Branes from Partial Spontaneous Breaking of Supersymmetry

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Abstract

We give a pedagogical introduction into the approach to superbranes based on the concept of partial breaking of global supersymmetry (PBGS). The main focus is put on the universal methods of constructing manifestly worldvolume supersymmetric Goldstone superfield actions of superbranes, proceeding from the general relationship between linear and nonlinear realizations of global supersymmetries. We illustrate this by a few simple examples of PBGS systems on a flat Minkowski background: $N = 1$ supermembrane and space-filling D2- and D3-branes. As more complicated examples, we present the PBGS superfield form of the worldvolume actions of AdS_4 supermembrane, as well as of 3-branes on the AdS_5 and $\mathrm{AdS}_5 \times S^1$ backgrounds related to each other by T-duality.

1. Introduction

One of the approaches to superbranes proceeds from the concept of partial breaking of global supersymmetry (PBGS) [1], [2]-[25]. In such a description the objects representing the physical worldvolume superbrane degrees of freedom are Goldstone superfields. The worldvolume supersymmetry acts on them as linear transformations and so is manifest. The rest of the full target supersymmetry is realized nonlinearly. After passing to components in the Goldstone superfield action and eliminating auxiliary fields, one recovers a "static-gauge" form of the appropriate Green-Schwarz-type action (in general, after a field redefinition.

While for the ordinary p-branes the worldvolume multiplets are scalar, analogous supermultiplets of Dp-branes are known to be vector, with the Born-Infeld dynamics for gauge fields (see [26] and refs. therein). So the corresponding PBGS actions should form a subclass of manifestly supersymmetric extensions [27]-[29] of the Born-Infeld (BI) action. The actions from this variety are characterized by the second nonlinearly realized hidden supersymmetry. The PBGS approach can be considered as an efficient tool for deducing such superextensions of the BI action. Until now, only

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superextensions of abelian BI theory were derived in this way [5, 7, 12, 13]. However, this approach could be useful in the non-abelian case too. [1]

In these lectures we explain, on a few instructive examples ($N = 1$ supermembrane, space-filling D2- and D3-branes), how the PBGS approach augmented with the general methods of the theory of nonlinear realizations [31] leads to a manifestly supersymmetric description of superbranes and superextensions of the BI theory in terms of worldvolume superfields. The superbrane or BI superfield Lagrangian density is identified with a proper component of some linear supermultiplet of the full underlying supersymmetry. This multiplet is subjected to covariant constraints which express all its components in terms of the Goldstone multiplet of the unbroken supersymmetry. The precise form of these constraints can be found using the general relationship between linear and nonlinear realizations of supersymmetries [32] adapted to the PBGS case in [17, 18].

Besides discussing superbranes on flat Minkowski backgrounds, we also describe three examples of applying the PBGS techniques to constructing Goldstone superfield actions for superbranes on some AdS backgrounds [22, 23].

2. N=1, D=4 supermembrane

2.1 $N = 1, D = 4$ supermembrane as a PBGS system. The supermembrane in $D = 4$ spontaneously breaks half of the $N = 1, D = 4$ supersymmetry and one translation. The set of generators of $N = 1$ $D = 4$ Poincaré superalgebra in the $d = 3$ notation is naturally split into the unbroken $\{Q_a, P_{ab}\}$ and broken $\{S_a, Z\}$ parts $(a, b = 1, 2)$. The basic anti-commutation relations in this notation read

$$\{Q_a, Q_b\} = \{S_a, S_b\} = P_{ab}, \qquad \{Q_a, S_b\} = \epsilon_{ab} Z \, . \tag{2.1}$$

As was argued in [11], for deriving manifestly covariant superfield equations describing the worldvolume dynamics of superbrane in the present case (and some other ones), it suffices to deal with a nonlinear realization of the superalgebra (2.1) itself, ignoring all generators of the automorphisms of (2.1). So we put all generators into the coset and associate the $N = 1$, $d = 3$ superspace coordinates $\left\{\theta^a, x^{ab}\right\}$ with Q_a, P_{ab}. The remaining coset parameters are Goldstone superfields, $\psi^a \equiv \psi^a(x, \theta),\ q \equiv q(x, \theta)$. A coset element g is defined by [2]

$$g = e^{x^{ab} P_{ab}} e^{\theta^a Q_a} e^{q Z} e^{\psi^a S_a} \, . \tag{2.2}$$

[1] Its “inborn” feature (as distinct, e.g., from the approach proceeding from gauge-fixed Green-Schwarz Dp-brane actions [30]) is the manifestly *linear* realization of unbroken supersymmetry.

[2] In our notation the coset parameters x^{ab} and q are imaginary, while θ^a and ψ^a are real.

Then one constructs the Cartan 1-forms

$$g^{-1}dg = \omega_Q^a Q_a + \omega_P^{ab} P_{ab} + \omega_Z Z + \omega_S^a S_a, \tag{2.3}$$

$$\omega_P^{ab} = dx^{ab} + \frac{1}{4}\,\theta^{(a}d\theta^{b)} + \frac{1}{4}\,\psi^{(a}d\psi^{b)},$$
$$\omega_Z = dq + \psi_a d\theta^a, \quad \omega_Q^a = d\theta^a, \quad \omega_S^a = d\psi^a \tag{2.4}$$

and the corresponding covariant derivatives

$$\mathcal{D}_{ab} = (E^{-1})_{ab}^{cd}\,\partial_{cd}\,, \qquad \mathcal{D}_a = D_a + \frac{1}{2}\,\psi^b D_a\psi^c\,\mathcal{D}_{bc}\,, \tag{2.5}$$

where

$$E_{ab}^{cd} = \frac{1}{2}\,(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c) + \frac{1}{4}\,(\psi^c\partial_{ab}\psi^d + \psi^d\partial_{ab}\psi^c)\,,$$
$$D_a = \frac{\partial}{\partial\theta^a} + \frac{1}{2}\,\theta^b\partial_{ab}\,, \qquad \{D_a, D_b\} = \partial_{ab}\,. \tag{2.6}$$

The set of Goldstone superfields $\{q(x,\theta), \psi^a(x,\theta)\}$ is reducible. Indeed, ψ_a appears inside the form ω_Z *linearly* and so it can be covariantly eliminated by imposing the following manifestly covariant inverse Higgs [33] constraint

$$\omega_Z|_{d\theta} = 0 \quad\Longrightarrow\quad \psi_a = \mathcal{D}_a q\;, \tag{2.7}$$

where $|_{d\theta}$ means the ordinary $d\theta$-projection of the form. Thus $q(x,\theta)$ is the only essential Goldstone superfield needed to describe the partial spontaneous breaking $N=1\,,\ D=4 \ \Rightarrow\ N=1\,,\ d=3$ within the coset scheme.

In order to get dynamical equations, we put an additional, manifestly covariant constraint on the superfield $q(x,\theta)$. It is a direct covariantization of the "flat" equation of motion:

$$D^a D_a q = 0 \quad\Longrightarrow\quad \mathcal{D}^a\mathcal{D}_a q = 0\;. \tag{2.8}$$

Eq. (2.8) coincides with the dynamical equation of the supermembrane in $D=4$ as it was given in [7]. It was derived there from the coset approach with the $D=4$ Lorentz group generators included, so (2.8) actually possesses the hidden covariance under the full $D=4$ Lorentz group $SO(1,3)$. For the bosonic field $q(x)\equiv q(x,\theta)|_{\theta=0}$ it yields the equation corresponding to the static-gauge form of the Nambu-Goto action for membrane in $D=4$.

Our next goal is to construct the corresponding invariant off-shell superfield action. We apply the systematic approach based on the relationship between linear and nonlinear realizations of supersymmetry [32]. The construction is quite similar to the one exploited in [18] in application to $d=2$ PBGS systems.

As a first step, we define a *linear* realization of the considered PBGS pattern $N=1, D=4 \to N=1, d=3$. From the $d=3$ point of view, it amounts to

$N=2 \to N=1$, with the $N=2, d=3$ Poincaré superalgebra given by the relations (2.1). The primary object of such a realization is the scalar chiral $N=2, d=3$ superfield $\Phi(x,\theta,\zeta)$, where $x^{ab}, \theta^a, \zeta^d$ are the $N=2, d=3$ superspace coordinates. It is assumed to have the following transformation property under the central charge operator Z:

$$Z\Phi = 1 \,. \tag{2.9}$$

This means that the central charge generator acts as shifts of Φ. Such a realization can be understood as the following specific coset realization of $N=2, d=3$ supersymmetry (2.1): one treats Φ as the coset parameter (Goldstone superfield) associated with Z, while the rest of coset parameters as the coordinates of $N=2, d=3$ superspace on which Φ "lives" (cf. similar $d=2$ realizations considered in [18]). With respect to the $N=1$ supersymmetry $\{P_{ab}, Q_a\}$, the superfield Φ is a collection of standard $N=1$ superfields in the expansion of Φ in ζ^a, while under the S-supersymmetry it transforms in the following way

$$\delta_\eta \Phi = -\eta^a \left(\frac{\partial}{\partial \zeta^a} - \frac{1}{2}\, \zeta^b \partial_{ab} - \theta_a Z \right) \Phi \,. \tag{2.10}$$

Respectively, the spinor covariant derivatives in this realization are given by

$$\hat{D}^\theta_a = \frac{\partial}{\partial \theta^a} + \frac{1}{2}\, \theta^b \partial_{ab} - \zeta_a Z = D_a - \zeta_a Z \,, \quad D^\zeta_a = \frac{\partial}{\partial \zeta^a} + \frac{1}{2}\, \zeta^b \partial_{ab} \,. \tag{2.11}$$

The covariant chirality condition reads

$$\left(\hat{D}^\theta_a - i D^\zeta_a \right) \Phi = 0 \quad \Longrightarrow \quad \begin{aligned} \Phi \;&=\; \phi - i\, \zeta^a D_a \phi + \frac{1}{4}\, \zeta^2 \left[D^2\, \phi + 2\, i \right] , \\ \phi \;&\equiv\; \phi(x,\theta)\,, \end{aligned} \tag{2.12}$$

where (2.9) was taken into account. Thus the complex $N=1$ superfield $\phi(x,\theta)$ accommodates the irreducible set of the $(4+4)$ off-shell component fields of $\Phi(x,\theta,\zeta)$. Its S-supersymmetry transformation directly stems from (2.10) and (2.9):

$$\delta_\eta \phi = \eta^a \theta_a + i\, \eta^a D_a \phi \,. \tag{2.13}$$

For the real superfields ρ and ϕ_0 defined by

$$\phi = \phi_0 + i\, \rho$$

we obtain the following transformation laws

$$\delta_\eta \rho = -i\, \eta^a \theta_a + \eta^a D_a \phi_0 \;, \quad \delta_\eta \phi_0 = -\eta^a D_a \rho \;. \tag{2.14}$$

The spinor superfield

$$\xi_a = i\, D_a \rho$$

transforms under the S-supersymmetry with an inhomogeneous shift

$$\delta_\eta \xi_a = \eta_a \left(1 - \frac{i}{2}\, D^2\phi_0\right) - \frac{i}{2}\, \eta^b \partial_{ab}\phi_0 \,, \tag{2.15}$$

and so can be viewed as the Goldstone fermion of linear realization of the same PBGS pattern $N = 2 \to N = 1, d = 3$. The field content of $\rho(x, \theta)$ coincides with that of $q(x, \theta)$, so ρ can be regarded as the $N = 1$ Goldstone superfield for the spontaneously broken Z-transformations (it is shifted under Z).

Besides the basic Goldstone superfield ρ, there still remains the superfield ϕ_0 possessing homogeneous transformation laws under both $N = 1, d = 3$ supersymmetries. Now we shall show that it can be eliminated in terms of ρ by imposing a nonlinear constraint which brings the considered linear realization into a nonlinear one related to the original nonlinear realization by a field redefinition. To this end, we apply the method of refs. [32], [17, 18] to the system of $N = 1$ superfields ξ_a, ϕ_0. Construct their *finite* S-supersymmetry transformation and replace, in the final expressions, the parameters η^a by the Goldstone superfields $\psi^a(x, \theta)$ of the original nonlinear realization (taken with the sign minus). The resulting objects

$$\begin{aligned}
\tilde{\xi}_a &= \xi_a - \psi_a \left(1 - \frac{i}{2}\, D^2\phi_0\right) + \frac{i}{2}\, \psi^d \partial_{ad}\phi_0 - \frac{1}{4}\, \psi^2 \partial_{ab}\xi^b \,, \\
\tilde{\phi}_0 &= \phi_0 - i\psi^a \xi_a + \frac{i}{2}\, \psi^2 \left(1 - \frac{i}{2}\, D^2\phi_0\right)
\end{aligned} \tag{2.16}$$

are homogeneously transformed under the S-supersymmetry (and under the Q one, of course). So it is the covariant condition to put them equal to zero

$$\tilde{\xi}_a = 0 \,, \qquad \tilde{\phi}_0 = 0 \,. \tag{2.17}$$

Using the nilpotency property $\psi^3 = 0$, it is easy to find that these equations amount to

$$(a)\;\; \psi^a = \frac{\xi^a}{1 - \frac{i}{2}\, D^2\phi_0} \,, \qquad (b)\;\; \phi_0 = \frac{i}{2}\, \frac{\xi^2}{1 - \frac{i}{2}\, D^2\phi_0} \,. \tag{2.18}$$

These relations coincide with those found in [7]. The first one is the equivalence relation between the nonlinear and linear realizations Goldstone fermions, while the second one expresses ϕ_0 in terms of ψ^a or ξ^a:

$$\phi_0 = \frac{i}{2}\, \frac{\psi^2}{1 - \frac{1}{4}\, D^2\psi^2} = \frac{i\,\xi^2}{1 + \sqrt{1 + D^2\xi^2}} \,. \tag{2.19}$$

In view of the transformation property (2.14) of ϕ_0, the integral

$$S \sim \int d^3x d^2\theta \, \phi_0 \tag{2.20}$$

is invariant under the whole $N = 2, d = 3$ supersymmetry, and so it is the sought off-shell action of the Goldstone superfield $\rho(x, \theta)$. It describes, in a manifestly worldvolume supersymmetric manner, the $N = 1, D = 4$ supermembrane in a flat background. It can equally be written through the initial chiral $N = 2$ superfield $\Phi(x, \theta, \zeta)$, eq. (2.12), as an integral over the full or chiral $N = 2, d = 3$ superspaces [19]. In such a representation the full $N = 2$, $d = 3$ supersymmetry (2.1) is manifest.

It can be shown [19] that the dynamical equation (2.8) postulated on the purely geometric grounds and the equation of motion following from the off-shell action (2.20) are equivalent to each other.

3. Space-filling D2-brane

As the second instructive example, we consider the "space-filling" D2-brane with the $N = 1$, $d = 3$ vector multiplet as the worldvolume one.

3.1 D2-brane dynamics from nonlinear realizations. Our starting point is the superalgebra (2.1) with $Z = 0$. The coset element g contains only one Goldstone superfield ψ^a, and the covariant derivatives are still given by (2.5). In the flat case the $d = 3$ vector multiplet is described by a $N = 1$ spinor superfield strength μ_a subjected to the Bianchi identity:

$$D^a \mu_a = 0 \quad \Longrightarrow \quad \begin{cases} D^2 \mu_a = -\partial_{ab} \mu^b \ , \\ \partial_{ab} D^a \mu^b = 0 \ . \end{cases}$$

Its equation of motion reads

$$D^2 \mu_a = 0 \ . \tag{3.1}$$

It was shown in [11] that the following manifestly covariant generalization of (3.1), (3.1) describes the D2-brane:

$$\text{(a)} \;\; \mathcal{D}^a \psi_a = 0 \ , \qquad \text{(b)} \;\; \mathcal{D}^2 \psi_a = 0 \ . \tag{3.2}$$

The reasoning was mainly based on the observation that the purely bosonic limit of (3.2) amounts to the following equation for the vector $V_{ab} \equiv \mathcal{D}_a \psi_b|_{\theta=0}$:

$$\left(\partial_{ac} + V_a^d V_c^f \partial_{df} \right) V_b^c = 0 \ . \tag{3.3}$$

This nonlinear but polynomial equation was shown to be a "disguised" form of the equations of the non-polynomial $d = 3$ BI action which is just the bosonic core of the superfield D2-brane PBGS action as was explicitly demonstrated in [7]. The passing to the standard form of the $d = 3$ BI equation is achieved by a field redefinition which is a bosonic limit of the superfield equivalence redefinition relating the nonlinear realization Goldstone fermion ψ_a to μ_a treated as the Goldstone fermion of a *linear* realization of the same PBGS pattern (see next Subsection). Using this equivalence, one may explicitly show, like in the supermembrane case, that the equations (3.2) are equivalent to the worldvolume superfield equation following from the off-shell D2-brane action given in [7].

3.2 Off-shell superfield D2-brane action. Now we shall re-derive the off-shell D2-action of ref. [7] by the same generic method which was applied above to construct the Goldstone superfield action of $N = 1, D = 4$ supermembrane. To define the appropriate linear realization of the considered PBGS pattern, one needs to embed the $N = 1, d = 3$ Maxwell superfield strength μ_a into a linear $N = 2, d = 3$ multiplet. The latter should have such a transformation law under the S-supersymmetry that μ_a transform with an inhomogeneous term $\sim \eta_a$ and so admit an interpretation as the Goldstone fermion of linear realization.

The appropriate $N = 2, d = 3$ supermultiplet was proposed in [16] as a deformation of the $N = 2, d = 3$ Maxwell multiplet (which is a dimensional reduction of the $N = 1, d = 4$ tensor multiplet). This deformed multiplet is described by a real $N = 2, d = 3$ superfield $W(x, \theta, \zeta)$ subjected to the following constraints

$$\text{(a)} \quad \left[(D)^2 - (D^\zeta)^2\right] W = -2\,i\,, \qquad \text{(b)} \quad D^a D^\zeta_a W = 0 \tag{3.4}$$

(this form of constraints can be obtained from the one given in [16] by choosing a specific frame with respect to the explicitly broken $U(1)$-automorphism symmetry and making an appropriate rescaling of W [3]).

The standard S-supersymmetry transformation law of W

$$\delta_\eta W = -\eta^a \left(\frac{\partial}{\partial \zeta^a} - \frac{1}{2}\,\zeta^b \partial_{ab} \right) W \tag{3.5}$$

implies the following transformation laws for the irreducible $N = 1$ superfield components of $W(x, \theta, \zeta)$, $\mu_a \equiv -i\, D^\zeta_a W|_{\zeta=0}$ and $w \equiv W|_{\zeta=0}$,

$$\text{(a)} \quad \delta_\eta \mu_a = \eta_a \left(1 - \frac{i}{2}\, D^2 w \right) + \frac{i}{2}\,\eta^b \partial_{ab} w\,, \quad \text{(b)} \quad \delta_\eta w = -i\,\eta^a \mu_a\,. \tag{3.6}$$

It is easy to check that eq. (3.6a) is consistent with the Bianchi identity (3.1) (which is none other than eq. (3.4b)). Just due to the presence of constant $U(1)_A$ breaking term in the r.h.s. of (3.4a), the $N = 1$ Maxwell superfield μ_a transforms inhomogeneously under the S-supersymmetry, and thus is recognized as the Goldstone fermion of the linear realization of the considered $N = 2 \to N = 1, d = 3$ PBGS pattern.

Like in the supermembrane case, the additional homogeneously transforming $N = 1$ superfield $w(x, \theta)$ can be traded for the Goldstone-Maxwell one μ_a by imposing nonlinear constraints the precise form of which is dictated by our generic method applied to the given system. As the first step,

[3] For the first time such a deformation of the $N = 1$, $d = 4$ tensor multiplet constraints was considered in [34] in the context of $N = 4$ superconformal mechanics.

one defines the superfields $\tilde{\mu}_a$ and $\tilde{w}$ as finite S-supersymmetry transforms of μ_a and w, with the supertranslations parameter η^a being replaced by $-\psi^a(x,\theta)$

$$\tilde{\mu}_a = \mu_a - \psi_a \left(1 - \frac{i}{2}\, D^2 w\right) - \frac{i}{2}\, \psi^b \partial_{ab} w - \frac{1}{4}\, \psi^2 \partial_{ab}\mu^b \,,$$
$$\tilde{w} = w + i\, \psi^a \mu_a - \frac{i}{2}\, \psi^2 \left(1 - \frac{i}{2}\, D^2 w\right). \tag{3.7}$$

These quantities homogeneously transform under all $N=2$, $d=3$ transformations and so one can covariantly equate them to zero

$$\tilde{\mu}_a = \tilde{w} = 0 \; . \tag{3.8}$$

From these covariant constraints one gets the equivalence relation between ψ^a and μ^a

$$\psi^a = \frac{\mu^a}{1 - \frac{i}{2}\, D^2 w} \;, \tag{3.9}$$

as well as the relation

$$w = -\frac{i}{2}\, \frac{\mu^2}{1 - \frac{i}{2}\, D^2 w} \; . \tag{3.10}$$

These are precisely the equations derived in [7] (up to a rescaling of w). They can be used to express w in terms of either ψ^a, or μ^a

$$w = -\frac{i}{2}\, \frac{\psi^2}{1 + \frac{1}{4}\, D^2 \psi^2} = -\frac{i\, \mu^2}{1 + \sqrt{1 - D^2 \mu^2}} \; . \tag{3.11}$$

This composite superfield is just the corresponding Goldstone superfield Lagrangian density,

$$S \sim \int d^3x\, d^2\theta\, w \;, \tag{3.12}$$

since, in virtue of the Bianchi identity (3.1), the $d^3x\, d^2\theta$ integral of the variation (3.6b) is vanishing, i.e. $\delta_\eta S = 0$.

The same superfield D2-brane action can be written in a manifestly $N=2$ supersymmetric form as an integral over the whole $N=2$ superspace, with either W^2 or the $N=2$, $d=3$ Fayet-Iliopoulos term as the Lagrangian densities (like in other PBGS cases, these two independent invariants are reduced to each other after passing to the nonlinear realization).

4. Space-filling D3-brane

As the last example of PBGS approach to branes on the flat background we consider the space-filling D3-brane in $d=4$. This system amounts to the PBGS pattern $N=2 \to N=1$ in $d=4$, with a nonlinear generalization of $N = 1$, $d = 4$ vector multiplet as the Goldstone multiplet [5, 6]. The off-shell superfield action for this system was constructed in [27, 5]. Here we explain, following ref. [11], how the corresponding dynamical equations can be derived directly from the coset approach, like in other cases considered in these lectures.

4.1 D3-brane superfield equations of motion from nonlinear realizations. Our starting point is the $N = 2$, $d = 4$ Poincaré superalgebra without central charges:

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}\ , \qquad \left\{S_\alpha, \bar{S}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}\ . \tag{4.1}$$

Assuming the $S_\alpha, \bar{S}_{\dot\alpha}$ supersymmetries to be spontaneously broken, we introduce the Goldstone superfields $\psi^\alpha(x,\theta,\bar\theta)$, $\bar\psi^{\dot\alpha}(x,\theta,\bar\theta)$ as the corresponding parameters of the following coset

$$g = e^{i\,x^{\alpha\dot\alpha}P_{\alpha\dot\alpha}}\ e^{i\,\theta^\alpha Q_\alpha + i\,\bar\theta_{\dot\alpha}\bar{Q}^{\dot\alpha}}\ e^{i\,\psi^\alpha S_\alpha + i\,\bar\psi_{\dot\alpha}\bar{S}^{\dot\alpha}}\ . \tag{4.2}$$

With the help of the corresponding Cartan forms one can define the covariant derivatives

$$\mathcal{D}_\alpha = D_\alpha - i\left(\bar\psi^{\dot\beta} D_\alpha \psi^\beta + \psi^\beta D_\alpha \bar\psi^{\dot\beta}\right)\mathcal{D}_{\beta\dot\beta}, \quad \mathcal{D}_{\alpha\dot\alpha} = \left(E^{-1}\right)^{\beta\dot\beta}_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\ , \tag{4.3}$$

where

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\,\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\ , \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\,\theta^\alpha\partial_{\alpha\dot\alpha}\ , \\ E^{\beta\dot\beta}_{\alpha\dot\alpha} &= \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} - i\,\psi^\beta\partial_{\alpha\dot\alpha}\bar\psi^{\dot\beta} - i\,\bar\psi^{\dot\beta}\partial_{\alpha\dot\alpha}\psi^\beta\ . \end{aligned} \tag{4.4}$$

Now we can write the covariant version of the constraints on ψ^α, $\bar\psi^{\dot\alpha}$ which define the superbrane generalization of $N = 1$, $d = 4$ vector multiplet, together with the covariant equations of motion for this system. They are a direct covariantization of the free $N = 1$, $d = 4$ Maxwell superfield strength constraints and equation of motion:

$$\text{(a)}\ \ \overline{\mathcal{D}}_{\dot\alpha}\psi_\alpha = 0\,,\ \ \mathcal{D}_\alpha\bar\psi_{\dot\alpha} = 0\,, \qquad \text{(b)}\ \ \mathcal{D}^\alpha\psi_\alpha = 0\,,\ \ \overline{\mathcal{D}}_{\dot\alpha}\overline{\psi}^{\dot\alpha} = 0\,. \tag{4.5}$$

Eqs. (4.5a) are a covariantization of the flat $N = 1$ chirality conditions while (4.5b) generalizes at once the $N = 1$ superfield strength Bianchi identity and equation of motion. As was argued in [11], this set of superfield equations is self-consistent and compatible with the algebra of the covariant derivatives (4.3). For the physical bosonic components of $\psi, \bar\psi$,

$$V^{\alpha\beta} \equiv \mathcal{D}^\alpha\psi^\beta|_{\theta=0}\ , \qquad \bar{V}^{\dot\alpha\dot\beta} \equiv \overline{\mathcal{D}}^{\dot\alpha}\bar\psi^{\dot\beta}|_{\theta=0}, \tag{4.6}$$

these superfield equations imply, in the purely bosonic limit, the following equations

$$\partial_{\alpha\dot\alpha}V^{\alpha\beta} - V^\gamma_\alpha\bar{V}^{\dot\gamma}_{\dot\alpha}\ \partial_{\gamma\dot\gamma}V^{\alpha\beta} = 0\,, \qquad \partial_{\alpha\dot\alpha}\bar{V}^{\dot\alpha\dot\beta} - V^\gamma_\alpha\bar{V}^{\dot\gamma}_{\dot\alpha}\ \partial_{\gamma\dot\gamma}\bar{V}^{\dot\alpha\dot\beta} = 0\,. \tag{4.7}$$

It was shown in [11] that, like the analogous equations (3.3) in the D2-brane case, these equations can be cast in the standard form of the $d=4$ BI theory equations augmented with the Bianchi identity for the Maxwell field strength.

Note that at the full superfield level the field redefinition which leads from the disguised form of the BI equations (4.7) to their "canonical" form corresponds to passing from the Goldstone fermions ψ_α, $\bar\psi_{\dot\alpha}$ to the standard Maxwell superfield strength $W_\alpha, \bar W_{\dot\alpha}$. The nonlinear action of [27, 5, 6] was written just in terms of this latter object. The equivalent form (4.5) of the equations of motion and Bianchi identity is advantageous in that it manifests the second (hidden) supersymmetry, being constructed out of the covariant objects.

4.2 Linear and nonlinear realizations of the $N=2 \to N=1$ PBGS. Now we wish to establish the correspondence just mentioned and to reproduce the off-shell BI action of [27, 5, 6] by applying the general techniques based on the relationship between linear and nonlinear realizations of PBGS, like in the previous Sections.

Our starting point is the $N=2, d=4$ Goldstone-Maxwell multiplet [14, 5, 15]. In the $N=2$ superspace $(x^{\alpha\dot\alpha}, \theta^\alpha_i, \bar\theta^{\dot\alpha i})$ it is defined by the following deformation [15] of the standard $N=2$ Maxwell superfield strength constraints

$$\text{(a)}\ \ D^{ik}W - \bar D^{ik}\bar W = i\,M^{(ik)}\,, \qquad \text{(b)}\ \ D^i_\alpha \bar W = \bar D_{\dot\alpha i} W = 0\,. \tag{4.8}$$

Here

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} - i\,\bar\theta^{\dot\alpha i}\partial_{\alpha\dot\alpha}\,, \qquad \bar D_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} + i\,\theta^\alpha_i\partial_{\alpha\dot\alpha}\,,$$
$$D^{ik} = D^{\alpha i}\,D^k_\alpha\,, \qquad \bar D^{ik} = \bar D^i_{\dot\alpha}\,\bar D^{\dot\alpha k}$$

and $M^{ik} = M^{ki}$ is a triplet of constants which explicitly break the automorphism $SU(2)_A$ of $N=2$ supersymmetry down to $U(1)_A$ and satisfy the pseudo-reality condition

$$\overline{(M^{ik})} = \epsilon_{in}\,\epsilon_{km}\,M^{nm}\,.$$

In components, the deformation (4.8a) amounts to the appearance of constant imaginary part $\sim M^{ik}$ in the isotriplet auxiliary field of $N=2$ Maxwell multiplet.

Now we pass to the $N=1$ superfield notation by relabelling the Grassmann coordinates and spinor derivatives as

$$\theta^\alpha_1 \equiv \theta^\alpha\,, \quad \theta^\alpha_2 \equiv \zeta^\alpha\,, \quad D^1_\alpha \equiv D_\alpha\,, \quad D^2_\alpha \equiv D^\zeta_\alpha\,.$$

In order to have the off-shell S-supersymmetry (acting as ζ-supertranslations) spontaneously broken while the Q-supersymmetry unbroken, we are led to choose the following frame with respect to the explicitly broken $SU(2)_A$

$$M^{12} = 0\,, \qquad M^{11} = M^{22} = m\,, \tag{4.9}$$

where m is a real constant. Like in the case of D2-brane it is fixed up to rescaling of W. A convenient choice is

$$m = -2\,.$$

It will be also convenient to choose the basis in $N=2$ superspace where the chirality with respect to the variable ζ^α is manifest

$$\bar{D}^{\zeta}_{\dot\alpha} = -\frac{\partial}{\partial\bar\zeta^{\dot\alpha}}\,, \qquad D^{\zeta}_{\alpha} = \frac{\partial}{\partial\zeta^{\alpha}} - 2\,i\,\bar\zeta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,. \tag{4.10}$$

In this basis, constraints (4.8) imply the following structure of the superfield $W(x,\theta,\zeta)$

$$W = i\,\phi + i\,\zeta^\alpha W_\alpha - i\,\frac{1}{2}\,\zeta^2\left(1+\frac{1}{2}\,\bar{D}^2\bar\phi\right), \tag{4.11}$$

where ϕ and W_α are chiral $N=1$ superfields

$$\bar{D}_{\dot\alpha}\phi = \bar{D}_{\dot\alpha}W_\alpha = 0\,, \tag{4.12}$$

and the fermionic superfield W_α obeys the $N=1$ Maxwell superfield strength constraint

$$D^\alpha W_\alpha + \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha} = 0\,. \tag{4.13}$$

The S-supersymmetry transformation of the $N=2$ superfield W

$$\delta_\eta W = -\left[\eta^\alpha\frac{\partial}{\partial\zeta^\alpha} + \bar\eta^{\dot\alpha}\left(\frac{\partial}{\partial\bar\zeta^{\dot\alpha}} + 2\,i\,\zeta^\alpha\partial_{\alpha\dot\alpha}\right)\right]W \tag{4.14}$$

implies the following ones for its $N=1$ superfield components ϕ and W_α

$$\begin{aligned} &\delta_\eta\phi = -(\eta W)\,, \qquad \delta_\eta\bar\phi = -(\bar{W}\bar\eta)\,, \\ &\delta_\eta W_\alpha = \eta_\alpha\left(1+\frac{1}{2}\,\bar{D}^2\bar\phi\right) + 2\,i\,\bar\eta^{\dot\alpha}\partial_{\alpha\dot\alpha}\phi\,, \qquad \delta_\eta\bar{W}_{\dot\alpha} = \overline{(\delta_\eta W_\alpha)}\,. \end{aligned} \tag{4.15}$$

The superfield W_α shows up an inhomogeneous shift $\sim \eta_\alpha$ (proportional to the $SU(2)_A$ breaking parameters) in its transformation, so it is the Goldstone fermion of the linear realization of the considered $N=2 \rightarrow N=1$, $d=4$ PBGS pattern (the Goldstone-Maxwell $N=1$ superfield).

Now we are prepared to start the algorithmic procedure of passing to the relevant nonlinear realization exemplified in the previous Sections. We construct the finite η-transformations of the superfields ϕ and W_α proceeding from the infinitesimal ones (4.15)

$$\{\,\phi(\eta)\;,\;\; W_\alpha(\eta)\,\} = \left(1+\delta_\eta+\frac{1}{2}\,\delta_\eta^2+\frac{1}{3!}\,\delta_\eta^3+\frac{1}{4!}\,\delta_\eta^4\right)\{\,\phi\;,\;\; W_\alpha\,\}\,, \tag{4.16}$$

then pull out the parameters $\eta_\alpha, \bar\eta_{\dot\alpha}$ to the left and replace them by the original nonlinear realization Goldstone fermions, $\eta_\alpha \rightarrow -\psi_\alpha,\ \bar\eta_{\dot\alpha} \rightarrow -\bar\psi_{\dot\alpha}$.

It is a matter of straightforward computation to check that the objects $\tilde{\phi} \equiv \phi(-\psi)$, $\tilde{W}_\alpha \equiv W_\alpha(-\psi)$ transform homogeneously (though nonlinearly) with respect to the η-transformations

$$\delta_\eta\{\, \tilde{\phi},\ \tilde{W}_\alpha \,\} = i\left(\psi^\alpha \bar{\eta}^{\dot{\alpha}} - \eta^\alpha \bar{\psi}^{\dot{\alpha}}\right) \partial_{\alpha\dot{\alpha}}\{\, \tilde{\phi},\ \tilde{W}_\alpha \,\}\,, \tag{4.17}$$

and behave as ordinary $N = 1$ superfields under the unbroken ϵ-supertranslations acting in the $N = 1$ superspace $(x, \theta, \bar{\theta})$. Hence, one can impose the covariant constraints

$$\tilde{\phi} = \tilde{W}_\alpha = 0\,. \tag{4.18}$$

The leading terms of the relations between ϕ, W_α and ψ_α implied by these constraints are as follows

$$\phi \;=\; -\frac{1}{2}\,\psi^2\left(1 + \frac{1}{2}\,\bar{D}^2\bar{\phi}\right) - i\,\psi^\alpha\bar{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi + \ldots\;, \tag{4.19}$$

$$\begin{aligned} W_\alpha \;=\; &\psi_\alpha\left(1 + \frac{1}{2}\,\bar{D}^2\bar{\phi}\right) + 2\,i\,\bar{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi \\ &+ i\left(\psi_\alpha\bar{\psi}^{\dot{\alpha}}\partial_{\beta\dot{\alpha}}W^\beta + \psi_\beta\bar{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}W^\beta\right) + \ldots\;. \end{aligned} \tag{4.20}$$

These equations can be treated in the same way as their analogs in the PBGS examples discussed above. One should firstly make use of the relation (4.20) (and its conjugate) to express ψ_α, $\bar{\psi}$ in terms of W_α, $\bar{W}_{\dot{\alpha}}$ and their x-derivatives, and then substitute these expressions into (4.19) and its conjugate, thus obtaining covariant relations between $\phi, \bar{\phi}$ and $W_\alpha, \bar{W}_{\dot{\alpha}}$. The latter should allow one to trade $\phi, \bar{\phi}$ for $W_\alpha, \bar{W}_{\dot{\alpha}}$. As the next step, one substitutes $\psi^2 = W^2\left(1 + \frac{1}{2}\,\bar{D}^2\bar{\phi}\right)^{-2} + \ldots$ into (4.19). One ends up with the simple relations

$$\phi = -\frac{1}{2}\,\frac{W^2}{1 + \frac{1}{2}\,\bar{D}^2\bar{\phi}}\;, \qquad \bar{\phi} = -\frac{1}{2}\,\frac{\bar{W}^2}{1 + \frac{1}{2}\,D^2\phi}\;, \tag{4.21}$$

which are just those postulated in [5] and derived from the nilpotency condition in [6]. An advantage of the present derivation is that it sets the direct relationship with the "canonical" nonlinear realization through the equations (4.19), (4.20). More details of this derivation can be found in [19].

As was shown in [5, 6] the chiral superfield ϕ is just the Goldstone superfield Lagrangian density for the $N = 2 \to N = 1$ PBGS (it is the Fayet-Iliopoulos term from the $N = 2$ perspective). It describes a $N = 1$ superextension of the $d = 4$ BI theory with the second hidden $N = 1$ supersymmetry, or, equivalently, the gauge-fixed space-filling D3-brane in a flat background.

For completeness, we quote here the solution of (4.21) [5]

$$\phi = -\frac{1}{2}\left\{ W^2 + \frac{1}{2}\,\bar{D}^2\,\frac{W^2\bar{W}^2}{1-\frac{1}{2}\,A+\sqrt{1-A+\frac{1}{4}\,B^2}}\right\}, \tag{4.22}$$

$$A \equiv \frac{1}{2}\left(D^2W^2+\bar{D}^2\bar{W}^2\right), \qquad B \equiv \frac{1}{2}\left(D^2W^2-\bar{D}^2\bar{W}^2\right). \tag{4.23}$$

Having at our disposal the explicit relations (4.19), (4.20) we can prove the equivalence between the equations of motion corresponding to the $N=2 \to N=1$ BI Lagrangian (4.22) and eqs. (4.5) proposed within the original nonlinear realization setting.

In the rest of these lectures we shall consider a few examples of using the PBGS approach for deriving manifestly worldvolume supersymmetric actions of superbranes on curved AdS-type backgrounds.

5. AdS$_4$ membrane from the coset approach

We start with the case of bosonic AdS$_4$ membrane. Whereas it was known how to derive the static-gauge Nambu-Goto action for the branes in the d-dimensional flat Minkowski background from the nonlinear realizations (coset) approach applied to the relevant Poincaré group [10, 20], no such a self-contained derivation existed for AdS branes. The algebra of the AdS$_4$ group $SO(2,3)$ in the $d=3$ spinor notation reads:

$$\begin{aligned}
&[M_{ab},M_{cd}] = \varepsilon_{ac}M_{bd}+\varepsilon_{ad}M_{bc}+\varepsilon_{bc}M_{ad}+\varepsilon_{bd}M_{ac} \equiv (M)_{ab,cd}\,,\\
&[K_{ab},K_{cd}] = -\,(M)_{ab,cd}\,,\ [M_{ab},K_{cd}] = (K)_{ab,cd}\,,\ [M_{ab},P_{cd}] = (P)_{ab,cd}\,,\\
&[K_{ab},D] = -2P_{ab}+2mK_{ab}\,,\quad [P_{ab},D] = -2mP_{ab}\,,\quad [P_{ab},P_{cd}] = 0\,,\\
&[K_{ab},P_{cd}] = -2\left(\varepsilon_{ac}\varepsilon_{bd}+\varepsilon_{bc}\varepsilon_{ad}\right)D - m\,(M)_{ab,cd}\,,\ (a,b,c,d=1,2)\,.
\end{aligned} \tag{5.1}$$

The contraction parameter m is proportional to the inverse AdS$_4$ radius, and

$$P^{\dagger}_{ab} = P_{ab}\,,\quad M^{\dagger}_{ab} = -M_{ab}\,,\quad K^{\dagger}_{ab} = -K_{ab}\,,\quad D^{\dagger} = D\,,\quad m^{\dagger} = -m\,. \tag{5.2}$$

The $SO(1,2)$ generators M_{ab} together with K_{ab} form the algebra of $SO(1,3)$. As $m \to 0$, (5.1) becomes the $d+1=4$ Poincaré algebra. Another basis may be defined as

$$\tilde{K}_{ab} = \frac{1}{m}\,K_{ab} - \frac{1}{2\,m^2}P_{ab}\,, \qquad \tilde{D} = \frac{1}{m}\,D\,, \tag{5.3}$$

which are the standard $d=3$ special conformal and dilatation generators:

$$\begin{aligned}
&\left[\tilde{K}_{ab},\tilde{K}_{cd}\right] = 0\,, \qquad \left[M_{ab},\tilde{K}_{cd}\right] = \left(\tilde{K}\right)_{ab,cd}\,,\\
&\left[\tilde{K}_{ab},\tilde{D}\right] = 2\tilde{K}_{ab}\,, \qquad \left[P_{ab},\tilde{D}\right] = -2P_{ab}\,,\\
&\left[\tilde{K}_{ab},P_{cd}\right] = -2\left(\varepsilon_{ac}\,\varepsilon_{bd}+\varepsilon_{bc}\,\varepsilon_{ad}\right)\tilde{D} - (M)_{ab,cd}\,.
\end{aligned} \tag{5.4}$$

In the basis (5.1) the $d=3$ Poincaré subalgebra $\propto (P_{ab}, M_{ab})$ is manifest (together with the manifest $so(1,3)$). The generators (P_{ab}, D) form the maximal solvable subalgebra of $so(2,3)$. Any AdS$_{d+1}$ algebra $so(2,d)$ can be written in the basis where the d -dimensional Poincaré algebra is manifest, the d-dimensional translation operator together with the dilatation generator form a solvable subalgebra and the $(d+1)$-dimensional Lorentz algebra $so(1,d)$ is manifest [35]. This basis, the particular case of which is (5.1), is very advantageous for treating AdS branes in the nonlinear realization approach.

Now we consider the coset $SO(2,3)/SO(1,2)$ parametrized by:

$$g = e^{x^{ab}P_{ab}}\, e^{q(x)D}\, e^{\Lambda^{ab}(x)K_{ab}}\ . \tag{5.5}$$

The parameters $x^{ab} = -(x^{ab})^\dagger$ and $q(x) = -q^\dagger(x)$ provide a specific parametrization of the coset $SO(2,3)/SO(1,3) \sim$ AdS$_4$, adapted to the above solvable-subgroup basis of $so(2,3)$. The vector field $\Lambda^{ab}(x) = (\Lambda^{ab}(x))^\dagger$ parametrizes the coset $SO(1,3)/SO(1,2)$. Its inclusion is necessary for deducing the AdS$_4$ membrane action from the coset approach. Taking into account that the parameters associated with P_{ab} are the $d=3$ space-time coordinates, the resulting nonlinear realization actually describes the spontaneous breaking of $SO(2,3)$ down to its $d=3$ Poincaré subgroup as the only linearly realized one.

The full set of the $SO(2,3)$ transformations of the coset parameters in (5.5) can be found by acting on (5.5) from the left by various $SO(2,3)$ group elements. The $d=3$ conformal transformations of the AdS$_4$ coordinates $(x^{ab}, q(x))$ are generated by $g_0 = e^{b^{ab}\tilde{K}_{ab}}$:

$$\delta x^{ab} = 4\left(x^2 b^{ab} - 2\, x^{cd} b_{cd}\, x^{ab}\right) - \frac{1}{2\,m^2}\, e^{4mq} b^{ab}\ , \quad \delta q = -\frac{4}{m}\, x^{ab} b_{ab}\ . \tag{5.6}$$

These transformations provide a specific nonlinear realization of the $d=3$ conformal group algebra, such that the Goldstone field $q(x)$ is present in the conformal transformation of x^{ab}. Just this realization underlies the AdS$_4$ membrane. The building-blocks in constructing the action are left-invariant Cartan one-forms:

$$g^{-1}dg = \omega_P \cdot P + \omega_D D + \omega_K \cdot K + \omega_M \cdot M\ . \tag{5.7}$$

For our purposes it suffices to know the expressions for ω_P^{ab} and ω_D:

$$\omega_P^{ab} \ = \ e^{-2mq}\left(dx^{ab} + \frac{4\lambda^{ab}\lambda_{cd}\, dx^{cd}}{1-2\lambda^2}\right) + \frac{2\lambda^{ab} dq}{1-2\lambda^2} \equiv E^{ab}_{cd}(q,\lambda)\, dx^{cd}, \tag{5.8}$$

$$\omega_D \ = \ \frac{1+2\lambda^2}{1-2\lambda^2}\left(dq + \frac{4\, e^{-2mq}\lambda_{ab}\, dx^{ab}}{1+2\lambda^2}\right),\ \lambda^{ab} \equiv \frac{\tanh\sqrt{2\Lambda^2}}{\sqrt{2\Lambda^2}}\,\Lambda^{ab}\ , \tag{5.9}$$

$$\lambda^2 = \lambda^{ab}\lambda_{ab}\ .$$

The field λ^{ab} can be traded for $q(x)$ by the covariant constraint [33]

$$\omega_D = 0 \implies \lambda_{ab} = -\frac{1}{2}\, e^{2mq}\, \frac{\partial_{ab} q}{1+\sqrt{1-\frac{1}{2}\, e^{4mq}(\partial q)^2}} \implies \tag{5.10}$$

$$E^{ab}_{cd}(q) = e^{-2mq}\delta^{(a}_{(c}\delta^{b)}_{d)} - \frac{1}{2}\, e^{2mq} \frac{1}{1+\sqrt{1-\frac{1}{2}\, e^{4mq}\,(\partial q)^2}} \partial^{ab} q \partial_{cd} q\,. \tag{5.11}$$

The simplest invariant is the covariant volume of the $d=3$ space, $\int d^3x \det E(q)$, and the correct invariant action vanishing for a constant q reads (up to a normalization factor)

$$\begin{aligned} S &= \int d^3x \left[e^{-6mq} - \det E(q)\right] \\ &= \int d^3x\, e^{-6mq}\Big(1 - \sqrt{1-\frac{e^{4mq}}{2}\, \partial^{ab} q\, \partial_{ab} q}\,\Big)\,. \end{aligned} \tag{5.12}$$

By construction, it possesses all symmetries of the AdS_4 space and in the limit $m=0$ goes into the static-gauge Nambu-Goto action for a membrane in $d=4$ Minkowski space. The term $\sim \int d^3x e^{-6mq}$ is $SO(2,3)$ invariant on its own right.

To see that the action (5.12) indeed describes a membrane embedded into the AdS_4 background, let us look at the induced distance defined as the square of $\omega^{ab}_P = E^{ab}_{cd}(q)\, dx^{cd}$:

$$ds^2 = \omega^{ab}_P\, \omega_{P\,ab} = e^{-4mq}\,(dx^{ab}\, dx_{ab}) - \frac{1}{2}\, dq\, dq\,. \tag{5.13}$$

Introducing $U = e^{-2mq}$ and rescaling $x^{ab} = \frac{1}{2\sqrt{2}m}\,\tilde{x}^{ab}$, one can rewrite (5.13) and (5.12), up to some overall constant factors, as

$$\begin{aligned} ds^2 &= U^2\,(d\tilde{x}^{ab}\, d\tilde{x}_{ab}) - \left(\frac{dU}{U}\right)^2, \\ S &= \int d^3\tilde{x}\, U^3 \left(1-\sqrt{1-\frac{(\tilde{\partial} U\cdot \tilde{\partial} U)}{U^4}}\right). \end{aligned} \tag{5.14}$$

Thus ds^2 is recognized as the standard invariant interval on AdS_4, while S as the $d=3$ analog of the Maldacena scale-invariant brane action on AdS_5 [36] (actually, of the scalar fields piece of the D3-brane action). A novel point as compared to the previous consideration [35, 37] is the explicit derivation of the AdS_4 membrane action from the coset approach. It can be straightforwardly extended to the case of $(d-1)$-brane in AdS_{d+1} [38, 39].

6. AdS$_4$ supermembrane

Our starting point will be the $N=1$ AdS$_4$ superalgebra $osp(1|4)$ in the following basis

$$
\begin{aligned}
&\{Q_a, Q_b\} = 2P_{ab}\,, \quad \{S_a, S_b\} = 2P_{ab} - 4m\,K_{ab}\,,\\
&\{Q_a, S_b\} = 2\,\varepsilon_{ab}D - 2\,mM_{ab}\,,\\
&[M_{ab}, Q_c] = \varepsilon_{ac}\,Q_b + \varepsilon_{bc}\,Q_a \equiv (Q)_{ab,c}\,,\\
&[M_{ab}, S_c] = (S)_{ab,c}\,,\\
&[K_{ab}, Q_c] = (S)_{ab,c}\,, \quad [K_{ab}, S_c] = -\,(Q)_{ab,c}\,, \quad [P_{ab}, Q_c] = 0\,,\\
&[P_{ab}, S_c] = -2m\,(Q)_{ab,c}\,, \quad [D, Q_a] = m\,Q_a\,, \quad [D, S_a] = -m\,S_a\,. \qquad (6.1)
\end{aligned}
$$

The generators Q_a, P_{ab}, M_{ab} form $N=1$, $d=3$ super Poincaré algebra. The passing to the conformal basis, besides the redefinitions (5.3), implies the rescaling $S_a = m\tilde{S}_a$, such that $\tilde{S}_a$ is the $d=3$ conformal supersymmetry generator. The advantage of the basis (6.1) is that it manifests the $N=1$, $d=3$ super Poincaré subalgebra of $osp(1|4)$ and still yields the $N=1$, $d=4$ super Poincaré algebra in the contraction limit $m=0$. The $N=1$, $d=3$ Poincaré supertranslations $\propto (Q_a, P_{ab})$ together with D form the maximal solvable supersubalgebra of $osp(1|4)$.

We wish to construct an $OSp(1|4)$ extension of the AdS$_4$ membrane action (5.12), such that it possesses a manifest $N=1, d=3$ supersymmetry extending the manifest $d=3$ Poincaré worldvolume invariance of (5.12), and reproduces the action of the flat $N=1, d=4$ supermembrane [7] in the limit $m=0$.

The construction of the AdS$_4$ supermembrane action as a Goldstone superfield action is not so straightforward as in the bosonic case. To construct the PBGS action of the AdS$_4$ supermembrane, we shall apply a curved-space generalization of the techniques developed in [40, 17, 21, 19] and exemplified in the previous lectures.

As a first step we need to define the appropriate analog of the PBGS linear realization. It turns out that in the AdS case it is already a sort of nonlinear realization, but with weaker nonlinearities as compared to the final nonlinear realization. As a natural superextension of the bosonic coset element (5.5) we choose

$$
g = e^{x^{ab}P_{ab}}\, e^{\theta^a Q_a}\, e^{\psi^a S_a}\, e^{u(z)D}\, e^{\Lambda^{ab}(z)K_{ab}}\,. \qquad (6.2)
$$

Here, the parameters $z \equiv \left(x^{ab}, \theta^a, \psi^a\right)$ are $N=2, d=3$ superspace coordinates, while $u = u(z)$ and $\Lambda^{ab}(z)$ are Goldstone superfields given on this superspace. The subspace spanned by the coordinate set $\zeta \equiv \left(x^{ab}, \theta^a\right)$ is the flat $N=1, d=3$ superspace in which $N=1, d=3$ Poincaré supertranslations $\propto (Q_a, P_{ab})$ are realized in a standard way:

$$
\delta x^{ab} = a^{ab} - \frac{1}{2}\left(\epsilon^a\theta^b + \epsilon^b\theta^a\right), \qquad \delta\theta^a = \epsilon^a\,. \qquad (6.3)
$$

These transformations correspond to the left shift of (6.2) by the element $g_0 = e^{a^{ab}P_{ab}}\, e^{\epsilon^a Q_a}$. The rest of the $OSp(1|4)$ transformations except for the $SO(1,2)$ rotations is nonlinearly realized on the coset coordinates, mixing the $N=2$ superspace coordinates with the Goldstone superfield $u(z)$. Acting on (6.2) from the left by the element $g_0 = e^{\eta^a S_a}$, we find the explicit form of the broken supersymmetry transformations

$$
\begin{aligned}
\delta x^{ab} &= 2\,m\left(\theta^a x^{bc} + \theta^b x^{ac}\right)\eta_c + \frac{1}{2}\,e^{4mu}\left(\psi^a\eta^b + \psi^b\eta^a\right) \\
&\quad + \frac{3}{2}\,m\,e^{4mu}\psi^2\left(\theta^a\eta^b + \theta^b\eta^a\right), \\
\delta\theta^a &= 4\,m\,x^{ac}\,\eta_c + m\,\theta^2\,\eta^a - 3\,m\,e^{4mu}\psi^2\eta^a\,, \\
\delta u &= 2\,\theta^a\eta_a\,, \qquad \delta\psi^a = \eta^a - 2\,m\left(\eta^b\theta_b\psi^a - \eta^a\theta^b\psi_b - \eta^b\theta^a\psi_b\right).
\end{aligned} \tag{6.4}
$$

As follows from (6.1), all bosonic transformations are actually contained in the closure of the supersymmetry transformations.

What we have at this stage, is a nonlinear realization of the $N=1$ AdS$_4$ supergroup on the $N=2$, $d=3$ Goldstone superfield $u(x,\theta,\psi)$:

$$
\delta^* u(x,\theta,\psi) = -\left(\delta x^{ab}\partial_{ab} + \delta\theta^a\partial^\theta_a + \delta\psi^a\partial^\psi_a\right)u(x,\theta,\psi) + 2\,\theta^a\eta_a\,. \tag{6.5}
$$

The first component in the θ,ψ expansion of u can be regarded as the Goldstone dilaton field discussed in the previous Section. The spinor derivative $D_a u$, where

$$
D_a = \frac{\partial}{\partial\theta^a} + \theta^b\partial_{ab}\,, \qquad \{D_a, D_b\} = 2\partial_{ab}\,, \tag{6.6}
$$

is shifted by η_a under the S -supersymmetry. This suggests that we actually face the 1/2 spontaneous breaking of the AdS$_4$ supersymmetry, with $D_a u|_{\psi=0}$ as the corresponding Goldstone fermionic $N=1$ superfield. However, u contains extra component fields having no Goldstone interpretation. To construct the minimal Goldstone multiplet, we resort to the method which was applied in [21] to $d=2$ PBGS systems and, in [19], to the flat-space $N=1, d=4$ supermembrane. Following the reasonings of [19] and keeping in mind that the scalar multiplets of $N=1$ AdS$_4$ supergroup are represented by chiral $N=1, d=4$ (or $N=2, d=3$) superfields, we regard the Goldstone superfield $u(z)$ to be *complex* and subject it to the covariant chirality constraint

$$
\left(\nabla^Q_a - i\nabla^S_a\right)u = 0 \tag{6.7}
$$

where $\nabla^Q_a u$ and $\nabla^S_a u$ are the $OSp(1|4)$ covariant spinor derivatives of $u(z)$ with respect to θ^a and ψ^a. For our purpose it is of no need to know their precise structure, what actually matters is that all the coefficients in the ψ expansion of $u(z)$ can be expressed by (6.7) in terms of $u(z)|_{\psi^a=0}$ and derivatives thereof. E.g., the $\psi^a = 0$ component of (6.7) expresses the first coefficient as

$$\frac{\partial u}{\partial \psi^a}|_{\psi=0} = -i\, e^{2mu} D_a u|_{\psi=0}\,. \tag{6.8}$$

Thus the complex $N = 1$, $d = 3$ superfield

$$u_0(x, \theta) \equiv q(x, \theta) + i\, \Phi(x, \theta)\ , \quad q^\dagger = -q\,, \quad \Phi^\dagger = -\Phi\ , \tag{6.9}$$

incorporates the full irreducible field content of the $N = 2$, $d = 3$ Goldstone chiral superfield $u(x, \theta, \psi)$. Its S-supersymmetry transformation reads

$$\begin{aligned}
\delta q &= Lq - e^{2mq}\, \eta^a \left[\sin(2m\Phi)\, D_a q + \cos(2m\Phi)\, D_a \Phi\right] + 2\, \eta^a \theta_a\ , \\
\delta \Phi &= L\Phi + e^{2mq}\, \eta^a \left[\cos(2m\Phi)\, D_a q - \sin(2m\Phi)\, D_a \Phi\right]\ .
\end{aligned} \tag{6.10}$$

The nonlinear realization we are facing at this step is still non-minimal. Besides the $N = 1$ superfield $q(x, \theta)$ which contains all Goldstone fields required by the 1/2 breaking of $OSp(1|4)$ down to its $N = 1, d = 3$ Poincaré subgroup there is an extra non-Goldstone $N = 1, d = 3$ superfield $\Phi(x, \theta)$. The last step is to express the latter in terms of q and its derivatives by imposing some nonlinear covariant constraint on $u_0(x, \theta)$, analogous to the constraints imposed in the flat case [7]. It reads

$$\Phi = \frac{e^{2mq} D^a\, q D_a q}{4 + e^{2mq}\, D^2 \Phi} \quad \Longleftrightarrow \quad \Phi = \frac{e^{2mq}\, D^a q\, D_a q}{2 + \sqrt{4 + e^{4mq} D^2 (D^b q\, D_b q)}}\ . \tag{6.11}$$

It can be directly checked to be covariant with respect to the transformations (6.10). From our superfield u_0 we can construct the invariant

$$S_2 = -\frac{1}{2\, i\, m} \int d^3x\, d^2\theta \left(e^{-4mu_0} - e^{4mu_0^\dagger}\right). \tag{6.12}$$

In view of the nilpotency of Φ defined by eq. (6.11), the final action takes the form

$$S_2 \sim \int d^3x\, d^2\theta\, \frac{e^{-2mq}\, D^a\, q D_a q}{2 + \sqrt{4 + e^{4mq} D^2 (D^b q\, D_b q)}}\ . \tag{6.13}$$

The action S_2 contains the kinetic term of $q(\zeta)$ and, in the limit $m \to 0$, reduces to the flat $N = 1, d = 4$ supermembrane PBGS action of [7]. After eliminating the auxiliary field $B = D^2 q|_{\theta=0}$, the bosonic part of S_2 coincides with (5.12).

We come to the conclusion that the Goldstone superfield action (6.13) is the natural superextension of the conformally-invariant AdS$_4$ membrane action (5.12). Besides being manifestly invariant under $N = 1, d = 3$ Poincaré supersymmetry, it is invariant under the nonlinearly realized part of $N = 1$ AdS$_4$ supersymmetry $OSp(1|4)$ which acts on the $N = 1, d = 3$ superworldvolume as the Goldstone superfield-modified $d = 3$ superconformal transformations. Thus it is a PBGS superfield form of the worldvolume action of $N = 1$ AdS$_4$ supermembrane.

7. 3-branes in super AdS$_5$ and AdS$_5 \times S^1$ backgrounds

We start with recalling how the PBGS $N=1$ L3-brane action and related to it via T-duality $N=1$ scalar 3-brane action in the flat Minkowski backgrounds can be deduced as the Goldstone superfield actions describing the one-half partial breaking of global $N=2$ Poincaré supersymmetry in $d=4$.

The first option corresponds to $N=1$ tensor multiplet as the Goldstone one [5, 6, 24]. The starting point is the $N=2$, $d=4$ Poincaré superalgebra with a real central charge D

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}\,, \quad \left\{S_\alpha, \bar{S}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}\,, \quad \left\{Q_\alpha, S_\beta\right\} = -\varepsilon_{\alpha\beta}\, D\,,$$
$$\left\{\bar{Q}_{\dot\alpha}, \bar{S}_{\dot\beta}\right\} = -\varepsilon_{\dot\alpha\dot\beta}\, D\,. \tag{7.1}$$

Here $Q_\alpha, \bar{Q}_{\dot\alpha}$ and $S_\alpha, \bar{S}_{\dot\alpha}$ are generators of the unbroken and broken $N=1$ supersymmetries, respectively. These generators and the 4-translation generator $P_{\alpha\dot\alpha}$ possess the standard commutation relations with the Lorentz $so(1,3)$ generators $(M_{\alpha\beta}, \bar{M}_{\dot\alpha\dot\beta})$:

$$\begin{aligned}
i\left[M_{\alpha\beta}, M_{\rho\sigma}\right] &= \varepsilon_{\alpha\rho}\, M_{\beta\sigma} + \varepsilon_{\alpha\sigma}\, M_{\beta\rho} + \varepsilon_{\beta\rho}\, M_{\alpha\sigma} + \varepsilon_{\beta\sigma}\, M_{\alpha\rho} \equiv \left(M\right)_{\alpha\beta,\rho\sigma}\,,\\
i\left[\bar{M}_{\dot\alpha\dot\beta}, \bar{M}_{\dot\rho\dot\sigma}\right] &= \left(\bar{M}\right)_{\dot\alpha\dot\beta,\dot\rho\dot\sigma}\,, \qquad i\left[M_{\alpha\beta}, P_{\rho\dot\rho}\right] = \varepsilon_{\alpha\rho}\, P_{\beta\dot\rho} + \varepsilon_{\beta\rho}\, P_{\alpha\dot\rho}\,,\\
i\left[\bar{M}_{\dot\alpha\dot\beta}, P_{\rho\dot\rho}\right] &= \varepsilon_{\dot\alpha\dot\rho}\, P_{\rho\dot\beta} + \varepsilon_{\dot\beta\dot\rho}\, P_{\rho\dot\alpha}\,,\\
i\left[M_{\alpha\beta}, Q_\gamma\right] &= \varepsilon_{\alpha\gamma}\, Q_\beta + \varepsilon_{\beta\gamma}\, Q_\alpha \equiv \left(Q\right)_{\alpha\beta,\gamma}\,,\\
i\left[M_{\alpha\beta}, S_\gamma\right] &= \left(S\right)_{\alpha\beta,\gamma}\,, \qquad i\left[\bar{M}_{\dot\alpha\dot\beta}, \bar{Q}_{\dot\gamma}\right] = \left(\bar{Q}\right)_{\dot\alpha\dot\beta,\dot\gamma}\,,\\
i\left[\bar{M}_{\dot\alpha\dot\beta}, \bar{S}_{\dot\gamma}\right] &= \left(\bar{S}\right)_{\dot\alpha\dot\beta,\dot\gamma}\,.
\end{aligned} \tag{7.2}$$

Then one introduces two $N=1$ superfields: a real one $L(x,\theta)$ subjected to the constraint

$$D^2 L = \bar{D}^2 L = 0\,, \tag{7.3}$$

and so describing a tensor $N=1$ supermultiplet, and a complex chiral superfield $F, \bar{F}$,

$$D_\alpha F = \bar{D}_{\dot\alpha} \bar{F} = 0\,. \tag{7.4}$$

Here

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\,\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\,\theta^\alpha\partial_{\alpha\dot\alpha}\,. \tag{7.5}$$

On these $N=1$ superfields one implements [5] the following off-shell representation of the full $N=2$ supersymmetry (7.1):

$$\delta L = -i\left(\eta^\alpha\theta_\alpha - \bar\eta_{\dot\alpha}\bar\theta^{\dot\alpha}\right) + \eta^\alpha D_\alpha \bar{F} - \bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha} F\,, \quad \delta F = -\eta^\alpha D_\alpha L\,,$$
$$\delta\bar{F} = \bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha} L \tag{7.6}$$

where $\eta_\alpha, \bar\eta_{\dot\alpha}$ are the infinitesimal transformation parameters associated with the generators $S_\alpha,\ \bar S_{\dot\alpha}$. It is a modification of the transformation law of $N=2$ tensor multiplet [41] written in terms of its $N=1$ superfield components.

One can construct the simplest invariant 'action' as follows

$$S = \frac{1}{4}\int d^4x\, d^2\bar\theta F + \frac{1}{4}\int d^4x\, d^2\theta \bar F\ . \tag{7.7}$$

To make it meaningful one should express the chiral supermultiplet $F, \bar F$ in terms of the Goldstone tensor multiplet L by imposing proper covariant constraints [5, 6]

$$F = -\frac{D^\alpha L\, D_\alpha L}{2 - D^2 \bar F} \qquad \bar F = -\frac{\bar D_{\dot\alpha} L\, \bar D^{\dot\alpha} L}{2 - \bar D^2 F} \quad \Longrightarrow \tag{7.8}$$

$$F = -\psi^2 + \frac{1}{2} D^2 \left[\frac{\psi^2 \bar\psi^2}{1 + \frac{1}{2} A_+ + \sqrt{1 + A_+ + \frac{1}{4}(A_-)^2}} \right], \tag{7.9}$$

$$\psi_\alpha \equiv D_\alpha L\,, \quad \bar\psi_{\dot\alpha} \equiv \bar D_{\dot\alpha} L\,, \quad A_\pm = \frac{1}{2}\left(D^2 \bar\psi^2 \pm \bar D^2 \psi^2 \right). \tag{7.10}$$

Finally, the action (7.7) becomes

$$\begin{aligned} S \ =\ & -\frac{1}{4}\int d^4x\, d^2\theta\, \bar\psi^2 - \frac{1}{4}\int d^4x\, d^2\bar\theta\, \psi^2 \\ & + \frac{1}{4}\int d^4x\, d^4\theta\, \frac{\psi^2 \bar\psi^2}{1 + \frac{1}{2} A_+ + \sqrt{1 + A_+ + \frac{1}{4}(A_-)^2}}\ . \end{aligned} \tag{7.11}$$

It is a nonlinear extension of the standard $N=1$ tensor multiplet action. In the bosonic sector it gives rise to the static-gauge Nambu-Goto action for L3-brane in $d=5$ Minkowski space, with one physical scalar of L being the transverse brane coordinate and another one represented by the notoph field strength. After dualizing L into a pair of conjugated chiral and antichiral $N=1$ superfields (the notoph strength is dualized into a scalar field) the PBGS form of the worldvolume action of super 3-brane in $d=6$ is reproduced [4].

Let us point out that the constraints (7.8) which play the central role in deriving the action (7.11) guarantee 5-dimensional Lorentz covariance [23].

Now we wish to generalize this flat superspace construction to the case of partial spontaneous breaking of the simplest AdS$_5$ supersymmetry $SU(2,2|1)$, that is $N=1$ superconformal group in $d=4$.

The superalgebra $su(2,2|1)$ contains $so(2,4) \oplus u(1)$ bosonic subalgebra with the generators $\left\{ P_{\alpha\dot\alpha}, M_{\alpha\beta}, \bar M_{\dot\alpha\dot\beta}, K_{\alpha\dot\alpha}, D \right\}$ and $\{J\}$ and eight supercharges $\left\{ Q_\alpha, \bar Q_{\dot\alpha}, S_\alpha, \bar S_{\dot\alpha} \right\}$. We choose the basis in a such way, that the generators

$K_{\alpha\dot\alpha}$ form $so(1,4)$ subalgebra together with the $d=4$ Lorentz generators $\left\{M_{\alpha\beta}, \bar M_{\dot\alpha\dot\beta}\right\}$. The rest of non-trivial (anti)commutators reads

$$
\begin{aligned}
& i\big[D, P_{\alpha\dot\alpha}\big] = mP_{\alpha\dot\alpha}\,, \qquad i\big[D, K_{\alpha\dot\alpha}\big] = 2P_{\alpha\dot\alpha} - mK_{\alpha\dot\alpha}\,, \\
& i\left[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}\right] = \varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,D - \frac{m}{2}\left(\varepsilon_{\alpha\beta}\,\bar M_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,M_{\alpha\beta}\right), \\
& \{Q_\alpha, S_\beta\} = -\varepsilon_{\alpha\beta}\,(D + i\,mJ) + mM_{\alpha\beta}\,, \qquad \left\{Q_\alpha, \bar Q_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}\,, \\
& \left\{S_\alpha, \bar S_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha} - 2mK_{\alpha\dot\alpha}\,,
\end{aligned}
$$

$$
\begin{aligned}
& i\,[D, Q_\alpha] = \frac{m}{2}\,Q_\alpha\,, && i\,[D, S_\alpha] = -\frac{m}{2}\,S_\alpha\,, \\
& [J, Q_\alpha] = \frac{3}{2}\,Q_\alpha\,, && [J, S_\alpha] = -\frac{3}{2}\,S_\alpha\,, \\
& i\,[K_{\alpha\dot\alpha}, Q_\beta] = -\,\varepsilon_{\alpha\beta}\bar S_{\dot\alpha}\,, && i\,[K_{\alpha\dot\alpha}, S_\beta] = \varepsilon_{\alpha\beta}\bar Q_{\dot\alpha}\,, \\
& i\,[P_{\alpha\dot\alpha}, S_\beta] = m\,\varepsilon_{\alpha\beta}\bar Q_{\dot\alpha}\,.
\end{aligned}
\tag{7.12}
$$

This basis is another example of the ‘AdS basis’ of conformal superalgebras [35, 37, 22, 42]. The parameter m has the meaning of the inverse AdS_5 radius, $m = R^{-1}$. In the limit $m = 0$ ($R = \infty$) one recovers from (7.12) the $N = 1, d = 5$ Poincaré superalgebra, with D becoming the 5th component of momenta. The generators J and $K_{\alpha\dot\alpha}, M_{\alpha\beta}, \bar M_{\dot\alpha\dot\beta}$ decouple and generate outer $u(1) \oplus so(1,4)$ automorphisms.

Our goal is to construct an AdS_5 version of the nonlinear realization (7.6), (7.8). The main hints which allowed us to do this are as follows. Firstly, we assert that this realization involves some modification of $N = 1$ tensor multiplet L and, as before, a pair of mutually conjugated $N = 1$ chiral and anti-chiral superfields $F, \bar F$ subjected to some generalization of (7.8). Second, in a close analogy with the flat case we require that the following ‘action’

$$
S \sim \int d^4x\, d^2\bar\theta F + \int d^4x\, d^2\theta \bar F \tag{7.13}
$$

is an invariant of the AdS_5 supersymmetry. Third, in the limit $m = 0$ our construction should reproduce the flat case outlined above. At last, it is sufficient to find the realization of conformal S supersymmetry, since the rest of $SU(2,2|1)$ transformations appears in the closure of S transformations with themselves and $N = 1$ Poincaré supersymmetry.

It turns out that this reasoning almost uniquely fixes the sought transformation laws and constraints (more details of the derivation are given in

[23]). These are

$$\delta^* \bar{F} = 6\, i\, m\theta^\alpha \eta_\alpha \bar{F} - \Delta x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\bar{F} + \Delta\theta^\alpha D_\alpha \bar{F} + i\, e^{-2mL}\bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha}L\,,$$
$$\delta^* F = -6\, i\, m\bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha} F - \Delta x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}F - \Delta\bar\theta^{\dot\alpha}\bar{D}_{\dot\alpha}F + i\, e^{-2mL}\eta^\alpha D_\alpha L\,,$$
$$\delta^* L = -\, i(\theta^\alpha\eta_\alpha - \bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}) - \Delta x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}L + \Delta\theta^\alpha D_\alpha L - \Delta\bar\theta^{\dot\alpha}\bar{D}_{\dot\alpha}L$$
$$- \, i\, e^{2mL}\left[\eta^\alpha D_\alpha\left(e^{2mL}\bar{F}\right) + \bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha}\left(e^{2mL}F\right)\right], \tag{7.14}$$
$$\frac{1}{m}\, D^2 e^{-2mL} = \frac{1}{m}\, \bar{D}^2 e^{-2mL} = 0\,, \qquad D_\alpha F = \bar{D}_{\dot\alpha}\bar{F} = 0\,, \tag{7.15}$$
$$F = -\frac{e^{-2mL}D^\alpha L\, D_\alpha L}{2 - e^{4mL}D^2\bar{F}}\,, \qquad \bar{F} = -\frac{e^{-2mL}\bar{D}_{\dot\alpha}L\,\bar{D}^{\dot\alpha}L}{2 - e^{4mL}\bar{D}^2 F}\,. \tag{7.16}$$

Here

$$\Delta x^{\alpha\dot\alpha} = 2\, i\, m\left(\eta_\beta x^{\beta\dot\alpha}\theta^\alpha + \bar\eta_{\dot\beta}x^{\alpha\dot\beta}\bar\theta^{\dot\alpha}\right) - m\left(\theta^2\eta^\alpha\bar\theta^{\dot\alpha} - \bar\theta^2\bar\eta^{\dot\alpha}\theta^\alpha\right),$$
$$\Delta\theta^\alpha = m\bar\eta_{\dot\alpha}x^{\alpha\dot\alpha} + im\left(\theta^2\eta^\alpha - \bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}\theta^\alpha\right),$$
$$\Delta\bar\theta^{\dot\alpha} = m\eta_\alpha x^{\alpha\dot\alpha} - im\left(\bar\theta^2\bar\eta^{\dot\alpha} - \theta^\alpha\eta_\alpha\bar\theta^{\dot\alpha}\right), \tag{7.17}$$

are the standard transformations of the $N = 1$ superspace coordinates with respect to the conformal supersymmetry.

In the limit $m = 0$ eqs. (7.14), (7.15) and (7.16) go, respectively, into (7.6), (7.3), (7.4) and (7.8). It can be checked that, on the surface of the nonlinear constraints (7.16), the off-shell transformations (7.14) are, first, compatible with the differential constraints (7.15) and, second, produce the whole $SU(2,2|1)$ symmetry when commuted among themselves and with $N = 1$ Poincaré supersymmetry. It is just due to the presence of the nonlinear mixed terms the transformations (7.14) constitute a realization of $SU(2,2|1)$ as the superisometry group of super AdS_5 background and correctly generalize the flat superspace realization (7.6). A striking difference between (7.6) and (7.14) lies in the fact that (7.6) close on $N = 2$ Poincaré superalgebra *before* imposing the constraints (7.8), while (7.14) define a closed supergroup structure only provided the constraints (7.16) are imposed from the very beginning. It is easy to check that (7.16) are covariant under (7.14).

Inspecting (7.14), one can be convinced that this realization corresponds to a half-breaking of the $SU(2,2|1)$ supersymmetry: the spinor derivatives of L are shifted by spinor parameters under the action of S supersymmetry, thus signaling that the latter is spontaneously broken. Broken are also D transformations (with $L|$ as the Goldstone field) and the $SO(1,4)/SO(1,3)$ transformations (with $\partial_{\alpha\dot\alpha}L|$ as the relevant 'Goldstone field').

Like their flat counterparts, the constraint (7.16) can be easily solved

$$F = -e^{-2mL}\psi^2 + \frac{1}{2}\, D^2 \left[\frac{\psi^2\bar{\psi}^2}{1+\frac{1}{2}\,A_+ + \sqrt{1+A_+ + \frac{1}{4}\,(A_-)^2}} \right], \qquad (7.18)$$

$$\psi_\alpha \equiv D_\alpha L\,, \qquad \bar{\psi}_{\dot\alpha} \equiv \bar{D}_{\dot\alpha} L\,, \qquad A_\pm = \frac{1}{2}\, e^{2mL}\left(D^2\bar{\psi}^2 \pm \bar{D}^2\psi^2\right). \quad (7.19)$$

Finally, the action (7.13) can be written in the form

$$\begin{aligned} S \;=\; & -\frac{1}{4}\int d^4x\, d^2\theta\, e^{-2mL}\bar{\psi}^2 - \frac{1}{4}\int d^4x\, d^2\bar{\theta}\, e^{-2mL}\psi^2 \\ & +\frac{1}{4}\int d^4x d^4\,\theta \frac{\psi^2\bar{\psi}^2}{1+\frac{1}{2}\,A_+ + \sqrt{1+A_+ + \frac{1}{4}\,(A_-)^2}}\,. \end{aligned} \qquad (7.20)$$

The first two terms in (7.20) are recognized as the action of the improved tensor $N=1$ superfield [43]. In the limit $m=0$ (7.20) converts into the flat superspace action (7.11).
Defining the bosonic components as

$$\phi = L|_{\theta=0}\ , \qquad \left[D_\alpha, \bar{D}_{\dot\alpha}\right] e^{-2mL}|_{\theta=0} = -2\, m\, V_{\alpha\dot\alpha}\ , \qquad (7.21)$$

where in virtue of (7.16)

$$\partial_{\alpha\dot\alpha} V^{\alpha\dot\alpha} = 0\,, \qquad (7.22)$$

the bosonic part of (7.20) proves to be

$$\begin{aligned} S_B = & \int d^4x e^{-4m\phi} \\ & \times \left[1 - \sqrt{1+\frac{1}{2}\, e^{6m\phi}V^2 - 2\, e^{2m\phi}(\partial\phi)^2 - e^{8m\phi}(V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi)^2} \right]. \end{aligned} \qquad (7.23)$$

It is a conformally-invariant extension of the static gauge Nambu-Goto action for L3-brane in $d=5$: the dilaton ϕ can be interpreted as a radial brane coordinate, while $V^{\alpha\dot\alpha}$ is the field strength of notoph which contributes one more scalar degree of freedom on shell. As is well known, $V^{\alpha\dot\alpha}$ can be dualized into an off-shell scalar by introducing the constraint (7.22) into the action with a Lagrange scalar multiplier and then eliminating $V^{\alpha\dot\alpha}$ by its algebraic equation of motion. Extending (7.23) as

$$S_B \quad \Longrightarrow \quad S_B^{dual} = S_B + \int d^4x\, \lambda\, \partial_{\alpha\dot\alpha} V^{\alpha\dot\alpha} \qquad (7.24)$$

and eliminating $V^{\alpha\dot\alpha}$, after some algebra we get

$$S_B^{dual} = \int d^4x\, |Z|^4 \left[1 - \sqrt{-\det\left(\eta_{\mu\nu} - \frac{2}{m^2}\frac{\partial_\mu Z^n\, \partial_\nu Z^n}{|Z|^4}\right)}\, \right], \tag{7.25}$$

where

$$Z^1 = r\, \cos\vartheta\,, \quad Z^2 = r\, \sin\vartheta\,, \quad r \equiv e^{-m\phi}\,, \quad \vartheta \equiv m\,\lambda\,,$$
$$\eta_{\mu\nu} = \mathrm{diag}\,(+ - - -)\,.$$

The action (7.25) is recognized as the $S^5 \to S^1$ reduction of the scalar part of the D3-brane action on $\mathrm{AdS}_5 \times S^5$ [36], that is the static-gauge Nambu-Goto action of scalar 3-brane on $\mathrm{AdS}_5 \times S^1$. The field ϑ can be shown to undergo a shift under the action of the $U(1)$ generator J, which justifies its interpretation as the S^1 angular variable.

The above duality transformation can be performed at the full superfield level. This results in $SU(2,2|1)$ invariant action of Goldstone chiral $N=1$ superfield which generalizes the action of [4, 6, 24] and describes a super 3-brane on $\mathrm{AdS}_5 \times S^1$ superbackground. [4] This dualization procedure is similar to the flat superspace one of [24]. Its details can be found in [23].

8. Outlook

In these lectures we overviewed, on a few simple instructive examples, basic features of the PBGS approach to superbranes, both for the flat and simple curved backgrounds. In particular, we demonstrated a universality of the method of constructing Goldstone superfield actions based on the general relationship between linear and nonlinear realizations of PBGS [32, 17, 18]. We left aside such interesting examples of PBGS as the $N=4 \to N=2$ and $N=8 \to N=4$ BI theories (super D3- and D6-branes in $D=6$ and $D=10$) [12, 13] which certainly offer new domains for applying the machinery expounded here.

A further work is also required in order to understand in full the links between the PBGS and superembedding [44] approaches. It would be tempting to understand linear realizations of the PBGS theories on the flat and curved superbackgrounds and their relationship to nonlinear realizations from the superembedding point of view.

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[4] This solves the problem posed in [25].

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