}}^\dagger K_1^{(d_k)} \\ \frac{B_2^{(d_k)}}{\beta^{(d_k)}} &= i\left({K_1^{(d_k)}}^\dagger K_0^{(d_k)} - {K_0^{(d_k)}}^\dagger K_1^{(d_k)}\right) \\ \frac{B_3^{(d_k)}}{\beta^{(d_k)}} &= {K_0^{(d_k)}}^\dagger K_0^{(d_k)} - {K_1^{(d_k)}}^\dagger K_1^{(d_k)}\end{aligned} \tag{2.59}$$

with $k = 1, 2$. If we take the transpose of the above observables we find that

$$\left[B_1^{(d_k)}\right]^T = B_1^{(d_k)}\,,\ \left[B_2^{(d_k)}\right]^T = -B_2^{(d_k)}\,,\ \left[B_3^{(d_k)}\right]^T = B_3^{(d_k)} \tag{2.60}$$

where the transpose accounts for a minus sign in the case of $B_2^{(d_k)}$ only. The same is also true for angular momentum operators, Pauli operators in particular. Now, let us calculate the mapping of the partial transpose of a state ρ, $\mathcal{M}_a \otimes \mathcal{M}_b\left(\rho^{(T_2)}\right)$ where ^{T_2} stands for partial transpose with respect to the second party:

$$\begin{aligned}\mathcal{M}_a \otimes \mathcal{M}_b\left(\rho^{T_2}\right) = \frac{1}{4}\Bigg[\mathbb{1}^{(4)} + \sum_{i=1}^{3}\frac{\left\langle B_i^{(d_1)}\right\rangle_\rho}{\beta^{(d_1)}}\sigma_i^{(1)} + \sum_{i=1}^{3}\frac{\left\langle\left[B_i^{(d_2)}\right]^{T_2}\right\rangle_\rho}{\beta^{(d_2)}}\sigma_i^{(2)} + \\ \sum_{i,j=1}^{3}\frac{\left\langle B_i^{(d_1)}\otimes\left[B_j^{(d_2)}\right]^{T_2}\right\rangle_\rho}{\beta^{(d_1)}\beta^{(d_2)}}\sigma_i^{(1)}\otimes\sigma_j^{(2)}\Bigg]\end{aligned} \tag{2.61}$$

where we have used the fact $\mathrm{Tr}\left[A^{T_2}B\right] = \mathrm{Tr}\left[AB^{T_2}\right]$ to do the following substitution:

$$\left\langle B_i^{(d_1)}\otimes B_j^{(d_2)}\right\rangle_{\rho^{T_2}} = \left\langle B_i^{(d_1)}\otimes\left[B_j^{(d_2)}\right]^{T_2}\right\rangle_\rho .$$

Taking eqs.(2.59) and the fact that Pauli operators verify the same equations, we find easily that:

$$\left\langle B_i^{(d_1)}\otimes\left[B_j^{(d_2)}\right]^{T_2}\right\rangle_\rho \sigma_i^{(1)}\otimes\sigma_j^{(2)} = \left\langle B_i^{(d_1)}\otimes B_j^{(d_2)}\right\rangle_\rho \sigma_i^{(1)}\otimes\left[\sigma_j^{(2)}\right]^{T_2}$$

which leads us to the conclusion that:

$$\mathcal{M}_a \otimes \mathcal{M}_b\left(\rho^{T_2}\right) = \left[\mathcal{M}_a \otimes \mathcal{M}_b\left(\rho\right)\right]^{T_2} \tag{2.62}$$

Which means that the PT (Partial Transpose) of a bipartite state ρ is mapped into the PT of its image $\mathcal{M}_a \otimes \mathcal{M}_b(\rho)$. Now, since our mapping preserves positivity, we can conclude that if $\rho^{T_2} \succeq 0$, ρ is PPT, then $\left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho \sigma_i^{(1)} \otimes \left[\sigma_j^{(2)}\right]^{T_2} \succeq 0$. Thus, we have proved that a PPT state is mapped into a PPT state.

In the general case of implementing the bipartite mapping via LOCC operations, it can be shown that PPT states are mapped into PPT states. For that we borrow two theorems form Ref.[145]:

- Any entangled (NPT) 2-qubit state is distillable
- PPT $d_1 \times d_2$ states with $d_1 \geq d_2 > 2$ cannot be distilled

If the mapping implemented by LOCC operations were able to map a PPT state into an NPT 2-qubit state, then the latter could be distilled implying the distillability of our PPT states contradicting the second theorem. Hence, any mapping implemented via LOCC operations should preserve PPT.

2.3.2 Positivity versus complete positivity

Here we will discuss briefly the utility of having a complete positive map versus a positive one. A positive map $\mathcal{M}$ will map any state into a state up to some normalization factor when considering that the system is isolated. However, when considering the system as a part of a bigger world, applying the mapping $\mathcal{M}$ to the system would mean applying $\mathcal{I} \otimes \mathcal{M}$ to the state of the system + environment where $\mathcal{I}$ accounts for the identity over the environment Hilbert space. Unless, $\mathcal{M}$ is completely positive, the mapping $\mathcal{I} \otimes \mathcal{M}$ is not guaranteed to be positive. Since quantum channels are implemented by applying some unitary to the system plus some environment and then tracing out the environment, complete positivity ensures the physicality of the mapped state.

An interesting observation is that the mapping $\mathcal{I} \otimes \mathcal{M}$ is positive for separable states if $\mathcal{M}$ is positive. Hence, if $\mathcal{M}$ is positive but not completely positive then, $\mathcal{I} \otimes \mathcal{M}$ can detect entanglement by mappping an entangled state to a non positive operator. This is a principle idea for entanglement criteria like PPT [83, 139].

Thus, for the entanglement detection, our mapping $\mathcal{M}_a \otimes \mathcal{M}_b$ (2.57) is required to be positive over the set of separable states only. This can be achieved iff $\mathcal{M}_a$ and $\mathcal{M}_b$ are positive. However, the normalization factor we have introduced in 2.1.4 $\beta^{(d_k)}$ guarantees

complete positivity. To ensure positivity, we can choose the normalization factor, as we discussed in Sec. 2.1.4, to be:

$$\alpha^{(d_k)} = \sup\{\left\langle B_1^{(d_k)}\right\rangle_\rho^2 + \left\langle B_2^{(d_k)}\right\rangle_\rho^2 + \left\langle B_3^{(d_k)}\right\rangle_\rho^2 \mid \rho \in \mathcal{B}\left(\mathcal{H}^{(d_k)}\right)\} \tag{2.63}$$

where $\alpha^{(d_k)} \leq \beta^{(d_k)}$. With this, we can finally write the desired mapping in the form:

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\left[\mathbb{1}^{(4)} + \sum_{i=1}^{3}\frac{\left\langle B_i^{(d_1)}\right\rangle_\rho}{\alpha^{(d_1)}}\sigma_i^{(1)} + \sum_{i=1}^{3}\frac{\left\langle B_i^{(d_2)}\right\rangle_\rho}{\alpha^{(d_2)}}\sigma_i^{(2)} + \sum_{i,j=1}^{3}\frac{\left\langle B_i^{(d_1)} \otimes B_j^{(d_2)}\right\rangle_\rho}{\alpha^{(d_1)}\alpha^{(d_2)}}\sigma_i^{(1)} \otimes \sigma_j^{(2)}\right] \tag{2.64}$$

The above mapping preserves separability and positivity of separable states. Hence, non positivity or entanglement in the image of some state ρ is an indication of entanglement of this state. Moreover, in the case where $\alpha^{(d_k)} \neq \beta^{(d_k)}$, PPT conservation is not guaranteed since such mapping is not guaranteed to be implementable with LOCC. Hence, in principle, this might be an advantage to detect PPT entangled states.

2.3.3 Sufficient entanglement criteria

We have thus far introduced the mapping from an arbitrary bipartite system into a 2-qubit system as a function of correlations of six observables, three for each party (2.64). From our previous discussion, since entanglement of the resulting 2-qubit state is sufficient for entanglement in the original state, it is sufficient to apply PPT criterion to the state $\mathcal{M}_a \otimes \mathcal{M}_b(\rho)$ (2.64). One can change the criterion simply by changing the choice of observables for each party.

An interesting question at this stage arises whether it is better to map into a 2×3 system where PPT, albeit complicated, is also a necessary and sufficient condition for entanglement. For this mapping to be done, we need three operators to map to a qubit and eight operators to map into a qutrit. One expects with the additional correlations needed for the mapping compared to mapping into a 2 qubits, more information is gained end better entanglement detection can be achieved. However, it can be shown that any NPT $d \times 2$ state can be mapped into an NPT 2×2 state. Hence all entangled states that can be detected via a mapping into a $d \times 2$ state, can be detected with a mapping into a 2×2 states.

As a choice of operators, we will use the example we introduced, for mapping a system into a single qubit using angular momentum operators (see section 2.1.5), to

derive entanglement criteria. The choice of angular momenta operators is justified by the simplicity of measuring their moments experimentally. However, as we will see their is a trade off between simplicity and efficiency to detect different entangled states. The last statement is expected for systems of large dimensionality as we will see next.

2.3.4 Entanglement criteria with angular momentum operators

Now, we consider the case of a 2-qudit state ρ in $\mathcal{B}(\mathcal{H}^{(d_1)} \otimes \mathcal{H}^{(d_2)})$ and use the mapping $\mathcal{M}_a \otimes \mathcal{M}_b$, where each $U_i (i = 1, 2)$ implements a mapping as the one given by Eq. (2.34). The resulting mapping $\mathcal{M}_a \otimes \mathcal{M}_b$ can then be written explicitly as follows:

$$\begin{aligned} \mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\Bigg[& \mathbb{1} + \frac{1}{j^{(d_1)}} \sum_{i=1}^{3} \langle J_i^{(d_1)} \otimes \mathbb{1} \rangle_\rho \sigma_i \otimes \mathbb{1} \\ & + \frac{1}{j^{(d_2)}} \sum_{i=1}^{3} \langle \mathbb{1} \otimes J_i^{(d_2)} \rangle_\rho \mathbb{1} \otimes \sigma_i + \\ & + \frac{1}{j^{(d_1)} j^{(d_2)}} \sum_{i,j=1}^{3} \langle J_i^{(d_1)} \otimes J_j^{(d_2)} \rangle_\rho \sigma_i \otimes \sigma_j \Bigg], \end{aligned} \tag{2.65}$$

where $J_i^{(d_k)}$ is the i-th $(i = 1, 2, 3)$ angular momentum component on the d_k-dimensional Hilbert space $\mathcal{H}^{(d_k)}$ $(k = 1, 2)$, with $d_k = 2j^{(d_k)} + 1$.

The mapping defined in Eq. (2.65) allows to obtain an operationally easy to implement entanglement witness based on second order correlations. This is achieved by using the following substitutions:

$$\begin{aligned} \langle \sigma_i \otimes \mathbb{1} \rangle_{\mathcal{M}_a \otimes \mathcal{M}_b(\rho)} &= \frac{\langle J_i^{(d_1)} \otimes \mathbb{1} \rangle_\rho}{j^{(d_1)}} \equiv \quad r_i^1, \\ \langle \mathbb{1} \otimes \sigma_i \rangle_{\mathcal{M}_a \otimes \mathcal{M}_b(\rho)} &= \frac{\langle \mathbb{1} \otimes J_i^{(d_2)} \rangle_\rho}{j^{(d_2)}} \equiv \quad r_i^2, \\ \langle \sigma_i \otimes \sigma_j \rangle_{\mathcal{M}_a \otimes \mathcal{M}_b(\rho)} &= \frac{\langle J_i^{(d_1)} \otimes J_j^{(d_2)} \rangle_\rho}{j^{(d_1)} j^{(d_2)}} \equiv \quad T_{ij}, \end{aligned} \tag{2.66}$$

which are direct consequences of Eq. (2.65). Therefore we consider the following form for the general mapped state:

$$\mathcal{M}_a \otimes \mathcal{M}_b(\rho) = \frac{1}{4}\left[\mathbb{1} + \vec{r}^1.\vec{\sigma} \otimes \mathbb{1} + \vec{r}^2.\mathbb{1} \otimes \vec{\sigma} + \sum_{i=1}^{3} T_{ii} \sigma_i \otimes \sigma_i \right], \tag{2.67}$$

where we assume that the rotations needed to diagonalize T_{ij} have been performed. We have introduced the vectors $\vec{r}^{\,j}$ $(j = 1, 2)$, which have as components the r_i^j $(i = 1, 2, 3)$ defined in Eq. (2.66), after the mentioned needed rotations.

Once we have the form of Eq. (2.65) or Eq. (2.67), the best entanglement witness that we can use is the Peres-Horodecki criterion [246] relying on the positiveness of the partial transpose. This criterion can be simplified by considering the geometric picture which was developed in Ref. [139]. In this paper, it was shown that for the states of the form given by Eq. (2.67), the vector $\vec{T} = \{T_{11}, T_{22}, T_{33}\} \in \mathbb{R}^3$ must lie within a tetrahedron with vertices $\{(-1,-1,-1), (-1,1,1), (1,-1,1), (1,1,-1)\}$, to fulfill the positiveness requirement. Each of the 4 vertices of this tetrahedron is reached when the 2-qubit state is one of the four Bell-states $|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$, $|\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$. In this picture, the separable states are those for which the vector $\vec{T} = \{T_{11}, T_{22}, T_{33}\} \in \mathbb{R}^3$ lies within the octahedron with vertices $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. This last property can be put in the following more compact form: for any separable state of the form given by Eq. (2.67), $\vec{T}$ verifies the inequality:

$$|T_{11}| + |T_{22}| + |T_{33}| \leq 1. \tag{2.68}$$

Using the definitions given by Eq. (2.66), Eq. (2.68) can be re-expressed as a 2-qudit entanglement criterion:
For any separable state ρ acting on $\mathcal{H}^{(d_1)} \otimes \mathcal{H}^{(d_2)}$, the vector $\{\langle J_i^{(d_1)} \otimes J_i^{(d_2)} \rangle_\rho : i = 1, 2, 3\}$ verifies the following inequality:

$$\begin{aligned} \left|\langle J_1^{(d_1)} \otimes J_1^{(d_2)} \rangle_\rho\right| + \left|\langle J_2^{(d_1)} \otimes J_2^{(d_2)} \rangle_\rho\right| + \\ + \left|\langle J_3^{(d_1)} \otimes J_3^{(d_2)} \rangle_\rho\right| \leq j^{(d_1)} j^{(d_2)}. \end{aligned} \tag{2.69}$$

Therefore, all states that violate inequality (2.69) lie outside the octahedron and are thus entangled. This 2-qudit entanglement criterion has the advantage of being very simple to test experimentally. One can notice that there are no 2-qudit states that are mapped to any of the 2-qubit Bell states. Therefore, the vertices of the tetrahedron do not belong to the image of our mapping. This can be proved by contradiction: consider that there exists a state ρ such that $\mathcal{M}_a \otimes \mathcal{M}_b(\rho)$ is one of the Bell states, say $|\Psi^+\rangle\langle\Psi^+|$. For this state we have:

$$\begin{aligned} &\mathrm{Tr}\left[|\Psi^+\rangle\langle\Psi^+|\sigma_3\right] = \ 1, \\ &\mathrm{Tr}\left[|\Psi^+\rangle\langle\Psi^+|\sigma_2\right] = -1, \ \mathrm{Tr}\left[|\Psi^+\rangle\langle\Psi^+|\sigma_1\right] = 1. \end{aligned} \tag{2.70}$$

From Eq. (2.70) and Eq. (2.65), we find $\langle J_3^{(d_1)} \otimes J_3^{(d_2)} \rangle_\rho = j^{(d_1)} j^{(d_2)}$ which in turn implies that the state ρ must be a pure state of the form $\alpha|j^{(d_1)}, j^{(d_2)}\rangle + \beta|-j^{(d_1)}, -j^{(d_2)}\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. For such states the other values of $\langle \sigma_i \rangle_{\mathcal{M}_a \otimes \mathcal{M}_b(\rho)}$ for $i = 1$ or $i = 2$ are zero and not 1 or -1, indeed $\langle J_1^{(d_1)} \otimes J_1^{(d_2)} \rangle_\rho = \langle J_2^{(d_1)} \otimes J_2^{(d_2)} \rangle_\rho = 0$ for $d_1 > 2$ or $d_2 > 2$. The same reasoning can be made for each Bell state.

To have an insight about the efficiency of our criterion to detect entanglement, we apply it to two known families of qudit states that have been extensively studied in Refs. [34, 47, 247, 36]. From now on, we suppose that the two qudits are of the same dimension, that is $d_1 = d_2 \equiv d = 2j + 1$. First, we recall the family of d^2 maximally entangled 2-qudit states $|\Omega_{kl}\rangle (k, l = 0, 1, \cdots, d-1)$ that generalize the four 2-qubit Bell states [34, 47, 247, 36]:

$$|\Omega_{kl}\rangle = W_{kl} \otimes \mathbb{1} |\Omega_{00}\rangle, \text{ with } |\Omega_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} |m, m\rangle, \tag{2.71}$$

where the d^2 W_{kl} operators acting on the first qudit are the Weyl operators defined as

$$W_{kl}|m\rangle = e^{\frac{i2\pi k(m-l)}{d}} |(m-l)_{\text{mod } d}\rangle. \tag{2.72}$$

These 2-qudit Bell states $P_{kl} = |\Omega_{kl}\rangle\langle\Omega_{kl}|$ are locally maximally mixed states, that is, by taking their partial trace one obtains the maximally mixed state $\frac{\mathbb{1}}{d}$. It is interesting to notice that they are mapped by Eq. (2.67) to a locally maximally mixed 2-qubit state. Indeed, we have:

$$\vec{r}_{kl}^{\,1} = \frac{\langle \vec{J} \otimes \mathbb{1} \rangle_{P_{kl}}}{j} = \vec{0} \quad \text{and} \quad \vec{r}_{kl}^{\,2} = \frac{\langle \mathbb{1} \otimes \vec{J} \rangle_{P_{kl}}}{j} = \vec{0}.$$

For such states, our simple criterion Eq. (2.69) is as strong as the PPT criterion applied to states given by Eq. (2.67) and detects all entangled locally maximally mixed states that are detected by the latter. We can thus say that every maximally entangled 2-qudit state $|\Omega_{kl}\rangle$ is detected by our criterion Eq. (2.69), even though these states are not mapped to 2-qubit Bell states. Instead, they are mapped to mixed states that are locally maximally mixed, that is, a convex sum (statistical mixture) of 2-qubit Bell states.

Examples

We proceed by exploring the statistical mixtures of maximally entangled 2-qudit pure states that can be detected by our criterion Eq. (2.69). We start by applying our criterion

to the so called Werner states which provide a good description of the effects of phase and depolarizing noise in maximally entangled states [141, 47]:

$$\rho_\alpha = \alpha|\Omega_{00}\rangle\langle\Omega_{00}| + \frac{1-\alpha}{d^2}\mathbb{1}, \text{ with } \frac{-1}{d^2-1} \leq \alpha \leq 1. \tag{2.73}$$

The bounds for the parameter α are such that ρ_α is positive. It is known [141] that ρ_α is entangled iff $\frac{1}{d+1} < \alpha \leq 1$. A straightforward application of our criterion Eq. (2.69) to ρ_α gives that if $\alpha j\,(j+1) > j^2$ then ρ_α is entangled. Therefore, our criterion can detect all the entangled states ρ_α for $\alpha \in [\frac{j}{(j+1)}, 1]$. Recalling that $d = 2j+1$, we realize that entangled states with $\frac{1}{2(j+1)} < \alpha \leq \frac{j}{(j+1)}$ are not detected.

As a more specific example, we now consider the 3-parameter family of 2-qudit states:

$$\begin{aligned}\rho_{\alpha,\beta,\gamma} &= \frac{1-\alpha-\beta-\gamma}{(2j+1)^2}\mathbb{1} + \alpha P_{00} \\ &+ \frac{\beta}{2j}\sum_{i=1}^{2j} P_{i0} + \frac{\gamma}{2j+1}\sum_{i=0}^{2j} P_{i1},\end{aligned} \tag{2.74}$$

where the P_{kl} are the projectors on the $|\Omega_{kl}\rangle$ states. This family of states is a generalization to arbitrary dimensional qudits of the 2-qutrit states originally introduced in Refs. [47, 247, 36] to study bound entanglement. Density matrices as given in Eq. (2.74) are interesting because their eigenvalues and those of their partial transpose can be explicitly expressed as a function of the parameters α, β and γ (see Appendix A.6). This allows to locate the set of PPT $\rho_{\alpha,\beta,\gamma}$ in the space spanned by $\alpha, \beta,$ and γ. To ensure positivity, the parameters $\alpha, \beta,$ and γ must verify the following inequalities (see Appendix A.6):

$$\begin{aligned}\frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0 \quad &, \alpha + \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0 \\ \frac{\beta}{2j} + \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0 \quad &, \frac{1-\alpha-\beta+2j\gamma}{(2j+1)^2} \geq 0.\end{aligned}$$

These inequalities define the interior of a tetrahedron. Now, applying our criterion Eq. (2.69), we obtain that all separable states $\rho_{\alpha,\beta,\gamma}$ given by Eq. (19) are such that:

$$\frac{|(j+1)(\alpha+\beta)+(j-2)\gamma| + (j+1)\,|2\alpha-\beta/j|}{3j} \leq 1. \tag{2.75}$$

All states $\rho_{\alpha,\beta,\gamma}$ for which the inequality Eq (2.75) is violated are therefore entangled.

In order to have an idea about the efficiency of criterion Eq. (2.69), we have calculated the eigenvalues of the PT of $\rho_{\alpha,\beta,\gamma}$ and thus obtained explicit conditions on $\alpha, \beta,$ and γ

for $\rho^{T_2}_{\alpha,\beta,\gamma}$ to be positive (see Appendix A.6). In Fig. 2.1, we present the tetrahedron of

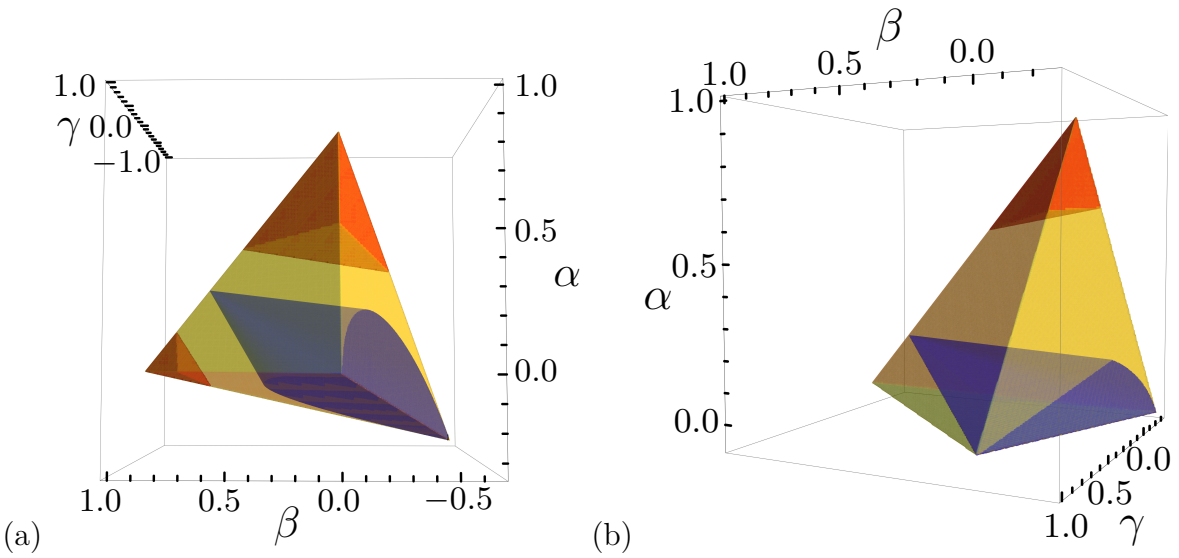


Figure 2.1: Geometrical representation of states $\rho_{\alpha,\beta,\gamma}$ [Eq. (2.74)] in parameter space for (a) $2j+1=3$ and (b) $2j+1=5$. All physical states lie inside the tetrahedron whereas the PPT states lie inside the blue region [a cone for the 2-qutrit case (a)]. Red regions depict entangled states detected using our criterion Eq. (2.69), whereas the yellow region hosts the non detected entangled states by this criterion.

positive states $\rho_{\alpha,\beta,\gamma}$ given by Eq. (2.74) in parameter space for the cases $j=1$ (Fig. 2.1.a) and $j=2$ (Fig. 2.1.b). The set of PPT states are depicted in blue, so the remaining volume of the tetrahedron corresponds to entangled states. We have represented in red the states that are detected by our criterion Eq. (2.69). We see clearly that it detects a significative part of the entangled states parametrized by Eq. (2.74). Nevertheless, comparing Fig. 2.1.a and Fig. 2.1.b, we note that this volume decreases when j increases from $j=1$ to $j=2$. However, different criteria which present a better scaling with dimension for this particular family of states can be found by changing the isometry U in Eq. (2.1) (or equivalently changing the corresponding A_i operators in Eq. (2.25), used to map each qudit to a qubit).

2.3.5 Mapping for N qudits and spin squeezing inequalities

Until now, we only have considered the entanglement of two qudits. Now, we adress the problem of detecting entanglement in a large N qudits system. An interesting consequence of our mapping is that it can be easily extended to map a system of N qudits to a system of N qubits. Indeed, by applying Eq. (2.34) individually to each qudit,

the separability among the parties is preserved. If we denote $L_\alpha = \frac{J_\alpha}{j} (\alpha = 1, 2, 3)$ and $L_0 = \mathbb{1}$, then the N-qubit mapped state corresponding to the N-qudit state ρ can be written as

$$\mathcal{M}_{\otimes^N U}(\rho) = \frac{1}{2^N} \sum_{\vec{k} \in \{0,1,2,3\}^N} \langle \otimes_{i=1}^N L_{k_i} \rangle_\rho \otimes_{i=1}^N \sigma_{k_i}. \tag{2.76}$$

An immediate consequence is that for any $\vec{k} \in \{0, 1, 2, 3\}^N$, we have

$$\langle \otimes_{i=1}^N L_{k_i} \rangle_\rho = \langle \otimes_{i=1}^N \sigma_{k_i} \rangle_{\mathcal{M}_{\otimes^N U}(\rho)}. \tag{2.77}$$

By using this property to compute first and second order correlations, we can provide an alternative derivation of the spin squeezing inequalities detecting N-qudit entanglement from the N-qubit ones introduced in Ref. [248]. It was shown in Ref. [201] that all separable N-qubit states satisfy the following inequalities:

$$\begin{aligned}
&\langle \hat{\mathcal{S}}_1^2 \rangle + \langle \hat{\mathcal{S}}_2^2 \rangle + \langle \hat{\mathcal{S}}_3^2 \rangle \leq \frac{N\,(N+2)}{4}, \\
&(\Delta \hat{\mathcal{S}}_1)^2 + (\Delta \hat{\mathcal{S}}_2)^2 + (\Delta \hat{\mathcal{S}}_3)^2 \geq \frac{N}{2}, \\
&\langle \hat{\mathcal{S}}_\alpha^{\mathrm{lsq}} \rangle + \langle \hat{\mathcal{S}}_\beta^{\mathrm{lsq}} \rangle - (N-1)(\widetilde{\Delta} \hat{\mathcal{S}}_\gamma)^2 \leq \frac{N(N-1)}{4}, \\
&\langle \hat{\mathcal{S}}_\alpha^{\mathrm{lsq}} \rangle - (N-1)[(\widetilde{\Delta} \hat{\mathcal{S}}_\beta)^2 + (\widetilde{\Delta} \hat{\mathcal{S}}_\gamma)^2] \leq \frac{N(N-1)}{4}.
\end{aligned} \tag{2.78}$$

where $\hat{\mathcal{S}}_\alpha = \frac{1}{2} \sum_{i=1}^N \sigma_\alpha^i$ is the collective spin operator in direction α. The indexes (α, β, γ) may assume any permutation of $(1, 2, 3)$ and the following definitions have been used:

$$\begin{aligned}
(\Delta \hat{\mathcal{S}}_\alpha)^2 &\equiv \langle \hat{\mathcal{S}}_\alpha^2 \rangle - \langle \hat{\mathcal{S}}_\alpha \rangle^2, \\
\langle \hat{\mathcal{S}}_\alpha^{\mathrm{lsq}} \rangle &\equiv \langle \tfrac{1}{4} \textstyle\sum_{i \neq j=1}^N \sigma_\alpha^i \otimes \sigma_\alpha^j \rangle, \\
(\widetilde{\Delta} \hat{\mathcal{S}}_\alpha)^2 &\equiv \langle \hat{\mathcal{S}}_\alpha^{\mathrm{lsq}} \rangle - \langle \hat{\mathcal{S}}_\alpha \rangle^2.
\end{aligned} \tag{2.79}$$

Using Eq. (2.77), we obtain, starting from the N-qubit inequalities Eqs. (2.78), the inequalities satisfied by all separable N-qudit states (N spins j). This is achieved with the following substitutions [248]:

$$\hat{\mathcal{S}}_\alpha \to \frac{1}{2j} \hat{\mathcal{J}}_\alpha, \quad \hat{\mathcal{S}}_\alpha^{\mathrm{lsq}} \to \frac{1}{4j^2} \hat{\mathcal{J}}_\alpha^{\mathrm{lsq}} \tag{2.80}$$

where $\hat{\mathcal{J}}_\alpha = \sum_{i=1}^{N} J_\alpha^i$ and $\hat{\mathcal{J}}_\alpha^{\mathrm{lsq}} = \frac{1}{4}\sum_{i\neq j=1}^{N} J_\alpha^i \otimes J_\alpha^j$. Therefore, the relation between the entanglement criterion for N-qubit and N-qudit systems, which was already considered in Ref. [248], can be thought as a simple consequence of the particular mapping explicitly given by Eqs. (2.25) or equivalently by Eq. (2.34). As a consequence, we have shown that the qudit entanglement revealed by the qudit spin squeezing inequalities can always be recast as qubit-like, or dichotomic, spin squeezing. Thus, qudit spin squeezing inequalities do not evidence entanglement of higher dimension than the qubit squeezing ones. Finally, we notice that by choosing different A_i operators in Eq. (2.1) we can expect to find new multipartite qudits entanglement criteria.

2.4 Conclusion

In conclusion, we have presented a general scheme to map qudits to qubits that can be used to define criteria to detect entanglement between qudits. We have applied this general scheme to provide a specific entanglement criterion based on the measurement of qudit–qudit correlations.

In addition, our results provide a way to classify multi-partite qudit entanglement according to its detectability through dichotomization. Finally, it opens the interesting question of what are the specific entanglement types, if any other than bound entanglement, that fail to be detected by our method.

3

Generalized spin squeezing inequalities for fluctuating particle number

In this chapter, we will focus in the detection of entanglement in multi-partite systems consisting of high number of particles. For such systems accessing and measuring individual is a prohibitive task and executing full state tomography is out of question. The entanglement criteria should be based on possible collective measurements and preferably ones that do not require high correlations that might prove impractical. We will focus on spin squeezing inequalities for a system of N spins. These inequalities are verified for separable states and only require measurement of first and second moments of collective spin components along three different directions. These inequalities can be further generalized to arbitrary collective operators by mapping each particle into a qubit as we have introduced in the previous chapter. This increases the possibilities to detect more entangled states than the squeezed states detected by spin squeezing inequalities. I will then introduce our main result which enables the detection of entanglement when the number of particles is not fixed, but can fluctuate. Particles number fluctuations are bound to occur in experiments and they can be put into two categories: classical and quantum fluctuations. Classical fluctuations can be related to experimental inaccuracies, where the number of particles varies from one state preparation to another. On the other hand, quantum fluctuations may appear when the state is not an eigenstate of the number operator. Accounting for such fluctuations will allow us to improve the bounds of spin squeezing inequalities and detect entangled states that are not detected with the original spin squeezing inequalities. This result has been recently published as an article

in Physical Review A:
"*Generalized spin-squeezing inequalities for particle number with quantum fluctuations*".
I. Saideh, S. Felicetti, T. Coudreau, P. Milman, and Arne Keller
Phys. Rev. A 94, 032312.

In the last part of this chapter, I present the work that I have performed in the framework of an ongoing collaboration with the "Quantum Dipolar Gazes" team at "Laboratoire de Physique de Laser" laboratory (Université Paris Nord). The main goal of this work was to bring entanglement criteria that could be implemented in their experiment. After a brief description of the experimental setup, which consists in a latice of optically trapped chromium atoms, I will present the results of numerical simulations aimed to model their experiment. These simulations are our first attempt to detect entanglement in such *dipolar gazes*. While these numerical results are not yet conclusive, they nevertheless show how entanglement can be difficult to detect, and can be considered as a starting point for future developpement.

3.1 Introduction

In the quest for quantum computers and quantum simulators, the ability to create, detect and characterize large scale entanglement in many body systems is one of the key points that has attracted a lot of interest in the last decade [249–251]. From a more fundamental perspective, the understanding of the entanglement properties and their manipulation at the macroscopic level is also of importance to understand the quantum to classical transition [252, 253]. It is worth noticing that due to the exponential growth of the Hilbert space dimension with the number of parties N, an exact numerical simulation of such systems with classical computer is not possible when N becomes of the order of some tens. In this context, providing theoretical tools for the experimental detection of entanglement is a necessity. Typically, such experiments involve interacting many body systems as cold atoms [254, 251], trapped ions [255–257] or photons [258, 259], and individual addressing or accessing each body individually is not possible. Accessible observables consist more often of collectives ones, expressed as the sum of local observables that in most cases are one body operators. The spatial component of a collective spin, sum of local spins, is such an instance of a global observable. If the global spin state is squeezed, that is, if the fluctuation of one of its component is sufficiently small compared to the expectation of the other components, then it can be shown that the N-spin systems is entangled [195, 198]. Tóth and collaborators [200, 201, 227, 228] have

generalized such an approach providing a set of inequalities that are fulfilled for all separable state of the N spins system and thus are able to detect entanglement when violated.

The original spin-squeezing inequalities [200, 201, 227] consider the number of particles N as a constant. In fact, N may undergo classical and/or quantum fluctuations. Classical fluctuations are due the presence of statistical mixtures of states with different N. In contrast, quantum fluctuations are given by coherent superposition of states corresponding to different number of particles. It is often argued, in the context of Bose-Einstein condensation [260], that coherent superposition of states corresponding to different number of particles are not allowed or can not give observable consequences. The proscription of such coherent superposition is often justified by an axiomatic superselection rule (SSR) which should be applied to massive particles but not to massless ones. Actually, this SSR is a consequence of the lack of a fixed absolute phase reference [261]. It has been pointed-out that such phase reference can be established allowing for instance the coherent quantum superposition of an atom and a molecule [262]. Quantum and classical particles number fluctuations have been considered in the context of quantum metrology [229], where the relation of quantum-enhanced parameter estimation and entanglement is investigated when the particles number is only known on average. Spin squeezing inequalities for fluctuating N have been considered in Ref [222] but the fluctuations of the total number of particles considered in that work were only classical (statistical) fluctuations and quantum fluctuations were not investigated.

In this work we generalize the original spin squeezing inequality of Ref. [228], by considering the situation of arbitrary particle number fluctuations, including quantum and/or classical ones. This generalization is important and necessary in many experimental situations even where the SSR applies. Such an interesting example can be found in Ref. [251] where a system of N spin 1 is considered as a systems of N spin-1/2 by projecting each spin 1 on the subspace spanned by two magnetic sub-levels. In this subspace, quantum fluctuations (and not only statistical ones) of the particle number are expected, and the validity of the original spin squeezing inequalities is not granted.

We first recall the original spin inequalities [228], and how they can be generalized using 3 collective operators A_1, A_2 and A_3, instead of the 3 components J_x, J_y and J_z of a collective spin. Finally, we present our new inequalities where particle number fluctuations are considered.

3.2 Original spin squeezing inequalities

Let us recall the original inequalities derived in Ref. [228] that are fulfilled by all separable states:

$$\tilde{\Delta}^2 J_x + \tilde{\Delta}^2 J_y + \tilde{\Delta}^2 J_z \geq -Nj^2 \tag{3.1a}$$

$$(N-1)\left[\tilde{\Delta}^2 J_k + \tilde{\Delta}^2 J_l\right] \geq \langle \tilde{J}_m^2 \rangle - N(N-1)j^2 \tag{3.1b}$$

$$\langle \tilde{J}_l^2 + \tilde{J}_m^2 \rangle - N(N-1)j^2 \leq (N-1)\tilde{\Delta}^2 J_k \tag{3.1c}$$

$$\langle \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2 \rangle \leq N(N-1)j^2, \tag{3.1d}$$

where l, m end k refer to different x, y or z component of the total spin operator $J_l = \sum_{i=1}^N J_l^{(i)}$, sum of the local spins operators $J_l^{(i)} = \bigotimes_{i'=1; i\neq i'}^N \mathbb{1}^{(i')} \otimes j_k^{(i)}$, where $\mathbb{1}^{(i')}$ denotes the identity operator and $j_k^{(i)}$ the k component of the spin in the one particle Hilbert space. The eigenvalues of $(\vec{j}^{(i)})^2$ are $j(j+1)$. As in Ref. [228], the notation $\tilde{J}_k^2$ means :

$$\tilde{J}_k^2 = J_k^2 - \sum_{i=1}^N (J_k^{(i)})^2 = \sum_{i\neq j=1}^N J_k^{(i)} J_k^{(j)} \tag{3.2}$$

and the modified variance is defined as $\tilde{\Delta}^2 J_k = \langle \tilde{J}_k^2 \rangle - \langle J_k \rangle^2$. The 4 inequalities Eqs. (3.1) can be written in the following compact form [227]:

$$(N-1)\sum_{k\notin\mathcal{I}} \tilde{\Delta}^2 J_k - \sum_{k\in\mathcal{I}} \langle \tilde{J}_k^2 \rangle \geq -N(N-1)j^2, \tag{3.3}$$

where $\mathcal{I}$ can be any subset of $\{x, y, z\}$ (including the empty set). Each inequality in Eqs. (3.1) is obtained by increasing the number of elements in $\mathcal{I}$ by one, starting from the empty set.

As it has been shown in Ref. [227], the vectorial character of the spin is not needed to obtain Eq. (3.3). Indeed, a set of 3 collective observables A_k where $k = 1, 2, 3$ can be used instead, each of them obtained as a sum of local observable as $A_k = \sum_{i=1}^N A_k^{(i)}$. To be able to derive inequalities as Eq. (3.3), it is only required that

$$\sum_{k=1}^3 \langle A_k^{(i)} \rangle^2 \leq \alpha^2; \forall i = 1, 2, \cdots, N, \tag{3.4}$$

which is satisfied by the spin operators $J_k^{(i)}$ with $\alpha = j$. Then, as it has been shown in Ref. [227], using the Cauchy-Schwartz inequality

$$\langle A_k \rangle^2 \leq N \sum_{i=1}^{N} \langle A_k^{(i)} \rangle^2 \tag{3.5}$$

and the concavity of the variance, we obtain the inequalities (3.3) where J_k is replaced by A_k, j is replaced by α and where $\mathcal{I}$ is any subset of $\{1, 2, 3\}$, including the empty set.

3.3 Fluctuations of particles number

Note that Eqs. (3.3) are derived for a fixed number of particles N. To generalize these equations to include quantum fluctuations of the particle number, we consider that we have N sites ($i = 1, 2, \cdots, N$), and that in each site there is one or zero particles. We define the local positive operator $\hat{N}^{(i)}$ giving the number of particle in site i; it has only two eigenvalues 0 or 1 corresponding to the absence or the presence of a particle. Hence, the collective operator $\hat{N} = \sum_{i=1}^{N} \hat{N}^{(i)}$ represents the total number of particles. Our main result is that all separable states fulfil the following inequalities:

$$\left(\langle \hat{N} \rangle - 1\right) \sum_{k \notin \mathcal{I}} \tilde{\Delta}^2 A_k - \sum_{k \in \mathcal{I}} \langle \tilde{A}_k^2 \rangle \geq -\langle \hat{N} \rangle \left(\langle \hat{N} \rangle - 1\right) \alpha^2 - \delta, \tag{3.6}$$

where δ is defined as

$$\delta = \tilde{\Delta}^2 A_1 + \tilde{\Delta}^2 A_2 + \tilde{\Delta}^2 A_3 + \alpha^2 \langle \hat{N} \rangle \tag{3.7}$$

and corresponds to the term added to Eq. (3.3) when N is replaced by $\langle \hat{N} \rangle$. That is, setting $\delta = 0$ and replacing $\langle N \rangle$ by the constant N in Eq. (3.6) give us Eq. (3.3). These inequalities are very convenient since they are as simple as the original ones. Indeed, to test their violation, the same type of measurements realized in the original inequalities for fixed particle number must be performed. Eq. (3.6) can also be written explicitly, by increasing the cardinality of $\mathcal{I}$:

$$\tilde{\Delta}^2 A_1 + \tilde{\Delta}^2 A_2 + \tilde{\Delta}^2 A_3 \geq -\alpha^2 \langle \hat{N} \rangle \tag{3.8a}$$

$$\left(\langle \hat{N} \rangle - 1\right) \left(\tilde{\Delta}^2 A_i + \tilde{\Delta}^2 A_j\right) - \langle \tilde{A}_k^2 \rangle \geq -\alpha^2 \langle \hat{N} \rangle \left(\langle \hat{N} \rangle - 1\right) - \delta \tag{3.8b}$$

$$\left(\langle \hat{N} \rangle - 1\right) \tilde{\Delta}^2 A_i - \langle \tilde{A}_j^2 \rangle - \langle \tilde{A}_k^2 \rangle \geq -\alpha^2 \langle \hat{N} \rangle \left(\langle \hat{N} \rangle - 1\right) - \delta \tag{3.8c}$$

$$\langle \tilde{A}_1^2 \rangle + \langle \tilde{A}_2^2 \rangle + \langle \tilde{A}_3^2 \rangle \leq \alpha^2 \langle \hat{N} \rangle \left(\langle \hat{N} \rangle - 1\right) + \delta \tag{3.8d}$$

We note that the first inequality is exactly the same as Eq. (3.1a) but with N replaced by $\langle N\rangle$. That is, we can replace N by $\langle N\rangle$ in Eq. (3.1a) and it remains a valid equation fulfilled by all separable states when N is not a constant. We also note that Eq. (3.8a) can be written as $\delta \geq 0$.

Now, if in a given experiment δ is found to be positive, then inequalities Eqs. (3.8b-d) are less tight than the original inequalities Eqs. (3.1b-d). Hence, a violation of Eqs. (3.1b-d) can appear without violating Eqs. (3.8b-d). In other words, The simple substitution of N by $\langle N\rangle$ in the original inequalities Eqs. (1b-d) can give false positive. This is why it is crucial to consider the term δ before to affirming entanglement detection. In the other case, when $\delta < 0$, both inequalities, Eq. (3.1a) or Eq. (3.8a), detect entanglement, but Eqs. (3.8b-d) becomes tighter than than the original ones, Eqs. (3.1b-d). Hence, in this case, the visibility of the violation is higher, which can represent an important advantage from the experimental point of view.

3.4 Proof

We give a sketch of the proof leaving the technical details in appendix A and B. The proof is done in two steps. In the first step, inequalities fulfilled by all product states $\rho = \bigotimes_{i=1}^{N} \rho^{(i)}$ are obtained, then in a second step we generalize them to all separable states using convexity arguments.

3.4.1 Inequalities for product states

For the first step, the main objective is to obtain a tighter inequality than the one obtained in Eq. (3.5) through the Cauchy-Schwartz inequality. For this, the main idea is to map each local state $\rho^{(i)}$, in the site i, to a spin 1, or a 3-level state $R^{(i)}$ in an auxiliary Hilbert space spanned by $|0^{(i)}\rangle, |1^{(i)}\rangle, |2^{(i)}\rangle$ states as follows :

$$R^{(i)} = n_i \left(\frac{{\sigma_0}^{(i)}}{2} + \frac{\left\langle A_1^{(i)} \right\rangle_\rho}{2\eta_i} {\sigma_x}^{(i)} + \frac{\left\langle A_2^{(i)} \right\rangle_\rho}{2\eta_i} {\sigma_y}^{(i)} + \frac{\left\langle A_3^{(i)} \right\rangle_\rho}{2\eta_i} \sigma_z \right) + (1 - n_i) \, |2^{(i)}\rangle\langle 2^{(i)}| \quad (3.9)$$

where $\sigma_0 = |0^{(i)}\rangle\langle 0^{(i)}| + |1^{(i)}\rangle\langle 1^{(i)}|$ is the projection operator on the qubit subspace spanned by the state $|0^{(i)}\rangle$ and $|1^{(i)}\rangle$ and $\sigma_k^{(i)} (k = x, y, z)$ are the Pauli matrices in the same subspace. The constant η_i is chosen as $\eta_i = \sqrt{\left\langle A_1^{(i)} \right\rangle_\rho^2 + \left\langle A_2^{(i)} \right\rangle_\rho^2 + \left\langle A_3^{(i)} \right\rangle_\rho^2}$, such that the term inside the braket in Eq. (3.9) is a pure state $|\Psi^{(i)}\rangle$. Therefore the state

$R^{(i)}$ can also be written as

$$R^{(i)} = n_i |\Psi^{(i)}\rangle\langle\Psi^{(i)}| + (1-n_i)|2^{(i)}\rangle\langle 2^{(i)}|, \tag{3.10}$$

where n_i represents the average occupation number of the particle in site i, that is $n_i = \left\langle N^{(i)} \right\rangle_\rho$. The mapping $\rho^{(i)} \to R^{(i)}$ can be interpreted in the following way: when there is a particle in site i, we map its state to a pure state $|\Psi^{(i)}\rangle$ such that $\langle\Psi^{(i)}|\sigma_k|\Psi^{(i)}\rangle = \frac{1}{\eta_i}\left\langle A_k^{(i)} \right\rangle_\rho$ and when there is no particle we attribute this event to state $|2^{(i)}\rangle$. Averaging over the occupation of the site i gives us the state $R^{(i)}$. Using techniques similar to those developed in Ref. [1] we can prove (see appendix B) that this mapping is completely positive and thus $R^{(i)}$ is indeed a state, that is a positive hermitian operator. Note that $\langle\sigma_x\rangle_{R^{(i)}} = n_i \dfrac{\left\langle A_1^{(i)} \right\rangle_\rho}{\eta_i}$. This relation is not exactly what we need. Indeed, if we sum over all sites i, $\langle\sum_{i=1}^N \sigma^{(i)}\rangle_{R^{(i)}}$ will not be simply related to the expectation of original collective operator $A_1 = \sum_{i=1}^N A_1^{(i)}$, because the pre-factor $\frac{n_i}{\eta_i}$ depends on the site i. It can be shown (see appendix A) that applying a rotation in the qubit subspace, we can obtain a new state $R^{(i)'}$ such that $\langle\sigma_x\rangle_{R^{(i)'}} = \dfrac{\left\langle A_1^{(i)} \right\rangle_\rho}{\alpha}$, where the factor α does not depends on the site i and is defined as $\alpha^2 = \sup_{\rho^{(i)}} \left[\sum_{k=1}^3 \left\langle A_k^{(i)} \right\rangle_{\rho^{(i)}}^2\right]$.

Now, we can consider the product state $R' = \bigotimes_{i=1}^N R^{(i)'}$ in the qutrit Hilbert space, and define collective spin operators: $S_k = \sum_{i=1}^N \sigma_k$, which verify the commutation relation $[S_k, S_l] = 2i\epsilon_{klm}S_m$. Using the Heisenberg inequality $\frac{1}{4}\left|\langle[A,B]\rangle_{R'}\right|^2 \leq (\Delta A)^2 (\Delta B)^2$ with $A = S_y$ and $B = S_z$, we can write

$$\left|\langle S_x\rangle_{R'}\right|^2 = \left|\sum_{i=1}^N \frac{\left\langle A_1^{(i)} \right\rangle_\rho}{\alpha}\right|^2 = \frac{\langle A_1\rangle_\rho^2}{\alpha^2} \leq (\Delta S_y)^2 (\Delta S_z)^2 . \tag{3.11}$$

In that way, we can obtain a tighter inequality for $\langle A_k\rangle_\rho^2$

$$\langle A_k\rangle_\rho^2 \leq \langle N\rangle_\rho \left(\sum_{i=1}^N \left\langle A_k^{(i)} \right\rangle_\rho^2\right) + \alpha^2 \langle N\rangle_\rho \left(\langle N\rangle_\rho - \sum_{i=1}^N \left\langle N^{(i)} \right\rangle_\rho^2\right). \tag{3.12}$$

Using similar techniques (see appendix A for more details), we obtain general inequalities fulfilled by all product states:

$$(\langle\hat{N}\rangle - 1)\sum_{k\notin\mathcal{I}} \tilde{\Delta}^2 A_k - \sum_{k\in\mathcal{I}} \langle\tilde{A}_k^2\rangle \geq -\langle\hat{N}\rangle(\langle\hat{N}\rangle - 1)\alpha^2 - \delta', \tag{3.13}$$

where δ' is given by

$$\delta' = \alpha^2\langle\hat{N}\rangle - \sum_{i=1}^{N}\sum_{k\in\mathcal{I};k\notin\mathcal{I}}\langle A_k^{(i)}\rangle^2 + \langle\hat{N}\rangle\sum_{i=1}^{N}\left(\sum_{k\in\mathcal{I};k\notin\mathcal{I}}\langle A_k^{(i)}\rangle^2 - \alpha^2\langle\hat{N}^{(i)}\rangle^2\right). \tag{3.14}$$

Eq. (3.3) is recovered when we replace A_k by J_k and $\langle\hat{N}\rangle$ by N, in this case $\alpha = j$, $\hat{N}^{(i)} = \mathbb{1}^{(i)}$ and $\sum_{k\in\mathcal{I};k\notin\mathcal{I}}\langle A_k^{(i)}\rangle^2 = j^2$, therefore $\delta' = 0$. The set of inequalities given by Eq. (3.13) are valid for any product state. The goal now is to generalize them for any separable state which can be written as a convex sum of product states.

3.4.2 Generalization to all separable states

The generalization of inequalities given by Eq. (3.13) to all separable states is not straightforward. To work around this difficulty, we look for an upper bound δ to δ', such that when δ' is replaced by δ in Eq. (3.13), the resulting inequalities are easily generalized to all separable states by convexity arguments.

In fact, the last term inside the brackets in Eq. (3.14) is negative, therefore $\delta' \leq \alpha^2\langle\hat{N}\rangle - \sum_{i=1}^{N}\sum_{k\in\mathcal{I};k\notin\mathcal{I}}\langle A_k^{(i)}\rangle^2$. In addition, from the definition of the modified variance $\tilde{\Delta}^2$ given by Eq. (3.2), it is not difficult to show that for product states we have

$$\tilde{\Delta}^2 A_k = -\sum_{i=1}^{N}\langle A_k^{(i)}\rangle^2_{\rho_{\text{prod}}}, \quad \rho_{\text{prod}} \text{ a product state.} \tag{3.15}$$

We thus obtain the following upper bound for δ':

$$\delta' \leq \alpha^2\langle\hat{N}\rangle + \tilde{\Delta}^2 A_1 + \tilde{\Delta}^2 A_2 + \tilde{\Delta}^2 A_3 = \delta, \tag{3.16}$$

which is the expression for δ, we have given previously in Eq. (3.7).

Finally, with this new upper bound, Eq. (3.13) becomes:

$$(\langle\hat{N}\rangle - 1)\sum_{k\notin\mathcal{I}}\tilde{\Delta}^2 A_k - \sum_{k\in\mathcal{I}}\langle\tilde{A}_k^2\rangle \geq -\langle\hat{N}\rangle(\langle\hat{N}\rangle - 1)\alpha^2 - \delta. \tag{3.17}$$

It turns out, that these inequalities which are valid for all product states can be generalized to all separable states by convexity (see appendix A).

3.5 Coordinate system independent form for dichotomic observables

Due to Heisenberg uncertainty principle, spin squeezing can not be achieved in all directions. The coordinate independent form of the spin squeezing inequalities [201, 228] allows to detect entanglement without knowing a-priori the direction where the squeezing is maximal.

To illustrate this point, let us recall the squeezing Hamiltonian [195] (see Ref. [216] for a review and the references therein):

$$H = \chi J_x^2 = \chi \sum_{i,j=1}^{N} J_x^{(i)} J_x^{(j)} \tag{3.18}$$

with χ being some coupling constant. The above squeezing Hamiltonian has been very well studied, both theoretically and experimentally, for a system of N spins 1/2 and is called one axis twisting Hamiltonian [195, 216]. The state $|\psi(t)\rangle = e^{-iJ_x^2\theta/2} \bigotimes_{i=1}^{N} |\frac{1}{2}^{(i)}\rangle$, where $\theta = 2\chi t$, is optimally squeezed along the direction lying in the x-y plane making an angle $\phi \approx \frac{1}{2}\arctan\left(N^{-1/3}\right)$ with the x-axis, for large N [195, 216]. This would suggest that in order to better detect the squeezing in such a state, using spin squeezing inequalities (3.1a-3.1d)[228], one needs to measure first and second moments of the rotated spin components $J_{x'} = \cos(\phi)J_x + \sin(\phi)J_y$, $J_{y'} = \cos(\phi)J_y - \sin(\phi)J_x$, and $J_{z'} = J_z$. The purpose of coordinate independent spin squeezing inequalities is to precisely optimally detect squeezing without knowing a-priori the optimal direction ϕ.

3.5.1 Coordinate system independent form of the spin-squeezing inequalities

In this section we recall the coordinate independent form of spin squeezing inequalities (3.1a-3.1d) [228] introduced in [201] for spin-1/2 and in [228] for general spin j. First,

one needs to define the following matrices [228]:

$$C_{ij} = \frac{1}{2}\langle J_i J_j + J_j J_i\rangle \tag{3.19}$$

$$\gamma_{ij} = C_{ij} - \langle J_i\rangle\langle J_j\rangle \tag{3.20}$$

$$Q_{ij} = \frac{1}{N}\sum_{k=1}^{N}\left(\frac{\left\langle J_i^{(k)}J_j^{(k)} + J_j^{(k)}J_i^{(k)}\right\rangle}{2} - \frac{j(j+1)\delta_{ij}}{3}\right) \tag{3.21}$$

$$\mathfrak{X} = (N-1)\gamma + C - N^2 Q \tag{3.22}$$

with δ_{ij} being the Kronecker delta function. Then, the inequalities inequalities (3.1a-3.1d) can be written in the following form [228]:

$$\mathrm{Tr}\left[\gamma\right] - Nj \geq 0 \tag{3.23a}$$

$$(N-1)\mathrm{Tr}\left[\gamma\right] - N(N-1)j + N^2\frac{j(j+1)}{3} - \lambda_{\max}\left(\mathfrak{X}\right) \geq 0 \tag{3.23b}$$

$$-\mathrm{Tr}\left[C\right] + Nj(Nj+1) - N^2\frac{j(j+1)}{3} + \lambda_{\min}\left(\mathfrak{X}\right) \geq 0 \tag{3.23c}$$

$$-\mathrm{Tr}\left[C\right] + Nj(Nj+1) \geq 0 \tag{3.23d}$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximum and minimum eigenvalues of A respectively. The key idea for the above inequalities is that $\mathfrak{X}$ is diagonalized via an orthogonal matrix $\mathfrak{O} \in O(3)$, i.e., $\mathfrak{X} = \mathfrak{O}\Lambda\mathfrak{O}^T$ with Λ a diagonal matrix. Hence, diagonalizing $\mathfrak{X}$ is equivalent to applying the following transformation: $\vec{J}' = \mathfrak{O}^T\mathfrak{J}$, $C' = \mathfrak{O}^T C\mathfrak{O}$, $\gamma' = \mathfrak{O}^T\gamma\mathfrak{O}$, and $Q' = \mathfrak{O}^T Q\mathfrak{O}$. Finally, we have $\mathrm{Tr}\left[C'\right] = \mathrm{Tr}\left[C\right]$ and $\mathrm{Tr}\left[\gamma'\right] = \mathrm{Tr}\left[\gamma\right]$ which represents the invariance of the quantities $\langle J_x^2\rangle + \left\langle J_y^2\right\rangle + \langle J_z^2\rangle$ and $(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2$, respectively, under rotations. Comparing the inequalities (3.23) with (3.1), one can show that there exists a direction for which some of the inequalities (3.1) is violated iff the corresponding inequalities (3.23) are violated [201, 228].

A natural question arises whether we could define our generalized inequalities (3.8) in a coordinate independent manner to simplify the task of finding operators A_i such that entanglement is detected. It turns out that we are able to define a coordinate independent version inequalities of (3.8) for a general class of dichotomic operators which we define now.

3.5.2 Dichotomic observables

A very important and popular choice of the operators A_i for the inequalities (3.8) are the dichotomic observables or spin-1/2 like operators. In this case, the Hilbert space of the single particle states is usually restricted to a 2-dimensional subspace of a two levels system. This restriction to a bi-dimensional subspace has been performed in Ref. [251] where a system of N spin-1 is considered as N spin-1/2 particles. The appeal of this choice is due to the fact that most of the entanglement criteria were originally derived for spin 1/2 systems. Notorious examples are CHSH inequalities [72] for the non-locality of a two spins-1/2 state and spin squeezing inequalities for N spin-1/2 [201].

Specifically, for each particle (i), consider only two magnetic levels states $|m_0^{(i)}\rangle$ and $|m_1^{(i)}\rangle$, among all the eigenstates $|m^{(i)}\rangle(\left|m^{(i)}\right| \leq j)$ of $j_z^{(i)}$. In the subspace spanned by these two states, we can define the Pauli operators as:

$$\sigma_x^{(i)} = |m_0^{(i)}\rangle\langle m_1^{(i)}| + |m_1^{(i)}\rangle\langle m_0^{(i)}| \tag{3.24}$$

$$\sigma_y^{(i)} = -i\left(|m_0^{(i)}\rangle\langle m_1^{(i)}| - |m_1^{(i)}\rangle\langle m_0^{(i)}|\right) \tag{3.25}$$

$$\sigma_z^{(i)} = |m_0^{(i)}\rangle\langle m_0^{(i)}| - |m_1^{(i)}\rangle\langle m_1^{(i)}| \tag{3.26}$$

and let us call $N^{(i)}$ the projector into this subspace spanned by $|m_0^{(i)}\rangle, |m_1^{(i)}\rangle$, that is:

$$N^{(i)} = |m_0^{(i)}\rangle\langle m_0^{(i)}| + |m_1^{(i)}\rangle\langle m_1^{(i)}|. \tag{3.27}$$

An elementary calculation shows that for any state $\rho^{(i)}$ acting on the single particle Hilbert space, we have:

$$\left\langle \sigma_x^{(i)} \right\rangle_{\rho^{(i)}}^2 + \left\langle \sigma_y^{(i)} \right\rangle_{\rho^{(i)}}^2 + \left\langle \sigma_z^{(i)} \right\rangle_{\rho^{(i)}}^2 \leq \left\langle N^{(i)} \right\rangle_{\rho^{(i)}}^2. \tag{3.28}$$

Since $N^{(i)}$ is a projector, it is positive and has two eigenvalues 0 and 1. Hence, from Eq. (3.28), if we choose the state $\rho^{(i)}$ to be a pure state in the subspace $|m_0^{(i)}\rangle, |m_1^{(i)}\rangle$, we find that

$$\alpha^2 = \sup_{\rho^{(i)}} \left[\sum_{k=1}^{3} \left\langle \sigma_k^{(i)} \right\rangle_{\rho^{(i)}}^2 \right] = 1. \tag{3.29}$$

We now define the collective operators A_i to be:

$$A_i = \frac{1}{2} \sum_{k=1}^{N} \sigma_i^{(k)}. \tag{3.30}$$

These three collectives observables A_i fulfill all the requirements to write the generalized spin squeezing inequalities (3.8).

The class of dichtotomic observables (3.30) can be extended in slightly more general manner. Instead of the two states $|m_0^{(i)}\rangle, |m_1^{(i)}\rangle$, we can consider two orthogonal projectors $P_0^{(i)}$ and $P_1^{(i)}$ such that

$$\text{rank}\left(P_0^{(i)}\right) = \text{rank}\left(P_1^{(i)}\right) = r. \tag{3.31}$$

Let us define $\mathcal{H}_0^{(i)}$, $\mathcal{H}_1^{(i)}$ to be the range of $P_0^{(i)}$ and $P_1^{(i)}$ respectively. Let $S^{(i)}$ be a linear map from $\mathcal{H}_1^{(i)}$ to $\mathcal{H}_0^{(i)}$ with singular values equal to 1, i.e. it can be written as

$$S^{(i)} = U^{(i)\dagger} \mathbb{1}^{(r)} V^{(i)}, \tag{3.32}$$

with $U^{(i)\dagger} U^{(i)} = \mathbb{1}^{(r)}$ and $V^{(i)\dagger} V^{(i)} = \mathbb{1}^{(r)}$.

Finally, let us define

$$S_-^{(i)} = P_0^{(i)} S^{(i)} P_1^{(i)}\,,\; S_+^{(i)} = S_-^{(i)\dagger}. \tag{3.33}$$

Now, we can generalize the operators A_i defined in Eq.(3.30) by defining the Pauli-like operators for each particle as:

$$\sigma_z^{(i)} = P_0^{(i)} - P_1^{(i)} \tag{3.34}$$

$$\sigma_x^{(i)} = S_-^{(i)} + S_+^{(i)} \tag{3.35}$$

$$\sigma_y^{(i)} = -i\left(S_-^{(i)} - S_+^{(i)}\right) \tag{3.36}$$

$$N^{(i)} = P_0^{(i)} + P_1^{(i)}. \tag{3.37}$$

With the above definitions, Eqs. (3.28, 3.29) are valid. Moreover, the commutation relations $\left[\sigma_i^{(l)}, \sigma_j^{()}\right] = 2i\epsilon_{ijk}\sigma_k^{(l)}$ still hold. Consequently, the operators defined in Eqs.(3.34) are the generators of SU(2) in the subspace $\mathcal{H}_0^{(i)} \oplus \mathcal{H}_1^{(i)}$, i.e. any rotation can be applied via the unitary $e^{i\vec{\hat{A}}.\vec{n}\theta}$, for some normalized vector $\vec{n}$ and real θ. In addition, since $\left[\hat{N}, A_i\right] = 0$, $\left\langle \hat{N} \right\rangle$ is invariant under such rotations and behaves simply as a scalar as in the usual spin squeezing inequalities. Hence, the additional quantity δ in our inequalities (3.8), that takes into account the fluctuations in particle number in the subspace of interest, is also invariant under $SU(2)$ transformations. Which allows us to follow the same steps as

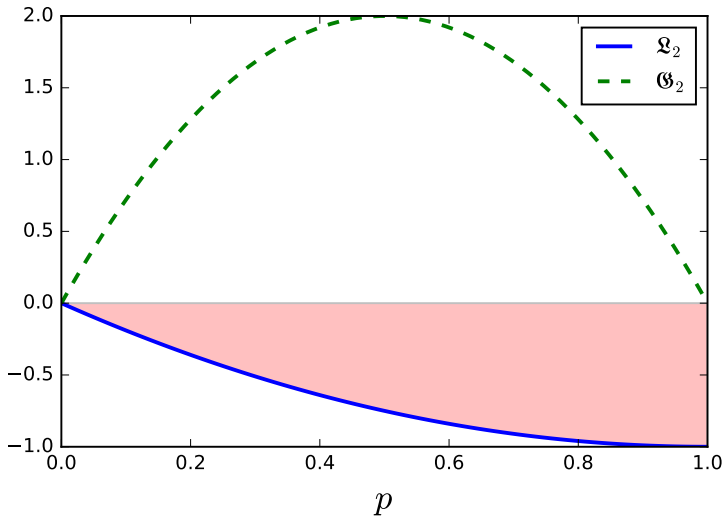


Figure 3.1: $\mathfrak{L}$ (3.46), in solid blue line, and $\mathfrak{G}$ (3.47), in dashed green line, calculated in the state $\rho(p)$ (3.44) as a function of p. The highlighted area represents the instances of p for which the inequality $\mathfrak{L}(p) \geq 0$ is violated.

in [201, 228] to define coordinate system independent inequalities:

$$\delta = \mathrm{Tr}\left[\gamma\right] - \left\langle \hat{N} \right\rangle /2 \geq 0 \tag{3.38a}$$

$$\delta + \left(\left\langle \hat{N} \right\rangle - 1\right) \mathrm{Tr}\left[\gamma\right] - \left\langle \hat{N} \right\rangle \left(\left\langle \hat{N} \right\rangle - 2\right)/2 - \lambda_{\max}\left(\mathfrak{X}\right) \geq 0 \tag{3.38b}$$

$$\delta - \mathrm{Tr}\left[C\right] - \left\langle \hat{N} \right\rangle /2 + \lambda_{\min}\left(\mathfrak{X}\right) \geq 0 \tag{3.38c}$$

$$\delta - \mathrm{Tr}\left[C\right] + \left\langle \hat{N} \right\rangle \left(\left\langle \hat{N} \right\rangle + 2\right)/4 \geq 0, \tag{3.38d}$$

where we have defined:

$$\begin{aligned} C_{ij} &= \frac{1}{2}\left\langle A_i A_j + A_j A_i \right\rangle, \\ \gamma_{ij} &= C_{ij} - \left\langle A_i \right\rangle \left\langle A_j \right\rangle \\ \mathfrak{X} &= (N-1)\gamma + C. \end{aligned} \tag{3.39}$$

As expected, comparing our inequalities (3.38) with the coordinate independent spin squeezing inequalities for $j = 1/2$ in Refs. [201, 228], they are quite similar except for replacing N with $\left\langle \hat{N} \right\rangle$ and the additional term δ. Simply replacing N with $\left\langle \hat{N} \right\rangle$ is not enough to obtain our inequalities as we will show later.

3.6 Generalized Sørensen-Mølmer criterion

As we mentioned in Chapter 1, Sørensen and Mølmer [220] showed that for all separable N j-spin states, the following inequality is verified:

$$(\Delta J_x)^2 \geq NjF_j\left(\frac{\langle J_z\rangle}{Nj}\right) \tag{3.40}$$

where

$$F_j(X) = \frac{1}{j}\min_{\langle J_z\rangle/j=X}(\Delta J_x)^2 \tag{3.41}$$

which is the minimum variance of J_x divided by j for a given value of $\langle J_z\rangle$. The function $F_j(X)$ can be computed numerically quite efficiently even for large spin-j.

The fact that this criterion requires only two operators is quite appealing, and we can generalize it in a similar fashion to what we did above. We can show that, see proof in appendix B.4:

$$\alpha^2\left\langle\hat{N}\right\rangle + \tilde{\Delta}^2 A_1 \geq 2\alpha^2\left\langle\hat{N}\right\rangle F_{\frac{1}{2}}\left(\frac{\langle A_2\rangle}{\alpha\left\langle\hat{N}\right\rangle}\right) = \frac{\langle A_2\rangle^2}{\left\langle\hat{N}\right\rangle} \tag{3.42}$$

where wa have used the fact $F_{\frac{1}{2}}(X) = \dfrac{X^2}{2}$. The reason we use the function $F_{\frac{1}{2}}$ is because we map into a qubit system in order to derive these criteria.

Comparing the above criterion with our main result (3.6), we can see that they are completely equivalent for the case where we choose two operators A_1 and A_2 only. This can be spotted more easily if we rewrite (3.6) in the following form:

$$\langle\hat{N}\rangle\left(\alpha^2\langle\hat{N}\rangle + \sum_{k\notin\mathcal{I}}\tilde{\Delta}^2 A_k\right) - \sum_{k\in\mathcal{I}}\langle A_k\rangle^2 \geq 0. \tag{3.43}$$

It is interesting to note that we have derived the same criteria (3.42) and (3.6) using two different methods. While we used a mapping to a qubit system for both, we did not use our improved bound (3.12) to derive (3.42). The improved bound (3.12) gave rise to the additional term δ in (3.6). This gives us confidence that the term δ is good upper bound of the term δ' defined in (3.14). Nevertheless, numerical investigation is needed to show if this is indeed the case.

3.7 Examples

In this section we compare, for two specific cases, the standard spin squeezing inequalities where N is replaced by its expectation $\langle N\rangle$ with our new inequalities. The first case illustrates the importance of the term δ in our inequalities. Indeed, we exhibit a separable mixed state that violates the original inequality, showing that the simple replacement of N by $\langle N\rangle$ can lead to false positive. In the second example, we study the detection of entanglement generated by the one axis twisting Hamiltonian (3.18) for $N = 5$ spin 1 system. We find that, when restricting to a subspace, our inequalities (3.38) show a clear advantage over spin squeezing inequalities (3.1). The latter show no violation at all, whereas, one of the inequalities (3.38) is violated indicating entanglement almost for all times of the evolution of the $N = 5$ spin 1 system.

3.7.1 Example I

We have shown, that through our special choice of operators A_i given by Eq. (3.30), our inequalities (3.8) and (3.38) can be obtained from spin squeezing inequalities for $j = 1/2$ [201, 228] by replacing N with $\left\langle \hat{N}\right\rangle$ and adding δ. In the following, we give a simple example to highlight the importance of the additional term δ. Let us consider the following separable mixed states for $N = 2$ spin $j = 1$:

$$\rho(p) = p\rho' + (1-p)|0^{(1)}\rangle\langle 0^{(1)}| \otimes |0^{(2)}\rangle\langle 0^{(2)}| \quad 0 \leq p \leq 1 \tag{3.44}$$

where:

$$\begin{aligned}\rho' = \frac{1}{2}\Big(&|-1^{(1)}\rangle\langle -1^{(1)}| \otimes |-1^{(2)}\rangle\langle -1^{(2)}| \\ &+ |1^{(1)}\rangle\langle 1^{(1)}| \otimes |1^{(2)}\rangle\langle 1^{(2)}|\Big)\end{aligned} \tag{3.45}$$

This state is clearly separable for any value of $0 \leq p \leq 1$. Now let us consider the inequality given by Eq. (3.1b) and let us replace N by $\left\langle \hat{N}\right\rangle$ where $\hat{N}$ is defined in Eq. (3.27) for the subspace $|-1^{(i)}\rangle, |1^{(i)}\rangle$. Then, it takes the following form: $\mathfrak{L}(p) \geq 0$, where

$$\begin{aligned}\mathfrak{L}(p) = &\left(\left\langle \hat{N}\right\rangle_{\rho(p)} - 1\right)\left(\tilde{\Delta}^2 A_x + \tilde{\Delta}^2 A_y\right) - \left\langle \tilde{A}_z^2\right\rangle_{\rho(p)} \\ &+ \frac{1}{4}\left\langle \hat{N}\right\rangle_{\rho(p)}\left(\left\langle \hat{N}\right\rangle_{\rho(p)} - 1\right)\end{aligned} \tag{3.46}$$

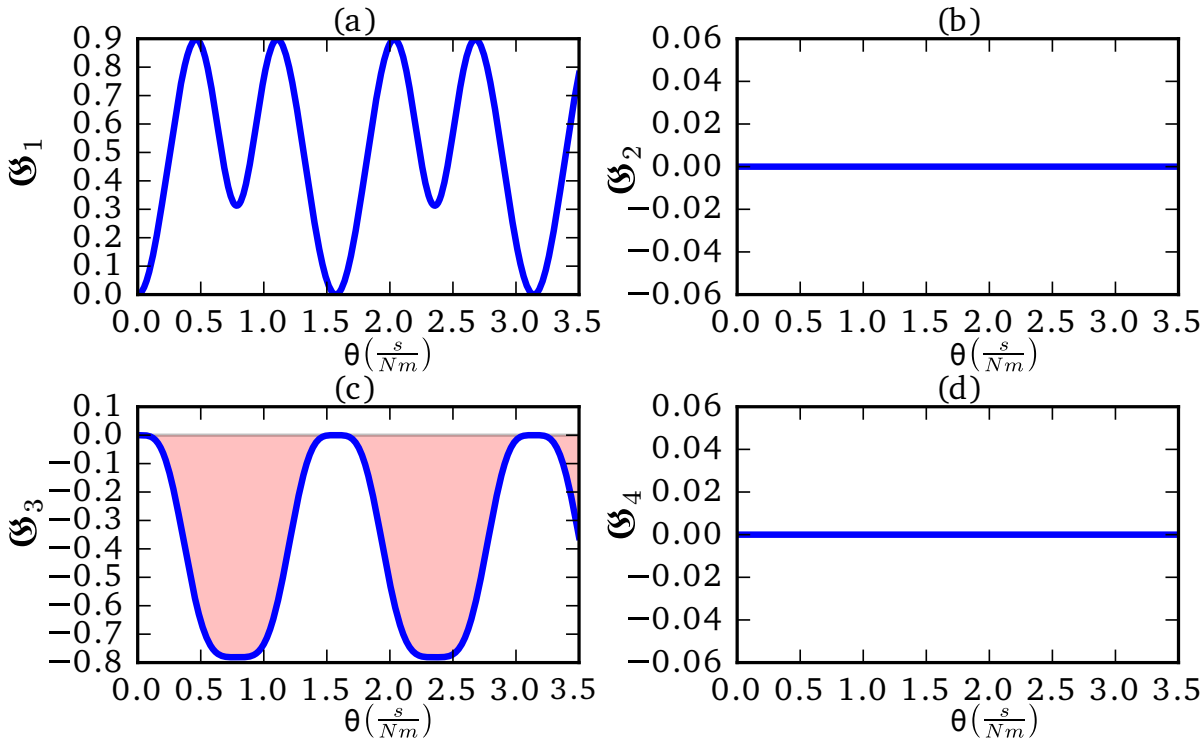


Figure 3.2: (a) to (d): left hand side of inequalities $\mathfrak{G}_1$ (3.38a) to $\mathfrak{G}_4$ (3.38d), respectively, calculated in the state $|\psi(\theta)\rangle$, defined in Eq. (3.48), as a function of θ for $N = 5$ spin-1 particles. Highlighted region shows the instances of θ for which inequality (3.38c) is violated.

Next, let us consider the correct form, i.e. Eq. (3.8b): $\mathfrak{G}(p) \geq 0$, where

$$\mathfrak{G}(p) = \mathfrak{L}(p) + \delta(p) \tag{3.47}$$

In figure Fig. 3.1, we plot both quantities $\mathfrak{G}(p)$ and $\mathfrak{L}(p)$ as a function of p. Inequality $\mathfrak{L}(p) \geq 0$ is violated for all p, but it is completely wrong to infer that the state is entangled. In contrast, our inequality $\mathfrak{G}(p) = \mathfrak{L}(p) + \delta(p) \geq 0$ is not violated, as expected. This example shows clearly the importance of the additional term δ when the number of particles is not constant.

3.7.2 Example II

As an illustrative example, we consider a system of N spin-$j = 1$ initialized in the product state $|\psi_0\rangle = \bigotimes_{i=1}^{N} |0^{(i)}\rangle$. Now let us calculate the left hand side of the inequalities (3.23)

for the state

$$|\psi(\theta)\rangle = e^{-iJ_x^2\theta/2} \bigotimes_{i=1}^{N} |0^{(i)}\rangle \tag{3.48}$$

of N spins $j = 1$. Let us call $\mathfrak{F}_1(\theta)$, $\mathfrak{F}_2(\theta)$, $\mathfrak{F}_3(\theta)$, and $\mathfrak{F}_4(\theta)$ to be the left hand side of inequalities (3.23a-3.23d) respectively. Numerical calculations show that these quantities are constant and positive. More precisely, one can verify that $\mathfrak{F}_1(\theta) = N$ and $\mathfrak{F}_i(\theta) = N(N-1)$ for $i = 2, 3, 4$, thus, spin squeezing inequalities (3.23) fail to detect entanglement in the state $|\psi(\theta)\rangle$. The constancy of the quantities $\mathfrak{F}_l(\theta)$, with $l = 1, \cdots, 4$, is due to the choice of the initial state $|\psi(0)\rangle = \bigotimes_{i=1}^{N} |0^{(i)}\rangle$ and the fact we have chosen $j = 1$. Non trivial evolution of the quantities $\mathfrak{F}_l(\theta)$ will occur for different initial states and different integer spin values $j = 2, 3, \cdots$.

However, if we choose different observables than the collective spin components, our generalized inequalities (3.8) can be violated inferring entanglement of the state $|\psi(\theta)\rangle$ for some θ. In particular, we will define dichotomic observables in the subspace $|-1^{(i)}\rangle, |1^{(i)}\rangle$, by setting $|m_0^{(i)}\rangle = |-1^{(i)}\rangle$ and $|m_1^{(i)}\rangle = |1^{(i)}\rangle$ in Eqs.(3.30, 3.24, 3.27), so that the N spin-1 particles can be seen as $\left\langle \hat{N} \right\rangle$ spin-1/2 particles.

Now, let us call $\mathfrak{G}_i(\theta) : i = 1, 2, 3, 4$ to be the left hand side of inequalities (3.38a-3.38d), respectively, calculated for the state $|\psi(\theta)\rangle$ (3.48). In Fig. 3.2, we plot $\mathfrak{G}_i(\theta)$ for $N = 5$, and we can see that $\mathfrak{G}_3(\theta)$ violates the inequality (3.38c), Fig. 3.2(c). Consequently, we show that the state $|\psi(\theta)\rangle$ is entangled, at least, when inequality (3.38c) is violated.

3.8 Entanglement detection in cold atom gaz

This work is the result of an ongoing collaboration with an experimental group "Quantum Dipolar Gazes" at "LPL" lab in Villetaneuse, France. The aim of this work is to find an experimental witness of entanglement which can be implemented in one of their systems. I will start by briefly describing their experimental setup and the different constraints. next, I will describe the model used for the numerical simulations and the different criteria applied using the simulation to certify presence of entanglement or to test the detection efficiency if it were to be tested experimentally.

3.8.1 Experimental setup

The experiment is described in [263], a chromium ^{52}Cr Bose Einstein condensate (BEC), of about 10^4 atoms, is created in a crossed-beam optical dipole trap as described in [264].

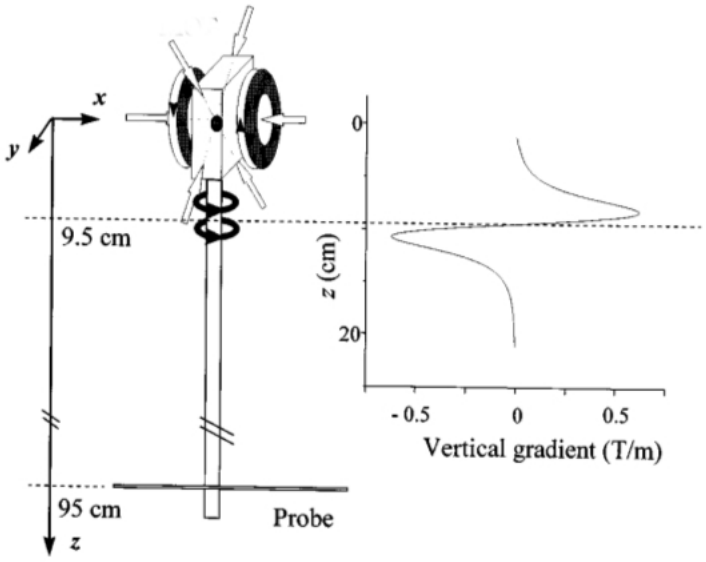


Figure 3.3: Schematic diagram of the experimental setup for Stern-Gerlach analysis, taken from Ref. [266].

Each atom possesses a spin $s = 3$ and is in the ground Zeeman state $m_s = -3$ when created in the BEC. The BEC is then loaded adiabatically into a 3D optical lattice with periodicity $\frac{\lambda}{2} \times (1, 1/\sin(\pi/8), 1/\cos(\pi/8))$ along x, y, z directions respectively, where $\lambda = 532$nm is the wavelength of the single mode laser used to generate the lattice [265]. The lattice energy band gaps are much larger than any other energy scales in the system (interactions, temperature, Zeeman shifts), and hence the atoms remain confined in the lowest energy band of the lattice. To achieve a regime of single atom per site, atoms sharing the same site are removed by dipolar relaxation at high magnetic field in the following way: atoms are first transferred to the highest energy Zeeman state $m_s = 3$, which induces losses selectively for doubly occupied sites as the released energy is larger than the lattice depth [263]. At the end, the magnetic field is lowered and the remaining atoms are transferred back to the lowest energy Zeeman level $m_s = -3$. After this preparation the system consists of a shell of about 4000 singly occupied sites close to unit filling. This allows to study dynamics related to Dipole Dipole Interactions DDI only and to ignore possible on site interactions. Moreover, the atoms have spin $s = 3$ which leads to rich physics expected for high spin lattice.

In order to initiate the dynamics due to DDI, the atoms in the $m_s = -3$ ground state are transferred to the excited Zeeman state $m_s = -2$ through an adiabatic Raman transfer. This transfer is performed by ramping up a quadratic light shift which leads to a level crossing between states $m_s = -3$ and $m_s = -2$, above which $m_s = -2$ becomes the Zeeman ground state. A weak two photon coupling between $m_s = -3$ and $m_s = -2$ ensures the transfer with efficiency of 80%.

Once the atoms are prepared in the $m_s = -2$ state, they will begin to interact through inter-site DDI. To follow the dynamics induced by this interaction, a snap shot of the different Zeeman $m_s = -3, \cdots, 3$ populations is obtained after a given time via a Stern-Gerlach measurement [266]. To perform this populations measurement, the atoms are released from the lattice and fall freely under gravity. During their free fall, they will pass through a region where a vertical magnetic gradient is applied and where atoms in different Zeeman state will experience different forces. Hence, the atoms will separate vertically according to their Zeeman state into distinct clouds. Finally, an horizontal laser beam measures the Zeeman levels populations via absorption imaging (see Fig. 3.3).

Hence, population measurements at different times of the evolution can be measured but no individual addressing of a single atom is possible for this experiment. Population analysis are the only collective measurements that we should rely on to calculate mean values and second moments of different collective observables.

3.8.2 Hamiltonian and model for numerical simulation

The dynamics we are interested in is due to Dipole Dipole interaction. DDI between two spins $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ separated by distance $\vec{r}$ is given by (see Fig 3.4)

$$\vec{V}_{dd}(\vec{r}) = d^2 \frac{\hat{J}^{(1)}.\hat{J}^{(2)} - 3\left(\hat{J}^{(1)}.\vec{u}_r\right)\left(\hat{J}^{(2)}.\vec{u}_r\right)}{r^3} \tag{3.49}$$

where $d^2 = \dfrac{\mu_0 (g j \mu_B)^2}{4\pi}$, $\vec{u}_r = \dfrac{\vec{r}}{r}$ the unit vector along the inter-spin axis, μ_0 is the magnetic permeability of the vacuum, μ_B the Bohr magneton and g the Lande factor. The above equation contain terms that conserve the total magnetization along the direction of the magnetic field, namely the terms $J_z^{(1)} J_z^{(2)}$, $J_+^{(1)} J_-^{(2)}$ and $J_-^{(1)} J_+^{(2)}$. Whereas the remaining terms do not and it was shown in Ref. [265] that these terms have a resonant character in a 3D lattice, and therefore are strongly suppressed at a low enough magnetic field $\vec{B}$ considered in the experiment of interest [263]. Hence, we can ignore the

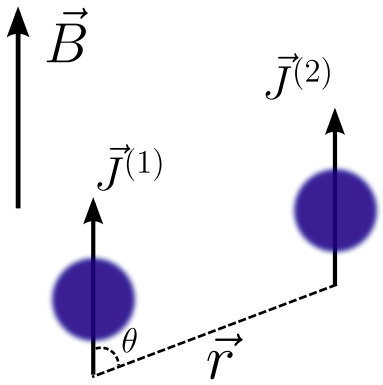


Figure 3.4: Two spins $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ interact due to DDI.

terms that do not conserve the total spin projection along the direction of the magnetic field. So the Hamiltonian is reduced to:

$$\vec{V}_{dd}^{\text{eff}}(\vec{r}) = \frac{d^2}{r^3}\left(1 - 3\frac{z^2}{r^2}\right)\left(\hat{J}_z^{(1)}\hat{J}_z^{(2)} - \frac{1}{4}\left(\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \hat{J}_-^{(1)}\hat{J}_+^{(2)}\right)\right) \tag{3.50}$$

which is a Heisenberg like Hamiltonian with z being the direction of the magnetic field.

Since the dependence of the DDI is $\frac{1}{r^3}$, and due to anisotropy of the lattice where the spacing between sites along y direction is almost 3 times that of the x and z direction, one could simplify the study of the above Hamiltonian of a 3D lattice into the study of 2D lattice.

So we finally write the desired Hamiltonian of 2D spin lattice to simulate as:

$$H = \sum_{i\neq j=1}^{N} V_{ij}\left(J_z^{(i)}J_z^{(j)} - \frac{1}{4}J_+^{(i)}J_-^{(j)} - \frac{1}{4}J_-^{(i)}J_+^{(j)}\right) \tag{3.51}$$

where N is the total number of spins and V_{ij} represent the interaction between spins on sites i and j and has a dependence on $\frac{1}{r^3}$ for the DDI. For our simulation we choose a uniform lattice with inter-site distance $\lambda/2$, where $\lambda = 532$nm is the wavelength of the single mode laser used to generate the lattice in [265], and we take V_{ij} to be

$$V_{ij} = \frac{d^2}{r_{ij}^3},$$

where d is defined in (3.49) and r_{ij} is the distance between the sites i and j of the lattice.

The form of the Hamiltonian (3.51) is not restrictive to a 2D lattice. However, due to the complexity and the exponential growth of the Hilbert space when increasing the number of spins N, we restrict the simulation to a 2D 3×3 lattice, where the state of the lattice is initialized in:

$$|\psi(0)\rangle = \bigotimes_{i=1}^{N} |-2\rangle$$

and then the exact dynamic is calculated, *i.e.*

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle \tag{3.52}$$

A direct calculation shows that $[H, Jz] = 0$ and that indeed the above Hamiltonian conserves the total magnetization along the z direction, where $J_z = \sum_i^N J_z^{(i)}$ is the total spin along the z direction.

Although 3×3 might seem insufficient, we see a general trend for the different quantities plotted as a function of time, as we will see later. when increasing the number from 2×2 to 2×3, fluctuations around a mean curve become smoother and almost disappear. When increasing the number from 2×3 to 3×3, fluctuations become even smaller. Which justifies that a 3×3 simulation captures the interesting features of that of the experiment.

The simulations are performed with a Python code I have written inspired by a code written in Mathematica by Paolo PEDRI a theoretician at "LPL" lab. The code is written with the help of the QuTiP (Quantum Toolbox in Python) library. I have optimized the generation of the basis states which is one of the more numerical demanding step.

3.8.3 Evidence for entanglement, Entropy of entanglement

Since the sate we are considering is a pure state, many entanglement criteria are easy to calculate numerically like entropy of entanglement. To apply this criterion, we need to partition the lattice into two parties A and B. For a separable state ρ_{AB}, one should have:

$$S_v\left(\rho_{AB}\right) \geq S_v\left(\rho_A\right) \tag{3.53}$$

where $S_v(\rho) = -\mathrm{Tr}\left[\rho \ln\left(\rho\right)\right]$ is the Von-Neumann entropy and $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$ is the state of the subsystem A given as the partial trace of ρ_{AB} over B.
For a pure state ρ_{AB}, the above separability criterion simplifies to :

$$S_v\left(\rho_A\right) = 0 \tag{3.54}$$

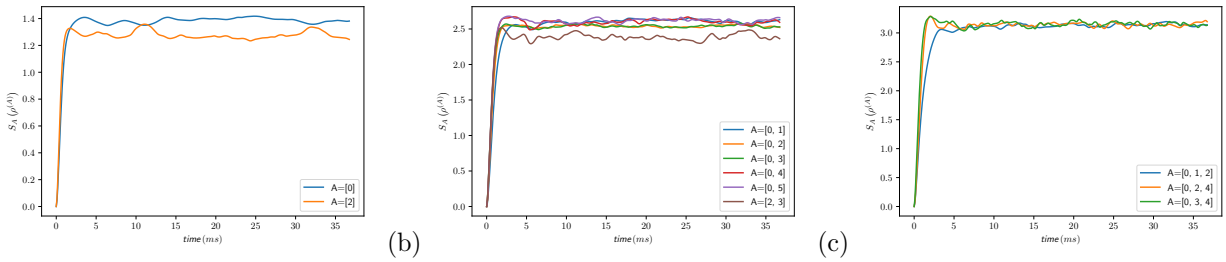


Figure 3.5: Entropy of subsystem A versus time for a 2×3 lattice where the subsystem A represents different lattice where the subsystem A represents different bipartitions. Case (a) Entropy for one site. (b) Entropy for sites two sites of the lattice, and (c) Entropy for 3 sites.

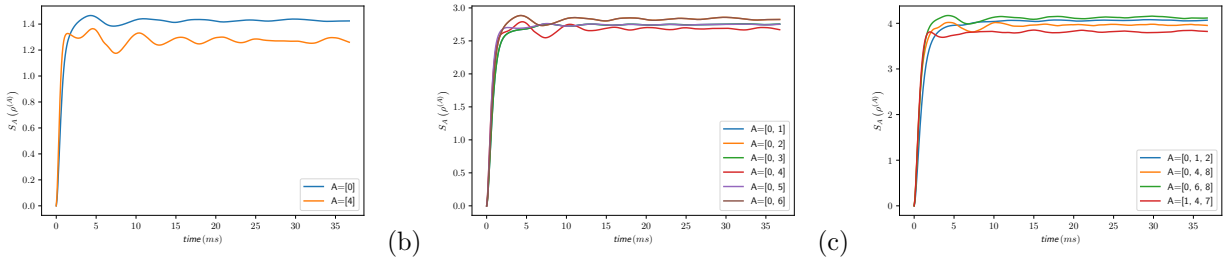


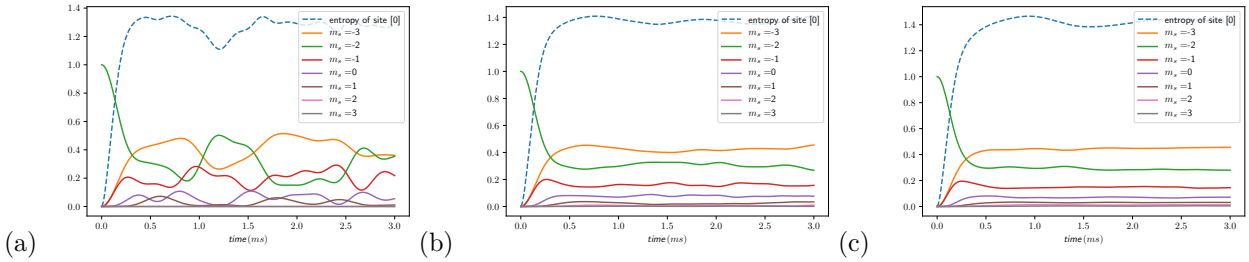
Figure 3.6: Entropy of subsystem A versus time for a 3×3 lattice where the subsystem A represents different bipartitions. Case (a) Entropy for one site. (b) Entropy for sites two sites of the lattice, and (c) Entropy for 3 sites.

which implies entanglement whenever $S_v(\rho_A) > 0$.

In figure 3.5 and figure 3.6, the entropy of subsystem A is plotted as a function of time for a 2×3 and 3×3 respectively. We see that $S_v(\rho_A) > 0$ regardless of the size of the bipartition and hence the state $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$ is entangled.

We also can see the similarities between the two cases 2×3 and 3×3, where entropy $S_v(\rho_A)$ approaches the same values depending on the size of the partition A. In both cases the entropy seems to increase until it reaches some kind of thermalization even for a small system. A more thorough analysis that takes into account the constancy of the total magnetization is lacking for the moment however the trend is pretty clear.

Finally, In figure 3.7, I plot entropy of site 1 and the evolution of the population of each zeeman level $m_s = -3, -2, \cdots, 3$ to for different lattice sizes 2×2, 2×3 and 3×3. As we have outlined earlier, the entropy of entanglement shows that the state is entangled for any time $t > 0$. We also notice the similarity for different lattice sizes



(a) (b) (c)

Figure 3.7: Entropy of site 1 and population of the different Zeeman levels versus time for a lattice of size:(a) 2×2 (b) 2×3 (c) 3×3.

where the fluctuations become smoother when increasing the lattice size.

These numerical shows that indeed, the state of the atome is entangled. However we recall that our goal is to bring a test that can be experimentally realized. Measuring the entropy of the a subsystem is challenging especially for the case of the present experiment [263]. That is why we study spin squeezing inequalities as a simple accessible alternative for entanglement detection.

3.8.4 Testing spin squeezing inequalities

Testing spin squeezing inequalities (3.1) for a spin-3 calculated for the entangled state $|\psi(t)\rangle$ showed no violation. Hence, these inequalities are blind to this entangled state. A similar result is obtained with Coordinate independent spin squeezing inequalities (3.23) which excludes the possibility to detect squeezing for different directions.

One might attempt to use a simpler criteria that does not involve all the first and second moments of total spin operators at once. for example, one can easily derive the following criterion (see appendix B.3):

$$\tilde{\Delta}^2 J_x + 4j^2 \left(\Delta J_z\right)^2 \geq 0 \tag{3.55}$$

which only involves two modified moments. The advantage of the above inequality is that it takes into account the fact that $|\psi(t)\rangle$ is an eigenstate of J_z; i.e $\left(\Delta J_z\right)^2 = 0$. If we plot the left hand side of the above inequality as a function of time Fig. 3.8, we clearly see a violation of the above inequality for different lattice sizes and it scales roughly like $0.2N$. However, it seems to be decreasing with increased lattice size.
However, this might seem contradictory that we are observing violation for our criterion,

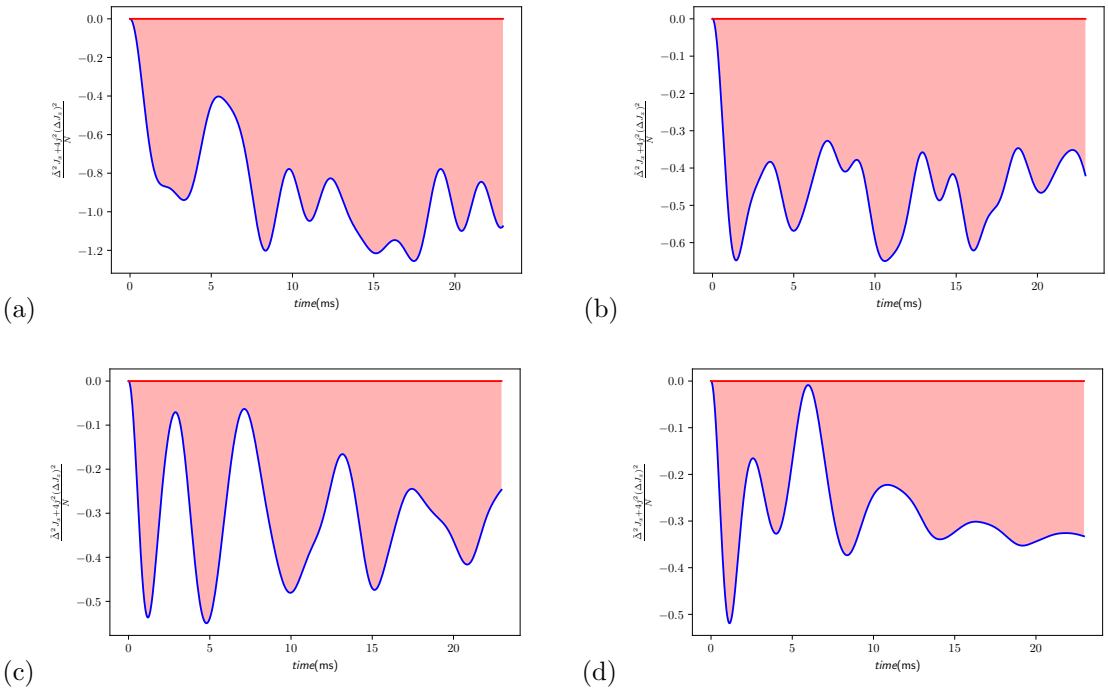


Figure 3.8: The blue curve shows the evolution of the quantity $\frac{(\tilde{\Delta}J_x)^2 + 4j^2(\Delta J_z)^2}{N}$ as a function of time, while the red area shows where the inequality $\frac{(\tilde{\Delta}J_x)^2 + 4j^2(\Delta J_z)^2}{N} \geq 0$ is violated starting with an initial state $|-2, -2, \cdots, -2\rangle$ for a lattice of size :(a) 2×2 (b) 2×3 (c) 3×3 and (d) 3×4.

which uses first and second moments, while spin squeezing inequlaities are satisfied 3.1. Although these inequalities do not guarantee optimality for finite number of spins N, they are optimal in the limit $N \to \infty$ [201, 228], in the sense that they cannot be outperformed by criteria based on first and second moments of spin operators. Thus, since no violation of inequalities (3.23) is observed, and we expect that to hold when N becomes larger than 3×4, we should expect that the above criterion, which detects entanglement for small N, will not give significant violation for very large N. Besides, the above simulation does not take into account experimental errors because of which $(\Delta J_z)^2$ will not be strictly zero but rather would scale like aN for some small a. In that case the above criterion would only detect entanglement for $a < \frac{0.5}{4j^2} \approx 0.0015$.

3.8.5 Testing with generalized spin squeezing inequalities

The general form of the new spin squeezing inequalities, gives it versatility. However, one is faced with a huge choice and we do not have, for the moment, a procedure to hone in on the perfect choice. Here, I provide one choice, inspired by the Hamiltonian and the initial state, that succeeds to detect entanglement for short times only.
We use dichotomic operators defined in (3.24), where we choose

$$|m_0^{(i)}\rangle = |-2^{(i)}\rangle, \quad |m_1^{(i)}\rangle = \frac{|-3^{(i)}\rangle + |-1^{(i)}\rangle}{\sqrt{2}}. \tag{3.56}$$

We can see why this is a relevant choice for short times if we look at the state

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle \approx (1 - itH)\,|\psi(0)\rangle.$$

We can see that applying H on the separable state $|\psi(0)\rangle = |-2\rangle^{\otimes N}$ will give us an entangled state of the form

$$\alpha|-2\rangle^{\otimes N} + \sum_{i\neq j} \beta_{i,j} \bigotimes_{k\neq i,j} |-2^{(k)}\rangle \otimes |-3^{(i)}\rangle \otimes |-1^{(j)}\rangle \tag{3.57}$$

such that the total magnetization is constant $-2N$.

In figure (3.9), we can see that indeed our generalized inequality (3.38c) is able to detect entanglement for short time in a window between 0 and 0.7 ms. Outside this window no violation is observed using this dichotomic operators we defined (3.56). To get a quantitative idea about the violation of inequality (3.38c), we rewrite in the equivalent form:

$$\frac{\delta - \left\langle \hat{N} \right\rangle/2 + \lambda_{\min}\left(\mathfrak{X}\right)}{\mathrm{Tr}\left[C\right]} \geq 1 \tag{3.58}$$

and then we plot for different lattice sizes. A very important observation can be seen in figure 3.9, is that the maximum violation is increasing when lattice size is increased, while the window of time, during which a violation can be observed, decreases.
Hence, we have a very good candidate for entanglement detection for short times. A better understanding of the state evolution over time might prove crucial for defining other entanglement witnesses for later times. Also, a method for measuring the involved quantities based on what can be done in the experiment is important and will be discussed in the next chapter.

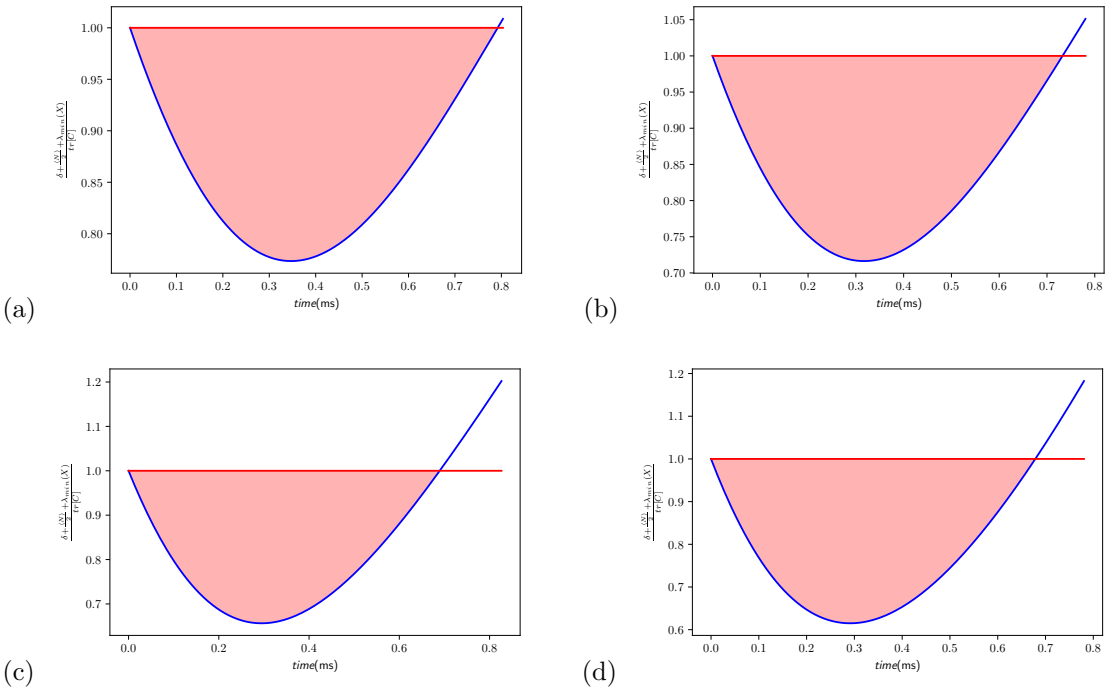


Figure 3.9: The blue curve shows the evolution of the quantity $\dfrac{\delta - \left\langle \hat{N} \right\rangle/2 + \lambda_{\min}(\mathfrak{X})}{\mathrm{Tr}\,[C]}$ as a function of time, while the red area shows where the inequality $\dfrac{\delta - \left\langle \hat{N} \right\rangle/2 + \lambda_{\min}(\mathfrak{X})}{\mathrm{Tr}\,[C]} \geq 1$ is violated starting with an initial state $|-2,-2,\cdots,-2\rangle$ for a lattice of size :(a) 2×2 (b) 2×3 (c) 3×3 and (d) 3×4.

3.9 Conclusion

We have generalized the spin squeezing inequalities in order to consider quantum fluctuations of the number of particles N. Our generalized inequalities can be obtained from the original ones by replacing N with its expectation value $\langle N \rangle$ and by adding a new term δ which is not more difficult to measure than the other terms forming the original inequalities. In the case where the measured observables are dichotomic, we have shown that we can define coordinates independent spin squeezing inequalities in the same way it had been defined previously for the original inequalities. The non conservation of the number of particles allows more flexibility in the set of observables to be used to test the inequalities. We have presented an example where such flexibility allows for the detection of an entangled state which was not detected by the original inequalities. We also warn that using the original inequalities, in a context where the number of particles

N fluctuates, by replacing N by its expectation value $\langle N \rangle$ can result in a violation for separable states, hence giving false positive.

We have also studied an experimental case where entanglement is present but undetected with the usual spin squeezing inequalities. The generalized spin squeezing inequalities detect entanglement for the short time scale. However, the thermalization of the state present a challenge which is to choose the proper operators for which we have no systematic way to choose. Another question one should keep in mind, is the ability to measure the chosen collective operators and their second moments necessary for the generalized spin squeezing inequalities. We will tackle this question in the next chapter where we will show that from population measurements along several directions one is able to calculate the mean value and second moments of any collective observable. Finally, despite the versatility the new inequalities present, choosing the best collective observables to detect entanglement compatible with experimental requirements is still a goal to aim for.

4

Perspectives and conclusion

In this chapter I will discuss some of the open questions and perspectives of the work presented in chapter 2 regarding the formalism introduced to map any system into a qubit and in chapter 3 regarding multipartite-entanglement detection and spin squeezing inequalities. The study presented in chapter 2 for mapping any system into a qubit and using this technique for entanglement detection is far from complete and still lacking in both theory and applications. For the bipartite mapping, we only studied the simplest case where the the mapping is implemented via LOCC with no classical communication. Despite its usefulness, one should explore all possible implementations with LOCC to derive entanglement criteria to succefully achieve all the goals of chapter 2. Whereas in chapter 3, with the generalization we introduced came a huge choice of operators. Such liberty is not useful if it does not give rise to observables that can be measured with relative ease. It is also interesting to explore other multipartite criteria in the context of particle number fluctuation. Here, we dig a little bit into these questions and possible directions.

4.1 Going beyond local operations with no communications

An immediate follow up to the mapping scheme introduced in chapter 2, is to try and study the structure of the mapping when rounds of classical communications can be performed. LOCC operations are difficult to describe mathematically in general, but one may expect to have a simplified description for LOCC operations in our specific case where the output is a 2-qubit state.

LOCC operations with no classical communications can be characterized using Kraus operators as follows

$$\mathcal{M}(\rho) = \sum_{i,j} K_i \otimes L_j \rho K_i^\dagger \otimes L_j^\dagger \tag{4.1}$$

with $\sum_i K_i^\dagger K_i = \mathbb{1}^{(d_1)}$ and $\sum_i L_i^\dagger L_i = \mathbb{1}^{(d_2)}$. we have shown in chapter 2 Eq.2.56 that it simplifies, when mapping to 2-qubits, to the following form:

$$\mathcal{M}(\rho) = \frac{1}{4} \sum_{i,j=0}^{3} \left\langle B_i^{(d_1)} \otimes B_j^{(d_2)} \right\rangle_\rho \sigma_i^{(1)} \otimes \sigma_j^{(2)} \tag{4.2}$$

where the observables $B_i^{(d_k)}$, $i = 1, 2, 3$ and $k = 1, 2$, are properly normalized to ensure complete positivity as mentioned in chapter 2 Eq. 2.57. Moreover, we have also shown Kraus operators in this case can be chosen of the following form (2.27), (A.12):

$$K_i = |0\rangle\langle e_i|\mathcal{K}_0 + |1\rangle\langle e_i|\mathcal{K}_1 \quad \text{for } 0 \leq i \leq 2d_1 - 1 \tag{4.3}$$

where $\{|e_i\rangle : 0 \leq i \leq 2d_1 - 1\}$ is an orthonormal basis of $\mathcal{H}^{(d_1)} \oplus \mathcal{H}^{(d_1)}$ the codomain of $\mathcal{K}_0$ and $\mathcal{K}_\infty$,

$$\mathcal{K}_0 = \begin{pmatrix} \sqrt{A} \\ 0 \end{pmatrix} \quad , \quad \mathcal{K}_1 = \begin{pmatrix} \sqrt{A}^{-1} B \\ \sqrt{C - B^\dagger A^{-1} B} \end{pmatrix} \tag{4.4}$$

and

$$A = \frac{\mathbb{1}^{(d_1)} + B_3^{(d_1)}}{2}, C = \frac{\mathbb{1}^{(d_1)} - B_3^{(d_1)}}{2}, B = \frac{B_1^{(d_1)} + iB_2^{(d_1)}}{2} \tag{4.5}$$

and the same can be done for the second party with the operators L_j.
Using this simple form, I will try to simplify the LOCC with one way classical communications for the case of mapping into two qubits.

4.1.1 Local operations with one way communication

Any CPTP map implemented with LOCC with one way communication can be written in the following form [40, 52]:

$$\mathcal{M}_{A\to B}(\rho) = \sum_{i,j} K_i' \left[\mathbb{1}^{(A)} \otimes \Lambda^{(i)}(\rho)\right] K_i'^\dagger \tag{4.6}$$

where $\Lambda^i(\cdot) = \sum_j L_j^i \cdot L_j^{i\,\dagger}$ and $K_i(L_i)$ are Kraus operators implemented by Alice(Bob) such that$\sum_i K_i'^\dagger K_i' = \mathbb{1}^{(d_1)}$ and $\sum_j L_j^{i\,\dagger} L_j^i = \mathbb{1}^{(d_2)}$. The subscript $A \to B$ means that the classical communication is made from Alice (the first party) to Bob (the second party).

From the above form, we can see that Alice performs her local operation $K'_i \cdot K'^{\dagger}_i$ first and sends a classical message to Bob to instruct him to apply the channel $\Lambda^i(\cdot) = \sum_j L^i_j \cdot L^{i\,\dagger}_j$ on his share of the state.

The set of operators representing a CPTP map is not unique. However, as Bob must wait for all operations made by Alice, before performing his own map, we have the freedom to choose the Kraus operators representing his map in the form given by Eq. (4.3):

$$L^i_j = |0\rangle\langle f_j|\mathcal{L}^i_0 + |1\rangle\langle f_j|\mathcal{L}^i_1 \quad \text{for } 0 \leq j \leq 2d_2 - 1 \tag{4.7}$$

On the contrary, we have not this freedom for Alice's Kraus operators. Alice needs to stick to her choice of Kraus operators K'_i in order to successfully achieve the overall mapping (4.6). In general, K'_i are not necessarily of the form (4.3), but rather some operations upon which depends the choice for the map implemented by Bob. However, if two sets of Kraus operators $\{K_i\}$ and $\{K'_i\}$ represent the same quantum channel, then there exists a unitary U, or an isometry when the number of Kraus operators of the two sets is not equal, such that [234]

$$K'_i = \sum_j U_{i,j} K_j \tag{4.8}$$

Let us call $D_1 = \max(2d_1, k_1)$ where d_1 is the dimension of the Hilbert space $\mathcal{H}^{(d_1)}$ and k_1 is the number of Kraus operators K'_i. Then, we can find Kraus operators of the form (4.3) that represent the same channel as K'_i:

$$K_i = |0\rangle\langle e_i|\mathcal{K}_0 + |1\rangle\langle e_i|\mathcal{K}_1 \quad \text{for } 0 \leq i \leq D_1 - 1 \tag{4.9}$$

where the operators $\mathcal{K}_0$ and $\mathcal{K}_1$ in (4.3) are extended to $D_1 \times d_1$ operators by padding them with zeros.The mapping (4.6) becomes:

$$\mathcal{M}_{A\to B}(\rho) = \sum_{i,j,k,l} U_{i,k} U^{\dagger}_{l,i} K_k \otimes L^i_j \rho K^{\dagger}_l \otimes L^{i\,\dagger}_j \tag{4.10}$$

with some simple manipulations, we can put the mapping into the following form:

$$\mathcal{M}_{A\to B}(\rho) = \sum_{m,m',n,n'=0,1} \mathcal{A}_{m,m',n,n'}(\rho)\, |m,n\rangle\langle m',n'| \tag{4.11}$$

where

$$\mathcal{A}_{m,m',n,n'}(\rho) = \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i\rangle\langle g_i| \mathcal{K}_m \otimes B_{n,n'}^i \right\rangle_\rho \tag{4.12}$$

$B_{n,n'}^i = {\mathcal{L}_{n'}^i}^{\dagger} \mathcal{L}_n^i$, and $\{|g_i\rangle\}$ is a set of vectors defined as:

$$|g_i\rangle = \sum_{k=1}^{D_1} U_{k,i}^{\dagger} |e_k\rangle \tag{4.13}$$

where U is the same unitary in (4.8). We show in App C.1 that for any basis $\{|g_i'\rangle\}_{i=1}^{D_1}$, we can write:

$$\mathcal{A}_{m,m',n,n'}(\rho) = \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i'\rangle\langle g_i'| \mathcal{K}_m \otimes B_{n,n'}^{\prime i} \right\rangle_\rho \tag{4.14}$$

where $B_{n,n'}^{\prime i}$ is a new set of operators obtained from $B_{n,n'}^i$ by the relation:

$$B_{n,n'}^{\prime i} = \sum_k |V_{k,i}|^2 B_{n,n'}^k \tag{4.15}$$

and V is the unitary operators such that:

$$|g_i\rangle = \sum_k V_{k,i} |g_k'\rangle. \tag{4.16}$$

With this freedom in mind, we proceed by writing the singular value decomposition of the operators $\mathcal{K}_m$:

$$\mathcal{K}_m = \sum_{k=1}^{d_1} \lambda_k^m |\zeta_k^m\rangle\langle\eta_k| \tag{4.17}$$

where $|\zeta_k^m\rangle$ is an eigenvector of $\mathcal{K}_m \mathcal{K}_m^{\dagger}$ in the Hilbert space $\mathcal{H}^{(d_1)} \oplus \mathcal{H}^{(d_1)}$, $\{|\eta_k\rangle\}_k$ are the mutual eigenvectors of $\mathcal{K}_0^{\dagger}\mathcal{K}_0 = \dfrac{\mathbb{1}^{(d_1)} + A_3^{(d_1)}}{2}$ and $\mathcal{K}_1^{\dagger}\mathcal{K}_1 = \dfrac{\mathbb{1}^{(d_1)} - A_3^{(d_1)}}{2}$, see (4.3). Now, if we call W^{01} the unitary matrix that verifies

$$|\zeta_i^1\rangle = \sum_k W_{k,i}^{01} |\zeta_k^0\rangle, \tag{4.18}$$

and write its spectral decomposition as:

$$W^{01} = \mathcal{W}^{\dagger} e^{iD} \mathcal{W}, \tag{4.19}$$

where $\mathcal{W}$ is a unitary matrix and D is a real diagonal one, we can choose the basis $\{|g_i'\rangle\}_{i=1}^{D_1}$ such that:

$$|g_i'\rangle = \sum_k e^{iD_{k,k}} \mathcal{W}_{k,i} |\zeta_k^0\rangle. \tag{4.20}$$

Hence, we have

$$|g_i'\rangle = \sum_k \mathcal{W}_{k,i} |\zeta_k^1\rangle. \tag{4.21}$$

From (4.20), (4.21) and (4.14) and the fact that $|\mathcal{W}_{k,i}| = \left|e^{iD_{k,k}} \mathcal{W}_{k,i}\right|$, we find that:

$$\mathcal{A}_{m,m',n,n'}(\rho) = \left\langle \mathcal{K}_{m'}^\dagger \sum_i |\zeta_i^m\rangle\langle\zeta_i^m| \mathcal{K}_m \otimes B_{n,n'}^{\prime\prime i} \right\rangle_\rho, \tag{4.22}$$

where, from (4.15),

$$B_{n,n'}^{\prime\prime i} = \sum_k |\mathcal{W}_{k,i}|^2 B_{n,n'}^{\prime k} \tag{4.23}$$

Finally, we use the fact

$$\mathcal{K}_{m'}^\dagger |\zeta_i^m\rangle\langle\zeta_i^m| \mathcal{K}_m = \mathcal{K}_{m'}^\dagger \mathcal{K}_m |\eta_i\rangle\langle\eta_i| \tag{4.24}$$

to derive the desired simple form of the LOCC mapping with one way communication:

$$\mathcal{M}(\rho) = \frac{1}{4} \sum_{i,j=0}^{3} \left\langle \sum_k B_i^{(d_1)} |\eta_k\rangle\langle\eta_k| \otimes B_j^{(d_2,k)} \right\rangle_\rho \sigma_i^{(1)} \otimes \sigma_j^{(2)} \tag{4.25}$$

where the observables $B_i^{(d_1)}$, $B_i^{(d_2,k)}$ for each $k = 1, \cdots, d_1$ where $i = 1, 2, 3$, are properly normalized to ensure complete positivity as mentioned in chapter 2. $|\eta_k\rangle$ are the eigenvectors of the operator $B_3^{(d_1)}$. We have thus far shown that any mapping to two qubits of the form (4.6) can be written in the above form (4.25). The converse is also true, hence we have characterized all possible LOCC with one way communication that maps to 2 qubits.

It is interesting to see that if we choose the same operators $B_i^{(d_2,k)} = B_i^{(d_2)}$, for every k, one recovers the mapping 4.2. This is a manifestation of the fact that the set of LOCC with no classical communication is contained in the set of LOCC with one way communication.

Going beyond one round of communication is difficult. However, as we have seen when one considers mapping to two qubits, it is a possibility that they would simplify. It would be very interesting to check if there is any collapse between the sets of LOCC

with n-rounds of classical communication because of this simplification.

As far as entanglement detection is concerned, we can check how separable operations are implemented via our formalism. These operations have the advantage of containing all LOCC operations and being simple to describe. Moreover, they are strictly contained in PPT preserving operations. Hence, they conserve separability since two qubit PPT states are separable. This might prove useful to provide stronger entanglement criteria than our current formalism for LOCC with no classical communication and is left for future investigations.

4.2 Generalized spin squeezing inequalities: entanglement depth and Measurability

4.2.1 Measurability of first and second moments of an arbitrary operator

In chapter 3 we have introduced the generalized spin squeezing inequalities for arbitrary operators A_i; $i = 1, \cdots, M$ and generalized number operator $\hat{N}$. In order for the arbitrary choice of operators to be of any use and represent a real freedom of choice, from an application point of view, one should ask the question whether such a quantity is measurable or not. One should keep in mind that the most appealing feature of spin squeezing inequalities is that they are based on easy to measure collective quantities. Moreover, the starting point of my study for these inequalities was the experimental constraints that prevents from using witnesses based on measurements of individual observables or collective observables that are not accessible with their present arsenal of experimental techniques.

I will be attempting to answer this question assuming the limitations, on what observables can be measured, of the experimental setup of [263] where the experimental techniques are typical for experiments involving BECs and cold atoms. In such experiments, we have a set of N atoms, each of which has spin-j. Population of each Zeeman level

$$P_m = \sum_{i=1}^{N} |m^{(i)}\rangle\langle m^{(i)}|, \quad m = -j, \cdots, j \tag{4.26}$$

can be measured in the z direction of the external magnetic field. In principle, the population measurement can also be performed in any other direction $\vec{\Theta} = (\alpha, \beta, \gamma)$,

characterized by the three Euler angles, by rotating the state of the system from $|\psi\rangle$ to $e^{-iJ_z\alpha}e^{-iJ_y\beta}e^{-iJ_z\gamma}|\psi\rangle$. From such measurements only, we would like to deduce the mean value and the second moment of an arbitrary collective operator

$$X = \sum_{m_1,m_2=-j}^{j} c_{m_1,m_2} C_{m_1,m_2} \tag{4.27}$$

where we have defined the collective operators

$$C_{m_r,m_s} = \sum_{i=1}^{N} |m_r^{(i)}\rangle\langle m_s^{(i)}| \quad m_r, m_s = -j, \cdots, j \tag{4.28}$$

where the superscript $i = 1, \cdots, N$ labels the particles and $m_r(m_s)$ labels the Zeeman levels of a single particle.

Measuring expectation value of arbitrary observable

In what follows I will show that it is possible to measure correlations between different Zeeman levels; i.e. mean value of operators of the form (4.28) from measuring population in different directions; i.e. measuring operators of the form:

$$\begin{aligned} P_{m_s}(\vec{\Theta}) &= \bigotimes_i U_i(\vec{\Theta})^\dagger \left(\sum_i |m_s^{(i)}\rangle\langle m_s^{(i)}| \right) \bigotimes_i U_i(\vec{\Theta}) \\ &= \sum_i \sum_{m_k,m_l} U^\dagger_{m_k,m_s}(\vec{\Theta}) U_{m_s,m_l}(\vec{\Theta}) |m_k^{(i)}\rangle\langle m_l^{(i)}| \end{aligned} \tag{4.29}$$

Where $U^\dagger_{m_s,m_l}(\vec{\Theta})$ is the (m_s, m_l) element of the unitary matrix corresponding to the rotation parametrized by the Euler angles $\vec{\Theta} = (\alpha, \beta, \gamma)$; more explicitly we write:

$$U_i(\vec{\Theta}) = e^{-i\gamma J_z^{(i)}} e^{-i\beta J_y^{(i)}} e^{-i\alpha J_z^{(i)}}$$

which is the same for every particle (i). Taking the expectation value of the above expression, we get:

$$\langle P_{m_s}(\vec{\Theta})\rangle = \sum_{m_k,m_l} U^\dagger_{m_k,m_s}(\vec{\Theta}) U_{m_s,m_l}(\vec{\Theta}) \langle C_{m_k,m_l}\rangle \tag{4.30}$$

Which can be rewritten in the equivalent form:

$$\langle P_{m_s}(\vec{\Theta})\rangle = \sum_{m_k \le m_l} 2Re\left(U^\dagger_{m_k,m_s}(\vec{\Theta}) U_{m_s,m_l}(\vec{\Theta}) \langle C_{m_k,m_l}\rangle \right) \tag{4.31}$$

Now we see that equation (4.31) is a system of $2j+1$ equations with $(2j+1)^2$ variables $x_{m_k,m_l} = Re\left(\langle C_{m_k,m_l}\rangle\right)$ and $y_{m_k,m_l} = Im\left(\langle C_{m_k,m_l}\rangle\right)$ with $m_k \leq m_l$. Thus, we only need $2j+1$ rotations $\vec{\Theta}_0, \cdots, \vec{\Theta}_{2j}$ to determine all correlations (4.28).

In the next step, we write the rotation matrix elements explicitly as a function of Euler angles:

$$U_{m_s,m_l}(\vec{\Theta}) = e^{-im_s\alpha} d_{m_s,m_l}(\beta) e^{-im_l\gamma}\,,\; U^{\dagger}_{m_k,m_s}(\vec{\Theta}) = e^{im_s\alpha} d_{m_s,m_k}(\beta) e^{im_k\gamma} \tag{4.32}$$

so we get :

$$U^{\dagger}_{m_k,m_s}(\vec{\Theta}) U_{m_s,m_l}(\vec{\Theta}) = e^{i(m_k-m_l)\gamma} d_{m_s,m_l}(\beta) d_{m_s,m_k}(\beta) \tag{4.33}$$

And we can see that the first rotation around the Z axis with angle α is not relevant in the equation (4.31), so we can set $\alpha = 0$. Choosing the γ angles of the $2j+1$ rotations to be

$$\gamma_n = \frac{2\pi n}{2j+1} \quad : n = 0, 1, \cdots, 2j \tag{4.34}$$

So the total rotation is $\vec{\theta}_n = (0, \beta, \gamma_n)$ and the idea is to optimize the inverse matrix as a function of β, the rotation angle around the y-axis.

This particular choice of angles (4.34), probably not the only choice, can simplify the system of equations (4.31). First, let us rewrite Eq (4.30) in the following form:

$$\langle P_{m_s}(\vec{\Theta_n})\rangle = \sum_{m=-2j}^{2j} \sum_{m_k} e^{-i\gamma_n m} d_{m_s,m_k+m}(\beta) d_{m_s,m_k}(\beta) \langle C_{m_k,m_k+m}\rangle \tag{4.35}$$

Where the subscript n represents the rotation $\vec{\theta}_n$.

Making use of the identity

$$\sum_{n=0}^{2j} e^{-i\gamma_n(k-l)} = (2j+1)\delta_{k,l}, \tag{4.36}$$

we can separate the above system of equations to smaller sets as a function of $m = -2j, \cdots, 2j$. More explicitly, multiplying both sides of the above equation by $e^{i\gamma_n m}$ and summing over n we get:

$$\frac{1}{2j+1} \sum_{n=0}^{2j} e^{i\gamma_n m} \langle P_{m_s}(\vec{\Theta_n})\rangle = \sum_{m_k} d_{m_s,m_k+m}(\beta) d_{m_s,m_k}(\beta) \langle C_{m_k,m_k+m}\rangle \tag{4.37}$$

Now, for fixed m, we see that it is a system of $2j+1$ equations (corresponding to different values of m_s with $2j+1-|m|$ variables $\langle C_{m_k,m_k+m}\rangle$. We want to choose an angle β such that the matrices

$$V(m)_{m_s,m_k} = d_{m_s,m_k}(\beta) d_{m_s,m_k+m}(\beta) \tag{4.38}$$

which are of dimension $(2j+1)\times(2j+1-|m|)$ have left inverse and such that the experimental errors are not amplified (at least not to infinity). The above matrix has a left inverse; i.e.$V(m)^{-1}V(m) = I$ and of dimension $(2j+1-|m|)\times(2j+1)$, of the following form:

$$V(m)^{-1}_{m_k,m_s} = (-1)^{m_k-m_s}\sum_J \frac{\langle j\ -m_s\, j\, m_s|J\,0\rangle\langle j\ -m_k\, j\, m_k+m|J\,m\rangle}{d^{(J)}_{0,m}(\beta)}, \tag{4.39}$$

where one can easily verify that,see App C.2,

$$P = V(m)^{-1}V(m) = I. \tag{4.40}$$

The relevant quantity to minimise is

$$\left\|V(m)^{-1}\right\|_\infty = Max_i \sum_j \left|V(m)^{-1}_{i,j}\right|. \tag{4.41}$$

That is because the error in measuring a quantity of the form $x_m = \sum_n A_{m,n}b_n$ is proportional to

$$\Delta x_m = \sum_n |A_{m,n}|\,\Delta b_n \leq \|A\|_\infty \Delta b; \tag{4.42}$$

where Δb represents the error in population measurement.

Next, set out to minimize $\|V(m)^{-1}\|_\infty$ as a function of β and we choose a value such that

$$\mathcal{D}(\beta) = \sum_{m=1}^{2j}\left\|V(m)^{-1}\right\|_\infty \tag{4.43}$$

is at its minimum. From Eq. (4.39),it is clear that such a value exists and verifies the following condition:

$$d^{(J)}_{0,m}(\beta) \neq 0 \quad ,\forall J \in \{0,\cdots,2j\}\forall m \in \{-2j,\cdots,2j\} \tag{4.44}$$

This concludes the proof for the possibility to extract correlations by measuring populations in $2j+1$ directions!

Please note that $\langle C_{m_k,m_k+m}\rangle = \langle C_{m_k+m,m_k}\rangle^*$, so we can get all correlations corresponding to $-m$ once correlations corresponding to m are calculated. That is why we do not need to include $\|V(-m)^{-1}\|_\infty$ with $m > 0$ in our study.

We can calculate an upper bound of $\mathcal{D}(\beta)$ (4.43):

$$\mathcal{D}(\beta) = \sum_{m=1}^{2j} \left\|V(m)^{-1}\right\|_\infty \leq \sqrt{2j+1} \sum_{m=1}^{2j} \left\|V(m)^{-1}\right\|_F = \mathcal{G}(\beta)$$

where $\|A\|_F = \sqrt{\mathrm{tr}\,(A^\dagger A)}$ is the Frobenius norm. From Eq. (4.39) and the identity:

$$\sum_m \langle j_1\, m\, j_2\, M - m | J_1\, M\rangle \langle j_1\, m\, j_2\, M - m | J_2\, M\rangle = \delta_{J_1,J_2}$$

we can calculate the upper bound $\mathcal{G}(\beta)$ to be:

$$\mathcal{G}(\beta) = \sqrt{2j+1} \sum_{m=1}^{2j} \sqrt{\sum_{J=|m|}^{2j} \frac{1}{\left(d^{(J)}_{0,m}(\beta)\right)^2}} \tag{4.45}$$

Finally we plot $\mathcal{D}(\beta)$ for spin $j = 3$ and we see that it reaches a minimum at $\beta \approx 1.4$ *rad*, see figure 4.1.

Measuring second moments of arbitrary observable

In this case we are looking to measure correlations of the form:

$$C_{m_p,m_q} C_{m_r,m_s} = \sum_i \delta_{m_q,m_r} |m_p^{(i)}\rangle\langle m_s^{(i)}| + \sum_{i\neq j} |m_p^{(i)}\rangle\langle m_q^{(i)}| \otimes |m_r^{(j)}\rangle\langle m_s^{(j)}| \tag{4.46}$$

from measuring second-order correlations of population in different directions; i.e. measuring operators of the form $P_{m_r}(\vec{\Theta})P_{m_s}(\vec{\Theta})$. Note that the correlations should be measured in the same direction $\vec{\Theta}$ and there are no correlations among different directions. This is because it is impossible to measure two directions simultaneously.

In a similar way to Eq. (4.30), we can write:

$$\begin{aligned}\langle P_{m_r}(\vec{\Theta})P_{m_s}(\vec{\Theta})\rangle = \sum_{m_k,m_l,m_p,m_q} & U^\dagger_{m_k,m_r}(\vec{\Theta})U_{m_r,m_l}(\vec{\Theta}) \\ & U^\dagger_{m_p,m_s}(\vec{\Theta})U_{m_s,m_q}(\vec{\Theta})\langle C_{m_k,m_l} C_{m_p,m_q}\rangle\end{aligned} \tag{4.47}$$

Which can be rewritten in the equivalent form:

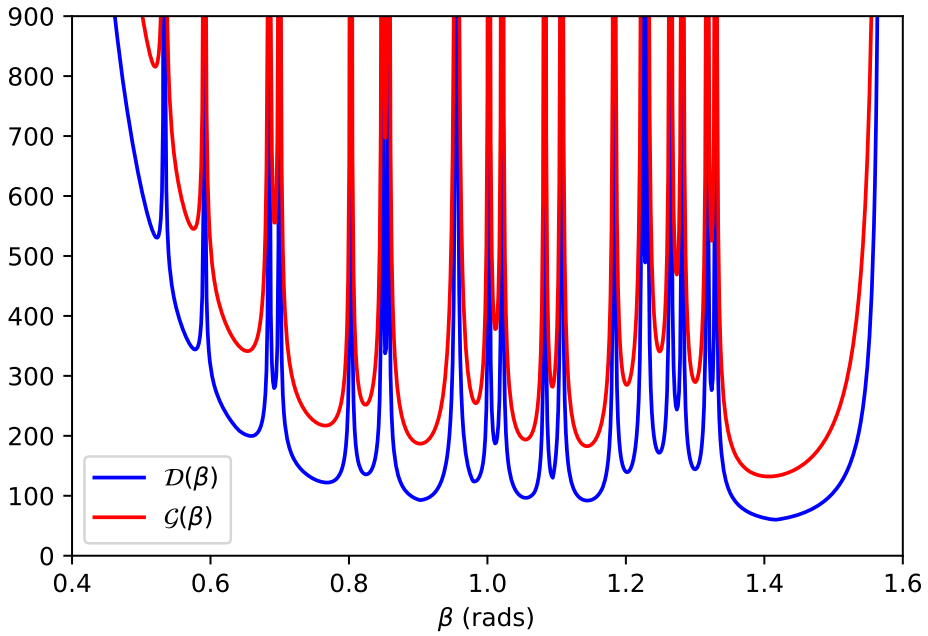


Figure 4.1: $\mathcal{D}(\beta)$ (4.43) in blue and its upper bound $\mathcal{G}(\beta)$ (4.45) in red as a function of β.

$$\langle P_{m_r}(\vec{\Theta})P_{m_s}(\vec{\Theta})\rangle = \sum_{m_k\leq m_q,m_l,m_p} 2Re\left(U^{\dagger}_{m_k,m_r}(\vec{\Theta})U_{m_r,m_l}(\vec{\Theta})\right. \\ \left. U^{\dagger}_{m_p,m_s}(\vec{\Theta})U_{m_s,m_q}(\vec{\Theta})\langle C_{m_k,m_l}C_{m_p,m_q}\rangle\right) \tag{4.48}$$

Now we see that equation (4.48) is a system of $(2j+1)^2$ equations with $(2j+1)^4$ variables $x_{m_k,m_l,m_p,m_q} = Re\left(\langle C_{m_k,m_l}C_{m_p,m_q}\rangle\right)$ and $y_{m_k,m_l,m_p,m_q} = Im\left(\langle C_{m_k,m_l}C_{m_p,m_q}\rangle\right)$ with $m_k \leq m_q$.

Hence, we need $(2j+1)^2$ rotations $\vec{\Theta}_0,\cdots,\vec{\Theta}_{(2j+1)^2-1}$, at least, to determine all correlations (4.46). Following the same steps as before, we choose the angle $\alpha = 0$. However, we choose γ angles of the $(2j+1)^2$ rotations to be:

$$\gamma_n = \frac{2\pi n}{(2j+1)^2} \quad : n = 0,1,\cdots,(2j+1)^2-1 \tag{4.49}$$

So the total rotation is $\vec{\theta}_n = (0,\beta,\gamma_n)$ and the idea is to optimize the inverse matrix as a function of β.

We proceed by writing:

$$\langle P_{m_r}(\vec{\Theta}_n)P_{m_s}(\vec{\Theta}_n)\rangle = \sum_{m_1,m_2=-2j}^{2j} \sum_{m_k,m_l} e^{-i\gamma_n(m_1+m_2)} d_{m_r,m_k+m_1}(\beta) d_{m_r,m_k}(\beta) \\ d_{m_s,m_l+m_2}(\beta) d_{m_s,m_l}(\beta) \langle C_{m_k,m_k+m_1} C_{m_l,m_l+m_2}\rangle \tag{4.50}$$

Where the subscript n represents the rotation $\vec{\theta}_n$. Next, we separate the above system of equations to smaller sets as a function of $m = m_1 + m_2 = -4j, \cdots, 4j$:

$$\frac{1}{(2j+1)^2} \sum_{n=0}^{(2j+1)^2-1} e^{i\gamma_n m} \langle P_{m_r}(\vec{\Theta}_n)P_{m_s}(\vec{\Theta}_n)\rangle = \\ \sum_{m_1} \sum_{m_k,m_l} d_{m_r,m_k+m_1}(\beta) d_{m_r,m_k}(\beta) d_{m_s,m_l+m-m_1}(\beta) \\ d_{m_s,m_l}(\beta) \langle C_{m_k,m_k+m_1} C_{m_l,m_l+m-m_1}\rangle \tag{4.51}$$

Now,for fixed m, we see it is a system of $(2j+1)^2$ equations (corresponding to different values of m_r, m_s) with

$$\sum_{m_1=\min(-2j,-2j+m)}^{\max(2j,2j+m)} (2j+1-|m_1|)\,(2j+1-|m-m_1|)$$

variables $\langle C_{m_k,m_k+m_1} C_{m_l,m_l+m_2}\rangle$. A comparison with Eq. (4.38), we see that:the matrix we want to find its left inverse is of the form:

$$\mathcal{V}(m) = \bigoplus_{m_1=\min(-2j,-2j+m)}^{\max(2j,2j+m)} V(m_1) \otimes V(m-m_1) \tag{4.52}$$

Since each of the matrices $V(m_1)$ has left inverse $V^{-1}(m_1)$ according to last section (4.39), we can easily find the left inverse of $\mathcal{V}(m)$ to be:

$$\mathcal{V}^{-1}(m) = \bigoplus_{m_1=\min(-2j,-2j+m)}^{\max(2j,2j+m)} V^{-1}(m_1) \otimes V^{-1}(m-m_1) \tag{4.53}$$

which concludes the proof.

We have thus far shown the possibility to measure second-order correlations (4.46) by measuring second order moments of operators of the form (4.47) in $(2j+1)^2$ directions. In fact,the above proof can be generalized to access n-order correlations from population n-order correlation measurements in $(2j+1)^n$ directions. Which, in turn, will allow the

measurement of moments up to the nth moment of an arbitrary collective operator of the form:

$$X = \sum_{m_k, m_l} c_{m_k,m_l} C_{m_k,m_l} \tag{4.54}$$

where $c_{m_k,m_l} \in \mathbb{C}$ and C_{m_k,m_l} is defined in Eq. (4.28).

At this point, it is important to note that the provided scheme becomes more unrealistic the higher spin-j is. For spin $j = 3$, measuring the second moments would require $(2j+1)^2 = 49$ rotations and hence a resolution better than $\frac{2\pi}{2 \times 49} \approx 0.06\,\text{rad} \approx 3°$. However, the above presented strategy is not optimal but rather shows that it is indeed possible, at least in theory, to extract the correlations. Different choices for γ and β might prove more advantageous. Another interesting question to look into is whether it is possible to extract a subset of the correlations (4.46) with less measurements.

4.2.2 Criteria for entanglement depth detection

The criteria we generalized in chapter 3 were true for fully separable states. When they detect a state as entangled, they are blind to what kind of entanglement we are dealing with. The way we did this generalization was to take entanglement criteria for N spin-$\frac{1}{2}$ and replace the particle number N by the expectation of the particle number operator $\left\langle \hat{N} \right\rangle$, the expectation of the collective spin component along the k-axis, $\langle S_k \rangle$, by $\langle A_k \rangle$ and the modified second moments $\left\langle \tilde{S}_k^2 \right\rangle$ by $\left\langle \tilde{A}_k^2 \right\rangle$. In addition, we had to add an additional term δ in order to generalize the spin squeezing inequalities (3.6). However, when generalizing Sørensen-Mølmer's criterion for N spin-$\frac{1}{2}$ particles, we did not need the additional term δ and it was sufficient to do the above mentioned substitutions. Next, we generalize Sørensen-Mølmer's criterion for k-producible states.

Sørensen-Mølmer criterion

As it turns out, we are able to generalize Sørensen-Mølmer's criterion for k-producible states of N spin-j particles:

$$(\Delta J_x)^2 \geq N j F_{kj}\left(\frac{\langle J_z \rangle}{Nj}\right) \tag{4.55}$$

in the following way:

$$\beta^2 \left\langle \hat{N} \right\rangle + \tilde{\Delta}^2 A_1 \geq 2\beta^2 \left\langle \hat{N} \right\rangle F_{\frac{k}{2}} \left(\frac{\langle A_2 \rangle}{\beta \left\langle \hat{N} \right\rangle} \right) \tag{4.56}$$

For the proof, see appendix C.3. Unlike the case $k = 1$, the normalization factor β is chosen to be $\sqrt{2}\alpha$ rather than α where $\alpha^2 = \sup_{\rho^{(i)}} \left[\sum_{k=1}^{2} \left\langle A_k^{(i)} \right\rangle_{\rho^{(i)}}^2 \right]$. This is because of the way we mapped the original system into an N-qubits system, the main step of the proof presented in appendix.C.3. Our choice is in no way unique nor optimal and it is quite plausible to find a better mapping that sharpens the normalization factor β in (4.56).

Another interesting point to investigate is when one has single particle operators $A_k^{(i)}$ with more than two eigenvectors. In this case, one should consider mapping to higher spin than $\frac{1}{2}$ to obtain a sharper inequality. One can conjecture that in this case one should obtain the function F_{kj} rather than $F_{\frac{k}{2}}$ in the right hand side of eq.(4.56). A rigorous way to do a mapping properly in the sense of chapter 2 to help prove it is a goal to aim for.

An interesting workaround though can be found in the case where the single particle operators have $2m$ eigenvectors. In that case, one can consider each particle as a collection of m two-level systems so one can directly replace $F_{\frac{k}{2}}$ by F_{km} in (4.56).

Finally, there exists other criteria that detect entanglement depth in terms of first and second moments of collective spin operators. For example, Duan [267] showed that for k-producible states of N qubits the following inequality

$$\frac{\langle J_x^2 + J_y^2 \rangle}{N\left(\frac{1}{4} + (\Delta J_z)^2\right)} \leq k + 1 \tag{4.57}$$

is verified. In [221], a generalization of Sørensen-Mølmer criterion [220] was presented in the following way:

$$(\Delta J_x)^2 \geq N j F_{kj} \left(\sqrt{\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj+1)}{N(N-k)j^2}} \right) \tag{4.58}$$

that holds for k-producible states of N spin-j particles. Along side with the above criterion Vitagliano et al. provided a stronger version of Duan's criterion (4.57) via a

series expansion of the function $F\left(\sqrt{.}\right)$. They also generalized (4.58) to the case where N presents a statistical fluctuations. However, in that case, higher moments are required.

An immediate follow up to our work will be to generalize these criteria in the sense of chapter 3. A heavier question to tackle would be finding the best operators to detect a specific kind of entanglement or finding a single criterion that detects as many entangled states as possible.

4.3 Conclusion

We conclude this chapter and the thesis by giving a brief summary of what has been presented in the previous chapters. The main objective of this thesis revolved around entanglement detection in high dimensional systems. This high dimensionality comes in two ways: bipartite systems with dimension higher than 2×3 or multipartite systems with high number of particles.

Chapter 2 was devoted to the problem of entanglement detection in bi-partite systems with high dimension. The main idea is to reduce the dimension by mapping each party to a 2-level system such that separability is conserved. Doing so, reduces the problem into studying entanglement of a 2×2 system for which PPT criterion is a necessary and sufficient condition that is easy to compute. We presented a general way of mapping any d-level system into a qubit states such that the Bloch vector of the resulting qubit is expressed in terms of the expectation value of three arbitrary operators A_k. We introduced a very simple way to normalize these operators to ensure complete positivity of the mapping. Hence, the convenience of our formalism.

Once mapping of a single particle into a qubit is defined, we introduced a very intuitive way of mapping a bipartite system into two qubits by applying a mapping to each subsystem. This way, separability is conserved and the resulting two qubit states can be written in terms of expectation values of the form $\left\langle A_k^{(d_1)} B_l^{(d_2)} \right\rangle$. Thus, applying the PPT criterion to this two qubit state give us an entanglement criterion in terms of $\left\langle A_k^{(d_1)} B_l^{(d_2)} \right\rangle$ calculated in the original state. Changing the two sets of operators $A_k^{(d_1)}$ and $B_l^{(d_2)}$ gives us different criteria. We studied a very simple case where we chose the sets of operators to be the spin operators and we showed that, in the multi-partite case, spin squeezing inequalities for N spin-j particles can be deduced from spin squeezing inequalities for N spin-$\frac{1}{2}$ particles using this mapping for each particle.

The simplicity of the way we map into two-qubit system follows from the simplicity of

LOCC operations with no classical communication. However, the formalism developed in chapter 2 allowed us to get a very simple description of LOCC with one-way classicla communication as a function of expectation values of simple bipartite observables rather than Kraus operators. This simplification was possible because we are considering the case of mapping into two-qubits rather than the general case. We expect this relative simplicity to carry over to describe LOCC with multiple round of communication to map into two qubits. However, the Kraus representation of such operations is more complex and a lot more involved. Obtaining a simple description of mapping any bipartite state into two qubit state via LOCC operations is quite important and can prove very important to the study of distillability. A major handicap in the study of distillability of NPT states is the absence of a simple mathematical description of LOCC operations with multiple-rounds of classical communication. Our formalism might give a fresh perspective to these studies.

Our use of the mapping in the multipartite case, where one uses operators defined on a subspace of the Hilbert space of a single particle, has led us to consider a unique situation where one has quantum fluctuations of particle number. Unlike the common experimental case where one has classical fluctuations of particle number among different realizations of a given experiment, using our mapping can give rise to coherent superposition among states with different particle number.
Dealing with this problem was the main objective of chapter 3, where we have generalized spin squeezing inequalities for arbitrary operators with a fluctuating particle number. While, spin squeezing inequalities have been generalized for the case of classical fluctuations of particles, these generalized inequalities involve higher moments and are no longer in terms of spin operators only. Our generalization is both simple and is in terms of the first and second moments of collective operators.
We have applied these inequalities to the experimental case of a Chromium lattice which can be seen as a system of spin-3 particles. We have shown, with a simulation, that our inequalities succeed to infer entanglement at short times. We have also shown that, in principle, it is possible to measure the involved collective operators to show this entanglement. Our method can be used for similar systems since it does not require complex manipulation of the system, but rather uses simple collective measurements and rotations.
Finally, we have shown that our arsenal of techniques used in this work are capable of generalizing some criteria of entanglement depth detection. Such criteria are extremely useful to systems with large number of particles and with proper choice of operators, one

might be able to witness better what kind of entanglement present in the system.

In our work, we have provided tools and criteria that give a lot of freedom . This freedom does not provide a satisfying answer on how to choose properly when faced with some kind of entanglement. We also still have no optimal set of criteria, for which is guaranteed to witness as much as possible of entangled states, yet. Dealing with these kind of questions, whilst very hard, is quite important and interesting at the same time to characterize large systems' entanglement.

Appendix A

Appendices of Chapter 2

A.1 Kraus representation of mapping (2.3)

Let us define $\{|e_i\rangle : 1 \leq i \leq D\}$ to be an orthonormal basis of $\mathcal{H}^{(D)}$. In order to find the Kraus representation of the mapping (2.7), we will make use of the conditions (2.15) and write the operators B_is as a function of the operators K_0 and K_1:

$$B_z = {K_0}^\dagger K_0 - {K_1}^\dagger K_1 \, , \, \mathbb{1} = {K_0}^\dagger K_0 + {K_1}^\dagger K_1 \, , \, B_+ = 2{K_0}^\dagger K_1 \tag{A.1}$$

Where we have defined:

$$B_+ = B_x + iB_y \quad , \quad B_+ = B_x - iB_y \tag{A.2}$$

Now let us rewrite the mapping of the state ρ in the following equivalent form:

$$\mathcal{M}(\rho) = \frac{\langle \mathbb{1} + B_z \rangle_\rho}{2} |0\rangle\langle 0| + \frac{\langle \mathbb{1} - B_z \rangle_\rho}{2} |1\rangle\langle 1| + \frac{\langle B_+ \rangle_\rho}{2} |1\rangle\langle 0| + \frac{\langle B_- \rangle_\rho}{2} |0\rangle\langle 1| \tag{A.3}$$

Now making use of the equations (A.1), we can write:

$$\begin{aligned} \mathcal{M}(\rho) = \mathrm{Tr}\left[K_0 \rho {K_0}^\dagger\right] |0\rangle\langle 0| \quad + \quad \mathrm{Tr}\left[K_1 \rho {K_1}^\dagger\right] |1\rangle\langle 1| \\ + \mathrm{Tr}\left[K_1 \rho {K_0}^\dagger\right] |1\rangle\langle 0| \quad + \quad \mathrm{Tr}\left[K_0 \rho {K_1}^\dagger\right] |0\rangle\langle 1| \end{aligned} \tag{A.4}$$

The above form, suggests the definition of the following Kraus operators:

$$K_i = |0\rangle\langle e_i| K_0 + |1\rangle\langle e_i| K_1 \quad : 1 \leq i \leq D \tag{A.5}$$

Now it is simple verification that $\sum_{i=1}^{D} K_i^\dagger K_i = \mathbb{1}^{(d)}$ and that:

$$\mathcal{M}(\rho) = \sum_{i=1}^{D} K_i \rho K_i^\dagger \tag{A.6}$$

which is the Kraus representation of the CP mapping (2.7).

A.2 Proof of theorem I 1

Let start proving that if the mapping $\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \left\langle \vec{B} \right\rangle_\rho . \frac{\vec{\sigma}}{2}$ is CP, then the condition 2.29 is verified. Since $\mathcal{M}$ is completely positive, we can write its Kraus representation [235, 230, 236]:

$$\mathcal{M}(\rho) = \sum_{i \in \mathcal{I}} \mathcal{K}_i \rho {\mathcal{K}_i}^\dagger \tag{A.7}$$

Where $\mathcal{I}$ is a countable set and $\mathcal{K}_i$ are linear operators from $\mathcal{H}$ the Hilbert space of the system in question to $\mathcal{H}^{(2)}$. Let us write the matrix elements of $\mathcal{M}(\rho)$ explicitly:

$$\begin{aligned}
\langle 0|\mathcal{M}(\rho)|0\rangle &= \textstyle\sum_i \langle 0|\mathcal{K}_i \rho {\mathcal{K}_i}^\dagger|0\rangle = \sum_i \left\langle {\mathcal{K}_i}^\dagger|0\rangle\langle 0|\mathcal{K}_i \right\rangle_\rho \\
\langle 1|\mathcal{M}(\rho)|1\rangle &= \textstyle\sum_i \langle 1|\mathcal{K}_i \rho {\mathcal{K}_i}^\dagger|1\rangle = \sum_i \left\langle {\mathcal{K}_i}^\dagger|1\rangle\langle 1|\mathcal{K}_i \right\rangle_\rho \\
\langle 0|\mathcal{M}(\rho)|1\rangle &= \textstyle\sum_i \langle 0|\mathcal{K}_i \rho {\mathcal{K}_i}^\dagger|1\rangle = \sum_i \left\langle {\mathcal{K}_i}^\dagger|1\rangle\langle 0|\mathcal{K}_i \right\rangle_\rho \\
\langle 1|\mathcal{M}(\rho)|0\rangle &= \textstyle\sum_i \langle 1|\mathcal{K}_i \rho {\mathcal{K}_i}^\dagger|0\rangle = \sum_i \left\langle {\mathcal{K}_i}^\dagger|0\rangle\langle 1|\mathcal{K}_i \right\rangle_\rho
\end{aligned} \tag{A.8}$$

comparing the above terms with the form of the mapping:

$$\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \left\langle \vec{B} \right\rangle_\rho . \frac{\vec{\sigma}}{2}$$

we can identify that:

$$\begin{aligned}
\frac{\mathbb{1} + B_z}{2} &= \sum_{i \in \mathcal{I}} {\mathcal{K}_i}^\dagger|0\rangle\langle 0|\mathcal{K}_i \quad , \quad \frac{\mathbb{1} - B_z}{2} = \sum_{i \in \mathcal{I}} {\mathcal{K}_i}^\dagger|1\rangle\langle 1|\mathcal{K}_i \\
\frac{B_x + iB_y}{2} &= \sum_{i \in \mathcal{I}} {\mathcal{K}_i}^\dagger|0\rangle\langle 1|\mathcal{K}_i \quad , \quad \frac{B_x - iB_y}{2} = \sum_{i \in \mathcal{I}} {\mathcal{K}_i}^\dagger|1\rangle\langle 0|\mathcal{K}_i
\end{aligned}$$

Now proving that the matrix

$$\mathcal{D} = \begin{pmatrix} \mathbb{1} + B_z & B_+ \\ B_- & \mathbb{1} - B_z \end{pmatrix} \tag{A.9}$$

is positive semi-definite is straightforward. For any $|\psi\rangle = |\psi_0\rangle \oplus |\psi_1\rangle \in \mathcal{H} \oplus \mathcal{H}$, we have:

$$\begin{aligned}\langle\psi|\mathcal{D}|\psi\rangle &= \langle\psi_0|\mathbb{1} + B_z|\psi_0\rangle + \langle\psi_1|\mathbb{1} - B_z|\psi_1\rangle + \langle\psi_0|B_+|\psi_1\rangle + \langle\psi_1|B_-|\psi_0\rangle \\ &= \sum_i \left|\langle\psi_0|{\mathcal{K}_i}^\dagger|0\rangle\right|^2 + \left|\langle\psi_1|{\mathcal{K}_i}^\dagger|1\rangle\right|^2 + 2\Re\left[\langle\psi_0|{\mathcal{K}_i}^\dagger|0\rangle\langle 1|\mathcal{K}_i|\psi_1\rangle\right] \\ &= \sum_i \left|\langle\psi_0|{\mathcal{K}_i}^\dagger|0\rangle + \langle\psi_1|{\mathcal{K}_i}^\dagger|1\rangle\right|^2 \geq 0\end{aligned}$$

Hence, we have proved the positivity semi-definiteness of (A.9).

Now let us assume that $\mathcal{D}$ is positive semi-definite and let us prove that the mapping $\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \left\langle \vec{B} \right\rangle_\rho . \frac{\vec{\sigma}}{2}$ is CP. Using the generalized Schur complements [268, 269], one can write the decomposition of the matrix $\mathcal{D}$ as [237, 238]:

$$\mathcal{D} = \begin{pmatrix} \mathbb{1} & 0 \\ B^\dagger A^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^\dagger A^{-1} B \end{pmatrix} \begin{pmatrix} \mathbb{1} & A^{-1}B \\ 0 & \mathbb{1} \end{pmatrix} \tag{A.10}$$

Where

$$A = \frac{\mathbb{1} + B_z}{2},\ C = \frac{\mathbb{1} - B_z}{2},\ B = B_+$$

and A^{-1} is the pseudo-inverse of A [239–242]. Since $\mathcal{D} \succeq 0$, theorem 2 implies that $A \succeq 0$ and $Q = C - B^\dagger A^{-1} B \succeq 0$ [237, 238]. Hence, we can $\sqrt{A}$ and $\sqrt{Q}$ exist and we can write:

$$\mathcal{D} = \begin{pmatrix} \sqrt{A} & 0 \\ B^\dagger \sqrt{A}^{-1} & \sqrt{Q} \end{pmatrix} \begin{pmatrix} \sqrt{A} & \sqrt{A}^{-1} B \\ 0 & \sqrt{Q} \end{pmatrix} = \begin{pmatrix} {K_0}^\dagger \\ {K_1}^\dagger \end{pmatrix} \begin{pmatrix} K_0 & K_1 \end{pmatrix} \tag{A.11}$$

Where we have defined

$$K_0 = \begin{pmatrix} \sqrt{A} \\ 0 \end{pmatrix} \quad , \quad K_1 = \begin{pmatrix} \sqrt{A}^{-1} B \\ \sqrt{Q} \end{pmatrix} \tag{A.12}$$

Thus, we have found operators K_0 and K_1 such that the conditions (2.15) are verified. Then, there exists an isometry U such that our mapping can be written as:

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}^{(D)}}\left[U\rho \otimes |0\rangle\langle 0|U^\dagger\right]$$

which is CP, completing the proof.

A.3 Proof of Eq. (2.25)

After we have set $K_0 = \frac{a}{\sqrt{2j}}$ and $K_1 = \frac{b}{\sqrt{2j}}$, we need to find A_is such that the following equations are verified:

$$\sqrt{j}\,(A_0 + A_3) = \frac{a}{\sqrt{2}} \quad , \quad \sqrt{j}\,(A_1 + iA_2) = \frac{b}{\sqrt{2}} \tag{A.13}$$

$$\vec{A}^\dagger A_0 + {A_0}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A} = 0 \tag{A.14}$$

We can already see that the conditions (A.13) are verified by the solutionj (2.25). One also could notice that conditions (A.13) and (A.14) actually imply the normalization condition (2.2). We can see that from the relation:

$$\mathbb{1} = \frac{a^\dagger a + b^\dagger b}{2j} = (A_0 + A_3)^\dagger (A_0 + A_3) + (A_1 + iA_2)^\dagger (A_1 + iA_2) \tag{A.15}$$

if (A.14) - especially $\left(\vec{A}^\dagger A_0 + {A_0}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A}\right)_3 = 0$ - is verified, then we find the normalization condition. Then All left is to verify the conditions (A.14).

Verifying the conditions (A.14) :

Verifying (2.11) We have :

$${A_1}^\dagger A_0 + {A_0}^\dagger A_1 = \frac{b^\dagger a^\dagger - ba}{4\sqrt{4j(j+1)}} + \frac{b^\dagger a}{8j} - \frac{ba^\dagger}{8(j+1)} + H.C. = \frac{b^\dagger a + a^\dagger b}{8j(j+1)}$$

$$i{A_2}^\dagger A_3 - i{A_3}^\dagger A_2 = \frac{b^\dagger a^\dagger - ba}{4\sqrt{4j(j+1)}} - \frac{b^\dagger a}{8j} + \frac{ba^\dagger}{8(j+1)} + H.C. = -\frac{b^\dagger a + a^\dagger b}{8j(j+1)}$$

And thus we have:

$$\left(\vec{A}^\dagger A_0 + {A_0}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A}\right)_1 = {A_1}^\dagger A_0 + {A_0}^\dagger A_1 + i{A_2}^\dagger A_3 - i{A_3}^\dagger A_2 = 0$$

Verifying (2.12) We have :

$${A_2}^\dagger A_0 + {A_0}^\dagger A_2 = i\left(\frac{b^\dagger a^\dagger + ba}{4\sqrt{4j(j+1)}} + \frac{b^\dagger a}{8j} + \frac{ba^\dagger}{8(j+1)}\right) + H.C. = i\frac{b^\dagger a - a^\dagger b}{8j(j+1)}$$

$$-iA_1{}^\dagger A_3 + iA_3{}^\dagger A_1 = -i\left(-\frac{b^\dagger a^\dagger + ba}{4\sqrt{4j(j+1)}} + \frac{b^\dagger a}{8j} + \frac{ba^\dagger}{8(j+1)}\right) + H.C. = -i\frac{b^\dagger a - a^\dagger b}{8j(j+1)}$$

And thus we have:

$$\left(\vec{A}^\dagger A_0 + A_0{}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A}\right)_2 = A_2{}^\dagger A_0 + A_0{}^\dagger A_2 - iA_1{}^\dagger A_3 + iA_3{}^\dagger A_1 = 0$$

Verifying (2.13) We have :

$$A_3{}^\dagger A_0 + A_0{}^\dagger A_3 = \frac{a^\dagger a^\dagger - aa}{4\sqrt{4j(j+1)}} + \frac{a^\dagger a}{8j} - \frac{aa^\dagger}{8(j+1)} + H.C. = \frac{a^\dagger a}{4j} - \frac{aa^\dagger}{4(j+1)} = \frac{a^\dagger a}{4j} - \frac{a^\dagger a + 1}{4(j+1)} = \frac{a^\dagger a - j}{4j(j+1)}$$

$$iA_1{}^\dagger A_2 - iA_2{}^\dagger A_1 = \frac{b^\dagger b^\dagger - bb}{4\sqrt{4j(j+1)}} + \frac{b^\dagger b}{8j} - \frac{bb^\dagger}{8(j+1)} + H.C. = \frac{b^\dagger b}{4j} - \frac{bb^\dagger}{4(j+1)} = \frac{b^\dagger b}{4j} - \frac{b^\dagger b + 1}{4(j+1)} = \frac{b^\dagger b - j}{4j(j+1)}$$

Where we have used $\left[a, a^\dagger\right] = 1$, $\left[b, b^\dagger\right] = 1$,and so we have:

$$\left(\vec{A}^\dagger A_0 + A_0{}^\dagger \vec{A} + i\vec{A}^\dagger \wedge \vec{A}\right)_3 = A_3{}^\dagger A_0 + A_0{}^\dagger A_3 + iA_1{}^\dagger A_2 - iA_2{}^\dagger A_1 = \frac{a^\dagger a + b^\dagger b - 2j}{4j(j+1)}$$

Where we have used $a^\dagger a + b^\dagger b = 2j$.

A.4 Proof of: $\|\Lambda\|_2 = 1 \Rightarrow \mathrm{Tr}\left[\mathcal{M}(\rho)^2\right] \leq 1$

We start by proving $\langle S_x\rangle^2_{|\psi\rangle} + \langle S_y\rangle^2_{|\psi\rangle} + \langle S_z\rangle^2_{|\psi\rangle} \leq 1$ for any pure state $|\psi\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_1$. We can write any normalized state as $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$ with $|\psi\rangle_{0(1)} \in \mathcal{H}_{0(1)}$ such that :$\langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle = 1$. Now calculating $\langle S_x\rangle_{|\psi\rangle}$ and $\langle S_y\rangle_{|\psi\rangle}$, we find:

$$\begin{aligned}\langle S_x\rangle_{|\psi\rangle} &= \langle\psi_0|\Lambda|\psi_1\rangle + \langle\psi_1|\Lambda^\dagger|\psi_0\rangle = 2Re\left[\langle\psi_0|\Lambda|\psi_1\rangle\right] \\ \langle S_y\rangle_{|\psi\rangle} &= -i\langle\psi_0|\Lambda|\psi_1\rangle + i\langle\psi_1|\Lambda^\dagger|\psi_0\rangle = 2Im\left[\langle\psi_0|\Lambda|\psi_1\rangle\right]\end{aligned}$$

As a result we get :

$$\langle S_x\rangle^2_{|\psi\rangle} + \langle S_y\rangle^2_{|\psi\rangle} = 4\left|\langle\psi_0|\Lambda|\psi_1\rangle\right|^2 \leq 4\langle\psi_0|\psi_0\rangle\langle\psi_1|\Lambda^\dagger\Lambda|\psi_1\rangle \leq 4\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \quad \text{(A.16)}$$

Where we have used Cauchy-Schwartz inequality and the fact that the largest eigenvalue of $\Lambda^\dagger\Lambda$ is $\lambda_{max} = \|\Lambda\|_2^2 = 1$.

we also have:

$$\langle S_z\rangle^2_{|\psi\rangle} = \left(\langle\psi_0|P_0|\psi_0\rangle - \langle\psi_1|P_1|\psi_1\rangle\right)^2 = \langle\psi_0|\psi_0\rangle^2 + \langle\psi_1|\psi_1\rangle^2 - 2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \quad \text{(A.17)}$$

Combining (A.16) and (A.17) we finally get:

$$\langle S_x\rangle^2_{|\psi\rangle}+\langle S_y\rangle^2_{|\psi\rangle}+\langle S_z\rangle^2_{|\psi\rangle} \leq \langle\psi_0|\psi_0\rangle^2+\langle\psi_1|\psi_1\rangle^2+2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \leq \left(\langle\psi_0|\psi_0\rangle+\langle\psi_1|\psi_1\rangle\right)^2 \leq 1 \quad \text{(A.18)}$$

Taking the special case where $\langle\psi_0|\psi_0\rangle = 1$ and $\langle\psi_1|\psi_1\rangle = 0$ we get the equality $\langle S_x\rangle^2_{|\psi\rangle} + \langle S_y\rangle^2_{|\psi\rangle} + \langle S_z\rangle^2_{|\psi\rangle} = 1$. Thus we conclude that :

$$\beta^2 = \sup\left\{\langle S_x\rangle^2_{|\psi\rangle} + \langle S_y\rangle^2_{|\psi\rangle} + \langle S_z\rangle^2_{|\psi\rangle} : (|\psi\rangle \in \mathcal{H}) \wedge (\langle\psi|\psi\rangle = 1)\right\} = 1 \quad \text{(A.19)}$$

Now we have established that the minimum value of β, such that the mapping (2.44) is CP, is 1. Now let use Theorem I 1 to prove it. In other words, let us prove that the matrix 2.31 is positive semi-definite.

A.5 Proof of complete positivity of the mapping (2.44)

We start by writing the singular value decomposition of the linear operator $P_0\Lambda P_1$ [270, 271]:

$$P_0\Lambda P_1 = \sum_n \sqrt{\lambda_n}|e_n\rangle\langle f_n| \quad \text{(A.20)}$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers such that

$$\max\{\lambda_n\}_{n\in\mathbb{N}} = 1$$

because of the condition $\|\Lambda\|_2 = 1$. $\{|e_n\rangle\}_{n\in\mathbb{N}}$ and $\{|f_n\rangle\}_{n\in\mathbb{N}}$ are orthonormal basis of $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. Then, we have:

$$\sum_n |e_n\rangle\langle e_n| = P_0 \quad , \quad \sum_n |f_n\rangle\langle f_n| = P_1 \quad \text{(A.21)}$$

Now let us prove that the matrix 2.31

$$\mathcal{D} = \begin{pmatrix} \mathbb{1} + S_z & 2P_0\Lambda P_1 \\ 2P_1\Lambda^\dagger P_0 & \mathbb{1} - S_z \end{pmatrix} \quad \text{(A.22)}$$

is positive semi-definite. For that we use again Theorem 2. Let us define

$$Q = \mathbb{1} - P_0 - P_1.$$

It is straightforward to show that

$$A = \mathbb{1} + S_z = Q + 2P_0 \succeq 0$$

and that:

$$\mathbb{1} - AA^{-1} = \mathbb{1} - (Q + 2P_0)\left(Q + \frac{1}{2}P_0\right) = P_1$$

Hence the second condition $(\mathbb{1} - AA^{-1})\, B_+ = 0$ is verified. Now let us check the final condition $C - {B_+}^\dagger A^{-1} B_+ \succeq 0$. To do so, let us calculate the quantity:

$${B_+}^\dagger A^{-1} B_+ = 4P_1\Lambda^\dagger P_0\left(Q + \frac{1}{2}P_0\right)P_0\Lambda P_1 = 2P_1\Lambda^\dagger P_0\Lambda P_1 = 2\sum_n \lambda_n |f_n\rangle\langle f_n|$$

From which we find:

$$C - {B_+}^\dagger A^{-1} B_+ = \mathbb{1} - S_z - 2\sum_n \lambda_n |f_n\rangle\langle f_n| = Q + 2\sum_n (1-\lambda_n)\,|f_n\rangle\langle f_n| \succeq 0$$

since $0 \leq \lambda_n \leq 1$. With this, we conclude the proof for positivity semi-definiteness of (A.22) which implies that the mapping Eq. (2.44) is C.P.

A.6 3-parameter family of 2-qudit states

A.6.1 Preliminaries

First, we recall the family of d^2 maximally entangled 2-qudit states $|\Omega_{kl}\rangle (k, l = 0, 1, \cdots d-1)$ that generalizes the four 2-qubit Bell states as introduced in Ref. [46]:

$$|\Omega_{kl}\rangle = W_{kl} \otimes \mathbb{1} |\Omega_{00}\rangle \text{ with } |\Omega_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} |m, m\rangle, \tag{A.23}$$

where the d^2 W_{kl} operators acting on the first qudit are the Weyl operators defined as:

$$W_{kl}|m\rangle = w^{k(m-l)}|(m-l)_{\text{mod } d}\rangle, \text{ where } w = e^{2\pi i/d}. \tag{A.24}$$

These operators verify the following orthogonality relation in the Hilbert-Schmidt norm:

$$\mathrm{Tr}\left[W_{kl}^{\dagger}W_{mn}\right] = d\ \delta_{km}\delta_{ln}. \tag{A.25}$$

These 2-qudit Bell states $P_{kl} = |\Omega_{kl}\rangle\langle\Omega_{kl}|$ are locally maximally mixed state, that is, their partial trace gives the maximally mixed state $\frac{\mathbb{1}}{d}$. Using the following relation [46]:

$$|j\rangle\langle k| = \frac{1}{d}\sum_{l=0}^{d-1} w^{lj}W_{j(k-j)}, \tag{A.26}$$

each state P_{kl} can be written in the basis $W_{kl}\otimes W_{mn}$ as follows [272, 46]:

$$P_{kl} = \frac{1}{d^2}\sum_{m,n=0}^{d-1} w^{ml-kn}W_{-m-n}^{\dagger}\otimes W_{m-n}^{\dagger}. \tag{A.27}$$

The operators $J_3,\ J_+,\ J_-$ can also be written in the Weyl basis:

$$\begin{aligned} J_3 &= \sum_{l=0}^{2j}\eta_z^l W_{l0}, \\ J_+ &= \sum_{l=0}^{2j}\eta_p^l w^{-l}W_{l,-1}\ ,\ J_- = \sum_{l=0}^{2j}\eta_p^l W_{l,1}, \end{aligned} \tag{A.28}$$

where we have defined

$$\begin{aligned} \eta_z^l &= \sum_{m=0}^{2j}\frac{m-j}{2j+1}w^{-ml}, \\ \eta_p^l &= \sum_{m=0}^{2j}\frac{\sqrt{j(j+1)-(m-j)(m-j+1)}}{2j+1}w^{-ml}. \end{aligned}$$

From Eqs. (A.27), (A.28), and (A.25), we get

$$\begin{aligned} \langle J_3\otimes J_3\rangle_{|\Omega_{kl}\rangle} &= \mathrm{Tr}\left[P_{kl}J_3\otimes J_3\right] = \sum_{m=0}^{2j}\eta_z^m\eta_z^{-m}w^{ml}, \\ \langle J_+\otimes J_+\rangle_{|\Omega_{kl}\rangle} &= \mathrm{Tr}\left[P_{kl}J_+\otimes J_+\right] = w^{-k}\sum_{m=0}^{2j}\eta_p^m\eta_p^{-m}w^{ml}, \\ \langle J_+\otimes J_-\rangle_{|\Omega_{kl}\rangle} &= \mathrm{Tr}\left[P_{kl}J_+\otimes J_-\right] = 0. \end{aligned}$$

Now, we have the tools that enable us to study the three-parameter family of 2-qudit states with $d = 2j + 1$ defined in Sec. 2.3.4:

$$\begin{aligned}\rho_{\alpha,\beta,\gamma} &= \frac{1-\alpha-\beta-\gamma}{(2j+1)^2}\mathbb{1} + \alpha P_{00} \\ &+ \frac{\beta}{2j}\sum_{i=1}^{2j} P_{i0} + \frac{\gamma}{2j+1}\sum_{i=0}^{2j} P_{i1}.\end{aligned} \tag{A.29}$$

For these states we have

$$\begin{aligned}\langle J_3 \otimes J_3\rangle_{\rho_{\alpha,\beta,\gamma}} &= \frac{j(j+1)\,(\alpha+\beta) + j(j-2)\gamma}{3}, \\ \langle J_1 \otimes J_1\rangle_{\rho_{\alpha,\beta,\gamma}} &= \frac{j(j+1)\,(\alpha-\beta/2j)}{3}, \\ \langle J_2 \otimes J_2\rangle_{\rho_{\alpha,\beta,\gamma}} &= -\frac{j(j+1)\,(\alpha-\beta/2j)}{3},\end{aligned}$$

so that the criterion

$$\begin{aligned}\left|\langle J_1^{(d_1)} \otimes J_1^{(d_2)}\rangle_\rho\right| &+ \left|\langle J_2^{(d_1)} \otimes J_2^{(d_2)}\rangle_\rho\right| \\ &+ \left|\langle J_3^{(d_1)} \otimes J_3^{(d_2)}\rangle_\rho\right| \leq j^{(d_1)} j^{(d_2)}\end{aligned}$$

introduced in the main text for separable states $\rho_{\alpha,\beta,\gamma}$ can be brought to the following form:

$$\frac{|(j+1)(\alpha+\beta) + (j-2)\gamma| + (j+1)\,|2\alpha - \beta/j|}{3j} \leq 1. \tag{A.30}$$

A.6.2 Positivity and partial transpose of $\rho_{\alpha,\beta,\gamma}$

From the definition in Eq. (A.29), $\rho_{\alpha,\beta,\gamma}$ is already in its diagonal form and we can easily identify its eigenvalues and eigenvectors. The eigenvalues are: $\frac{1-\alpha-\beta-\gamma}{(2j+1)^2} + \frac{\gamma}{2j+1}$ with degeneracy $2j+1$, $\frac{1-\alpha-\beta-\gamma}{(2j+1)^2} + \frac{\beta}{2j}$ with degeneracy $2j$, $\frac{1-\alpha-\beta-\gamma}{(2j+1)^2} + \alpha$ with degeneracy 1, and $\frac{1-\alpha-\beta-\gamma}{(2j+1)^2}$ with degeneracy $(2j+1)(2j-1)$. To ensure the positivity of $\rho_{\alpha,\beta,\gamma}$, each of these eigenvalues must be positive, hence we obtain the positivity conditions stated in Sec. 2.3.4:

$$\begin{aligned}&\frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0, \qquad \alpha + \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0, \\ &\frac{\beta}{2j} + \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \geq 0, \qquad \frac{1-\alpha-\beta+2j\gamma}{(2j+1)^2} \geq 0.\end{aligned}$$

As for the partial transpose (PT) of $\rho_{\alpha,\beta,\gamma}$, it was shown in Ref. [272] that for a state of the form $\rho = \sum_{k,l=0}^{2j} C_{kl} P_{kl}$, its partial transpose can be written as:

$$\rho^{T_B} = \bigoplus_{m=0}^{2j} B_m \quad : B_m^\dagger = B_m, \tag{A.31}$$

where, the matrix elements of B_m are defined as [272]:

$$\langle s|B_m|t\rangle = \frac{1}{2j+1} \sum_{k=0}^{2j} C_{s,s+t-m} w^{k(s-t)} : s,t = 0,\cdots,2j. \tag{A.32}$$

It was also shown [272] that for integer values of j, all matrices B_m are unitarily equivalent, while for j half-integer, there are two classes of unitarily equivalent matrices, the class of B_m for even m and the class for odd m.

In what follows, we will restrict ourselves to the case of j integer and we will calculate the elements of the matrix B_0. The case of half-integer j can be easily obtained based on the calculation of the matrix B_0.

From Eqs. (2.74) and (A.32), we find that the matrix B_0 has the following form:

$$B_0 = \begin{pmatrix} b_{00} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & b_{11} & 0 & \cdots & 0 & \kappa \\ \vdots & 0 & \ddots & & \iddots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \iddots & & b_{2j-1,2j-1} & 0 \\ 0 & \kappa & 0 & \cdots & 0 & b_{2j,2j} \end{pmatrix}, \tag{A.33}$$

with

$$\begin{aligned} b_{00} = \langle 0|B_0|0\rangle &= \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} + \frac{\alpha+\beta}{2j+1}, \\ b_{j+1,j+1} = \langle j+1|B_0|j+1\rangle &= \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} + \frac{\gamma}{2j+1}, \\ b_{k,k} = \langle k|B_0|k\rangle &= \frac{1-\alpha-\beta-\gamma}{(2j+1)^2} \\ &: k \in \{0,\cdots,2j\} \setminus \{0, j+1\}, \\ \kappa = \langle s|B_0|t\rangle &= \frac{\alpha - \beta/2j}{2j+1} \qquad : s+t = 0 \text{ and } s \neq t, \\ \langle s|B_0|t\rangle &= 0 \qquad\qquad\qquad\qquad : s \neq t \text{ and} s+t \neq 0. \end{aligned}$$

The characteristic polynomial of B_0 can be easily computed which allows to calculate the eigenvalues of B_0. As $\rho^{T_B}_{\alpha,\beta,\gamma}$ is the direct sum of matrices B_m that are unitarily equivalent to B_0, then the eigenvalues of B_0 are also those of $\rho^{T_B}_{\alpha,\beta,\gamma}$. We distinguish 2 cases: for $j = 1$ and $j > 1$. For $j = 1$, the 3 eigenvalues $\lambda_i (i = 1, 2, 3)$ are:

$$\begin{aligned}
\lambda_1 &= \frac{1 + 2j(\alpha + \beta) - \gamma}{(1 + 2j)^2}, \\
\lambda_2 &= \frac{1}{2}\left(\frac{\gamma}{1 + 2j} - \frac{2(-1 + \alpha + \beta + \gamma)}{(1 + 2j)^2}\right. \\
&\quad \left. - \sqrt{\frac{(-2j\alpha + \beta)^2 + j^2\gamma^2}{j^2(1 + 2j)^2}}\right), \\
\lambda_3 &= \frac{1}{2}\left(\frac{\gamma}{1 + 2j} - \frac{2(-1 + \alpha + \beta + \gamma)}{(1 + 2j)^2} + \right. \\
&\quad \left. \sqrt{\frac{(-2j\alpha + \beta)^2 + j^2\gamma^2}{j^2(1 + 2j)^2}}\right),
\end{aligned}$$

with degeneracy 3 each. Whereas for $j > 1$, the 4 eigenvalues $\lambda^+_i (i = 1, 2, \cdots, 5)$ are:

$$\begin{aligned}
\lambda^+_1 &= \frac{1 + 2j(\alpha + \beta) - \gamma}{(1 + 2j)^2}, \\
\lambda^+_2 &= \frac{-\beta - 2j(-1 - 2j\alpha + 2\beta + \gamma)}{2j(1 + 2j)^2}, \\
\lambda^+_3 &= \frac{\beta - 2j(-1 + 2(1 + j)\alpha + \gamma)}{2j(1 + 2j)^2}, \\
\lambda^+_4 &= \frac{1}{2}\left(\frac{\gamma}{1 + 2j} - \frac{2(-1 + \alpha + \beta + \gamma)}{(1 + 2j)^2} - \sqrt{\frac{(-2j\alpha + \beta)^2 + j^2\gamma^2}{j^2(1 + 2j)^2}}\right), \\
\lambda^+_5 &= \frac{1}{2}\left(\frac{\gamma}{1 + 2j} - \frac{2(-1 + \alpha + \beta + \gamma)}{(1 + 2j)^2} + \sqrt{\frac{(-2j\alpha + \beta)^2 + j^2\gamma^2}{j^2(1 + 2j)^2}}\right)
\end{aligned}$$

with degeneracy $2j+1$, $(2j+1)(j-1)$, $(2j+1)(j-1)$, $(2j+1)$, and $(2j+1)$ correspondingly. Thus by imposing positivity on the above eigenvalues, we get the set of conditions for the state $\rho_{\alpha,\beta,\gamma}$ to be PPT. This is how we compute the blue region on Fig. 2.1 (a) and (b).

Appendix B

Appendices of chapter 3

B.1 Proof of Eq. (3.6)

In this appendix, we present in detail the different steps to derive our main inequality (3.6). As mentioned in the main text, we proceed to the proof in two steps. Firstly, we start by proving the inequality (3.6) for product states. Next, we generalize the inequality for mixed state by convexity argument. Before proceeding to the two parts of the proof, let us rewrite the inequality (3.6) and its different ingredients:

$$\left(\langle \hat{N} \rangle - 1\right) \sum_{k \notin \mathcal{I}} \tilde{\Delta}^2 A_k - \sum_{k \in \mathcal{I}} \langle \tilde{A}_k^2 \rangle \geq -\langle \hat{N} \rangle \left(\langle \hat{N} \rangle - 1\right) \alpha^2 - \delta, \tag{B.1}$$

with $\alpha^2 = \sup_{\rho^{(i)}} \left[\sum_{k=1}^{3} \left\langle A_k^{(i)} \right\rangle_{\rho^{(i)}}^2\right]$ and $\hat{N} = \sum_{i=1}^{N} N^{(i)}$ represents the particle number operator as explained in the main text. We choose the operator $N^{(i)}$ to be positive and to verify the following inequality:

$$\frac{\sum_{k=1}^{3} \left\langle A_k^{(i)} \right\rangle_{\rho^{(i)}}^2}{\alpha^2} \leq \left\langle N^{(i)} \right\rangle_{\rho^{(i)}}^2 \leq 1 \tag{B.2}$$

for any state $\rho^{(i)}$ acting on the single particle Hilbert space. One can always find a positive operator $N^{(i)}$ such that Eq. (B.2) is verified, since one can always choose $N^{(i)}$ to be the identity in the single particle Hilbert space. Finally, we recall the expression for δ (3.16):

$$\delta = \alpha^2 \langle \hat{N} \rangle + \tilde{\Delta}^2 A_1 + \tilde{\Delta}^2 A_2 + \tilde{\Delta}^2 A_3 \tag{B.3}$$

B.1.1 Proof of Eq. (3.6) for product states

As we have outlined in the main text, our main improvement comes from deriving a new bound for $\langle A_i \rangle^2$ better than the standard one

$$\langle A_i \rangle^2 \leq N \sum_{k=1}^{N} \left\langle A_i^{(k)} \right\rangle^2 . \tag{B.4}$$

The previous inequality can be obtained directly from Cauchy-Schwartz inequality. However, it can also be obtained in a different way using the Heisenberg uncertainty inequality as follows. First, for the sake of illustration, consider that $A_i = S_x = \sum_{i=1}^{N} \sigma_x^{(i)}$ and let $|\psi\rangle$, the product state $|\psi\rangle = \bigotimes_{i=1}^{N} |\psi^{(i)}\rangle$ of N spin-1/2. Starting from the Heisenberg uncertainty inequality:

$$\langle S_x \rangle^2 \leq (\Delta S_y)^2 (\Delta S_z)^2 , \tag{B.5}$$

we can apply a rotation $U^{(i)} = e^{i \frac{\sigma_x^{(i)} \theta_i}{2}}$ to each spin such that

$$\left\langle \sigma_y^{(i)} \right\rangle_{U^{(i)}|\psi^{(i)}\rangle} = 0, \left\langle \sigma_x^{(i)} \right\rangle_{U^{(i)}|\psi^{(i)}\rangle} = \left\langle \sigma_x^{(i)} \right\rangle_{|\psi^{(i)}\rangle} .$$

Since $|\psi\rangle$ is a pure product state, we have:

$$\left\langle \sigma_z^{(i)} \right\rangle^2_{U^{(i)}|\psi^{(i)}\rangle} = 1 - \left\langle \sigma_x^{(i)} \right\rangle^2_{|\psi^{(i)}\rangle} .$$

Then, a straightforward calculation, in the rotated state, would yield $(\Delta S_y)^2 = N$ and $(\Delta S_z)^2 = \sum_{i=1}^{N} \left\langle \sigma_x^{(i)} \right\rangle^2_{|\psi^{(i)}\rangle}$, hence:

$$\langle S_x \rangle^2_{|\psi\rangle} \leq N \sum_{i=1}^{N} \left\langle \sigma_x^{(i)} \right\rangle^2_{|\psi^{(i)}\rangle} \tag{B.6}$$

which is the same inequality than Eq. (B.4).

It is the above reasoning that motivates the mapping of the original product state $\rho = \bigotimes \rho^{(i)}$ to the the product state of N spin 1, $R = \bigotimes_{i=1}^{N} R^{(i)}$, where:

$$R^{(i)} = n_i |\Psi^{(i)}\rangle\langle\Psi^{(i)}| + (1 - n_i)|2^{(i)}\rangle\langle 2^{(i)}|, \tag{B.7}$$

$n_i = \left\langle N^{(i)} \right\rangle_{\rho^{(i)}}$, and $|\Psi^{(i)}\rangle$ is a pure state defined, in the subspace spanned by $|0^{(i)}\rangle, |1^{(i)}\rangle$, as:

$$|\Psi^{(i)}\rangle\langle\Psi^{(i)}| = \frac{\sigma_0{}^{(i)}}{2} + \frac{\left\langle A_1^{(i)} \right\rangle_\rho}{2\eta_i}\sigma_x{}^{(i)} + \frac{\left\langle A_2^{(i)} \right\rangle_\rho}{2\eta_i}\sigma_y{}^{(i)} + \frac{\left\langle A_3^{(i)} \right\rangle_\rho}{2\eta_i}\sigma_z \tag{B.8}$$

where $\sigma_0 = |0^{(i)}\rangle\langle 0^{(i)}| + |1^{(i)}\rangle\langle 1^{(i)}|$ and $\sigma_k^{(i)} (k = x, y, z)$ are the Pauli matrices in the same subspace. The constant η_i is chosen as $\eta_i = \sqrt{\left\langle A_1^{(i)} \right\rangle_\rho^2 + \left\langle A_2^{(i)} \right\rangle_\rho^2 + \left\langle A_3^{(i)} \right\rangle_\rho^2}$, to ensure the purity of the state $|\Psi^{(i)}\rangle$.

Inequality for $\langle A_i \rangle^2$

Following the same reasoning as above, We first apply the following unitary $|2^{(i)}\rangle\langle 2^{(i)}| + e^{-i\theta_i \sigma_x^{(i)}/2}$ to the state $R^{(i)}$ Eq. (B.7). After applying the unitary, we get the following state:

$$r^{(i)} = n_i |\Phi^{(i)}\rangle\langle\Phi^{(i)}| + (1 - n_i)\, |2^{(i)}\rangle\langle 2^{(i)}|, \tag{B.9}$$

where $|\Phi^{(i)}\rangle = e^{-i\theta_i \sigma_x^{(i)}/2} |\Psi^{(i)}\rangle$. We choose θ_i such that:

$$\begin{aligned}\langle\Phi^{(i)}|\sigma_y^{(i)}|\Phi^{(i)}\rangle \equiv \cos\theta_i \langle\Psi^{(i)}|\sigma_y^{(i)}|\Psi^{(i)}\rangle \\ - \sin\theta_i \langle\Psi^{(i)}|\sigma_z^{(i)}|\Psi^{(i)}\rangle = 0\end{aligned} \tag{B.10}$$

which can be achieved with the choice

$$\cos\theta_i = \frac{\langle\Psi^{(i)}|\sigma_z^{(i)}|\Psi^{(i)}\rangle}{\sqrt{\langle\Psi^{(i)}|\sigma_z^{(i)}|\Psi^{(i)}\rangle^2 + \langle\Psi^{(i)}|\sigma_y^{(i)}|\Psi^{(i)}\rangle^2}},$$

$$\sin\theta_i = \frac{\langle\Psi^{(i)}|\sigma_y^{(i)}|\Psi^{(i)}\rangle}{\sqrt{\langle\Psi^{(i)}|\sigma_z^{(i)}|\Psi^{(i)}\rangle^2 + \langle\Psi^{(i)}|\sigma_y^{(i)}|\Psi^{(i)}\rangle^2}}$$

Since $\left[\sigma_x^{(i)}, e^{-i\theta_i \sigma_x^{(i)}/2}\right] = 0$, we have:

$$\langle\Phi^{(i)}|\sigma_x^{(i)}|\Phi^{(i)}\rangle = \langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle. \tag{B.11}$$

Now, because $\frac{\eta_i}{\alpha n_i} \leq 1$ (B.2), there exists an angle ξ_i such that:

$$\langle\Phi^{(i)}|e^{i\xi_i \sigma_y^{(i)}/2} \sigma_x^{(i)} e^{-i\xi_i \sigma_y^{(i)}/2}|\Phi^{(i)}\rangle = \frac{\eta_i}{\alpha n_i}\langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle. \tag{B.12}$$

Applying the unitary $|2^{(i)}\rangle\langle 2^{(i)}| + e^{-i\xi_i\sigma_y^{(i)}/2}$ to the state $r^{(i)}$, it becomes:

$$R'^{(i)} = n_i|\Psi'^{(i)}\rangle\langle\Psi'^{(i)}| + (1-n_i)\,|2^{(i)}\rangle\langle 2^{(i)}|, \tag{B.13}$$

where $|\Psi'^{(i)}\rangle = e^{-i\xi_i\sigma_y^{(i)}/2}|\Phi^{(i)}\rangle$. Since $\left[\sigma_y^{(i)}, e^{-i\xi_i\sigma_y^{(i)}/2}\right] = 0$, we have:

$$\langle\Psi'^{(i)}|\sigma_y^{(i)}|\Psi'^{(i)}\rangle = \langle\Phi^{(i)}|\sigma_y^{(i)}|\Phi^{(i)}\rangle = 0. \tag{B.14}$$

And because the state $|\Psi'^{(i)}\rangle$ is pure we have:

$$\langle\Psi'^{(i)}|\sigma_x^{(i)}|\Psi'^{(i)}\rangle^2 + \langle\Psi'^{(i)}|\sigma_y^{(i)}|\Psi'^{(i)}\rangle^2 + \langle\Psi'^{(i)}|\sigma_z^{(i)}|\Psi'^{(i)}\rangle^2 = 1$$

i.e.,

$$\langle\Psi'^{(i)}|\sigma_z^{(i)}|\Psi'^{(i)}\rangle^2 = 1 - \frac{\eta_i^2}{\alpha^2 n_i^2}\langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle^2 \tag{B.15}$$

Finally we are in position to apply the Heisenberg uncertainty principle for the operators S_x, S_y, S_z in the state $R' = \bigotimes R'^{(i)}$:

$$\left|\langle S_x\rangle_{R'}\right|^2 \leq \left(\Delta S_y\right)^2\left(\Delta S_z\right)^2. \tag{B.16}$$

For product states $R' = \bigotimes_{i=1}^N R'^{(i)}$, we have:

$$\begin{aligned}
\left(\Delta S_y\right)^2 &= \sum_{i=1}^N \left(\Delta\sigma_y^{(i)}\right)^2 = \sum_{i=1}^N \left\langle\left(\sigma_y^{(i)}\right)^2\right\rangle_{R'^{(i)}} - \left\langle\sigma_y^{(i)}\right\rangle_{R'^{(i)}}^2 \\
&= \sum_{i=1}^N n_i - \sum_{i=1}^N n_i^2\langle\Psi'^{(i)}|\sigma_y^{(i)}|\Psi'^{(i)}\rangle^2 \\
&= \sum_{i=1}^N n_i.
\end{aligned} \tag{B.17}$$

The same calculation for S_z will give:

$$\begin{aligned}
\left(\Delta S_z\right)^2 &= \sum_{i=1}^N n_i - \sum_{i=1}^N n_i^2\langle\Psi'^{(i)}|\sigma_z^{(i)}|\Psi'^{(i)}\rangle^2 \\
&= \sum_{i=1}^N n_i - n_i^2 + \frac{\eta_i^2}{\alpha^2}\langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle^2.
\end{aligned} \tag{B.18}$$

We also have:

$$\langle S_x\rangle_{R'} = \sum_{i=1}^N \frac{\eta_i}{\alpha}\langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle, \tag{B.19}$$

but from Eq. (B.7), we have:

$$\langle\Psi^{(i)}|\sigma_x^{(i)}|\Psi^{(i)}\rangle = \frac{\left\langle A_x^{(i)}\right\rangle_\rho}{\eta_i}. \tag{B.20}$$

Using all the above the inequality (B.16), can be simplified to obtain the desired form:

$$\langle A_x\rangle_\rho^2 \leq \sum_{i=1}^N n_i \left(\sum_{i=1}^N \left\langle A_x^{(i)}\right\rangle_\rho^2\right) + \alpha^2 \sum_{i=1}^N n_i \left(\sum_{i=1}^N n_i - \sum_{i=1}^N n_i^2\right), \tag{B.21}$$

which is exactly the inequality Eq. (3.12), since $\left\langle \hat{N}\right\rangle = \sum_{i=1}^N n_i$.

Inequality for $\langle A_i\rangle_\rho^2 + \langle A_j\rangle_\rho^2$

One might suggest adding the two inequalities (3.12) for the quantities $\langle A_i\rangle_\rho^2$ and $\langle A_j\rangle_\rho^2$. But we can derive a tigther inequality by considering the following mapping of the form (B.7):

$$\begin{aligned} R^{(i)} = \;& n_i \left(\frac{{\sigma_0}^{(i)}}{2} + \frac{\left\langle A_x'^{(i)}\right\rangle_\rho}{2\eta_i}{\sigma_x}^{(i)} + \frac{\left\langle A_y'^{(i)}\right\rangle_\rho}{2\eta_i}{\sigma_y}^{(i)} + \frac{\left\langle A_z'^{(i)}\right\rangle_\rho}{2\eta_i}\sigma_z\right) \\ + \;& (1-n_i)\,|2^{(i)}\rangle\langle 2^{(i)}| \end{aligned} \tag{B.22}$$

where we have chosen: $\left\langle A_x'^{(i)}\right\rangle_\rho = \sqrt{\left\langle A_x^{(i)}\right\rangle_\rho^2 + \left\langle A_y^{(i)}\right\rangle_\rho^2}$, $\left\langle A_y'^{(i)}\right\rangle_\rho = 0$ and $\left\langle A_z'^{(i)}\right\rangle_\rho = \left\langle A_z^{(i)}\right\rangle_\rho$. Then we apply the inequality (3.12) for $\left\langle A_x'^{(i)}\right\rangle_\rho$ and we get:

$$\begin{aligned} \langle A_x'\rangle_\rho^2 = \;& \left(\textstyle\sum_{i=1}^N \sqrt{\left\langle A_x^{(i)}\right\rangle_\rho^2 + \left\langle A_y^{(i)}\right\rangle_\rho^2}\right)^2 \\ \leq \;& \langle N\rangle_\rho \left(\textstyle\sum_{i=1}^N \left\langle A_x^{(i)}\right\rangle_\rho^2 + \left\langle A_y^{(i)}\right\rangle_\rho^2\right) \\ & + \alpha^2 \langle N\rangle_\rho \left(\langle N\rangle_\rho - \textstyle\sum_{i=1}^N \left\langle N^{(i)}\right\rangle_\rho^2\right). \end{aligned} \tag{B.23}$$

Using Cauchy-Schwartz inequality

$$x_i x_j + y_i y_j \leq \sqrt{x_i^2 + y_i^2}\sqrt{x_j^2 + y_j^2}$$

we obtain:

$$\left(\sum_{i=1}^{N}\left\langle A_x^{(i)}\right\rangle_\rho\right)^2+\left(\sum_{i=1}^{N}\left\langle A_y^{(i)}\right\rangle_\rho\right)^2\leq\left(\sum_{i=1}^{N}\sqrt{\left\langle A_x^{(i)}\right\rangle_\rho^2+\left\langle A_y^{(i)}\right\rangle_\rho^2}\right)^2 \tag{B.24}$$

and we finally get:

$$\begin{aligned}\langle A_i\rangle_\rho^2+\langle A_j\rangle_\rho^2 \quad &\leq \langle N\rangle_\rho\left(\textstyle\sum_{i=1}^{N}\left\langle A_i^{(i)}\right\rangle_\rho^2+\left\langle A_j^{(i)}\right\rangle_\rho^2\right)\\ &+\alpha^2\langle N\rangle_\rho\left(\langle N\rangle_\rho-\textstyle\sum_{i=1}^{N}\left\langle N^{(i)}\right\rangle_\rho^2\right).\end{aligned} \tag{B.25}$$

Following the same steps, we can prove in general that:

$$\begin{aligned}\sum_{k\in\mathcal{I}}\langle A_k\rangle_\rho^2 \quad &\leq \quad \langle N\rangle_\rho\sum_{i=1}^{N}\sum_{k\in\mathcal{I}}\left\langle A_k^{(i)}\right\rangle_\rho^2\\ &+ \quad \alpha^2\langle N\rangle_\rho\left(\langle N\rangle_\rho-\sum_{i=1}^{N}\left\langle N^{(i)}\right\rangle_\rho^2\right)\end{aligned} \tag{B.26}$$

Where $\mathcal{I}$ is any subset of $\{1,2,\cdots,M\}$, and

$$\frac{\sum_{k=1}^{M}\left\langle A_k^{(i)}\right\rangle_{\rho^{(i)}}^2}{\alpha^2}\leq\left\langle n^{(i)}\right\rangle_{\rho^{(i)}}^2\leq 1 \tag{B.27}$$

$$\alpha^2=\sup_{\rho^{(i)}}\left[\sum_{k=1}^{M}\left\langle A_k^{(i)}\right\rangle_{\rho^{(i)}}^2\right]. \tag{B.28}$$

Notice that in the case of angular momentum operators $M=3$, as in the main text. With inequality Eq.(B.26), we have all the ingredients needed to derive inequality Eq.(3.6).

Proof of Eq. (3.6) for product states

Let $\mathcal{I}\subseteq\{1,\cdots,M\}$ including the empty set ϕ. We have the following equalities for product states:

$$\begin{aligned}\left\langle\tilde{A}_k^2\right\rangle \quad &\equiv \quad \left\langle A_k^2\right\rangle-\sum_i\left\langle A_k^{(i)2}\right\rangle=\sum_{i\neq j}\left\langle A_k^{(i)}A_k^{(j)}\right\rangle\\ &= \quad \sum_{i\neq j}\left\langle A_k^{(i)}\right\rangle\left\langle A_k^{(j)}\right\rangle=\langle A_k\rangle^2-\sum_{i=1}^{N}\left\langle A_k^{(i)}\right\rangle^2\end{aligned} \tag{B.29}$$

$$\tilde{\Delta}^2A_k \quad \equiv \quad \left\langle\tilde{A}_k^2\right\rangle-\langle A_k\rangle^2=-\sum_{i=1}^{N}\left\langle A_k^{(i)}\right\rangle^2. \tag{B.30}$$

From Eq. (B.29) and Eq. (B.26), we get:

$$\begin{aligned}\sum_{k\in\mathcal{I}}\left\langle \tilde{A}_k^2\right\rangle_\rho &\leq \left(\langle N\rangle_\rho - 1\right)\left(\textstyle\sum_{i=1}^N\sum_{k\in\mathcal{I}}\left\langle A_k^{(i)}\right\rangle_\rho^2\right)\\ &\quad +\alpha^2\langle N\rangle_\rho\left(\langle N\rangle_\rho - \textstyle\sum_{i=1}^N\left\langle N^{(i)}\right\rangle_\rho^2\right).\end{aligned} \tag{B.31}$$

Now we have all the ingredients to derive the desired inequality. From (B.31) and (B.30) we get:

$$\begin{aligned}(\left\langle\hat{N}\right\rangle_\rho - 1)\sum_{k\notin\mathcal{I}}\tilde{\Delta}^2A_k - \sum_{k\in\mathcal{I}}\left\langle\tilde{A}_k^2\right\rangle &\geq -\alpha^2\langle N\rangle_\rho\left(\langle N\rangle_\rho - \textstyle\sum_{i=1}^N\left\langle N^{(i)}\right\rangle_\rho^2\right)\\ &\quad -\left(\left\langle\hat{N}\right\rangle_\rho - 1\right)\left(\textstyle\sum_{i=1}^N\left\langle A_x^{(i)}\right\rangle_\rho^2 + \left\langle A_y^{(i)}\right\rangle_\rho^2 + \left\langle A_z^{(i)}\right\rangle_\rho^2\right)\end{aligned} \tag{B.32}$$

The above inequality is hard to extend for mixed states, that's why we put it in a more convenient form and we simplify it further using $\left\langle A_x^{(i)}\right\rangle_\rho^2 + \left\langle A_y^{(i)}\right\rangle_\rho^2 + \left\langle A_z^{(i)}\right\rangle_\rho^2 \leq \alpha^2\left\langle N^{(i)}\right\rangle_\rho^2$ to finally get:

$$\left\langle\hat{N}\right\rangle_\rho\sum_{k\notin\mathcal{I}}\tilde{\Delta}^2A_k + \sum_{k\in\mathcal{I}}\tilde{\Delta}^2A_k - \sum_{k\in\mathcal{I}}\left\langle\tilde{A}_k^2\right\rangle \geq -\alpha^2\langle N\rangle_\rho^2. \tag{B.33}$$

Proof of Eq. (3.6) for mixed separable states

Let us consider the most general separable state as a convex mixture of pure product states:

$$\rho = \sum_l \lambda_l\rho_l \quad : \lambda_l \geq 0,\ \sum_l\lambda_l = 1,\ \rho_l^2 = \rho_l \tag{B.34}$$

Then we have the following inequality :

$$\begin{aligned}\sum_{k\notin\mathcal{I}}\tilde{\Delta}^2A_k + \alpha^2\langle N\rangle_\rho &\underset{\text{concavity of variance}}{\geq} \textstyle\sum_l\lambda_l\left(\sum_{k\notin\mathcal{I}}\tilde{\Delta}_l^2A_k + \alpha^2\langle N\rangle_{\rho_l}\right)\\ &\underset{Eq.\ (B.33)}{\geq} \textstyle\sum_l\lambda_l\sum_{k\in\mathcal{I}}\dfrac{\langle A_k\rangle_{\rho_l}^2}{\langle N\rangle_{\rho_l}},\end{aligned} \tag{B.35}$$

where we have used the definition of the modified moments $\tilde{\Delta}^2A_k = \left\langle\tilde{A}_k^2\right\rangle - \langle A_k\rangle^2$ for the right hand side. Next, we use the convexity of the function $f(x,y) = \dfrac{x^2}{y}$ over $\mathbb{R}\times(0,\infty]$. The convexity of $f(x,y)$ can be shown by considering its Hessian matrix

$H_{i,j} = \dfrac{\partial^2 f(x_1, x_2)}{\partial x_i \partial x_j}$:

$$H = \begin{pmatrix} \dfrac{2}{y} & -\dfrac{2x}{y^2} \\ -\dfrac{2x}{y^2} & \dfrac{2x^2}{y^3} \end{pmatrix} \tag{B.36}$$

which eigenvalues are $\{0, \dfrac{2(x^2+y^2)}{y^3}\}$. Both eigenvalues being positive for any $(x, y) \in \mathbb{R} \times (0, \infty]$, we can conclude $f(x, y) = \dfrac{x^2}{y}$ is convex. From the convexity of $f(x, y)$, we obtain a lower bound of Eq. (B.35):

$$\sum_l \lambda_l \sum_{k\in\mathcal{I}} \frac{\langle A_k\rangle^2_{\rho_l}}{\langle N\rangle_{\rho_l}} \geq \sum_{k\in\mathcal{I}} \frac{\langle A_k\rangle^2_{\rho}}{\langle N\rangle_{\rho}} = \frac{\sum_{k\in\mathcal{I}} \left\langle \tilde{A}_k^2 \right\rangle_\rho - \sum_{k\in\mathcal{I}} \tilde{\Delta}^2 A_k}{\langle N\rangle_\rho} \tag{B.37}$$

completing the proof of inequality Eq. (3.6) for any separable state.

B.2 Mapping to a qutrit Eq (3.9)

Here we will consider only a special case of mappings to a qutrit, where the image is a mixed state of a spin-1/2 like state, in the subspace $|0\rangle, |1\rangle$, and the state $|2\rangle$ as in Eq (3.9). The starting point is the mapping that maps every spin-j state to the following spin-1/2 state:

$$\mathcal{M}(\rho) = \frac{\mathbb{1}}{2} + \frac{\langle A_x\rangle_\rho}{2\eta}\sigma_x + \frac{\langle A_y\rangle_\rho}{2\eta}\sigma_y + \frac{\langle A_z\rangle_\rho}{2\eta}\sigma_z, \tag{B.38}$$

where $\eta = \sqrt{\langle A_x\rangle^2_\rho + \langle A_y\rangle^2_\rho + \langle A_z\rangle^2_\rho}$. The above mapping is a completely positive mapping that can be written as [1]

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}_D}\left[U\rho \otimes |0\rangle\langle 0| U^\dagger\right], \tag{B.39}$$

where $|0\rangle$ is a reference state in the qubit subspace and U is an isometry that can always be written as:

$$U : \mathcal{H}_{2s+1} \otimes \mathcal{H}_2 \to \mathcal{H}_D \otimes \mathcal{H}_2 : U = \sum_{i=0}^{4} \mathcal{A}_i \otimes \sigma_i \,;\, \sigma_0 = \mathbb{1}, \tag{B.40}$$

where the dimension of $\mathcal{H}_D$ satisfies $D \geq 2s+1$. Since the three Gell-Mann matrices $\Lambda^s_{1,2}$, $\Lambda^a_{1,2}$ and Λ_1 [273] are the Pauli operators in the subspace $\{|0\rangle, |1\rangle\}$ of the qutrit, we

can define the following mapping:

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}_D}\left[\mathcal{U}\rho \otimes (\beta|0\rangle\langle 0| + (1-\beta)|2\rangle\langle 2|)\,\mathcal{U}^\dagger\right], \tag{B.41}$$

where

$$\mathcal{U} : \mathcal{H}_{2s+1} \otimes \mathcal{H}_3 \to \mathcal{H}_D \otimes \mathcal{H}_3 : \mathcal{U} = \sum_{i=0}^{4} \mathcal{A}_i \otimes \sigma_i + \mathcal{I} \otimes |2\rangle\langle 2|, \tag{B.42}$$

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1|\,,\ \sigma_1 = \Lambda^s_{1,2}\,,\ \sigma_2 = \Lambda^a_{1,2}\,,\ \sigma_3 = \Lambda_1$$

$$\mathcal{I} : \mathcal{H}_{2s+1} \to \mathcal{H}_D :\ \mathcal{I}^\dagger\mathcal{I} = \mathbb{1}_{2s+1} \tag{B.43}$$

an arbitrary isometry from $\mathcal{H}_{2s+1}$ to $\mathcal{H}_D$, and $0 < \beta < 1$ is some positive number which, in the mapping of interest (3.9), was set to be $\beta_i = n_i = \left\langle N^{(i)} \right\rangle_\rho$. One can easily verify that $\mathcal{U}$ is an isometry, i.e., $\mathcal{U}^\dagger\mathcal{U} = \mathbb{1}_{2s+1} \otimes \mathbb{1}_3$ and that the resulting mapping can be written as:

$$\begin{aligned}\mathcal{M}(\rho) = \beta\left(\frac{\sigma_0}{2} + \frac{\langle A_x\rangle_\rho}{2\eta}\sigma_x + \frac{\langle A_y\rangle_\rho}{2\eta}\sigma_y + \frac{\langle A_z\rangle_\rho}{2\eta}\sigma_z\right)\\ + (1-\beta)\,|2\rangle\langle 2|.\end{aligned} \tag{B.44}$$

Hence we have proven that the mapping (3.9) is completely positive.

B.3 Simplified squeezing criterion

Here we present a simple proof of the following inequality valid for all separable states:

$$\tilde{\Delta}^2 J_x + 4j^2\,(\Delta J_z)^2 \geq 0 \tag{B.45}$$

Let us assume that the state is pure and separable;i.e. it can be written as:$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_N\rangle$. then we have the following:

$$\left\langle \tilde{J}_k^2 \right\rangle \equiv \sum_{m\neq n} \left\langle J_k^{(m)} j_k^{(n)} \right\rangle \underset{separability}{=} \sum_{m\neq n} \left\langle J_k^{(m)} \right\rangle \left\langle J_k^{(n)} \right\rangle = \langle J_k\rangle^2 - \sum_n \left\langle J_k^{(n)} \right\rangle^2 \tag{B.46}$$

We also have:

$$(\Delta J_k)^2 = \sum_n (\Delta J_k^{(n)})^2 \tag{B.47}$$

Now, let us consider the quantity $\sum_n \left\langle J_x^{(n)} \right\rangle^2$. From Heisenberg uncertainty principle:

$$\frac{\left\langle J_x^{(n)} \right\rangle^2}{4} \leq (\Delta j_y^{(n)})^2 (\Delta j_z^{(n)})^2 \leq j^2 (\Delta j_z^{(n)})^2 \tag{B.48}$$

where I have used $(\Delta j_y^{(n)})^2 = \left\langle (J_y^{(n)})^2 \right\rangle - \left\langle J_y^{(n)} \right\rangle^2 \leq \left\langle (J_y^{(n)})^2 \right\rangle \leq j^2$. From (B.48) and (B.47), we get the upper bound:

$$\sum_n \left\langle J_x^{(n)} \right\rangle^2 \leq 4j^2 \sum_n (\Delta J_z^{(n)})^2 \leq 4j^2 (\Delta J_z)^2 \tag{B.49}$$

And for the lower bound, we minimize the function $\sum_n \left\langle J_x^{(n)} \right\rangle^2$ with the condition $\sum_n \left\langle J_x^{(n)} \right\rangle = \langle J_x \rangle$. Then, we find that $\sum_n \left\langle j_x^{(n)} \right\rangle^2$ reaches its minimal value when $\left\langle J_x^{(n)} \right\rangle = \frac{\langle J_x \rangle}{N}$ and thus we find that:

$$\sum_n \left\langle j_x^{(n)} \right\rangle^2 \geq \frac{\langle J_x \rangle^2}{N} \tag{B.50}$$

From (B.46), (B.49) and (B.50), we finally have:

$$\langle J_x \rangle^2 - 4j^2 (\Delta J_z)^2 \leq \left\langle \tilde{J}_x^2 \right\rangle \leq \frac{(N-1)}{N} \langle J_x \rangle^2 \tag{B.51}$$

where the left hand side inequality gives us the desired form

$$\tilde{\Delta}^2 J_x + 4j^2 (\Delta J_z)^2 \geq 0 \tag{B.52}$$

which can be generalized by concavity of variance to any separable state.

B.4 Generalized Sørensen-Mølmer criterion

We proceed in a similar fashion to proving the generalized spin squeezing inequalities. For product state of the form $\bigotimes_i \rho^{(i)}$, we map each state $\rho^{(i)}$ into the following qubit state:

$$|\Psi^{(i)}\rangle\langle\Psi^{(i)}| = \frac{{\sigma_0}^{(i)}}{2} + \frac{\left\langle A_1^{(i)} \right\rangle_\rho}{2\eta_i} {\sigma_1}^{(i)} + \frac{\left\langle A_2^{(i)} \right\rangle_\rho}{2\eta_i} {\sigma_2}^{(i)} + \frac{\left\langle A_3^{(i)} \right\rangle_\rho}{2\eta_i} \sigma_3 \tag{B.53}$$

with $\eta_i = \sqrt{\left\langle A_1^{(i)}\right\rangle_\rho^2 + \left\langle A_2^{(i)}\right\rangle_\rho^2 + \left\langle A_3^{(i)}\right\rangle_\rho^2}$. Next, we perform rotations around the y axis and then around the x axis to obtain the state $r^{(i)} = |\Phi^{(i)}\rangle\langle\Phi^{(i)}|$ such that:

$$\left\langle \sigma_1^{(i)}\right\rangle_{r^{(i)}} = \frac{\left\langle A_1^{(i)}\right\rangle_{\rho^{(i)}}}{\alpha\sqrt{n_i}}\,,\ \left\langle \sigma_2^{(i)}\right\rangle_{r^{(i)}} = \frac{\left\langle A_2^{(i)}\right\rangle_{\rho^{(i)}}}{\alpha n_i}, \tag{B.54}$$

where, $n_i = \left\langle n^{(i)}\right\rangle_\rho$. Finally, we recall that for product states we have:

$$\tilde{\Delta}^2 A_k = -\sum_i \left\langle A^{(i)}\right\rangle_{\rho^{(i)}}^2 . \tag{B.55}$$

Now, we can proceed to prove the inequality (3.42):

$$\left\langle \hat{N}\right\rangle + \frac{\tilde{\Delta}^2 A_k}{\alpha^2} = \sum_i n_i \Delta^2_{r^{(i)}}\sigma_1^{(i)} \geq \sum_i 2n_i F_{\frac{1}{2}}\left(\left\langle \sigma_2^{(i)}\right\rangle_{r^{(i)}}\right) \tag{B.56}$$

where we have used the relation verified by spin-$\frac{1}{2}$ operators

$$\frac{\Delta^2_{r^{(i)}}\sigma_1^{(i)}}{4} \geq \frac{1}{2}F_{\frac{1}{2}}\left(\left\langle \sigma_2^{(i)}\right\rangle_{r^{(i)}}\right).$$

Since the function $F_{\frac{1}{2}}(.)$ is convex, we find that:

$$\left\langle \hat{N}\right\rangle + \frac{\tilde{\Delta}^2 A_k}{\alpha^2} \geq 2\left\langle \hat{N}\right\rangle \sum_i \frac{n_i}{\left\langle \hat{N}\right\rangle} F_{\frac{1}{2}}\left(\frac{\left\langle A_2^{(i)}\right\rangle_{\rho^{(i)}}}{n_i\alpha}\right) \geq 2\left\langle \hat{N}\right\rangle F_{\frac{1}{2}}\left(\frac{\langle A_2\rangle}{\alpha\left\langle \hat{N}\right\rangle}\right) \tag{B.57}$$

completing the proof of (3.42) for product states. It is then straightforward to prove for mixed separable states using the concavity of the variance and convexity of the function $F_{\frac{1}{2}}(.)$.

Appendix C

Appendices of chapter 4

C.1 Proof of equation (4.14)

We start by calculating the quantities $\mathcal{A}_{m,m',0}(\rho)$ and $\mathcal{A}_{m,m',1}(\rho)$ defined as:

$$\begin{aligned}\mathcal{A}_{m,m',0}(\rho) &= \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i\rangle\langle g_i| \mathcal{K}_m \otimes B_0^i \right\rangle_\rho \\ \mathcal{A}_{m,m',1}(\rho) &= \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i\rangle\langle g_i| \mathcal{K}_m \otimes B_1^{\prime i} \right\rangle_\rho\end{aligned}$$

where we have defined :

$$B_0^i = \frac{B_{n,n'}^i + {B_{n,n'}^i}^{\dagger}}{2}\,,\; B_1^i = \frac{B_{n,n'}^i - {B_{n,n'}^i}^{\dagger}}{2i} \tag{C.1}$$

such that we have

$$\mathcal{A}_{m,m',n,n'}(\rho) = \mathcal{A}_{m,m',0}(\rho) + i\mathcal{A}_{m,m',1}(\rho)\,. \tag{C.2}$$

Next, we define the operators X_n^i and Y_n^i, for $n = 0, 1$, such that:

$$B_n^i = {X_n^i}^{\dagger} X_n^i - {Y_n^i}^{\dagger} Y_n^i, \tag{C.3}$$

where ${X_n^i}^{\dagger} X_n^i (-{Y_n^i}^{\dagger} Y_n^i)$ represents the positive(negative) part of the hermitian operator B_n^i. This allows us to introduce the following:

$$\begin{aligned}\mathcal{A}_{m,m',n,x}(\rho) &= \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i\rangle\langle g_i| \mathcal{K}_m \otimes {X_n^i}^{\dagger} X_n^i \right\rangle_\rho \\ \mathcal{A}_{m,m',n,y}(\rho) &= \left\langle \mathcal{K}_{m'}^{\dagger} \sum_i |g_i\rangle\langle g_i| \mathcal{K}_m \otimes {Y_n^i}^{\dagger} Y_n^i \right\rangle_\rho\end{aligned}$$

so that :

$$\mathcal{A}_{m,m',n}(\rho) = \mathcal{A}_{m,m',n,x}(\rho) - \mathcal{A}_{m,m',n,y}(\rho). \tag{C.4}$$

Now, we focus on the quantity $\mathcal{A}_{m,m',n,x}(\rho)$ which we rewrite in an equivalent way as:

$$\mathcal{A}_{m,m',n,x}(\rho) = \sum_i \mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)} \otimes \mathcal{X}_n^i\right) \rho \left(\mathcal{K}_{m'}^\dagger |g_i\rangle\langle g_i| \mathcal{K}_m \otimes \mathcal{X}_n^{i\,\dagger}\right)\right] \tag{C.5}$$

where we have introduced the operators

$$\mathcal{X}_n^i = \left(0, \cdots, X_n^i, \cdots, 0\right)^T \tag{C.6}$$

which are of dimension $d_2 \times D_1 d_2$. such that:

$$\mathcal{X}_n^{i\,\dagger} \mathcal{X}_n^j = \delta_{i,j} \mathcal{X}_n^{i\,\dagger} \mathcal{X}_n^i = X_n^{i\,\dagger} X_n^i \tag{C.7}$$

The above property allows us to write:

$$\mathcal{A}_{m,m',n,x}(\rho) = \sum_{i,j} \mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)} \otimes \mathcal{X}_n^i\right) \rho \left(\mathcal{K}_{m'}^\dagger |g_i\rangle\langle g_j| \mathcal{K}_m \otimes \mathcal{X}_n^{j\,\dagger}\right)\right] \tag{C.8}$$

Next, let us define the operators $\mathcal{X}_n'^i$ as:

$$\mathcal{X}_n'^{\,i} = \sum_{k=1}^{D_1} V_{i,k} \mathcal{X}_n^k \tag{C.9}$$

where V is the unitary such that:

$$|g_i\rangle = \sum_k V_{k,i} |g_k'\rangle \tag{C.10}$$

Consequently, we have the following equality:

$$\sum_i |g_i\rangle \otimes \mathcal{X}_n^{\,i} = \sum_i |g_i'\rangle \otimes \mathcal{X}_n'^i \tag{C.11}$$

which gives us:

$$\mathcal{A}_{m,m',n,x}(\rho) = \sum_{i,j} \mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)} \otimes \mathcal{X}_n'^i\right) \rho \left(\mathcal{K}_{m'}^\dagger |g_i'\rangle\langle g_j'| \mathcal{K}_m \otimes \mathcal{X}_n'^{j\,\dagger}\right)\right] \tag{C.12}$$

All left is to show that:

$$\begin{aligned}\mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)}\otimes\mathcal{X}_n'^{i}\right)\rho\left(\mathcal{K}_{m'}^{\dagger}|g_i'\rangle\langle g_j'|\mathcal{K}_m\otimes\mathcal{X}_n'^{j\dagger}\right)\right]=\\ \delta_{i,j}\mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)}\otimes\mathcal{X}_n'^{i}\right)\rho\left(\mathcal{K}_{m'}^{\dagger}|g_i'\rangle\langle g_i'|\mathcal{K}_m\otimes\mathcal{X}_n'^{i\dagger}\right)\right]\end{aligned}\tag{C.13}$$

To do so, we calculate the trace explicitly:

$$\begin{aligned}&\mathrm{Tr}\left[\left(\mathbb{1}^{(d_1)}\otimes\mathcal{X}_n'^{i}\right)\rho\left(\mathcal{K}_{m'}^{\dagger}|g_i'\rangle\langle g_j'|\mathcal{K}_m\otimes\mathcal{X}_n'^{j\dagger}\right)\right]=\\ &\sum_{i_1=1}^{d_1}\sum_{i_2=1}^{D_1}\sum_{j_1=1}^{d_2}\langle e_{i_1},\chi_{i_2,j_1}|\left(\mathbb{1}^{(d_1)}\otimes\mathcal{X}_n'^{i}\right)\rho\left(\mathcal{K}_{m'}^{\dagger}|g_i'\rangle\langle g_i'|\mathcal{K}_m\otimes\mathcal{X}_n'^{i\dagger}\right)|e_{i_1},\chi_{i_2,j_1}\rangle\end{aligned}\tag{C.14}$$

where we have defined the basis $|\chi_{j,j_1}\rangle$ as:

$$|\chi_{j,j_1}\rangle=\frac{1}{\alpha_{j,j_1}}\sum_{k=1}^{D_1}V_{k,j}|e_k\rangle\otimes\left(X_n^k\right)^{-1^{\dagger}}|f_{j_1}\rangle\tag{C.15}$$

with the basis $|e_{i_2},f_{j_1}\rangle$ chosen such that:

$$\mathcal{X}_n'^{i\,\dagger}|e_{i_2},f_{j_1}\rangle=\delta_{i,i_2}X_n^{i\,\dagger}|f_{j_1}\rangle\tag{C.16}$$

and $\left(X_n^k\right)^{-1}$ being the pseudo inverse of X_n^k. If $|f_{j_1}\rangle$ is in the kernel of X_n^k, then we simply replace $\left(X_n^k\right)^{-1^{\dagger}}|f_{j_1}\rangle$ by $|f_{j_1}\rangle$ in (C.15). Now, from eqs. (C.15), (C.16) and (C.9), we find:

$$\mathcal{X}_n'^{i\dagger}|\chi_{i_2,j_1}\rangle=\frac{1}{\alpha_{i_2,j_1}}\sum_{k=1}^{D_1}V_{k,i_2}{V_{k,i}}^{*}|f_{j_1}\rangle=\delta_{i,i_2}\mathcal{X}_n'^{i\dagger}|\chi_{i_2,j_1}\rangle\tag{C.17}$$

Plugging this relation into (C.14), we find the relation (C.13). and

$$\mathcal{X}_n'^{i\dagger}\mathcal{X}_n'^{i}=\sum_k|V_{k,i}|^2\,X_n^{k\dagger}X_n^k\tag{C.18}$$

we finally reach the conclusion that

$$\mathcal{A}_{m,m',n,x}(\rho)=\sum_i\left\langle\mathcal{K}_{m'}^{\dagger}|g_i'\rangle\langle g_i'|\mathcal{K}_m\otimes\sum_k|V_{k,i}|^2\,X_n^{k\dagger}X_n^k\right\rangle_{\rho}.\tag{C.19}$$

The same calculations can be made for $\mathcal{A}_{m,m',n,y}$ which in turn , along side with relations (C.2) and (C.4), gives us Eq. (4.14).

C.2 Proof of equation (4.40)

First, we use the property of Wigner d-mtrix $d^{(J)}_{p,q}(\beta) = (-1)^{p-q} d^{(J)}_{-p,-q}(\beta)$ and the following identity:

$$d^{(j_1)}_{m_r,m_k}(\beta) d^{(j_2)}_{m_s,m_l}(\beta) = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1\, m_r\, j_2\, m_s | J\, m_r + m_s\rangle \langle j_1\, m_k\, j_2\, m_l | J\, m_k + m_l\rangle d^{(J)}_{m_r+m_s,m_k+m_l}(\beta),$$

to write (4.38) as:

$$V(m)_{m_s,m_k} = (-1)^{m_s-m_k} \sum_J \langle j\ -m_s\, j\, m_s | J\, 0\rangle \langle j\ -m_k\, j\, m_k + m | J\, m\rangle d^J_{0,m}(\beta) \tag{C.20}$$

Using Eqs.(C.20),(4.39), we see that

$$P_{m_k,m_l} = (-1)^{m_k-m_l} \sum_{m_s,J_1,J_2} \langle j\ -m_s\, j\, m_s | J_1\, 0\rangle \langle j\ -m_s\, j\, m_s | J_2\, 0\rangle \langle j\ -m_k\, j\, m_k + m | J_1\, m\rangle \langle j\ -m_l\, j\, m_l + m | J_2\, m\rangle \frac{d^{(J_2)}_{0,m}(\beta)}{d^{(J_1)}_{0,m}(\beta)}$$

and using the identity:

$$\sum_{m_s} \langle j\, m_s\, j\ -m_s | J_1\, 0\rangle \langle j\, m_s\, j\ -m_s | J_2\, 0\rangle = \delta_{J_1,J_2},$$

the above equation simplifies into:

$$P_{m_k,m_l} = (-1)^{m_k-m_l} \sum_J \langle j\ -m_k\, j\, m_k + m | J\, m\rangle \langle j\ -m_l\, j\, m_l + m | J\, m\rangle = \delta_{m_k,m_l}$$

Where we have used the following Clebsch-Gordan identity:

$$\sum_J \langle j_1\, m_1\, j_2\, n_1 | J\, m_1 + n_1\rangle \langle j_1\, m_2\, j_2\, n_2 | J\, m_2 + n_2\rangle = \delta_{m_1,m_2}\delta_{n_1,n_2}.$$

C.3 Proof of inequality (4.56)

We start the proof for product k-producible states; i.e. states of the form:

$$\rho = \rho^{(k_1)} \otimes \rho^{(k_2)} \otimes \cdots \otimes \rho^{(k_m)} \quad : \frac{N}{k} \leq m \leq N \tag{C.21}$$

with $\rho^{(k_i)}$ being a state of $k_i \leq k$ particles such that $\sum_{i=1}^{m} k_i = N$. Next, we map each state $\rho^{(k_i)}$ into the k_i qubits state $\zeta^{(k_i)}$ where the single particle mapping is defined:

$$\mathcal{M}(\eta) = \frac{\mathbb{1}}{2} + \frac{\left\langle A_1^{(i)} \right\rangle}{2\sqrt{2}\alpha}\sigma_1 + \frac{c}{2\sqrt{2}}\sigma_2. \tag{C.22}$$

where c is a real number such that $c \leq 1$ and $\alpha^2 = \sup_{\rho^{(i)}} \left[\sum_{k=1}^{2} \left\langle A_k^{(i)} \right\rangle_{\rho^{(i)}}^2\right]$. One can check easily that the above mapping is CPTP. As we have shown in Chapter 2, complete positivity of (C.22) is equivalent to the positivity of the following matrix:

$$\begin{pmatrix} \mathbb{1}^{(i)} & \frac{A_1^{(i)}}{\sqrt{2}\alpha} + i\frac{c\mathbb{1}^{(i)}}{\sqrt{2}} \\ \frac{A_1^{(i)}}{\sqrt{2}\alpha} - i\frac{c\mathbb{1}^{(i)}}{\sqrt{2}} & \mathbb{1}^{(i)} \end{pmatrix} \tag{C.23}$$

which is equivalent to the positivity of

$$\mathbb{1}^{(i)} - \frac{\left(A_1^{(i)}\right)^2}{2\alpha^2} - \frac{c^2}{2}\mathbb{1}^{(i)}. \tag{C.24}$$

From the min-max theorem, the maximum eigenvalue of $\left(A_1^{(i)}\right)^2$,λ_M, verifies

$$\lambda_M = \sup_{\rho^{(i)}} \left[\left\langle A_1^{(i)} \right\rangle_{\rho^{(i)}}^2\right] \leq \alpha^2 \tag{C.25}$$

And since $c \leq 1$ we conclude that the matrix (C.24) is positive semi-definite and that indeed the mapping (C.22) is CPTP. Now, we choose c to be the following convenient value:

$$c^{(k_i)} = \frac{\left\langle A_2^{(k_i)} \right\rangle_{\rho^{(k_i)}}}{\alpha n_i} \tag{C.26}$$

where we have introduced $n_i = \left\langle \hat{N}^{(k_i)} \right\rangle_{\rho^{(k_i)}}$ such that $\left\langle \hat{N} \right\rangle_{\rho} = \sum_{i=1}^{m} n_i$ and we have used the fact

$$\frac{\left\langle A_l^{(k_i)} \right\rangle}{\alpha} \leq n_i \leq k_i$$

to ensure that $c^{(k_i)} \leq 1$. The resulting N-qubits' state is a k-producible state of the form:

$$\zeta = \zeta^{(k_1)} \otimes \zeta^{(k_2)} \otimes \cdots \otimes \zeta^{(k_m)} \tag{C.27}$$

Next, we consider the collective spin operators:

$$\frac{\hat{S}_k}{2} = \frac{1}{2}\sum_{i=1}^{N} \sigma_k^{(i)}. \tag{C.28}$$

From the form of the mapping (C.22), one can easily verify the following relations:

$$\left\langle \hat{S}_2^{\ (k_i)} \right\rangle_{\zeta^{(k_i)}} = k_i \frac{c^{(k_i)}}{\sqrt{2}} = \frac{k_i \left\langle A_2^{(k_i)} \right\rangle_{\rho^{(k_i)}}}{n_i \sqrt{2}\alpha}, \left\langle \tilde{\Delta}^2 \hat{S}_1^{(k_i)} \right\rangle_{\zeta^{(k_i)}} = \frac{\left\langle \tilde{\Delta}^2 A_1^{(k_i)} \right\rangle_{\rho^{(k_i)}}}{2\alpha^2} \tag{C.29}$$

Let us call $\beta = \sqrt{2}\alpha$, so that we have:

$$\frac{\tilde{\Delta}^2 A_1}{\beta^2} = \sum_i \frac{\tilde{\Delta}^2 A_l^{(k_i)}}{\beta^2} = \sum_i \tilde{\Delta}^2 \hat{S'}_1^{(k_i)} \geq \sum_i \frac{n_i}{k_i} \left(\Delta \hat{S'}_1^{(k_i)} \right)^2 - \sum_i n_i \tag{C.30}$$

where we have used $\sum_{l=1}^{k_i} \left\langle \left(\sigma_1^{(k_i)} \right)^2 \right\rangle_{\zeta^{(k_i)}} = k_i$. Thus we have,

$$\left\langle \hat{N} \right\rangle + \frac{\tilde{\Delta}^2 A_1}{\beta^2} \geq 2 \left\langle \hat{N} \right\rangle \sum_i \frac{n_i}{\left\langle \hat{N} \right\rangle} F_{\frac{k}{2}} \left(\frac{\left\langle \hat{S'}_2^{(k_i)} \right\rangle_{\zeta^{(k_i)}}}{k_i} \right) \tag{C.31}$$

where we have introduced the relation:

$$\left(\Delta \hat{S'}_1^{(k_i)} \right)^2 \geq 2k_i F_{\frac{k_i}{2}} \left(\frac{\left\langle \hat{S'}_2^{(k_i)} \right\rangle_{\zeta^{(k_i)}}}{k_i} \right) \geq 2k_i F_{\frac{k}{2}} \left(\frac{\left\langle \hat{S'}_2^{(k_i)} \right\rangle_{\zeta^{(k_i)}}}{k_i} \right) \tag{C.32}$$

At last, from the convexity of the function $F_{\frac{k}{2}}(.)$, on gets:

$$\left\langle \hat{N} \right\rangle + \frac{\tilde{\Delta}^2 A_1}{\beta^2} \geq 2 \left\langle \hat{N} \right\rangle F_{\frac{k}{2}} \left(\sum_i \frac{n_i}{k_i \left\langle \hat{N} \right\rangle} \left\langle \hat{S'}_2^{(k_i)} \right\rangle_{\zeta^{(k_i)}} \right) = 2 \left\langle \hat{N} \right\rangle F_{\frac{k}{2}} \left(\frac{\langle A_2 \rangle}{\beta \left\langle \hat{N} \right\rangle} \right) \tag{C.33}$$

completing the proof for product k-producible states. This can be easily extended to mixed k-producible states using the concavity of the variance and the convexity of $F_{\frac{k}{2}}(.)$.

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ÉCOLE DOCTORALE
Ondes et Matière (EDOM)

Titre : Intrication dans des systèmes quantiques de grandes dimensions

Mots clés : Intrication, information quantique, témoin d'intrication, états comprimés

Résumé : La détection de l'intrication est une étape indispensable dans le contexte de l'information et du calcul quantique. Cette tâche importante s'est avérée difficile pour les systèmes quantiques de grandes dimensions supérieures à 2×3, auquel cas il existe des conditions nécessaires et suffisantes bien établies.
Notre approche consiste à réduire la dimensionnalité du problème. Pour ce faire, on transforme, localement, chaque sous-système en un qubit sans créer de l'intrication. Le mapping est exprimé en fonction des valeurs moyennes de trois opérateurs arbitraires dans l'état original. Nous donnons des conditions nécessaires et suffisantes pour que cette transformation soit valide du point de vue physique.Nous exploitons ce formalisme pour dériver des critères d'intrication pour des systèmes bipartites ou multipartites sur la base des critères existants pour les qubits. En transformant localement chaque sous-système, tel que la séparabilité est conservée, l'application de critères d'intrication connus pour les qubits à l'état résultant induit automatiquement des critères d'intrication en fonction d'opérateurs utilisés pour réaliser la transformation.
Pour le cas multipartite, on s'intéresse aux critères d'intrication basés sur des mesures d'observables collectives. Après avoir transformé l'état du système en un état multipartite de qubits, on applique les inégalités de compression de spin. Cependant, lorsqu'on applique notre formalisme à ce cas, il est possible d'obtenir une superposition cohérente d'états avec des nombres de particules différentes. Par conséquent, nous avons dû prendre en compte les fluctuations quantiques et/ou classiques que l'opérateur du nombre de particules peut présenter. Nous avons obtenu une forme généralisée des inégalités de compression de spin pour un nombre de particules fluctuant et pour des observables collectifs arbitraires. Nous avons appliqué nos résultats à un système d'atomes de chrome ultrafroids piégés dans un réseau optique, en collaboration avec l'équipe Gazes Dipolaires Quantiques du laboratoire LPL de l'université Paris 13. Nous avons montré, à l'aide d'une simulation numérique, que nos inégalités sont capables de détecter l'intrication à l'aide d'opérateurs collectifs mesurables en utilisant des techniques accessibles dans ce type de dispositif.

Title : Entanglement in high dimensional quantum systems

Keywords : Entanglement, entanglement witness, spin squeezing, quantum information

Abstract : Entanglement detection is crucial and a necessity in the context of quantum information and quantum computation. This important task has proven to be quite hard for quantum systems of dimensions higher than 2×3, in which case, there exists well established necessary and sufficient conditions like Peres-Horodecki criterion.
To tackle this challenge, we introduce a mathematical framework to reduce the problem to entanglement in a two qubit system. This is done by mapping each subsystem locally into a qubit without increasing entanglement. The mapping is expressed in terms of expectation values of three arbitrary operators in the original state. We give necessary and sufficient conditions for such mapping to be valid from a physical point of view, providing thence a versatile tool for dimension reduction in various applications.
Our main use of this formalism is as a gateway to derive entanglement criteria for bipartite or multi-partite systemas based on existing ones derived for qubit systems. By mapping each subsystem locally into a qubit, such that seprability is preserved, applying entanglement criteria known for qubits on the resulting state automatically gives us entanglement criteria in terms of the chosen operators used to implement the mapping.
For the multi-partite case, we focus on spin squeezing inequalities for qubits to derive entanglement criteria for general systems based on collective observables. However, when applying our formalism to this case, an interesting situation arises where one is able to obtain coherent superposition of multi-partite qubit states with different particle number. Hence, to derive better entanglement criteria, we had to consider quantum and/or classical fluctuations that may be exhibited by the particle number operator. We derive a generalized form of spin squeezing inequalities for fluctuating particle number in terms of arbitrary collective operators. We applied our results to study entanglement in a system of ultra-cold Chromium atoms trapped in a bi-dimensional optical lattice in collaboration with the Quantum Dipolar Gazes team in LPL at Paris 13 university. We showed, in a numerical simulation, that our generalized inequalities are able to detect entanglement in their system using collective operators. Moreover, we show that such observables can be measured using available techniques.

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