A mathematical theory for deterministic quantum mechanics

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Abstract.   Classical, *i.e.* deterministic theories underlying quantum mechanics are considered, and it is shown how an apparent quantum mechanical Hamiltonian can be defined in such theories, being the operator that generates evolution in time. It includes various types of interactions. An explanation must be found for the fact that, in the real world, this Hamiltonian is bounded from below. The mechanism that can produce exactly such a constraint is identified in this paper. It is the fact that not all classical data are registered in the quantum description. Large sets of values of these data are assumed to be indistinguishable, forming *equivalence classes*. It is argued that this should be attributed to information loss, such as what one might suspect to happen during the formation and annihilation of virtual black holes.

    The nature of the equivalence classes follows from the positivity of the Hamiltonian. Our world is assumed to consist of a very large number of subsystems that may be regarded as approximately independent, or weakly interacting with one another. As long as two (or more) sectors of our world are treated as being independent, they all must be demanded to be restricted to positive energy states only. What follows from these considerations is a unique definition of energy in the quantum system in terms of the periodicity of the limit cycles of the deterministic model.

1. Introduction

A natural way to interpret Quantum Mechanics is the following assumption: There is some set of deterministic equations of motion that lead to chaotic behavior, even in the state we normally call the vacuum. At large scales, which here means any scale large compared to the Planck length, the solutions of these equations can only be handled statistically. A scheme to handle the statistical properties of these solutions might well be what is called Quantum Mechanics today. To get some feeling of how such a scheme might work in practice, we have to start with studying some simple models. Most of these are far too simple to necessitate a statistical treatment, so that the apparatus of operators acting in a Hilbert space might seem to be unnecessary. But one can see how a formalism could be set up, and this is the aim of the present investigation.

    We start with *pre-quantization*. The probability distribution W is then re-written as the

absolute square of a wave function. The wave function obeys all the axioms of conventional quantum mechanics, allowing us to perform all the mathematical tricks known from Quantum Mechanics and Quantum Field Theory, such as group representation theory and renormalization group transformations.

We suspect that our world can be understood by starting from a pre-quantized classical, or 'ontological', system. However, a serious difficulty is then encountered: one indeed gets Quantum Mechanics, but the Hamiltonian is not naturally bounded from below. If time would be assumed to be discrete, the Hamiltonian eigenvalues would turn out to be periodic, so one might limit oneself to eigenvalues E with $0 \leq E < 2\pi/\delta t$, where δt is the duration of a fundamental time step, but then the choice of a vacuum state is completely ambiguous, unlike the situation in the real world that one might want to mimic. If time is continuous, the Hamiltonian eigenvalues tend to spread over the real line, from $-\infty$ to ∞.

In realistic theories, one therefore must impose a "superselection rule", projecting out a subspace of Hilbert space where all energies are non-negative. How exactly to do this will be described here. At first sight, the freedom to choose phase factors in wave functions allows one to make such a selection without loss of generality. This observation, however, is not the solution to the positivity problem of the Hamiltonian, since positivity must also be demanded to hold for subsystems, and when such systems interact, the suppression of negative energy states might cause the violation of unitarity, or locality, or both.

The work described in this paper was reported about earlier [1]. We derive the plausibility of our assumptions from first principles. First, the formalism is displayed in Section 2. Deterministic systems are shown to be accessible by quantum mechanical procedures, although this does not turn them into acceptable quantum mechanical models just yet, because the Hamiltonian is not bounded from below. Then, we briefly discuss how interactions can occur in a quantum mechanical sense (Section 3). We use the empirically known fact that the Hamiltonians are all bounded from below both before introducing the interaction and after having included the interaction. This necessitates our introduction of equivalence classes (Section 4), such that neither the quantum mechanical nor the macroscopic observer can distinguish the elements within one equivalence class, but they can distinguish the equivalence classes.

This procedure is necessary in particular when two systems are considered together *prior* to considering any interaction. We are led to the discovery that, besides the Hamiltonian, there must be a classical quantity E that also corresponds to energy, and is absolutely conserved as well as positive (Section 5). It allows us to define the equivalence classes. We end up discovering a precise definition of the quantum wave function for a classical system (both amplitude and phase), and continue our procedure from there.

Physical and intuitive arguments were displayed in Ref. [2]. In that paper, it was argued that any system with information loss tends to show periodicity at small scales, and quantization of orbits. It was also argued that some lock-in mechanism was needed to relate the Hamiltonian with an ontologically observable quantity E that is bounded from below. The lock-in mechanism was still not understood; here however we present the exact mathematical treatment and its relation to information loss. Interaction can be introduced in a rather direct manner (Section 3), by assuming energy not to be directly additive, but then it is difficult to understand how different energy sectors of the theory can be related to one another.

A more satisfactory picture emerges if one realizes that energy is not directly locally observable, but determined by the periods of the limit cycles. This is explained in Section 7. We think that this interpretation is imperative, and it sheds an interesting new light on the phenomenon we call quantum mechanics. After a discussion of our results (Section 8) an appendix follows in which we discuss the 'random automaton'. It allows us to estimate the

distribution of its limit cycles, though we immediately observe that the quantum models it generates are not realistic because the energy will not be an extensive quantity. The deterministic models that might reproduce observed quantum field systems must be very special.

2. Variables, beables and changeables

Time might be defined as a discrete variable or a continuous one, but this distinction is not essential for our present considerations.

The important thing is that, if in a deterministic theory, a tiny subsection of our universe is considered in isolation, then it will necessarily go into a periodic orbit. By itself, this remark is already non-trivial; one might have thought that, with only two or more continuous degrees of freedom, non-periodic orbits could easily arise. We will argue below that, if quantum mechanical behavior is desired, then the system will have to obey non-Newtonian equations of motion, and thereby evolve towards stable attractors; these must be periodic.

In a periodic orbit, a variable φ can be found such that the equation is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega \; ; \qquad \varphi \in [\, 0, \, 2\pi \,] \; . \tag{2.1}$$

This evolution equation is associated to the *quantum* Hamiltonian

$$H \; = \; \omega p \; = \; -i\omega \frac{\partial}{\partial \varphi} \; = \; \omega L_3 \; = \; \omega m \; , \quad m = 0, \pm 1, \pm 2, \cdots \; . \tag{2.2}$$

This is very similar to the Hamiltonian of the quantum harmonic oscillator,

$$H \; = \; \omega \, n \; , \qquad n = 0, +1, +2, \cdots \; , \tag{2.3}$$

where, momentarily, we ignore the immaterial additive constant $\frac{1}{2}\omega$. Of course, the quantum harmonic oscillator has only non-negative eigenvalues for the energy. Quantum harmonic oscillators seem to be characteristic for isolated sections of the universe in a quantum world. If our philosophy is to make sense, we have to explain what happened to the negative energy states.

One possibility is to interpret the negative energy states as the *bra* states. One can do this, but it hardly affects our problem. We will need to explain why direct transitions between positive and negative energy states cannot occur.

For the time being, we assume that the negative energy states are projected away somehow (a more satisfactory explanation will follow later). At this point, we note that, in a deterministic theory, there are two kinds of operators.

Assume that there are many periodic variables $\varphi^i(t)$. Any observable quantity, being a function of φ^i, not depending on operators such as $p_i = -i\partial/\partial\varphi^i$, and therefore commuting with all φ^i, will be called a *beable*. Through the time dependence of φ^i, the beables will depend on time as well. Any pair of beables, A and B, will commute with one another at all times:

$$[A(t_1), \, B(t_2)] = 0 \; , \quad \forall \, t_1, t_2 \; . \tag{2.4}$$

A *changeable* is an operator not commuting with at least one of the φ^i's. Thus, the operators p_i and the Hamiltonian H are changeables. Using beables and changeables as operators [2], we can employ all standard rules of quantum mechanics to describe a classical system.

3. Interactions

Interactions can be introduced in two distinct ways. Consider two (or more) periodic variables,

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \omega_i \varphi^i + \varepsilon f^i(\vec{\varphi}) \; ; \qquad H^{\mathrm{int}} = \varepsilon (f^i p_i + i \, g(\vec{\varphi})). \tag{3.5}$$

Here, the function $f(\vec{\varphi})$ does not have to be conservative:

$$\partial_i f^i \neq 0 \; , \tag{3.6}$$

in which case we have to add the function $g(\vec{\varphi}) = -\frac{1}{2}\partial_i f^i$ in Eq. (3.5). Perturbation theory can now be applied in the usual fashion by expanding in ε, and computing $\langle n|H^{\mathrm{int}}|m\rangle$.

The second way to introduce interactions is displayed in Fig. 1, for the two-variable case.
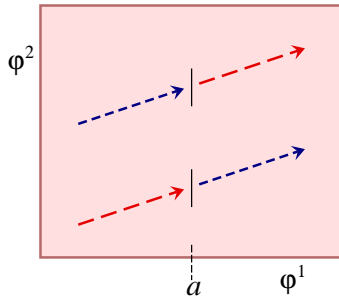
Figure 1. Interaction of type 2. There are two 'slits' in the two-variable plane. If the system crosses one slit, it hops to the same position at the other slit and *vice versa*. If the slit is small, this is a tiny perturbation.

Consider a discrete variable $s = 0, 1$ indicating whether the system is at the top or at the bottom of the figure. Write the hopping operator as

$$\sigma_x \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; . \tag{3.7}$$

There are two variables, $\varphi^1(t)$ and $\varphi^2(t)$. The first of these moves without perturbation:

$$\varphi^1(t) = \varphi^1(0) + \omega_1 t \; . \tag{3.8}$$

The other variable has angular velocity ω_2, but it hops whenever $\varphi^1(t)$ takes the value a (see Fig. 1) while φ^2 happens to be within the range of the slits, indicated by a function $\theta(\varphi^2)$ which is 1 if φ^2 hits one of the slits, and 0 otherwise.

Now, take as our total Hamiltonian

$$\begin{aligned} H \;&=\; H_0 + H^{\mathrm{int}} \; , \\ H_0 &= \omega_1 p_1 + \omega_2 p_2 \; , \qquad H^{\mathrm{int}} = A \, (\sigma_x - 1) \, \delta(\varphi^1 - a) \, \theta(\varphi^2) \; , \end{aligned} \tag{3.9}$$

where A is a constant, to be determined shortly.

Use the fact that the eigenvalues of σ_x are ± 1, and

$$e^{\pm\frac{1}{2}\pi i} = \pm i \; ; \quad \text{so} \qquad e^{\frac{1}{2}\pi i\sigma_x} = i\sigma_x \; ; \qquad e^{\frac{1}{2}\pi i(\sigma_x - 1)} = \sigma_x \; , \tag{3.10}$$

to derive

$$
\begin{aligned}
\exp\left(-i\int_{t_1}^{t_2} H\,\mathrm{d}t\right) &= \exp\left(-iA\,\theta(\varphi^2)(\sigma_x - 1)\tfrac{1}{\omega_1}\right) \\
&= \sigma_x \quad \text{if} \quad
\begin{cases}
A = \tfrac{1}{2}\pi\omega_1 \;, \\
\theta(\varphi^2) = 1 \;,
\end{cases}
\end{aligned}
\tag{3.11}
$$

where t_1 is chosen just before the slit is passed, and t_2 just after that moment. Thus, choosing in Eq. (3.9), $A = \tfrac{1}{2}\pi\omega_1$ gives us the correct hamiltonian to describe the slit.

Taking $|n_1, n_2\rangle$ to be the eigen modes of the unperturbed Hamiltonian H_0, one can now compute the matrix elements

$$
\langle n_1, n_2| H^{\mathrm{int}} |m_1, m_2\rangle \;,
\tag{3.12}
$$

which will be small if the slits are tiny.

However, these matrix elements will be unequal to zero for the entire range of values of n_i and m_i, so that positive energy states and negative energy states will be mixed. This problem also emerges in the interaction Hamiltonian (3.5). Is there no way out of this problem? Could one choose the interactions in such a way that the positive and negative energy states do not mix? At first sight, there might be a way. Suppose we have many slits. The function $\theta(\varphi^2)$ then takes a complicated form, switching from zero or one and back many times. The matrix elements (3.12) are obtained as Fourier transforms of $\theta(\varphi^2)$. Could one perhaps choose θ such that the unwanted Fourier coefficients cancel out to zero? Closer inspection of this question yields a negative result. Suppose one would replace $\theta(\varphi^2)$ by a smeared function $f(\varphi^2)$, whose Fourier transform is constrained by demanding superselection rules for the sign of the energy. We found that this smearing would not produce the desired amplitudes, but instead, give us mixed states! Of course, the smearing is a classical smearing, and not a smearing of the quantum amplitudes, so this would not give us an acceptable quantum theory.

Interaction must be introduced in a much more delicate way, and in any case, the superselection of the positive energy sector must be attributed to some more fundamental phenomenon in the theory.

# 4. Two (or more) harmonic oscillators; a relative-time uncertainty relation

As was explained briefly at the beginning of Section 2, we expect that, when two periodic variables interact, again periodic motion will result. What is needed for this is information loss. Periodicity will then again result. We will then again work on the superselection problem. Thus, our strategy will be now *first* to find the mechanism that selects out only the positive energy states, and only then we switch on the interaction.

In Fig. 2, the states are listed for two harmonic oscillators combined. Let their frequencies be ω_1 and ω_2. For later convenience, we re-inserted the ground state energies , so

$$
E = E_1 + E_2 = \omega_1(n_1 + \tfrac{1}{2}) + \omega_2(n_2 + \tfrac{1}{2}) \;.
\tag{4.13}
$$

The kets $|n_1, n_2\rangle = |n_1\rangle|n_2\rangle$ have $n_1 \geq 0$ and $n_2 \geq 0$, so they occupy the quadrant labelled I in Fig. 2. The bra states, in view of their time dependence, occupy the quadrant labelled III. The other two quadrants contain states with mixed positive and negative energies. Those

must be projected away. If we would keep those states, then any interaction between the two oscillators would result in inadmissible mixed states, in disagreement with what we know of ordinary quantum mechanics. So, although keeping the bra states is harmless because total energy is conserved anyway, the mixed states must be removed. This is very important, because now we see that the joint system cannot be regarded as a direct product. Some of the states that would be allowed classically, must be postulated to disappear. We now ask what this means in terms of the two periodic systems that we thought were underlying the two quantum harmonic oscillators.
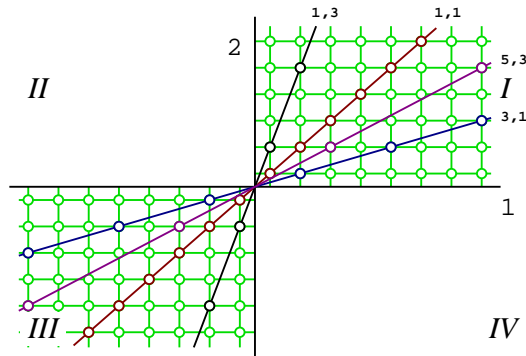
Figure 2. Combining two harmonic oscillators. Tilted lines show sequences of spectral states again associated to harmonic oscillators. For further explanation, see text.

The removal of the mixed energy states has one curious consequence. Let $E = E_1 + E_2$. Suppose we do admit the states where either both energies are positive or both energies are negative. This gives us the constraint

$$E_1 \cdot E_2 \geq 0 \ \rightarrow \ |E_1 + E_2| \geq |E_1 - E_2| \ . \tag{4.14}$$

A simplified notation for the position-momentum uncertainty relation for these systems is the time-energy uncertainty[1]

$$E \cdot \delta t = \tfrac{1}{2} \ . \tag{4.15}$$

Now rewriting

$$E_1 \cdot \delta t_1 + E_2 \cdot \delta t_2 = \tfrac{1}{2}\Big((E_1 + E_2)\delta(t_1 + t_2) + (E_1 - E_2)\delta(t_1 - t_2) \Big) \ , \tag{4.16}$$

and combining with (4.14), we get

$$\delta(t_1 + t_2) \leq \delta(t_1 - t_2) \ . \tag{4.17}$$

In conclusion: *the average time (position) is always more precisely defined than the relative time (position).* Or, in other words, the positive energy superselection rule may come about when the relative position of two variables is allowed to fluctuate due to some information-destroying mechanism, while the average position, or time variable, is maintained to be more

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[1] As is well-known, any time-energy uncertainty relation must be handled with care. Here, it should really be interpreted as a position-momentum uncertainty, which is practically the same thing for systems with a constant velocity.

accurately defined. At the end of this section, we will see a more quantitative description of this phenomenon.

It is surprising that the spectrum of combined states will still be discrete. The classical, non interacting system would only be periodic if the two frequencies have a rational ratio: $p\,\omega_1 - q\,\omega_2 = 0$, where p and q are relative primes. The smallest period would be $T = 2\pi q/\omega_1 = 2\pi p/w_2$, so that we would expect equally spaced energy levels with spacings $\omega_1/q = \omega_2/p$. Indeed, at high energies, we do get such spacings also in the quantum system, with increasing degeneracies, but at lower energies many of these levels are missing. If the frequencies have an irrational ratio, the period of the classical system is infinite, and so a continuous spectrum would have to be expected.

When two quantum harmonic oscillators are considered together, this does not happen. The spectrum is always discrete. In Fig. 2, it is indicated how to avoid having missing states and variable degeneracies. We see that actually full series of equally spaced energy levels still exist:

At any given choice of a pair of odd relative primes p and q, we have a unique series of bra- and ket states with energies $\omega_{pq}(n + \frac{1}{2})$, *with* $\omega_{pq} = p\,\omega_1 + q\,\omega_2$.

It is easy to see that these sequences are not degenerate, that all odd relative prime pairs of integers $(p,\,q)$ occur exactly once, and that all states are represented this way:

$$E_{n_1, n_2} = (n_1 + \tfrac{1}{2})\,\omega_1 + (n_2 + \tfrac{1}{2})\,\omega_2 = (n + \tfrac{1}{2})(p\,\omega_1 + q\,\omega_2)\ ;\qquad(4.18)$$

$$\frac{2n_1 + 1}{2n_2 + 1} = \frac{p}{q}\ .\qquad(4.19)$$

Some of these series are shown in the Figure.

We see that, in order to reproduce the quantum mechanical features, that is, to avoid the unphysical states where one energy is positive and the other negative, we have to combine two periodic systems in such a way that a new set of periodic systems arises, with frequencies ω_{pq}. Only then can one safely introduce interactions of some form. Conservation of total energy ensures that the bra and ket states cannot mix. States where one quantum oscillator would have positive energy and one has negative energy, have been projected out.

But how can such a rearrangement of the frequencies come about in a pair of classical periodic systems? Indeed, why are these frequencies (4.18) so large, and why are they labelled by odd relative primes? In Fig. 3 the periodicities are displayed in configuration space, $\{\varphi^1,\ \varphi^2\}$. The combined system evolves as indicated by the arrows. The evolution might not be periodic at all. Consider now the $(5,\ 3)$ mode. We can explain its short period $T_{53} = 2\pi/\omega_{53}$ only by assuming that the points form equivalence classes, such that different points within one equivalence class are regarded as forming the same 'quantum' state. If all points on the lines shown in Fig. 3 (the ones slanting downwards) form one equivalence class, then this class evolves with exactly the period of the oscillator whose frequency is ω_{53}.

In principle, this equivalence class can be formed in one of two ways: the information concerning the location of a point on this line is lost, either because there is an inherent information loss mechanism, implying that two different states may actually evolve to become the same state, or it could simply be that this information cannot be transmitted to macroscopically observable quantities. One could imagine a renormalization group technique that relates microscopic states to states at much larger distance scales. Not all data are being faithfully transmitted in the procedure. This latter option will later be dismissed as being impractical; it is highly revealing to assume explicit information loss.

For the time being, imagine that information loss takes place by means of processes that are random, uncontrollable or impossible to follow in detail, that cause our data point to fluctuate
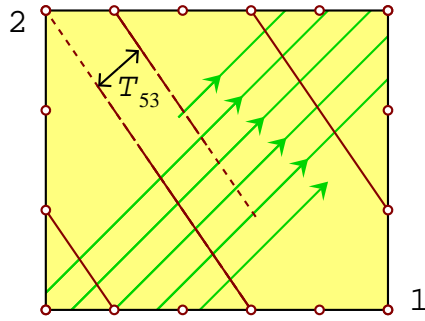
Figure 3. The equivalence class for oscillators in the case $(p, q) = (5, 3)$. Lines with arrows pointing right and up: time trajectories of individual points. Solid and broken lines going downwards: (part of) the $(5, 3)$ equivalence class at $t = 0$. For further explanation, see text.

along the line of its equivalence class. The line itself moves with the deterministic speed of the original oscillators.

Observe in Figure 3 that, in case $(p, q) = (5, 3)$, due to these fluctuations, five points of system 1 alone now form a single equivalence class, and three points in system 2. This is because we have assigned 5 quanta of energy to system 1 for every three quanta of energy of system 2. More generally, we could represent this situation with the wave function

$$\psi_{pq} = e^{i\,(n+\frac{1}{2})\,\left(p\,\phi^{1} + q\,\phi^{2} - \omega_{pq}t\right)}\,e^{-\frac{1}{2}\,i(\phi^{1} + \phi^{2})}\,, \tag{4.20}$$

where both variables $\phi^{1,2}$ were taken to be periodic with periods 2π. The (p, q) equivalence classes appear to be defined by the condition

$$p\,\phi^{1} + q\,\phi^{2} \;=\; \text{Constant}\,, \tag{4.21}$$

and this means that the n-dependent part of the wave function (4.20) has a the same phase all over the entire equivalence class, if we may assume that the second term in Eq. (4.20), arising from the vacuum fluctuations $\frac{1}{2}\,\omega\,t$, may be ignored.

To describe the equivalence classes it is helpful to introduce time variables t_a for the subsystems $a = 1, 2, \cdots$ in terms of their *unperturbed* evolution law, $q_a = \omega_a t_a$. Then, writing $E_1 = p\,\omega_1$, $E_2 = q\,\omega_2$, one can characterize the equivalence classes as

$$E_1\,\delta\,t_1 + E_2\,\delta\,t_2 \;=\; 0\,, \tag{4.22}$$

which means that the reactions that induce information loss cause q_a to speed up or slow down by an amount $\pm\delta\,t_a$ obeying this equation. One can easily generalize this result for many coexisting oscillators. They must form equivalence classes such that fluctuating time differences occur that are only constrained by

$$\sum_{a} E_a\,\delta\,t_a = 0\,, \tag{4.23}$$

which also are the collections of points that have the same phase in their quantum wave functions. We conclude that, in the ontological basis $\{|\,\vec{\phi}\,\rangle\}$, *all states* $|\,\vec{\phi}\,\rangle$ *which have the same phase in the wave function* $\langle\,\vec{\phi}\,|\psi\rangle$ (apart from a fixed, time independent term), *form one complete equivalence class.*

5. Energy and Hamiltonian

In the previous section, it was derived that the energies of the various oscillators determine the shape of the equivalence classes that are being formed. However, this would require energy to be a beable, as defined in Section 2. Of course, the Hamiltonian, being the generator of time evolution, cannot be a beable. It is important to notice here, that the parameters p and q defining the equivalence classes as in Section 4, are not exactly the energies of ϕ^1 and ϕ^2; the Hamiltonian eigenvalues are

$$H_1 = (n + \tfrac{1}{2})p\,\omega_1 \; ; \qquad H_2 = (n + \tfrac{1}{2})q\,\omega_2 \; , \tag{5.24}$$

with a common integral multiplication factor $n + \tfrac{1}{2}$. This n indeed defines the Hamiltonian of the orbit of the equivalence class. Generalizing this, the relation between the energies E_i in Eqs. (4.22) and (4.23) and the Hamiltonian H, is

$$H = (n + \tfrac{1}{2})E \; , \tag{5.25}$$

where n defines the evolution of a single clock that monitors the evolution of the entire universe.

Now that the relative primes p and q have become beables, we may allow for the fact that the periods of ϕ^1 and ϕ^2 depend on p and q as a consequence of some non-trivial interaction. But there is more. We read off from Fig. 3, that p points on the orbit of ϕ^1 in fact belong to the same equivalence class. Assuming that the systems 1 and 2 that we started off with, had been obtained again by composing other systems, we must identify these points. But this forces us to redefine the original periods by dividing these by p and q, respectively, and then we end up with two redefined periodic systems that are combined in the one and only allowed way:

$$p = q = 1 \; . \tag{5.26}$$

Only a single line in Fig. 2 survives: the diagonal.

The picture that emerges is the following. We are considering a collection of variables ϕ^a, each being periodic with different periods $T_a = 2\pi/\omega_a$. They each are associated with a positive beable E_a, such that $E_a = \omega_a$. The interactions will be such that the total energy $E = \sum_a E_a$ is conserved. Now as soon as these variables are observed together (even if they do not interact), an uncontrollable mixing mechanism takes place in such a way that the variables are sped up or slowed down by time steps δt_a obeying Eq. (4.23), so that, at any time t, all states obeying

$$\sum_a E_a t_a = \Big(\sum_a E_a\Big)t \; , \tag{5.27}$$

form one single equivalence class.

The evolution and the mixing mechanism described here are entirely classical, yet we claim that such a system turns into an acceptable quantum mechanical theory when handled probabilistically.

6. The total Hamiltonian

Consider two systems, labelled by an index $a = 1, 2$. System a is characterized by a variable $\phi^a \in [0, 2\pi)$ and a discrete index $i = 1, \cdots, N_a$, which is a label for the spectrum of states the system can be in.

The frequencies are characterized by the values $E_a^i = \omega_a^i$, so that the periods are $T_a^i = 2\pi/\omega_a^i$. Originally, as in Section 4, we had $\omega_a^i = p_a\omega_a$, where $p_1 = p$ and $p_2 = q$ were relative primes (and both odd), but the periods ω_a are allowed to depend on p_a, so it makes more sense to choose a general spectrum to start with.

The non-interacting parts of the Hamiltonians of the two systems, responsible for the evolution of each, are described by

$$H_a^0|n_a,\, i\rangle = (n_a + \tfrac{1}{2})E_a^i|n_a,\, i\rangle \;, \tag{6.28}$$

where the integer $n_a = -\infty, \cdots, +\infty$ is the changeable generating the motion along the circle with angular velocity ω_a. We have

$$n_a = -i\partial/\partial\phi^a \;. \tag{6.29}$$

The total Hamiltonian describing the evolution of the combined, unperturbed, system is not $H_1^0 + H_2^0$, but

$$H_{\mathrm{tot}}^0 = (n_{\mathrm{tot}} + \tfrac{1}{2})(E_1^i + E_2^j) \;, \tag{6.30}$$

where i, j characterize the states 1 and 2, but we have a single periodic variable $\phi^{\mathrm{tot}} \in [\,0,\, 2\pi)$, and

$$n_{\mathrm{tot}} = -i\partial/\partial\phi^{\mathrm{tot}} \;. \tag{6.31}$$

In view of Eq. (4.21), which here holds for $p = q = 1$, we can define

$$\phi^{\mathrm{tot}} = \phi^1 + \phi^2 \;, \tag{6.32}$$

while $\phi^1 - \phi^2$ has become invisible. We can also say,

$$n_1 = n_2 = n_{\mathrm{tot}} \;. \tag{6.33}$$

An interacting system is expected to have perturbed energy levels, so that its Hamiltonian should become

$$H^0 + H^{\mathrm{int}} = (n_{\mathrm{tot}} + \tfrac{1}{2})(E_1^i + E_2^j + \delta E^{ij}) \;, \tag{6.34}$$

where δE^{ij} are correction terms depending on both i and j. This is realized simply by demanding that the beables E_1^i and/or E_2^j get their correction terms straight from the other system. This is an existence proof for interactions in this framework, but, at first sight, it appears not to be very elegant. It means that the velocity ω_1^{ij} of one variable ϕ^1 depends on the state j that the other variable is in, but no matrix diagonalization is required. Indeed, we still have no transitions between the different energy states i. It may seem that we have to search for a more general interaction scheme. Instead, the scheme to be discussed next differs from the one described in this section by the fact that the energies E cannot be read off directly from the state a system is in, even though they are beables. The indices i, j are locally unobservable, and this is why we usually work with superimposed states.

7. Limit cycles

Consider the evolution following from a given initial configuration at $t = t_0$, having an energy E. Let us denote the state at time t by $F(t)$. The equivalence classes $|\psi(t)\rangle$ defined in Section 4 are such that the state $F(t)$ is equivalent to the state $F(t + \Delta t)$, where
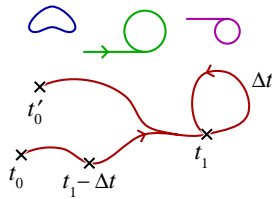
$$\Delta t = h/E \ , \tag{7.35}$$

in which h is Planck's constant, or:

$$|\psi(t)\rangle \overset{\text{def}}{=} \{F(t + n\Delta t), \ \forall n|_{n \geq n_1}\} \ , \tag{7.36}$$

for some n_1. Let us now assume that the equivalence indeed is determined by information loss. That the states in Eq. (7.36) are equivalent then means that there is a smallest time t_1 such that

$$F(t_1 + n\Delta t) = F(t_1) \ , \ \forall n \geq 0 \ . \tag{7.37}$$

Thus, the system ends in a *limit cycle* with period Δt.

Figure 4. Configuration space showing the limit cycles of an evolving system, indicating the times t_0, t_1 and the period Δt of a limit cycle. The points at t_0 and t_0' form one equivalence class.

One now may turn this observation around. A closed system that can only be in a finite number of different states, making transitions at discrete time intervals, would necessarily evolve back into itself after a certain amount of time, thus exhibiting what is called a Poincaré cycle. If there were no information loss, these Poincaré cycles would tend to become very long, with a periodicity that would increase exponentially with the size of the system. If there is information loss, for instance in the form of some dissipation effect, a system may eventually end up in Poincaré cycles with much shorter periodicities. Indeed, time does not have to be discrete in that case, and the physical variables may form a continuum; there could be a finite set of stable orbits such that, regardless the initial configuration, any orbit is attracted towards one of these stable orbits; they are the limit cycles. The energy of a state is then simply defined to be given by Eq. (7.35), or $E \overset{\text{def}}{=} h/P$, where $P = \Delta t$ is the period of the limit cycle, and h is Planck's constant.

Since this period coincides with the period of the wave function, we now deduce a physical interpretation of the phase of the wave function: *The phase of a wave function* (in the frame of energy eigenstates) *indicates where in the limit cycle the state will be.*

In general, we will have a superposition of many possibilities, and so we add to this the interpretation of the amplitude of the wave function: the absolute value of the amplitude in the frame of energy eigenstates indicates the probability that a particular limit cycle will be reached. Thus, we have reached the exact physical meaning of a quantum wave function.

We identified any deterministic system having information loss, with a quantum mechanical system evolving with a Hamiltonian defined by Eq. (5.25). However, in order to obtain a realistic model, one has to search for a system where the energy is extrinsic. With this, we mean that the universe consists of subsystems that are weakly coupled. Uncoupled systems are described as in Section 4; weakly coupled systems must be such that the limit cycles of the combination (12) of two systems (1) and (2) must have periods P_{12} obeying

$$\frac{1}{P_{12}} \approx \frac{1}{P_1} + \frac{1}{P_2} \ , \tag{7.38}$$

which approaches the exact identity in the limit of large systems being weakly coupled. This is the energy conservation law.

8. Discussion

When we attempt to regard quantum mechanics as a deterministic system, we have to face the problem of the positivity of the Hamiltonian, as was concluded earlier in Refs. [2],[3],[4]. There, also, the suspicion was raised that information loss is essential for the resolution of this problem. In this paper, the mathematical procedures have been worked out further, and we note that the deterministic models that we seek must have short limit cycles, obeying Eq. (7.35). Short limit cycles can easily be obtained in cellular automaton models with information loss, but the problem is to establish the addition rule (7.38), which suggests the large equivalence classes defined by Eq. (4.21). We think that the observations made in this paper are an important step towards the demystification of quantum mechanics.

We found that the energy eigenstates of a quantum system correspond to the limit cycles of the deterministic model. If P is the period of the limit cycle, then the energy E of this state is $E = h/P$ (see Eq. 7.35).

In models with more or less random evolution laws, one can guess the distribution of the periods of the limit cycles. In the Appendix of Ref. [1], the distribution of limit cycles with periods $\Delta t = P$ is derived for a "completely random" model, which we define to be a model where the mapping $F(t) \to F(t+1)$ is chosen completely randomly for every $F(t)$, independently of how many other states $F'(t)$ might map into the same state $F(t+1)$. It is found that the distribution of the cycles may then be expected to be logarithmic, which leads to a logarithmic energy spectrum: the energy eigenstates are a Poisson distribution on a logarithmic scale:

$$\varrho(E)\mathrm{d}E = \mathrm{d}E/E = \mathrm{d}\log E \ , \tag{8.39}$$

with cutoffs at

$$E_{\min} = \mathcal{O}(h/\sqrt{2\mathcal{N}}) \ , \qquad E_{\max} = \mathcal{O}(1/\delta t) \ , \tag{8.40}$$

where $\mathcal{N} = \mathcal{O}(e^V)$ is the total number of possible states and δt is some cutoff in time that our system may have. This is not a realistic energy spectrum for a quantum field theory, so we must conclude that realistic models will have to be far from random.

Cellular automaton models can be written down that show a rapid convergence towards small limit cycles, starting from any state $F(0)$. Conway's "game of life" [5] is an example, although that also features 'glider solutions', which are structures that are periodic, but they move forward when released in an empty region, so that they are not limit cycles in the strict

sense. It must be emphasized, however, that Conway's game of life will not serve as a model generating quantum mechanics. In a model generating quantum mechanics, the vacuum state is the state with the *longest* limit cycle, since it has the lowest energy. Thus, the empty state in Conway's game of life would carry more energy than its glider solutions.

The limit cycles of a random model are too long, those of cellular automata such as Conway's Game of Life are too short. In the real universe we have a small number of massless particle species, and many more massive ones, generating a rich spectrum of energies all very low compared to the Planck energy. If we assume that the Planck time would be Nature's natural time scale, then we observe that there must be many limit cycles whose periods are very long compared to the Planck time. Our universe appears to be built in such a way that, as soon as several of these cyclic limiting solutions are allowed to interact, new limit cycles will be reached with shorter periods, due to information loss.

In cellular automaton models, one might be able to mimic such a situation best by introducing a nearly conserved, positive quantity resembling energy, which can be seen statistically to decrease slowly on the average, so that the most chaotic initial states relax into more organized states that can easily end up in a limit cycle. The more chaotic the initial state, the smaller the period of its eventual limit cycle is expected to be, but there are many special initial states with very long limit cycles: the low energy states.

States of interest, with which we might attempt to describe the universe as we know it, must be very far away from any limit cycle. They are also far away from the strictly stationary eigenstates of the Hamiltonian. This means that we *do not yet know* which of the numerous possible limit cycles our universe will land into. This is why we normally use wave functions that have a distribution of amplitudes in the basis of the Hamiltonian eigenmodes. The squares of these amplitudes indicate the probability that any particular limit cycle will be reached. Also note that, according to General Relativity, taking into account the negative energies in the gravitational potentials, the total energy of the universe should vanish, which means that the entire universe might never settle for any limit cycle, as is indeed suggested by what we know of cosmology today: the universe continues to expand. The limit cycles mentioned in this paper refer to idealized situations where small sections of the universe are isolated from the rest, so as to be able to define their energies exactly. Only when a small part of the universe is sealed off from the rest, it is destined to end up in a limit cycle.

It may be of importance to note that our definition of energy, as being the inverse of the period of the limit cycle, supplies us with an absolute scale of energy: it is not allowed to add a constant to it. Moreover, the energy E in Eq. (5.25), as opposed to the Hamiltonian H, is a beable. Thus it is allowed to couple it to gravity by imposing Einstein's equations. If indeed the vacuum has a limit cycle with a large period, it carries a very low energy, and this is why we suspect that the true resolution of the cosmological constant problem [6] will come from deterministic quantum mechanics rather than some symmetry principle [7]. Earlier, the cosmological constant has been considered in connection with deterministic quantum mechanics by Elze [8]. The fact that the observed cosmological constant appears to be non-vanishing implies that a finite volume V of space will have a largest limit cycle with period

$$P = \frac{8\pi\hbar G}{\Lambda V} \; , \tag{8.41}$$

which is of the order of a microsecond for a volume of a cubic micron. If Λ were negative we would have had to assume that gravity does not exactly couple to energy.

Lorentz transformations and general coordinate transformations have not been considered in this paper. Before doing that, we must find models in which the Hamiltonian is indeed

extensive, that is, it can be described as the integral of an Hamiltonian density $T_{00}(\vec{x}, t)$ over 3-space, as soon as the integration volume element $\mathrm{d}^3\vec{x}$ is taken to be large compared to the 'Planck volume'. When that is achieved, we will be only one step away from generating locally deterministic quantum field theories.

What can be said from what we know presently, is that a particle with 4-momentum p_μ must represent an equivalence class that contains all translations $x^\mu \to x^\mu + \Delta x^\mu$ with $p_\mu \Delta x^\mu = n\,h$, where n is an integer. Note that a limit cycle having this large transformation group as an invariance group is hard to imagine, which probably implies that this particular limit cycle will take an infinite time to be established. Indeed, a particle in a fixed momentum state occupies the entire universe, and we already observed that the entire universe will never reach a limit cycle.

In the real world, we have only identified the observable quantum states, which we now identify with the equivalence classes of ontological states. We note that the physical states of the Standard Model in fact are also known to be gauge equivalence classes, Local gauge transformations modify our description of the dynamical variables, but not the physical observations. It is tempting to speculate that these gauge equivalence classes (partly) coincide with the equivalence discussed in this paper, although our equivalence classes are probably a lot larger, which may mean that many more local gauge symmetries are still to be expected.

It is even more tempting to include here the gauge equivalence classes of General Relativity: perhaps local coordinate transformations are among the dissipative transitions. In this case, the underlying deterministic theory might not be invariant under local coordinate transformations, and here also one may find novel approaches towards the cosmological constant problem and the apparent flatness of our universe.

Our reason for mentioning virtual black holes being sources of information loss might require further explanation. Indeed, the *quantum mechanical* description of a black hole is not expected to require information loss (in the form of quantum decoherence); it is the corresponding classical black hole that we might expect to play a role in the ontological theory, and that is where information loss is to be expected, since classical black holes do not emit Hawking radiation. As soon as we turn to the quantum mechanical description in accordance to the theory explained in this paper, a conventional, fully coherent quantum description of the black hole is expected.

Although we do feel that this paper is bringing forward an important new approach towards the interpretation of Quantum Mechanics, there are many questions that have not yet been answered. One urgent question is how to construct explicit models in which energy can be seen as extrinsic, that is, an integral of an energy density over space. A related problem is how to introduce weak interactions between two nearly independent systems. Next, one would like to gain more understanding of the phenomenon of (destructive) interference, a feature typical for Quantum Mechanics while absent in other statistical theories.

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