IOP Publishing Phys. Scr. 99 (2024) 055209 https://doi.org/10.1088/1402-4896/ad35f1

Physica Scripta

PAPER

RECEIVED
11 August 2023

REVISED
1 March 2024

ACCEPTED FOR PUBLICATION
20 March 2024

PUBLISHED
2 April 2024

Bound states of relativistic spinless particles in a mix of circularly symmetric vector and scalar harmonic oscillators

V B Mendrot[1], A S de Castro[1] and P Alberto[2,*]

[1] Departamento de Física, Unesp—Universidade Estadual Paulista, Guaratinguetá, Brazil
[2] CFisUC, Departamento de Física, Universidade de Coimbra, Coimbra, Portugal
[*] Author to whom any correspondence should be addressed.

E-mail: vitor.mendrot@unesp.br, antonio.castro@unesp.br and pedro.alberto@uc.pt

Keywords: Klein–Gordon equation, circular symmetry, relativistic harmonic oscillator

Abstract

We study the dynamics of relativistic spinless particles moving in a plane when there is circular symmetry. The general formalism for solving the Klein–Gordon equation in cylindrical coordinates for such systems is presented, as well as the conserved observables and the corresponding quantum numbers. We look for bound solutions of the corresponding Klein–Gordon equation when one has vector and scalar circularly symmetric harmonic oscillator potentials. Both positive and negative bound solutions are considered when there is either equal vector and scalar potentials or symmetric vector and scalar potentials, and it is shown how both cases are related through charge conjugation. We compute the non-relativistic limit for those cases, and show that for symmetric scalar and vector potentials the limit does not exist in the first order of an harmonic oscillator frequency, recovering a known result from the Dirac equation with the same kind of potentials.

1. Introduction

The harmonic oscillator stands out as one of the most important potentials arising in both relativistic and non-relativistic quantum systems. One particular feature is that it can describe bound quantum states in regions around local minima potentials with complex shapes [1]. The fact that in general it allows for analytical solutions for the wave function and the energy spectrum also enables to extract the essential physical features of the quantum system under study. Its applications range from quantum field theory of elementary particles to describing some molecule degrees of freedom and many kinds of physical lattices [2].

Concerning the Klein–Gordon theory, which models the relativistic behavior of spinless quantum particles, the harmonic oscillator problem can be set up in different ways because of the different varieties of Lorentz structures [3]. One has the Lorentz four-vector potential $V^{\mu} = (V_v, \boldsymbol{V})$, which among other applications, is used to model the electromagnetic field interaction, introduced by the minimal coupling $p^{\mu} \rightarrow p^{\mu} - V^{\mu}$; and the Lorentz scalar potential V_s, established by its coupling to the mass term $m \rightarrow m + V_s$. In the case of vector interactions based on electromagnetic origin, the four-vector components in $3 + 1$ dimensions are connected to the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, by the expressions

$$\boldsymbol{E} = -\boldsymbol{\nabla} V_v - \frac{\partial \boldsymbol{V}}{\partial t}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{V}. \tag{1}$$

One should note that the denominations 'scalar' and 'vector' for electromagnetic potentials used in textbooks have often a different meaning than the one used here. In our case, V_v (vector) is the time component of a Lorentz four-vector potential whereas V_s is a Lorentz scalar potential. In this paper, as explained below, we consider a mix of those scalar and vector potentials, so they cannot *per se* be related to electromagnetic potentials. If one would have only the V_v potential, then the corresponding electric field would be $\boldsymbol{E} = -\boldsymbol{\nabla} V_v$ which, for a harmonic oscillator potential considered below, would be a linear radial vector field.

In discussions regarding the non-relativistic limits of the harmonic oscillator problem, plenty of attention has also been given to a third possible coupling, which originally appeared in the context of the Dirac theory, the so-called Dirac oscillator [4]. This inspired research on the Klein–Gordon oscillator coupling [5, 6] or non-minimal vector coupling, given by the prescription $\boldsymbol{p}^2 \rightarrow (\boldsymbol{p} + im\omega\boldsymbol{r}) \cdot (\boldsymbol{p} - im\omega\boldsymbol{r})$. This interaction model has been employed to study noncommutative spaces [7, 8], topological defects [9], the realization of bound states, [10] and violation of the Lorentz symmetry [11], to cite some examples. In [12], it was presented a modified D-dimensional Klein–Gordon equation featuring a non-minimal vector interaction which incorporates the Klein–Gordon oscillator as a particular case.

In recent studies there is a particular interest in relativistic quantum systems in which the potentials involved are the time component of a vector coupling V_v and the scalar coupling V_s. The realization of bound states for this mix has been explored for a myriad of potentials. However, in the Klein–Gordon theory, analytical solutions have been found only for a restrict group of potentials. By making a convenient approximation to the centrifugal term, the Hulthén potential has analytical bound solutions for s-waves and $l \neq 0$ waves [13–15]. Usage of supersymmetric quantum mechanics and hypergeometric equations yields analytical bound states for the pseudoharmonic oscillator potential [16]. The hypergeometric approach also has analytical solutions for the $V(r) = Ar^{-2} + Br^{-1}$ potential [17]. Potentials given by trigonometric functions such as the ring-shaped harmonic oscillator [18], the ring-shaped Kratzer-type [19], the $V_0\tanh^2(r/d)$ [20], as well as the s-wave states for the symmetrical double-well model [21] and the Rosen-Morse-type [22] also give rise to bound states. Lastly, there are bound states for exponential type potentials, such as the s-wave solutions for five-parameter exponential-type potential [23].

The majority of these works assume that $V_v = V_s$. For the Dirac theory, when the vector and scalar potentials are such that $V_v = \pm V_s + C$, being C an arbitrary constant (actually zero for potentials going to zero at infinity), two additional $SU(2)$ symmetries arise: the spin symmetry, for $V_v = V_s + C$, and the pseudospin symmetry, for $V_v = -V_s + C$ [24]. In the Dirac equation, the spin (pseudospin) symmetry account for the suppression of the spin–orbit coupling term for the upper (lower) component of the Dirac spinor. The resulting equations for these components are actually Klein–Gordon equations. These symmetries play an important role in describing the nucleon single particle structure of heavy nuclei and can describe the suppression of spin-orbit in mesons with a heavy quark. A review of these symmetries and its applications can be found in [25]. It can also be shown that for both Klein–Gordon and Dirac theories, solutions for the $V_v = V_s$ case also provides the solutions for the $V_v = -V_s$ case, and vice-versa. This can be done using the charge conjugation operation and the chiral transformation as well [26].

In the present work we are interested in the configuration of a mix of vector V_v and scalar V_s potentials, both having the shape of a two-dimensional harmonic oscillator. There are plenty of physical systems in which the particle dynamics is essentially confined to a fixed plane. Such motion is usually described by considering a $2+1$ $-$dimensional world instead of the usual $3+1-$dimensional space-time. For the Klein–Gordon equation, this procedure is equivalent to a $3+1-$dimensional system such that its wave function has zero linear momentum (has eigenvalue zero of the corresponding operator) along the direction perpendicular to the plane of motion. The resulting wave function depends only on the coordinates in the plane, exactly like in the $2+1-$dimensional world.

In reference [27] this setup was studied. However, it only dealt with the particle states with the condition $V_v = V_s > 0$, and the normalization condition used is not suitable for the relativistic theory. In the present work, we analyze both particle and antiparticle solutions, discuss also the $V_v = -V_s$ condition as well as the non-relativistic limits for both cases. We find that, to first order in the frequency over mass expansion, there is no non-relativistic limit for negative energy states when $V_v = V_s < 0$ and to positive energy states when $V_v = -V_s > 0$. This result has already been obtained for the Dirac equation with harmonic oscillator potentials [28]. In this way, we are able to get new results that can be used in studying two-dimensional relativistic quantum systems for which pseudospin symmetry and the consideration of anti-particles states is important. As referred before, this particular symmetry is relevant to study nuclear systems, where pseudospin has been extensively studied, including for anti-nucleon systems and for both spherical and deformed harmonic oscillator mean-fields [24]. The use of Klein–Gordon equation for these fermion systems in pseudospin symmetry conditions is warranted for analysing the energy spectrum, as was shown in [29]. The application to a two-dimensional problem of these results can be done if the there would be no (or almost no) dynamics in a particular space direction.

The paper is organized as follows. In section 2 we will present the Klein–Gordon equation in $3+1$ dimensions for circularly symmetric systems in cylindrical coordinates and discuss its properties. In section 3 the harmonic oscillator problem will be completely solved by mapping it to a non-relativistic problem which solutions are known. Section 4 will be devoted to discussing the relation between the $V_v = +V_s$ configuration and the $V_v = -V_s$ one and the respective non-relativistic limits. The conclusions will be left to section 5.

2. Circularly symmetric Klein–Gordon equation

The time-independent Klein–Gordon equation for a particle embedded in scalar V_s and vector V_v potentials can be expressed as (with $\hbar = c = 1$)

$$[(\varepsilon - V_v)^2 + \nabla^2 - (m + V_s)^2]\varphi(\boldsymbol{r}) = 0, \tag{2}$$

where the charge density takes the form $(\varepsilon - V_v)|\varphi|^2/m$. The charge conjugation operation is achieved by the substitutions $\varepsilon \to -\varepsilon$, $V_v \to -V_v$ and $V_s \to +V_s$. We will consider the motion essentially restricted to a plane for circularly symmetric potentials, in which case the cylindrical coordinates are a natural choice. The wave function can then be written as

$$\varphi_l(\rho, \phi) = \frac{U_l(\rho)}{\sqrt{\rho}}\Phi_l(\phi), \tag{3}$$

where ρ is the radial distance from the z-axis, and ϕ is the angle formed with the reference axis in this plane (the x-axis in Cartesian coordinates). The fixed plane coincides with the xy-plane. The functions Φ_l are eigenfunctions of $L_z = -i\partial/\partial\phi$ with eigenvalues $l = 0, \pm 1, \pm 2, \ldots$:

$$\Phi_l(\phi) = \frac{1}{\sqrt{2\pi}}e^{il\phi}, \tag{4}$$

Since the potentials V_s and V_v depend only on ρ, U_l obeys the radial equation

$$-\frac{1}{2\mathrm{m}}\frac{d^2U_l}{d\rho^2} + \left(\frac{V_s^2 - V_v^2}{2\mathrm{m}} + \frac{\varepsilon V_v + mV_s}{\mathrm{m}} + \frac{l^2 - 1/4}{2\mathrm{m}\rho^2}\right)U_l = \frac{\varepsilon^2 - m^2}{2\mathrm{m}}U_l. \tag{5}$$

The normalization condition can be obtained from the charge density and is expressed as

$$\int_0^\infty d\rho(\varepsilon - V_v)\,|U_l|^2 = \pm m, \tag{6}$$

where the plus and minus signs on the right-hand side refer to positively or negatively charged states, respectively.

For the formation of analytic bound-state solutions, the role of the sum and the difference of the vector and scalar potentials is crucial, as will be shown. Because of that, it is convenient to analyze the problem in terms of the sum and difference potentials $V_\Sigma = V_v + V_s$ and $V_\Delta = V_v - V_s$. Then, we can write the radial equation as

$$-\frac{1}{2\mathrm{m}}\frac{d^2U_l}{d\rho^2} + \left[\frac{(\varepsilon + m)V_\Sigma + (\varepsilon - m)V_\Delta - V_\Sigma V_\Delta}{2\mathrm{m}} + \frac{l^2 - 1/4}{2\mathrm{m}\rho^2}\right]U_l = \frac{\varepsilon^2 - m^2}{2\mathrm{m}}U_l. \tag{7}$$

It is worth noting that $V_\Sigma \to -V_\Delta$, $V_\Delta \to -V_\Sigma$ and $\varepsilon \to -\varepsilon$ under charge conjugation. Equation (7) reveals the possibility of bound-state solutions when either V_Δ or V_Σ are zero. For confining potentials this is the case with $|\varepsilon| > m$, when (i) $V_\Delta = 0$ and $\lim_{\rho\to\infty}V_\Sigma = \pm\infty$ (respectively when $\varepsilon > m$ and $\varepsilon < m$); (ii) $V_\Sigma = 0$ and $\lim_{\rho\to\infty}V_\Delta = \pm\infty$ (respectively when $\varepsilon > m$ and $\varepsilon < m$). One the other hand, for potentials going to zero when $\rho \to \infty$ one has bound-state solutions with $|\varepsilon| < m$ only when $V_\Delta = 0$ and $V_\Sigma < 0$, $\varepsilon > 0$, or $V_\Sigma = 0$ and $V_\Delta > 0$, $\varepsilon < 0$.

For small binding energies, $\varepsilon \simeq m$, we obtain the Schrödinger-like equation

$$-\frac{1}{2\mathrm{m}}\frac{d^2U_l}{d\rho^2} + \left[\frac{\varepsilon - m}{2\mathrm{m}}V_\Delta + V_\Sigma\left(1 - \frac{V_\Delta}{2\mathrm{m}}\right) + \frac{l^2 - 1/4}{2\mathrm{m}\rho^2}\right]U_l \simeq (\varepsilon - m)U_l. \tag{8}$$

For $\varepsilon \simeq -m$, the equation we get is

$$-\frac{1}{2\mathrm{m}}\frac{d^2U_l}{d\rho^2} + \left[\frac{\varepsilon + m}{2\mathrm{m}}V_\Sigma - V_\Delta\left(1 + \frac{V_\Sigma}{2\mathrm{m}}\right) + \frac{l^2 - 1/4}{2\mathrm{m}\rho^2}\right]U_l \simeq (\varepsilon + m)U_l. \tag{9}$$

3. The harmonic oscillator problem

We explore a combination of vector and scalar harmonic potentials $V_v = K_v\rho^2/2$ and $V_s = K_s\rho^2/2$, such that V_Σ and V_Δ are harmonic potentials as well, with coefficients $K_\Sigma = K_v + K_s$ and $K_\Delta = K_v - K_s$. To derive precise analytic solutions for the radial equation, we selectively set either K_Σ or K_Δ to zero, which stands for the cases where $V_v = -V_s$ and $V_v = V_s$, respectively. Consequently, the problem transforms into an exactly solvable scenario, described by a Schrödinger-like equation featuring an effective singular harmonic oscillator potential. By setting V_Δ to zero, one gets $K_v = K_s$, thus $K_\Sigma = 2K_v$ (we could as well have written $K_\Sigma = 2K_s$), and we achieve the following radial equations, derived from (7):

$$-\frac{1}{2\mathrm{m}}\frac{d^2U_l}{d\rho^2}+\left[\frac{(\varepsilon+m)K_v\rho^2}{2\mathrm{m}}+\frac{l^2-1/4}{2\mathrm{m}\rho^2}\right]U_l=\frac{\varepsilon^2-m^2}{2\mathrm{m}}U_l. \tag{10}$$

Otherwise, analytical solutions would not be obtainable, due to a ρ^4 term appearing in the equation.

Notably, the form of equation (10) is akin to that of the nonrelativistic harmonic oscillator equation. Additionally, from equation (6), it is evident that the condition $|\int_0^\infty d\rho\ V_v\ \ |U_l|^2|<\infty$ is less restrictive than $\int_0^\infty d\rho\ |U_l|^2<\infty$ in the case of a V_v being an harmonic oscillator potential, indicating that solutions to the nonrelativistic problem can be mapped into the normalizable solutions of the relativistic problem. From the study by Nogueira et al [30], the radial equation for a particle under a harmonic oscillator potential is given by (in their notation):

$$\frac{d^2U}{dr^2}+\left[-\left(\frac{M\omega}{\hbar}\right)^2r^2-\frac{S^2-1/4}{r^2}+\frac{2ME}{\hbar^2}\right]U=0, \tag{11}$$

with

$$E_{n_r S}=\omega(2n_r+1+S),\quad n_r=0,1,2,\ldots, \tag{12}$$

and eigenfunctions expressed in terms of the generalized Laguerre polynomials [31]

$$U_{n_r S}(\rho)=A_{n_r S}\rho^{1/2+S}e^{-M\omega\rho^2/2}L_{n_r}^{(S)}(M\omega\rho^2). \tag{13}$$

The parameters M, E, ω and S are then related here to m, ε, K_v and l in (10) through the following relations:

$$(M\omega)^2=K_v(\varepsilon+m),\quad S^2-\frac{1}{4}=l^2-1/4,\quad 2ME=\varepsilon^2-m^2. \tag{14}$$

Thus, we have

$$U_{n|l|}=A_{n|l|}\rho^{1/2+|l|}e^{-\gamma\rho^2/2}L_{(n-|l|)/2}^{(|l|)}(\gamma\rho^2), \tag{15}$$

$$\pm\sqrt{4K_v(\varepsilon_n+m)}(n+1)=\varepsilon_n^2-m^2, \tag{16}$$

where $n=2n_r+|l|$, $|l|\leqslant n$, $A_{n|l|}$ is the normalization constant and

$$\gamma\equiv\sqrt{K_v(\varepsilon+m)}. \tag{17}$$

Relations (14) provide additional conditions for the eigenvalues. The first relation implies that K_v is greater or less than zero if and only if ε is greater or less than $-m$, while the third relation restricts the energies to $|\varepsilon|>m$. Combining these two conditions into a single expression, we conclude that $\mathrm{sgn}(\varepsilon)=\mathrm{sgn}(K_v)$. In addition, it should be noted that the negative sign in the left-hand side of equation (16) is extraneous and should be disregarded. To derive an explicit expression for the energy spectrum, we begin by squaring equation (16) and making the following change of variables:

$$\epsilon=3\frac{\varepsilon}{m}, \tag{18}$$

$$A=\frac{27}{8}K_v\frac{(n+1)^2}{m^3}, \tag{19}$$

which are subject to the condition

$$\mathrm{sgn}(\epsilon)=\mathrm{sgn}(A). \tag{20}$$

The resulting equation is

$$\left(\frac{\epsilon}{3}-1\right)^2\left(\frac{\epsilon}{3}+1\right)=\frac{32}{27}A. \tag{21}$$

Our objective now is not only to solve this equation but also to determine which solutions are relevant to our original problem. It should be noted that squaring the energy spectrum equation leads to double the number of solutions.

To gain insight into the possible solutions of equation (21), we can employ Descartes' rule of signs [32]. For $\epsilon>3$, there is only one feasible value with $A>0$, while $\epsilon<-3$ supports also one possible value with $A<0$. Now it is suitable to cast ϵ as

$$\epsilon=1+2(\chi+\chi^{-1}), \tag{22}$$

in which

$$\chi=[2A-1+2\sqrt{A(A-1)}]^{1/3}. \tag{23}$$

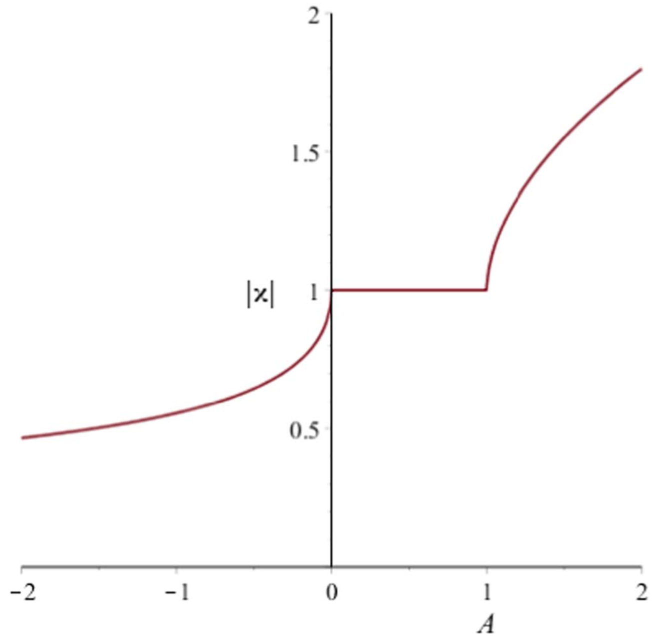

Figure 1. Modulus of χ as a function of A.

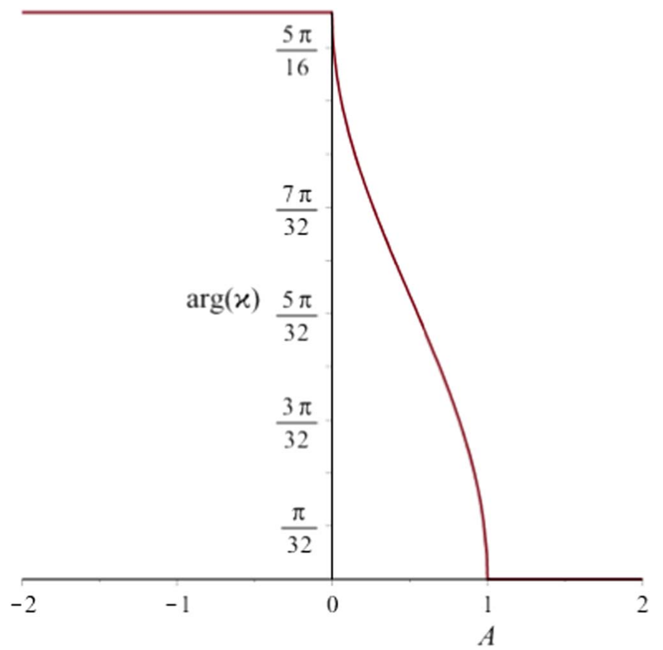

Figure 2. Argument of χ as a function of A taking the principal cubic root.

We are going now to analyze χ. We can start our search by studying the modulus as well as the argument of χ as a function of A for the principal cubic root, as displayed in figures 1 and 2 respectively. Note that the other two roots are obtained by adding $2\pi/3$ and $4\pi/3$ to the principal value argument.

As suggested by the figures 1 and 2, we have divided our analysis into three intervals: $A \geqslant 1, 0 \leqslant A < 1$ and $A < 0$. In the first case, the principal argument is always null, which leads us to conclude that only by setting this value, one can obtain real energy values. Therefore, for $A \geqslant 1$, we can express this as

$$\epsilon = 1 + 2(|\chi| + |\chi|^{-1}). \tag{24}$$

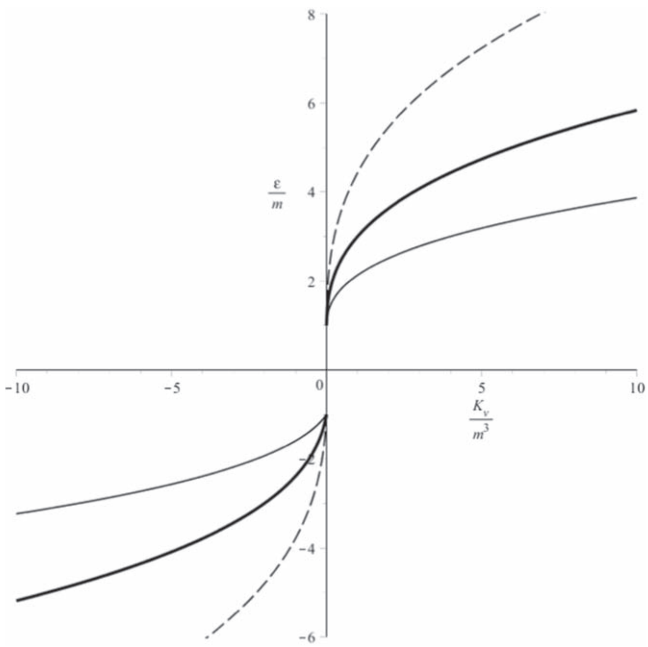

Figure 3. Spectrum as a function of the interaction strength for the first three principal quantum numbers. The fine line corresponds to $n = 0$, the thick line to $n = 1$, and the discontinuous line to $n = 2$.

Moving on to the interval $0 \leqslant A < 1$, the modulus will always be equal to one, enabling us to state that

$$\epsilon = 1 + 4\cos(\arg(\chi)). \tag{25}$$

To fulfill the condition (20) it is necessary to consider the principal argument, which falls within the interval $]0, \pi/3]$.

For the remaining case, in which $A < 0$, we write

$$\epsilon = 1 - 2(|\chi| + |\chi|^{-1}), \tag{26}$$

as only the second argument, which is equal to π, leads to real energy values satisfying the condition (20).

To summarize the above analysis, we obtain the energy spectrum, returning to the original variables and coefficients:

$$\frac{\varepsilon_n}{m} = \frac{1}{3} + \frac{2}{3}\,\mathrm{sgn}(K_v)(\chi_n^{1/3} + \chi_n^{-1/3}), \tag{27}$$

with

$$\chi_n = \mathrm{sgn}(K_v)[2A_n - 1 + 2\sqrt{A_n(A_n - 1)}], \quad A_n = \frac{27}{8}K_v\frac{(n+1)^2}{m^3}. \tag{28}$$

Applying (6), the normalized eigenfunctions can be expressed as

$$\varphi_{nl}(\rho, \phi) = \sqrt{\frac{4m\gamma^{|l|+2}}{\pi[2\gamma\varepsilon - K_v(n+1)]}\frac{\Gamma\left(\frac{n-|l|}{2}+1\right)}{\Gamma\left(\frac{n+|l|}{2}+1\right)}}\rho^{|l|}e^{-\gamma\rho^2/2}L^{(|l|)}_{(n-|l|)/2}(\gamma\rho^2)e^{il\phi}. \tag{29}$$

The degeneracy of the n-th energy level is $n + 1$. Figure 3 shows the first three levels, corresponding to $n = 0$, 1, 2 of the obtained spectrum as a function of the interaction strength. Although the spectrum for different signs of K_v may look anti-symmetrical around the origin, the change of sign of K_v in (28) reveals that this is not the case.

By considering the condition $|K_v| \ll m^3$, we will expand the spectrum (27) to obtain the non-relativistic limit (zeroth order term) and the first order relativistic correction. Thus, we get

$$\varepsilon - m \approx (n+1)\sqrt{\frac{2K_v}{\mathrm{m}}} - (n+1)^2\frac{K_v}{2m^2} + \ldots, \quad K_v > 0, \tag{30}$$

$$\varepsilon + m \approx (n+1)^2\frac{K_v}{m^2} + \ldots, \quad K_v < 0. \tag{31}$$

The non-existence of the first order term in the non-relativistic approximation when $K_v < 0$ also appears in the Dirac equation [33]. Indeed, as can be seen from (8), if $K_v < 0$, the potential is not binding in the non-relativistic limit, and therefore we only see higher relativistic corrections for the spectrum.

4. Charge conjugation

Applying to solutions (27) and (29) the charge conjugation operation [3], which, as mentioned before, performs the changes $V_\Sigma \to -V_\Delta$, $V_\Delta \to -V_\Sigma$ and $\varepsilon \to -\varepsilon$, yields the corresponding results to a system in which K_Σ is taken to be null and $V_\Delta = K_\Delta \rho^2/2$, $K_\Delta = 2K_v$ having the opposite sign of the former K_Σ, which in practical terms, reverses the sign of K_v. We obtain

$$\frac{\varepsilon_n}{m} = \frac{1}{3} - \frac{2}{3}\,\mathrm{sgn}(K_v)(\chi_n^{1/3} + \chi_n^{-1/3}), \tag{32}$$

$$\chi_n = -\mathrm{sgn}(K_v)[2A_n - 1 + 2\sqrt{A_n(A_n - 1)}], \tag{33}$$

$$A_n = -\frac{27}{8}K_v\frac{(n+1)^2}{m^3}. \tag{34}$$

and

$$\varphi_{nl}(\boldsymbol{r}) = \sqrt{\frac{4\mathrm{m}\tilde{\gamma}^{|l|+2}}{\pi[2\tilde{\gamma}\varepsilon + K_v(n+1)]}\frac{\Gamma\left(\frac{n-|l|}{2}+1\right)}{\Gamma\left(\frac{n+|l|}{2}+1\right)}}\,\rho^{|l|}e^{-\frac{1}{2}\tilde{\gamma}\rho^2}L_{(n-|l|)/2}^{(|l|)}(\tilde{\gamma}\rho^2)e^{-il\phi}, \tag{35}$$

where

$$\tilde{\gamma} \equiv \sqrt{-K_v(\varepsilon + m)}. \tag{36}$$

Furthermore, we also obtain the non-relativistic limits to be

$$\varepsilon - m \approx (n+1)^2\frac{K_v}{m^2} + \ldots, \quad K_v > 0, \tag{37}$$

$$\varepsilon + m \approx -(n+1)\sqrt{-\frac{2K_v}{m}} + (n+1)^2\frac{K_v}{2\mathrm{m}^2} + \ldots, \quad K_v < 0. \tag{38}$$

Here we see that for $K_v > 0$ the non-relativistic limit does not yield bound-state solutions, which is evident from (9).

Referring again to the sum and difference potentials coefficients, we can say in general terms that the charge conjugation for $K_\Sigma < 0$, yielding the solution with $K_\Delta > 0$, is exactly the same as changing $K_\Sigma > 0$ into $K_\Delta > 0$ and $m \to -m$ in equations (27) and (28). This is similar to the chiral transformation applied in the Dirac equation (see [26] for details).

If we make the correspondence between the positive energy charge conjugated solutions ($K_v > 0$) to the pseudospin symmetry solutions for the harmonic oscillator potential in 3 + 1 Dirac equation [28] we see that most of the features of the 3 + 1 Dirac spectrum are reproduced in the 2 + 1 Klein–Gordon equation with the same potential, namely the fact that one does not have a non-relativistic limit to the lowest order of the oscillator frequency (proportional to the square root of K_v).

5. Conclusions

In this work we have presented a complete study of the Klein–Gordon equation in a mix of vector and scalar circularly symmetric harmonic oscillator potentials. We cast the equation in terms of the sum and difference potentials $V_\Sigma = V_v + V_s$ and $V_\Delta = V_v - V_s$ and analyzed the possible configurations in which bound-state solutions could be analitically achieved. Furthermore we discussed the relevant quantum numbers and the correct normalization of these states.

We developed the complete resolution for the case in which $V_\Delta = 0$ and $V_\Sigma = K_v\rho^2$. The eigenfunctions and eigenvalues were obtained by mapping the radial equation in a nonrelativistic framework (Schrödinger equation), whose general solutions are known, to the relativistic case. We were able to show that the eigenfunctions are given in terms of the generalized Laguerre polynomials and solved the irrational equation for the energy eigenvalue which was provided by the mapping. The non-relativistic expansion was shown for both $K_v > 0$ and $K_v < 0$ cases. In the latter case, the first order term in the expansion in the oscillator frequency vanishes, showing that in this order the potential is not binding.

By performing the charge conjugation operation on the results of the $V_\Delta = 0$ ($K_v < 0$) configuration we obtain the solutions of the radial two-dimensional Klein–Gordon equation for $V_\Sigma = 0$ and $V_\Delta = K_v\rho^2$ ($K_v > 0$). This configuration of harmonic oscillator potentials was not previously discussed in the literature, and has a striking similarity with the solution of the radial 3 + 1 Dirac equation for the lower component radial function in pseudospin conditions (i.e. $V_\Sigma = 0$). In particular, also in that case there is no lowest order term in the non-relativistic expansion of the binding energy. In reference [29] it was shown that in spin or pseudospin conditions and for central potentials, the 3 + 1 Dirac equation and the corresponding Klein–Gordon equation with the same potentials have the same energy spectrum. The results obtained in this paper strongly suggest that the same is also true for radial (circularly symmetric) solutions of the 2 + 1 Dirac and Klein–Gordon equations.

Acknowledgments

Grant 09 126/2019-3, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil. The study was partially financed through national funds by FCT - Fundação para a Ciência e Tecnologia, I.P. in the framework of the projects UIDB/04564/2020 and UIDP/04564/2020, with DOI identifiers 10.54499/UIDB/04564/2020 and 10.54499/UIDP/04564/2020, respectively. as well as by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior—Brasil (Capes) - Finance Code 001. PA would like to thank the Universidade Estadual Paulista, Guaratinguetá Campus, for supporting his stays in its Physics Department. AC would like to thank Universidade de Coimbra for supporting his stays at its Physics Department.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

A S de Castro https://orcid.org/0000-0001-8802-8806
P Alberto https://orcid.org/0000-0002-2754-7642

References

[1] Griffiths D J 2005 *Introduction to Quantum Mechanics* II edn (Prentice Hall)
[2] Cohen-Tannoudji C, Diu B and Laloë F 2005 *Quantum Mechanics* vol 1 (John Wiley & Sons)
[3] Wachter A 2011 *Relativistic Quantum Mechanics* (Springer Science+Business Media B.V.)
[4] Moshinsky M and Szczepaniak A 1989 The Dirac oscillator *J. Phys. A: Math. Gen.* **22** L817–19
[5] Bruce S and Minning P 1993 The Klein-Gordon oscillator *Il Nuovo Cimento* **106** 711–3
[6] Dvoeglazov V V 1994 Comment on The Klein-Gordon oscillator *Il Nuovo Cimento* **107** ed S Bruce and P Minning 1411–7
[7] Mirza B and Mohadesi M 2004 The Klein-Gordon and the Dirac oscillators in a noncommutative space *Commun. Theor. Phys.* **42** 664–68
[8] Jian-Hua W, Kang L and Sayipjamal D 2008 Klein-Gordon oscillators in noncommutative phase space* *Chin. Phys.*C **32** 803–06
[9] Carvalho J, de Carvalho A M M, Cavalcante E and Furtado C 2016 Klein-Gordon oscillator in Kaluza-Klein theory *Eur. Phys. J.*C **76** 365–73 Eur. Phys. J.
[10] Rao N A and Kagali B A 2007 Energy profile of the one-dimensional Klein-Gordon oscillator *Phys. Scr.* **77** 015003–06015003
[11] Ahmed F 2022 Relativistic quantum oscillator model under the effects of the violation of Lorentz symmetry by an arbitrary fixed vector field *Europhys. Lett.* **138** 20001
[12] Garcia M G, de Castro A S, Castro L B and Alberto P 2017 New solutions of the D-dimensional Klein-Gordon equation via mapping onto the nonrelativistic one-dimensional Morse potential *Ann. Phys.* **378** 88–99
[13] Qiang W-C, Zhou R-S and Gao Y 2007 Any l-state solutions of the Klein-Gordon equation with the generalized Hulthén potential *Phys. Lett.* A **371** 201–4
[14] Chen C-Y, Sun D-S and Lu F-L 2007 Approximate analytical solutions of Klein-Gordon equation with Hulthén potentials for nonzero angular momentum *Phys. Lett.* A **370** 219–21
[15] Chen G 2004 Spinless particle in the generalized Hulthén potential *Mod. Phys. Lett.*A **19** 2009–12
[16] Gang C, Zi-Dong C and Zhi-Mei L 2004 Bound states of the Klein-Gordon and Dirac equation for scalar and vector pseudoharmonic oscillator potentials *Chin. Phys.* **13** 279–82
[17] Qiang W-C 2003 Bound states of the Klein-Gordon and Dirac equations for potential $V(r) = Ar^{-2} - Br^{-1}$ *Chin. Phys.* **12** 1054–04
[18] Qiang W-C 2003 Bound states of Klein-Gordon equation for ring-shaped harmonic oscillator scalar and vector potentials *Chin. Phys.* **12** 136–39
[19] Qiang W-C 2004 Bound states of the Klein-Gordon equation for ring-shaped Kratzer-type potential *Chin. Phys.* **13** 575–78
[20] Qiang W-C 2004 Bound states of the Klein-Gordon and Dirac equations for potential $V_0 tanh^2(r/d)$ *Chin. Phys.*B **13** 571–4
[21] Zhao X-Q, Jia C-S and Yang Q-B 2005 Bound states of relativistic particles in the generalized symmetrical double-well potential *Phys. Lett.* A **337** 189–96
[22] Yi L-Z, Diao Y-F, Liu J-Y and Jia C-S 2004 Bound states of the Klein-Gordon equation with vector and scalar Rosen-Morse-type potentials *Phys. Lett.* A **333** 212–7

[23] Diao Y-F, Yi L-Z and Jia C-S 2004 Bound states of the Klein-Gordon equation with vector and scalar five-parameter exponential-type potentials *Phys. Lett.* A **332** 157–67
[24] Ginocchio J N 2005 Relativistic symmetries in nuclei and hadrons *Phys. Rep.* **414** 165–261
[25] Liang H, Meng J and Zhou S-G 2015 Hidden pseudospin and spin symmetries and their origins in atomic nuclei *Phys. Rep.* **570** 1–84 Hidden pseudospin and spin symmetries and their origins in atomic nuclei
[26] de Castro A S, Alberto P, Lisboa R and Malheiro M 2006 Relating pseudospin and spin symmetries through charge conjugation and chiral transformations: The case of the relativistic harmonic oscillator *Phys. Rev.*C **73** 054309
[27] Qiang W-C 2004 Bound states of two-dimensional relativistic harmonic oscillators *Chin. Phys.* **13** 283–86
[28] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 Pseudospin symmetry and the relativistic harmonic oscillator *Phys. Rev.*C **69** 024319
[29] Alberto P, de Castro A S and Malheiro M 2007 Spin and pseudospin symmetries and the equivalent spectra of relativistic spin-1/2 and spin-0 particles *Phys. Rev.*C **75** 047303
[30] Nogueira P H F and de Castro A S 2016 From the generalized Morse potential to a unified treatment of the D-dimensional singular harmonic oscillator and singular Coulomb potentials *J. Math. Chem.* **54** 1783–91
[31] Arfken G B 1985 *Mathematical Methods for Physicists* (Academic Press, Inc.)
[32] Meserve B E 1982 *Fundamental Concepts of Algebra* (Dover Publications, Inc.)
[33] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 Pseudospin symmetry and the relativistic harmonic oscillator *Phys. Rev.*C **69** 024319