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Reframing Classical Mechanics: An AKSZ Sigma Model Perspective

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Abstract: The path-integral re-formulation due to E. Gozzi, M. Regini, M. Reuter, and W. D. Thacker of Koopman and von Neumann's original operator formulation of a classical Hamiltonian system on a symplectic manifold M is identified as a gauge slice of a one-dimensional Alexandrov–Kontsevich–Schwarz–Zaboronsky sigma model with target $T^*(T[1]M \times \mathbb{R}[1])$.

Keywords: classical mechanics; koopman-von neumann formulation; classical path integral; AKSZ sigma model

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1. Introduction

In the early days of quantum mechanics, before the arrival of the path-integral formalism, Koopman and von Neumann (KvN) [1–3], motivated by improving their understanding of classical ergodicity, reformulated classical mechanics on a symplectic manifold M as quantum mechanics on T^*M with Hamiltonian linear in momenta and complex wave function on M given by the square root of the classical density function modulo phase, whose fate remains an interesting open problem in KvN mechanics. Over half a century later, an equivalent path-integral formulation was given by E. Gozzi, M. Reuter, and W. D. Thacker (GRT) [4,5].

On the other hand, the Alexandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) formalism [6] provides a natural framework for the deformation quantization of graded geometries appearing in the context of symplectic mechanics and gauge field theories. It was initially applied for the quantization of topological systems, but is also suitable to the description of dynamical systems with local degrees of freedom, provided these systems are described in terms of an exterior Cartan-integrable system; see, e.g., [7–9] for discussions.

In this paper, we will show that the AKSZ formalism enables us to reframe in a very natural way the path-integral formulation of classical mechanics starting from the operatorial formulation by Koopman and von Neumann (KvN) [1–3], and was further developed in [4,5]. For that purpose, we will be studying the dynamics of a classical system whose phase space corresponds to the symplectic manifold (M, ω) and focus on its cotangent bundle T^*M, which is also symplectic. We will further subject T^*M to a system of first-class constraints that will provide us with a natural extension of the Koopman–von Neumann (KvN) reformulation of classical mechanics on M. More precisely, as we shall see, the link between KvN and AKSZ consists of the fact that the rewriting due to a series of works by Gozzi, Reuter, and Thacker (GRT) [4,5] (and thereafter by Gozzi and Regini [10]) of the pull-back operation on M along a symplectomorphism generated by a Hamiltonian vector field during a fixed time t, by means of time slicing of the path integral over particle configurations on $T^*T[1]M$, can be recovered by gauge fixing of a one-dimensional AKSZ sigma model with the target

$$T^*\left(T[1]M \times \mathbb{R}[1]\right). \tag{1}$$

In [10], the authors worked out all the different transformations of the fields appearing in the path-integral formulation of classical mechanics, which we will briefly discuss in Section 2, to show that they form the cotangent bundle $T^*T[1]M$ of the reversed-parity tangent bundle of the phase space M. In our case, we show that this identification is quite natural from the viewpoint of a one-dimensional sigma model, opening up the possibility of generalizing the construction by replacing $\mathbb{R}$ in (1) with a general group G. Most importantly, our observation would facilitate novel ways of looking at the symmetries and conservation laws of classical mechanics from the viewpoint of the rich geometric and topological structure of the AKSZ sigma models. The possibility of a connection between KvN formalism and geometric quantization as envisaged previously in [11,12] can now be further explained from the vantage point of AKSZ sigma models.

Our plan for this paper is the following: In Section 2, we begin with a brief review of the Koopman–von Neumann (KvN) formulation [1–3] of classical mechanics, as well its path-integral reformulation due to Gozzi, Reuter, and Thacker (GRT) in [4,5,10]. Section 3 will then elaborate on the AKSZ action for the worldline of a particle. Finally, in Section 4, we show the exact connection between the AKSZ sigma model and the GRT model and conclude with a summary of our results and future outlook in Section 5.

2. KvN and Classical Path-Integral Formulation

Classical mechanics and quantum mechanics are developed on the basis of two completely different mathematical paradigms. Contrary to the geometric approach of classical mechanics, the description of quantum mechanics is more algebraic in nature[1]. A state in classical mechanics can be viewed as a point on a symplectic manifold, the phase space, which is by definition endowed with a Lie bracket, the Poisson bracket. Any individual observable can then be described as some real-valued function on this symplectic manifold, associated with a Hamiltonian vector field, generating individual flows on this manifold. For example, the flow corresponding to the Hamiltonian H would describe the time evolution of the system. On the opposite side, the algebraic language of quantum mechanics revolves around the construction of a Hilbert space. Each physical state is described by a ray in the Hilbert space, and observables are the self-adjoint linear operators defined on the Hilbert space. The Lie algebra structure appears by taking commutators between different observables; i.e., it comes through via the associative product defined by the composition of operators acting on the Hilbert space.

During the era of 1930s, several attempts were made to reconcile these two languages. Perhaps with this early motivation, Koopman and von Neumann reformulated classical mechanics to associate it with a Hilbert space of complex and square-integrable functions similar to its quantum mechanical counterparts. Analogous to quantum mechanics, one can also associate complex classical wave functions with a classical mechanical system, of course, with some caveats on which we will not focus in the current context and instead refer to [18].

Without losing any generality, let us start with an one-dimensional system with phase space density $\rho(q, p, t)$, which can be interpreted as the probability density of finding a particle at point q with momentum p exactly at time t with the measure $\int dq\, dp$. Liouville's theorem states that this density has the same property as an incompressible fluid that the phase space volume $\int \rho(q,p,t)\,dq\,dp$ remains constant and the corresponding continuity equation becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \dot{q}\frac{\partial \rho}{\partial q} + \dot{p}\frac{\partial \rho}{\partial p} = 0\,. \tag{2}$$

One can now use Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}\,, \quad \dot{p} = -\frac{\partial H}{\partial q}\,, \tag{3}$$

to show that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p}\,. \tag{4}$$

One can easily generalize the discussion to a dynamical system with n degrees of freedom in configuration space, corresponding to a phase space of dimension $2n$. Defining the Liouville operator as

$$\hat{\mathcal{L}} = -i\,\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + i\,\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{5}$$

one can rewrite (4) as

$$i\,\frac{\partial \rho}{\partial t} = \hat{\mathcal{L}}\,\rho\,. \tag{6}$$

In the following, we will use the notation $z = (z^a) = (q^1,\ldots,q^n,p_1,\ldots,p_n)$ for the dynamical variables in phase space. The basic postulates of Koopman and von Neumann formalism are the following:

(1) The existence of a complex function $\psi(z,t)$, which obeys the same dynamical equation as $\rho(z,t)$, i.e.,

$$i\,\frac{\partial \psi(z,t)}{\partial t} = \hat{\mathcal{L}}\,\psi(z,t)\,; \tag{7}$$

(2) $\psi(z,t)$ is L^2 normalizable; i.e., its norm with respect to the following scalar product is finite

$$\langle \psi|\varphi\rangle_t = \int d^{2n}z\ \psi(z,t)^*\varphi(z,t)\,. \tag{8}$$

Equation (7) can then be thought of as the analogue of Schrödinger's equation in quantum mechanics. The Hilbert space spanned by the functions $\psi(z,t)$ can then be considered as the Hilbert space for classical mechanics. The postulate of the scalar product ensures a proper definition of the Hilbert space and imposes the norm squared of the states to be

$$\langle \psi|\psi\rangle = \int d^{2n}z\ \psi(z,t)^*\psi(z,t)\,. \tag{9}$$

With this definition of scalar product, one can further show that $\hat{\mathcal{L}}$ is a Hermitian operator[2] such that

$$\langle \psi|\hat{\mathcal{L}}\,\varphi\rangle = \langle \hat{\mathcal{L}}\,\psi|\varphi\rangle\,. \tag{10}$$

The Hermitian character of $\hat{\mathcal{L}}$ ensures that $\langle \psi|\psi\rangle$ remains conserved during the evolution. Therefore, one can now consistently interpret

$$\psi^*(z,t)\psi(z,t) = \rho(z,t) \tag{11}$$

as the density probability function, and note that the Liouville theorem (6) can be derived starting from the postulate (7) of KvN mechanics itself. As evident from (6), although the classical wave function $\psi(z,t)$ is complex, the evolution of its phase is completely independent from its modulus, unlike the situation in quantum mechanics. We will keep the implications of this observation and further comparisons of KvN mechanics and quantum mechanics for the excellent review in [18], and move on to the path-integral approach. More details of the following discussion can be found in [10,18].

In [19], the author prescribes a simple way to introduce a path-integral formulation for classical mechanics. Unlike quantum mechanics, where each path is weighted by a probability $\exp(\frac{i}{\hbar}S)$ with S being the action of the path considered, in classical mechanics, only the classical path between two fixed end points is allowed to have weight 1, while all the others are weighted to zero. Nevertheless, Hamilton's variational principle considers all these virtual paths as well, only one being realized in the classical world as the one that extremizes the action. One can think of the classical analogue of the propagator, i.e., the probability of finding a classical particle at a point z in phase space at some time t, if it was initially at the point z_0 at the time t_0, as follows:

$$K(z,t|z_0^a,t_0) = \delta^{2n}\left(z^a - z_{cl}^a(t;z_0,t_0)\right), \tag{12}$$

where, by $z_{cl}^a(t;z_0,t_0)$, we denote the classical solution of the Hamiltonian equations of motion

$$\dot{z}^a = \pi^{ab}\partial_b H, \tag{13}$$

given the initial condition $z|_{t=t_0} = z_0^a$, where $\pi = (\pi^{ab})$ is the inverse of the symplectic matrix.

Slicing up the time interval $[t,t_0]$ into $N+1$ equal intervals δt, and denoting the time in each interval as t_i with $z_i = z(t_i)$ and $t_{N+1} = t$, one can write the delta distribution in (12) as

$$\delta^{2n}\left(z^a - z_{cl}^a(t;z_0,t_0)\right) = \left(\prod_{i=1}^{N}\int dz_i\right)\delta^{2n}\left(z_{N+1} - z_{cl}(t_{N+1};z_N,t_N)\right)\ldots\,\delta^{2n}\left(z_1 - z_{cl}(t_1;z_0,t_0)\right). \tag{14}$$

Using (13) and having in mind the limit where $N\to\infty$, so that the interval $\delta t = t_{i+1} - t_i$ goes to zero, each of the delta distributions above ($j \in \{0,1,\ldots,N\}$) can be rewritten as

$$\delta^{2n}\left(z_{j+1} - z_{cl}(t_{j+1};z_j,t_j)\right) = \prod_{a=1}^{2n}\delta\left(\dot{z}^a - \pi^{ab}\partial_b H\right)|_{t=t_j}\ \det[\partial_t\delta_b^a - \pi^{ac}\partial_c\partial_b H]|_{t=t_j}, \tag{15}$$

where we have made use of the standard formula $\delta^{2n}(z^a - z_*^a) = \delta^{2n}(g^a(z))\left|\det(\frac{\partial g^a}{\delta z^b})\right|$, where $g^a(z) = \delta t\left(\frac{z_{i+1}^a - z_i^a}{\delta t} - \pi^{ac}\partial_c H(z_i)\right)$. At this level of formality, the absolute value of the determinant is dropped. Collecting all these definitions together and taking the $N\to\infty$ limit[3], one can rewrite

$$\delta^{2n}\left(z^a - z_{cl}^a(t;z_0,t_0)\right) = \int_{z_0}^{z}\mathcal{D}\mathcal{Z}\ \tilde{\delta}(\dot{z}^a - \pi^{ab}\partial_b H)\det(\partial_t\delta_b^a - \pi^{ac}\partial_c\partial_b H), \tag{16}$$

in a form of path integral in phase space, where the symbol $\tilde{\delta}$ indicates a functional definition for the product of the infinite number of delta function coming from (14) in the limit $N\to\infty$.

One can then exponentiate both factors under the path integral in (16) by introducing $2n$ variables λ_a and a total of $4n$ anti-commuting variables $(\bar{c}_a, c^a)$ through the simple relations

$$\begin{aligned}
\tilde{\delta}(\dot{z}^a - \pi^{ab}\partial_b H) &= \int\mathcal{D}\lambda\ \exp\left[i\int_{t_0}^{t}\mathrm{d}t'\,\lambda_a(t')\left(\dot{z}^a - \pi^{ab}\partial_b H\right)\right],\\
\det(\partial_t\delta_b^a - \pi^{ac}\partial_c\partial_b H) &= \int\mathcal{D}c\mathcal{D}\bar{c}\exp\left[\int_{t_0}^{t}dt'\bar{c}_a(t')(\partial_{t'}\delta_b^a - \pi^{ac}\partial_c\partial_b H)c^b(t')\right].
\end{aligned} \tag{17}$$

The final result is that the propagator in classical mechanics can be represented as the path integral

$$K(z,t|z_0,t_0) = \int_{z_0}^{z} \mathcal{D}\mathcal{Z}\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\bar{c}\, \exp\left[i\int_{t_0}^{t} dt'\, \mathcal{L}\right], \tag{18}$$

with the Lagrangian $\mathcal{L}$ being

$$\mathcal{L} = \lambda_a \dot{z}^a + i\bar{c}_a \dot{c}^a - \lambda_a \pi^{ab}\partial_b H - i\bar{c}_a \pi^{ad}(\partial_d\partial_b H)c^b\,. \tag{19}$$

The first two terms provide one with a symplectic structure. The rest give the extended Hamiltonian $\mathcal{H}$ as [10],

$$\mathcal{H} = \lambda_a \pi^{ab}\partial_b H + i\bar{c}_a \pi^{ad}(\partial_d\partial_b H)c^b. \tag{20}$$

Hence, starting from the original $2n$ dimensional phase space with coordinates z^a, one arrives at an $8n$ dimensional extended phase space with coordinates $(z^a, \lambda_a, c^a, \bar{c}_a)$, where each of the paths in the path-integral formulation is weighted by a factor of $\exp[\,i\tilde{S}\,] = \exp[i\int dt\,\mathcal{L}]$, which, by construction, reproduces all the standard results of classical mechanics. In the series of works pioneered by E. Gozzi in [19], the authors have explicitly searched for the geometric meaning of this $8n$ dimensional space, which at this point seems like an abstraction over the usual notions of the symplectic formulation of classical mechanics. In the following sections, we will show how this apparent abstraction of the extended $8n$ dimensional phase space can be understood through a one-dimensional AKSZ sigma model. Together with the equations of motion derived from $\mathcal{L}$

$$\begin{aligned}
\dot{z}^a &= \pi^{ab}\partial_b H, \\
\dot{c}^a &= \pi^{ac}\partial_c\partial_b H\, c^b, \\
\dot{\bar{c}}_b &= -\bar{c}_a\, \pi^{ac}\partial_c\partial_b H, \\
\dot{\lambda}_b &= -\pi^{ac}\partial_c\partial_b H\, \lambda_a - i\,\bar{c}_a\, \pi^{ac}\partial_c\partial_d\partial_b H\, c^d,
\end{aligned} \tag{21}$$

and the transformations of each of these new fields under symplectic diffeomorphisms of z^a, the authors in [10,18] correctly concluded that the phase space spanned by the $8n$ variables $(z^a, \lambda_a, c^a, \bar{c}_a)$ is $T^*T[1]M$, where M is the symplectic manifold coordinatized by the original $2n$ variables z^a. We remark that, here and in the rest of the paper, we work in Darboux coordinates, only allowing for canonical transformations instead of all the possible diffeomorphisms of M.

3. Worldline Model: AKSZ Approach

In this section, we show that the identification of a phase space $T^*T[1]M$ automatically follows from the AKSZ treatment of a classical particle.

3.1. Constraints from Lie Algebra Actions

Instead of considering an unconstrained particle evolving though an Hamilton flow, as we did in the previous section, here, we consider a constrained particle whose phase space corresponds to a symplectic manifold $(\mathscr{M}, \omega)$, subject to a system of first-class constraints T_I, which define a representation of a Lie algebra $\mathfrak{g}$ on the algebra of functions $\mathscr{C}^\infty(\mathscr{M})$. In other words, we assume that the constraints T_I assemble into an equivariant moment map $T : \mathscr{M} \longrightarrow \mathfrak{g}^*$, meaning that the functions T_I are obtained as $T_I(x) := \langle T(x), \mathrm{t}_I \rangle_{\mathfrak{g}}$ for $x \in \mathscr{M}$ and $\{\mathrm{t}_I\}$ a basis of $\mathfrak{g}$, and where $\langle -, - \rangle_{\mathfrak{g}}$ denotes the canonical pairing between $\mathfrak{g}$ and its linear dual $\mathfrak{g}^*$. Such a system can be encoded in the degree 0 symplectic $\mathcal{Q}$-manifold $\mathscr{M} \times T^*\mathfrak{g}[1] \cong \mathscr{M} \times (\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1])$, with the cohomological vector field

$$\mathcal{Q} := \{\Theta, -\} \qquad \text{where} \qquad \Theta = c^I \left(T_I - \tfrac{1}{2} f_{IJ}{}^K c^J \mathcal{P}_K\right), \tag{22}$$

and where $\{c^I\}$ are degree $+1$ coordinates on $\mathfrak{g}[1]$ and $\{\mathcal{P}_I\}$ are their momenta, i.e., degree -1 coordinates in the fiber directions of $T^*\mathfrak{g}[1]$, and $\{-,-\}$ is the sum of the Poisson brackets on $\mathscr{M}$ and the canonical Poisson bracket on $T^*\mathfrak{g}[1]$. The algebra of functions on this graded manifold is isomorphic to the complex

$$\wedge(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathscr{C}^\infty(\mathscr{M}), \tag{23}$$

equipped with the differential

$$\mathcal{Q} = \delta_{\mathfrak{g}} + \delta, \tag{24}$$

where

$$\delta_{\mathfrak{g}} = c^I \left(\{T_I, -\} + f_{IJ}{}^K \mathcal{P}_K \tfrac{\partial}{\partial \mathcal{P}_J} - \tfrac{1}{2} f_{IJ}{}^K c^J \tfrac{\partial}{\partial c^K}\right) \tag{25}$$

is the Chevalley–Eilenberg differential on the module $\wedge\mathfrak{g} \otimes \mathscr{C}^\infty(\mathscr{M})$, and

$$\delta = T_I \tfrac{\partial}{\partial \mathcal{P}_I} \tag{26}$$

is the Koszul differential. In other words, it is nothing but the Batalin–Fradkin–Vilkovisky (BFV) [20–22] and Batalin–Vilkovisky (BV) [23,24] extensions of the Becchi–Rouet–Stora–Tyutin (BRST) [25–27] complex associated with the constrained system described by $(\mathscr{M}, \{T_I\})$. Note that the AKSZ sigma model having the BFV–BRST phase space of a constrained system was first introduced and discussed in [28]. For reviews on the BV, BFV, and BRST approaches to gauge theories, see, e.g., [29–35].

3.2. AKSZ Action

Let us assume that the symplectic form on $\mathscr{M}$ is exact, i.e., that we can write

$$\omega = \mathrm{d}\vartheta, \tag{27}$$

for some one-form $\vartheta \in \Omega^1(\mathscr{M})$. The source manifold Σ of the sigma model we shall consider is the worldline of the classical particle. One can write the AKSZ action associated with $\mathscr{M} \times T^*\mathfrak{g}[1]$ in terms of "super-maps"

$$T[1]\Sigma \longrightarrow \mathscr{M} \times T^*\mathfrak{g}[1], \tag{28}$$

whose components give rise to the "super-fields"[4]

$$\boldsymbol{x}^\mu(\tau) := x^\mu(\tau) + \theta\, \pi^{\mu\nu}(x)\, x^+_\nu(\tau) \quad \boldsymbol{e}^I(\tau) := c^I(\tau) + \theta\, e^I(\tau), \quad \boldsymbol{c}_I(\tau) := e^+_I(\tau) + \theta\, c^+_I(\tau) \tag{29}$$

where τ and θ are respectively the even and the odd coordinates on $T[1]\Sigma$ and the bivector $\pi^{\mu\nu}(x)$ is the inverse of the symplectic form $\omega_{\mu\nu}$ of $\mathscr{M}$. The classical fields, of ghost number 0, consists of a map $x : \Sigma \to \mathscr{M}$ from the worldline Σ to the target space $\mathscr{M}$ and a Lagrange multiplier $e = e^I\, \mathrm{t}_I$, which takes values in $\mathfrak{g}$. It can be thought of as an einbein, or a gauge field on the worldline Σ, whose gauge parameters give rise to the ghosts $c = c^I\, \mathrm{t}_I$ of ghost number 1 and $\mathfrak{g}$-valued. The corresponding antifields x^+, e^+ and c^+ are of ghost number -1, -1 and -2, respectively.

The AKSZ action is then given by

$$S_{\mathsf{AKSZ}}[\boldsymbol{x}, \boldsymbol{e}, \boldsymbol{c}] = \int_{T[1]\Sigma} \left(\vartheta_\mu(\boldsymbol{x})\, \boldsymbol{d}_\Sigma \boldsymbol{x}^\mu + \boldsymbol{c}_I\, \boldsymbol{d}_\Sigma \boldsymbol{e}^I - T_I(\boldsymbol{x})\, \boldsymbol{e}^I + \tfrac{1}{2} f_{IJ}{}^K \boldsymbol{e}^I \boldsymbol{e}^J \boldsymbol{c}_K\right), \tag{30}$$

where

$$\boldsymbol{d}_\Sigma = \theta \, \frac{\partial}{\partial \tau} \,, \tag{31}$$

is the homological vector field corresponding to the de Rham differential on Σ, and the integration over $T[1]\Sigma$ is defined as

$$\int_{T[1]\Sigma} (-) = \int_\Sigma \mathrm{d}\tau \int \mathrm{d}\theta (-) \,, \tag{32}$$

with the piece $\int \mathrm{d}\theta(-)$ being the Berezin integral. In the AKSZ scheme, selecting the top-form component amounts to restricting to the ghost number zero piece of the integrand. Performing the Berezin integral yields

$$S_{\mathsf{AKSZ}}[x,e,c] = \int_\Sigma \mathrm{d}\tau \left(\vartheta_\mu(x)\, \dot{x}^\mu - \langle T(x), e\rangle_{\mathfrak{g}} - x^+_\mu \, c^I \{T_I, x^\mu\} + \langle e^+, \dot{c} + [e,c]\rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c,c]\rangle_{\mathfrak{g}} \right) , \tag{33}$$

where we write τ for the worldline coordinate; the dot over a field denotes its derivative with respect to τ, e.g., $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$; and $\langle -,-\rangle_{\mathfrak{g}}$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The classical piece of this action, obtained by setting all fields of a ghost number different than zero, is simply given by

$$S_{\mathsf{cl.}}[x,e] = \int_\Sigma \vartheta_\mu(x)\, \dot{x}^\mu - e^I \, T_I(x) \,, \tag{34}$$

and is invariant (modulo a boundary term) under the gauge symmetry

$$\delta_\epsilon x^\mu = \epsilon^I \{T_I, x^\mu\} \,, \qquad \delta_\epsilon e^I = \dot{\epsilon}^I + f_{JK}{}^I \, e^J \, \epsilon^K \,, \tag{35}$$

with the gauge parameter $\epsilon \in \mathscr{C}^\infty(\Sigma, \mathfrak{g})$. The corresponding equations of motion read

$$\dot{x}^\mu = e^I \{T_I, x^\mu\} \,, \qquad\qquad T_I = 0 \,, \tag{36}$$

In plain words, the trajectories are constrained on the surface defined by the zeroes of the moment map, and evolve along the fundamental vector fields of the $\mathfrak{g}$-action.

3.3. Gauge Fixing

Let us single out a direction in $\mathfrak{g}$, denoted by the value D, and choose the gauge wherein only this component of the einbein is fixed to 1 and all others to zero. To do so, we should add the non-minimal sector to the AKSZ action,

$$S_{\mathsf{non-min.}}[b, \bar{c}^+] = \int_\Sigma \mathrm{d}\tau \, b_I \, \bar{c}^{+I} \,, \tag{37}$$

where both b_I and $\bar{c}^{+I}$ have the ghost number 0, and encode this choice of gauge through the gauge fixing fermion

$$\Psi[\bar{c}, e] = \int_\Sigma \mathrm{d}\tau \, \bar{c}_I \, (e^I - \delta^I_D) \,. \tag{38}$$

Its variation with respect to each field fixes the value of the corresponding antifield, which in our case yields

$$\bar{c}^{+I} = \frac{\delta\Psi}{\delta\bar{c}_I} = e^I - \delta^I_D \,, \qquad\qquad e^+_I = \frac{\delta\Psi}{\delta e^I} = \bar{c}_I \,, \tag{39a}$$

$$x^+_\mu = \frac{\delta\Psi}{\delta x^\mu} = 0 \qquad\qquad c^{+I} = \frac{\delta\Psi}{\delta c_I} = 0 \,, \tag{39b}$$

so that the gauge fixed action becomes

$$S_{\text{g.f.}}[x,e,c,\bar{c},b] = \int_\Sigma \mathrm{d}\tau \Big[\vartheta_\mu(x)\,\dot{x}^\mu - e^I\,T_I(x) + \bar{c}_I\,(\dot{c}^I + f_{JK}{}^I\,e^J\,c^K) + b_I\,(e^I - \delta^I_D)\Big]\,. \tag{40}$$

Upon integrating out the Lagrange multiplier b_I, one finds

$$S_{\text{g.f.}}[x,c,\bar{c}] = \int_\Sigma \mathrm{d}\tau\,\Big[\vartheta_\mu(x)\,\dot{x}^\mu - \mathcal{H}(x) + \bar{c}_I\,(\dot{c}^I + \rho_J{}^I\,c^J)\Big]\,, \tag{41}$$

where we re-named $T_D(x) \equiv \mathcal{H}(x)$ and $f_{D\,J}{}^I \equiv \rho_J{}^I$.

3.4. Generic First-Class Constraints

Suppose that the constraints T_I do not come from a moment map for some Lie algebra, but are instead generic first-class constraints,

$$\{T_I, T_J\} = C_{IJ}{}^K\,T_K\,, \tag{42}$$

where $C_{IJ}{}^K \equiv C_{IJ}{}^K(x)$ are structure *functions*; i.e., they may depend non-trivially on the phase space coordinates x^μ. In this more general case, the target space BRST charge Θ receives corrections,

$$\Theta(x,c,\mathcal{P}) = c^I\,T_I - \tfrac{1}{2}\,C_{JK}{}^I\,c^I c^J \mathcal{P}_K + \dots\,, \tag{43}$$

where the dots denote the term of higher order in ghost momenta $\mathcal{P}$. We can nevertheless write the corresponding AKSZ action

$$S_{\mathsf{AKSZ}}[x,e,c] = \int_\Sigma \mathrm{d}\tau\,\Big[\vartheta_\mu(x)\,\dot{x}^\mu + e^+_I\,\dot{c}^I - x^+_\mu\,\pi^{\mu\nu}\,\tfrac{\delta}{\delta x^\nu}\Theta(x,c,e^+) - e^I\,\tfrac{\delta}{\delta c^I}\Theta(x,c,e^+) - c^+_I\,\tfrac{\delta}{\delta e^+_I}\Theta(x,c,e^+)\Big]\,. \tag{44}$$

The same gauge fixing as in the previous paragraph can be implemented to give

$$S_{\text{g.f.}}[x,e,c,\bar{c},b] = \int_\Sigma \mathrm{d}\tau\Big[\vartheta_\mu(x)\,\dot{x}^\mu - e^I\,\tfrac{\partial}{\partial c^I}\Theta(x,c,\bar{c}) + \bar{c}_I\,\dot{c}^I + b_I\,(e^I - \delta^I_D)\Big]\,, \tag{45}$$

which can be further simplified by integrating out b_I,

$$S_{\text{g.f.}}[x,e,c,\bar{c},b] = \int_\Sigma \mathrm{d}\tau\Big[\vartheta_\mu(x)\,\dot{x}^\mu - \tfrac{\delta}{\delta c^D}\Theta(x,c,\bar{c}) + \bar{c}_I\,\dot{c}^I\Big]\,. \tag{46}$$

4. Recovering the GRT Formulation

In order to make contact with the Lagrangian (19) and the other results of [4] (Section 3) reviewed in Section 2, it appears that one must consider the cotangent bundle of the phase space of our original system, i.e., $\mathscr{M} = T^*M$, where M is the original symplectic manifold with local coordinates $(z^a)_{a=1,\dots,2n}$, as this would account for the classical fields (z^a,λ_a), where λ_a are conjugated to z^a in T^*M. In other words, we have the decomposition

$$(x^\mu)_{\mu=1,\dots,4n} = (z^a,\lambda_a)_{a=1,\dots,2n}\,. \tag{47}$$

On top of that, the symplectic potential on $\mathscr{M}$ is taken to be canonical, $\vartheta = \vartheta_\mu(x)\mathrm{d}x^\mu = \lambda_a\,\mathrm{d}z^a$, which does lead to the kinetic term $\int_\Sigma \mathrm{d}\tau\,\lambda_a\,\dot{z}^a$. To account for the interactions in the action, as a result of a gauged fixed AKSZ action as described above, we find that one should use a BFV–BRST charge of the form

$$\Theta(z,\lambda,c,\mathcal{P}) = c^D\Big(\lambda_a\,\pi^{ab}\,\partial_b H(z) + c^b\,\mathcal{P}_a\,\pi^{ac}\,\partial_c\partial_b H(z)\Big) + \dots\,, \tag{48}$$

where we recall the notation $\partial_a \equiv \frac{\partial}{\partial z^a}$, and where the dots denote terms that are independent of c^D and that ensure that $\{\Theta, \Theta\} = 0$.

This suggests that the constrained system described by the sought-for BRST charge Θ is determined by choosing the constraints

$$T_I = (T_D\,, T_a)\,, \quad \text{i.e.,} \quad I = (D\,, a)\,, \tag{49}$$

where

$$T_D = \lambda_a X^a_H(z)\,, \qquad T_a = \lambda_a\,, \qquad \text{with} \qquad X^a_H(z) = \{z^a, H\}_M = \pi^{ab}\,\partial_b H(z)\,, \tag{50}$$

which verify

$$\{T_D, T_a\}_{T^*M} = C_{Da}{}^b\, T_b\,, \qquad \text{with} \qquad C_{Da}{}^b = \partial_a X^b_H\,. \tag{51}$$

These constraints are all first class, in accordance with our working assumption, and the structure functions are nothing but the first derivatives of the components of the Hamiltonian vector field of the Hamiltonian H. A direct computation shows that the BRST charge

$$\begin{aligned}\Theta(z, \lambda, c, \mathcal{P}) &= c^D\, T_D + c^a\, T_a - C_{Da}{}^b\, c^D c^a\, \mathcal{P}_b \\ &= c^a\, \lambda_a + c^D\, \pi^{ab}\, \lambda_a\, \partial_b H(z) - c^D c^b\, \pi^{ac}\, \partial_b \partial_c H\, \mathcal{P}_a\,,\end{aligned} \tag{52}$$

with the previously defined constraints and structure functions, does indeed satisfy $\{\Theta, \Theta\} = 0$.

Therefore, we have shown that the action of [4] (Section 3) reproduced in the exponential in Equation (18) is recovered from a gauge fixed AKSZ model in one dimension, whose target space is associated with the system of first-class constraints $\{T_D\,, T_a\}$ on T^*M (namely, the Lagrangian (19) is recovered by plugging (52) in the gauge-fixed action (46) for a first-class constrained system). This is the main result of the present paper.

Let us discuss these constraints. The easiest ones are $T_a = \lambda_a$, which identify the constraint surface as a submanifold of the phase space T^*M. In fact, T_D does not specify further the constraint surface, as it vanishes already on $M \equiv \{(z^a, \lambda_a = 0)\} \subset T^*M$. Recall however that in the presence of first-class constraints, one is interested in the reduced phase space, that is, the quotient of the constraint surface by the action of the distribution generated by the first-class constraints. This is where T_D becomes relevant for us, as quotienting M by its action yields the set of classical trajectories (the flows generated by the Hamiltonian H) as the reduced phase space of our model.

Constraints from the Shifted Tangent Bundle

Let us re-derive this constrained system from a different perspective. Suppose we are given a symplectic manifold M and an Hamiltonian $H \in \mathscr{C}^\infty(M)$. The latter defines an action on M of $\mathbb{R}$, which viewed a Lie group with addition as its multiplication rule, whose fundamental vector field is thus the associated Hamiltonian vector field $X_H = \{H, -\}$. The integral curves of this vector field, which are nothing but the classical trajectories of this mechanical system, correspond to the orbits of $\mathbb{R}$ on M. Therefore, the set of classical solutions can be identified with the quotient $M/\mathbb{R}$, the set of the aforementioned orbits.

In order to recover the space of classical solutions from a one-dimensional AKSZ sigma model, to be identified with the previous model, we should find a BFV description of this space, i.e., identify the symplectic $\mathcal{Q}$-manifold of degree 0 encoding the space of classical trajectories as the result of a coisotropic Weinstein reduction. In other words, let us look for a constrained system, with only first-class constraints, such that the orbits of the gauge symmetry generated by the latter on the constraint surface is isomorphic to the set of classical trajectories.

We have recalled that the space of classical solutions can be thought of as the set of orbits of the $\mathbb{R}$-action generated by the Hamiltonian H, on the phase space M. Therefore, we should find a way to recover the latter as a constraint surface, in another symplectic manifold. One simple manner to do so is to consider the cotangent bundle T^*M, with coordinates (z^a, λ_b), where z^a are coordinates on M and λ_a the associated momenta, i.e., the coordinates along the fibers of T^*M. The original manifold M can then be recovered as a constraint surface defined by

$$M = \big\{(z^a, \lambda_a) \in T^*M \mid \lambda_a = 0\big\}\,, \tag{53}$$

or more geometrically, by identifying M as the zero section

$$\zeta:\; M \longhookrightarrow T^*M\,, \tag{54}$$

of its cotangent bundle $T^*M \twoheadrightarrow M$. We can lift the $\mathbb{R}$ action to T^*M, where it becomes Hamiltonian, generated by the cotangent lift of X_H. To summarize, this reasoning leads us to considering the same system of first-class constraints as we proposed before, that is,

$$T_a := \lambda_a\,, \qquad T_D = X_H^a \lambda_a = \pi^{ab}\,\lambda_a\,\partial_b H\,. \tag{55}$$

The first ones, T_a, identify M as the constraint surface in T^*M, while the last one, T_D, corresponds to the $\mathbb{R}$-action lifted to T^*M.

At this point, we can make two observations. First, the Hamiltonian vector fields associated with T_a obviously form an integrable distribution, as they span the tangent bundle of M at any point, and hence the Lie algebroid associated with it is simply TM. Second, the $\mathbb{R}$-action also defines a Lie algebroid, as any action of a Lie algebra on a manifold does,[5] denoted $M \rtimes \mathbb{R}$, whose underlying vector bundle is the trivial one, $M \times \mathbb{R}$, and which has only a non-trivial anchor in the guise of the fundamental vector X_H generating the action of $\mathbb{R}$. Both TM and $M \rtimes \mathbb{R}$ are Lie algebroids over M, and hence, so is their direct (or Whitney) sum, which we shall denote with

$$E := TM \rtimes \mathbb{R}\,. \tag{56}$$

Any Lie algebroid famously gives rise to a $\mathcal{Q}$-manifold [37], so in our case, $E[1]$ is a graded manifold with coordinates z^a of degree 0, corresponding to coordinates on M, and degree 1 coordinates c^a and c^D corresponding to coordinates along the fibers of TM and $M \rtimes \mathbb{R}$, respectively, giving rise to ghosts on the worldline. The cohomological vector field making $E[1]$ into a $\mathcal{Q}$-manifold reads

$$\mathcal{Q}_E = c^a\,\tfrac{\partial}{\partial z^a} + c^D\,\pi^{ab}\,\partial_b H\,\tfrac{\partial}{\partial z^a} - c^D\,c^b\,\pi^{ac}\,\partial_b\partial_c H\,\tfrac{\partial}{\partial c^a}\,, \tag{57}$$

in this coordinate system. The cotangent bundle of this $\mathcal{Q}$-manifold

$$T^*E[1] = T^*\big(T[1]M \rtimes \mathbb{R}[1]\big)\,, \tag{58}$$

defines the symplectic $\mathcal{Q}$-manifold encoding the BFV description of the classical trajectories we discussed, in accordance with the results and observations of [38]. Recalling from (22) that the momentum of c^a is $\mathcal{P}_a$, the cotangent lift of $\mathcal{Q}_E$ is given by

$$\Theta = c^a\,\lambda_a + c^D\pi^{ab}\,\lambda_a\,\partial_b H - c^D c^b \pi^{ac}\,\partial_b\partial_c H\,\mathcal{P}_a\,, \tag{59}$$

which exactly reproduces the BFV–BRST charge (52) defining the AKSZ sigma model, with target space $T^*E[1]$, and whose gauge fixing reproduces the GRT one [4], as we have shown.

5. Conclusions

This paper offers a way to rethink classical mechanics as a gauge fixed AKSZ sigma model. The way we showed this is to start from the GRT reformulation of KvN classical mechanics.

In the GRT reformulation of classical mechanics, one considers a simple classical system evolving in phase space along a Hamiltonian flow. The authors of [4,5,10] proposed a path-integral prescription for the classical system, for which the price to pay is the introduction of additional fields. They showed that, for the classical motion of the system with n degrees of freedom in configuration space, one has to introduce a total of $8n$ fields to ensure the consistency of the path integral and to reproduce the expected classical trajectories in the phase space of dimension $2n$. As observed in [10], these $8n$ variables span $T^*T[1]M$, where M is the $2n$ dimensional phase space for the system under consideration.

In this paper, we considered the worldline of a particle constrained by a set of first class constraints, and wrote the AKSZ action corresponding to that constrained particle. We showed that the gauge-fixed version of the AKSZ action, for a suitable choice of target space and constraints spelled out in Section 4, reproduces the action that dictates the GRT path-integral formulation of classical mechanics.

We then reinterpreted our AKSZ sigma model as the BFV description of the constrained system that was designed to reproduce the GRT formulation of a classical, unconstrained, dynamical system in a phase space M. The reduced phase space of this constraint system on T^*M, which consists of the set of classical trajectories of the original mechanical system encoded by M and the Hamiltonian H, is recovered by taking the quotient of M by the distribution associated with the Lie algebroid $TM \rtimes \mathbb{R}$. In particular, the last factor $\mathbb{R}$ accounts for the flow generated by the Hamiltonian H of the original system. These observations confirm our claim that a classical system, whose phase space corresponds to the symplectic manifold M, is equivalent to a gauge fixed one-dimensional AKSZ sigma model with target space $T^*(T[1]M \times \mathbb{R}[1])$.

This simple yet intriguing mapping between a classical system and a gauge-fixed AKSZ sigma model opens up interesting avenues of research. One direct application of this mapping would be to start with a constrained classical system, and look for its AKSZ counterpart. In particular, one could consider the case of first-class constraints generated by the action of a Lie group G on M, whose BFV–BRST description leads to an AKSZ model with target space $T^*(M \rtimes \mathfrak{g}[1])$, where $\mathfrak{g}$ is the Lie algebra of G. In light of the previous treatment of an unconstrained classical system, one could expect that the relevant target space be of the form $T^*\big(T[1]C \rtimes (\mathbb{R}[1] \oplus \mathfrak{g}[1])\big)$, where C is the constraint surface defined by the first-class constraints.

As another direction of research, one can study higher-dimensional sigma models and look for an effective classical mechanical system equivalent to it. Another interesting avenue would be to understand the connection between geometric quantization (see [13–15] for original references, and, e.g., [16,17]) and KvN mechanics, as shown by [11,12] in more detail now in the light of AKSZ sigma models.

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Notes

1. Note that geometric quantization stems in part from noticing this gap of approach, and aims at bridging it [11–17].
2. Assuming $\psi(z,t)$ to behave in such a way that $\psi(z,t)|_{z\to\infty} = 0$.
3. The normalization factor arising due to this limit can be absorbed into the path-integral measure $\mathcal{DZ}$.
4. By "super-maps" we mean maps between graded manifolds that *do not* preserve the degree. This is instrumental in recovering the set of fields, ghosts, and antifields in the AKSZ formalism, see, e.g., [36] (Section 4.9.1).
5. Indeed, given a Lie algebra $\mathfrak{g}$ that acts on a smooth manifold M via $\rho : \mathfrak{g} \longrightarrow \Gamma(TM)$, one can endow the trivial bundle $M \times \mathfrak{g} \twoheadrightarrow M$ with a structure of Lie algebroid as follows. First, notice that sections of this trivial bundle are nothing but $\mathfrak{g}$-valued functions on M, i.e., $\Gamma(M \times \mathfrak{g}) \cong \mathscr{C}^\infty(M) \otimes \mathfrak{g}$. The anchor of the Lie algebroid is therefore simply given by the $\mathscr{C}^\infty(M)$-linear extension of the $\mathfrak{g}$-action ρ. Explicitly, for a section $\psi \in \Gamma(M \times \mathfrak{g})$ written as $\psi = \psi^a(x)\,\mathrm{t}_a$ with $\{\mathrm{t}_a\}$ a basis of $\mathfrak{g}$, one defines the action of the anchor on it via
$$\rho_\psi := \psi^a(x)\,\rho_{\mathrm{t}_a} = \psi^a(x)\,\rho_a{}^\mu(x)\partial_\mu\,.$$
The Lie bracket is defined as
$$[\psi_1,\psi_2]_{M\times\mathfrak{g}} = \big(\psi_1^a\rho_a{}^\mu\partial_\mu\psi_2^c - \psi_2^a\rho_a{}^\mu\partial_\mu\psi_1^c + \psi_1^a\psi_2^b f_{ab}{}^c\big)\mathrm{t}_c\,,$$
which can be thought of as a "twist" of the Lie bracket of $\mathfrak{g}$ by the action of the latter on $\mathscr{C}^\infty(M)$.

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