Einstein Manifold of SO(p,q) with a New Parabolic Subgroup and Invariant Differential Operators

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Abstract. In the present paper we continue the project of systematic explicit construction of invariant differential operators. On the example of the non-compact group $SO(p, 9-p)$ (for $p = 5, 6$) we give the classification multiplets of indecomposable elementary representations induced from a *new* (relative to earlier considerations) choice of parabolic subgroup MAN so that the factor-group G/M is an Einstein manifold. This classification includes the data for the relevant invariant differential operators.

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory, (for a recent review, cf. e.g., [2]). Thus, it is important for the applications in physics to study systematically such operators.

In a recent paper [3] (see also [4], [5]) we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on one particular group $SO(p, 9-p)$, which is very interesting because it has as subgroup $M = SO(3) \times SO(3)$ so that the factor G/M is an Einstein manifold [6].

The present paper is organized a follows. In section 2 we give the preliminaries, actually recalling and adapting facts from [3], [7]. In Section 3 we specialize to the $SO(p, 9-p)$ cases. In Section 4 we present our results on the multiplet classification of the representations and intertwining differential operators between them.

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2. Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [8, 9].

Let ν be a (non-unitary) character of A', $\nu \in \mathcal{A}'^*$, let μ fix an irreducible representation D^μ of M' on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [10]. (These are called *generalized principal series representations* (or *limits thereof*) in [11].) Their spaces of functions are:

$$\mathcal{C}_\chi \ = \ \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) \ = \ e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \tag{1}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left* regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') \ = \ \mathcal{F}(g^{-1}g') \ , \quad g, g' \in G \ . \tag{2}$$

Further, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf. [11]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of M' with the same Casimirs.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^\mathbb{C}$, where $\Lambda \in (\mathcal{H}^\mathbb{C})^*$, $\mathcal{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathcal{G}^\mathbb{C}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [12]. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $\mathcal{G}$ from $\mathcal{G}^\mathbb{C}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [12, 13]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^\mathbb{C}$, $m \in \mathbb{N}$, such that the BGG [14] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \ = \ m \ , \quad \beta^\vee \equiv 2\beta/(\beta, \beta) \ . \tag{3}$$

When (3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [15]. More explicitly, [12], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then there exists [12] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \ : \ \mathcal{C}_{\chi(\Lambda)} \ \longrightarrow \ \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \ = \ \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (1).

3. The non-compact Lie algebra so(p,9-p)

Let $\mathcal{G} \ = \ so(p, 9-p)$ [16], $5 \le p \le 8$. For all p these algebras have discrete series representations (quaternionic for $p = 5, 6$, holomorphic for $p = 7$). The maximal compact subalgebra is $\mathcal{K} \cong so(p) \oplus so(9-p)$, $\dim_{\mathbb{R}} \mathcal{Q} = 9p - p^2$, $\dim_{\mathbb{R}} \mathcal{N}^\pm = 10p - p^2 - 9$. The split rank is equal to $9-p$, while $\mathcal{M}_0 = so(2p-9)$.

The Dynkin diagram of $so(9,\mathbb{C})$ and $so(5,4)$ is:

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\circ} \text{———} \underset{\alpha_3}{\circ} \Rightarrow \underset{\alpha_4}{\circ} \tag{6}$$

The Dynkin diagram of $so(6,3)$ is:

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\circ} \text{———} \underset{\alpha_3}{\circ} \Rightarrow \underset{\alpha_4}{\bullet} \tag{7}$$

The Dynkin diagram of $so(7,2)$ is:

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\circ} \text{———} \underset{\alpha_3}{\bullet} \Rightarrow \underset{\alpha_4}{\bullet} \tag{8}$$

The Dynkin diagram of $so(8,1)$ is:

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\bullet} \text{———} \underset{\alpha_3}{\bullet} \Rightarrow \underset{\alpha_4}{\bullet} \tag{9}$$

where by convention the black dots represent the subdiagram of $\mathcal{M}_0$.

We would like to work with the non-minimal parabolic subalgebra determined by:

$$\mathcal{M} \ = \ \begin{array}{ll} sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) & \text{for } so(5,4) \\ sl(2,\mathbb{R}) \oplus so(3) & \text{for } so(6,3) \end{array} \tag{10}$$

since the corresponding compact quotient $SO(9)/(SO(3) \times SO(3))$ is an Einstein manifold [6]. By general considerations we should have as possible embedding $\mathcal{M}_0 \subset \mathcal{M}$. Thus, we are restricted to consider the cases $p = 5, 6$.

The Satake diagram [17] for $\mathcal{G}$ for both $p = 5, 6$ may be chosen to be:

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\bullet} \text{———} \underset{\alpha_3}{\circ} \Rightarrow \underset{\alpha_4}{\bullet} \tag{11}$$

where the black dots represent the roots corresponding to $\mathcal{M}$. This Satake diagram was used in [18].

Again for both $p = 5, 6$ there is one more choice of Satake diagram corresponding to different embedding and roots of $\mathcal{M}$:

$$\underset{\alpha_1}{\bullet} \text{---} \underset{\alpha_2}{\circ} \text{---} \underset{\alpha_3}{\circ} \Rightarrow \underset{\alpha_4}{\bullet} \tag{12}$$

In the case of $so(5,4)$ there is one more choice of Satake diagram:

$$\underset{\alpha_1}{\bullet} \text{---} \underset{\alpha_2}{\circ} \text{---} \underset{\alpha_3}{\bullet} \Rightarrow \underset{\alpha_4}{\circ} \tag{13}$$

This Satake diagram was used in [19].

In the present paper we shall consider $\mathcal{G} = so(p, 9-p)$ for both $p = 5, 6$ and Satake diagram (12).

Further, we need the root system of the algebra $so(9, \mathbb{C})$. With Dynkin diagram enumerating the simple roots α_i as in (6),(7) the positive roots are:

$$\begin{aligned} \alpha_{ij} &= \epsilon_i - \epsilon_j,\ 1 \le i < j \le 4 \\ \beta_{ij} &= \epsilon_i + \epsilon_j,\ 1 \le i < j \le 4 \\ \epsilon_k&,\ 1 \le k \le 4 \end{aligned} \tag{14}$$

where ϵ_k are a standard orthonormal set $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

The simple roots are given as follows:

$$\begin{aligned} \gamma_1 &= \alpha_{12} = \epsilon_1 - \epsilon_2,\ \gamma_2 = \alpha_{23} = \epsilon_2 - \epsilon_3, \\ \gamma_3 &= \alpha_{34} = \epsilon_3 - \epsilon_4,\ \gamma_4 = \epsilon_4 \end{aligned} \tag{15}$$

The roots in terms of the simple roots are explicitly given:

$$\begin{aligned} \alpha_{12} &= \gamma_1,\ \ \alpha_{13} = \gamma_1 + \gamma_2 = \gamma_{12},\ \ \alpha_{14} = \gamma_1 + \gamma_2 + \gamma_3 = \gamma_{13}, \\ \alpha_{23} &= \gamma_2,\ \ \alpha_{24} = \gamma_2 + \gamma_3 = \gamma_{23},\ \ \alpha_{34} = \gamma_3 \\ \beta_{12} &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4,\ \beta_{13} = \gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4, \\ \beta_{14} &= \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4,\ \beta_{23} = \gamma_2 + 2\gamma_3 + 2\gamma_4, \\ \beta_{24} &= \gamma_2 + \gamma_3 + 2\gamma_4,\ \beta_{34} = \gamma_3 + 2\gamma_4, \end{aligned} \tag{16a}$$

$$\epsilon_j = \gamma_j + \cdots + \gamma_4, \quad j = 1, 2, 3, 4, \tag{16b}$$

where the 12 roots in (16a) are the long roots, while the four roots in (16b) are the short roots. Note that (16b) gives also the inverse formula for ϵ_k in terms of γ_j. We note also that the highest root of the root system is $\tilde{\alpha} = \beta_{12}$.

Further we need the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^\vee), \quad i = 1, \ldots, 4, \tag{17}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters [20]:

$$m_\beta \equiv (\Lambda + \rho, \beta^\vee), \tag{18}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters (for the non-simple roots) are given in terms of the Dynkin labels explicitly as follows:

$$\begin{aligned}
&m_{12} = m_1 + m_2,\ m_{13} = m_1 + m_2 + m_3, \quad m_{23} = m_2 + m_3, \qquad (19)\\
&\hat{m}_{12} \ = \ m_{14,23} = m_1 + 2m_2 + 2m_3 + m_4,\\
&\hat{m}_{13} \ = \ m_{14,3} = m_1 + m_2 + 2m_3 + m_4,\\
&\hat{m}_{14} = m_{14} \ = \ m_1 + m_2 + m_3 + m_4,\ \hat{m}_{23} \ = \ m_{24,3} = m_2 + 2m_3 + m_4,\\
&\hat{m}_{24} = m_{24} \ = \ m_2 + m_3 + m_4,\ \hat{m}_{34} = m_{34} \ = \ m_3 + m_4,\\
&\tilde{m}_1 = m_{14,13} = 2m_1 + 2m_2 + 2m_3 + m_4,\ \tilde{m}_2 = m_{24,23} = 2m_2 + 2m_3 + m_4,\\
&\tilde{m}_3 = m_{34,3} = 2m_3 + m_4, \quad \tilde{m}_4 = m_4,
\end{aligned}$$

where $m_{ij} \equiv m_{\gamma_{ij}}$, $\hat{m}_{ij} \equiv m_{\beta_{ij}}$, $\tilde{m}_k \equiv m_{\epsilon_k}$.

As stated we shall work with the case (12). Then the roots α_1, α_4 are $\mathcal{M}$-compact, while all other roots are $\mathcal{M}$-non-compact. The differential intertwining operators that give the multiplets correspond to the $\mathcal{M}$-noncompact roots.

We shall induce our representations from the parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, where $\dim \mathcal{A} = 2$, $\dim \mathcal{N} = 14$. Thus, we label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi \ = \ \{\, n_1\, , n_2\, ;\, c_1\, , c_2 \}\ , \qquad (20)$$

where the first two entries are labels of the finite-dimensional irreps of $\mathcal{M}$, thus $n_j \in \mathbb{N}$, while the last two entries of χ label the characters of $\mathcal{A}$. The explicit connection for the choice in (13) is:

$$n_1 = m_1,\ n_2 = m_4,\ c_1 = -m_{12,2},\ c_2 = -m_{34,3} \qquad (21)$$

4. Multiplets

We first give the classification of the multiplets of the **Main type**, which contain maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations. These multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$, i.e., they will be labelled by the four positive Dynkin labels $m_i \in \mathbb{N}$. We know from [4] that the number of ERs in the multiplets is equal to the following ratio of numbers of elements of Weyl groups:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| \ = \ 2^4 4!/2^2 \ = \ 96 \qquad (22)$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp. The multiplets may be given in the following way using (21):

$$\begin{aligned}
\chi_0^\pm \ &= \ \{\, m_1\, , m_4\, ; \mp(m1 + 2m_2)\, , \mp(2m_3 + m_4)\, \},\ \Lambda_0^+ = \Lambda_1^+ - m_3\beta_{34} = \Lambda_2^+ - m_2\beta_{12}\, , \qquad (23)\\
\chi_1^\pm \ &= \ \{\, m_1\, , m_4 + 2m_3\, ; \mp m_{13,23}\, , \mp m_4\, \},\ \Lambda_1^- = \Lambda_0^- - m_3\gamma_3\, ,\\
\chi_2^\pm \ &= \ \{\, m_{12}\, , m_4\, ; \mp(\mu_1 - m_2)\, , \mp m_{24,23}\, \},\ \Lambda_2^- = \Lambda_0^- - m_2\gamma_2\\
\chi_3^\pm \ &= \ \{\, m_1\, , m_4 + 2m_3\, ; \mp m_{14,24}\, , \pm m_4\, \},\ \Lambda_3^- = \Lambda_1^- - m_4\epsilon_3\\
\chi_4^\pm \ &= \ \{\, m_{12}\, , m_4 + 2m_3 + 2m_2\, ; \mp m_{13,3}\, , \mp m_4\, \},\ \Lambda_4^- = \Lambda_1^- - m_2\gamma_{23}\\
\chi_5^\pm \ &= \ \{\, m_{13}\, , m_4 + 2m_3\, ; \mp(m_1 - m_{23})\, , \mp m_{24,23}\, \},\ \Lambda_5^- = \Lambda_2^- - m_3\gamma_{23}\\
\chi_6^\pm \ &= \ \{\, m_2\, , m_4\, ; \pm m_{12,1}\, , \mp m_{14,13}\, \},\ \Lambda_6^- = \Lambda_2^- - m_1\gamma_{12}\\
\chi_7^\pm \ &= \ \{\, m_1\, , m_4\, ; \mp m_{14,24,3,3}\, , \pm m_{34,3}\, \},\ \Lambda_7^- = \Lambda_3^- - m_3\beta_{34}\\
\chi_8^\pm \ &= \ \{\, m_{12}\, , m_{24,23}\, ; \mp m_{14,34}\, , \pm m_4\, \},\ \Lambda_8^- = \Lambda_3^- - m_2\gamma_{23}\\
\chi_9^\pm \ &= \ \{\, m_2\, , m_{14,13}\, ; \mp m_{23,3}\, , \mp m_4\, \},\ \Lambda_9^- = \Lambda_4^- - m_1\gamma_{13}\\
\chi_{10}^\pm \ &= \ \{\, m_{13}\, , m_4 + 2m_3 + 2m_2\, ; \pm(m_3 - m_1 - m_2)\, , \mp m_{34,3}\, \},\ \Lambda_{10}^- = \Lambda_4^- - m_3\gamma_2
\end{aligned}$$

$$
\begin{aligned}
\chi^\pm_{11} &= \{\, m_{23}\,, m_4+2m_3\,; \pm m_{13,1}\,, \mp m_{14,13}\,\},\ \Lambda^-_{11}=\Lambda^-_5 - m_1\gamma_{12}\\
\chi^\pm_{12} &= \{\, m_{14}\,, m_4+2m_3\,; \pm (m_{24}-m_1)\,, \mp m_{24,23}\,\},\ \Lambda^-_{12}=\Lambda^-_5 - m_4\epsilon_2\\
\chi^\pm_{13} &= \{\, m_{12}\,, m_4\,; \mp m_{14,24,34,3}\,, \pm m_{24,23}\,\},\ \Lambda^-_{13}=\Lambda^-_7 - m_2\beta_{23}\\
\chi^\pm_{14} &= \{\, m_{13}\,, m_4+2m_3+2m_2\,; \mp m_{14,34,3}\,, \pm_{34,3}\,\},\ \Lambda^-_{14}=\Lambda^-_8 - m_3\beta_{23}\\
\chi^\pm_{15} &= \{\, m_2\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{24,34}\,, \pm m_4\,\},\ \Lambda^-_{15}=\Lambda^-_8 - m_1\gamma_{13}\\
\chi^\pm_{16} &= \{\, m_{23}\,, m_4+2m_3+2m_2+2m_1\,; \pm (m_3-m_2)\,, \mp m_{34,3}\,\},\ \Lambda^-_{16}=\Lambda^-_9 - m_3\gamma_2\\
\chi^\pm_{17} &= \{\, m_{14}\,, m_4+2m_3+2m_2\,; \pm (m_{34}-m_{12})\,, \mp m_{34,3}\,\},\ \Lambda^-_{17}=\Lambda^-_{10} - m_4\epsilon_2\\
\chi^\pm_{18} &= \{\, m_3\,, m_4+2m_3+2m_2\,; \pm m_{13,12}\,, \mp m_{14,24}\,\},\ \Lambda^-_{18}=\Lambda^-_{11} - m_2\gamma_{13}\\
\chi^\pm_{19} &= \{\, m_{24}\,, m_4+2m_3\,; \pm m_{14,1}\,, \mp m_{14,13}\,\},\ \Lambda^-_{19}=\Lambda^-_{12} - m_1\gamma_{12}\\
\chi^\pm_{20} &= \{\, m_{14,3}\,, m_4\,; \pm (m_{24,3}-m_1)\,, \mp m_{24,23}\,\},\ \Lambda^-_{20}=\Lambda^-_{12} - m_3\beta_{24}\\
\chi^\pm_{21} &= \{\, m_2\,, m_4\,; \mp m_{14,14,23,3}\,, \pm m_{14,13}\,\},\ \Lambda^-_{21}=\Lambda^-_{13} - m_1\beta_{13}\\
\chi^\pm_{22} &= \{\, m_{13}\,, m_4+2m_3\,; \mp m_{14,24,23}\,, \pm m_{24,23}\,\},\ \Lambda^-_{22}=\Lambda^-_{13} - m_3\gamma_{23}\\
\chi^\pm_{23} &= \{\, m_{23}\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{24,34,3}\,, \pm m_{34,3}\,\},\ \Lambda^-_{23}=\Lambda^-_{14} - m_1\gamma_{13}\\
\chi^\pm_{24} &= \{\, m_{14}\,, m_4+2m_3+2m_2\,; \mp m_{14,3,3}\,, \pm m_{34,3}\,\},\ \Lambda^-_{24}=\Lambda^-_{14} - m_4\epsilon_2\\
\chi^\pm_{25} &= \{\, m_3\,, m_4+2m_3+2m_2+2m_1\,; \pm m_{23,2}\,, \mp m_{24,23}\,\},\ \Lambda^-_{25}=\Lambda^-_{16} - m_2\gamma_{12}\\
\chi^\pm_{26} &= \{\, m_{14,3}\,, m_4+2m_3+2m_2\,; \mp (m_{12}-m_4)\,, \mp m_4\,\},\ \Lambda^-_{26}=\Lambda^-_{17} - m_3\beta_{23}\\
\chi^\pm_{27} &= \{\, m_{24}\,, m_4+2m_3+2m_2+2m_1\,; \pm (m_{34}-m_2)\,, \mp m_{34,3}\,\},\ \Lambda^-_{27}=\Lambda^-_{17} - m_1\gamma_{13}\\
\chi^\pm_{28} &= \{\, m_{34}\,, m_4+2m_3+2m_2\,; \pm m_{14,12}\,, \mp m_{14,13}\,\},\ \Lambda^-_{28}=\Lambda^-_{18} - m_4\epsilon_2\\
\chi^\pm_{29} &= \{\, m_{24,3}\,, m_4\,; \pm m_{14,1,3}\,, \mp m_{14,13}\,\},\ \Lambda^-_{29}=\Lambda^-_{19} - m_3\beta_{24}\\
\chi^\pm_{30} &= \{\, m_{14,23}\,, m_4\,; \pm (m_{34,3}-m_1)\,, \mp m_{34,3}\,\},\ \Lambda^-_{30}=\Lambda^-_{20} - m_2\beta_{23}\\
\chi^\pm_{31} &= \{\, m_{23}\,, m_{34,3}\,; \mp m_{14,14,23}\,, \pm m_{14,13}\,\},\ \Lambda^-_{31}=\Lambda^-_{21} - m_3\gamma_{23}\\
\chi^\pm_{32} &= \{\, m_3\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{24,24,3}\,, \pm m_{24,23}\,\},\ \Lambda^-_{32}=\Lambda^-_{23} - m_2\beta_{13}\\
\chi^\pm_{33} &= \{\, m_{14}\,, m_4+2m_3\,; \mp m_{14,23,23}\,, \pm m_{24,23}\,\},\ \Lambda^-_{33}=\Lambda^-_{22} - m_4\epsilon_2\\
\chi^\pm_{34} &= \{\, m_{24}\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{24,3,3}\,, \pm m_{34,3}\,\},\ \Lambda^-_{34}=\Lambda^-_{24} - m_1\gamma_{13}\\
\chi^\pm_{35} &= \{\, m_{14,3}\,, m_4+2m_3+2m_2\,; \pm m_{12,4}\,, \mp m_4\,\},\ \Lambda^-_{35}=\Lambda^-_{24} - m_3\gamma_2\\
\chi^\pm_{36} &= \{\, m_{14,23}\,, m_4+2m_3\,; \pm (m_4-m_1)\,, \mp m_4\,\},\ \Lambda^-_{36}=\Lambda^-_{26} - m_2\beta_{24}\\
\chi^\pm_{37} &= \{\, m_{24,3}\,, m_4+2m_3+2m_2+2m_1\,; \pm (m_4-m_2)\,, \mp m_4\,\},\ \Lambda^-_{37}=\Lambda^-_{27} - m_3\beta_{23}\\
\chi^\pm_{38} &= \{\, m_{34}\,, m_4+2m_3+2m_2+2m_1\,; \pm m_{24,2}\,, \mp m_{24,23}\,\},\ \Lambda^-_{38}=\Lambda^-_{25} - m_4\epsilon_2\\
\chi^\pm_{39} &= \{\, m_{34}\,, m_4+2m_3+2m_2\,; \pm m_{14,13,3}\,, \mp m_{14,13}\,\},\ \Lambda^-_{39}=\Lambda^-_{28} - m_3\beta_{12}\\
\chi^\pm_{40} &= \{\, m_{24,3}\,, m_4\,; \pm m_{14,13,2}\,, \mp m_{14,13}\,\},\ \Lambda^-_{40}=\Lambda^-_{29} - m_2\beta_{12}\\
\chi^\pm_{41} &= \{\, m_{14,23}\,, m_4\,; \pm m_{1,34,3}\,, \mp m_{34,3}\,\},\ \Lambda^-_{41}=\Lambda^-_{30} - m_1\beta_{12}\\
\chi^\pm_{42} &= \{\, m_3\,, m_{24,23}\,; \mp m_{14,14,3}\,, \pm m_{14,13}\,\},\ \Lambda^-_{42}=\Lambda^-_{31} - m_2\gamma_{13}\\
\chi^\pm_{43} &= \{\, m_{24}\,, m_{34,3}\,; \mp m_{14,13,23}\,, \pm m_{14,13}\,\},\ \Lambda^-_{43}=\Lambda^-_{31} - m_4\epsilon_2\\
\chi^\pm_{44} &= \{\, m_{34}\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{24,23,3}\,, \pm m_{24,23}\,\},\ \Lambda^-_{44}=\Lambda^-_{32} - m_4\epsilon_2 = \Lambda^-_{34} - m_2\beta_{13}\\
\chi^\pm_{45} &= \{\, m_{14,3}\,, m_4\,; \mp m_{14,23,2}\,, \pm m_{24,23}\,\},\ \Lambda^-_{45}=\Lambda^-_{33} - m_3\beta_{24}\\
\chi^\pm_{46} &= \{\, m_{14,23}\,, m_4+2m_3\,; \pm m_{1,4}\,, \mp m_4\,\},\ \Lambda^-_{46}=\Lambda^-_{36} - m_1\beta_{12} = \Lambda^-_{35} - m_2\beta_{24}\\
\chi^\pm_{47} &= \{\, m_{24,3}\,, m_4+2m_3+2m_2+2m_1\,; \mp m_{2,4}\,, \pm m_4\,\},\ \Lambda^-_{47}=\Lambda^-_{34} - m_3\gamma_2
\end{aligned}
$$

where we have used for the numbers m_β the same compact notation as in (19) for the roots β. We have shown embeddings between Verma modules only for some weights for the lack of space.

The ERs in the multiplet are related also by intertwining Knapp-Stein integral operators [21]. In our parametrization these operators, denoted $G^\pm$, intertwine the pairs $\mathcal{C}_{\chi^\mp}$ as follows:

$$G^\pm \ :\ \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm} \tag{24}$$

Some remarks are in order. The ER $\mathcal{C}_{\chi_0^-}$ contains as its minimal irreducible subspace the finite-dimensional irrep $\mathcal{E}_0$ parametrized by the four Dynkin labels $[m_1, m_2, m_3, m_4]$. The ER $\mathcal{C}_{\chi_0^+}$ contains as its minimal irreducible subspace the infinite-dimensional quaternionic discrete series irrep $\mathcal{D}_0$ parametrized by the Dynkin labels $[m'_1, m'_2, m'_3, m'_4] = [m_1\,, -m_{12}\,, -m_{34}\,, m_4]$.

Furthermore, the Knapp-Stein operator G^+ acting on $\mathcal{C}_{\chi_0^-}$ annihilates $\mathcal{E}_0$ and its image is $\mathcal{D}_0$. Analogously, the operator G^- acting on $\mathcal{C}_{\chi_0^+}$ annihilates $\mathcal{D}_0$ and its image is $\mathcal{E}_0$. The subspace $\mathcal{E}_0$ is also annihilated by the intertwining differential operators $\mathcal{D}_{m_2,\gamma_2}$, $\mathcal{D}_{m_3,\gamma_3}$ acting from $\mathcal{C}_{\chi_0^-}$ to $\mathcal{C}_{\chi_2^-}$, $\mathcal{C}_{\chi_1^-}$, resp.

Reduced multiplets of type M1

$$\begin{aligned}
\chi_0^{\pm,1} &= \{\,0\,,m_4\,;\mp 2m_2\,,\mp(2m_3+m_4)\,\},\ \Lambda_0^+ = \Lambda_1^+ - m_3\beta_{34} = \Lambda_2^+ - m_2\beta_{12}\,, \\
\chi_1^{\pm,1} &= \{\,0\,,m_4+2m_3\,;\mp m_{23,23}\,,\mp m_4\,\},\ \Lambda_1^- = \Lambda_0^- - m_3\gamma_3\,, \\
\chi_2^{\pm,1} &= \{\,m_2\,,m_4\,;\pm m_2\,,\mp m_{24,23}\,\},\ \Lambda_2^- = \Lambda_0^- - m_2\gamma_2 \\
\chi_3^{\pm,1} &= \{\,0\,,m_4+2m_3\,;\mp m_{24,24}\,,\pm m_4\,\},\ \Lambda_3^- = \Lambda_1^- - m_4\epsilon_3 \\
\chi_4^{\pm,1} &= \{\,m_2\,,m_4+2m_3+2m_2\,;\mp m_{23,3}\,,\mp m_4\,\},\ \Lambda_4^- = \Lambda_1^- - m_2\gamma_{23} \\
\chi_5^{\pm,1} &= \{\,m_{23}\,,m_4+2m_3\,;\pm m_{23}\,,\mp m_{24,23}\,\},\ \Lambda_5^- = \Lambda_2^- - m_3\gamma_{23} \\
\chi_7^{\pm,1} &= \{\,0\,,m_4\,;\mp m_{24,24,3,3}\,,\pm m_{34,3}\,\},\ \Lambda_7^- = \Lambda_3^- - m_3\beta_{34} \\
\chi_8^{\pm,1} &= \{\,m_2\,,m_{24,23}\,;\mp m_{24,34}\,,\pm m_4\,\},\ \Lambda_8^- = \Lambda_3^- - m_2\gamma_{23} \\
\chi_{10}^{\pm,1} &= \{\,m_{23}\,,m_{24,23}\,;\pm(m_3-m_2)\,,\mp m_{34,3}\,\},\ \Lambda_{10}^- = \Lambda_4^- - m_3\gamma_2 \\
\chi_{12}^{\pm,1} &= \{\,m_{24}\,,m_4+2m_3\,;\pm m_{24}\,,\mp m_{24,23}\,\},\ \Lambda_{12}^- = \Lambda_5^- - m_4\epsilon_2 \\
\chi_{13}^{\pm,1} &= \{\,m_2\,,m_4\,;\mp m_{24,24,34,3}\,,\pm m_{24,23}\,\},\ \Lambda_{13}^- = \Lambda_7^- - m_2\beta_{23} \\
\chi_{14}^{\pm,1} &= \{\,m_{23}\,,m_4+2m_3+2m_2\,;\mp m_{24,34,3}\,,\pm_{34,3}\,\},\ \Lambda_{14}^- = \Lambda_8^- - m_3\beta_{23} \\
\chi_{17}^{\pm,1} &= \{\,m_{24}\,,m_4+2m_3+2m_2\,;\pm(m_{34}-m_2)\,,\mp m_{34,3}\,\},\ \Lambda_{17}^- = \Lambda_{10}^- - m_4\epsilon_2 \\
\chi_{18}^{\pm,1} &= \{\,m_3\,,m_4+2m_3+2m_2\,;\pm m_{23,2}\,,\mp m_{24,24}\,\},\ \Lambda_{18}^- = \Lambda_5^- - m_2\gamma_{13} \\
\chi_{20}^{\pm,1} &= \{\,m_{24,3}\,,m_4\,;\pm(m_{24,3}-m_1)\,,\mp m_{24,23}\,\},\ \Lambda_{20}^- = \Lambda_{12}^- - m_3\beta_{24} \\
\chi_{22}^{\pm,1} &= \{\,m_{23}\,,m_4+2m_3\,;\mp m_{24,24,23}\,,\pm m_{24,23}\,\},\ \Lambda_{22}^- = \Lambda_{13}^- - m_3\gamma_{23} \\
\chi_{24}^{\pm,1} &= \{\,m_{24}\,,m_4+2m_3+2m_2\,;\mp m_{24,3,3}\,,\pm m_{34,3}\,\},\ \Lambda_{24}^- = \Lambda_{14}^- - m_4\epsilon_2 \\
\chi_{26}^{\pm,1} &= \{\,m_{24,3}\,,m_4+2m_3+2m_2\,;\mp(m_2-m_4)\,,\mp m_4\,\},\ \Lambda_{26}^- = \Lambda_{17}^- - m_3\beta_{23} \\
\chi_{27}^{\pm,1} &= \{\,m_{24}\,,m_4+2m_3+2m_2\,;\pm(m_{34}-m_2)\,,\mp m_{34,3}\,\},\ \Lambda_{27}^- = \Lambda_{17}^- - m_1\gamma_{13} \\
\chi_{28}^{\pm,1} &= \{\,m_{34}\,,m_4+2m_3+2m_2\,;\pm m_{24,2}\,,\mp m_{24,13}\,\},\ \Lambda_{28}^- = \Lambda_{18}^- - m_4\epsilon_2 \\
\chi_{30}^{\pm,1} &= \{\,m_{24,23}\,,m_4\,;\pm m_{34,3}\,,\mp m_{34,3}\,\},\ \Lambda_{30}^- = \Lambda_{20}^- - m_2\beta_{23} \\
\chi_{32}^{\pm,1} &= \{\,m_3\,,m_4+2m_3+2m_2\,;\mp m_{24,24,3}\,,\pm m_{24,23}\,\},\ \Lambda_{32}^- = \Lambda_{14}^- - m_2\beta_{13} \\
\chi_{33}^{\pm,1} &= \{\,m_{24}\,,m_4+2m_3\,;\mp m_{24,23,23}\,,\pm m_{24,23}\,\},\ \Lambda_{33}^- = \Lambda_{22}^- - m_4\epsilon_2 \\
\chi_{35}^{\pm,1} &= \{\,m_{24,3}\,,m_4+2m_3+2m_2\,;\pm m_{2,4}\,,\mp m_4\,\},\ \Lambda_{35}^- = \Lambda_{24}^- - m_3\gamma_2 \\
\chi_{36}^{\pm,1} &= \{\,m_{24,23}\,,m_4+2m_3\,;\pm m_4\,,\mp m_4\,\},\ \Lambda_{36}^- = \Lambda_{26}^- - m_2\beta_{24} \\
\chi_{39}^{\pm,1} &= \{\,m_{34}\,,m_4+2m_3+2m_2\,;\pm m_{24,23,3}\,,\mp m_{24,23}\,\},\ \Lambda_{39}^- = \Lambda_{28}^- - m_3\beta_{12} \\
\chi_{40}^{\pm,1} &= \{\,m_{24,3}\,,m_4\,;\pm m_{24,23,2}\,,\mp m_{24,23}\,\},\ \Lambda_{40}^- = \Lambda_{20}^- - m_2\beta_{12} \\
\chi_{44}^{\pm,1} &= \{\,m_{34}\,,m_4+2m_3+2m_2\,;\mp m_{24,23,3}\,,\pm m_{24,23}\,\},\ \Lambda_{44}^- = \Lambda_{32}^- - m_4\epsilon_2 \\
\chi_{45}^{\pm,1} &= \{\,m_{24,3}\,,m_4\,;\mp m_{24,23,2}\,,\pm m_{24,23}\,\},\ \Lambda_{45}^- = \Lambda_{33}^- - m_3\beta_{24}
\end{aligned} \tag{25}$$

Reduced multiplets of type M2

$$\begin{aligned}
\chi_0^{\pm,2} &= \{\,m_1\,,m_4\,;\mp m_1\,,\mp(2m_3+m_4)\,\},\ \Lambda_0^+ = \Lambda_1^+ - m_3\beta_{34}\,, \\
\chi_1^{\pm,2} &= \{\,m_1\,,m_4+2m_3\,;\mp m_{1,3,3}\,,\mp m_4\,\},\ \Lambda_1^- = \Lambda_0^- - m_3\gamma_3\,, \\
\chi_3^{\pm,2} &= \{\,m_1\,,m_4+2m_3\,;\mp m_{1,34,34}\,,\pm m_4\,\},\ \Lambda_3^- = \Lambda_1^- - m_4\epsilon_3 \\
\chi_5^{\pm,2} &= \{\,m_{1,3}\,,m_4+2m_3\,;\mp(m_1-m_3)\,,\mp m_{34,3}\,\},\ \Lambda_5^- = \Lambda_2^- - m_3\gamma_{23} \\
\chi_6^{\pm,2} &= \{\,0\,,m_4\,;\pm m_{1,1}\,,\mp m_{1,34,1,3}\,\},\ \Lambda_6^- = \Lambda_0^- - m_1\gamma_{12} \\
\chi_7^{\pm,2} &= \{\,m_1\,,m_4\,;\mp m_{1,34,34,3,3}\,,\pm m_{34,3}\,\},\ \Lambda_7^- = \Lambda_3^- - m_3\beta_{34}
\end{aligned} \tag{26}$$

$$\begin{aligned}
\chi_9^{\pm,2} &= \{\, 0\,, m_{1,34,1,3}\,; \mp m_{3,3}\,, \mp m_4 \,\},\ \Lambda_9^- = \Lambda_4^- - m_1\gamma_{13} \\
\chi_{11}^{\pm,2} &= \{\, m_3\,, m_4 + 2m_3\,; \pm m_{1,3,1}\,, \mp m_{1,34,1,3} \,\},\ \Lambda_{11}^- = \Lambda_5^- - m_1\gamma_{12} \\
\chi_{12}^{\pm,2} &= \{\, m_{1,34}\,, m_4 + 2m_3\,; \pm(m_{34} - m_1)\,, \mp m_{34,3} \,\},\ \Lambda_{12}^- = \Lambda_5^- - m_4\epsilon_2 \\
\chi_{14}^{\pm,2} &= \{\, m_{1,3}\,, m_4 + 2m_3\,; \mp m_{1,34,34,3}\,, \pm_{34,3} \,\},\ \Lambda_{14}^- = \Lambda_3^- - m_3\beta_{23} \\
\chi_{15}^{\pm,2} &= \{\, 0\,, m_4 + 2m_3 + 2m_1\,; \mp m_{34,34}\,, \pm m_4 \,\},\ \Lambda_{15}^- = \Lambda_3^- - m_1\gamma_{13} \\
\chi_{16}^{\pm,2} &= \{\, m_3\,, m_4 + 2m_3 + 2m_1\,; \pm m_3\,, \mp m_{34,3} \,\},\ \Lambda_{16}^- = \Lambda_9^- - m_3\gamma_2 \\
\chi_{19}^{\pm,2} &= \{\, m_{34}\,, m_4 + 2m_3\,; \pm m_{1,34,1}\,, \mp m_{1,34,1,3} \,\},\ \Lambda_{19}^- = \Lambda_{12}^- - m_1\gamma_{12} \\
\chi_{20}^{\pm,2} &= \{\, m_{1,34,3}\,, m_4\,; \pm(m_{34,3} - m_1)\,, \mp m_{34,3} \,\},\ \Lambda_{20}^- = \Lambda_{12}^- - m_3\beta_{24} \\
\chi_{21}^{\pm,2} &= \{\, 0\,, m_4\,; \mp m_{1,34,1,34,3,3}\,, \pm m_{1,34,1,3} \,\},\ \Lambda_{21}^- = \Lambda_7^- - m_1\beta_{13} \\
\chi_{23}^{\pm,2} &= \{\, m_3\,, m_4 + 2m_3 + 2m_1\,; \mp m_{34,34,3}\,, \pm m_{34,3} \,\},\ \Lambda_{23}^- = \Lambda_{14}^- - m_1\gamma_{13} \\
\chi_{24}^{\pm,2} &= \{\, m_{1,34}\,, m_4 + 2m_3\,; \mp m_{1,34,3,3}\,, \pm m_{34,3} \,\},\ \Lambda_{24}^- = \Lambda_{14}^- - m_4\epsilon_2 \\
\chi_{26}^{\pm,2} &= \{\, m_{1,34,3}\,, m_4 + 2m_3\,; \mp(m_1 - m_4)\,, \mp m_4 \,\},\ \Lambda_{26}^- = \Lambda_{17}^- - m_3\beta_{23} \\
\chi_{27}^{\pm,2} &= \{\, m_{34}\,, m_4 + 2m_3 + 2m_1\,; \pm m_{34}\,, \mp m_{34,3} \,\},\ \Lambda_{27}^- = \Lambda_{17}^- - m_1\gamma_{13} \\
\chi_{29}^{\pm,2} &= \{\, m_{34,3}\,, m_4\,; \pm m_{1,34,1,3}\,, \mp m_{1,34,1,3} \,\},\ \Lambda_{29}^- = \Lambda_{19}^- - m_3\beta_{24} \\
\chi_{31}^{\pm,2} &= \{\, m_3\,, m_{34,3}\,; \mp m_{1,34,1,34,3}\,, \pm m_{1,34,1,3} \,\},\ \Lambda_{31}^- = \Lambda_{21}^- - m_3\gamma_{23} \\
\chi_{34}^{\pm,2} &= \{\, m_{34}\,, m_4 + 2m_3 + 2m_1\,; \mp m_{34,3,3}\,, \pm m_{34,3} \,\},\ \Lambda_{34}^- = \Lambda_{24}^- - m_1\gamma_{13} \\
\chi_{35}^{\pm,2} &= \{\, m_{1,34,3}\,, m_4 + 2m_3\,; \pm m_{1,4}\,, \mp m_4 \,\},\ \Lambda_{35}^- = \Lambda_{24}^- - m_3\gamma_2 \\
\chi_{37}^{\pm,2} &= \{\, m_{34,3}\,, m_4 + 2m_3 + 2m_1\,; \pm m_4\,, \mp m_4 \,\},\ \Lambda_{37}^- = \Lambda_{27}^- - m_3\beta_{23} \\
\chi_{39}^{\pm,2} &= \{\, m_{34}\,, m_4 + 2m_3\,; \pm m_{1,34,1,3,3}\,, \mp m_{1,34,1,3} \,\},\ \Lambda_{39}^- = \Lambda_{28}^- - m_3\beta_{12} \\
\chi_{41}^{\pm,2} &= \{\, m_{1,34,3}\,, m_4\,; \pm m_{1,34,3}\,, \mp m_{34,3} \,\},\ \Lambda_{41}^- = \Lambda_{20}^- - m_1\beta_{12} \\
\chi_{43}^{\pm,2} &= \{\, m_{34}\,, m_{34,3}\,; \mp m_{1,34,1,3,3}\,, \pm m_{1,34,1,3} \,\},\ \Lambda_{43}^- = \Lambda_{31}^- - m_4\epsilon_2 \\
\chi_{45}^{\pm,2} &= \{\, m_{1,34,3}\,, m_4\,; \mp m_{1,34,3}\,, \pm m_{34,3} \,\},\ \Lambda_{45}^- = \Lambda_{33}^- - m_3\beta_{24} \\
\chi_{47}^{\pm,2} &= \{\, m_{34,3}\,, m_4 + 2m_3 + 2m_1\,; \mp m_4\,, \pm m_4 \,\},\ \Lambda_{47}^- = \Lambda_{34}^- - m_3\gamma_2
\end{aligned}$$

The ER $\mathcal{C}_{\chi_0^{+,2}}$ contains as its minimal irreducible subspace an infinite-dimensional quaternionic discrete series irrep $\mathcal{D}_0^2$ parametrized by the Dynkin labels $[m_1', m_2', m_3', m_4'] = [m_1\,, -m_1\,, -m_{34}\,, m_4]$.

Reduced multiplets of type M3

$$\begin{aligned}
\chi_0^{\pm,3} &= \{\, m_1\,, m_4\,; \mp(m1 + 2m_2)\,, \mp m_4 \,\},\ \Lambda_0^+ = \Lambda_2^+ - m_2\beta_{12}\,, \qquad (27) \\
\chi_2^{\pm,3} &= \{\, m_{12}\,, m_4\,; \mp(\mu_1 - m_2)\,, \mp m_{2,4,2} \,\},\ \Lambda_2^- = \Lambda_0^- - m_2\gamma_2 \\
\chi_3^{\pm,3} &= \{\, m_1\,, m_4\,; \mp m_{12,4,2,4}\,, \pm m_4 \,\},\ \Lambda_3^- = \Lambda_0^- - m_4\epsilon_3 \\
\chi_4^{\pm,3} &= \{\, m_{12}\,, m_4 + 2m_2\,; \mp m_{12}\,, \mp m_4 \,\},\ \Lambda_4^- = \Lambda_0^- - m_2\gamma_{23} \\
\chi_6^{\pm,3} &= \{\, m_2\,, m_4\,; \pm m_{12,1}\,, \mp m_{12,4,12} \,\},\ \Lambda_6^- = \Lambda_2^- - m_1\gamma_{12} \\
\chi_8^{\pm,3} &= \{\, m_{12}\,, m_{2,4,2}\,; \mp m_{12,4,4}\,, \pm m_4 \,\},\ \Lambda_8^- = \Lambda_3^- - m_2\gamma_{23} \\
\chi_9^{\pm,3} &= \{\, m_2\,, m_{12,4,12}\,; \mp m_2\,, \mp m_4 \,\},\ \Lambda_9^- = \Lambda_4^- - m_1\gamma_{13} \\
\chi_{12}^{\pm,3} &= \{\, m_{12,4}\,, m_4\,; \pm(m_{2,4} - m_1)\,, \mp m_{2,4,2} \,\},\ \Lambda_{12}^- = \Lambda_2^- - m_4\epsilon_2 \\
\chi_{13}^{\pm,3} &= \{\, m_{12}\,, m_4\,; \mp m_{12,4,2,4,4}\,, \pm m_{2,4,2} \,\},\ \Lambda_{13}^- = \Lambda_3^- - m_2\beta_{23} \\
\chi_{15}^{\pm,3} &= \{\, m_2\,, m_4 + 2m_2 + 2m_1\,; \mp m_{2,4,4}\,, \pm m_4 \,\},\ \Lambda_{15}^- = \Lambda_8^- - m_1\gamma_{13} \\
\chi_{17}^{\pm,3} &= \{\, m_{12,4}\,, m_4 + 2m_2\,; \pm(m_4 - m_{12})\,, \mp m_4 \,\},\ \Lambda_{17}^- = \Lambda_4^- - m_4\epsilon_2 \\
\chi_{18}^{\pm,3} &= \{\, 0\,, m_4 + 2m_2\,; \pm m_{12,12}\,, \mp m_{12,4,2,4} \,\},\ \Lambda_{18}^- = \Lambda_{11}^- - m_2\gamma_{13} \\
\chi_{19}^{\pm,3} &= \{\, m_{2,4}\,, m_4\,; \pm m_{12,4,1}\,, \mp m_{12,4,12} \,\},\ \Lambda_{19}^- = \Lambda_{12}^- - m_1\gamma_{12} \\
\chi_{21}^{\pm,3} &= \{\, m_2\,, m_4\,; \mp m_{12,4,12,4,2}\,, \pm m_{12,4,12} \,\},\ \Lambda_{21}^- = \Lambda_{13}^- - m_1\beta_{13} \\
\chi_{24}^{\pm,3} &= \{\, m_{12,4}\,, m_4 + 2m_2\,; \mp m_{12,4}\,, \pm m_4 \,\},\ \Lambda_{24}^- = \Lambda_8^- - m_4\epsilon_2 \\
\chi_{25}^{\pm,3} &= \{\, 0\,, m_4 + 2m_2 + 2m_1\,; \pm m_{2,2}\,, \mp m_{2,4,2} \,\},\ \Lambda_{25}^- = \Lambda_9^- - m_2\gamma_{12}
\end{aligned}$$

$$
\begin{aligned}
\chi_{27}^{\pm,3} &= \{\, m_{2,4}\,, m_4 + 2m_2 + 2m_1\,; \pm(m_4 - m_2)\,, \mp m_4 \,\},\ \Lambda_{27}^- = \Lambda_{17}^- - m_1\gamma_{13} \\
\chi_{28}^{\pm,3} &= \{\, m_4\,, m_4 + 2m_2\,; \pm m_{12,4,12}\,, \mp m_{12,4,12} \,\},\ \Lambda_{28}^- = \Lambda_{18}^- - m_4\epsilon_2 \\
\chi_{30}^{\pm,3} &= \{\, m_{12,4,2}\,, m_4\,; \pm(m_4 - m_1)\,, \mp m_4 \,\},\ \Lambda_{30}^- = \Lambda_{12}^- - m_2\beta_{23} \\
\chi_{32}^{\pm,3} &= \{\, 0\,, m_4 + 2m_2 + 2m_1\,; \mp m_{2,4,2,4}\,, \pm m_{2,4,2} \,\},\ \Lambda_{32}^- = \Lambda_{23}^- - m_2\beta_{13} \\
\chi_{33}^{\pm,3} &= \{\, m_{12,4}\,, m_4\,; \mp m_{12,4,2,2}\,, \pm m_{2,4,2} \,\},\ \Lambda_{33}^- = \Lambda_{13}^- - m_4\epsilon_2 \\
\chi_{34}^{\pm,3} &= \{\, m_{2,4}\,, m_4 + 2m_2 + 2m_1\,; \mp m_{2,4}\,, \pm m_4 \,\},\ \Lambda_{34}^- = \Lambda_{24}^- - m_1\gamma_{13} \\
\chi_{38}^{\pm,3} &= \{\, m_4\,, m_4 + 2m_2 + 2m_1\,; \pm m_{2,4,2}\,, \mp m_{2,4,2} \,\},\ \Lambda_{38}^- = \Lambda_{25}^- - m_4\epsilon_2 \\
\chi_{40}^{\pm,3} &= \{\, m_{2,4}\,, m_4\,; \pm m_{12,4,12,2}\,, \mp m_{12,4,12} \,\},\ \Lambda_{40}^- = \Lambda_{19}^- - m_2\beta_{12} \\
\chi_{41}^{\pm,3} &= \{\, m_{12,4,2}\,, m_4\,; \pm m_{1,4}\,, \mp m_4 \,\},\ \Lambda_{41}^- = \Lambda_{30}^- - m_1\beta_{12} \\
\chi_{42}^{\pm,3} &= \{\, 0\,, m_{2,4,2}\,; \mp m_{12,4,12,4}\,, \pm m_{12,4,12} \,\},\ \Lambda_{42}^- = \Lambda_{21}^- - m_2\gamma_{13} \\
\chi_{43}^{\pm,3} &= \{\, m_{2,4}\,, m_4\,; \mp m_{12,4,12,2}\,, \pm m_{12,4,12} \,\},\ \Lambda_{43}^- = \Lambda_{21}^- - m_4\epsilon_2 \\
\chi_{44}^{\pm} &= \{\, m_4\,, m_4 + 2m_2 + 2m_1\,; \mp m_{2,4,2}\,, \pm m_{2,4,2} \,\},\ \Lambda_{44}^- = \Lambda_{32}^- - m_4\epsilon_2 = \Lambda_{34}^- - m_2\beta_{13}
\end{aligned}
$$

The ER $\mathcal{C}_{\chi_0^{+,3}}$ contains as its minimal irreducible subspace an infinite-dimensional quaternionic discrete series irrep $\mathcal{D}_0^3$ parametrized by the Dynkin labels $[m_1', m_2', m_3', m_4'] = [m_1\,, -m_{12}\,, -m_4\,, m_4]$.

Reduced multiplets of type M4

$$
\begin{aligned}
\chi_0^{\pm,4} &= \{\, m_1\,, 0\,; \mp(m1 + 2m_2)\,, \mp 2m_3 \,\},\ \Lambda_0^+ = \Lambda_1^+ - m_3\beta_{34} = \Lambda_2^+ - m_2\beta_{12}\,, \qquad (28) \\
\chi_1^{\pm,4} &= \{\, m_1\,, 2m_3\,; \mp m_{13,23}\,, 0 \,\},\ \Lambda_1^- = \Lambda_0^- - m_3\gamma_3\,, \\
\chi_2^{\pm,4} &= \{\, m_{12}\,, 0\,; \mp(\mu_1 - m_2)\,, \mp m_{23,23} \,\},\ \Lambda_2^- = \Lambda_0^- - m_2\gamma_2 \\
\chi_4^{\pm,4} &= \{\, m_{12}\,, 2m_3 + 2m_2\,; \mp m_{13,3}\,, 0 \,\},\ \Lambda_4^- = \Lambda_1^- - m_2\gamma_{23} \\
\chi_5^{\pm,4} &= \{\, m_{13}\,, 2m_3\,; \mp(m_1 - m_{23})\,, \mp m_{23,23} \,\},\ \Lambda_5^- = \Lambda_2^- - m_3\gamma_{23} \\
\chi_6^{\pm,4} &= \{\, m_2\,, 0\,; \pm m_{12,1}\,, \mp m_{13,13} \,\},\ \Lambda_6^- = \Lambda_2^- - m_1\gamma_{12} \\
\chi_7^{\pm,4} &= \{\, m_1\,, 0\,; \mp m_{13,23,3,3}\,, \pm m_{3,3} \,\},\ \Lambda_7^- = \Lambda_1^- - m_3\beta_{34} \\
\chi_8^{\pm,4} &= \{\, m_{12}\,, m_{23,23}\,; \mp m_{13,3}\,, 0 \,\},\ \Lambda_8^- = \Lambda_1^- - m_2\gamma_{23} \\
\chi_9^{\pm,4} &= \{\, m_2\,, m_{13,13}\,; \mp m_{23,3}\,, 0 \,\},\ \Lambda_9^- = \Lambda_4^- - m_1\gamma_{13} \\
\chi_{10}^{\pm,4} &= \{\, m_{13}\,, 2m_3 + 2m_2\,; \pm(m_3 - m_1 - m_2)\,, \mp m_{3,3} \,\},\ \Lambda_{10}^- = \Lambda_4^- - m_3\gamma_2 \\
\chi_{11}^{\pm,4} &= \{\, m_{23}\,, 2m_3\,; \pm m_{13,1}\,, \mp m_{13,13} \,\},\ \Lambda_{11}^- = \Lambda_5^- - m_1\gamma_{12} \\
\chi_{13}^{\pm,4} &= \{\, m_{12}\,, 0\,; \mp m_{13,23,3,3}\,, \pm m_{23,23} \,\},\ \Lambda_{13}^- = \Lambda_7^- - m_2\beta_{23} \\
\chi_{14}^{\pm,4} &= \{\, m_{13}\,, 2m_3 + 2m_2\,; \mp m_{13,3,3}\,, \pm_{3,3} \,\},\ \Lambda_{14}^- = \Lambda_8^- - m_3\beta_{23} \\
\chi_{15}^{\pm,4} &= \{\, m_2\,, 2m_3 + 2m_2 + 2m_1\,; \mp m_{23,3}\,, 0 \,\},\ \Lambda_{15}^- = \Lambda_8^- - m_1\gamma_{13} \\
\chi_{16}^{\pm,4} &= \{\, m_{23}\,, 2m_3 + 2m_2 + 2m_1\,; \pm(m_3 - m_2)\,, \mp m_{3,3} \,\},\ \Lambda_{16}^- = \Lambda_9^- - m_3\gamma_2 \\
\chi_{18}^{\pm,4} &= \{\, m_3\,, 2m_3 + 2m_2\,; \pm m_{13,12}\,, \mp m_{13,23} \,\},\ \Lambda_{18}^- = \Lambda_{11}^- - m_2\gamma_{13} \\
\chi_{19}^{\pm,4} &= \{\, m_{23}\,, 2m_3\,; \pm m_{13,1}\,, \mp m_{13,13} \,\},\ \Lambda_{19}^- = \Lambda_5^- - m_1\gamma_{12} \\
\chi_{20}^{\pm,4} &= \{\, m_{13,3}\,, 0\,; \pm(m_{23,3} - m_1)\,, \mp m_{23,23} \,\},\ \Lambda_{20}^- = \Lambda_5^- - m_3\beta_{24} \\
\chi_{21}^{\pm,4} &= \{\, m_2\,, 0\,; \mp m_{13,13,23,3}\,, \pm m_{13,13} \,\},\ \Lambda_{21}^- = \Lambda_{13}^- - m_1\beta_{13} \\
\chi_{22}^{\pm,4} &= \{\, m_{13}\,, 2m_3\,; \mp m_{13,23,23}\,, \pm m_{23,23} \,\},\ \Lambda_{22}^- = \Lambda_{13}^- - m_3\gamma_{23} \\
\chi_{23}^{\pm,4} &= \{\, m_{23}\,, 2m_{13}\,; \mp m_{23,33,3}\,, \pm m_{33,3} \,\},\ \Lambda_{23}^- = \Lambda_{14}^- - m_1\gamma_{13} \\
\chi_{25}^{\pm,4} &= \{\, m_3\,, 2m_{13}\,; \pm m_{23,2}\,, \mp m_{23,23} \,\},\ \Lambda_{25}^- = \Lambda_{16}^- - m_2\gamma_{12} \\
\chi_{26}^{\pm,4} &= \{\, m_{13,3}\,, 2m_3 + 2m_2\,; \mp m_{12}\,, 0 \,\},\ \Lambda_{26}^- = \Lambda_{10}^- - m_3\beta_{23} \\
\chi_{27}^{\pm,4} &= \{\, m_{23}\,, 2m_{13}\,; \pm(m_3 - m_2)\,, \mp m_{3,3} \,\},\ \Lambda_{27}^- = \Lambda_{10}^- - m_1\gamma_{13} \\
\chi_{29}^{\pm,4} &= \{\, m_{23,3}\,, 0\,; \pm m_{13,1,3}\,, \mp m_{13,13} \,\},\ \Lambda_{29}^- = \Lambda_{19}^- - m_3\beta_{24} \\
\chi_{30}^{\pm,4} &= \{\, m_{13,23}\,, 0\,; \pm(m_{3,3} - m_1)\,, \mp m_{3,3} \,\},\ \Lambda_{30}^- = \Lambda_{20}^- - m_2\beta_{23} \\
\chi_{31}^{\pm,4} &= \{\, m_{23}\,, 2m_3\,; \mp m_{13,13,23}\,, \pm m_{13,13} \,\},\ \Lambda_{31}^- = \Lambda_{21}^- - m_3\gamma_{23}
\end{aligned}
$$

$$\begin{aligned}
\chi_{32}^{\pm,4} &= \{\, m_3\,, 2m_{13}\,; \mp m_{23,23,3}\,, \pm m_{23,23}\,\},\ \Lambda_{32}^- = \Lambda_{23}^- - m_2\beta_{13}\\
\chi_{34}^{\pm,4} &= \{\, m_{23}\,, 2m_{13}\,; \mp m_{23,3,3}\,, \pm m_{3,3}\,\},\ \Lambda_{34}^- = \Lambda_{24}^- - m_1\gamma_{13}\\
\chi_{35}^{\pm,4} &= \{\, m_{13,3}\,, 2m_{23}\,; \pm m_{12}\,, 0\,\},\ \Lambda_{35}^- = \Lambda_{24}^- - m_3\gamma_2\\
\chi_{36}^{\pm,4} &= \{\, m_{13,23}\,, 2m_3\,; \mp m_1\,, 0\,\},\ \Lambda_{36}^- = \Lambda_{26}^- - m_2\beta_{24}\\
\chi_{37}^{\pm,4} &= \{\, m_{23,3}\,, 2m_{13}\,; \mp m_2\,, 0\,\},\ \Lambda_{37}^- = \Lambda_{27}^- - m_3\beta_{23}\\
\chi_{39}^{\pm,4} &= \{\, m_3\,, 2m_3 + 2m_2\,; \pm m_{13,13,3}\,, \mp m_{13,13}\,\},\ \Lambda_{39}^- = \Lambda_{28}^- - m_3\beta_{12}\\
\chi_{40}^{\pm,4} &= \{\, m_{23,3}\,, 0\,; \pm m_{13,13,2}\,, \mp m_{13,13}\,\},\ \Lambda_{40}^- = \Lambda_{29}^- - m_2\beta_{12}\\
\chi_{41}^{\pm,4} &= \{\, m_{13,23}\,, 0\,; \pm m_{1,3,3}\,, \mp m_{3,3}\,\},\ \Lambda_{41}^- = \Lambda_{30}^- - m_1\beta_{12}\\
\chi_{42}^{\pm,4} &= \{\, m_3\,, m_{23,23}\,; \mp m_{13,13,3}\,, \pm m_{13,13}\,\},\ \Lambda_{42}^- = \Lambda_{31}^- - m_2\gamma_{13}\\
\chi_{45}^{\pm,4} &= \{\, m_{13,3}\,, 0\,; \mp m_{13,23,2}\,, \pm m_{23,23}\,\},\ \Lambda_{45}^- = \Lambda_{33}^- - m_3\beta_{24}\\
\chi_{46}^{\pm,4} &= \{\, m_{13,23}\,, 2m_3\,; \pm m_1\,, 0\,\},\ \Lambda_{46}^- = \Lambda_{36}^- - m_1\beta_{12} = \Lambda_{35}^- - m_2\beta_{24}\\
\chi_{47}^{\pm,4} &= \{\, m_{23,3}\,, 2m_3 + 2m_2 + 2m_1\,; \mp m_2\,, 0\,\},\ \Lambda_{47}^- = \Lambda_{34}^- - m_3\gamma_2
\end{aligned}$$

Finally, we note a doubly reduced multiplet, from which we show only the ER:

$$\chi_{00}^{+} = [m_1\,, -m_1\,, -m_4\,, m_4] \tag{29}$$

which contains another quaternionic discrete series representation.

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