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Article

Continuity Equation of Transverse Kähler Metrics on Sasakian Manifolds

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Abstract: We introduce the continuity equation of transverse Kähler metrics on Sasakian manifolds and establish its interval of maximal existence. When the first basic Chern class is null (resp. negative), we prove that the solution of the (resp. normalized) continuity equation converges smoothly to the unique η-Einstein metric in the basic Bott–Chern cohomological class of the initial transverse Kähler metric (resp. first basic Chern class). These results are the transverse version of the continuity equation of the Kähler metrics studied by La Nave and Tian, and also counterparts of the Sasaki–Ricci flow studied by Smoczyk, Wang, and Zhang.

Keywords: Sasakian manifold; basic Chern class; continuity equation; transverse Kähler metric; η-Einstein metric

MSC: 53C25; 35J60; 32W20; 58J05

1. Introduction

The Sasakian manifold introduced by Sasaki [1] is the odd dimension counterpart of the Kähler manifold and the natural intersection of Cauchy–Riemann (frequently abbreviated as CR), contact and Riemannian geometry. It plays an important role in Riemannian geometry, algebraic geometry, and physics, such as string theory [2] and anti-de Sitter/conformal field theory (frequently abbreviated as Ads/CFT) correspondence in which the Sasaki–Einstein metric is useful (see more details in [3] and the references therein).

We refer the reader to Boyer and Galicki [4], which includes many papers and references about all kinds of differential geometric aspects of Sasakian manifolds.

There are many transverse counterparts of the famous results in Kähler geometry on Sasakian manifolds, which is called the basic global analysis on Sasakian manifolds. These results include the transverse Calabi–Yau theorem [5] (see also [6,7]), the existence of canonical metrics on Sasakian manifolds [8], the (general) Frankel conjecture on Sasakian manifolds [9,10], Sasaki–Einstein metrics and stability on Sasakian manifolds [11,12], and so on. We refer the reader to [13–21] and references therein for more results for basic global analysis on Sasakian manifolds.

Recently, La Nave and Tian [22] introduced the continuity equation of Kähler metrics as an alternative to the Kähler–Ricci flow in carrying out the analytic minimal model program [23,24]. There are many developments [25–34] in this field.

On the other hand, the Sasaki–Ricci flow [7], a transverse version of the Kähler–Ricci flow, is now very useful in the research of Sasaki geometry, which leads to many developments [35–42].

Motivated by the continuity equation of Kähler metrics [22] and the Sasaki–Ricci flow [7], we study the continuity equation of the transverse Kähler metric on Sasaki manifolds

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$$\omega^\dagger(t) = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger(t)), \tag{1}$$

where $\omega_0^\dagger$ is an initial transverse Kähler metric, and $\mathrm{Ric}^\dagger(\omega^\dagger(t))$ defined by (79) is the basic Chern–Ricci form of a family of transverse Kähler metrics $\omega^\dagger(t)$. All the terms in this section can be found in Section 2.

We first prove the maximum time existence of the solution to the continuity Equation (1) of transverse Kähler metrics as follows:

Theorem 1. *Let $(M, \xi, \phi, \eta, g_0)$ be a Sasaki manifold with $\dim_{\mathbb{R}} M = 2n+1$ and $\omega_0^\dagger$ the transverse Kähler form associated to g_0 given by (24). Then, there exists a unique family of transverse Kähler forms $\omega^\dagger(t)$ satisfying*

$$\omega^\dagger(t) = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger(t)), \quad \omega^\dagger(t) >_{\mathrm{b}} 0, \quad t \in [0, T), \tag{2}$$

where T is defined by

$$T := \sup\Big\{ t > 0 : \{\omega_0^\dagger\} - tc_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) >_{\mathrm{b}} 0 \text{ in } H^{1,1}_{\mathrm{BC,b}}(M, \mathbb{R}) \Big\}. \tag{3}$$

In particular, if $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi))$ is transverse nef, then $T = +\infty$.

Theorem 1 is a transverse version of the interval of maximum existence of the solution of the continuity equation of Kähler metrics studied by La Nave and Tian [22] (Theorem 1.1) (see Sherman and Weinkove [43] in the Hermitian case and Li and Zheng [44] in the almost Hermitian case), and a counterpart of the Sasaki–Ricci flow [7].

Then, we have the following convergence results for the solution to the continuity Equation (1) of transverse Kähler metrics.

Theorem 2. *Let $(M, \xi, \phi, \eta, g_0)$ be a Sasaki manifold with $\dim_{\mathbb{R}} M = 2n+1$ and $\omega_0^\dagger$ the transverse Kähler form associated to g_0 given by (24). If $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) <_{\mathrm{b}} 0$, then the solution of the normalized continuity equation of transverse Kähler metrics*

$$(1+t)\omega^\dagger(t) = \omega_0^\dagger - t\,\mathrm{Ric}(\omega^\dagger(t)) \tag{4}$$

converges smoothly to a limit, which is the unique η-Einstein metric.

If $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) = 0$, then the solution of the continuity equation of transverse Kähler metrics (2) converges smoothly to a limit, which is the unique η-Einstein metric in $[\omega_0^\dagger]$.

Theorem 2 is a transverse version of convergence of the solution of the continuity equation of Kähler metrics to the Kähler–Einstein metric when the first Chern class is negative or null implied in La Nave and Tian [22] (Theorem 1.2), and a counterpart of the Sasaki–Ricci flow [7].

When the Sasakian manifold M is regular, its base space B of the Boothby–Wang foliation given by [45] is a Hodge manifold (see [46] (Theorem 4)). In this special case, the continuity equation of transverse Kähler metrics can be reduced to the continuity equation of Kähler metrics on B studied in La Nave and Tian [22]. In the general case, the base space B would be very wild and has no manifold structure, and hence it is meaningful to consider the continuity equation of transverse Kähler metrics on Sasakian manifolds from this viewpoint.

The method in this paper is modified from the (almost) complex case [22,44,47] (we refer the reader to [48]). Given enough details of the preliminaries of Sasakian manifolds, such as the foliated local coordinates (see Section 2), we find that many calculations at one fixed point are the same as the ones in the complex case. Hence, we can omit some very complicated calculations, which can be found in [47] (we refer the reader to [48]) and give more details, such as in the proof of Proposition 1, which are omitted in [22] for convenience.

The outline of this paper is as follows: In Section 2, we give some details of basic concepts about (almost) contact and Sasakian manifolds. We prove Theorem 1 and Theorem 2 in Section 3 and Section 4, respectively.

2. Preliminaries

In this section, we provide some preliminaries about contact manifolds, which will be used in the following (see, for example, [4,17,49,50]):

Let (M, g) be a Riemannian manifold with $\dim_{\mathbb{R}} M = n$ and a Riemannian metric g. Let ∇ denote the Levi–Civita connection of g. Then, the curvature R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \forall\, X,Y,Z \in \Gamma(TM), \tag{5}$$

where $\Gamma(\bullet)$ denote the set of smooth sections of vector bundle $\bullet$. We also use the notation

$$R(X,Y,Z,W) := g(R(X,Y)Z, W), \quad \forall\, W,X,Y,Z \in \Gamma(TM). \tag{6}$$

The Ricci curvature Ric of ∇ is defined by

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(Z \mapsto R(Z,X)Y), \quad \forall\, X,Y \in \Gamma(TM). \tag{7}$$

2.1. Almost Contact Metric Manifolds

Let M be a manifold with $\dim_{\mathbb{R}} M = 2n+1$ and ϕ, ξ, η be a tensor field of type $(1,1)$, a vector field and a 1 form on M, respectively. If (ϕ, ξ, η) satisfies

$$\eta(\xi) = 1, \tag{8}$$

$$\phi^2(X) = -X + \eta(X)\xi, \quad \forall\, X \in \Gamma(TM), \tag{9}$$

then M is said to have an almost contact structure (ϕ, ξ, η) and is called an almost contact manifold. It follows from (8), (9), and [50] (Proposition V-1.1) that

$$\phi(\xi) = 0, \tag{10}$$

$$\eta \circ \phi(X) = 0, \quad \forall\, X \in \Gamma(TM), \tag{11}$$

$$\mathrm{rank}\phi = 2n. \tag{12}$$

Also M admits a Riemannian metric g such that (see, for example, [50] (Proposition V-1.2))

$$\eta(X) = g(X, \xi), \quad \forall\, X \in \Gamma(TM), \tag{13}$$

$$g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y), \quad \forall\, X,Y \in \Gamma(TM). \tag{14}$$

In this case, M is said to have an almost contact metric structure (ϕ, ξ, η, g) and is called an almost contact metric manifold.

We use the notation

$$g^\dagger(X,Y) := g(\phi X, \phi Y), \quad \forall\, X,Y \in \Gamma(TM).$$

It follows from (9), (11), and (14) that

$$g^\dagger(\phi X, \phi Y) = g^\dagger(X,Y) = g(\phi X, \phi Y), \tag{15}$$

$$g^\dagger(\phi X, Y) = g(\phi X, Y) = -g(X, \phi Y), \quad \forall\, X,Y \in \Gamma(TM). \tag{16}$$

From (15) and (16), we can define a 2 form $\omega^\dagger$ by

$$\omega^\dagger(X,Y) := g^\dagger(\phi X, Y) = -\omega^\dagger(Y,X), \quad \forall\, X,Y \in \Gamma(TM). \tag{17}$$

This form $\omega^\dagger$ is determined uniquely by $g^\dagger$ and vice versa. In what follows, we will not distinguish them.

It follows from (8) that ξ is nowhere vanishing, and generates a 1-dimensional subbundle L_ξ of the tangent bundle TM. There is a 1-dimensional foliation $\mathcal{F}_\xi$ on the almost contact manifold (M, ϕ, ξ, η), which is called the characteristic foliation associated to L_ξ.

We call η the characteristic 1 form, and define a horizontal subbundle $\mathcal{D}$ of TM by

$$\mathcal{D}_p := \mathrm{Ker}\eta_p, \quad \forall\, p \in M.$$

Therefore, one obtains a decomposition of the tangent bundle TM given by

$$TM = \mathcal{D} \oplus L_\xi,$$

and an exact sequence of vector bundles

$$0 \longrightarrow L_\xi \longrightarrow TM \stackrel{\pi}{\longrightarrow} \nu(\mathcal{F}_\xi) \longrightarrow 0,$$

where $\nu(\mathcal{F}_\xi) := TM/L_\xi$, which is called the normal bundle of the foliation $\mathcal{F}_\xi$.

There is a smooth vector bundle isomorphism σ given by

$$\sigma :\, \nu(\mathcal{F}_\xi) \longrightarrow \mathcal{D} \tag{18}$$

such that $\pi \circ \sigma = \mathrm{Id}_{\nu(\mathcal{F}_\xi)}$.

It follows that ϕ induces a splitting

$$\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1},$$

where $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$ are eigenspaces of ϕ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. We call $(\mathcal{D}, \phi_{\restriction \mathcal{D}})$ an almost CR structure.

A p form ϖ on M is called basic if

$$\iota_X \varpi = 0 \quad \text{and} \quad \mathcal{L}_X \varpi = \iota_X \mathrm{d}\varpi = 0, \quad \forall\, X \in \Gamma(L_\xi), \tag{19}$$

where ι_X is the inner product defined by

$$(\iota_X \varpi)(X_1, \cdots, X_{p-1}) = \varpi(X, X_1, \cdots, X_{p-1}), \quad \forall\, X_1, \cdots, X_{p-1} \in \Gamma(TM),$$

and $\mathcal{L}_X$ is the Lie derivative given by

$$\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X.$$

Here, we recall that $\mathrm{d}\varpi$ is defined by

$$\begin{aligned} &(\mathrm{d}\varpi)(X_0, X_1, \cdots, X_p) \\ =& \sum_{k=0}^{p} (-1)^k X_k(\varpi(X_0, X_1 \cdots, \widehat{X_k}, \cdots, X_p)) \\ &+ \sum_{k<\ell} (-1)^{k+\ell} \varpi([X_k, X_\ell], X_0, X_1 \cdots, \widehat{X_k}, \cdots, \widehat{X_\ell}, \cdots, X_p) \end{aligned} \tag{20}$$

for each $X_0, X_1, \cdots, X_p \in \Gamma(TM)$.

Note that a basic 0 form (i.e., basic function) means that a function $u \in C^1(M, \mathbb{R})$ satisfying $\xi(u) \equiv 0$. We use the notation that

$$C^k_{\mathrm{B}}(M, \mathbb{R}) := \Big\{ u \in C^k(M, \mathbb{R}) :\, \xi(u) = 0 \Big\}, \quad k \in \mathbb{N}^* \cup \{\infty\}. \tag{21}$$

Let $\bigwedge_{\mathrm{B}}^{p}$ denote the sheaf of germs of basic p forms, $\Omega_{\mathrm{B}}^{p} := \Gamma(M, \bigwedge_{\mathrm{B}}^{p})$ the set of global sections of $\bigwedge_{\mathrm{B}}^{p}$. Since the exterior differential preserves basic forms, we set $\mathrm{d}_{\mathrm{B}} := \mathrm{d}_{\restriction \Omega_{\mathrm{B}}^{p}}$ with $\mathrm{d}_{\mathrm{B}}^{2} = 0$. Note that Ω_{B}^{p} is $C_{\mathrm{B}}^{\infty}(M, \mathbb{R})$ module.

2.2. Contact Metric Manifolds

A contact form η on a manifold M with $\dim_{\mathbb{R}} M = 2n+1$ is a 1 form with

$$\eta \wedge (\mathrm{d}\eta)^{n} \neq 0 \tag{22}$$

everywhere on M. This yields that M is orientable. It follows from [4] (Lemma 6.1.24) that there exists a unique vector field ξ such that

$$\begin{aligned} 1 =& \eta(\xi), \\ 0 =& (\mathrm{d}\eta)(\xi, X), \quad \forall X \in \Gamma(TM), \end{aligned} \tag{23}$$

and this vector field ξ is called the characteristic vector field or the Reeb vector field.

For this contact manifold M, there exists (see, for example, [50] (Theorem V-2.1), (19) and (23)) an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\omega^{\dagger}(X, Y) = g^{\dagger}(\phi X, Y) = g(\phi X, Y) = \frac{1}{2}(\mathrm{d}\eta)(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{24}$$

which is a d_{B}-closed basic 2 form.

The almost contact metric structure (ϕ, ξ, η, g) constructed from a contact form η is called a contact metric structure associated to η and M is called a contact metric manifold.

For a contact metric manifold (M, ϕ, ξ, η, g), if ξ is a Killing vector field with respect to g, then (ϕ, ξ, η, g) is called a K-contact structure and M is called a K-contact manifold.

Let (M, ϕ, ξ, η, g) be a contact metric manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, we take

$$\mathrm{dvol}_{g} := \frac{\eta \wedge (\omega^{\dagger})^{n}}{n!} = \frac{1}{2^{n} n!} \eta \wedge (\mathrm{d}\eta)^{n} \tag{25}$$

as the Riemannian volume form, and define the transverse Hodge star operator $*^{\dagger}$ in terms of the usual Hodge star operator $*$ by (see, for example, [4] (Formula (7.2.2)) and [17])

$$*^{\dagger} : \Omega_{\mathrm{B}}^{p} \to \Omega_{\mathrm{B}}^{2n-p}, \quad *^{\dagger}\varphi \mapsto *(\eta \wedge \varphi) = (-1)^{p} \iota_{\xi}(*\varphi). \tag{26}$$

The adjoint $\delta_{\mathrm{B}} : \Omega_{\mathrm{B}}^{p} \to \omega_{\mathrm{B}}^{p-1}$ of d_{B} is given by

$$\delta_{\mathrm{B}} := - *^{\dagger} \circ \mathrm{d}_{\mathrm{B}} \circ *^{\dagger}. \tag{27}$$

The basic Laplacian $\Delta_{\mathrm{d}_{\mathrm{B}}}$ is defined in terms of d_{B} and its adjoint δ_{B} by

$$\Delta_{\mathrm{d}_{\mathrm{B}}} := \mathrm{d}_{\mathrm{B}} \circ \delta_{\mathrm{B}} + \delta_{\mathrm{B}} \circ \mathrm{d}_{\mathrm{B}}. \tag{28}$$

2.3. Normality of Almost Contact Manifolds

Let us recall the notation of normality of the almost contact structure (see [50] (Section V-3) and [4] (Section 6.5)).

Let (M, ϕ, ξ, η) be an almost contact manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, we define

$$\begin{aligned} 4\mathcal{N}_{\phi}(X, Y) :=& \phi^{2}[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \\ \mathcal{N}^{(1)}(X, Y) :=& 4\mathcal{N}_{\phi}(X, Y) + (\mathrm{d}\eta(X, Y))\xi, \\ \mathcal{N}^{(2)}(X, Y) :=& (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X), \\ \mathcal{N}^{(3)}(X) :=& (L_{\xi}\phi)(X), \end{aligned}$$

$$\mathcal{N}^{(4)}(X) := (L_{\xi}\eta)(X), \quad \forall\, X,\, Y \in \Gamma(TM).$$

If $\mathcal{N}^{(1)}$ vanishes, so does $\mathcal{N}^{(i)}$ for $i = 2, 3, 4$ (see, for example, [4] (Lemma 6.5.10)).

If $(M, \phi, \xi, \eta,\, g)$ is a contact metric manifold, then one can infer that $\mathcal{N}^{(2)} = \mathcal{N}^{(4)} = 0$.

A contact metric manifold (M, ϕ, ξ, η, g) is a K-contact if, and only if, $\mathcal{N}^{(3)} = 0$ (see, for example, [4] (Proposition 6.5.12)).

For a smooth manifold M, we denote by $\hat{M} := \mathbb{R}_+ \times M$ the cone on M, where $\mathbb{R}_+$ is the set of positive real numbers with coordinate r. We shall identify M with $\{1\} \times M$.

Let $(M,\, \phi,\, \xi,\, \eta)$ be an almost contact manifold with $\dim_{\mathbb{R}} M = 2n + 1$. Then, we define (see more details in [17]) as an almost complex structure J on the tangent bundle $T\hat{M}$ of the cone by

$$JX = \phi X - \eta(X)(r\partial_r), \quad J(r\partial_r) = \xi, \quad \forall\; X \in \Gamma(M), \tag{29}$$

where $r\partial_r := r(\partial/\partial r)$ is the Liouville (or Euler) vector field, and a Hermitian metric $g_{\hat{M}}$ given by

$$g_{\hat{M}} = \mathrm{d}r \otimes \mathrm{d}r + r^2 g.$$

That is, there holds

$$J^2 = -\mathrm{Id}_{T\hat{M}},$$

and that $g_{\hat{M}}$ is a Riemannian metric such that

$$g_{\hat{M}}(JX, JY) = g_{\hat{M}}(X, Y), \quad \forall\, X, Y \in \Gamma(T\hat{M}).$$

Then, the fundamental 2 form $\omega_{\hat{M}}$ (sometimes $\omega_{\hat{M}}$ is also called the Kähler form) associated to $g_{\hat{M}}$ is defined by

$$\omega_{\hat{M}}(X, Y) = g_{\hat{M}}(JX, Y), \quad \forall\; X, Y \in \Gamma(T\hat{M}). \tag{30}$$

The 2 form $\omega_{\hat{M}}$ is determined uniquely by $g_{\hat{M}}$ and vice versa.

It follows from (16), (29), and (30) that

$$\omega_{\hat{M}} = r\mathrm{d}r \wedge \eta + r^2 \omega^{\dagger}. \tag{31}$$

The Nijenhuis tensor $\mathcal{N}_J$ of this almost complex structure is defined by

$$4\mathcal{N}_J(X, Y) = J^2[X, Y] + [JX, JY] - J[JX,\, Y] - J[X,\, JY], \quad \forall\, X, Y \in \Gamma(T\hat{M}). \tag{32}$$

If the Nijenhuis tensor is integrable, i.e., $\mathcal{N}_J \equiv 0$, then the almost contact structure (ϕ, ξ, η) is called normal.

We know that (see, for example, [4] (Theorem 6.5.9)) the almost contact structure $(\phi,\, \xi,\, \eta)$ of M is normal if, and only if, $4\mathcal{N}_\phi = -\mathrm{d}\eta \otimes \xi$; and, if, and only if, (see for example [51])

(a) $[\Gamma(\mathcal{D}^{0,1}), \Gamma(\mathcal{D}^{0,1})] \subset \Gamma(\mathcal{D}^{0,1})$, i.e., the almost CR structure $(\mathcal{D}, \phi_{\restriction \mathcal{D}})$ is integrable;

(b) $[\xi, \Gamma(\mathcal{D}^{0,1})] \subset \Gamma(\mathcal{D}^{0,1})$, i.e., $\mathcal{N}^{(3)} \equiv 0$.

A normal contact metric structure $(\phi,\, \xi,\, \eta,\, g)$ on M is called a Sasakian structure, and M with this structure is called a Sasakian manifold and g is called a Sasaki metric.

A contact metric manifold (M, ϕ, ξ, η, g) is Sasakian if, and only if, its metric cone

$$(\hat{M}, J, g_{\hat{M}})$$

is Kähler.

Indeed, it follows from (24) and (31) that

$$\omega_{\hat{M}} = r\mathrm{d}r \wedge \eta + r^2 \omega_{\dagger}$$

$$
\begin{aligned}
=& r\mathrm{d}r\wedge\eta + r^2\frac{1}{2}(\mathrm{d}\eta)\\
=& \frac{1}{2}\mathrm{d}(r^2\eta),
\end{aligned}
$$

which is d-closed, as desired.

For any p form ϖ on the almost contact manifold $(M,\phi,\xi,\eta,\,g)$, we can define

$$
(\phi\varpi)(X_1,\cdots,X_p) := (-1)^p\varpi(\phi X_1,\cdots,\phi X_p), \quad \forall\ X_1,\cdots,X_p\in\Gamma(TM). \tag{33}
$$

If $\mathcal{N}^{(3)}=0$, then we have

$$
\phi[\xi,X] = [\xi,\phi X], \quad \forall\ X\in\Gamma(TM). \tag{34}
$$

This yields that if ϖ is a basic p form, then so is $\phi\varpi$, and that $\phi^2\varpi=-\varpi$. Then, we have $\bigwedge_{\mathrm{B}}^1\otimes_{\mathbb{R}}\mathbb{C} := \bigwedge_{\mathrm{B}}^{1,0}+\bigwedge_{\mathrm{B}}^{0,1}$ and

$$
\Omega_{\mathrm{B}}^1\otimes_{\mathbb{R}}\mathbb{C} := \Omega_{\mathrm{B}}^{1,0}+\Omega_{\mathrm{B}}^{0,1},
$$

where $\Omega_{\mathrm{B}}^{1,0}$ and $\Omega_{\mathrm{B}}^{0,1}$ are eigenspaces of ϕ with eigenvalues $-\sqrt{-1}$ and $\sqrt{-1}$, respectively. We also denote that $\bigwedge_{\mathrm{B}}^{p,q} := \bigwedge^p\left(\bigwedge_{\mathrm{B}}^{1,0}\right)\otimes\bigwedge^q\left(\bigwedge_{\mathrm{B}}^{0,1}\right)$, and

$$
\Omega_{\mathrm{B}}^{p,q} := \bigwedge^p\left(\Omega_{\mathrm{B}}^{1,0}\right)\otimes\bigwedge^q\left(\Omega_{\mathrm{B}}^{0,1}\right).
$$

Then, we have $\bigwedge_{\mathrm{B}}^p\otimes_{\mathbb{R}}\mathbb{C}=\bigoplus_{r+s=p}\bigwedge_{\mathrm{B}}^{r,s}$, and

$$
\Omega_{\mathrm{B}}^p\otimes_{\mathbb{R}}\mathbb{C} = \bigoplus_{r+s=p}\Omega_{\mathrm{B}}^{r,s}.
$$

It is easy to find a local frame basis

$$
\theta^1,\cdots,\theta^n
$$

of $\bigwedge_{\mathrm{B}}^{1,0}$ and a local frame basis

$$
e_1,\cdots,e_n
$$

of $\mathcal{D}^{1,0}$ such that

$$
\theta^i(e_j)=\delta^i_j, \quad \phi e_i=\sqrt{-1}e_i, \quad 1\le i,\,j\le n.
$$

Note that

$$
\xi,e_1,\cdots,e_n,\bar{e}_1,\cdots,\bar{e}_n
$$

is a local frame basis of $TM\otimes_{\mathbb{R}}\mathbb{C}$ with dual

$$
\eta,\theta^1,\cdots,\theta^n,\bar{\theta}^1,\cdots,\bar{\theta}^n.
$$

We have

$$
\begin{aligned}
g^\dagger =& g^\dagger_{i\bar{j}}(\theta_i\otimes\bar{\theta}_j+\bar{\theta}_j\otimes\theta_i),\\
\omega^\dagger =& = \sqrt{-1}g^\dagger_{i\bar{j}}(\theta^i\otimes\bar{\theta}^j-\bar{\theta}^j\otimes\theta^i) = \sqrt{-1}g^\dagger_{i\bar{j}}\theta^i\wedge\bar{\theta}^j,
\end{aligned} \tag{35}
$$

where $g^\dagger_{i\bar{j}} := g(\phi e_i,\phi\bar{e}_j)=g(e_i,\bar{e}_j)$.

We set

$$
\begin{aligned}
[\xi,e_i] =& C^0_{0i}\xi + C^k_{0i}e_k,\\
[e_i,e_j] =& C^0_{ij}\xi + C^k_{ij}e_k + C^{\bar{k}}_{ij}\bar{e}_k,\\
[e_i,\bar{e}_j] =& C^0_{i\bar{j}}\xi + C^k_{i\bar{j}}e_k + C^{\bar{k}}_{i\bar{j}}\bar{e}_k,
\end{aligned}
$$

since $\mathcal{N}^{(3)} = 0$ yields that $[\xi, e_i]^{(0,1)} = 0$. Then, we obtain

$$d\theta^k = d_B\theta^k = -\frac{1}{2}C^k_{ij}\theta^i \wedge \theta^j - \frac{1}{2}\overline{C^{\bar{k}}_{ij}}\bar{\theta}^i \wedge \bar{\theta}^j - C^k_{i\bar{j}}\theta^i \wedge \bar{\theta}^j, \tag{36}$$

since $d\theta^k$ is also basic.

From (36), we can split the basic exterior differential operator,

$$d_B : \Omega^\bullet_B \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Omega^{\bullet+1}_B \otimes_{\mathbb{R}} \mathbb{C},$$

into four components (see, for example, [52] for the almost complex case)

$$d_B = A_B + \partial_B + \bar{\partial}_B + \overline{A}_B \tag{37}$$

with

$$\begin{aligned} A_B &: \Omega^{\bullet,\bullet}_B \longrightarrow \Omega^{\bullet+2,\bullet-1}_B, \\ \partial_B &: \Omega^{\bullet,\bullet}_B \longrightarrow \Omega^{\bullet+1,\bullet}_B, \\ \bar{\partial}_B &: \Omega^{\bullet,\bullet}_B \longrightarrow \Omega^{\bullet,\bullet+1}_B, \\ \overline{A}_B &: \Omega^{\bullet,\bullet}_B \longrightarrow \Omega^{\bullet-1,\bullet+2}_B. \end{aligned}$$

In terms of these components, the condition $d_B^2 = 0$ can be written as

$$\begin{aligned} 0 &= A_B^2, \\ 0 &= \partial_B A_B + A_B\partial_B, \\ 0 &= A_B\bar{\partial}_B + \partial_B^2 + \bar{\partial}_B A_B, \\ 0 &= A_B\overline{A}_B + \partial_B\bar{\partial}_B + \bar{\partial}_B\partial_B + \overline{A}_B A_B, \\ 0 &= \partial_B\overline{A}_B + \bar{\partial}_B^2 + \overline{A}_B\partial_B, \\ 0 &= \overline{A}_B\bar{\partial}_B + \bar{\partial}_B\overline{A}_B, \\ 0 &= \overline{A}_B^2. \end{aligned}$$

For any basic function $\varphi \in C^2_B(M, \mathbb{R})$, from (20) and (33), a direct computation yields

$$\begin{aligned} (d_B\phi d_B\varphi)(e_i, e_j) &= -2\sqrt{-1}[e_i, e_j]^{(0,1)}(\varphi), \\ (d_B\phi d_B\varphi)(\bar{e}_i, \bar{e}_j) &= 2\sqrt{-1}[\bar{e}_i, \bar{e}_j]^{(1,0)}(\varphi), \\ (d_B\phi d_B\varphi)(e_i, \bar{e}_j) &= 2\sqrt{-1}\Big(e_i\bar{e}_j(\varphi) - [e_i, \bar{e}_j]^{(0,1)}(\varphi)\Big), \end{aligned}$$

where $[e_i, e_j]^{(0,1)}$ means the projection of $[e_i, e_j]$ to $\mathcal{D}^{0,1}$.

A direct calculation shows that

$$\sqrt{-1}\partial_B\bar{\partial}_B\varphi = \frac{1}{2}(d_B\phi d_B\varphi)^{(1,1)} = \sqrt{-1}\Big(e_i\bar{e}_j(\varphi) - [e_i, \bar{e}_j]^{(0,1)}(\varphi)\Big)\theta^i \wedge \bar{\theta}^j.$$

We also deduce

$$\mathcal{N}_\phi = -\Re\Big(C^{\bar{k}}_{ij}(\theta^i \wedge \theta^j) \otimes \bar{e}_k\Big) - \frac{1}{4}\Re\Big(\eta([e_i, e_j])(\theta^i \wedge \theta^j) \otimes \xi\Big) + \frac{1}{4}\eta([e_i, \bar{e}_j])(\theta^i \wedge \bar{\theta}^j) \otimes \xi,$$

since $\mathcal{N}^{(3)} = 0$ means $C^{\bar{k}}_{0i} = 0$.

For the almost contact metric manifold (M,ϕ,ξ,η,g) with $\mathcal{N}^{(i)}=0$, $i=2,3,4$ (e.g., the K-contact manifold), we have $C^0_{0i}=C^0_{ij}=0$, and hence,

$$\mathcal{N}_\phi=-\Re\Big(C^{\bar{k}}_{ij}(\theta^i\wedge\theta^j)\otimes\bar{e}_k\Big)+\frac{1}{4}\eta([e_i,\bar{e}_j])(\theta^i\wedge\bar{\theta}^j)\otimes\xi.$$

If the almost contact metric manifold (M,ϕ,ξ,η,g) is normal (e.g., the Sasakian manifold), then there also holds $C^{\bar{k}}_{0i}=C^{\bar{k}}_{ij}=0$. This yields that

$$\begin{aligned}
\mathrm{d_B}=&\partial_\mathrm{B}+\bar{\partial}_\mathrm{B},\\
0=&\partial_\mathrm{B}\bar{\partial}_\mathrm{B}+\bar{\partial}_\mathrm{B}\partial_\mathrm{B},\\
\mathcal{N}_\phi=&\frac{1}{4}\eta([e_i,\bar{e}_j])(\theta^i\wedge\bar{\theta}^j)\otimes\xi,\\
\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B}\varphi=&\frac{1}{2}\mathrm{d_B}\phi\mathrm{d_B}(\varphi),\quad\forall\ \varphi\in C^2_\mathrm{B}(M,\mathbb{R}).
\end{aligned}\tag{38}$$

For each $\varphi\in C^2_\mathrm{B}(M,\mathbb{R})$, we can define

$$\begin{aligned}
\Delta_\mathrm{B}\varphi:=&\frac{n\mathrm{d_B}\phi\mathrm{d_B}\varphi\wedge(\omega^\dagger)^{n-1}}{2(\omega^\dagger)^n}\\
=&\frac{n\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B}\varphi\wedge(\omega^\dagger)^{n-1}}{(\omega^\dagger)^n}\\
=&(g^\dagger)^{\bar{j}i}\Big(e_i\bar{e}_j(\varphi)-[e_i,\bar{e}_j]^{(0,1)}(\varphi)\Big),
\end{aligned}\tag{39}$$

where $\Big((g^\dagger)^{\bar{j}i}\Big)$ satisfies

$$(g^\dagger)_{j\bar{k}}(g^\dagger)^{\bar{k}i}=\delta^i_j,\quad 1\le i,j\le n.$$

At the point $\mathbf{x}_0$, where φ attains its local maximum (resp. local minimum), there holds (see, for example, [53] for the almost complex case)

$$\Big(e_i\bar{e}_j(\varphi)-[e_i,\bar{e}_j]^{(0,1)}(\varphi)\Big)(\mathbf{x}_0)\tag{40}$$

is a semi-negative definite, denoted by ≤ 0, (resp. semi-positive definite, denoted by $\ge$0) Hermitian matrix.

If (M,ϕ,ξ,η,g) is a Sasakian manifold, then we have

$$\Delta_\mathrm{B}\varphi=-\frac{1}{2}\Delta_{\mathrm{d_B}}\varphi;$$

otherwise, the difference of these two operators is another operator with order no larger than one (see [54,55]).

It follows from [50] (Theorem V-5.1) or [4] (Theorem 7.3.16) that an almost contact metric manifold (M,ϕ,ξ,η,g) is a Sasakian manifold if, and only if,

$$(\nabla_X\phi)Y=g(Y,\xi)X-g(X,Y)\xi,\quad\forall\ X,Y\in\Gamma(TM).\tag{41}$$

Let (M,g) be a Riemannian manifold with $\dim_\mathbb{R}M=2n+1$ admitting a unit Killing vector field ξ. Then, one can infer from [50] (Theorems V-3.1, V-5.1 and V-5.2) or [4] (Proposition 7.3.17) that M is a Sasakian manifold if, and only if,

$$R(X,\xi)Y=g(Y,\xi)X-g(X,Y)\xi,\quad\forall\ X,Y\in\Gamma(TM).\tag{42}$$

If the Ricci curvature of a K-contact manifold (M,ϕ,ξ,η,g) is of the form

$$\mathrm{Ric}(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),\quad\forall\ X,Y\in\Gamma(TM);(a/b:constants)\tag{43}$$

with a and b being constant, then M is called an η-Einstein manifold.

If the Ricci curvature of a Sasaki manifold (M, ϕ, ξ, η, g) with $\dim_{\mathbb{R}} M = 2n+1$ is of the form

$$\mathrm{Ric}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad \forall\, X, Y \in \Gamma(TM) \tag{44}$$

for $a, b \in C^\infty(M, \mathbb{R})$, then a and b are constant with $a + b = 2n$ and hence M is an η-Einstein manifold (see [50] (Proposition V-5.4)).

If the Ricci curvature of a Sasaki manifold (M, ϕ, ξ, η, g) with $\dim_{\mathbb{R}} M = 2n+1$ is of the form

$$\mathrm{Ric}(X,Y) = ag(X,Y), \quad \forall\, X, Y \in \Gamma(TM) \tag{45}$$

for a a constant, then M is called a Sasaki–Einstein metric and it follows from (42) that $a = 2n$.

2.4. Local Coordinates on Sasakian Manifolds

Let $(M,\ \phi,\ \xi,\ \eta,\ g)$ be a compact Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, it follows from [51] (Theorem 1) that there is a foliated atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ given by

$$\varphi_\alpha = (x^{(\alpha)}, z_1^{(\alpha)}, \cdots, z_n^{(\alpha)}) : U_\alpha \to (-a, a) \times V_\alpha \subset \mathbb{R} \times \mathbb{C}^n$$

such that

$$f_\beta = \tau_{\beta\alpha} \circ f_\alpha, \quad \text{on} \quad U_\alpha \cap U_\beta \neq \emptyset,$$

where

$$f_\alpha :\ U_\alpha \to V_\alpha$$

is the natural composition of projection

$$\pi_\alpha :\ (-a, a) \times V_\alpha \to V_\alpha$$

and φ_α, and $\{\tau_{\alpha\beta}\}$ is a family of bi-holomorphic maps given by

$$\tau_{\beta\alpha} :\ f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta), \quad U_\alpha \cap U_\beta \neq \emptyset$$

satisfying the cocycle conditions

$$\tau_{\gamma\alpha} = \tau_{\gamma\beta} \circ \tau_{\beta\alpha}, \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset.$$

Furthermore, on such a foliated coordinate patch $(U; x, z_1, \cdots, z_n)$, one infers that

1. the Reeb vector field ξ can be written as

$$\xi = \partial_x := \frac{\partial}{\partial x};$$

2. the contact form η is given by

$$\begin{aligned} \eta =& \mathrm{d}x - \sqrt{-1}\sum_{i=1}^n K_i \mathrm{d}z_i + \sqrt{-1}\sum_{j=1}^n K_{\bar{j}} \mathrm{d}\bar{z}_j, \\ \omega^\dagger =& \frac{1}{2}\mathrm{d}\eta = \sqrt{-1} K_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \end{aligned}$$

where

$$K_i := \frac{\partial K}{\partial z_i}, \quad K_{\bar{j}} := \frac{\partial K}{\partial \bar{z}_j}, \quad K_{i\bar{j}} := \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}$$

and

$$K :\ V \to \mathbb{R}$$

is a smooth basic function, i.e., a smooth function K satisfies $\xi(K) = 0$;

3. the Sasaki metric g is given by

$$
\begin{aligned}
g =&\eta \otimes \eta + \sum_{i,j=1}^{n} K_{i\bar{j}}(\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j + \mathrm{d}\bar{z}_j \otimes \mathrm{d}z_i) \\
=&\eta \otimes \eta + g^{\dagger}.
\end{aligned}
$$

4. the tensor field ϕ is written as

$$
\begin{aligned}
\phi =&\sqrt{-1}\sum_{i=1}^{n}\left(\partial_i + \sqrt{-1}K_i\partial_x\right) \otimes \mathrm{d}z_i \\
&- \sqrt{-1}\sum_{j=1}^{n}\left(\partial_{\bar{j}} - \sqrt{-1}K_{\bar{j}}\partial_x\right) \otimes \mathrm{d}\bar{z}_j,
\end{aligned} \tag{46}
$$

where

$$
\partial_i := \frac{\partial}{\partial z_i}, \quad \partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}_j};
$$

5. one can choose (see more details in [17]) some basic smooth function

$$
K: V \to \mathbb{R}
$$

such that

$$
\xi(K) = 0, \quad \partial_{\mathrm{B}}K(p) = \bar{\partial}_{\mathrm{B}}K(p) = 0, \quad i, j = 1, \cdots, n.
$$

6. since M is compact, it can be covered by finite foliated local coordinate charts

$$
\{(U_\alpha, \varphi_\alpha)\}
$$

each of which is diffeomorphic to $(-\varepsilon_0, \varepsilon_0) \times B_2(\mathbf{0})$ with $\varepsilon_0 > 0$ fixed, where

$$
B_2(\mathbf{0}) \subset \mathbb{C}^n
$$

is the ball centered at origin with radius 2, and on $B_2(\mathbf{0})$, we have

$$
C^{-1}\delta_{ij} \leq \omega^{\dagger} \leq C\delta_{ij}
$$

with a uniform constant C. Moreover,

$$
\left\{(\tilde{U}_\alpha, \varphi_\alpha)\right\}
$$

still covers M, where

$$
\tilde{U}_\alpha := \varphi_\alpha^{-1}((-\varepsilon_0/2, \varepsilon_0/2) \times B_1(\mathbf{0})).
$$

Now, we see that $(\mathbb{R}_+ \times U; r, x, z_i)$ is a local coordinate patch of $\hat{M} := \mathbb{R}_+ \times M$. We have

$$
J\partial_x = - r\partial_r, \tag{47}
$$

$$
J\partial_r = \frac{1}{r}\partial_x, \tag{48}
$$

$$
\begin{aligned}
J\partial_i =&\sqrt{-1}\left(\partial_i + \sqrt{-1}K_i\partial_x\right) + \sqrt{-1}K_i r\partial_r \\
=&\sqrt{-1}\left(\partial_i + \sqrt{-1}K_i\partial_x\right) - \sqrt{-1}K_i J\partial_x, \quad 1 \leq i \leq n,
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
J\partial_{\bar{j}} =& - \sqrt{-1}\left(\partial_{\bar{j}} - \sqrt{-1}K_{\bar{j}}\partial_x\right) - \sqrt{-1}K_{\bar{j}} r\partial_r \\
=& - \sqrt{-1}\left(\partial_{\bar{j}} - \sqrt{-1}K_{\bar{j}}\partial_x\right) + \sqrt{-1}K_{\bar{j}} J\partial_x, \quad 1 \leq j \leq n.
\end{aligned} \tag{50}
$$

We set

$$
\begin{aligned}
e_0 :=& -\frac{\sqrt{-1}}{2}\Big(\partial_x - \sqrt{-1}J\partial_x\Big) && (51)\\
=& -\frac{\sqrt{-1}}{2}\Big(\partial_x + \sqrt{-1}r\partial_r\Big),\\
e_{\bar{0}} :=& \bar{e}_0 = \frac{\sqrt{-1}}{2}\Big(\partial_x - \sqrt{-1}r\partial_r\Big), && (52)\\
e_i :=& \partial_i + \sqrt{-1}K_i\partial_x, \quad 1 \le i \le n, && (53)\\
e_{\bar{i}} :=& \bar{e}_i = \partial_{\bar{i}} - \sqrt{-1}K_{\bar{i}}\partial_x, \quad 1 \le i \le n. && (54)
\end{aligned}
$$

It follows from (46)–(50) that

$$
\begin{aligned}
Je_i =& \sqrt{-1}e_i, \quad i = 0, 1, 2, \cdots, n, && (55)\\
\phi e_i =& \sqrt{-1}e_i, \quad 1 \le i \le n, && (56)\\
\phi e_{\bar{i}} =& -\sqrt{-1}e_{\bar{i}}, \quad 1 \le i \le n, && (57)\\
0 =& [e_0, e_i] = [e_0, e_{\bar{i}}], \quad 1 \le i \le n, && (58)\\
0 =& [\xi, e_i] = [\xi, e_{\bar{i}}], \quad 1 \le i \le n, && (59)\\
0 =& [e_i, e_j] = [e_{\bar{i}}, e_{\bar{j}}], \quad 1 \le i, j \le n, && (60)\\
-2\sqrt{-1}K_{i\bar{j}}\partial_x =& [e_i, e_{\bar{j}}], \quad 1 \le i, j \le n. && (61)
\end{aligned}
$$

For any p form ϑ, we define

$$
(J\vartheta)(X_1, \cdots, X_p) := (-1)^p \vartheta(JX_1, \cdots, JX_p), \quad \forall X_1, \cdots, X_p \in \Gamma(T\hat{M}).
$$

Then, a direct calculation yields that

$$
\begin{aligned}
J(\mathrm{d}z_i) =& -\sqrt{-1}\mathrm{d}z_i, \quad 1 \le i \le n,\\
J(\mathrm{d}\bar{z}_j) =& \sqrt{-1}\mathrm{d}\bar{z}_j, \quad 1 \le j \le n,\\
J(\mathrm{d}x) =& -\frac{1}{r}\mathrm{d}r + K_i\mathrm{d}z_i + K_{\bar{j}}\mathrm{d}\bar{z}_j && (62)\\
=& -\frac{1}{r}\mathrm{d}r + \mathrm{d}K,\\
J(\mathrm{d}r) =& r\mathrm{d}x - \sqrt{-1}rK_i\mathrm{d}z_i + \sqrt{-1}rK_{\bar{j}}\mathrm{d}\bar{z}_j && (63)\\
=& r\mathrm{d}x + rJ\mathrm{d}K\\
=& r\eta.
\end{aligned}
$$

We set

$$
z_0 := \log r - K + \sqrt{-1}x.
$$

A direct calculation, together with (62) and (63), yields that

$$
\begin{aligned}
\mathrm{d}z_0 =& \mathrm{d}\log r - \mathrm{d}K + \sqrt{-1}\mathrm{d}x && (64)\\
=& \sqrt{-1}\Big(\mathrm{d}x + \sqrt{-1}J\mathrm{d}x\Big),\\
J\mathrm{d}z_0 =& J\Big(\mathrm{d}\log r - \mathrm{d}K + \sqrt{-1}\mathrm{d}x\Big) && (65)\\
=& \mathrm{d}x + J\mathrm{d}K - J\mathrm{d}K + \sqrt{-1}J\mathrm{d}x\\
=& \mathrm{d}x + \sqrt{-1}(-\mathrm{d}\log r + \mathrm{d}K)\\
=& -\sqrt{-1}\mathrm{d}\Big(\log r - K + \sqrt{-1}x\Big)\\
=& -\sqrt{-1}\mathrm{d}z_0.
\end{aligned}
$$

It follows that $(\mathbb{R}_+ \times V; z_0, z_1, \cdots, z_n)$ is a local coordinate patch of $\hat{M}$, which is first proved by [20] (Lemma 2.1). One can deduce from (47)–(50) that $(\mathbb{R}_+ \times V; z_0, z_1, \cdots, z_n)$ is not a holomorphic local coordinate patch of $\hat{M}$. See [49] (Lemma 1) and [17] for more details.

2.5. Transverse Positivity on Sasakian Manifolds

Let (M, ϕ, ξ, η, g) be a compact Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, we fix a canonical orientation $\eta \wedge (\mathrm{d}\eta)^n$ and recall the concepts of transverse positivity on Sasakian manifolds. See [56] and [57] (Chapter III) for the complex case and [17,58] for more details of transverse positivity.

In the local coordinates, a real continuous basic $(1,1)$ form

$$v = \sqrt{-1} v_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j \tag{66}$$

is transverse positive at $\mathbf{x}_0$ (denote it by $\geq_{\mathrm{b}} 0$) if, and only if, $(v_{i\bar{j}}(\mathbf{x}_0))$ is a semi-positive Hermitian matrix with $\xi(v_{i\bar{j}}) \equiv 0$ and we denote $\det v := \det(v_{i\bar{j}})$.

One can call a real continuous basic $(1,1)$ form v strictly transverse positive at $\mathbf{x}_0$ (denoted by $>_{\mathrm{b}} 0$) if the Hermitian matrix $(v_{i\bar{j}}(\mathbf{x}_0))$ is a positive definite Hermitian matrix with $\xi(v_{i\bar{j}}) \equiv 0$.

For each $\varphi \in C^2_{\mathrm{B}}(M, \mathbb{R})$, if φ attains its maximum (resp. minimum) at $\mathbf{x}_0$, then it follows from (40) that

$$\left(\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi\right)(\mathbf{x}_0) \leq_{\mathrm{b}} 0 \text{ (resp. } \geq_{\mathrm{b}} 0). \tag{67}$$

If a real basic $(1,1)$ form v is d_{B}-closed strictly transverse positive on M, then it is called a transverse Kähler form.

Let $[\alpha] \in H^{1,1}_{\mathrm{BC,b}}(M, \mathbb{R})$ (see (75) for basic Bott–Chern cohomology), where α is a smooth d_{B}-closed real $(1,1)$ basic form. Then, we say that

- $[\alpha]$ is transverse Kähler if it contains a representative which is a transverse Kähler form, i.e., if there is a smooth basic function φ such that

$$\alpha + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi >_{\mathrm{b}} \varepsilon\omega^{\dagger}$$

  on M for some $\varepsilon > 0$.
- $[\alpha]$ is transverse nef if for each $\varepsilon > 0$ there is a smooth basic function φ_ε such that

$$\alpha + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\varepsilon >_{\mathrm{b}} -\varepsilon\omega^{\dagger} \tag{68}$$

  holds on M.

The set of all the transverse Kähler classes (resp. transverse nef classes) is denoted by $\mathcal{K}_{M,\mathrm{b}}$ (resp. $\mathcal{N}_{M,\mathrm{b}}$). The transverse Kähler cone $\mathcal{K}_{M,\mathrm{b}}$ is an open and convex cone inside $H^{1,1}_{\mathrm{BC,b}}(M, \mathbb{R})$. Furthermore, there holds (see [58]) that

$$\mathcal{K}_{M,\mathrm{b}} \cap (-\mathcal{K}_{M,\mathrm{b}}) = \{0\}, \quad \mathcal{N}_{M,\mathrm{b}} \cap (-\mathcal{N}_{M,\mathrm{b}}) = \{0\}$$

and that

$$\mathcal{N}_{M,\mathrm{b}} = \overline{\mathcal{K}_{M,\mathrm{b}}}.$$

2.6. Transverse Kähler Structures on Sasakian Manifolds

Let (M, ϕ, ξ, η, g) be a Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, $(\mathcal{D}, \phi_{\restriction\mathcal{D}}, \mathrm{d}\eta)$ gives M a transverse Kähler structure with transverse Kähler form $\omega^{\dagger} := \frac{1}{2}\mathrm{d}\eta$ and transverse Kähelr metric $g^{\dagger}$ defined by (24). We have the relationship that $g = \eta \otimes \eta + g^{\dagger}$.

Given a Sasakian structure (ϕ, ξ, η, g) on M and any $u \in C^{\infty}_{\mathrm{B}}(M, \mathbb{R})$, we define

$$\tilde{\eta} := \eta + \sqrt{-1}(\bar{\partial}_{\mathrm{B}} - \partial_{\mathrm{B}})u. \tag{69}$$

If
$$\mathrm{d}\tilde{\eta} = \mathrm{d}\eta + 2\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}u >_{\mathrm{b}} 0$$
and
$$\tilde{\eta} \wedge (\mathrm{d}\tilde{\eta})^n \neq 0,$$
then $(\xi, \tilde{\eta}, \tilde{\phi}, \tilde{g})$ is also a Sasaki structure homologous to (ξ, η, ϕ, g) (see [4]) on M, where
$$\tilde{\phi} := \phi - \xi \otimes \sqrt{-1}\big((\bar{\partial}_{\mathrm{B}} - \partial_{\mathrm{B}})u\big) \circ \phi, \tag{70}$$
and
$$\tilde{g} := \frac{1}{2}(\mathrm{d}\tilde{\eta}) \circ (\mathrm{Id} \otimes \tilde{\phi}) + \tilde{\eta} \otimes \tilde{\eta}. \tag{71}$$
These deformations fix the Reeb field ξ and change (η, ϕ, g) and hence $\mathcal{D}$; however, the quotient vector bundle $\nu(\mathcal{F}_\xi)$ is invariant.

We equip $\nu(\mathcal{F}_\xi)$ with a Kähler structure $(\mathbf{I}, \sigma^* g^\dagger)$, where the complex structure $\mathbf{I}$ invariant under the deformations above is given by
$$\mathbf{I}(X) := \pi \circ \phi \circ \sigma(X), \quad \forall\, X \in \nu(\mathcal{F}_\xi). \tag{72}$$

2.7. Vector Bundles on Sasakian Manifolds

Let us recall the concepts of transverse vector bundles in [14] originated from [59] and [60,61]. The transverse vector bundle is sometimes also called a foliated vector bundle. Here, we follow [17] and the readers can find more details in [13,17] and references therein.

Let M be a smooth manifold over $\mathbb{R}$ and $S \subset TM$ an involutive subbundle of TM, which means that
$$[X, Y] \in \Gamma(S), \quad \forall\, X, Y \in \Gamma(S).$$
Then, the partial connection ∇^0 of a vector bundle $E \to M$ with respect to S is defined by
$$\nabla^0 : \Gamma(S) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla^0_Z s,$$
satisfying
$$\begin{aligned} \nabla^0_{X+uY} s &= \nabla^0_X s + u\nabla^0_Y s, \\ \nabla^0_X(s + ut) &= \nabla^0_X s + (Xu)t + u\nabla^0_X t, \quad \forall\, X, Y \in \Gamma(S), \forall\, s, t \in \Gamma(E), \forall\, u \in C^\infty(M, \mathbb{R}). \end{aligned}$$
Let (M, ϕ, ξ, η, g) be a Sasakian manifold with $\dim_{\mathbb{R}} M = 2n + 1$. Then, a transverse complex vector bundle on M is a pair of (E, ∇^0), where E is a smooth complex vector bundle on M and ∇^0 is a partial connection with respect to L_ξ.

A transverse Hermitian metric h on (E, ∇^0) is a smooth Hermitian metric on E, which can be preserved by ∇^0, i.e.,
$$\nabla^0 h = 0,$$
where we also denote by ∇^0 the induced connection on the dual bundle $E^* \otimes \bar{E}^*$.

We say that (E, ∇^0, h) is a transverse complex Hermitian vector bundle.

We define
$$\Gamma_{\mathrm{B}}(E) := \Big\{t \in \Gamma(E) : \nabla^0 t = 0\Big\}.$$
A transverse holomorphic vector bundle on M is a pair of (E, ∇^0, ∇^1), where (E, ∇^0) is a transverse complex vector bundle and ∇^1 is a flat partial connection of E with respect to $(L_\xi \otimes_{\mathbb{R}} \mathbb{C}) \oplus \mathcal{D}^{0,1}$, which is an involutive subbundle and coincides with ∇^0 with respect to L_ξ.

We call $(E, \nabla^0, \nabla^1, h)$ a transverse holomorphic Hermitian vector bundle.

For a transverse holomorphic Hermitian vector bundle $(E, \nabla^0, \nabla^1, h)$, the adapted Chern connection, denoted by ∇^{C}, on E is the unique connection which preserves h and coincides with ∇^1 with respect to $(L_\xi \otimes_{\mathbb{R}} \mathbb{C}) \oplus \mathcal{D}^{0,1}$. Let $\mathcal{K}(E,h)$ denote the curvature of the connection ∇^{C}. Then, the first Chern–Ricci form $\mathrm{Ric}(E,h)$ is given by

$$\mathrm{Ric}(E,h) := \sqrt{-1}\mathrm{tr}(\mathcal{K}(E,h) : \Gamma(E) \to \Gamma(E)),$$

which is a basic real $(1,1)$ form. Hence, we call $\mathrm{Ric}(E,h)$ the basic first Chern–Ricci form.

It follows from the Chern–Weil theory [62] that the Chern form $c_j(E,h)$ of the transverse holomorphic Hermitian vector bundle (E,∇^0,∇^1,h) is defined by

$$\det\left(\mathrm{Id}_E + \frac{\sqrt{-1}}{2\pi}\mathcal{K}(E,h)\right) = \sum_{j\geq 0} c_j(E,h), \tag{73}$$

where $c_j(E,h)$ is a closed basic real (j,j) form for $j \geq 0$. Hence, we call $c_j(E,h)$ the basic Chern form for $1 \leq j \leq r$.

It follows from (73) and [17] (Equality (2,24)) that

$$2\pi c_1(E,h) = \mathrm{Ric}(E,h) = -\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} \log\det h. \tag{74}$$

We call that

$$\begin{aligned} c_j^{\mathrm{BC,b}}(E) =& \{c_j(E,h)\} \in H^{j,j}_{\mathrm{BC,b}}(M,\mathbb{R}) \\ :=& \frac{\{\mathrm{d_B}\text{-closed basic real } (j,j) \text{ forms}\}}{\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\{\text{basic real } (j-1,j-1) \text{ forms}\}} \end{aligned} \tag{75}$$

is the jth basic Chern class of E.

It follows from [7] that $\left(\nu(\mathcal{F}_\xi),\nabla^{\mathrm{B}},\bar{\partial}_{\mathrm{B}},\sigma^*g^\dagger\right)$ is a transverse holomorphic Hermitian bundle, where ∇^{B} is the Bott connection defined by

$$\nabla^{\mathrm{B}}_\xi V := \pi([\xi,\sigma(V)]), \quad \forall\, V \in \Gamma(\nu(\mathcal{F}_\xi)).$$

The adapted Chern connection $\nabla^\dagger$ on $\left(\nu(\mathcal{F}_\xi),\nabla^{\mathrm{B}},\bar{\partial}_{\mathrm{B}},\sigma^*g^\dagger\right)$ is defined by

$$\nabla^\dagger_X V := \begin{cases} \pi(\nabla_X\sigma(V)), & \text{if } X \in \Gamma(\mathcal{D}); \\ \nabla^{\mathrm{B}}_\xi V, & \text{if } X = \xi. \end{cases} \tag{76}$$

It follows that $\{\eta, \mathrm{d}z^i, \mathrm{d}\bar{z}^j\}$ is the dual basis of $\{\partial_x, e_i, e_{\bar{j}}\}$. We deduce that $\{\pi(e_i)\}_{1\leq i\leq n}$ is a basic holomorphic basis of $\Gamma_{\mathrm{B}}(U,\nu(\mathcal{F}_\xi))$, and that $\bigwedge^{1,0}_{\mathrm{B}}$ is dual to $\nu(\mathcal{F}_\xi)$.

In the following, we will use the induced connection, also denoted by $\nabla^\dagger$, on $\bigwedge^{1,0}_{\mathrm{B}}$.

For later use, let us fix some notations.

$$\nabla^\dagger_i := \nabla^\dagger_{e_i}, \quad \nabla^\dagger_{\bar{j}} := \nabla^\dagger_{e_{\bar{j}}}, \quad \nabla^\dagger_i \mathrm{d}z_k = -\Gamma^k_{ij}\mathrm{d}z_j, \quad R^\dagger(e_i,e_{\bar{j}})\mathrm{d}z_\ell := -R^\dagger_{i\bar{j}k}{}^{\ell}\mathrm{d}z_k.$$

A direct calculation yields that (see for example [7])

$$\nabla^\dagger_{\partial_x}\mathrm{d}z_k = 0, \quad \Gamma^k_{ij} = (g^\dagger)^{\bar{q}k}\partial_i(g^\dagger)_{j\bar{q}}, \quad R^\dagger_{i\bar{j}k}{}^{\ell} = -\partial_{\bar{j}}\Gamma^\ell_{ik}, \quad R^\dagger_{i\bar{j}k\bar{\ell}} = R^\dagger_{i\bar{j}k}{}^{p}(g^\dagger)_{p\bar{\ell}}$$

such that

$$\overline{R^\dagger_{i\bar{j}k\bar{\ell}}} = R^\dagger_{j\bar{i}\ell\bar{k}}, \quad R^\dagger_{i\bar{j}k\bar{\ell}} = R^\dagger_{k\bar{j}i\bar{\ell}} = R^\dagger_{i\bar{\ell}k\bar{j}} = R^\dagger_{k\bar{\ell}i\bar{j}}, \tag{77}$$

where $((g^\dagger)^{\bar{q}k})$ is the inverse matrix of $((g^\dagger)_{j\bar{\ell}})$.

We remind that

$$R^\dagger_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}g^\dagger_{k\bar{\ell}} + (g^\dagger)^{\bar{q}p}\left(\partial_i g^\dagger_{k\bar{q}}\right)\left(\partial_{\bar{j}}g^\dagger_{p\bar{\ell}}\right). \tag{78}$$

From (38) and (74) (see more details in [17] (Equality (2.27))), the basic Chern–Ricci form of $\nu(\mathcal{F}_\xi)$ given by

$$\begin{aligned}\mathrm{Ric}^\dagger(\omega^\dagger) :=&\mathrm{Ric}(\nu(\mathcal{F}_\xi, \sigma^* g^\dagger) \\ =&-\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} \log\det((g^\dagger)_{i\bar{j}}) \\ =&-\frac{\sqrt{-1}}{2}\mathrm{d}_{\mathrm{B}}\phi\mathrm{d}_{\mathrm{B}}\log\det((g^\dagger)_{i\bar{j}}) \\ \in& 2\pi c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi))\end{aligned} \tag{79}$$

is a $\mathrm{d_B}$-closed real basic $(1,1)$ form. Sometimes, we also use the notation

$$\mathrm{Ric}^\dagger(\omega^\dagger) = -\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\log(\omega^\dagger)^n. \tag{80}$$

If

$$\mathrm{Ric}^\dagger(\omega^\dagger) = c\omega^\dagger$$

for some constant $c \in \mathbb{R}$, then g is called a transverse Einstein metric.

Let (M, ϕ, ξ, η, g) be a Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, it follows from [4] (Theorem 7.3.12) that a Sasaki metric g is a transverse Einstein metric if, and only if, g is an η-Einstein metric.

As explained in Section 2.6, in the following, by saying transverse Einstein metric, it means the transverse Kähler metric $g^\dagger$ or $\omega^\dagger$ for convenience.

For a basic $(1,0)$ form $a = a_\ell \mathrm{d}z^\ell$, its covariant derivative $\nabla_i^\dagger a_\ell$ is defined by

$$\nabla_i^\dagger a_\ell := \partial_i a_\ell - \Gamma_{i\ell}^p a_p. \tag{81}$$

Then, one infers that

$$[\nabla_i^\dagger, \nabla_{\bar{j}}^\dagger] a_\ell = - R^{\dagger}_{i\bar{j}\ell}{}^{p} a_p, \tag{82}$$

$$[\nabla_i^\dagger, \nabla_{\bar{j}}^\dagger] a_{\bar{m}} = R^{\dagger}_{i\bar{j}}{}^{\bar{q}}{}_{\bar{m}} a_{\bar{q}}, \tag{83}$$

where

$$R^{\dagger}_{i\bar{j}}{}^{\bar{q}}{}_{\bar{m}} = R^{\dagger}_{i\bar{j}p}{}^{\ell}(g^\dagger)^{\bar{q}p}(g^\dagger)_{\ell\bar{m}}.$$

For each $u \in C^4_{\mathrm{B}}(M, \mathbb{R})$, one defines

$$u_i := \partial_i u = \nabla_i^\dagger u, \tag{84}$$

$$u_{\bar{j}} := \partial_{\bar{j}} u = \nabla_{\bar{j}}^\dagger u, \tag{85}$$

$$u_{i\bar{j}} := \partial_{\bar{j}}\partial_i u = \nabla_{\bar{j}}^\dagger \nabla_i^\dagger u. \tag{86}$$

Using (82) and (83), one infers the following commutation formulae:

$$[\nabla_i^\dagger, \nabla_j^\dagger] u = 0, \tag{87}$$

$$[\nabla_i^\dagger, \nabla_{\bar{j}}^\dagger] u = 0, \tag{88}$$

$$\nabla_\ell^\dagger u_{i\bar{q}} - \nabla_i^\dagger u_{\ell\bar{q}} = 0, \tag{89}$$

$$\nabla_{\bar{m}}^\dagger u_{p\bar{j}} - \nabla_{\bar{j}}^\dagger u_{p\bar{m}} = 0, \tag{90}$$

$$\nabla_\ell^\dagger u_{i\bar{j}} - \nabla_{\bar{j}}^\dagger \nabla_\ell^\dagger u_i = - R^{\dagger}_{\ell\bar{j}i}{}^{p} u_p, \tag{91}$$

$$\nabla_{\bar{m}}^\dagger \nabla_\ell^\dagger u_{i\bar{j}} - \nabla_{\bar{j}}^\dagger \nabla_i^\dagger u_{\ell\bar{m}} = R^{\dagger}_{\ell\bar{m}i}{}^{p} u_{p\bar{j}} - R^{\dagger}_{i\bar{j}\ell}{}^{p} u_{p\bar{m}}. \tag{92}$$

3. Proof of Theorem 1

In order to prove Theorem 1, we need to prove that (2) is equivalent to a transverse Monge–Ampere equation on M. For each given $\hat{T} \in (0, T)$ with T defined by (3), there exists $\varphi \in C^\infty_B(M, \mathbb{R})$ such that

$$\omega_0^\dagger - \hat{T}\mathrm{Ric}^\dagger(\omega_0^\dagger) + \sqrt{-1}\partial_B\bar{\partial}_B\varphi >_b 0. \tag{93}$$

We set

$$\Omega^\dagger := (\omega_0^\dagger)^n e^{\varphi/\hat{T}}.$$

Then, it follows from (80) that

$$\mathrm{Ric}^\dagger(\Omega^\dagger) := -\sqrt{-1}\partial_B\bar{\partial}_B \log\Omega^\dagger = \mathrm{Ric}^\dagger(\omega_0^\dagger) - \frac{\sqrt{-1}}{\hat{T}}\partial_B\bar{\partial}_B\varphi, \tag{94}$$

which, together with (93), yields that

$$\omega_0^\dagger - \hat{T}\mathrm{Ric}^\dagger(\Omega^\dagger) >_b 0. \tag{95}$$

It follows from the convexity of the space of Hermitian matrices that

$$\omega_0^\dagger - t\sqrt{-1}\mathrm{Ric}^\dagger(\Omega^\dagger) >_b 0, \quad \forall\, t \in [0, \hat{T}].$$

Proposition 1. *Let (M, ϕ, ξ, η, g) be a compact Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, for $t \in [0, \hat{T}]$ fixed, there exists a transverse Kähler metric $\omega^\dagger$ with $\omega^\dagger = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger)$ if, and only if, there exists a smooth basic function $u \in C^\infty_B(M, \mathbb{R})$ satisfying*

$$\log\frac{\left(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_B\bar{\partial}_B u\right)^n}{\Omega^\dagger} - u = 0, \quad \omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_B\bar{\partial}_B u >_b 0. \tag{96}$$

Proof. We use the ideas from [22,43,44] in the (almost) complex case. For the 'only if' direction, we suppose that $\omega^\dagger$ with

$$\omega^\dagger := \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger).$$

We define a smooth basic function u by

$$u = \log\frac{(\omega^\dagger)^n}{\Omega^\dagger}.$$

It follows from (80) and (94) that

$$\mathrm{Ric}^\dagger(\omega^\dagger) - \mathrm{Ric}^\dagger(\Omega^\dagger) = -\sqrt{-1}\partial_B\bar{\partial}_B \log\frac{(\omega^\dagger)^n}{\Omega^\dagger} = -\sqrt{-1}\partial_B\bar{\partial}_B u, \tag{97}$$

which yields that

$$\omega^\dagger = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger) = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_B\bar{\partial}_B u >_b 0,$$

as desired.

For the 'if' direction, if the smooth basic function u satisfies (96), then a direct calculation, together with (80) and (94), yields that

$$\omega^\dagger := \omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_B\bar{\partial}_B u$$

satisfies

$$\omega^\dagger = \omega_0^\dagger - t\mathrm{Ric}^\dagger(\omega^\dagger).$$

□

Let us use Proposition 1 to prove the uniqueness of solutions to the continuity Equation (2).

Corollary 1. *Let (M, ϕ, ξ, η, g) be a compact Sasakian manifold with $\dim_{\mathbb{R}} M = 2n + 1$. Then, if $\omega_1^\dagger$ and $\omega_2^\dagger$ are two transverse Kähler metrics solving the continuity Equation (2) for the same t in $[0, T)$, then $\omega_1^\dagger = \omega_2^\dagger$.*

Proof. For $t = 0$, there is nothing to prove.

For $t \in (0, T)$, it is sufficient to prove the uniqueness of the solutions of (96) by Proposition 1.

We suppose that both u_1 and u_2 are the solutions to (96). We set $\theta := u_1 - u_2$. Then, it follows from (96) that

$$\begin{aligned} &\log \frac{(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u_1)^n}{(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u_2)^n} \\ =&\log \frac{(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u_2 + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\theta)^n}{(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger) + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u_2)^n} = \theta. \end{aligned} \tag{98}$$

Applying (67) to θ in (98) at the points where θ attains its maximum and minimum yields that $\theta \equiv 0$, as desired. □

When $t = 0$, (96) is trivially solved by

$$u_0 = \log \frac{(\omega_0^\dagger)^n}{\Omega^\dagger}. \tag{99}$$

Fix $t \in (0, \hat{T}]$. We define a new smooth basic function

$$G = \log \frac{\Omega^\dagger}{(\hat{\omega}_t^\dagger)^n},$$

where $\hat{\omega}_t^\dagger$ is a transverse Kähler form given by

$$\hat{\omega}_t^\dagger := \omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega^\dagger).$$

Then, (96) can be rewritten as

$$F(u, t) := \log \frac{\left(\hat{\omega}_t^\dagger + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u\right)^n}{(\hat{\omega}_t^\dagger)^n} - u - G = 0, \quad \hat{\omega}_t^\dagger + t\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} u >_{\mathrm{b}} 0. \tag{100}$$

We set

$$\mathscr{T} := \left\{ t \in [0, \hat{T}] : \text{(100) has a solution} \right\}.$$

It follows from (99) that $0 \in \mathscr{T}$.

Lemma 1. *The set $\mathscr{T}$ is open.*

Proof. Assume that (u_t, t) satisfies $F(u_t, t) = 0$. The Fréchet derivative in the direction of u at (u_t, t) is given by

$$(D_u F)_{(u_t, t)} \varphi = \begin{cases} -\varphi, & \text{if } \quad t = 0, \\ t\hat{\Delta}_{\mathrm{B},t}\varphi - \varphi, & \text{if } \quad t \neq 0, \end{cases} \tag{101}$$

where

$$\hat{\Delta}_{\mathrm{B},t}\varphi := \frac{n\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi \wedge (\hat{\omega}_t^\dagger)^{n-1}}{(\hat{\omega}_t^\dagger)^n}, \quad \forall\, \varphi \in C_{\mathrm{B}}^\infty(M, \mathbb{R}). \tag{102}$$

For $t > 0$ fixed, $(D_u F)_{u_t,t}$ is a bijection. Indeed, since $\hat{\omega}_t^\dagger$ is a transverse Kähler metric, both $t\hat{\Delta}_{\mathrm{B},t}$ and $(D_u F)_{u_t,t}$ are strictly transverse elliptic operators. For each smooth basic function $\varphi \in \ker(D_u F)_{u_t,t}$, at the point $\mathbf{x}_{\max}$ (resp. $\mathbf{x}_{\min}$), where φ attains its maximum (resp. minimum), one infers from (67) that

$$-\varphi(\mathbf{x}_{\max}) \geq 0 \ (\text{resp.} - \varphi(\mathbf{x}_{\min}) \leq 0), \tag{103}$$

i.e., $\varphi \equiv 0$. This yields that $\ker(D_u F)_{u_t,t} = \{0\}$, i.e., $(D_u F)_{u_t,t}$ is injective.

Since $(D_u F)_{u_t,t}$ is a self-adjoint operator, its index is zero. Hence, the injectivity of $(D_u F)_{u_t,t}$ implies that it is also surjective.

Since $(D_u F)_{u_t,t}$ is bijective, the implicit function theorem yields that $\mathscr{T}$ is open at the point t.

For $t = 0$, we have to work carefully since $(D_u F)_{u_0,0}$ is degenerate. We use the ideas from [22] and give more details for convenience.

We set

$$u_t := u_0 + t^{-1} w.$$

Then, (100) can be rewritten as

$$\log \frac{(\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega) + t\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} u_0 + \sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} w)^n}{(\omega_0^\dagger)^n} = t^{-1} w \tag{104}$$

with

$$\omega_0^\dagger - t\mathrm{Ric}^\dagger(\Omega) + t\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} u_0 + \sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} w >_\mathrm{b} 0.$$

We expand (104) at $t = 0$ (see [63]) (7.4 of Chapter 7) to obtain

$$\Delta_{\mathrm{B},0}(w) - t^{-1} w = -t\mathrm{tr}_{\omega_0^\dagger}\mathrm{Ric}^\dagger(\Omega^\dagger) - t\Delta_{\mathrm{B},0}(u_0) + Q((A_i^p)), \tag{105}$$

where

$$A_i^p := t\Big(\mathrm{Ric}^\dagger(\Omega^\dagger) + \sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} u_0\Big)_{i\bar{j}} (g_0^\dagger)^{\bar{j}p} + (\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} w)_{i\bar{j}} (g_0^\dagger)^{\bar{j}p},$$

and

$$\Delta_{\mathrm{B},0}(w) = \frac{n\sqrt{-1}\partial_\mathrm{B}\bar{\partial}_\mathrm{B} w \wedge (\omega_0^\dagger)^{n-1}}{(\omega_0^\dagger)^n}.$$

Here, we write $\omega_0^\dagger$ as

$$\omega_0^\dagger = \sqrt{-1}(g_0^\dagger)_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

with

$$(g_0^\dagger)^{\bar{j}p}(g_0^\dagger)_{i\bar{j}} = \delta_i^p, \quad 1 \leq i, p \leq n,$$

and $Q(\mathbf{a})$ denotes a polynomial in $\mathbf{a}$ starting with quadratic terms.

Proposition 2. *There exists a uniform constant C such that for each $f \in C_\mathrm{B}^{\frac{1}{3}}(M, \mathbb{R})$, there is a basic function v satisfying*

$$\Delta_{\mathrm{B},0} v - t^{-1} v = tf, \quad t^{-1}\|v\|_{C_\mathrm{B}^0} + \|v\|_{C_\mathrm{B}^{2,\frac{1}{3}}} \leq Ct^{\frac{2}{3}}\|f\|_{C_\mathrm{B}^{\frac{1}{3}}}. \tag{106}$$

Proof of Proposition 2. Since in the local coordinate given in Section 2.4, Equation (106) is a standard elliptic PDE of second order, we can use the standard PDE theory. It follows from the maximum principle that

$$|v| \leq \|f\|_{L^\infty} t^2. \tag{107}$$

We obtain from (107) and the interior $W^{2,p}$ estimates (see, for example, [64]) (Theorem 4.2 of Chapter 3) that

$$\|v\|_{W^{2,p}} \leq C\Big(\|tf + t^{-1}v\|_{L^p} + \|v\|_{L^p}\Big) \leq C\|f\|_{L^\infty} t. \tag{108}$$

From (108) and the basic Sobolev embedding theorem in [13] (Theorem 2.16) (see for example [63] (Section 3 of Chapter 2) for ordinary version), we obtain

$$\begin{aligned}\|v\|_{C_B^{1,\alpha}} \leq & C_{n,p}\|v\|_{W^{2,p}} \leq C_{n,p}\Big(\|tf + t^{-1}v\|_{L^p} + \|v\|_{L^p}\Big) \\ \leq & C_{n,p}\|f\|_{L^\infty} t, \quad \alpha = 1 - \frac{2n}{p} \in (0,1].\end{aligned} \tag{109}$$

If $|x_1 - x_2| \leq t$, then we infer from (109) with $\alpha = \frac{1}{3}$ that

$$\frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^{\frac{1}{3}}} \leq \|v\|_{C_B^1}|x_1 - x_2|^{\frac{2}{3}} \leq C\|f\|_{L^\infty} t^{\frac{5}{3}}. \tag{110}$$

If $|x_1 - x_2| > t$, then one obtains from (107) that

$$\frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^{\frac{1}{3}}} \leq 2\|v\|_{L^\infty} t^{-\frac{1}{3}} \leq C\|f\|_{L^\infty} t^{\frac{5}{3}}. \tag{111}$$

It follows from (107), (110) and (111) that

$$\|v\|_{C_B^{\frac{1}{3}}} \leq C\|f\|_{L^\infty} t^{\frac{5}{3}}. \tag{112}$$

From (107), (112) and the interior Schauder estimate (see, for example, [64] (Theorem 4.3 of Chapter 2)), one can obtain

$$\|v\|_{C_B^{2,\frac{1}{3}}} \leq C\left(\|tf + t^{-1}v\|_{C_B^{\frac{1}{3}}} + \|v\|_{L^\infty}\right) \leq C\|f\|_{C_B^{\frac{1}{3}}} t^{\frac{2}{3}}. \tag{113}$$

Then, Proposition 2 follows from (107) and (113). □

We need the standard iteration as follows. We set $w_0 = 0$ and

$$A := \left\|\mathrm{tr}_{\omega_0^\dagger}\mathrm{Ric}^\dagger(\Omega^\dagger) + \Delta_{B,0}(u_0) + \frac{1}{t}Q((E_i^p))\right\|_{C_B^{\frac{1}{3}}}, \tag{114}$$

where

$$E_i^p := t\Big(\mathrm{Ric}^\dagger(\Omega^\dagger) + \sqrt{-1}\partial_B\bar{\partial}_B u_0\Big)_{i\bar{j}}(g_0^\dagger)^{\bar{j}p}.$$

We construct w_ℓ for $\ell \geq 1$ by solving the equation

$$\Delta_0(w_\ell) - t^{-1}w_\ell = -t\mathrm{tr}_{\omega_0^\dagger}\mathrm{Ric}^\dagger(\Omega^\dagger) - t\Delta_{B,0}(u_0) + Q((E_i^p + (w_{\ell-1})_{i\bar{j}}(g_0^\dagger)^{\bar{j}p})). \tag{115}$$

One infers from the claim that

$$\|w_1\|_{C_B^{2,\frac{1}{3}}} \leq C(1+A)t^{\frac{2}{3}}.$$

If $w_{\ell-1}$ satisfies

$$\|w_{\ell-1}\|_{C_B^{2,\frac{1}{3}}} \leq C(1+A)t^{\frac{2}{3}},$$

then for sufficiently small t, the right side of (115) can be bounded by At. Hence, the claim above implies that

$$\|w_\ell\|_{C_B^{2,\frac{1}{3}}} \le C(1+A)t^{\frac{2}{3}}.$$

We can deduce from (115) that

$$\Delta_{B,0}((w_{\ell+1}-w_\ell)) - t^{-1}(w_{\ell+1}-w_\ell) = Q((E_i^p + (w_\ell)_{i\bar{j}}(g_0^\dagger)^{\bar{j}p})) - Q((E_i^p + (w_{\ell-1})_{i\bar{j}}(g_0^\dagger)^{\bar{j}p})). \tag{116}$$

Since the right side of (116) is $(w_\ell - w_{\ell-1})_{i\bar{j}}(g_0^\dagger)^{\bar{j}p}t^{-\frac{1}{3}}t$ times a bounded function for t small enough by the definition of Q, it follows from the claim above and (116) that

$$\|w_{\ell+1}-w_\ell\|_{C_B^{2,\frac{1}{3}}} \le Ct^{\frac{1}{3}}\|w_\ell - w_{\ell-1}\|_{C_B^{2,\frac{1}{3}}}. \tag{117}$$

It follows from (117) and the Arzelà–Ascoli theorem that for sufficiently small t, there exists a subsequence $\{w_{\ell_p}\}$ which converges to w in C_B^2-topology, which solves (105). This, together with the Evans–Krylov theory [65–67] (see also [68]) and the Schauder theory (see, for example, [69]), shows the solvability of (104) for t sufficiently small (i.e., the openness at $0 \in \mathscr{T}$), which completes the proof of Lemma 1. □

One infers from Lemma 1 that there exists a sufficiently small $T_0 > 0$ such that $[0, 2T_0] \subset \mathscr{T}$. Hence, Theorem 1 follows from Proposition 1, Lemma 1 and the following result which asserts the closeness of $\mathscr{T}$:

Theorem 3. *Let (M, ϕ, ξ, η, g) be a compact Sasakian manifold with $\dim_{\mathbb{R}} M = 2n+1$. Then, for each $F \in C_B^\infty(M, \mathbb{R})$ and $\frac{1}{\lambda} \in [T_0, \hat{T}]$ $(T_0 > 0)$ a constant, there holds*

$$\|\varphi\|_{C_B^k(M,g)} \le C_k, \quad k = 1, 2, 3, \cdots, \tag{118}$$

where C_k is a uniform constant depending only on the background data and k. Here, $\varphi \in C_B^\infty(M, \mathbb{R})$ is a solution to

$$\log \frac{(\omega^\dagger + \sqrt{-1}\partial_B\bar{\partial}_B\varphi)^n}{(\omega^\dagger)^n} = \lambda\varphi + F, \quad \omega^\dagger + \sqrt{-1}\partial_B\bar{\partial}_B\varphi >_b 0, \tag{119}$$

where $\omega^\dagger = \sqrt{-1}g_{i\bar{j}}^\dagger dz_i \wedge d\bar{z}_j$ is an initial transverse Kähler metric.

Proof. The Equation (119) is a transverse version of [47,70], and hence we can sketch the following:

C^0-estimate

$$\|\varphi\|_{L^\infty} \le C_0. \tag{120}$$

follows from the maximum principle directly.

C^2-estimate

$$C^{-1}\omega^\dagger <_b \tilde{\omega}^\dagger <_b C\omega^\dagger \tag{121}$$

with a uniform constant $C > 0$ follows from

$$\mathrm{tr}_{\omega^\dagger}\tilde{\omega}^\dagger \le \frac{1}{(n-1)!}\left(\mathrm{tr}_{\tilde{\omega}^\dagger}\omega^\dagger\right)^{n-1}\frac{(\tilde{\omega}^\dagger)^n}{(\omega^\dagger)^n} \tag{122}$$

which is proved in [47] (we refer the reader to [48] (Formula (3.30))) and the key estimate

$$\mathrm{tr}_{\omega^\dagger}\tilde{\omega}^\dagger \le C. \tag{123}$$

To prove (123), we write

$$\tilde{\omega}^\dagger := \omega^\dagger + \sqrt{-1}\partial_B\bar{\partial}_B\varphi = \sqrt{-1}\tilde{g}_{i\bar{j}}^\dagger dz_i \wedge d\bar{z}_j$$

and

$$\tilde{\Delta}_{\mathrm{B}} u := \frac{n\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}} \wedge (\tilde{\omega}^{\dagger})^{n-1}}{(\tilde{\omega}^{\dagger})^{n}} = (\tilde{g}^{\dagger})^{\bar{j}i}\partial_i\partial_{\bar{j}}u, \quad \forall\, u \in C^{\infty}_{\mathrm{B}}(M, \mathbb{R}).$$

Note that

$$n = \mathrm{tr}_{\tilde{\omega}^{\dagger}}\tilde{\omega}^{\dagger} = \mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger} + \tilde{\Delta}_{\mathrm{B}}\varphi. \tag{124}$$

A direct calculation (see, for example, [48] (Proposition 2.4)), together with (78) and (81), yields that

$$\begin{aligned}\partial_i\partial_{\bar{j}}\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) =& R^{\dagger}_{i\bar{j}r\bar{s}}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} - \tilde{R}^{\dagger}_{i\bar{j}p\bar{q}}(g^{\dagger})^{\bar{q}p} + (\tilde{g}^{\dagger})^{\bar{s}r}(g^{\dagger})^{\bar{q}p}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right) \\ =& R^{\dagger}_{i\bar{j}r\bar{s}}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} - \tilde{R}^{\dagger}_{p\bar{q}i\bar{j}}(g^{\dagger})^{\bar{q}p} + (\tilde{g}^{\dagger})^{\bar{s}r}(g^{\dagger})^{\bar{q}p}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right),\end{aligned} \tag{125}$$

where $\nabla^{\dagger}$ and $R^{\dagger}$ (resp. $\tilde{R}^{\dagger}$) denote the basic Chern connection and its curvature of $\omega^{\dagger}$ (resp. $\tilde{\omega}^{\dagger}$), and for the second equality, we use (77).

It follows from (80), (97) and (119) that

$$\tilde{R}^{\dagger}_{i\bar{j}} - R^{\dagger}_{i\bar{j}} = -\lambda\partial_i\partial_{\bar{j}}\varphi - \partial_i\partial_{\bar{j}}F, \tag{126}$$

where $R^{\dagger}_{i\bar{j}}$ (resp. $\tilde{R}^{\dagger}_{i\bar{j}}$) is the basic Chern–Ricci form of the basic Chern connection of $\omega^{\dagger}$ (resp. $\tilde{\omega}^{\dagger}$).

We can deduce (125) and (126) that

$$\begin{aligned}\tilde{\Delta}_{\mathrm{B}}\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) =& (\tilde{g}^{\dagger})^{\bar{j}i}\partial_i\partial_{\bar{j}}\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) \\ =& R^{\dagger}_{i\bar{j}r\bar{s}}(\tilde{g}^{\dagger})^{\bar{j}i}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} - \tilde{R}^{\dagger}_{p\bar{q}}(g^{\dagger})^{\bar{q}p} \\ &+ (g^{\dagger})^{\bar{q}p}(\tilde{g}^{\dagger})^{\bar{j}i}(\tilde{g}^{\dagger})^{\bar{s}r}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right) \\ =& R^{\dagger}_{i\bar{j}r\bar{s}}(\tilde{g}^{\dagger})^{\bar{j}i}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} \\ &+ (g^{\dagger})^{\bar{q}p}(\tilde{g}^{\dagger})^{\bar{j}i}(\tilde{g}^{\dagger})^{\bar{s}r}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right) \\ &+ \mathrm{tr}_{\omega^{\dagger}}\mathrm{Ric}^{\dagger}(\omega^{\dagger}) - \lambda(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger} - n) - \Delta_{\mathrm{B}}F.\end{aligned} \tag{127}$$

One infers from (127) that

$$\begin{aligned}\tilde{\Delta}_{\mathrm{B}}\log\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) =& \frac{1}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}\left(\tilde{\Delta}_{\mathrm{B}}\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) - \frac{\left|\partial_{\mathrm{B}}(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger})\right|^2_{\tilde{\omega}^{\dagger}}}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}\right) \\ =& \frac{1}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}\left((g^{\dagger})^{\bar{q}p}(\tilde{g}^{\dagger})^{\bar{j}i}(\tilde{g}^{\dagger})^{\bar{s}r}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right) - \frac{\left|\partial_{\mathrm{B}}(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger})\right|^2_{\tilde{\omega}^{\dagger}}}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}\right) \\ &+ \frac{1}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}R^{\dagger}_{i\bar{j}r\bar{s}}(\tilde{g}^{\dagger})^{\bar{j}i}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} \\ &+ \frac{1}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}}\left(\mathrm{tr}_{\omega^{\dagger}}\mathrm{Ric}^{\dagger}(\omega^{\dagger}) - \lambda(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger} - n) - \Delta_{\mathrm{B}}F\right).\end{aligned} \tag{128}$$

We know the key inequality proved in [47] (we refer the reader to [48] (Formula (2.19)))

$$(g^{\dagger})^{\bar{q}p}(\tilde{g}^{\dagger})^{\bar{j}i}(\tilde{g}^{\dagger})^{\bar{s}r}\left(\nabla^{\dagger}_i\tilde{g}_{p\bar{s}}\right)\left(\nabla^{\dagger}_{\bar{j}}\tilde{g}_{r\bar{q}}\right) - \frac{\left|\partial_{\mathrm{B}}(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger})\right|^2_{\tilde{\omega}^{\dagger}}}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}} \geq 0. \tag{129}$$

We denote by $-B_0$ with $B_0 > 0$ the infimum of

$$\frac{R^{\dagger}(X, \bar{X}, Y, \bar{Y})}{g^{\dagger}(X, \bar{X})g^{\dagger}(Y, \bar{Y})}, \quad \forall\, X, Y \in \Gamma(\mathcal{D}^{1,0}).$$

Then, we have

$$R^{\dagger}_{i\bar{j}r\bar{s}}(\tilde{g}^{\dagger})^{\bar{j}i}(g^{\dagger})^{\bar{s}p}(g^{\dagger})^{\bar{q}r}\tilde{g}^{\dagger}_{p\bar{q}} \geq -B_0\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right)\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right). \tag{130}$$

It follows from (128)–(130) that

$$\tilde{\Delta}_{\mathrm{B}}\log\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) \geq -B_0\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right) - \lambda - \frac{C}{\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}} \tag{131}$$

for a uniform constant $C > 0$.

From the Cauchy–Schwarz inequality, we deduce

$$\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right)\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right) \geq n^2. \tag{132}$$

It follows from (131) and (132) that

$$\tilde{\Delta}_{\mathrm{B}}\log\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) \geq -B\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right) - \lambda, \tag{133}$$

with a uniform constant $B > 0$.

We set

$$Q := \log\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) - A\varphi$$

with $A = B + 1$.

Then, it follows from (124) and (133) that

$$\tilde{\Delta}_{\mathrm{B}}Q \geq \mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger} - \lambda - An. \tag{134}$$

At the point $\mathbf{x}_0$ where Q attains its maximum, it follows from (67) that

$$\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right)(\mathbf{x}_0) \leq C \tag{135}$$

for a uniform constant $C > 0$.

It follows from (122) and (135) that

$$\begin{aligned}\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right)(\mathbf{x}_0) \leq& \frac{1}{(n-1)!}\left(\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right)(\mathbf{x}_0)\right)^{n-1}\frac{(\tilde{\omega}^{\dagger})^n}{(\omega^{\dagger})^n}(\mathbf{x}_0)\\ =& \frac{1}{(n-1)!}\left(\left(\mathrm{tr}_{\tilde{\omega}^{\dagger}}\omega^{\dagger}\right)(\mathbf{x}_0)\right)^{n-1}e^{\lambda\varphi(\mathbf{x}_0)+F(\mathbf{x}_0)}\\ \leq& Ce^{\lambda\varphi(\mathbf{x}_0)}.\end{aligned} \tag{136}$$

Then, (123) follows from (120), (136) and the following

$$\log\left(\mathrm{tr}_{\omega^{\dagger}}\tilde{\omega}^{\dagger}\right) = Q + A\varphi \leq Q(\mathbf{x}_0) + A\varphi \leq \log C + \lambda\varphi(\mathbf{x}_0) + A\varphi - A\varphi(\mathbf{x}_0).$$

Given (120) and (121), the $C^{2,\alpha}$-estimate for some $0 < \alpha < 1$ follows from the Evans–Krylov theory [65–67] (see also [68]).

Differentiating the equations and applying the Schauder theory (see, for example, [69]), we then obtain uniform a priori C^k_{B} estimates for all $k \geq 3$. □

4. Proof of Theorem 2

In this section, we prove Theorem 2.

Proof of Theorem 2. Case 1: $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) <_{\mathrm{b}} 0$. Let $\underline{\omega}^{\dagger}(t)$ be the solution to (1). Then,

$$\omega^{\dagger}(t) := \frac{\underline{\omega}^{\dagger}(t)}{1+t}$$

is the solution of (4), and hence it follows from Theorem 1 that the solution of (4) always exists.

Fix the strictly positive transverse form $\eta \in -2\pi c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi))$. Then, it follows from the basic $\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}$-lemma (see for example [13]) that there exists $F \in C_{\mathrm{B}}^{\infty}(M, \mathbb{R})$ such that

$$\mathrm{Ric}(\omega_0^{\dagger}) + \eta = \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}F.$$

We fix $\Omega^{\dagger} = e^F(\omega_0^{\dagger})^n$ and have

$$\mathrm{Ric}^{\dagger}(\Omega^{\dagger}) = -\eta <_{\mathrm{b}} 0. \tag{137}$$

We define

$$\hat{\omega}^{\dagger}(t) := \frac{1}{1+t}\omega_0^{\dagger} + \frac{t}{1+t}\eta, \quad \forall\, t > 0, \tag{138}$$

which is a strictly positive transverse Kähler metric and equivalent to $\omega_0^{\dagger}$, i.e.,

$$C^{-1}\omega_0^{\dagger} <_{\mathrm{b}} \hat{\omega}^{\dagger}(t) <_{\mathrm{b}} C^{-1}\omega_0^{\dagger}, \quad \forall\, t \geq 0 \tag{139}$$

for a uniform constant $C > 0$.

Proposition 3. *The continuity Equation* (4) *is equivalent to*

$$\begin{cases} t\log\frac{\left(\hat{\omega}^{\dagger}(t)+\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi\right)^n}{\Omega^{\dagger}} = (1+t)\varphi, \\ \tilde{\omega}^{\dagger} := \hat{\omega}^{\dagger}(t) + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi >_{\mathrm{b}} 0, \quad \varphi \in C_{\mathrm{B}}^{\infty}(M, \mathbb{R}). \end{cases} \tag{140}$$

Proof of Proposition 3. If $\omega^{\dagger}(t)$ is the solution of (4), then we define a smooth basic function φ by

$$t\log\frac{(\omega^{\dagger}(t))^n}{\Omega^{\dagger}} = (1+t)\varphi. \tag{141}$$

It follows from (80), (97), (137), and (141) that

$$t\mathrm{Ric}^{\dagger}(\omega^{\dagger}(t)) - t\mathrm{Ric}^{\dagger}(\Omega^{\dagger}) = \omega_0^{\dagger} - (1+t)\omega^{\dagger}(t) + t\eta = (1+t)\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi,$$

i.e.,

$$\omega^{\dagger}(t) = \frac{\omega_0^{\dagger}}{1+t} + \frac{t}{1+t}\eta + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi = \hat{\omega}^{\dagger}(t) + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi >_{\mathrm{b}} 0. \tag{142}$$

On the other hand, if $\varphi \in C_{\mathrm{B}}^{\infty}(M, \mathbb{R})$ is the solution of (140), then it follows from (80), (137), and (141) that

$$\omega^{\dagger}(t) := \hat{\omega}^{\dagger}(t) + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi$$

solves (4). This completes the proof of Proposition 3. □

Case 2: $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) = 0$. We fix $\Omega^{\dagger} = e^F(\omega_0^{\dagger})^n$ with $F \in C_{\mathrm{B}}^{\infty}(M, \mathbb{R})$ such that

$$\mathrm{Ric}^{\dagger}(\Omega^{\dagger}) = 0. \tag{143}$$

Then, (96) reads as

$$\log\frac{\left(\omega_0^{\dagger} + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi\right)^n}{(\omega_0^{\dagger})^n} = \frac{1}{t}\varphi + F, \quad \varpi^{\dagger} := \omega_0^{\dagger} + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi >_{\mathrm{b}} 0. \tag{144}$$

We write

$$\varpi^{\dagger} := \sqrt{-1}h_{i\bar{j}}^{\dagger}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \tag{145}$$

with $(h^\dagger)^{\bar{j}i}$ such that

$$h^\dagger_{k\bar{j}}(h^\dagger)^{\bar{j}i} = \delta^i_k, \quad 1 \le i,k \le n. \tag{146}$$

The uniqueness of the η-Einstein metric is proved in [5] (see also [7]).

For case 1, where $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) <_{\mathrm{b}} 0$, we only need to consider (140) when $t \ge 1$, and rewrite it as

$$\begin{cases} \log \frac{\left(\hat{\omega}^\dagger(t)+\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi\right)^n}{\Omega^\dagger} = \frac{1+t}{t}\varphi, \\ \tilde{\omega}^\dagger := \hat{\omega}^\dagger(t) + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi >_{\mathrm{b}} 0, \quad \varphi \in C^\infty_{\mathrm{B}}(M,\mathbb{R}). \end{cases} \tag{147}$$

Note that

$$\frac{1+t}{t} \in [1,2], \quad \forall\, t \ge 1.$$

Then, it follows from Theorem 3 and (139) that we have a priori estimates in (118). One infers from the Arzelà–Ascoli theorem that there exists a sequence $\{t_i\}$ with $\lim_{i\to+\infty} t_i = +\infty$ such that $\{\varphi(t_i)\}$ converges smoothly to $\varphi_\infty \in C^\infty_{\mathrm{B}}(M,\mathbb{R})$. This, together with (121) and (139), yields that

$$\begin{cases} \log \frac{\left(\eta+\sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty\right)^n}{\Omega^\dagger} = \varphi_\infty, \\ \tilde{\omega}^\dagger_\infty := \eta + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty >_{\mathrm{b}} 0, \quad \varphi_\infty \in C^\infty_{\mathrm{B}}(M,\mathbb{R}). \end{cases} \tag{148}$$

It follows from (80), (97), (137) and (148) that

$$\mathrm{Ric}^\dagger(\tilde{\omega}^\dagger_\infty) = \mathrm{Ric}^\dagger(\Omega^\dagger) - \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty = -\eta - \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty = -\tilde{\omega}^\dagger_\infty.$$

Because of the uniqueness of the η-Einstein metric with the negative basic first Chern class, one infers that $\varphi(t)$ converges smoothly to φ_∞ as $t \to \infty$. $\square$

For case 2, where $c_1^{\mathrm{BC,b}}(\nu(\mathcal{F}_\xi)) = 0$, we only need consider (144) on $M \times [1,+\infty)$. The maximum principle yields that

$$\frac{1}{t} \sup_{M\times[1,+\infty)} |\varphi| \le C \tag{149}$$

for the uniform constant $C > 0$.

The C^0-estimate in [5] originated from [47], together with (149), yields that there exists a uniform constant $C > 0$, which depends only on the initial data on M and $\sup_{M\times[1,+\infty)} |\frac{\varphi}{t} + F|$ such that

$$\sup_{x,y\in M} |\varphi(x,t) - \varphi(y,t)| \le C, \quad \forall\, t \in [1,+\infty). \tag{150}$$

The same argument as in the proof of Theorem 3 yields that

$$\left(\mathrm{tr}_{\omega^\dagger}\varpi^\dagger\right)(x,t) \le Ce^{A\left(\varphi(x,t)-\inf_{y\in M}\varphi(y,t)\right)}, \quad \forall\,(x,t) \in M \times [1,+\infty). \tag{151}$$

The C^2-estimate

$$C^{-1}\omega^\dagger_0 <_{\mathrm{b}} \varpi^\dagger <_{\mathrm{b}} C\omega^\dagger_0 \tag{152}$$

on $M \times [1,+\infty)$ follows from (122), (150) and (151).

Given (150) and (152), the $C^{2,\alpha}$-estimate for some $0 < \alpha < 1$ follows from the Evans–Krylov theory [65–67] (see also [68]).

Differentiating the equations and applying the Schauder theory (see, for example, [69]), we then obtain uniform a priori C^k_{B} estimates

$$\|\varphi(t)\|_{C^k_{\mathrm{B}}(M,\mathbb{R})} \le C_k, \quad k \ge 3, \quad \forall\, t \in [1,+\infty). \tag{153}$$

One infers from the Arzelà–Ascoli theorem and (153) that there exists a sequence $\{t_i\}$ with $\lim_{i\to+\infty} t_i = +\infty$ such that $\{\varphi(t_i)\}$ converges smoothly to $\varphi_\infty \in C^\infty_{\mathrm{B}}(M,\mathbb{R})$. This, together with (144), yields that

$$\log \frac{\left(\omega_0^\dagger + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty\right)^n}{(\omega_0^\dagger)^n} = F, \quad \varpi_\infty^\dagger := \omega_0^\dagger + \sqrt{-1}\partial_{\mathrm{B}}\bar{\partial}_{\mathrm{B}}\varphi_\infty >_{\mathrm{b}} 0. \tag{154}$$

It follows from (80), (143), (152), and (154) that

$$\mathrm{Ric}^\dagger(\varpi_\infty^\dagger) = 0.$$

Because of the uniqueness of the η-Einstein metric with null basic first Chern class in $[\omega_0^\dagger]$, one infers that $\varphi(t)$ converges smoothly to φ_∞ as $t \to \infty$.

5. Examples and Further Discussion

There are many examples with null and negative first basic Chern class. These examples include Reid's list of 95 K3 surfaces (see [71,72]), as well as all the examples of both null and negative Sasaki structures in [4] (Section 10.3). Furthermore, as discussed in [4] (Section 11.8), it is easy to construct examples of Sasaki manifolds with negative first basic Chern class in arbitrary odd dimensions. We refer the reader to [4] (Example 11.8.7, Theorems 11.8.8 and 11.8.9 in Section 11.8) for details of these examples.

As in El Kacimi Alaoui [5], we point out that our argument works directly on a compact oriented, taut, transverse Kähler foliated manifold with complex codimension n.

We refer the reader to [13,17] for preliminary details of compact oriented, taut, transverse Hermitian foliated manifolds with complex codimension n. Then, we can also use the methods originated by [73–75] (see also [17]) to study the continuity equation of transverse Hermitian metrics and transverse Gauduchon metrics, which are transverse versions of Sherman and Weinkove [43] for the continuity equation of Hermitian metrics and Zheng [76] for the continuity equation of Gauduchon metrics.

Explaining the emergence of the classical space-time geometry in some limit of a more fundamental, microscopic description of nature is a central problem in any quantum theory of gravity. In principle, the gauge/gravity-correspondence provides a framework where this problem can be addressed. This is a holographic correspondence relating a supergravity theory in five-dimensional Anti-deSitter space to a strongly coupled superconformal gauge theory on its 4-dimensional flat Minkowski boundary. In particular, the classical geometry should emerge from some quantum state of the dual gauge theory. In [77], the authors confirm this by showing how the classical metric emerges from a canonical state in the dual gauge theory. In particular, they obtain approximations to the Sasaki–Einstein metric (which is also the η-Einstein metric as mentioned in Section 2) underlying the supergravity geometry. We refer the reader to [77] for more details.

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