Generalized geometry and non-symmetric gravity

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Generalized geometry provides the framework for a systematic approach to non-symmetric metric gravity theory and naturally leads to an Einstein-Kalb-Ramond gravity theory with totally anti-symmetric contortion. The approach is related to the study of the low-energy effective closed string gravity actions.

Keywords: Non-symmetric metric; torsion connection; Koszul formula; generalized geometry.

1. Motivation

A proposal for a gravity theory with non-symmetric metric began with an idea of Einstein to unify gravity and electromagnetism (Refs. 1, 2). In general relativity, the metric of the Riemannian manifold is a symmetric bilinear form. Interpreted as an invertible map from the tangent to the cotangent space, it is natural to allow also an anti-symmetric part. While the original hope of Einstein that the anti-symmetric part of a non-symmetric metric tensor may be directly related to the electromagnetic force turned out to be incorrect, there is nevertheless phenomenological interest in non-symmetric gravity theories. Damour *et al.* (Ref. 3) discussed the problems associated with the construction of non-symmetric gravity theories, where theories were typically in need for treatment of ghost terms. There have since been numerous studies on the topic (Refs. 4, 5, 6). On the other hand, generalized geometry (Refs. 7, 8), which incorporates symmetries of string theory (T-duality, B-transform) and spacetime geometry (diffeomorphisms) seems to offer a well-suited geometrical framework for string theory as well as non-symmetric gravity theories. Generalized geometry as an extension of Riemannian geometry can reproduce the Einstein-Hilbert and supergravity actions (Refs. 9, 10). In the present work, we consider an alternative approach that naturally incorporates torsion. For a recent work on metric connections with skew torsion in Riemannian geometry, see Ref. 11. An alternative approach to Einstein-Hilbert type actions using structures of generalized geometry can be found in Ref. 12.

2. Background setup in generalized geometry

We consider a vector bundle $E = TM \bigoplus T^*M$. The elements of the space of sections of the vector bundle are formal sums $e = X + \zeta \in \Gamma(E)$, where X is a vector field and ζ is a one-form. We have a natural pairing

$$\langle X+\zeta, Y+\eta\rangle = i_X\eta + i_Y\zeta\,, \tag{1}$$

which is symmetric and non-degenerate. The signature of the pairing is (d, d), where d are the spacetime dimension of TM and T^*M respectively. The pairing is invariant under $O(d, d)$ transformations. We have also a Dorfman bracket

$$[X+\zeta, Y+\eta]_{\rm D} = [X,Y]_{\rm Lie} + \mathcal{L}_X\eta - i_Y d\zeta\,, \tag{2}$$

where $[X, Y]_{\rm Lie}$ is the Lie bracket of vector fields. Finally there is an anchor map $a : E \to TM$ that maps from the vector bundle being considered here to the tangent bundle. Thus we define a Courant algebroid: $(E, \langle\ ,\ \rangle, [\ ,\ ]_{\rm D}, a)$, with the following properties:
for a function $f \in C^\infty(M)$ and elements $e_1, e_2 \in \Gamma(E)$, the Dorfman bracket $[\ ,\ ]_{\rm D}$: $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfies the Leibniz rule

$$[e_1, fe_2]_{\rm D} = f[e_1,e_2]_{\rm D} + (a(e_1)f)e_2 \tag{3}$$

and Jacobi identity

$$[e_1,[e_2,e_3]_{\rm D}]_{\rm D} = [[e_1,e_2]_{\rm D},e_3]_{\rm D} + [e_2,[e_1,e_3]_{\rm D}]_{\rm D}\,. \tag{4}$$

The anchor map a obeys the homomorphism property

$$a\left([e_1,e_2]_{\rm D}\right) = \left[a(e_1),a(e_2)\right]_{\rm Lie}\,, \tag{5}$$

while the pairing $\langle\ ,\ \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ exhibits the following properties,

$$a(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2]_{\rm D},e_3\rangle + \langle e_2,[e_1,e_3]_{\rm D}\rangle \tag{6}$$

and

$$a^\dagger d\langle e_1,e_2\rangle = [e_1,e_2]_{\rm D} + [e_2,e_1]_{\rm D}\,, \tag{7}$$

where $a^\dagger : T^*M \to E^* \simeq E$. From (7), it is obvious that the Dorfman bracket is not a Lie bracket as it is not anti-symmetric.

The following are examples of $O(d, d)$ transformations.
B-transform:

$$e^B\begin{pmatrix}V\\ \zeta\end{pmatrix} = \begin{pmatrix}1 & 0\\ B^T & 1\end{pmatrix}\begin{pmatrix}V\\ \zeta\end{pmatrix} \tag{8}$$

$$B : TM \to T^*M \quad \text{where} \quad B \in \Omega^2(M) \text{ is a two-form.}$$

This orthogonal transformation is well known in string theory and will be a central object in our current study.
β-transform:

$$e^{\beta}\begin{pmatrix} V \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & \beta^T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V \\ \zeta \end{pmatrix} \tag{9}$$

$$\beta : T^*M \to TM \quad \text{where} \quad \beta \in \mathfrak{X}^2(M) \text{ is a bi-vector.}$$

We refer interested readers to Ref. 13 for an application in non-commutative geometry.
Diffeomorphism:

$$O_N\begin{pmatrix} V \\ \zeta \end{pmatrix} = \begin{pmatrix} N^T & 0 \\ 0 & N^{-1} \end{pmatrix}\begin{pmatrix} V \\ \zeta \end{pmatrix} \tag{10}$$

$$N : TM \to TM \quad \text{where} \quad N|_{p\in M} \in GL(d).$$

3. Deformations

Within the context of satisfying the Courant algebroid properties (3)-(7), we propose the following deformations

$$\langle e_1, e_2\rangle \to \langle e_1, e_2\rangle^{'} = \langle e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)\rangle \tag{11}$$

$$[e_1, e_2]_{\mathrm{D}} \to [e_1, e_2]^{'}_{\mathrm{D}} = e^{-\mathcal{G}}[e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)]_{\mathrm{D}}\,, \tag{12}$$

for elements $e_1 = X + \zeta,\ e_2 = Y + \eta$. We have introduced here a non-symmetric metric $\mathcal{G} = g + B$, which is composed of a symmetric g and an anti-symmetric B as an invertible map $\mathcal{G} : TM \to T^*M$ and $e^{\mathcal{G}} : E \to E :\ e^{\mathcal{G}}(e) = e + \mathcal{G}\,(a(e), -)$.

4. Computations

The deformations (11) and (12) preserve the Courant algebroid properties.
Given elements $X+\zeta$ and $Y+\eta$, deformation (11) corresponds to the pairing being deformed with the symmetric metric g,

$$\langle X+\zeta, Y+\eta\rangle^{'} = \langle X+\zeta, Y+\eta\rangle + 2g\,(X, Y)\ , \tag{13}$$

while for the deformed Dorfman bracket, it is straightforward to compute from (12) by using the definition of Dorfman bracket that

$$\begin{aligned} & [X+\zeta, Y+\eta]^{'}_{\mathrm{D}} \\ = \ & [X+\zeta, Y+\eta]_{\mathrm{D}} + X\mathcal{G}(Y,-) - Y\mathcal{G}(X,-) + d\mathcal{G}(X,Y) \\ & -\mathcal{G}(Y,[X,-]_{\mathrm{Lie}}) - \mathcal{G}([X,Y]_{\mathrm{Lie}},-) + \mathcal{G}(X,[Y,-]_{\mathrm{Lie}}) \\ = \ & [X+\zeta, Y+\eta]_{\mathrm{D}} + 2g(\nabla X, Y)\,. \end{aligned} \tag{14}$$

We find that the bracket is twisted by a connection ∇ in which the non-symmetric metric $\mathcal{G} = g + B$ is encoded. From (14),

$$\begin{aligned} 2g(\nabla_Z X, Y) = \; & X\mathcal{G}(Y,Z) - Y\mathcal{G}(X,Z) + Z\mathcal{G}(X,Y) \\ & -\mathcal{G}(Y,[X,Z]_{\mathrm{Lie}}) - \mathcal{G}([X,Y]_{\mathrm{Lie}},Z) + \mathcal{G}(X,[Y,Z]_{\mathrm{Lie}}) \end{aligned} \tag{15}$$

is a generalized version of the Koszul formula that involves torsion, while the original formula defines the torsion-free Levi-Civita connection. Note the unusual ordering of arguments in (15). From the generalized Koszul formula (15), we compute the torsion connection

$$g(\nabla_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} H(X,Y,Z)\,, \tag{16}$$

where $H(= dB)$ is an anti-symmetric 3-form and ∇^{LC} is the Levi-Civita connection. For contortion

$$K(X,Y,Z) = \frac{1}{2}\left(g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)\right)\,, \tag{17}$$

we can deduce from the deformed property (6) that

$$2K(X,Y,Z) = H(X,Y,Z) = g(T(X,Y),Z)\,. \tag{18}$$

On the other hand, from the deformed version of property (7), we have metricity of the connection

$$g(\nabla_X Y, Z) + g(\nabla_X Z, Y) = Xg(Y,Z)\,. \tag{19}$$

5. Results

The connection that appears in the deformation is found to be

$$g \circ \nabla = g \circ \nabla^{LC} + K \tag{20}$$

with contortion K. We find that the correspondingly deformed equation (6) gives us a totally anti-symmetric contortion $K = H/2 = dB/2$. The contortion is closed under the deformed Jacobi identity (4), whereas the deformed equation (7) gives us the metricity condition, see Ref. 14 for further computational details and results.

Having the connection (20), we compute the Ricci tensor in components

$$R_{jl} = R^{LC}_{jl} - \frac{1}{2}\nabla^{LC}_i H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m\,. \tag{21}$$

It turns out to be non-symmetric due to the anti-symmetric second term. When (21) is treated as a vacuum field equation, that is, let $R_{jl} = 0$, we have the corresponding non-symmetric gravity action

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N}\int d^d x \sqrt{-g}\left(R^{LC} - \frac{1}{12} H_{ijk}H^{ijk}\right)\,, \tag{22}$$

where G_N is Newton's gravitational constant in d dimensions. Varying with respect to g and B implies the field equations:

$$\frac{\delta S}{\delta g^{lj}} = 0\,, \quad \frac{\delta S}{\delta B^{lj}} = 0 \qquad \Rightarrow \qquad R_{jl} = 0\,. \tag{23}$$

Note the ordering of the indices, which is only a matter of convention.

6. Discussions

In string theory, the non-linear sigma model on worldsheet Σ, with worldsheet metric $\gamma^{\mu\nu}$ for $\mu, \nu = 0, 1$,

$$\begin{aligned} S_{\rm nlsm} = \; & \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\gamma} \\ & (\gamma^{\mu\nu} g_{mn}(X)\, \partial_\mu X^m \partial_\nu X^n \\ & + i\epsilon^{\mu\nu} B_{mn}(X)\, \partial_\mu X^m \partial_\nu X^n \\ & + \alpha' \phi(X) R_{(\gamma)})\,, \end{aligned} \tag{24}$$

where $m, n = 0, 1, \cdots, 25$, in 26-dimensional spacetime, describes the string propagation in background fields: metric g, Kalb-Ramond B and dilaton ϕ. Beta functions

$$\beta_{\mu\nu}(g) = \alpha' R^{LC}_{\mu\nu} - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} + 2\alpha' \nabla_\mu \nabla_\nu \phi \tag{25}$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \phi H_{\lambda\mu\nu} \tag{26}$$

$$\beta_{\mu\nu}(\phi) = -\frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_\mu \phi \nabla^\mu \phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}\,, \tag{27}$$

which follow from (24) are required to vanish in order to preserve the Weyl invariance in string theory as a quantum theory. The low-energy closed bosonic string action (Ref. 15)

$$S_{\rm eff} = \frac{1}{2\kappa^2} \int d^{26} X \sqrt{-g}\, e^{-2\phi} \left(R^{LC} - \frac{1}{12} H_{abc} H^{abc} + 4 g^{ab} \partial_a \phi\, \partial_b \phi \right) \tag{28}$$

has been derived as the effective string action that gives the equations of motion, which are equal to the vanishing beta functions (25), (26) and (27).

We notice that our non-symmetric Ricci tensor (21) contains the beta functions (25) and (26) and our non-symmetric gravity action (22) resembles the closed string effective action (28) without dilaton. Our action (22) is effectively an action, where its equations of motion describe a Ricci flow. A closely related work has previously appeared in Ref. 16 in the scope of supergravity.

While in string theory, the spacetime dimension of (28) is determined during the derivation of the beta function (27), the dimension d of our similarly Ricci-flat spacetime theory (22) is unrestricted. Recall that our Courant algebroid deformations involve only the tangent bundle. Interestingly, the combination of $g + B$ in the non-linear sigma model (24), which was a motivation in the pursuit of non-symmetric gravity theory, appears to be at equal footing in our deformations (11) and (12). Deforming the Einstein-Hilbert action has led us to an Einstein-Kalb-Ramond theory (Ref. 17).

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