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Aspects of Penrose Limits and Spacetime Singularities

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Aspects of Penrose Limits and Spacetime Singularities

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Felix Kessler

To my parents

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coordonnées de Fermi, limites de Penrose, singularités d'espace-temps, théorie des cordes

Contents

CHAPTER 1

Introduction

Together with a young collaborator, I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the non-linear field equations can show us more, or rather limit us more, than we have believed up till now.

Albert Einstein to his friend Max Born 1936

Plane waves or more generally plane gravitational waves are an old playground of general relativity. Their history dates back to an early work by Brinkmann 1923 *"On Riemann spaces conformal to flat space"* [**1**] and a subsequent publication by the same author about *"Einstein spaces which are mapped conformally on each other"* [**2**]. However, they have been rediscovered since then many times most prominently by Einstein and (the young collaborator above) Rosen 1937 in their work *"On gravitational waves"* [**3**]. They were brought to new attention in the late 1950s by Bondi, Pirani and Robinson [**4**] and others. A comprehensive discussion and classification in four space-time dimensions was given by Ehlers and Kundt [**5**].

Their vacuum representatives, being the simplest cousins to the plane waves of Maxwell's electrodynamics, helped to establish the long-disputed interpretation of general relativity in terms of a non-linear second order wave-equation, early postulated by Einstein 1916 but repudiated later 1936 in *"Do gravitational waves exist"*[**6**]. In particular, belonging to the class of Petrov type N backgrounds they are directly associated with long range/weak field transversely polarised gravitational radiation (decaying like $O(r^{-1})$). Apart from their dominant role in the linearised theory their physical relevance is primarily due to the class of exact wave-like vacuum solutions covering all of space-time and the description of physically important wave phenomena as e.g. gravitational shocks. Quite remarkable in this context is the property of vacuum plane waves or more generally plane gravitational waves with parallel rays (pp-waves) sharing the same wave vector field (geodesic null congruence, covariantly constant null vector) that their linear superpositions form new vacuum solutions.

However this is only one in a long row of appealing properties. Just to mention a few, plane waves are the only known counterexample [**7**] in four dimensions to the Hadamard conjecture [**8**], stating that Huygens principle applies only in Minkowski space. As was pointed out by Penrose [**9**] they belong to the simplest examples of space-times which are not globally hyperbolic. Moreover, although all their curvature scalars vanish identically, plane waves are the simplest models describing singular tidal forces. Last, not least, it is another discovery made by Penrose [**10**] which will primarily interest us in this thesis, namely that each space-time has a plane wave limit.

Quite generally plane waves enjoy the remarkable property of being in many respects "calculable" and it is predominantly this quality which makes them (enforced by their physical relevance) one of the most important toy model classes in the modern literature where in many respects they act as the "harmonic oscillator" of gravity.

In the string theory context for instance is has long been known that they provide exactly *solvable* models. The term "solvable" here means that it is possible to explicitly write down the solutions to the classical string equations of motion, perform a canonical quantisation, determine the spectrum of the Hamilton operator and calculate the scattering amplitudes. Moreover, it is a general property of (vacuum) pp-waves that all loop quantum corrections vanish identically. As a consequence there are many *exact* string theory plane wave backgrounds supported by R-R form as well as NS-NS gauge potentials receiving essentially no α' corrections. In this context they complement other exactly solvable and exact models as e.g. flat Minkowski space and its different compact and non-compact orbifolds as well as compact and non-compact Wess-Zumino-Witten backgrounds and their orbifolds supported by NS-NS gauge potentials and a dilaton.

Renewed interest in this subject was generated by the discovery of the exactly solvable maximally supersymmetric Blau-Figueroa-O'Farrill-Hull-Papadopoulos plane wave solution of type IIB string theory [**11**]. This RR-background completed the set of already known existing backgrounds enjoying maximal supersymmetry in ten space-time dimensions, namely Minkowski space and $\mathrm{AdS}_5 \times S^5$ in close analogy to the eleven dimensional plane wave discovered by Kowalski-Glikman [**12**] completing the maximally supersymmetric supergravity solutions, that is Minkowski space, $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$.

The existence of these maximally supersymmetric plane wave backgrounds at first sight seemed to be somewhat miraculous. In particular, it was intriguing that they had to be treated on the same footing as flat space and the solutions of the $\mathrm{AdS} \times S$ variety. While the latter can be interpreted as near horizon limits of fundamental brane solutions, no limiting procedure of a similar kind was known for the former. This changed however with the discovery [**13**] that these plane waves can be interpreted as the Penrose limit [**10**] of the corresponding $\mathrm{AdS} \times S$ solutions, or to be more precise as the generalised Penrose-Güven limit [**14**] which includes also a limit of the supergravity gauge fields. This picture was finally completed with the realisation [**15**] that the number of supersymmetries never decreases in this limit (hence, enforcing the existence of the maximally supersymmetric plane waves), the classification of all $\mathrm{AdS}\times S$ space-times and supergravity brane solutions as well as the interpretation of the Penrose limit in terms of a large tension limit for all types of branes.

In consideration of the AdS/CFT correspondence the next obvious step in this development was to find the dual to the Penrose limit on the conformal field theory side, the Berenstein-Maldacena-Nastase (BMN) limit and to use the desirable features inherited from the plane wave structure to recast the resulting field theory into a simple matrix model [**16**]. After this jump-start much attention has been dedicated to the investigation of the more tractable BMN plane wave/CFT correspondence [**17**] as well as the implications of the limiting procedures [**18, 19**] leading to new insights regarding the integrable structures on both sides of the duality.

Simultaneously, there was a series of publications dealing with the geometric properties of the Penrose limit per se. While its original formulation primarily based on an invariant object, namely a null geodesic congruence as well as a conformal rescaling of the metric, it also involved a complicated sequence of coordinate transformations clouding its intrinsically covariant character. After some initial remarks related to this subject in the seminal publication [**13**] a completely covariant characterisation without involving any limit was finally given in [**20**].

Beside the Penrose limit, the general properties of the limiting plane waves were studied. In particular, special attention was given to homogeneous plane waves, whose complete classification was given in [**21**]. In addition to the completely regular time-independent homogeneous plane waves, comprising the maximally supersymmetric plane waves above, there exists a (possibly singular) time-dependent branch, enjoying an additional scale invariance. Solvable string models for the regular [**22**] and singular variety [**23**] were discussed. Soon later it was shown [**24**] that M-theory plane waves with additional supersymmetries are always homogeneous and quite surprisingly possibly of the (regular) time-dependent type.

The singular homogeneous plane waves were further singled out by the realisation [**20, 25**] that they always describe the Penrose limit of a large class of physically reasonable space-time singularities, namely the class of Szekerez-Iyer metrics fulfilling the Dominant Energy Condition (DEC). This "universality" of the Penrose limit is quite remarkable for two reasons. First of all, it promotes these plane waves directly from mere toy models to the rank of realistic metric approximations of true space-time singularities. Second, the necessity to impose the DEC in the present context is very intriguing as this excludes a simple technical explanation in favour of a physically convincing interpretation. As mentioned above, the Penrose limit contains vital information about geodesic congruences and it is the intimate relation between these and some energy condition forming a major building block of the celebrated singularity theorems by means of the Raychaudhuri equation.

The present thesis is divided into two parts. The first one is devoted to open questions directly related to the geometrisation and interpretation of the Penrose limit. Accordingly, in chapter 2 we provide the technical prerequisites concerning Penrose limits, plane waves and string theory in these backgrounds.

The necessary equipment at hand we go to work in chapter 3 where we identify the Penrose limit [**20**] as the lowest order of a covariant metric expansion around a null geodesic, i.e. extend the result obtained in [**20**]. To this end we introduce the null analogue to *Fermi coordinates* which are usually based on an expansion around a timelike geodesic. Eventually, this will lead us to what we dubbed a *Penrose-Fermi expansion* of the metric. In addition to its aesthetically appealing properties this novel prescription comes with different technical advantages which we will discuss.

Building on these results in chapter 4 we compare the Penrose-Fermi expansion of the metric on the one hand with a Riemann coordinate expansion of the string embedding variables on the other in order to clarify the usual statement that exact string theory in a Penrose limit plane wave background might be interpreted as a lowest order approximation to string theory in the original background. Indeed, as we will see both expansions agree order by order after imposing the light cone gauge which acts as a pivotal point between space-time and world-sheet expansions in this context.

In the second part of this thesis we embed the observed universality for Penrose limits of various space-time singularities in the broader context of different space-time probes. To this end in chapter 5 we shortly review the results obtained in [**20, 25**] underlining the interpretation in terms of null geodesic congruences.

Then in chapter 6 we replace the null congruence by what we consider as their nearest relative, namely a massless scalar field. After observing a completely analogous universal behaviour in this context, we concentrate on related topics as the uniqueness of time evolution near space-time singularities.

We conclude in chapter 7 with a short outlook and discussion of open questions.

Three publications are embedded into this thesis and form the larger part of its body, namely chapters 3, 4 and 6 are identical to [**26**], [**27**] and [**28**] respectively up to minor changes w.r.t. layout, (page, section, equation and citation) numbering and in some rare cases symbols, which were adjusted in favour of overall readability. For the same reason we also combined the different tables of content and the bibliographies. All alterations as well as errata are indicated individually at the end of each chapter/publication.

CHAPTER 2

Plane Waves and Penrose Limits

This chapter provides some of the most necessary background material about plane waves and Penrose limits needed in this thesis. We begin in section 1 with a technical description of plane waves concentrating on the properties of direct relevance to the subsequent chapters 3, 4 and 5. Then in section 2 we briefly discuss how bosonic string theory in these backgrounds reduces to the ubiquitous time dependent harmonic oscillator equations. Finally, in section 3 we describe the Penrose limit. Starting from its historical, non-manifestly covariant formulation we finally conclude this chapter with its modern, manifestly covariant form given in terms of the null geodesic deviation equation.

1. Plane Waves

1.1. Plane Waves as a Subclass of Brinkmann Metrics. The starting point for our brief discussion of plane waves will be the class of space-times admitting a covariantly constant (parallel) null vector field. The pragmatic reasoning behind this choice lies in its property, to be established in section 2, of being the largest class admitting simultaneously the conformal and the light-cone gauge in string theory, and we accordingly postpone a short discussion of the interpretation of such backgrounds as *gravitational waves* (in four space-time dimensions) to Appendix A section 4.

As the light-cone gauge is a space-time non-covariant relation the answer to the question regarding the (additional) availability of the conformal gauge depends not only on the background metric but also on the set of space-time coordinates one is working with. As we will see it is in the affirmative iff the parallel null vector k is a coordinate vector of the coordinate system we use, i.e.

$$k = \partial_v, \quad k^\mu = \delta^\mu_v, \quad k_\mu = g_{\mu v}. \tag{2.1}$$

If k^μ is nowhere vanishing such a coordinate system exists in general and (2.1) simply means that we use the parameter along the integral curves of k^μ as the coordinate v. Our first task will be to find the corresponding metric representation, which in four dimensions is due to Brinkmann. To this end we decompose the equation stating k^μ being parallel $\nabla_\mu k_\nu = 0$ into its symmetric and antisymmetric part

$$\begin{aligned} \nabla_{(\mu} k_{\nu)} &= \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \\ \nabla_{[\mu} k_{\nu]} &= \nabla_\mu k_\nu - \nabla_\nu k_\mu = 0. \end{aligned} \tag{2.2}$$

The first relation is just the Killing equation and the second, which is actually a metric independent statement

$$\nabla_\mu k_\nu - \nabla_\nu k_\mu = \partial_\mu k_\nu - \partial_\nu k_\mu = 0, \tag{2.3}$$

turns out to be the usual integrability condition, stating that k_μ is a gradient of a potential, i.e. there is function on our manifold (with D space-time dimensions) $u(x^\mu)$ s.t.

$$k_\mu = g_{\mu v} = \partial_\mu u. \tag{2.4}$$

The conditions on ∂_v being null and Killing finally imply

$$k_v = g_{vv} = 0 \tag{2.5}$$

and

$$\partial_v g_{\mu\nu} = 0 \tag{2.6}$$

respectively. Apart from (2.4-2.6) there are no further constraints and therefore changing from the generic x^μ-coordinates to $\{u, v, y^i\}$, $i \in \{1, ..., d = D-2\}$ we finally get the desired *Brinkmann form* of a metric admitting a covariantly constant null vector field ∂_v

$$ds^2 = 2du(dv + A(u, y^i)du + A_i(u, y^k)dy^i) + g_{ij}(u, y^k)dy^i dy^j. \tag{2.7}$$

There are still enough residual coordinate transformations left to eliminate A and A_i in favour of g_{ij} **[29]** leading to the *Rosen form*

$$ds^2 = 2dudv + g_{ij}(u, y^k)dy^i dy^j \tag{2.8}$$

which will become important in chapter 4 section 8. At the moment however we are primarily interested in a subclass of these metrics characterised by a transverse metric $\bar{g}_{ij}$ depending on u only

$$d\bar{s}^2 = 2dudv + \bar{g}_{ij}(u)dy^i dy^j. \tag{2.9}$$

This is the desired *plane wave* written down in *Rosen coordinates*. However, these are not the coordinates plane waves are usually discussed, among other reasons because typically in Rosen coordinates the metric exhibits spurious coordinate singularities where $\bar{g}_{ij}$ degenerates (cf. section 3.1). Historically, precisely these coordinate singularities lead Rosen 1937 to the erroneous conclusion that there were no exact plane waves filling all space-time **[30]**:

> *It is found that all non-trivial solutions of these equations contain singularities, so that one must conclude that strictly plane polarised waves of finite amplitude, in contrast to cylindrical waves, cannot exist in the general theory of relativity.*

However as was pointed out much later 1958 by Bondi, Pirani and Robinson **[4]**:

> *In effect, Rosen did not distinguish sufficiently between coordinate singularities and physical singularities, which could in principle, be detected experimentally.*

To avoid these spurious singularities and also to understand the origin of the term "plane" let us reconsider the general form (2.7). While one can eliminate A and A_i in favour of g_{ij} the converse is not true (in arbitrary space-time dimension), i.e. triviality of the transverse metric ($g_{ij}(u, y^k) = \delta_{ij}$), can not be generically achieved by a mere coordinate transformation. Therefore, this restriction again amounts to the selection of a true subset of Brinkmann metrics

$$ds^2 = 2du(dv + A(u, y^k)du^2 + A_i(u, y^k)dy^i) + \delta_{ij}dy^i dy^j \tag{2.10}$$

which are called *plane fronted waves with parallel rays*, or *pp-waves* for short. Note that the term "plane-fronted" now naturally refers to the *wave fronts* $u = const.$ being flat ("planar") and the term "parallel rays" to the existence of the parallel null vector ∂_v[1].

We will now show that plane waves are a true subset of pp-waves, namely locally there exists a coordinate transformation relating (2.9) to a special form of (2.10). Is is clear

[1] Note that in the $D = 4$ literature the term pp-wave usually refers to vacuum solutions

that, in order to transform the non-flat transverse metric $\bar{g}_{ij}(u)$ in Rosen coordinates to the flat transverse metric of a pp-wave, one has to change coordinates according to

$$x^a = \bar{E}^a_i y^i, \quad a \in \{1, ..., d\} \tag{2.11}$$

where $\bar{E}^a_i$ is a vielbein for $\bar{g}_{ij}(u)$, i.e.

$$\bar{g}_{ij} = \bar{E}^a_i \bar{E}^b_j \delta_{ab}. \tag{2.12}$$

Denoting the inverse vielbein by $\bar{E}^i_a$, one finds

$$\bar{g}_{ij} dy^i dy^j = (dx^a - \dot{\bar{E}}^a_i \bar{E}^i_c x^c du)(dx^b - \dot{\bar{E}}^b_j \bar{E}^j_d x^d du)\delta_{ab}. \tag{2.13}$$

Thus, this generates the flat transverse metric in favour of a du^2-term quadratic in the x^a as well as a $dudx^a$-term linear in the x^a. Although this is already a pp-wave we can do a bit better if we demand $\bar{E}^i_a$ to fulfil the symmetry condition[2]

$$\dot{\bar{E}}_{ai} \bar{E}^i_b = \dot{\bar{E}}_{bi} \bar{E}^i_a. \tag{2.14}$$

Then a shift in the v-coordinate

$$v = x^- + \frac{1}{2} \dot{\bar{E}}_{ai} \bar{E}^i_b x^a x^b \tag{2.15}$$

cancels the linear term. Plugging everything together we find that the coordinate transformation

$$u = x^+, \quad v = x^- + \frac{1}{2} \dot{\bar{E}}_{ai} \bar{E}^i_b x^a x^b, \quad y^i = \bar{E}^i_a x^a \tag{2.16}$$

leads from the plane wave in Rosen coordinates to

$$d\bar{s}^2 = 2dx^+ dx^- + A_{ab}(x^+) x^a x^b du^2 + \delta_{ab} dx^a dx^b, \tag{2.17}$$

with the *wave-profile*

$$A_{ab} = \ddot{\bar{E}}_{ai} \bar{E}^i_b. \tag{2.18}$$

The x^A, $A \in \{+, -, a\}$ are called *Brinkmann coordinates* for the plane wave. The fact that they cover the space-time completely is one of the many advantages they have over Rosen coordinates. We will get to know another virtue of Brinkmann coordinates in the following section, namely that they encode all the curvature information of the plane wave on the level of the metric in a purely algebraic way.

1.2. Curvature of Plane Waves. It is easy to see that there is essentially only one non-vanishing component of the Riemann curvature tensor of a plane wave metric, namely

$$\bar{R}_{+a+b} = \bar{E}^i_a \bar{E}^j_b \bar{R}_{iuju} = -A_{ab}, \tag{2.19}$$

and thus, the metric is flat iff $A_{ab} = 0$.

As the Weyl tensor is just the traceless part of the Riemann tensor,

$$\bar{C}_{+a+b} = -(A_{ab} - \frac{1}{d}\delta_{ab} \operatorname{Tr} A), \tag{2.20}$$

it vanishes iff A_{ab} is pure trace,

$$A_{ab}(x^+) = A(x^+)\delta_{ab}, \tag{2.21}$$

in particular, for $d = 1$, every plane wave is conformally flat.

[2]This can always be achieved and fixes the $\bar{E}$ up to u-independent orthonormal rotations.

Direct inspection shows that all curvature invariants (completely contracted polynomials of the Riemann tensor and its covariant derivatives) of plane waves vanish identically, as it is simply impossible to soak up the different +-indices. Another, quite elegant argument **[31, 32]** relies only on the scaling behaviour of different generally covariant quantities and the symmetries of the plane wave metric. As it is also sheds some light on the Penrose limit (cf. the discussion at the end of section 3.1), to be discussed below, we included it in Appendix A section 6.

Furthermore, because of the null (or chiral) structure of the metric, there is only one non-trivial component of the Ricci tensor,

$$\bar{R}_{++} = -\delta^{ab} A_{ab} \equiv -\operatorname{Tr} A. \tag{2.22}$$

Thus in Brinkmann coordinates the vacuum Einstein equations reduce to a simple algebraic condition on A_{ab} (regardless of its x^+-dependence), namely that it has to be traceless. This also implies the somewhat unexpected fact that the vacuum plane wave solutions sharing the same parallel null vector, i.e. with an equal set of Brinkmann coordinates, form a linear vector space, a quality they have in common with the usual plane wave solutions of electrodynamics.

Even more surprising is the following property. Any generally covariant symmetric second rank tensor $T_{\mu\nu}[\bar{g}]$ (here we mean functional dependence on the plane wave metric $\bar{g}$ and arbitrarily high derivatives thereof) is a linear combination of the metric and the Ricci tensor. Thus the only not automatically satisfied field equations are of the form

$$a\bar{g}_{\mu\nu} + b\bar{R}_{\mu\nu} = 0. \tag{2.23}$$

The proof relying on similar arguments as in **[31, 32]** can be found in **[33]**, where this property was actually shown for any metric with the plane wave symmetry group. Note that equation (2.23) gives a simple explanation for the robustness of plane wave solutions under quantum corrections. For further reading about this interesting topic we refer to **[34]** and references therein.

A simple example of a vacuum (i.e. Ricci flat) plane wave metric in four space-time dimensions is

$$d\bar{s}^2 = 2dx^+dx^- + ((x^1)^2 - (x^2)^2)(dx^+)^2 + (dx^1)^2 + (dx^2)^2, \tag{2.24}$$

or, more generally,

$$d\bar{s}^2 = 2dx^+dx^- + [A_{11}(x^+)((x^1)^2 - (x^2)^2) + 2A_{12}(x^+)x^1x^2](dx^+)^2 + (dx^1)^2 + (dx^2)^2 \tag{2.25}$$

for arbitrary functions $A_{11}(x^+)$ and $A_{12}(x^+)$. This reflects the two polarisation states or degrees of freedom of a four-dimensional graviton. Evidently, this generalises to arbitrary dimensions: the degrees of freedom of the traceless matrix $A_{ab}(x^+)$ correspond precisely to those of a transverse traceless symmetric tensor, i.e. the graviton.

When the Ricci tensor is non-zero (A_{ab} has non-vanishing trace), then plane waves solve the Einstein equations with null matter or null fluxes, i.e. with an energy-momentum tensor $\bar{T}_{\mu\nu}$ whose only non-vanishing component is $\bar{T}_{++}$,

$$\bar{T}_{\mu\nu} = \rho(x^+)\delta_{\mu+}\delta_{\nu+}. \tag{2.26}$$

Simple examples are null Maxwell fields $a(x^+)$ with field strength

$$F = dx^+ \wedge a'(x^+) \tag{2.27}$$

and their higher-rank generalisations appearing in supergravity. Note that physical matter (with positive energy density) corresponds to $\bar{R}_{++} > 0$ or $\operatorname{Tr} A < 0$.

2. String Equations of Motion in Plane Wave Backgrounds

2.1. A Lightning Review of the Conformal Gauge. The Polyakov action for a string moving in a curved background is given by

$$S[X,h] = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{h}h^{ij}g_{\mu\nu}(X)\partial_i X^\mu \partial_j X^\nu, \tag{2.28}$$

where $X^\mu(\tau,\sigma)$ are the embedding variables of the string world-sheet, parametrised by the coordinates (τ,σ) and endowed with the metric h_{ij}, into the D-dimensional target space with metric $g_{\mu\nu}$. The dynamical variables are h_{ij} and X^μ. The variation of the action w.r.t. the embedding variables leads to the e.o.m.

$$0 = \pi\alpha'\frac{\delta S}{\delta X^\rho} = \eta^{ij}\tilde{\nabla}_i\partial_j X^\rho = \partial_i(\sqrt{h}h^{ij}\partial_j X^\rho) + \sqrt{h}h^{ij}\Gamma^\rho_{\mu\nu}\partial_i X^\mu\partial_j X^\nu, \tag{2.29}$$

where $\tilde{\nabla}$ denotes the covariant derivative w.r.t. h_{ij} and $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols of $g_{\mu\nu}$.

The e.o.m. for h_{ij} imply the vanishing of the world-sheet energy-momentum tensor

$$0 = \frac{4\pi\alpha'}{\sqrt{h}}\frac{\delta S}{\delta h^{ij}} = T_{ij} = g_{\mu\nu}\left(\partial_i X^\mu\partial_j X^\nu - \frac{1}{2}h_{ij}(h^{kl}\partial_k X^\mu\partial_l X^\nu)\right). \tag{2.30}$$

The conformal and world-sheet diffeomorphism invariance of (2.28) lead to the usual constraints of vanishing trace and divergence of T_{ij}, i.e.

$$h^{ij}T_{ij} = T^i_{\ i} = 0, \quad \tilde{\nabla}_i T^{ij} = 0. \tag{2.31}$$

Recalling that $\bar{h}_{ij} = g_{\mu\nu}\partial_i X^\mu\partial_j X^\nu$ is the metric induced on the world-sheet we can rewrite this as

$$\bar{h}_{ij} - \frac{1}{2}h_{ij}(h^{kl}\bar{h}_{kl}) = 0, \tag{2.32}$$

or equivalently

$$h_{ij}\sqrt{h} = \bar{h}_{ij}\sqrt{\bar{h}}. \tag{2.33}$$

We see that the e.o.m. restrict h_{ij} to be proportional to the induced metric $\bar{h}_{ij}$ whereas the conformal factor relating the two stays out of the game as a direct consequence of the conformal invariance of the action (2.28). Equation (2.33) can be used to eliminate h_{ij} from (2.28) leading to the classical minimal surface Nambu-Goto string action

$$S_{NG}[X] = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\bar{h}}. \tag{2.34}$$

Usually starting from (2.28) one uses the world-sheet diffeomorphisms to gauge h_{ij} to a suitable form choosing an adequate coordinate system on the world-sheet. The most common choice are so called *isothermal coordinates* in which h_{ij} becomes proportional to the flat metric

$$h_{ij} = e^{\phi(\tau,\sigma)}\eta_{ij}. \tag{2.35}$$

The existence of such a coordinate system is easily proved **[35]** in the Lorentzian case[3]. In this *conformal gauge* the Polyakov action (2.28) reduces to

$$S[X] = \frac{1}{2\pi\alpha'}\int d^2\sigma g_{\mu\nu}(X)\partial^i X^\mu\partial_i X^\nu. \tag{2.36}$$

[3]For the Euclidean case isothermal coordinates can be derived from the solution of the Beltrami equations **[36]**.

Consistently, this action reproduces the same e.o.m. as the direct insertion of (2.35) into (2.29) namely the Lorentzian version of an *harmonic map*

$$\eta^{ij}\nabla_i\partial_j X^\rho = \eta^{ij}\partial_i\partial_j X^\rho + \eta^{ij}\Gamma^\rho_{\mu\nu}\partial_i X^\mu \partial_j X^\nu = 0. \tag{2.37}$$

Moreover, the constraints imposed by the conformal gauge (2.35) are

$$\begin{aligned} h_{\tau\tau} + h_{\sigma\sigma} &= g_{\mu\nu}(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu) = 0 \\ h_{\tau\sigma} = h_{\sigma\tau} &= g_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0, \end{aligned} \tag{2.38}$$

and one easily shows that they are equivalent to the vanishing of the world-sheet energy-momentum tensor (2.30).

2.2. Compatibility of Lightcone and Conformal Gauge in Brinkmann Metrics. The first step in the canonical light-cone formalism of strings and membranes is to impose the *light-cone gauge*, i.e. to identify the world-sheet time τ with a null coordinate, i.e. a function u on the space-time manifold whose gradient $g^{\mu\nu}\nabla_\nu u$ is a null vector. However, for general space-times it is not possible to impose the light-cone together with the conformal gauge but rather as announced in section 1.1 it is precisely the Brinkmann class (2.7) where both are simultaneously available [**37**]. To see this, first recall that the conformal gauge leaves us with a residual gauge freedom of changing τ and σ by solutions to the two-dimensional wave equation. Thus, to impose the light-cone in addition to the conformal gauge one needs a null coordinate u fulfilling the wave equation along each solution $X^\mu(\tau,\sigma)$ of (2.37), i.e.

$$\eta^{ij}\nabla_i\partial_j u = \eta^{ij}(\partial_i X^\mu)\nabla_\mu(\partial_j X^\nu)\partial_\nu u = \eta^{ij}(\partial_i X^\mu)(\partial_j X^\nu)\nabla_\mu\partial_\nu u = 0. \tag{2.39}$$

As the string e.o.m. are second order we can arbitrarily chose $\partial_\tau X^\mu$ and $\partial_\sigma X^\nu$ (up to the constraints (2.38)) in each space-time point. Thus, we see that actually ($D > 2$)

$$\nabla_\mu\partial_\nu u = 0. \tag{2.40}$$

Therefore, in order to impose both gauges the space-time must admit a covariantly constant null vector $k^\mu = g^{\mu\nu}\partial_\nu u$. As we have seen in the previous section the property of being a gradient then follows automatically and thus we established that the conformal and the light-cone gauge can be imposed together iff the space-time is described by a Brinkmann metric.

Indeed, the conformally gauged string e.o.m. in a Brinkmann background (2.7) for the embedding coordinate $U(\tau,\sigma)$ is just the wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)U(\tau,\sigma) = 0 \tag{2.41}$$

as demanded and we can therefore use the residual symmetry to impose

$$U(\tau,\sigma) = P_V\tau. \tag{2.42}$$

Then the transverse string e.o.m. reduce to

$$\begin{aligned} (-\partial_\tau^2 + \partial_\sigma^2)Y^a + P_V g^{ab}(\partial_b A_c - \partial_c A_b)\dot{Y}^c - P_V^2 g^{ab}\partial_u A_b + P_V^2 g^{ab}\partial_b A \\ - P_V g^{ab}\partial_u g_{bc}\dot{Y}^c + \Gamma^a_{bc}(g_{de})(-\dot{Y}^b\dot{Y}^c + Y^{b\prime}Y^{c\prime}) = 0 \end{aligned} \tag{2.43}$$

whereas the e.o.m. for V (not appearing in (2.43)) follow directly from the constraints

$$\begin{aligned} \dot{V} + A + P_V A_i\dot{Y}^i + \frac{1}{2P_V}g_{ij}(\dot{Y}^i\dot{Y}^j + Y^{i\prime}Y^{j\prime}) = 0 \\ V' + A_i Y^{i\prime} + \frac{1}{P_V}g_{ij}\dot{Y}^i Y^{j\prime} = 0, \end{aligned} \tag{2.44}$$

which, depending linearly on V, can be solved directly for it. Thus, V becomes an *auxiliary field*. We will come back to this issue in the special case of plane waves in chapter 4 section 3. At the moment we simply note that for Brinkmann metrics, working in the combined light-cone and conformal gauge it suffices to look at the e.o.m. for the transverse fields given by (2.43).

For a plane wave in Rosen coordinates (2.9) these equations reduce to the somewhat unenlightening

$$(-\partial_\tau^2 + \partial_\sigma^2)Y^a - P_V g^{ab}\partial_u g_{bc}\dot{Y}^c = 0, \tag{2.45}$$

while on the other hand in Brinkmann coordinates (2.17) we simply get a massive wave equation

$$(-\partial_\tau^2 + \partial_\sigma^2)X^a + P_-^2 A_{ab}(P_-\tau)X^b = 0 \tag{2.46}$$

with a *time dependent mass matrix* given by the waveprofile A_{ab}. We see that bosonic string theory in plane wave backgrounds reduces to a *free field theory* and thus can be quantised using similar methods as in flat space.

In particular, expanding $X^a(\tau,\sigma)$ in Fourier modes

$$X^a(\tau,\sigma) = \sum_n X_n^a(\tau)e^{in\sigma}, \tag{2.47}$$

one obtains decoupled harmonic oscillator equations for the individual modes

$$\ddot{X}_n^a + (n^2\delta_{ab} - P_-^2 A_{ab}(P_-\tau))X_n^b = 0. \tag{2.48}$$

Obviously, for $n = 0$ these are nothing else than the (linear) transverse geodesic equations of motion in the plane wave background.

We conclude this section with two remarks. First, in a plane wave or more generally a Brinkman metric, the string or geodesic equations of motions reduce to a simple $D-2 = d$ dimensional, Riemannian mechanical system. This in some sense is the inverse of the Eisenhart procedure (see **[38]** and references therein) interpreting a d-dimensional, Riemannian mechanical system in terms of geodesic motion in a D dimensional pseudo-Riemannian space, i.e. the Brinkann metric.

Second, the elimination of the V-field from the e.o.m. described above in a very simplified way can actually be formulated in a very elegant and concise manner for general *p-branes* using the Hamilton (or Routh) formalism. After fixing the light-cone gauge the condition on the light-cone Hamiltonian (motivated by a subsequent quantisation) to be polynomial in the dynamical fields (being tantamount to the elimination of V and P_V from the canonical e.o.m.) restricts the background metric to be of the form **[39]**

$$ds^2 = C(u,y^k)\left(2du(dv + A(u,y^i)du + A_i(u,y^k)dy^i) + g_{ij}(u,y^k)dy^i dy^j\right), \tag{2.49}$$

that is conformal to the Brinkmann metric (2.7) with ∂_- still being Killing[4]. Then in the string case the conformal gauge (2.38) has to be modified, for example

$$h_{\tau\tau} + h_{\sigma\sigma}C^2(U,Y^k) = 0, \quad h_{\tau\sigma} = h_{\sigma\tau} = 0. \tag{2.50}$$

[4]Similarly, restrictions can be derived for *p+1-form* background gauge fields.

3. The Penrose Limit

3.1. Historical Prescription. We now come to the *Penrose limit*, whose physical interpretation is described by Penrose 1976 as follows [**10**]:

> *We envisage a succession of observers travelling in the space-time M whose world lines approach the null geodesic γ more and more closely; so we picture these observers as travelling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x^0 along γ. (Without clock recalibration a degenerate space-time metric would result.) In the limit the observers measure the space-time to have the plane wave structure W_γ.*

In other words, the Penrose limit can be understood as a boost accompanied by an adequate uniform rescaling of the coordinates in such a way that the affine parameter along the null geodesic remains invariant.

To implement this procedure in practice, one chooses some null geodesic γ of a given space-time with metric $g_{\mu\nu}$ and then locally, i.e. in a certain neighbourhood of a segment of γ, introduces an adapted coordinate system

$$x^\mu \to (\tilde{u}, \tilde{v}, \tilde{y}^k), \tag{2.51}$$

s.t. γ is given by $\tilde{u} = \tau$ and $\tilde{v} = \tilde{y}^i = 0$ and the metric takes the *Penrose form*

$$ds^2_\gamma = 2d\tilde{v}(d\tilde{u} + B(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{v} + B_i(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{y}^i) + g_{ij}(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{y}^i d\tilde{y}^j \tag{2.52}$$

being characterised by $g_{\tilde{u}\tilde{v}} = 1$ and $g_{\tilde{u}\tilde{u}} = g_{\tilde{u}i} = 0$. These are $D = d + 2$ coordinate conditions suggesting that generically any metric can locally be written in this way. A first hint to see that this is actually true is to realise that ∂_u describes a twist-free congruence of null geodesics[5] and the insight that the construction of such a coordinate system is therefore tantamount to embedding the original null geodesic into this congruence. Then, as was suggested in [**40**], one might try to describe the congruence, i.e. the coordinate system using the Hamilton-Jacobi formalism. The general construction/proof which we included in Appendix A section 3 was finally given in [**25**].

Before we continue let us note that Rosen coordinates for a plane wave (2.9) are a special case of adapted coordinates (2.52). This explains why they suffer generically from spurious coordinate singularities which occur precisely in *conjugate points* of the congruence, i.e. in places where different null geodesics intersect.

The adapted coordinate system being constructed we perform the boost

$$(\tilde{u}, \tilde{v}, \tilde{y}^k) \to (\lambda^{-1}\tilde{u}, \lambda\tilde{v}, \tilde{y}^k). \tag{2.53}$$

Obviously, trying to take the infinite boost limit $\lambda \to 0$ without recalibrating ones coordinates (clocks and measuring rods) results in a singular metric. To prevent this, one has

[5]The Brinkmann form (2.7) which is based on the parallel null coordinate vector ∂_v is a special case of the Penrose form (2.52) based on the twist-free, null and geodesic coordinate vector ∂_u upon interchanging u and v. The relation becomes obvious if one notes that the condition of being parallel implies being twist- (as well as shear- and expansion-)free and geodetic as well as being Killing, cf. Appendix A section 4

to uniformly rescale the coordinates as

$$(\tilde{u},\tilde{v},\tilde{y}^k)\to(\lambda\tilde{u},\lambda\tilde{v},\lambda\tilde{y}^k). \tag{2.54}$$

The net effect is given by the asymmetric scaling of the coordinates

$$(\tilde{u},\tilde{v},\tilde{y}^k)\to(\tilde{u},\lambda^2\tilde{v},\lambda\tilde{y}^k), \tag{2.55}$$

leaving the affine parameter $\tilde{u}=u$ invariant. Writing this as

$$(\tilde{u},\tilde{v},\tilde{y}^k)=(u,\lambda^2 v,\lambda y^k), \tag{2.56}$$

we obtain a one-parameter family of (isometric) metrics

$$ds^2_\gamma\to ds^2_{\gamma,\lambda}, \tag{2.57}$$

where $ds^2_{\gamma,\lambda}$ is the metric ds^2_γ in the coordinates (u,v,y^i),

$$\begin{aligned} ds^2_{\gamma,\lambda} = 2\lambda^2 dv(du+\lambda^2 B(u,\lambda^2 v,\lambda y^k)dv+\lambda B_i(u,\lambda^2 v,\lambda y^k)dy^i) \\ +\lambda^2 g_{ij}(u,\lambda^2 v,\lambda y^k)dy^i dy^j. \end{aligned} \tag{2.58}$$

Simultaneously, we perform a conformal rescaling of the metric,

$$ds^2_{\gamma,\lambda}\to\lambda^{-2}ds^2_{\gamma,\lambda}, \tag{2.59}$$

leading to

$$\begin{aligned} \lambda^{-2}ds^2_{\gamma,\lambda} = 2dv(du+\lambda^2 B(u,\lambda^2 v,\lambda y^k)dv+\lambda B_i(u,\lambda^2 v,\lambda y^k)dy^i) \\ +g_{ij}(u,\lambda^2 v,\lambda y^k)dy^i dy^j. \end{aligned} \tag{2.60}$$

Now the combined infinite boost and large volume limit $\lambda\to 0$ results in a well-defined and non-degenerate metric $\bar{g}_{\mu\nu}$,

$$\text{Penrose Limit:}\quad d\bar{s}^2 = \lim_{\lambda\to 0}\lambda^{-2}ds^2_{\gamma,\lambda} \tag{2.61}$$

$$= 2dudv+\bar{g}_{ij}(u)dy^i dy^j, \tag{2.62}$$

where $\bar{g}_{ij}(u)=g_{ij}(u,0,0)$ denotes simply the restriction of the transverse metric g_{ij} to the null geodesic γ. This is a plane wave in Rosen coordinates (2.9) which we can transform to Brinkmann coordinates (2.17) using (2.16).

Before we proceed let us shortly comment on some technicalities of the procedure. First, it is easy to see (cf. Appendix A section 7) that the conformal transformation (2.59) leads to a rescaling of an arbitrary elementary curvature scalar, i.e. a contracted product of covariant derivatives of the Riemann curvature tensor, with a positive power of λ. Thus, polynomial curvature scalars vanish in the Penrose limit, while (2.56) renders the limit of the metric itself finite. This is in some sense a "kinematic" variant of the argument given in [**31, 32**] where the "fixpoint" properties of a generic plane wave under the combined action of (2.56,2.59) are used to show that all its curvature scalars vanish (cf. Appendix A section 6). In particular, it does not use the information that the Penrose limit always leads to a plane wave.

It is obvious, that similar to the scalars the majority of the curvature components of the original metric has to die away during the Penrose limit in favour of the simple curvature structure of the plane wave (cf. section 1.2). A more elaborate discussion of this *peeling off*-behaviour for the Weyl tensor in four space-time dimensions can be found in [**41**]. We will give a detailed and fully covariant discussion for the Riemann tensor in arbitrary space-time dimension [**26**] in chapter 3 section 8.

Second, whatever the nature of the transverse coordinates $\tilde{y}^k$ may have been before taking the limit, e.g. compact angular coordinates, because of the large volume limit the final y^k have infinite range. Moreover, any points that were originally at a finite distance of the null geodesic are pushed off to infinity in the Penrose limit and only an infinitesimal neighbourhood of the null geodesic survives.

Third, the absence of (constant) $g_{\tilde{u}\tilde{u}}$- and $g_{\tilde{u}i}$-terms from the metric in the adapted coordinate system is crucial for the Penrose limit to exist as such terms (generically) scale as λ^{-2} and λ^{-1} respectively. However in veiling the covariant and global properties of the Penrose limit the usage of this same coordinate system is also responsible for the main shortcomings of the construction. As we will see in the next section the Penrose limit can be completely defined in terms of certain frame components of the Riemann tensor of the original metric and all along the null geodesic.

3.2. Manifestly Covariant Description of the Penrose Limit. A completely covariant characterisation and definition of the Penrose limit which does in addition not require taking any limit was given in [**20**], where a relation between the wave profile $A_{ab}(x^+)$ of the Penrose limit plane wave and certain components of the Riemann curvature tensor of the original metric was established.

The general idea to this new characterisation still roots deeply in the historical prescription of Penrose and might be shortly described as follows. The covariant ingredients of the original Penrose costruction are the central null geodesic as well as an arbitrary surrounding twist-free null geodesic congruence. We have seen that due to the large volume limit the Penrose limit plane wave contains only information of an infinitesimal neighbourhood around the geodesic w.r.t. the original space-time. It is however a well-known fact that infinitesimally geodesic congruences are described by the *geodesic deviation* or *Jacobi equation*

$$\frac{D^2}{D\tau^2}\xi^\mu = R^\mu{}_{\nu\lambda\rho}|_\gamma \dot{x}^\nu \dot{x}^\lambda \xi^\rho, \tag{2.63}$$

where τ (here $\tau = \tilde{u} = u = x^+$) is the affine parameter of the central geodesic γ, $R^\mu{}_{\nu\lambda\rho}|_\gamma$ the restriction of the Riemann curvature tensor to γ and $\xi(\tau)$ a vector field along γ "pointing" to nearby geodesics via the exponential map.

Therefore the first step taken in [**20**] is to consider the components $R^i{}_{\tilde{u}j\tilde{u}}$ of the curvature tensor of the Penrose metric (2.52) which enters into the geodesic deviation equation (2.63) of the corresponding null geodesic congruence. As one might check these are the only components surviving the Penrose limit, the others getting peeled-off while λ goes to zero (see chapter 4 section 8 and the remarks at the end of section 3.1).

Inspection shows that

$$R^i{}_{\tilde{u}j\tilde{u}} = -(\partial_{\tilde{u}}\Gamma^i{}_{j\tilde{u}} + \Gamma^i{}_{k\tilde{u}}\Gamma^k{}_{j\tilde{u}}) \tag{2.64}$$

does not depend on the coefficients B and B_i of the original metric (2.52) and only involves $\tilde{u}$-derivatives of g_{ij}. From this one directly infers that these components of the Riemann curvature tensor are trivially related to those of the Penrose limit metric

$$\bar{R}^i{}_{uju} = R^i{}_{\tilde{u}j\tilde{u}}|_\gamma. \tag{2.65}$$

Then one introduces a pseudo-orthonormal frame E^A_μ, $A = (+,-,a)$ for the metric (2.52),

$$ds^2 = 2E^+E^- + \delta_{ab}E^aE^b, \tag{2.66}$$

which is parallel along the null geodesic congruence,

$$\nabla_{\tilde{u}} E^A_\mu = 0 \tag{2.67}$$

with tangent component $E_+ = \partial_{\tilde{u}}$. One easily checks that E_a has the form

$$E_a = E^i_a \partial_i + E^{\tilde{u}}_a \partial_{\tilde{u}}, \tag{2.68}$$

where E^a_i is a vielbein for $g_{ij}(\tilde{u}, \tilde{v}, \tilde{y}^k)$ satisfying

$$\dot{E}_{ai} E^i_b = \dot{E}_{bi} E^i_a. \tag{2.69}$$

Also this condition is independent of B, B_i and only involves $\tilde{u}$-derivatives of E^a_i and as above it follows that the vielbeins $\bar{E}^a_i$ of the Penrose limit metric satisfying the symmetry condition (2.14) are nothing else but the parallel-propagated (2.69) vielbeins of the full metric by restriction to the null geodesic γ,

$$\bar{E}^a_i = E^a_i|_\gamma. \tag{2.70}$$

In particular, this insight provides a geometric interpretation of the, so far somewhat mysterious, symmetry condition (2.14) that arose in the coordinate transformation between Rosen and Brinkmann coordinates.

Finally, combining (2.19) with (2.65) and (2.70), and using (2.68) one finds the key result that the wave profile $A_{ab}(x^+)$ of the Penrose limit metric is

$$A_{ab}(x^+) = -(E^i_a E^j_b R_{i+j+})|_\gamma. \tag{2.71}$$

This however allows for a fully covariant characterisation and definition of the Penrose limit. Namely, given a null geodesic γ, one constructs a pseudo-orthonormal parallel propagated coframe (E_+, E_-, E_a) with $E_+ = \partial_u$ tangent to the null geodesic and E_- characterised by $g(E_-, E_-) = 0$ and $g(E_+, E_-) = 1$. Then the Penrose limit is the plane wave metric given by the wave profile

$$A_{ab}(x^+) = -R_{a+b+}|_\gamma, \tag{2.72}$$

which is determined uniquely up to x^+-independent orthogonal transformations.

Again a few remarks are in order. First, note that equation (2.72) frees us of all the redundancies in the original formulation of the Penrose limit (i.e. the global information of the arbitrary twist-free congruence) showing precisely which covariant information about the original metric it actually contains. Moreover (2.72) gives the physical interpretation of this data. Namely, $A_{ab}(x^+)$ can be characterised as the *transverse null geodesic deviation matrix* [**42**, Section 4.2] of the original metric,

$$\frac{d^2}{(dx^+)^2}\xi^a = -(E^i_a E^j_b R_{i+j+})|_\gamma \xi^b = A_{ab}(x^+)\xi^b, \tag{2.73}$$

with ξ^a the transverse part of the geodesic deviation vector. For a detailed derivation of this equation we refer to Appendix A section 1. In a nutshell, starting from (2.63) one can use the parallel frame condition (2.67) ($u = x^+$) to replace the covariant by ordinary derivatives and restricts the solutions $\xi^A = (\xi^+, \xi^-, \xi^a)$ to be orthogonal to $\gamma = E_+$[6], i.e. sets the non-orthogonal component to zero, $\xi^- = 0$ ($Z^- = 0$ in the notation of [**42**, Section 4.2]). Note that this is a valid (and physically motivated as well as soft) truncation of the e.o.m. Indeed as one can easily see using the affine parametrisation of the congruence $\dot{\xi}^- = 0$, i.e. this component is a pure parallel transport. The parallel component ξ^+ decouples from the transverse e.o.m. for the ξ^a. We will come back to this issue in a more stringy context in chapter 4 section 3.

[6] This is not a natural decomposition and relies on the additional structure of the frame E^A_μ

Together with (2.19) equation (2.73) implies that geodesic deviation along the selected null geodesic in the original space-time is identical to null geodesic deviation in the corresponding Penrose limit plane wave metric. Moreover, comparison of (2.73) with (2.48) (for $n = 0$) shows that the geodesic deviation equation in a plane wave background is formally equivalent to the exact geodesic equations of motion. It is not hard to see that this directly generalises to strings (arbitrary n). This however already implies the key observation of chapter 4, namely that string deviation in the original background is mirrored one-to-one by the exact string equations of motion in the Penrose limit plane wave **[27]**.

Second, in assigning directly the Penrose limit plane wave profile $A_{ab}(x^+)$ to the initial data $(g_{\mu\nu}, \gamma)$ without making any appeal to (only locally defined) Penrose adapted coordinates (2.52) equation (2.72) also shows that the Penrose limit can be defined all along the null geodesic.

Last not least (2.72) constitutes the missing link in the commuting diagram (2.74) below, which summarises the results obtained so far.

$$\begin{array}{ccccc} (g_{\mu\nu},\gamma) & \stackrel{(2.51)}{\longrightarrow} & ds^2_\gamma \ (2.52) & \stackrel{(2.57)}{\longrightarrow} & ds^2_{\gamma,\lambda} \quad (2.58) \\ \downarrow^{(2.72)} & & & & \downarrow^{(2.59)} \\ A_{ab}(x^+) & \stackrel{(2.16)}{\longleftarrow} & d\bar{s}^2 \ (2.62) & \stackrel{(2.61)}{\longleftarrow} & \lambda^{-2} ds^2_{\gamma,\lambda} \ (2.60) \end{array} \tag{2.74}$$

Though this is a very complete and satisfying picture of the Penrose limit one piece of the puzzle is still missing. Note that using Penrose's adapted coordinates one can easily calculate arbitrarily high orders in λ simply by expanding (2.60). On the other hand we are still lacking a covariant version of such an expansion, whose lowest order is given by the Penrose limit plane wave in Brinkmann coordinates (2.17). An expansion with precisely these criteria is the topic of the next chapter **[26]**.

CHAPTER 3

Fermi Coordinates and Penrose Limits

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We propose a formulation of the Penrose plane wave limit in terms of null Fermi coordinates. This provides a physically intuitive (Fermi coordinates are direct measures of geodesic distance in space-time) and manifestly covariant description of the expansion around the plane wave metric in terms of components of the curvature tensor of the original metric, and generalises the covariant description of the lowest order Penrose limit metric itself, obtained in [**20**]. We describe in some detail the construction of null Fermi coordinates and the corresponding expansion of the metric, and then study various aspects of the higher order corrections to the Penrose limit. In particular, we observe that in general the first-order corrected metric is such that it admits a light-cone gauge description in string theory. We also establish a formal analogue of the Weyl tensor peeling theorem for the Penrose limit expansion in any dimension, and we give a simple derivation of the leading (quadratic) corrections to the Penrose limit of $\mathrm{AdS}_5 \times S^5$.

1. Introduction

Following the observations in [**11, 43, 44, 13, 16**] regarding the maximally supersymmetric type IIB plane wave background, its relation to the Penrose limit of $\mathrm{AdS}_5 \times S^5$, and the corresponding BMN limit on the dual CFT side[1], the Penrose plane wave limit construction [**10**] has attracted a lot of attention. This construction associates to a Lorentzian space-time metric $g_{\mu\nu}$ and a null-geodesic γ in that space-time a plane wave metric,

$$(3.1) \quad (ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \gamma) \quad \rightarrow \quad d\bar{s}^2_\gamma = 2dx^+dx^- + A_{ab}(x^+)x^a x^b dx^{+2} + \delta_{ab}dx^a dx^b \ ,$$

the right hand side being the metric of a plane wave in Brinkmann coordinates, characterised by the wave profile $A_{ab}(x^+)$.

The usual definition of the Penrose limit [**10, 14, 15**] is somewhat round-about and in general requires a sequence of coordinate transformations (to adapted or Penrose coordinates, from Rosen to Brinkmann coordinates), scalings (of the metric and the adapted coordinates) and limits.[2] And even though general arguments about the covariance of

[1] see e.g. [**17**] for a review and further references.

[2] For sufficiently simple metrics and null geodesics it is of course possible to devise more direct ad hoc prescriptions for finding a Penrose limit.

the Penrose limit [**15**] show that there is of course something covariant lurking behind that prescription, after having gone through this sequence of operations one has probably pretty much lost track of what sort of information about the original space-time the Penrose limit plane wave metric actually encodes.

This somewhat unsatisfactory state of affairs was improved upon in [**20, 25**]. There it was shown that the wave profile $A_{ab}(x^+)$ of the Penrose limit metric can be determined from the original metric without taking any limits, and has a manifestly covariant characterisation as the matrix

$$A_{ab}(x^+) = -R_{a+b+}|_{\gamma(x^+)} \tag{3.2}$$

of curvature components (with respect to a suitable frame) of the original metric, restricted to the null geodesic γ. This will be briefly reviewed in section 2.

The aim of the present paper is to extend this to a covariant prescription for the expansion of the original metric around the Penrose limit metric, i.e. to find a formulation of the Penrose limit which is such that

- to lowest order one directly finds the plane wave metric in Brinkmann coordinates, with the manifest identification (3.2);
- higher order corrections are also covariantly expressed in terms of the curvature tensor of the original metric.

We are thus seeking analogues of Brinkmann coordinates, the covariant counterpart of Rosen coordinates for plane waves, for an arbitrary metric. We will show that this is provided by Fermi coordinates based on the null geodesic γ. Fermi normal coordinates for *timelike* geodesics are well known and are discussed in detail e.g. in [**45, 46**]. They are natural coordinates for freely falling observers since, in particular, the corresponding Christoffel symbols vanish along the entire worldline of the observer (geodesic), thus embodying the equivalence principle.

In retrospect, the appearance of Fermi coordinates in this context is perhaps not particularly surprising. Indeed, it has always been clear that, in some suitable sense, the Penrose limit should be thought of as a truncation of a Taylor expansion of the metric in directions transverse to the null geodesic, and that the full expansion of the metric should just be the complete transverse expansion. The natural setting for a covariant transverse Taylor expansion are Fermi coordinates, and thus what we are claiming is that the precise way of saying "in some suitable sense" is "in Fermi coordinates".

In order to motivate this and to understand how to generalise Brinkmann coordinates, in section 3 we will begin with some elementary considerations, showing that Brinkmann coordinates are null Fermi coordinates for plane waves. Discussing plane waves from this point of view, we will also recover some well known facts about Brinkmann coordinates from a slightly different perspective.

In section 4 we introduce null Fermi coordinates in general, adapting the construction of timelike Fermi coordinates in [**46**] to the null case. These coordinates $(x^A) = (x^+, x^{\bar{a}})$ consist of the affine parameter x^+ along the null geodesic γ and geodesic coordinates $x^{\bar{a}}$ in the transverse directions. We also introduce the covariant transverse Taylor expansion of a function, which takes the form

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(E^{\mu_1}_{\bar{a}_1} \dots E^{\mu_n}_{\bar{a}_n} \nabla_{\mu_1} \dots \nabla_{\mu_n} f\right)(x^+)\, x^{\bar{a}_1} \dots x^{\bar{a}_n} \quad , \tag{3.3}$$

where E^μ_A is a parallel frame along γ. As an application we show that the coordinate transformation from arbitrary adapted coordinates (i.e. coordinates for which the null geodesic γ agrees with one of the coordinate lines) to Fermi coordinates is nothing other than the transverse Taylor expansion of the coordinate functions in terms of Fermi coordinates.

In section 5, we discuss the covariant expansion of the metric in Fermi coordinates in terms of components of the Riemann tensor and its covariant derivatives evaluated on the null geodesic. We explicitly derive the expansion of the metric up to quadratic order in the transverse coordinates and show that the result is the exact null analogue of the classical Manasse-Misner result **[47]** in the timelike case, namely

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ &- \left[R_{+\bar{a}+\bar{b}}\, x^{\bar{a}}x^{\bar{b}}(dx^+)^2 + \tfrac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) + \tfrac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}})\right] \\ &+ \mathcal{O}(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}) \end{aligned} \tag{3.4}$$

where $(x^{\bar{a}}) = (x^-, x^a)$ and all the curvature components are evaluated on γ. The expansion up to quartic order in the transverse coordinates is given in section 9.1.

In section 6, we show how to implement the Penrose limit in Fermi coordinates. To that end we first discuss the behaviour of Fermi coordinates under scalings $g_{\mu\nu} \to \lambda^{-2}g_{\mu\nu}$ of the metric. Since Fermi coordinates are geodesic coordinates, measuring invariant geodesic distances, Fermi coordinates will scale non-trivially under scalings of the metric, and we will see that the characteristic asymmetric scaling of the coordinates that one performs in whichever way one does the Penrose limit arises completely naturally from the very definition of Fermi coordinates. Combining this with the expansion of the metric of section 5, we then obtain the desired covariant expansion of the metric around its Penrose limit.

The expansion to $\mathcal{O}(\lambda)$, for which knowledge of the expansion of the metric in Fermi coordinates to cubic order is required, reads

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \\ &+ \lambda\left[-2R_{+a+-}\, x^a x^-(dx^+)^2 - \tfrac{4}{3}R_{+bac}\, x^b x^c (dx^+dx^a) - \tfrac{1}{3}R_{+a+b;c}\, x^a x^b x^c (dx^+)^2\right] \\ &+ \mathcal{O}(\lambda^2)\ . \end{aligned} \tag{3.5}$$

where the first line is the Penrose limit metric (3.1). In particular, if the characteristic covariantly constant null vector $\partial/\partial x^-$ of (3.1) is such that it remains Killing at first order it is actually covariantly constant and the first-order corrected metric is that of a pp-wave which is amenable to a standard light-cone gauge description in string theory **[37]**. Moreover, in general the above metric is precisely such that it admits a modified light-cone gauge in the sense of **[48]**. The expansion to $\mathcal{O}(\lambda^2)$ is given in section 9.2.

We illustrate the formalism in section 7 by giving a quick derivation of the second order corrections to the Penrose limit of $\mathrm{AdS}_5 \times S^5$. These corrections have been calculated before in other ways **[49, 50, 51]**, and the point of this example is not so much to advocate the Fermi coordinate prescription as the method of choice to do such calculations (even though it is geometrically appealing and transparent in general, and the calculation happens to be extremely simple and purely algebraic in this particular case). Rather, the interest is more conceptual and lies in the precise identification of the corrections that have already been calculated (and subsequently been used in the context of the BMN correspondence) with particular components of the curvature tensor of $\mathrm{AdS}_5 \times S^5$.

In section 8 we return to the general structure of the λ-expansion of the metric. The leading non-trivial contribution to the metric is the λ^0-term R_{a+b+} (3.2) of the Penrose limit, and higher order corrections involve other frame components of the Riemann tensor, each arising with a particular scaling weight λ^w. In the four-dimensional case it was shown in [**41**], using the Newman-Penrose formalism, that the complex Weyl scalars Ψ_i, $i = 0, \ldots, 4$ scale as λ^{4-i}. This is formally analogous to the scaling of the Ψ_i as $(1/r)^{5-i}$ with the radial distance, the peeling theorem [**52, 53, 54**] of radiation theory in general relativity. We will show that the present covariant formulation of the Penrose limit significantly simplifies the analysis of the peeling property in this context (already in dimension four) and, using the analysis in [**55, 56, 57**] of algebraically special tensors and the (partial) generalised Petrov classification of the Weyl tensor in higher dimensions, allows us to establish an analogous result in any dimension.

We hope that the covariant null Fermi normal coordinate expansion of the metric developed here will provide a useful alternative to the standard Riemann normal coordinate expansion, in particular, but not only, in the context of string theory in plane wave backgrounds and perturbations around such backgrounds.

2. Lightning Review of the Penrose Limit

The traditional systematic construction of the Penrose limit [**10, 14, 15**] involves the following steps:

(1) First one introduces Penrose coordinates $(\tilde{u}, \tilde{v}, \tilde{y}^k)$ adapted to the null geodesic γ (see [**25**] for a general construction), in which the metric takes the form

$$ds^2_\gamma = 2d\tilde{v}(d\tilde{u} + B(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{v} + B_i(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{y}^i) + g_{ij}(\tilde{u}, \tilde{v}, \tilde{y}^k)d\tilde{y}^i d\tilde{y}^j \quad . \tag{3.6}$$

Here the original null-geodesic γ is the curve $(\tilde{u}, 0, 0)$ with affine parameter $\tilde{u}$, embedded into the congruence $(\tilde{u}, \tilde{v}_0, \tilde{y}^i_0)$ of null geodesics labelled by the constant values $(\tilde{v}_0, \tilde{y}^i_0)$, $i = 1, \ldots, d$, of the transverse coordinates.

(2) Next one performs an asymmetric rescaling of the coordinates,

$$(\tilde{u}, \tilde{v}, \tilde{y}^k) = (u, \lambda^2 v, \lambda y_k) \quad , \tag{3.7}$$

accompanied by an overall rescaling of the metric, to obtain the one-parameter family of metrics

$$\begin{aligned} \lambda^{-2} ds^2_{\gamma,\lambda} = \; & 2dv(du + \lambda^2 B(u, \lambda^2 v, \lambda y^k)dv + \lambda B_i(u, \lambda^2 v, \lambda y^k)dy^i) \\ & + g_{ij}(u, \lambda^2 v, \lambda y^k)dy^i dy^j \quad . \end{aligned} \tag{3.8}$$

(3) Now taking the combined infinite boost and large volume limit $\lambda \to 0$ results in a well-defined and non-degenerate metric $\bar{g}_{\mu\nu}$,

$$\text{Penrose Limit}: \; d\bar{s}^2_\gamma = \lim_{\lambda \to 0} \lambda^{-2} ds^2_{\gamma,\lambda} \tag{3.9}$$

$$= 2dudv + \bar{g}_{ij}(u)dy^i dy^j \quad , \tag{3.10}$$

where $\bar{g}_{ij}(u) = g_{ij}(u, 0, 0)$ is the restriction of g_{ij} to the null geodesic γ. This is the metric of a plane wave in Rosen coordinates.

(4) One then transforms this to Brinkmann coordinates $(x^A) = (x^+, x^-, x^a)$, $a = 1, \ldots, d$, via

$$(u, v, y^k) = (x^+, x^- + \tfrac{1}{2}\dot{\bar{E}}_{ai}\bar{E}^i_{\ b}x^a x^b, \bar{E}^k_{\ a}x^a) \tag{3.11}$$

where $\bar{E}^a_i$ is a vielbein for $\bar{g}_{ij}$, i.e. $\bar{g}_{ij} = \bar{E}^a_i \bar{E}^b_j \delta_{ab}$, required to satisfy the symmetry condition $\dot{\bar{E}}_{ai}\bar{E}^i_{\ b} = \dot{\bar{E}}_{bi}\bar{E}^i_{\ a}$. In these coordinates the plane wave metric takes the canonical form

$$d\bar{s}^2_\gamma = 2dx^+dx^- + A_{ab}(x^+)x^ax^bdx^{+2} + \delta_{ab}dx^adx^b \quad , \tag{3.12}$$

with $A_{ab}(x^+)$ given by [**21**]

$$A_{ab} = \ddot{\bar{E}}_{ai}\bar{E}^i_{\ b} \quad . \tag{3.13}$$

While this is, in a nutshell, the construction of the Penrose limit metric, the above definition looks rather round-about and non-covariant and manages to hide quite effectively the relation between the original data $(g_{\mu\nu}, \gamma)$ and the resulting plane wave metric. In principle taking the Penrose limit amounts to assigning the wave profile A_{ab} to the initial data $(g_{\mu\nu}, \gamma)$,

$$(g_{\mu\nu}, \gamma) \quad \rightarrow \quad A_{ab} \quad . \tag{3.14}$$

This certainly begs the question if there is not a more direct (and geometrically appealing) route from $(g_{\mu\nu}, \gamma)$ to A_{ab} which elucidates the precise nature of the Penrose limit and the extent to which it encodes generally covariant properties of the original space-time.

Indeed, as shown in [**20, 25**], there is. Given the affinely parametrised null geodesic $\gamma = \gamma(u)$, the tangent vector $E^\mu_+ = \dot{\gamma}^\mu$ is (by definition) parallel transported along γ. We extend this to a pseudo-orthonormal parallel transported frame $(E^\mu_A) = (E^\mu_+, E^\mu_-, E^\mu_a)$ along γ. Thus, in terms of the dual coframe (E^A_μ), the metric restricted to γ can be written as

$$ds^2|_\gamma = 2E^+E^- + \delta_{ab}E^aE^b \quad . \tag{3.15}$$

The main result of [**20**] is the observation that the wave profile $A_{ab}(x^+)$ of the associated Penrose limit metric is nothing other than the matrix

$$A_{ab}(x^+) = -R_{a+b+}|_{\gamma(x^+)} \tag{3.16}$$

of frame curvature components of the original metric, evaluated at the point $\gamma(x^+)$.

Modulo constant $SO(d)$-rotations this is independent of the choice of parallel frame and provides a manifestly covariant characterisation of the Penrose limit plane wave metric which, moreover, does not require taking any limits. The geometric significance of $A_{ab}(x^+)$ is that it is the transverse null geodesic deviation matrix along γ [**42**, Section 4.2] of the original metric,

$$\frac{d^2}{du^2}\xi^a = A_{ab}(u)\xi^b \quad , \tag{3.17}$$

with Z the transverse geodesic deviation vector. Since the only non-vanishing curvature components of the Penrose limit plane wave metric $d\bar{s}^2_\gamma$ in Brinkmann coordinates (3.12) are

$$\bar{R}_{a+b+} = -A_{ab} \quad , \tag{3.18}$$

this implies that geodesic deviation along the selected null geodesic in the original space-time is identical to null geodesic deviation in the corresponding Penrose limit plane wave metric and shows that it is precisely this information about tidal forces in the original metric that the Penrose limit encodes (while discarding all other information about the original metric).

Let us now consider higher order terms in the expansion of the original metric about the Penrose limit. To that end we return to (3.8) and expand in a power-series in λ. To $\mathcal{O}(\lambda)$ one has

$$\begin{aligned}\lambda^{-2}ds^2_{\gamma,\lambda} &= 2dudv + \bar{g}_{ij}(u)dy^i dy^j \\ &\quad + \lambda\left(2\bar{B}_i(u)dy^i dv + y^k\bar{g}_{ij,k}(u)dy^i dy^j\right) + \mathcal{O}(\lambda^2)\end{aligned} \tag{3.19}$$

where, as before, an overbar denotes evaluation on the null geodesic, i.e. $\bar{g}_{ij,k}(u) = g_{ij,k}(u,0,0)$ etc. We see that in the expansion of the metric in Penrose coordinates these higher order terms are not covariant (e.g. the $\bar{g}_{ij,k}$ are Christoffel symbols).

This raises the question if there is a different way of implementing the Penrose limit which is such that all terms in the λ-expansion of the metric are covariant expressions in the curvature tensor of the original metric.

A ham-handed way to approach this issue would be to seek a λ-dependent (and analytic in λ) coordinate transformations that extends the transformation from Rosen to Brinkmann coordinates and, applied to the above expansion of the metric, results in order by order covariant expressions. However, first of all this strategy puts undue emphasis on the coordinate transformation that relates Penrose coordinates to the new coordinates, rather than on the expansion of the metric itself. Secondly, even if one happens to find a solution to the problem in this way, in all likelihood one will in the end have discovered a coordinate system that is sufficiently natural to have been discoverable by other, less brute-force, means as well. Indeed, we will see in sections 5 and 6, without having to go through the explicit coordinate transformation from Penrose coordinates, that all this is accomplished by Fermi coordinates adapted to the null geodesic γ.

3. Brinkmann Coordinates are Null Fermi Coordinates

In this section we will discuss Brinkmann coordinates for plane waves from (what will turn out to be) the point of view of Fermi coordinates. The considerations in this section are elementary, but they serve as a motivation for the subsequent general discussion of Fermi coordinates. Moreover, we find it illuminating to recover some well known facts about Brinkmann coordinates and their relation to Rosen coordinates from this perspective.

First of all, we note that a particular solution of the null geodesic equation in Brinkmann coordinates is the curve $\gamma(u) = (u, 0, 0)$ with affine parameter $u = x^+$ (in the Penrose limit context this is obviously just the original null geodesic γ). Along this curve all the Christoffel symbols of the metric are zero (the a priori non-vanishing Christoffel symbols are linear and quadratic in the x^a and thus vanish for $x^a = 0$). This is the counterpart of the usual statement for Riemann normal coordinates that the Christoffel symbols are zero at some chosen base-point. Here we have a geodesic of such base-points.

Next we observe that the straight lines

$$x^A(s) = (x_0^+, sx^-, sx^a) \tag{3.20}$$

connecting a point $(x_0^+, 0, 0)$ on γ to the point (x_0^+, x^-, x^a) are also geodesics. In the standard plane wave terminology these are spacelike or null geodesics with zero lightcone momentum, $p_- = x^{+\prime}(s) = 0$, a prime denoting an s-derivative. Thus the coordinate lines of x^- and x^a are geodesics, while x^+ labels the original null geodesic γ. These are the characteristic and defining properties of null Fermi coordinates.

There is also a Fermi analogue of the Riemann normal coordinate expansion of the metric in terms of the Riemann tensor and its covariant derivatives. In the special case of plane waves we have, combining (3.12) with (3.18),

$$d\bar{s}^2 = 2dx^+dx^- + \delta_{ab}dx^a dx^b - \bar{R}_{a+b+}(x^+)x^a x^b dx^{+2} \quad . \tag{3.21}$$

Thus in this case the expansion of the metric terminates at quadratic order.

We can also understand (and rederive) the somewhat peculiar coordinate transformation (3.11) from Rosen to Brinkmann coordinates from this point of view. Thus this time we begin with the metric

$$d\bar{s}^2 = 2dudv + \bar{g}_{ij}(u)dy^i dy^j \tag{3.22}$$

of a plane wave in Rosen coordinates and introduce a pseudo-orthonormal frame $\bar{E}^\mu_A$,

$$\bar{E}_+ = \partial_u \quad , \quad \bar{E}_- = \partial_v \quad , \quad \bar{E}_a = \bar{E}^i_a \partial_i \tag{3.23}$$

where $\bar{E}^a_i(u)$ is a vielbein for $\bar{g}_{ij}(u)$. Demanding that this frame be parallel propagated along the null geodesic congruence, $\bar{\nabla}_u \bar{E}^\mu_A = 0$, imposes the condition

$$\partial_u \bar{E}^i_a + \tfrac{1}{2}\bar{g}^{ij}\partial_u \bar{g}_{jk}\bar{E}^k_a = 0 \quad \Leftrightarrow \quad \dot{\bar{E}}_{ai}\bar{E}^i_{\ b} = \dot{\bar{E}}_{bi}\bar{E}^i_{\ a} \quad , \tag{3.24}$$

which is thus the geometric significance of the symmetry condition appearing in the transformation from Rosen to Brinkmann coordinates.

Now we consider geodesics $x^\mu(s)$ emanating from γ, i.e. $(u(0), v(0), y^i(0)) = (u_0, 0, 0)$, with the further initial condition that $x^{\mu\prime}(s = 0)$ have no component tangent to γ, i.e. vanishing scalar product with E_-,

$$0 = \bar{g}_{\mu\nu}(u_0)x^{\mu\prime}(0)\bar{E}^\nu_-(u_0) = u'(0) \quad . \tag{3.25}$$

Then the Euler-Lagrange equations following from

$$L = u'v' + \tfrac{1}{2}\bar{g}_{ij}y^{i\prime}y^{j\prime} \tag{3.26}$$

imply that

(1) the conserved lightcone momentum p_v is zero, $p_v = u' = 0$, so that $u(s) = u_0$;
(2) the transverse coordinates $y^i(s)$ evolve linearly with s, $y^i(s) = y^{i\prime}(0)s$;
(3) the solution for $v(s)$ is $v(s) = v'(0)s + \frac{1}{4}\dot{\bar{g}}_{ij}(u_0)y^{i\prime}(0)y^{j\prime}(0)s^2$.

One now introduces the geodesic coordinates $(x^{\bar{a}}) = (x^-, x^a)$ by the condition that the geodesics be straight lines, i.e. via

$$x^{\bar{a}} = \bar{E}^{\bar{a}}_\mu x^{\mu\prime}(0)s \quad . \tag{3.27}$$

Substituting this into the above solution of the geodesic equations one finds

$$y^i(s) = \bar{E}^i_a x^a \quad , \quad v(s) = x^- + \tfrac{1}{4}\dot{\bar{g}}_{ij}\bar{E}^i_a\bar{E}^j_b x^a x^b \quad , \tag{3.28}$$

which, together with $u = x^+$, is precisely the coordinate transformation (3.11) from Rosen coordinates x^μ to Brinkmann coordinates x^A. Finally we note that, as we will explain in section 4, this transformation can also be regarded as the covariant Taylor expansion of the x^μ in the quasi-transverse variables $x^{\bar{a}}$. Here and in the following we use the terminology that "transverse" refers to the variables x^a and "quasi-transverse" to the variables $(x^{\bar{a}}) = (x^-, x^a)$.

4. Null Fermi Coordinates: General Construction

We now come to the general construction of Fermi coordinates associated to a null geodesic γ in a space-time with Lorentzian metric $g_{\mu\nu}$. Along γ we introduce a parallel transported pseudo-orthonormal frame E^A_μ,

$$ds^2|_\gamma = 2E^+E^- + \delta_{ab}E^aE^b \quad , \tag{3.29}$$

with $E^\mu_+ = \dot{\gamma}^\mu$, the overdot denoting the derivative with respect to the affine parameter. As in the previous section, we now consider geodesics $\beta(s) = (x^\mu(s))$ emanating from γ, i.e. with $\beta(0) = x_0 \in \gamma$, that satisfy

$$g_{\mu\nu}(x_0)x^{\mu\prime}(0)E^\nu_-(x_0) \equiv x^{\mu\prime}(0)E^+_\mu(x_0) = 0 \quad . \tag{3.30}$$

In comparison with the standard timelike case, we note that the double role played by the tangent vector E_0 to the timelike geodesic, as the tangent vector and as the vector to which the connecting geodesics $\beta(s)$ should be orthogonal, is in the null case shared among the two null vectors E_+ (the tangent vector) and E_- (providing the condition on $\beta(s)$).

Then the Fermi coordinates $(x^A) = (x^+, x^-, x^a)$ of the point $x = \beta(s)$ are defined by

$$(x^A) = (x^+, x^{\bar{a}} = sE^{\bar{a}}_\mu(x_0)x^{\mu\prime}(0)) \tag{3.31}$$

where $\gamma(x^+) = x_0$ and $\bar{a} = (-, a)$. We note that these definitions imply that

$$E^{\bar{a}}_\mu(x_0)x^{\mu\prime}(0) = \left.\frac{\partial x^{\bar{a}}}{\partial s}\right|_{s=0} = \left.\frac{\partial x^{\bar{a}}}{\partial x^\mu}\right|_{s=0} x^{\mu\prime}(0) \tag{3.32}$$

and

$$\left.\frac{\partial x^\mu}{\partial x^+}\right|_\gamma = \dot{\gamma}^\mu = E^\mu_+ \tag{3.33}$$

so that on γ the Fermi coordinates are related to the original coordinates x^μ by

$$\left.\frac{\partial x^A}{\partial x^\mu}\right|_\gamma = E^A_\mu \quad , \quad \left.\frac{\partial x^\mu}{\partial x^A}\right|_\gamma = E^\mu_A \quad . \tag{3.34}$$

Thus we see that Fermi coordinates are uniquely determined by a choice of parallel pseudo-orthonormal frame along the null geodesic γ. How unique is this choice? Let us first consider the case of timelike Fermi coordinates. In this case, there is a frame (E_0, E_k), $k = 1, \ldots, n = d+1$, with $E_0 = \dot{\gamma}$ tangent to the timelike geodesic. Evidently, therefore, the parallel frame is unique up to constant $SO(d+1)$ rotations of the spatial frame E_k. Consequently, the spatial Fermi coordinates x^k, constructed exactly as above, are unique up to these constant rotations.

In the lightlike case, $SO(d+1)$ is deformed to the semi-direct product of transverse $SO(d)$-rotations of the E_a (which have the obvious corresponding effect on the transverse Fermi coordinates x^a) and the Abelian group $\simeq \mathbb{R}^d$ of null rotations about E_+ which acts as

$$(E_+, E_-, E_a) \mapsto (E_+, E_- - \omega^a E_a - \tfrac{1}{2}\delta_{ab}\omega^a\omega^b E_+, E_a + \omega_a E_+) \quad , \tag{3.35}$$

where $(\omega^a) \in \mathbb{R}^d$ are constant parameters. Since the corresponding action on the relevant components $E^{\bar{a}}$ of the dual frame is

$$(E^-, E^a) \mapsto (E^-, E^a + \omega^a E^-) \quad , \tag{3.36}$$

this action of constant null rotations on the frame induces the transformation

$$(x^-, x^a) \mapsto (x^-, x^a + \omega^a x^-) \tag{3.37}$$

of the Fermi coordinates. Thus null Fermi coordinates are unique up to constant transverse rotations and shifts of the x^a by x^-. This should, in particular, be compared and contrasted with the ambiguity

$$\tilde{y}^k \mapsto \tilde{y}'^k = \tilde{y}'^k(\tilde{y}^k, \tilde{v}) \tag{3.38}$$

of Penrose coordinates (3.6), which consists of d functions of $(d+1)$ variables rather than $d(d+1)/2$ constant parameters.

For many (in particular more advanced) purposes it is useful to rephrase the above construction of Fermi coordinates in terms of the Synge world function $\sigma(x, x_0)$ **[45, 46]**. For a point x in the normal convex neighbourhood of x_0, i.e. such that there is a unique geodesic β connecting x to x_0, with $\beta(0) = x_0$ and $\beta(s) = x$, $\sigma(x, x_0)$ is defined by

$$\sigma(x, x_0) = \tfrac{1}{2} s \int_0^s dt\, g_{\mu\nu}(\beta(t)) x^{\mu\prime}(t) x^{\nu\prime}(t) \tag{3.39}$$

(this is half the geodesic distance squared between x and x_0). Since, up to the prefactor s, $\sigma(x, x_0)$ is the classical action corresponding to the Lagrangian $L = (1/2) g_{\mu\nu} x^{\mu\prime} x^{\nu\prime}$, standard Hamilton-Jacobi theory implies that

$$\sigma_\mu(x, x_0) \equiv \frac{\partial}{\partial x_0^\mu} \sigma(x, x_0) = -s g_{\mu\nu}(x_0) x^{\nu\prime}(0) \;\;, \tag{3.40}$$

as well as

$$\sigma(x, x_0) = \tfrac{1}{2} g_{\mu\nu}(x_0) \sigma^\mu(x, x_0) \sigma^\nu(x, x_0) \;\;. \tag{3.41}$$

In particular, this way of writing things makes it more transparent that something as innocuous looking as $x^{\mu\prime}(0)$ is actually a bitensor, namely not just a vector at x_0 but also a scalar at x.

Thus we can also summarise the construction (3.30,3.31) of Fermi coordinates in the following way: given $x_0 \in \gamma$, the condition

$$\sigma^\mu(x, x_0) E^+_\mu(x_0) = 0 \tag{3.42}$$

selects those points x that can be connected to x_0 by a geodesic with no initial component along γ. Locally around γ this foliates the space-time into hypersurfaces Σ_{x_0} pseudo-orthogonal to γ. For $x \in \Sigma_{x_0}$, its quasi-transverse Fermi coordinates $x^{\bar{a}}$ are then defined by

$$x^{\bar{a}} = -\sigma^\mu(x, x_0) E^{\bar{a}}_\mu(x_0) \;\;. \tag{3.43}$$

Conversely, for $x \in \Sigma_{x_0}$, the $\sigma^\mu(x, x_0)$ can be expressed in terms of the Fermi coordinates of x (using $E^\mu_A E^A_\nu = \delta^\mu_\nu$) as

$$\sigma^\mu(x, x_0) = E^\mu_A E^A_\nu \sigma^\nu(x, x_0) = -E^\mu_{\bar{a}} x^{\bar{a}} \;\;. \tag{3.44}$$

It now follows from the Hamilton-Jacobi equation (3.41) that the geodesic distance squared of a point $x = (x^+, x^-, x^a)$ to $x_0 = (x^+, 0, 0)$ is

$$2\sigma(x, x_0) = \sigma^\mu(x, x_0) E^A_\mu(x_0) E^\nu_A(x_0) \sigma_\nu(x, x_0) = \delta_{ab} x^a x^b \;\;. \tag{3.45}$$

The $\sigma^\mu = \sigma^\mu(x, x_0)$ also appear naturally in the manifestly covariant Taylor expansion of a function $f(x)$ around x_0,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\sigma^{\mu_1} \ldots \sigma^{\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} f\right)(x_0) \;\;. \tag{3.46}$$

This can e.g. be seen by beginning with the ordinary Taylor expansion of $f(x) = f(\beta(s))$, regarded as a function of the single variable s, around $s = 0$, and using the geodesic equation to convert resulting second derivatives of $x^\mu(s)$ into first derivatives. There is an analogous covariant Taylor expansion for higher-rank tensor fields [**46**] which, in addition to the above component-wise covariant expansion, also involves parallel transport from x_0 to x.

If we want to expand f not around a point x_0 but only in the directions quasi-transverse to a geodesic γ with $\gamma(x^+) = x_0$, we can use the parallel frame to project out the direction tangential to γ. Indeed, for $x \in \Sigma_{x_0}$ we can use (3.44) to express σ^μ in terms of the quasi-transverse Fermi coordinates $x^{\bar{a}}$. Plugging this into (3.46), one obtains

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(E^{\mu_1}_{\bar{a}_1} \dots E^{\mu_n}_{\bar{a}_n} \nabla_{\mu_1} \dots \nabla_{\mu_n} f\right)(x^+)\, x^{\bar{a}_1} \dots x^{\bar{a}_n} \quad . \tag{3.47}$$

This is a Taylor expansion in the quasi-transverse Fermi coordinates $(x^{\bar{a}}) = (x^-, x^a)$, with the full dependence on x^+ retained.

When $f(x)$ is itself a coordinate function, $f(x) = x^\mu$, say, then $\nabla_{\mu_1} f = \delta^\mu_{\mu_1}$ and, for $n \geq 2$,

$$\nabla_{(\mu_1} \dots \nabla_{\mu_n)} f = -\nabla_{(\mu_1} \dots \nabla_{\mu_{n-2}} \Gamma^\mu{}_{\mu_{n-1}\mu_n)} \equiv -\Gamma^\mu{}_{(\mu_1 \dots \mu_n)} \tag{3.48}$$

(the covariant derivatives act only on the lower indices) are the generalised Christoffel symbols. Provided that $\{x^\mu\}$ is an adapted coordinate system, in the sense that γ coincides with one of its coordinate lines (Penrose coordinates (3.6) are a special case of this), this gives us on the nose the coordinate transformation between such adapted coordinates and Fermi coordinates,

$$x^\mu(x^+, x^{\bar{a}}) = x^\mu(x^+) + E^\mu_{\bar{a}_1}(x^+) x^{\bar{a}_1} - \sum_{n=2}^{\infty} \left(\Gamma^\mu{}_{(\mu_1 \dots \mu_n)} E^{\mu_1}_{\bar{a}_1} \dots E^{\mu_n}_{\bar{a}_n}\right)(x^+)\, x^{\bar{a}_1} \dots x^{\bar{a}_n} \quad . \tag{3.49}$$

Thus the coordinate transformation between adapted and Fermi coordinates is nothing other than the quasi-transverse Taylor expansion of the adapted coordinates.

While formally the above equation is correct for an arbitrary coordinate system, it is less explicit if the coordinate system is not adapted since x^+, the coordinate along the geodesic, is then non-trivially related to the x^μ.

In the special case of Rosen coordinates for plane waves, the above expansion is finite and reduces to the standard result (3.11,3.28). To see this e.g. for the Rosen coordinate v, one calculates

$$v(x^+, x^-, x^a) = v(x^+) + \left(\bar{E}^\mu_{\bar{a}} \partial_\mu v\right)(x^+)\, x^{\bar{a}} + \frac{1}{2} \left(\bar{E}^\mu_{\bar{a}} \bar{E}^\nu_{\bar{b}} \nabla_\mu \partial_\nu v\right)(x^+)\, x^{\bar{a}} x^{\bar{b}} \tag{3.50}$$

with all higher order terms vanishing, and uses that on the geodesic $v = 0$, that $\bar{E}^v_- = 1$, $\bar{E}^v_a = 0$ (3.23), and that the only non-trivial $\Gamma^v_{\mu\nu}$ is $\Gamma^v_{ij} = -\frac{1}{2}\dot{\bar{g}}_{ij}$, to find yet again

$$v = x^- + \tfrac{1}{4}\dot{\bar{g}}_{ij} \bar{E}^i_a \bar{E}^j_b x^a x^b \quad . \tag{3.51}$$

5. Expansion of the Metric in Null Fermi Coordinates

We will now discuss the metric in Fermi coordinates, given by an expansion in the quasi-transverse Fermi coordinates $x^{\bar{a}}$.

First of all it follows from (3.29) and (3.34) that to zero'th order, i.e. restricted to the null geodesic γ at $x^{\bar{a}} = 0$, the metric is the flat metric.

Moreover, there are no linear terms in the metric, i.e. the Christoffel symbols restricted to γ are zero (the main characteristic of Fermi coordinates in general). To see this, note that the geodesic equation applied to the geodesic straight lines

$$(x^A(s)) = (x^+, x^{\bar{a}}(s) = v^{\bar{a}} s) \tag{3.52}$$

implies

$$\frac{d^2}{ds^2} x^A(s) + \Gamma^A{}_{BC} \frac{d}{ds} x^B(s) \frac{d}{ds} x^C(s) = 0 \quad \Rightarrow \quad \Gamma^A{}_{\bar{b}\bar{c}}(x^+, v^{\bar{a}} s)\, v^{\bar{b}} v^{\bar{c}} = 0 \ . \tag{3.53}$$

Since at $s = 0$ this has to be true for all $v^{\bar{a}}$, we conclude that

$$\Gamma^A{}_{\bar{b}\bar{c}}|_\gamma = 0 \ . \tag{3.54}$$

Moreover, since the frames E^A_μ are parallel propagated along γ, it follows that in Fermi coordinates

$$\nabla_+ E^A_{\mu=B} = \nabla_+ \delta^A_{\mu=B} = 0 \quad \Rightarrow \quad \Gamma^A{}_{B+}|_\gamma = 0 \ . \tag{3.55}$$

Together, these two results imply that all Christoffel symbols are zero along γ,

$$\Gamma^A{}_{BC}|_\gamma = 0 \ . \tag{3.56}$$

To determine the quadratic term in the expansion of the metric, we need to look at the derivatives of the Christoffel symbols. Differentiating (3.56) along γ one finds

$$\Gamma^A{}_{BC,+}|_\gamma = 0 \ . \tag{3.57}$$

From the definition of the Riemann tensor

$$R^A{}_{BCD} = \Gamma^A{}_{BD,C} - \Gamma^A{}_{BC,D} + \Gamma^A{}_{CE}\Gamma^E{}_{BD} - \Gamma^A{}_{DE}\Gamma^E{}_{BC} \tag{3.58}$$

it now follows that

$$\Gamma^A{}_{B+,C}|_\gamma = R^A{}_{BC+}|_\gamma \ . \tag{3.59}$$

To calculate the derivatives $\Gamma^A{}_{\bar{b}\bar{c},\bar{d}}$, we now use the fact all the symmetrised first derivatives of the Christoffel symbols are zero,

$$\Gamma^A{}_{(\bar{b}\bar{c},\bar{d})}|_\gamma = 0 \ . \tag{3.60}$$

This follows e.g. from applying the Taylor expansion (3.49) for adapted coordinates to the Fermi coordinates themselves: all higher order terms in that expansion, whose coefficients are the above symmetrised derivatives of the Christoffel symbols, have to vanish. Incidentally, the required vanishing of the quadratic terms in the expansion (3.49) provides another argument for the vanishing (3.54) of the $\Gamma^A{}_{\bar{b}\bar{c}}|_\gamma$.

We can now calculate (with hindsight)

$$(R^A{}_{\bar{b}\bar{c}\bar{d}} + R^A{}_{\bar{c}\bar{b}\bar{d}})|_\gamma = (\Gamma^A{}_{\bar{b}\bar{d},\bar{c}} - \Gamma^A{}_{\bar{b}\bar{c},\bar{d}} + \Gamma^A{}_{\bar{c}\bar{d},\bar{b}} - \Gamma^A{}_{\bar{c}\bar{b},\bar{d}})|_\gamma \tag{3.61}$$

and use (3.60) to conclude that

$$\Gamma^A{}_{\bar{b}\bar{c},\bar{d}}|_\gamma = -\tfrac{1}{3}(R^A{}_{\bar{b}\bar{c}\bar{d}} + R^A{}_{\bar{c}\bar{b}\bar{d}})|_\gamma \ . \tag{3.62}$$

Since we now have all the derivatives of the Christoffel symbols on γ, we equivalently know all the second derivatives $g_{AB,CD}|_\gamma$ of the metric, namely

$$\begin{aligned} g_{AB,C+}|_\gamma &= 0 \\ g_{++,\bar{c}\bar{d}}|_\gamma &= 2R_{+\bar{c}\bar{d}+}|_\gamma \\ g_{+\bar{b},\bar{c}\bar{d}}|_\gamma &= -\tfrac{2}{3}(R_{+\bar{c}\bar{b}\bar{d}} + R_{+\bar{d}\bar{b}\bar{c}})|_\gamma \\ g_{\bar{a}\bar{b},\bar{c}\bar{d}}|_\gamma &= -\tfrac{1}{3}(R_{\bar{c}\bar{a}\bar{d}\bar{b}} + R_{\bar{c}\bar{b}\bar{d}\bar{a}})|_\gamma \ . \end{aligned} \tag{3.63}$$

Thus the expansion of the metric to quadratic order is

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ &\left[R_{+\bar{a}+\bar{b}}\, x^{\bar{a}}x^{\bar{b}}(dx^+)^2 + \tfrac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) + \tfrac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}})\right] \\ &+ \mathcal{O}(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}) \end{aligned} \tag{3.64}$$

where all the curvature components are evaluated on the null geodesic. This is the precise null analogue of the Manasse-Misner result [**47, 46**] in the timelike case, i.e. Fermi coordinates associated to a timelike geodesic.

In the timelike case, the expansion of the metric to fourth order was determined in [**58**]. The calculations in [**58**], based on repeated differentiation and expansion of the geodesic and geodesic deviation equations associated to $\gamma(u)$ and $\beta(s)$ and expressing the results in terms of components of the Riemann tensor and its covariant derivatives, are straightforward in principle but somewhat tedious in practise. They can be simplified a bit by using, as we have done above, the symmetrised derivative identities following from (3.49) instead of the geodesic deviation equations. Either way, some care is required in translating and adapting the intermediate steps in these calculations to the null case (cf. the comment in section 9.1). However, as far as we can tell (and we have performed numerous checks), the final results for the expansion of the metric in the timelike and null case are just related by the simple index relabelling $(0,k) \leftrightarrow (+,\bar{a})$, where (x^0, x^k) are the Fermi coordinates in the timelike case, with x^0 proper time along the timelike geodesic. In its full glory, the expansion to quartic order (which we will require later on) is given in section 9.1.

6. Covariant Penrose Limit Expansion via Fermi Coordinates

We now come to the heart of the matter, namely the description of the Penrose limit in Fermi coordinates. Let us first investigate how Fermi coordinates transform under scalings of the metric. Thus we consider the scaling

$$g_{\mu\nu} \to g_{\mu\nu}(\lambda) = \lambda^{-2} g_{\mu\nu} \quad . \tag{3.65}$$

First of all we note that γ continues to be a null geodesic for the rescaled metric. The scaling of the metric evidently requires a concomitant scaling of the parallel pseudo-orthonormal frame along γ, $E^A \to E^A(\lambda)$, which must be such that

$$2\lambda^{-2}E^+E^- + \lambda^{-2}\delta_{ab}E^aE^b = 2E^+(\lambda)E^-(\lambda) + \delta_{ab}E^a(\lambda)E^b(\lambda) \quad . \tag{3.66}$$

Consequently, for the transverse components $E^a(\lambda)$ we have (up to rotations)

$$E^a(\lambda) = \lambda^{-1}E^a \quad . \tag{3.67}$$

In order to determine the transformation of the $E^{\pm}(\lambda)$, we recall that in the construction of the Fermi coordinates the component E_+ is fixed to be the tangent vector to γ, independently of the metric, $E_+^\mu = \dot{\gamma}^\mu$. This requirement determines uniquely

$$E^+(\lambda) = E^+ \quad , \quad E^-(\lambda) = \lambda^{-2}E^- \quad , \tag{3.68}$$

which is related by a boost to the symmetric choice $E^{\pm}(\lambda) = \lambda^{-1}E^{\pm}$. To determine the Fermi coordinates, we note that

$$\sigma^\mu(x,x_0) = \tfrac{1}{2}sg^{\mu\nu}(x_0)\frac{\partial}{\partial x_0^\nu}\int_0^s dt\; g_{\rho\sigma}(\beta(t))x^{\rho\prime}(t)x^{\sigma\prime}(t) = -sx^{\mu\prime}(0) \tag{3.69}$$

is scale invariant. Thus the Fermi coordinates $x^A(\lambda)$ are

$$\begin{aligned} x^+(\lambda) &= x^+ \\ x^-(\lambda) &= -\sigma^\mu E^-_\mu(\lambda) = \lambda^{-2}x^- \\ x^a(\lambda) &= -\sigma^\mu E^a_\mu(\lambda) = \lambda^{-1}x^a \ . \end{aligned} \tag{3.70}$$

Writing this as

$$(x^+, x^-, x^a) = (x^+(\lambda), \lambda^2 x^-(\lambda), \lambda x^a(\lambda)) \ , \tag{3.71}$$

we see that here the asymmetric rescaling of the coordinates, which is completely analogous to that imposed "by hand" in Penrose coordinates[3],

$$(\tilde{u}, \tilde{v}, \tilde{y}^k) = (u, \lambda^2 v(\lambda), \lambda y^k(\lambda)) \tag{3.72}$$

arises naturally and automatically from the very definition of Fermi coordinates.

To now implement the Penrose limit,

- one can either start with the expansion (3.64,3.94) of the unscaled metric in its Fermi coordinates, multiply by λ^{-2} and express the metric in terms of the scaled Fermi coordinates, i.e. make the substitution (3.71);
- or one takes the expansion of the rescaled metric in its Fermi coordinates $x^A(\lambda)$ and then replaces in that expansion each $x^A(\lambda)$ by the original x^A.

Which point of view one prefers is a matter of taste and depends on whether one thinks of the scale transformation actively, as acting on space-time, or passively on measuring rods. The net effect is the same.

Let us now look at the effect of this operation on the metric (3.64,3.94), using the language appropriate to the first point of view to determine the powers of λ with which each term in (3.94) appears. There is thus an overall λ^{-2}, and each x^a or dx^a contributes a λ whereas x^- and dx^- gives a λ^2 contribution.[4] The first consequence of this is that the flat metric is of order λ^0, the overall λ^{-2} being cancelled by a λ^2 from either one dx^- or two dx^a's. Moreover, precisely one of the quadratic terms in (3.64) also gives a contribution of order λ^0, namely $R_{a+b+}x^a x^b (dx^+)^2$, the λ^{-2} being cancelled by the quadratic term in the x^a's. Thus the metric to order λ^0 is

$$ds^2_{\lambda^0} = 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \ . \tag{3.73}$$

Comparison with (3.12, 3.16) or (3.21) shows that this is precisely the Penrose limit along γ of the original metric,

$$ds^2_{\lambda^0} = d\bar{s}^2_\gamma \qquad \text{(Penrose Limit)} \tag{3.74}$$

obtained here directly in Brinkmann coordinates.

Moreover the expansion to quartic order in (3.94) is sufficient to give us the covariant expansion of the metric around its Penrose limit to order λ^2 (a quintic term would scale at least as $\lambda^{-2}\lambda^5 = \lambda^3$). Explicitly, the $\mathcal{O}(\lambda)$ term is

$$ds^2_{\lambda^1} = -2R_{+a+-}\, x^a x^- (dx^+)^2 - \tfrac{4}{3}R_{+bac}\, x^b x^c (dx^+ dx^a) - \tfrac{1}{3}R_{a+b+;c}\, x^a x^b x^c (dx^+)^2 \tag{3.75}$$

[3]Here we have explicitly indicated the λ-dependence of the new coordinates that we suppressed for notational simplicity in (3.7).

[4]Alternatively, for the counting from the second point of view, one uses the fact that the coordinate components $\mathcal{R}(g)_{\alpha_1\cdots\alpha_n\alpha\beta}$ of the "vertices" $\mathcal{R}(g)_{\bar{a}_1\cdots\bar{a}_n AB}x^{\bar{a}_1}\ldots x^{\bar{a}_n}$ appearing in the expansion of the metric $g_{AB}dx^A dx^B$ scale like the metric, $\mathcal{R}(g(\lambda)) = \lambda^{-2}\mathcal{R}(g)$. This can be checked explicitly for the terms written in (3.94) and in general follows from the fact that the expansion of the metric $g_{\mu\nu}(\lambda)$ in its Fermi coordinates $x^A(\lambda)$ must be λ^{-2} times the expansion of $g_{\mu\nu}$ in its Fermi coordinates x^A.

and the expansion to $\mathcal{O}(\lambda^2)$ is given in section 9.2.

One characteristic property of the lowest order (Penrose limit) metric is the existence of the covariantly constant null vector $\partial_- \equiv \partial/\partial x^-$. We see from the above that ∂_- continues to be null at $\mathcal{O}(\lambda)$. Actually this property is guaranteed to persist up to and including $\mathcal{O}(\lambda^3)$, since a $(dx^-)^2$-term in the metric will scale at least with a power $\lambda^{-2}\lambda^2\lambda^4 = \lambda^4$ (such a term arises e.g. from the last term in (3.64) with $\bar{a} = \bar{b} = -$ and $\bar{c} = c, \bar{d} = d$).

Moreover, we see that ∂_- remains Killing to $\mathcal{O}(\lambda)$ provided that $R_{+a+-} = 0$. If that condition is satisfied, actually something more is true. Namely ∂_- remains covariantly constant and the metric is that of a pp-wave (plane-fronted wave with parallel rays), whose general form is

$$ds^2_{\mathrm{pp}} = 2dx^+dx^- + \delta_{ab}dx^a dx^b + 2A(x^+, x^a)(dx^+)^2 + 2A_b(x^+, x^a)(dx^+dx^b) \ . \tag{3.76}$$

As shown in [**37**], this is precisely the condition for string theory in a curved background to admit a standard (conformal gauge for the world-sheet metric h_{rs}) light-cone gauge $X^+(\sigma,\tau) = p_-\tau$.

More interestingly, perhaps, in general the metric to $\mathcal{O}(\lambda)$ is precisely such that it admits a modified light cone gauge $h^{00} = -1$ and $X^+(\sigma,\tau) = p_-\tau$ [**48**]. Indeed, the conditions on the metric g_{AB} (we do not consider the conditions on the dilaton) found in [**48**] in order for X^- to have an explicit representation on the transverse Fock space

$$g_{-+} = 1 \quad , \quad g_{-\bar{a}} = 0 \quad , \quad \partial_-^2 g_{AB} = 0 \ , \tag{3.77}$$

(see [**59, 60**] for a discussion of the case $g_{-+} \neq 1$), and for X^- to be auxiliary, $g_{-\bar{a}} = 0$, are satisfied by the $\mathcal{O}(\lambda)$ metric (3.73, 3.75).

7. Example: $\mathrm{AdS}_5 \times S^5$

We will now illustrate the formalism introduced above by giving a simple purely algebraic derivation of the Penrose limit expansion of the $\mathrm{AdS}_5 \times S^5$ metric to $\mathcal{O}(\lambda^2)$. These terms have been calculated before in different ways [**49, 50, 51**]. In the present framework, the identification of these corrections with certain components of the curvature tensor of $\mathrm{AdS}_5 \times S^5$ is manifest.

Thus consider the unit (curvature) radius metric[5] of that space-time, a null geodesic γ, with $E_\pm$ the lightcone components of the corresponding parallel frame. Let us consider the case that γ has a non-vanishing component along the sphere (i.e. non-zero angular momentum). Then, due to the product structure of the metric, the components of E_+ along S^5 and AdS_5 are geodesic, and since E_+ is null they are of opposite norm squared α^2. Thus we have the decomposition

$$E_\pm = \tfrac{1}{\sqrt{2}}\alpha^{\pm 1}(E_9 \pm E_0) \tag{3.78}$$

where E_0 and E_9 are normalised and geodesic in AdS_5 and S^5 respectively. Without loss of generality we can (and will) assume $\alpha = 1$ because we can either perform a boost now or the coordinate transformation $x^\pm \to \alpha^{\pm 1}x^\pm$ later to achieve this. We now extend E_0 and E_9 to parallel orthonormal frames along γ in AdS_5 and S^5,

$$\begin{aligned} ds^2_{\mathrm{AdS}} &= \eta_{\tilde{A}\tilde{B}}E^{\tilde{A}}E^{\tilde{B}} = -(E^0)^2 + \delta_{\tilde{a}\tilde{b}}E^{\tilde{a}}E^{\tilde{b}} \\ ds^2_S &= \delta_{AB}E^A E^B = \quad (E^9)^2 + \delta_{ab}E^a E^b \ . \end{aligned} \tag{3.79}$$

[5] We can restrict to unit radius since we have already implemented the large volume limit via the λ-expansion.

Here $\tilde{A}, \tilde{B}, \ldots = 0, \ldots, 4$, while $a, b, \ldots = 5, \ldots, 8$ etc. Since both spaces are maximally symmetric, the frame components of the curvature tensor are

$$R_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = -(\eta_{\tilde{A}\tilde{C}}\eta_{\tilde{B}\tilde{D}} - \eta_{\tilde{A}\tilde{D}}\eta_{\tilde{B}\tilde{C}}) \quad , \quad R_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} \ , \tag{3.80}$$

and therefore the only non-vanishing frame components in the parallel frame $(E_\pm, E_{\tilde{a}}, E_a)$ along γ are

$$\begin{aligned} R_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = -(\delta_{\tilde{a}\tilde{c}}\delta_{\tilde{b}\tilde{d}} - \delta_{\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}}) \qquad & R_{abcd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \\ R_{+a+b} = R_{-a-b} = \quad R_{+a-b} = \quad & R_{-a+b} = \tfrac{1}{2}\delta_{ab} \\ R_{+\tilde{a}+\tilde{b}} = R_{-\tilde{a}-\tilde{b}} = -R_{+\tilde{a}-\tilde{b}} = & -R_{-\tilde{a}+\tilde{b}} = \tfrac{1}{2}\delta_{\tilde{a}\tilde{b}} \end{aligned} \tag{3.81}$$

We now have all the information we need to determine the Penrose limit and the higher order corrections. For the Penrose limit we immediately find, from (3.73), the result[6]

$$ds^2_{\lambda^0} = 2dx^+dx^- + dx^2 + d\tilde{x}^2 - \tfrac{1}{2}(x^2 + \tilde{x}^2)(dx^+)^2 \ . \tag{3.82}$$

This is of course the standard result [**13, 16**], namely the maximally supersymmetric BFHP plane wave [**11**].

On symmetry grounds and/or because the curvature tensors are covariantly constant, all the $\mathcal{O}(\lambda)$-corrections (3.75) to the Penrose limit are identically zero in this case. Actually, (3.81) shows that to any order only even numbers of transverse indices $(a, b, \ldots)$ or $(\tilde{a}, \tilde{b}, \ldots)$ can appear in the expansion of the metric, and thus all odd order corrections $\mathcal{O}(\lambda^{2n+1})$ to the metric are identically zero.

For the $\mathcal{O}(\lambda^2)$-corrections, displayed in (3.97), one finds non-zero contributions from the second, fourth and fifth terms in square brackets as well as from the term quadratic in the Riemann tensor, and one can read off the result

$$\begin{aligned} ds^2 = {} & 2dx^+dx^- + dx^2 + d\tilde{x}^2 - (x^2 + \tilde{x}^2)(dx^+)^2 \\ & + \lambda^2 \left[-\tfrac{2}{3}(x^2 - \tilde{x}^2)(dx^+dx^-) - \tfrac{1}{3}(x^2dx^2 - (xdx)^2) + \tfrac{1}{3}(\tilde{x}^2d\tilde{x}^2 - (\tilde{x}d\tilde{x})^2) \right. \\ & \left. + \tfrac{2}{3}x^-(xdx - \tilde{x}d\tilde{x})dx^+ + \tfrac{1}{6}((x^2)^2 - (\tilde{x}^2)^2)(dx^+)^2 \right] + \mathcal{O}(\lambda^4) \end{aligned} \tag{3.83}$$

While this may not be the world's nicest metric, at least every term in this metric has a clear geometric interpretation in terms of the Riemann tensor of the original AdS $\times$ S metric. This metric can be simplified somewhat, perhaps at the expense of geometric clarity, by the λ-dependent coordinate transformation

$$x^- = w^-(1 - \frac{\lambda^2}{6}(y^2 - z^2)) \ , \ x^a = y^a(1 - \frac{\lambda^2}{12}y^2) \ , \ x^{\tilde{a}} = z^a(1 + \frac{\lambda^2}{12}z^2) \ , \tag{3.84}$$

which has the effect of removing the explicit x^- from the metric and eliminating the radial xdx and $\tilde{x}d\tilde{x}$ terms. Performing only the x^--transformation, and neglecting terms of $\mathcal{O}(\lambda^4)$, the metric takes the form

$$\begin{aligned} ds^2 = {} & 2dx^+dw^- + dx^2 + d\tilde{x}^2 - (x^2 + \tilde{x}^2)(dx^+)^2 \\ & + \frac{\lambda^2}{3} \left[-3(x^2 - \tilde{x}^2)(dx^+dw^-) - (x^2dx^2 - (xdx)^2) + (\tilde{x}^2d\tilde{x}^2 - (\tilde{x}d\tilde{x})^2) \right. \\ & \left. + ((x^2)^2 - (\tilde{x}^2)^2)(dx^+)^2 \right] + \mathcal{O}(\lambda^4) \ . \end{aligned} \tag{3.85}$$

[6] Here and in the following we use a short-hand notation, $\tilde{x}^2 = \delta_{\tilde{a}\tilde{b}}x^{\tilde{a}}x^{\tilde{b}}$, $xdx = \delta_{ab}x^a dx^b$, etc.

With $w^- \to -2x^-$ and $\lambda \to 1/R$, R the radius, this agrees with the metric found in [**49**]. The subsequent transformation $(x^a, x^{\tilde{a}}) \to (y^a, z^a)$ leads to the metric

$$
\begin{aligned}
ds^2 &= 2dx^+dw^- + dy^2 + dz^2 - (y^2+z^2)(dx^+)^2 \\
&+ \frac{\lambda^2}{2}\left[(y^4 - z^4)(dx^+)^2 - 2(y^2-z^2)dx^+dw^- + z^2dz^2 - y^2dy^2\right] \ ,
\end{aligned} \tag{3.86}
$$

which, with $w^- \to x^-$, is identical to the metric found in [**50, 51**] (via a coordinate transformation similar to (3.84) before taking the Penrose limit) and studied there from the point of view of the BMN correspondence [**16**].

8. A Peeling Theorem for Penrose Limits

In section 6 we have seen that the leading non-trivial contribution to the metric in a series expansion in the scaling parameter λ arises at $\mathcal{O}(\lambda^0)$ from the R_{a+b+} component of the Riemann tensor. And, more generally, we have essentially already seen (and used) there, although we did not phrase it that way, that under a rescaling

$$
g_{\mu\nu} \to g(\lambda)_{\mu\nu} = \lambda^{-2} g_{\mu\nu} \tag{3.87}
$$

of the metric, effectively the components R_{ABCD} of the Riemann tensor restricted to the null geodesic scale as

$$
R_{ABCD}(g(\lambda)) = \lambda^{-2+w_A+w_B+w_C+w_D} R_{ABCD}(g) \tag{3.88}
$$

where the weights are

$$
(w_+, w_-, w_a) = (0, 2, 1) \ . \tag{3.89}
$$

The resulting scaling weights $w = -2 + w_A + w_B + w_C + w_D$ of the frame components of the Riemann tensor are summarised in the table below.

λ^0	λ^1	λ^2	λ^3	λ^4
R_{a+b+}	R_{+-+a}, R_{+abc}	$R_{+-+-}, R_{+a-b}, R_{+-ab}, R_{abcd}$	R_{+-a-}, R_{-abc}	R_{-a-b}

It is also not difficult to see that the leading scaling weight of a component of the Riemann (Weyl) tensor at a point x not on γ is identical to that on γ,

$$
R_{ABCD}(x_0) = \mathcal{O}(\lambda^w) \quad \Rightarrow \quad R_{ABCD}(x) = \mathcal{O}(\lambda^w) \ . \tag{3.90}
$$

To be specific, in this equation we let both $R_{ABCD}(x_0)$ and $R_{ABCD}(x)$ refer to frame components at the respective points (since the generalised Petrov classification [**55, 56, 57**] we will employ below refers to such components), the frame at x being obtained by parallel transport of the standard frame at x_0 along the unique geodesic connecting x and x_0.

The statement (3.90) is intuitively obvious since moving away from γ involves more insertions of quasi-transverse coordinates $x^{\bar{a}}$ and thus, upon scaling of the coordinates, higher powers of λ. One can base a formal argument along these lines on the covariant Taylor expansion of a tensor. However, for present purposes it is enough to note that the expansion of a tensor at a point $x = (x^+, \lambda^2 x^-, \lambda x^a)$ around the point $x_0 = (x^+, 0, 0)$ is tantamount to an expansion in non-negative powers of λ. The same is true for the frames and this establishes (3.90). This argument also shows that the statement (3.90) as such is also valid for Fermi coordinate rather than frame components since they agree at x_0 and differ by higher powers of λ at x.

We will now establish the relation of the above results to the peeling property of the Weyl tensor in the Penrose limit context. This was first analysed in the four-dimensional $d=2$ case in [**41**], where it was shown that the complex Weyl scalars Ψ_i, $i=0,\dots,4$ scale as λ^{4-i}, the $\mathcal{O}(\lambda^0)$-term Ψ_4 corresponding to the type N Penrose limit components C_{a+b+}.

In higher dimensions $d>2$, instead of complex Weyl scalars (one complex transverse dimension) one has $SO(d)$-tensors of the transverse rotation group, and the appropriate framework is then provided by the analysis in [**55, 56, 57**]. There the primary classification of the Weyl tensor (according to principal or Weyl type) is based on the boost weight of a frame component of a tensor under the boost

$$(E_+,E_-)\to(\alpha^{-1}E_+,\alpha E_-) \tag{3.91}$$

Evidently, the individual boost weights b_A are

$$(b_+,b_-,b_a)=(-1,+1,0)\ . \tag{3.92}$$

Comparison with (3.89) shows that $b_A=w_A-1$, and thus the relation between w and the boost weight $b=\sum b_A$ of the Riemann or Weyl tensor is

$$b=\sum_A(w_A-1)=w-2\in\{-2,-1,0,1,2\}\ . \tag{3.93}$$

In particular, the characterisation in terms of the scaling weight w is equivalent to that in terms of boost weights, and a component with boost weight b scales as λ^{b+2}.

According to the generalised Petrov classification in [**55, 56, 57**], the component characterising the alignment property of type N has the lowest boost weight $b=-2$, thus scales as λ^0, as we already know from the Penrose limit, type III has $b=-1$, etc.[7] Thus, generalising the result of [**41**], we have established that the scaling properties (scaling weights) of the frame components of the Weyl tensor are strictly correlated with their algebraic properties. This can be regarded as a formal analogue, in the Penrose limit context, of the standard peeling theorem [**52, 53, 54**] of radiation theory in general relativity which describes the algebraic properties of the coefficients of the Weyl tensor in a large distance $1/r$ expansion.

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9. Higher Order Terms

9.1. Expansion of the Metric in Fermi Coordinates to Quartic Order. As mentioned in section 5, the expansion of the metric in null Fermi coordinates follows the pattern of the expansion in the timelike case, determined to quartic order in [**58**]. Thus

[7] In comparing with [**55, 56, 57**], one should note that there the metric decays along the null geodesic (connecting an interior point to conformal infinity) whereas here this decay occurs in the directions quasi-transverse to the null geodesic. Thus their C_{0i0j} correspond to our C_{-a-b} etc.

one has[8]

$$
\begin{aligned}
ds^2 = {} & 2dx^+dx^- + \delta_{ab}dx^a dx^b \\
& - R_{+\bar{a}+\bar{b}}\, x^{\bar{a}}x^{\bar{b}}(dx^+)^2 - \tfrac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) - \tfrac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}}) \\
& - \tfrac{1}{3}R_{+\bar{a}+\bar{b};\bar{c}}\, x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}(dx^+)^2 - \tfrac{1}{4}R_{+\bar{b}\bar{a}\bar{c};\bar{d}}\, x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}(dx^+dx^{\bar{a}}) \\
& \qquad - \tfrac{1}{6}R_{\bar{a}\bar{c}\bar{b}\bar{d};\bar{e}}\, x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}(dx^{\bar{a}}dx^{\bar{b}}) \\
& + \left(\tfrac{1}{3}R_{+\bar{a}A\bar{b}}R^A{}_{\bar{c}+\bar{d}} - \tfrac{1}{12}R_{+\bar{a}+\bar{b};\bar{c}\bar{d}}\right) x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}(dx^+)^2 \\
& + \left(\tfrac{2}{15}R_{+\bar{b}A\bar{c}}R^A{}_{\bar{d}\bar{a}\bar{e}} - \tfrac{1}{15}R_{+\bar{b}\bar{a}\bar{c};\bar{d}\bar{e}}\right) x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}(dx^+dx^{\bar{a}}) \\
& + \left(\tfrac{2}{45}R_{A\bar{c}\bar{a}\bar{d}}R^A{}_{\bar{e}\bar{b}\bar{f}} - \tfrac{1}{20}R_{\bar{a}\bar{f}\bar{b}\bar{c};\bar{d}\bar{e}}\right) x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}x^{\bar{f}}(dx^{\bar{a}}dx^{\bar{b}}) \\
& + \mathcal{O}(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}x^{\bar{e}})
\end{aligned}
\tag{3.94}
$$

However, the actual calculation of the fourth and higher order terms requires a closer inspection. For example, to determine the metric at quartic order, one needs to express the third derivatives of the Christoffel symbols in terms of Riemann tensors. One such identity is

$$
\begin{aligned}
\Gamma^A_{++,\bar{a}\bar{b}\bar{c}} = {} & R^A{}_{+(\bar{a}|+;|\bar{b}\bar{c})} + R^A{}_{(\bar{a}\bar{b}|+;|\bar{c})+} - R^A{}_{+(\bar{a}|B}R^B{}_{|\bar{b}|+|\bar{c})} \\
& - 3R^A{}_{B(\bar{a}|+}R^B{}_{|\bar{b}|+|\bar{c})} + 3R^A{}_{(\bar{a}|B|\bar{b}|}R^B{}_{+|\bar{c})+} - 2R^A{}_{(\bar{a}|\bar{p}|\bar{b}|}R^{\bar{p}}{}_{+|\bar{c})+}
\end{aligned}
\tag{3.95}
$$

As written, this identity is correct both in the null and (with the substitution $(+,\bar{a}) \to (0,k)$) in the timelike case, whereas the expression given in [**58**, eq.(33)],

$$
\begin{aligned}
\Gamma^\mu_{00,klm} = {} & R^\mu{}_{0(k|0;|lm)} + R^\mu{}_{(kl|0;|m)0} - R^\mu{}_{0(k|\kappa}R^\kappa{}_{|l|0|m)} \\
& - 3R^\mu{}_{\kappa(k|0}R^\kappa{}_{|l|0|m)} + R^\mu{}_{(k|p|l|}R^p{}_{0|m)0}
\end{aligned}
\tag{3.96}
$$

is valid only in the timelike case (where it agrees with (3.95)).

9.2. Expansion around the Penrose Limit to $\mathcal{O}(\lambda^2)$. The covariant Fermi coordinate expansion of the Penrose limit to $\mathcal{O}(\lambda^2)$ is

$$
\begin{aligned}
ds^2 = {} & 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \\
& + \lambda\left[-2R_{+a+-}\, x^a x^-(dx^+)^2 - \tfrac{4}{3}R_{+bac}\, x^b x^c(dx^+dx^a) - \tfrac{1}{3}R_{+a+b;c}\, x^a x^b x^c (dx^+)^2\right] \\
& + \lambda^2\left[-R_{+-+-}\, x^- x^-(dx^+)^2 - \tfrac{4}{3}R_{+b-c}\, x^b x^c(dx^+dx^-) - \tfrac{4}{3}R_{+-ac}\, x^- x^c(dx^+dx^a)\right. \\
& - \tfrac{4}{3}R_{+ba-}\, x^b x^-(dx^+dx^a) - \tfrac{1}{3}R_{acbd}\, x^c x^d(dx^a dx^b) - \tfrac{2}{3}R_{+a+-;c}\, x^a x^- x^c(dx^+)^2 \\
& - \tfrac{1}{3}R_{+a+b;-}\, x^a x^b x^-(dx^+)^2 - \tfrac{1}{4}R_{+bac;d}\, x^b x^c x^d(dx^+dx^a) \\
& \left. + \left(\tfrac{1}{3}R_{+aAb}R^A_{c+d} - \tfrac{1}{12}R_{+a+b;cd}\right) x^a x^b x^c x^d (dx^+)^2\right] \\
& + \mathcal{O}(\lambda^3)
\end{aligned}
\tag{3.97}
$$

Determining the expansion to $\mathcal{O}(\lambda^3)$ would require knowledge of the quintic terms in the expansion of the metric in Fermi coordinates.

[8]In the second line, the Manasse-Misner result [**47, 46**], we have corrected a misprint in [**58**].

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Subsequent Alterations

The Penrose coordinates U, V, Y^i have been changed to $\tilde{u}$, $\tilde{v}$ and $\tilde{y}^i$ respectively.

The metric components a and b_i introduced in equation (3.6) have been changed to $2B$ and B_i respectively.

The frame components of the deviation vector Z^a introduced in equation (3.17) have been changed to ξ^a.

The metric components A and B_a of the pp-wave in Brinkmann coordinates in equation (3.76) have been changed to $2A$ and A_a respectively.

Erratum

In the discussion following equation (3.76) we state that this is the most general form of a metric admitting simultaneously the conformal and the lightcone gauge. However as was shown in chapter 2 section 2.2 this is not true as the metrics with this property form the even larger Brinkmann class (2.7).

CHAPTER 4

Penrose Limits vs String Expansions

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We analyse the relation between two a priori quite different expansions of the string equations of motion and constraints in a general curved background, namely one based on the covariant Penrose-Fermi expansion of the metric $g_{\mu\nu}$ around a Penrose limit plane wave associated to a null geodesic γ, and the other on the Riemann coordinate expansion in the exact metric $g_{\mu\nu}$ of the string embedding variables around the null geodesic γ. Starting with the observation that there is a formal analogy between the exact string equations in a plane wave and the first order string equations in a general background, we show that this analogy becomes exact provided that one chooses the background string configuration to be the null geodesic γ itself. We then explore the higher-order correspondence between these two expansions and find that for a general curved background they agree to all orders provided that one works in Fermi coordinates and in the light-cone gauge. Requiring moreover the conformal gauge restricts one to the usual class of (Brinkmann) backgrounds admitting simultaneously the light-cone and the conformal gauge, without further restrictions.

1. Introduction

After the initial developments [**11, 43, 44, 13**] related to the discovery of the maximally supersymmetric IIB plane wave and its connection with the Penrose limit [**10, 14**], much effort has, in the wake of the seminal BMN paper [**16**], understandably gone into exploring the consequences of these ideas in the context of the AdS/CFT correspondence, eventually leading to deep new insights into the integrable structures underlying the theories on both sides of the correspondence. Some of these developments are described e.g. in [**17, 18, 19**].

Along a different line, in a series of papers [**15, 21, 22, 20, 25, 26**] we have explored various aspects of the geometry and physics of plane waves and Penrose limits *per se*, also with the expectation that these results will eventually lead to further insights into the gauge theory – geometry correspondence. In particular, in [**20, 25**] we provided a geometrically transparent and covariant characterisation of the Penrose limit map $(g_{\mu\nu}, \gamma) \mapsto A_{ab}$ that associates to a space-time metric $g_{\mu\nu}$ and a null geodesic γ the wave profile A_{ab} characterising the Penrose limit plane wave metric $ds^2 = 2dx^+dx^- + A_{ab}(x^+)x^ax^b(dx^+)^2 + \delta_{ab}\, dx^adx^b$.

Namely, the $A_{ab}(x^+) = -R_{+a+b}(x^+)$, which are the only non-vanishing coordinate components of the curvature tensor of the plane wave, are at the same time simply certain frame components of the curvature tensor of the original metric $g_{\mu\nu}$, restricted to the null geodesic γ (with affine parameter x^+) along which the Penrose limit is taken.

In [**26**], we used Fermi coordinates based on the null geodesic γ to generalise the above result to an all order covariant expansion of a metric around its Penrose limit (covariant in the sense that all the higher order terms are also expressed in terms of the Riemann tensor of the original metric and its derivatives). In the following we will refer to this expansion as the *Penrose-Fermi expansion* of a metric.

Within this clear geometric setting it is now possible to address questions regarding the relation between the dynamics of various objects in the original metric and its Penrose limit. In particular, the above geometric interpretation of the Penrose limit can be reinterpreted as providing an answer to the

Question: What is the interpretation of the geodesic equation in the Penrose limit plane wave (associated to the metric $g_{\mu\nu}$ and a null geodesic γ) in terms of the original data $(g_{\mu\nu}, \gamma)$?

Answer: It is simply the transverse geodesic deviation equation for $(g_{\mu\nu}, \gamma)$.

It is then natural to next ask the same question for strings rather than for particles.[1]

Question: What is the interpretation of the string equations of motion in the Penrose limit plane wave in terms of the string equations of motion in the original metric $g_{\mu\nu}$?

Thinking about this, one quickly realises that this will have to be related to a (first order) expansion of the string embedding variables $X^\mu(\tau, \sigma)$ around the null geodesic $\gamma(\tau)$, the latter regarded as a string background solution of the equations of motion and constraints in the original space-time with metric $g_{\mu\nu}$.

Thus, in more general terms what this amounts to is a comparison of two apparently quite different expansions of the string equations in a curved background, an expansion of the metric itself (the Penrose-Fermi expansion of $g_{\mu\nu}$) on the one hand, and an expansion of the string embedding variables around a background string configuration (but in the exact metric $g_{\mu\nu}$) on the other.

In order to be able to assess what the advantages (or perhaps drawbacks) are of choosing a null geodesic as a (somewhat degenerate) string background configuration, we have found it useful to begin the discussion with an analysis of the expansion of the string equations around a non-degenerate string background configuration $X_B^\mu(\tau, \sigma)$. This is, of course, largely classical material, the Riemann coordinate expansion of the two-dimensional sigma-model having been discussed at length e.g. in [**61**], and we briefly recall this (and adapt it to the present setting) in section 2 and section 7.1.

The principal difference to the discussion of [**61**] is that, in addition to the sigma-model equations of motion we also have to deal with the string constraints. Then the main observation of this section is that, to first order in an expansion around a classical string configuration $X_B(\tau, \sigma)$ in an arbitrary curved background, these constraints allow one to explicitly solve for the tangential fluctuations and to completely eliminate them from the equations of motion for the true dynamical transverse degrees of freedom. While

[1] In a similar spirit, in [**28**] we showed that scalar field probes of space-time singularities exhibit a universal behaviour that is strictly analogous to that of massless particle probes (i.e. the Penrose limit) uncovered in [**20, 25**].

the result as such may not be surprising (it is essentially a consequence of world-sheet diffeomorphism invariance), our presentation is aimed at highlighting the analogies and differences with strings in the conformal and light-cone gauge in plane wave backgrounds.

We pursue this analogy in section 3, where we observe first of all that the main difference between the first order and plane wave equations of motion for the true dynamical transverse degrees of freedom is due to the extrinsic curvature of the background string X_B. We then argue that this difference disappears, and that the analogy becomes perfect, when one chooses the background string configuration to be a null geodesic, $X_B(\tau,\sigma) \to \gamma(\tau)$. The result of this section can then be summarised as the answer to the question posed above.

ANSWER: The exact transverse string equations in the first order Penrose-Fermi expansion of the metric $g_{\mu\nu}$ around γ, i.e. in the Penrose limit plane wave metric associated to $g_{\mu\nu}$ and γ, are equivalent to the transverse first-order string expansion equations around a null geodesic γ in the original background $g_{\mu\nu}$.

Finally, in section 4 we address the

QUESTION: What can one say about the correspondence between the string expansion on the one hand and the Penrose-Fermi expansion on the other, established to first order in section 3, at higher orders?

This boils down to a comparison of two different prescriptions for how to describe the locus of nearby strings in terms of geodesic distance (namely via Riemann or Fermi coordinates). We show that demanding all order equivalence of the two expansions is tantamount to the requirement that the string be comoving with the null geodesic, and these geometric considerations then lead to the

ANSWER: Provided that one works in Fermi coordinates and in the light-cone gauge, these two expansions agree to all orders.

This combined light-cone (world-sheet) and Fermi (space-time) gauge (i.e. writing the metric in Fermi coordinates) is, a priori, always available. Frequently, however, the light-cone gauge is imposed in conjunction with the conformal gauge, and this requires a metric that has a parallel null vector, as well as a coordinate system in which this is a coordinate vector ∂_v [**37**]. We show (section 8) that for all such metrics the latter requirement is actually compatible with the Fermi gauge. Since for this class of metrics canonical quantisation becomes particularly tractable in the light-cone and conformal gauge, this makes this all order equivalence especially appealing.

These results provide us with what seems to be a satisfactory overall geometric picture of the relation between string dynamics in a general curved background and in the Penrose-Fermi expansion of that background around its Penrose limit plane wave metric.

We should also note here in passing that the idea of basing a string expansion on an expansion around a geodesic is as such of course not new. Such an expansion was e.g. considered (to first order) in [**62, 63**], primarily for specific examples of metric backgrounds, and using (for reasons we do not fully comprehend) timelike instead of null geodesics. An expansion based on null geodesics was considered in [**64, 65, 66**], in the context of tensionless strings. While formally similar, our treatment of this expansion is quite different, both technically (using in an essential way the manifestly covariant Riemann and Fermi coordinate expansions) and in spirit. E.g. we argued in [**13, 15**] that the Penrose limit is most naturally understood as a particular large tension $\alpha' \to 0$ limit, and in the present context the Riemann coordinate (derivative) expansion we employ can, as usual, be translated into an α' expansion.

2. Covariant String Expansion Around a Regular String Background Solution

Our point of departure is the Polyakov action

$$S[X,h] = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{h}h^{ij}g_{\mu\nu}(X)\partial_i X^\mu\partial_j X^\nu, \tag{4.1}$$

for a string moving in the D-dimensional curved space-time background described by the metric $g_{\mu\nu}$, with $X^\mu = X^\mu(\tau,\sigma)$ the string embedding variables corresponding to the target space coordinates x^μ, and h_{ij} the world-sheet metric. Throughout this paper, with the exception of section 4, we work in the conformal gauge $h_{ij} = e^\phi\eta_{ij}$, leading to the sigma-model action (the conformal factor e^ϕ drops out of all subsequent equations)

$$S[X] = \frac{1}{2\pi\alpha'}\int d^2\sigma g_{\mu\nu}(X)\partial^i X^\mu\partial_i X^\nu. \tag{4.2}$$

The equations of motion (e.o.m.)

$$\nabla^i\partial_i X^\mu = \partial^i\partial_i X^\mu + \Gamma^\mu_{\nu\lambda}(X)\partial^i X^\nu\partial_i X^\lambda = 0 \tag{4.3}$$

have to be supplemented by the constraints

$$g_{\mu\nu}(X)\partial_\pm X^\mu\partial_\pm X^\nu = 0, \tag{4.4}$$

written here in world-sheet light-cone coordinates $\sigma^\pm = (\sigma\pm\tau)/\sqrt{2}$.

We will now expand the action covariantly around a background string solution X_B^μ of (4.3). The standard technique for this is the Riemann coordinate expansion $X^\mu = X_B^\mu + \xi^\mu$ discussed in detail in the sigma-model context in **[61]** and briefly recalled in section 7.

For the time being, in order to compare the Riemann coordinate expansion with the Penrose limit, we are only interested in the lowest non-trivial order of this expansion. The e.o.m. for the expansion fields ξ^μ (most readily obtained by expanding and then varying the action) are

$$\nabla_i\nabla^i\xi^\lambda + R^\lambda{}_{\mu\rho_1\nu}\partial_i X_B^\mu\partial^i X_B^\nu\xi^{\rho_1} = 0. \tag{4.5}$$

The corresponding first-order constraints are calculated by expanding (4.4) accordingly (4.49), and read

$$g_{\mu\nu}\nabla_\pm\xi^\mu\partial_\pm X_B^\nu = 0. \tag{4.6}$$

It is now convenient to introduce a frame $E_A^\mu(X_B)$ along the world-sheet. The components tangential to the world-sheet E_i^μ, $i \in \{+,-\}$ or $\{\tau,\sigma\}$, are chosen to be the derivatives along the coordinate lines of the conformal gauge coordinate system, viewed as the stringy generalisation of the geodesic affine parameter, i.e.

$$E_i = \partial_i, \quad E_i^\mu = \partial_i X_B^\mu, \tag{4.7}$$

completeted by an orthonormal frame E_a^μ, $a \in \{2,...,D-1\}$ (determined up to transverse orthogonal frame rotations), such that

$$g_{\mu\nu}E_i^\mu E_j^\nu = \bar{h}_{ij}, \quad g_{\mu\nu}E_i^\mu E_a^\nu = 0, \quad g_{\mu\nu}E_a^\mu E_b^\nu = \delta_{ab}. \tag{4.8}$$

Thus $\bar{h}_{ij}$ is the induced metric on the classical world-sheet background (constrained to be conformally flat by the conformal gauge condition). The string e.o.m (4.3) can now simply be written as $\nabla^i E_i = 0$, replacing the auto-parallelity condition $\nabla_\tau E_\tau = 0$ of a geodesic. They can be supplemented by the integrability conditions $\epsilon^{ij}\nabla_i E_j = 0$, which are due to the fact that the E_i are coordinate vectors. In terms of the world-sheet light-cone coordinates $\sigma^{\pm}$, these two equations can then be written in the condensed and useful form

$$\nabla_{\pm} E_{\mp} = 0. \tag{4.9}$$

After decomposition of the expansion fields into their tangential and transverse components,

$$\xi^{\mu} = \xi^A E^{\mu}_A = \xi^i E^{\mu}_i + \xi^a E^{\mu}_a, \tag{4.10}$$

one can reformulate the action, e.o.m. and the constraints in frame component form. Using (4.8) and (4.9), we find for the latter

$$\bar{h}_{+-}\partial_{\pm}\xi^{\mp} - g_{\mu\nu}E^{\mu}_a \nabla_{\pm} E^{\nu}_{\pm}\xi^a = 0. \tag{4.11}$$

These constraints can be solved for the (tangential, longitudinal) light-cone components $\xi^{\pm}$, up to the residual gauge freedom $\xi^{\pm} \to \xi^{\pm} + f^{\pm}(\sigma^{\pm})$. Therefore their e.o.m. must be satisfied identically by virtue of the constraints. Indeed, after a lengthy calculation we find that the tangential components of (4.5) are just the derivatives of (4.11), i.e.

$$\partial_{\mp}(\bar{h}_{+-}\partial_{\pm}\xi^{\mp} - g_{\mu\nu}E^{\mu}_a \nabla_{\pm} E^{\nu}_{\pm}\xi^a) = 0. \tag{4.12}$$

Furthermore, since the tangential components $\xi^{\pm}$ appear in the transverse components of the e.o.m. (4.5)

$$\begin{aligned} &\partial_+\partial_-\xi^a + g_{\mu\nu}\left(E^{a\mu}\nabla_+ E^{\nu}_+\partial_-\xi^+ + E^{a\mu}\nabla_+ E^{\nu}_b\partial_-\xi^b + E^{a\mu}\nabla_- E^{\nu}_-\partial_+\xi^- + E^{a\mu}\nabla_- E^{\nu}_b\partial_+\xi^b\right) \\ &\qquad + \frac{1}{2}g_{\mu\nu}\left(E^{a\mu}\nabla_+\nabla_- E^{\nu}_b\xi^b + E^{a\mu}\nabla_-\nabla_+ E^{\nu}_b\xi^b\right) + \frac{1}{2}R^a{}_{+b-}\xi^b + \frac{1}{2}R^a{}_{-b+}\xi^b = 0 \end{aligned} \tag{4.13}$$

only via their derivatives $\partial_{\pm}\xi^{\mp}$, we can use the constraints (4.11) to completely eliminate them. One then finds the purely transverse e.o.m.

$$\begin{aligned} \Big(\frac{1}{2}\partial^i\partial_i\delta^a_b \; &+ g_{\mu\nu}E^{a\mu}\nabla^i E^{\nu}_b\partial_i + g_{\mu\nu}\nabla^i E^{a\mu}\nabla_i E^{\nu}_b - (g_{\mu\nu}\nabla^i E^{a\mu}E^{\nu}_c)(g_{\lambda\kappa}E^{c\lambda}\nabla_i E^{\kappa}_b) \\ &+ \frac{1}{2}g_{\mu\nu}E^{a\mu}\nabla^i\nabla_i E^{\nu}_b + \frac{1}{2}R^{ai}{}_{bi}\Big)\,\xi^b = 0. \end{aligned} \tag{4.14}$$

Thus we have shown that, to first order in an expansion around a classical string configuration X_B in an arbitrary curved background, the tangential fluctuations can be explicitly solved for and eliminated from the e.o.m. for the true dynamical transverse degrees of freedom by virtue of the constraints (4.11).

We conclude this section with two comments on these observations:

(1) First of all, the fact that the tangential components $\xi^{\pm}$ can, in principle, be eliminated to first order is of course related to the underlying world-sheet diffeomorphism invariance. The crucial point here is that (4.11) shows how they can explicitly, and thus in practice, be eliminated in the already partially gauge fixed (conformal gauge) theory. This should be contrasted with the world-sheet covariant approach, e.g. based on the Nambu-Goto action, in which the tangential components, identified to first order with generators of world-sheet diffeomorphisms, can be set to zero (or drop out of the equations) by virtue of the world-sheet diffeomorphism invariance (for a geometrically transparent discussion of these issues see e.g. **[67, 68]**). However, this is no longer possible (or true) at higher orders in the expansion, which, in contrast to the first order, encode information

beyond mere infinitesimal deviations of nearby strings, and thus are not (to the same extent) susceptible to world-sheet diffeomorphisms. Thus if one wants to go to higher orders (as we will eventually do in section 4), the simplest way to control the world-sheet diffeomorphisms is to start with a gauge fixed action and to then simply expand it together with the constraints, exactly as we have done here to first order.

(2) Secondly, this is evidently quite reminiscent of the standard treatment of strings in the light-cone gauge, available for plane wave (or more general pp-wave or Brinkmann metric) backgrounds [**37**]. We will pursue this analogy in the subsequent section. To that end it will be useful to rewrite (4.14) in a manner that makes the underlying geometric structure more manifest, by introducing the gauge covariant derivative w.r.t. transverse frame rotations D_i and the extrinsic curvature of the world-sheet K^a_{ij},

$$D_i\xi^a = \partial_i\xi^a + g_{\mu\nu}E^{a\mu}\nabla_i E^\nu_b\xi^b, \qquad K^a_{ij} = g_{\mu\nu}E^\mu_i\nabla_j E^{\nu a} \quad . \tag{4.15}$$

In terms of these, (4.14) can be written more transparently as (see e.g. [**67**])

$$\bar{h}^{ij}D_iD_j\xi^a + \bar{h}^{ij}\bar{h}^{kl}K^a_{ik}K_{jlb}\xi^b + \bar{h}^{ij}R^a{}_{jbi}\xi^b = 0. \tag{4.16}$$

3. Transition from Strings to Null Geodesics as Background Fields

As mentioned above, the explicit elimination of the light-cone degrees of freedom $\xi^\pm$ from the first order string expansion by virtue of the constraints is strikingly reminiscent of the string e.o.m. in a Penrose limit expansion of the metric whose first order is the plane wave metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 2dx^+dx^- - R_{+a+b}(x^+)x^ax^bdx^+dx^+ + \delta_{ab}dx^adx^b. \tag{4.17}$$

Imposing the conformal gauge, the e.o.m. for $X^+(\tau,\sigma)$ is just the free wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^+ = 0, \tag{4.18}$$

and one can fix the residual world-sheet diffeomorphism invariance by choosing the light-cone gauge $X^+(\tau,\sigma) = \tau$. In this gauge, X^- is determined by the constraints

$$\begin{aligned} &\dot{X}^- - \frac{1}{2}R_{a+b+}X^aX^b + \frac{1}{2}\delta_{ab}(\dot{X}^a\dot{X}^b + X^{a\prime}X^{b\prime}) = 0 \\ &X^{-\prime} + \delta_{ab}\dot{X}^aX^{b\prime} = 0, \end{aligned} \tag{4.19}$$

and its e.o.m.

$$(\partial_\tau^2 - \partial_\sigma^2)X^- + 2R_{+ab+}\partial_\tau X^aX^b + \frac{1}{2}\partial_+R_{+ab+}X^aX^b = 0 \tag{4.20}$$

is then, as in section 2, identically satisfied by virtue of the constraints. The e.o.m. for the remaining transverse variables X^a are simply

$$(-\partial_\tau^2 + \partial_\sigma^2)X^a - R^a{}_{+b+}(\tau)X^b = 0. \tag{4.21}$$

Now these equations are quite similar to the transverse equations of motion (4.14,4.16), the difference between the two being mainly due to the complicated extrinsic curvature information of the background string X_B encoded in the second term of (4.16).

Thinking of strings as probes of the background geometry, one is tempted to say that the complicated (extrinsic) geometry of the probe itself obscures or contaminates the background geometry. This becomes most obvious in flat space where the first order string expansion equations about an excited string look much more complicated than the exact string equations themselves. On top of that, for generic curved backgrounds it is typically very hard to find even one exact solution X_B of the non-linear string e.o.m.

It is thus legitimate to ask if there is not a better way to perform a string expansion, one which rids us of all the (for present purposes largely superfluous) geometric information encoded in the extrinsic geometry of the string. Of course the first guess is to try a simpler background X_B, ideally an object with vanishing extrinsic curvature satisfying the exact string e.o.m. and constraints. All of these conditions are satisfied by choosing $X_B(\tau,\sigma) = \gamma(\tau)$ to be a null geodesic since

- for $X_B(\tau,\sigma) = \gamma(\tau)$, the e.o.m. (4.3) reduce to the geodesic equation;
- the constraints (4.4) reduce to the condition that this geodesic be null;
- the extrinsic curvature (4.15) of $\gamma(\tau)$ vanishes, since a geodesic extremises proper time.

The validity of the first two statements is obvious. As regards the third claim, note that in general an extremal submanifold is characterised by the vanishing of the trace of the extrinsic curvature. For a one-dimensional object this is equivalent to vanishing of the extrinsic curvature itself, the condition $K^a_{\tau\tau} = 0$ being just another way of writing the geodesic equation.

As we will see in the following, this choice of background will remedy all the shortcomings mentioned above and, in the end, lead to a first order string expansion equation of the form (4.21).

First of all we need to address the issue how to formulate the string expansion around this somewhat degenerate (because σ-independent) string background $X_B(\tau,\sigma) = \gamma(\tau)$. It turns out that simply making the replacement $X^\mu_B \to \gamma^\mu$, while retaining the τ and σ-dependence of ξ, so that e.g.

$$\partial_\tau X^\mu_B = \dot\gamma^\mu \qquad \partial_\sigma X^\mu_B = \partial_\sigma \gamma^\mu = 0 \qquad \nabla_\sigma \xi^\mu = \partial_\sigma \xi^\mu, \tag{4.22}$$

yields valid expansions of the action, constraints and the e.o.m. Therefore we get from (4.5) the e.o.m.

$$(-\nabla^2_\tau + \partial^2_\sigma)\xi^\lambda - R^\lambda{}_{\mu\rho_1\nu}\dot\gamma^\mu\dot\gamma^\nu\xi^{\rho_1} = 0 \tag{4.23}$$

while the constraints (4.6) reduce to

$$g_{\mu\nu}\nabla_\tau\xi^\mu\dot\gamma^\nu = 0 \qquad\qquad g_{\mu\nu}\partial_\sigma\xi^\mu\dot\gamma^\nu = 0. \tag{4.24}$$

Using the geodesic equation of motion, these constraints can be integrated to $g_{\mu\nu}\xi^\mu\dot\gamma^\nu = c$ with some constant c. We will now show that this constant can be set to zero. Assume a general solution $\xi(\tau,\sigma)$ of the e.o.m. (4.23) and the constraints (4.24), and consider the shifted expansion vector $\tilde\xi(\tau,\sigma) = \xi(\tau,\sigma) - c\xi_0(\tau)$, where $\xi_0(\tau)$ satisfies the ordinary geodesic deviation equation with respect to γ, and is normalised according to $g_{\mu\nu}\xi^\mu_0\dot\gamma^\nu = 1$. Then $\tilde\xi^\mu$ still satisfies the e.o.m. (4.23), but the constraint is

$$g_{\mu\nu}\tilde\xi^\mu\dot\gamma^\nu = 0. \tag{4.25}$$

In the following we consider two solutions of the first order string expansion to be equivalent if they differ only by a solution of the mere geodesic deviation equation, corresponding essentially just to a rigid displacement of the background geodesic, and consistently set $c = 0$.

Further simplifications arise after introduction of a parallel transported quasi-orthonormal frame E^A_μ (with $E^\mu_+ = \dot\gamma^\mu$) along the null geodesic γ, as in (4.50), since one then has, expanding $\xi^\mu = \xi^A E^\mu_A$ in this basis, $\nabla_\tau\xi^\mu = (\partial_\tau\xi^A)E^\mu_A$, so that all covariant derivatives can be replaced by partial derivatives acting on the frame components. Hence in frame components the e.o.m. (4.23) are simply

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^A + R^A{}_{+B+}\xi^B = 0, \tag{4.26}$$

while the choice $c = 0$ (4.25) is tantamount to $\xi^-(\tau,\sigma) = 0$. This condition is strictly analogous to the standard condition one imposes in the construction of the transverse geodesic deviation matrix [**42**] ($Z^- = 0$ in the notation of [**25**, section 2.1]). Thus, for the individual frame components one finds

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^+ = -R^+{}_{+B+}\xi^B = -R^+{}_{+-+}\xi^- - R^+{}_{+a+}\xi^a = -R^+{}_{+a+}\xi^a \tag{4.27}$$

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^- = -R^-{}_{+B+}\xi^B \equiv 0 \tag{4.28}$$

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^a = -R^a{}_{+B+}\xi^B = -R^a{}_{+-+}\xi^- - R^a{}_{+b+}\xi^b = -R^a{}_{+b+}\xi^b. \tag{4.29}$$

In particular, the transverse equations (4.29) are now identical to the exact transverse string equations (4.21) in a plane wave background. As regards the equation for ξ^-, on the other hand, comparison with the exact equation (4.20) shows that $\xi^- = 0$ is only a solution to the e.o.m. to lowest order in the Riemann expansion - consistent with the fact that in the scaling (4.57) leading to the Penrose plane wave limit X^- is treated as higher order relative to the X^a.

We conclude that the exact transverse string equations in the first order Penrose-Fermi expansion of the metric $g_{\mu\nu}$ around γ, i.e. in the Penrose limit plane wave metric associated to $g_{\mu\nu}$ and γ, are equivalent to the transverse first-order string equations obtained by expanding the string embedding fuctions around a null geodesic γ in the original background $g_{\mu\nu}$.

4. The Correspondence to All Orders

To what degree and for which metric/geodesic backgrounds can we expect the correspondence between the string expansion and the Penrose-Fermi expansion, which we established above to first order, to be valid at higher orders? To answer this question it is worthwhile to take a step back and compare the geometric set-up in both cases. Although the underlying interpretation is that of an expansion of the embedding variables on the one hand, and of the metric on the other, in the end it all reduces to a different prescription for how to describe the locus of nearby strings in terms of geodesic distance. This is mirrored by the different adapted coordinate systems used, i.e. Riemann vs. Fermi coordinates.

The Riemann coordinates ξ^+, ξ^- and ξ^a, used as the embedding variables in the string expansion, describe the instantaneous distance to a lightlike particle $\gamma(\tau)$. The somewhat awkward feature of this coordinate system (in the present context) is that, as this particle moves along γ, these coordinates changes (differentiably) with the affine parameter, i.e. with time.

The Penrose-Fermi expansion, on the other hand, is based on Fermi coordinates x^+, x^- and x^a adapted to the null geodesic γ [**26**]. In Fermi coordinates, one measures distance w.r.t. the null geodesic as a one-dimensional object. To this end space-time is foliated into transverse hypersurfaces which are parametrised by the affine parameter, promoted to the Fermi coordinate $x^+ = \tau$, and covered with $D-1$ dimensional, time-independent Riemann coordinates x^- and x^a around the intersection point of geodesic and hypersurface.

At a given but fixed time $\tau = \tau_0$, the position of the string is described by

$$X^\mu(\tau_0,\sigma) = \gamma^\mu(\tau_0) + \Delta X^\mu(\xi((\tau_0,\sigma)), \tag{4.30}$$

and generically $\xi^\mu(\tau_0,\sigma)$ will not lie in the corresponding transverse hypersurface, because the string is not comoving with the null geodesic. In that case, the first construction (Riemann coordinates), in which one simply has $X^\mu(\tau_0,\sigma) = \gamma^\mu(\tau_0) + \xi^\mu(\tau_0,\sigma)$ (4.44), is more convenient and efficient than the Fermi construction, as it accounts for the free movement of the string in space-time.

However, this discussion also shows that both approaches should agree completely if the string is actually confined to comove with the null geodesic. To make this more precise, note that comovement in terms of Fermi coordinates is equivalent to

$$X^+(\tau,\sigma) = \tau, \tag{4.31}$$

i.e. precisely the light-cone gauge condition, whereas in the Riemann string expansion it simply means

$$\xi^+(\tau,\sigma) = 0. \tag{4.32}$$

Now, by construction the transverse Fermi coordinates $(x^{\bar{a}}) = (x^-, x^a)$ are equal to the remaining transverse Riemann coordinates (4.51),

$$x^{\bar{a}} = \xi^{\bar{a}}_{\gamma(\tau)}. \tag{4.33}$$

Thus for comoving strings (light-cone gauge), the two prescriptions to measure the locus of the string, namely transverse distance from the geodesic, indeed agree. In that special case it is enlighting to recalculate the manifest covariant form of the string expansion using Fermi and not Riemann coordinates. As we will show, this significantly simplifies the identification of the tensorial structures at intermediate steps of the calculation, and demonstrates that Fermi coordinates are the ideal reference system to describe the perturbative string expansion in the light-cone gauge.

To see this, recall first that in Riemann coordinates one has the simple relationship $X^\mu(\gamma,\xi) = \gamma^\mu + \xi^\mu$ for the embedding variable, while the expression for its τ-derivative is more complicated (esentially because Riemann coordinates are anchored at a fixed base-point and thus change as one moves along γ) and given by the infinite series (4.46,4.47).

In Fermi coordinates, on the other hand, the initial expression for the expansion of $X^A(\gamma,\xi)$ is somewhat more complicated, being given by the infinite series (4.55), but since this expression holds along the entire null geodesic, no new terms are generated when taking the τ-derivative (4.56). The simple (but crucial) observation is now that, upon using (4.32), this expansion (4.55) collapses to the simple result

$$X^A(\gamma,\xi) = \delta^A_+\tau + \delta^A_{\bar{a}}\xi^{\bar{a}}, \tag{4.34}$$

in accordance with (4.31) and (4.33) and the statement that on the transverse hypersurface $\xi^+ = 0$ through the event $\gamma(\tau)$ Fermi coordinates are identical to Riemann coordinates around $\gamma(\tau)$. Moreover, as a Fermi expression, (4.34) is valid not only at a certain time τ but all along γ. Therefore its time derivative does not include new terms and one simply has

$$\partial_\tau X^A(\gamma,\xi) = \delta^A_+ + \delta^A_{\bar{a}}\partial_\tau\xi^{\bar{a}}. \tag{4.35}$$

as well as (evidently)

$$\partial_\sigma X^A(\gamma,\xi) = \delta^A_{\bar{a}}\partial_\sigma\xi^{\bar{a}}. \tag{4.36}$$

Thus, provided that one imposes the light-cone gauge one can simultaneously use the attractive features of Riemann and Fermi coordinates, i.e. one can eat one's cake and have it too, and the covariant expansions of $X^A(\tau,\sigma)$ (4.34) and its derivatives (4.35,4.36) become as simple as they could possibly be.

Moreover, by virtue of the identification (4.33), the tranverse $\xi^+ = 0$ Riemann coordinate expansion (4.45) of the metric in terms of $\xi^{\bar{a}}$ is equivalent to the expansion (4.54) of the metric in Fermi coordinates.

Note that, in order to arrive at this conclusion, we only needed to impose the space-time diffeomorphism gauge condition that the metric be written in Fermi coordinates as well as the world-sheet diffeomorphism light-cone gauge condition $X^+ = \tau$. This is always possible.

Putting everything together, we conclude that in this combined light-cone (world-sheet) and Fermi (space-time) gauge, the expansion of the string e.o.m. around the null geodesic γ becomes identical, to all orders, actually term by term, to the light-cone gauged string theory e.o.m. in the Fermi coordinate expansion of the metric. Since the expansions agree term by term, this conclusion is valid both for the ordinary Fermi expansion (4.54) as well as for the Penrose-Fermi expansion (4.58) of the metric (whose lowest order term is the Penrose limit plane wave) because the latter is in essence just a reordering of the former.

Frequently, the light-cone gauge is imposed in conjunction with the conformal gauge, and this imposes strong constraints on the background geometry which lead to the usual simplifications in the subsequent canonical quantisation. It is well known that the metrics for which the light-cone gauge can be imposed in addition to the conformal gauge are metrics of the Brinkmann form (4.59) admitting a parallel null vectorfield ∂_v [**37**]. Thus, if we insist on the conformal gauge (depending on the form of the metric, there may also be other suitable gauge choices leading to a tractable canonical formalism, see e.g. [**48**]), we need to understand for which Brinkmann metrics we can introduce Fermi cooordinates compatible with the above Brinkmann form. In section 8 we establish the optimal result along these lines, namely that demanding the Fermi gauge, associated with any one of a spacetime filling congruence of null geodesics, imposes no further restrictions on the metric beyond those required by the light-cone and conformal gauge alone.

5. Example: Riemann Expansion of the Plane Wave String Equations

To illustrate the above argument regarding the equivalence of the Riemann and Penrose-Fermi expansions, as a simple example we reconsider the plane wave in Brinkmann coordinates (4.17). These Brinkmann coordinates are Fermi coordinates for the central null geodesic $x^+ = \tau, x^{\bar{a}} = 0$, and the exact string e.o.m. and constraints, given in (4.18)-(4.21), are at most quadratic in the transverse fields $X^{\bar{a}}$. Their Riemann coordinate expansion, on the other hand, is *a priori* given by an infinite series. Thus our claim that these two expansions are (term by term) equivalent may at first appear to be puzzling.

To see what is going on, let us take a closer look at the second order Riemann coordinate string expansion of the e.o.m. (4.48) around the null geodesic. Using the rules (4.22), one finds

$$(4.37)\quad (-\nabla_\tau^2 + \partial_\sigma^2)\xi^\lambda - R^\lambda{}_{\mu\rho_1\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu\xi^{\rho_1} - 2R^\lambda{}_{\rho_1\rho_2\mu}\dot{\gamma}^\mu\nabla_\tau\xi^{\rho_1}\xi^{\rho_2} - \frac{1}{2}\left[\nabla_{\rho_1}R^\lambda{}_{\mu\rho_2\nu} + \nabla_\mu R^\lambda{}_{\rho_1\rho_2\nu}\right]\dot{\gamma}^\mu\dot{\gamma}^\nu\xi^{\rho_1}\xi^{\rho_2} + \mathcal{O}((\xi)^3) = 0.$$

Evaluating these in frame components, using the fact that for a plane wave the only nonvanishing component of the Riemann tensor is $R_{a+b+}(\tau)$, and after imposing the light-cone gauge $\xi^+ = 0$ (4.32), one finds the e.o.m.

$$\begin{aligned}(\partial_\tau^2 - \partial_\sigma^2)(\xi^+ = 0) + \mathcal{O}((\xi)^3) &= 0\\ (\partial_\tau^2 - \partial_\sigma^2)\xi^- + 2R_{+ab+}\partial_\tau \xi^a \xi^b + \frac{1}{2}\partial_+ R_{+ab+}\xi^a\xi^b + \mathcal{O}((\xi)^3) &= 0\\ (\partial_\tau^2 - \partial_\sigma^2)\xi^a + R^a{}_{+b+}\xi^b + \mathcal{O}((\xi)^3) &= 0\end{aligned} \tag{4.38}$$

and similarly the constraints

$$\begin{aligned}\dot{\xi}^- - \frac{1}{2}R_{a+b+}\xi^a\xi^b + \frac{1}{2}\delta_{ab}(\dot{\xi}^a\dot{\xi}^b + \xi^{a\prime}\xi^{b\prime}) + \mathcal{O}((\xi)^3) &= 0\\ \xi^{-\prime} + \delta_{ab}\dot{\xi}^a\xi^{b\prime} + \mathcal{O}((\xi)^3) &= 0.\end{aligned} \tag{4.39}$$

These equations are identical to the standard e.o.m. and constraints in Brinkmann/Fermi coordinates provided that all the higher order $\mathcal{O}((\xi)^{n\geq 3})$ terms in the Riemann expansion vanish. Thus the result of section 4 tells us that these terms have to be identically zero.

As a check on this geometric reasoning, in this case one can also establish the absence of these higher order terms in the Riemann coordinate expansion directly, by using some elementary combinatorial considerations similar to the kinds of arguments that are used to show **[29]** that plane wave (or pp-wave) backgrounds are exact solutions of string theory. Namely, as $\xi^+ = 0$, there are at most two contravariant + indices, stemming from $\dot{\gamma} = E^+$. An initial R_{+a+b} contributes two covariant indices. Each additional power of the Riemann tensor adds another two covariant + indices (since contractions are only possible over transverse indices), and each covariant derivative adds at least one, namely the +-derivative (the others add two as can be seen by direct inspection of the Christoffel symbols). One covariant + might be a free contravariant − index (in the e.o.m. for ξ^-). Thus, denoting by r the number of Riemannn tensors and by d the number of derivatives, we find the condition

$$2r + d - 1 \leq 2. \tag{4.40}$$

This implies that only terms with $r \leq 1$ and $d \leq 1$ can contribute, thus providing an alternative argument to the effect that the higher order terms in the expansion (4.38) are zero.

6. Outlook

For practical applications, the key consequence of our work is the observation that in the combined Fermi/light-cone gauge, the naive expansion of the string coordinates (4.34) and their derivatives (4.35,4.36) is manifestly covariant. This should provide additional insight into, and significant simplification of, calculations performed e.g. in the AdS/CFT context (e.g. by extending the Fermi expansion of $AdS_5 \times S^5$ **[26]** to a string theory expansion).

Applications of this procedure are, however, not limited to the Penrose limit AdS/CFT context. For example, it was noted in **[69, 70]** that the Penrose-Fermi expansion developed in **[26]**, with γ interpreted as a photon trajectory, provides the ideal setting for performing certain QED calculations (like vacuum polarisation) in a curved background. It was also remarked there that it would be interesting to perform analogous calculations in string theory. We expect the formalism that we have developed in this paper, a stringy generalisation of **[26]**, to be useful for that purpose.

The results obtained here also shed light on the propagation of strings in curved (and singular) backgrounds. For example, some of the observations in **[71]** regarding the string propagation through a big crunch / big bang singularity (namely that in the neighbourhood of such a cosmological singularity the string equations reduce to those in a plane wave) can be understood as a particular manifestation of the more general phenomenon that we have described here, since the plane wave in question is precisely the kind of singular homogeneous plane wave **[21]** that was shown in **[20, 25]** to arise generically as the Penrose limit of a space-time singuarity.

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7. Taylor Expansion in Riemann and Fermi Coordinates

7.1. Riemann Expansion. The covariant expansion of a general space-time tensor using Riemann coordinates is discussed in detail in **[61]**. Here we can restrict ourselfes to the embedding variables and the metric. First note that a coordinate difference $\Delta x^\mu = x^\mu - x^\mu_B$ of (nearby) points on the curved space-time manifold is an object whose transformation under space-time diffeomorphisms is not well defined. Thus a naive Taylor expansion in Δx^μ is bound to produce correct but nevertheless non-covariant equations. To circumvent this difficulty one can reparametrise $\Delta x^\mu(\xi)$ by a vector ξ sitting at x_B by means of the exponential map

$$x^\mu(x_B, \xi) = x^\mu_B + \Delta x^\mu(\xi) = (\mathrm{Exp}_{x_B}(\xi))^\mu. \tag{4.41}$$

As ξ^μ transforms as a vector, the ordinary Taylor expansion of the metric in terms of ξ^μ,

$$g_{\mu\nu}(x_B + \Delta x(\xi)) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial \xi^{\rho_1}} \cdots \frac{\partial}{\partial \xi^{\rho_n}} g_{\mu\nu}(x_B) \xi^{\rho_1} \cdots \xi^{\rho_n}, \tag{4.42}$$

has to be covariant, i.e. the coefficients can be re-expressed in terms of the curvature tensor and its covariant derivatives. Note, however, that in a general coordinate system the definition via the exponential map leads to a rather complicated dependence of $\Delta x(\xi)$ on ξ, namely

$$x^\mu(x_B, \xi) = x^\mu_B + \Delta x^\mu(\xi) = x^\mu_B + \xi^\mu - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^\mu_{\rho_1 \cdots \rho_n} \xi^{\rho_1} \cdot \xi^{\rho_n}, \tag{4.43}$$

where $\Gamma^\mu_{\rho_1 \cdots \rho_n} = \nabla_{\rho_1} \ldots \nabla_{\rho_{n-2}} \Gamma^\mu_{\rho_{n-1} \rho_n}$ and ∇_ρ means covariant differentiation w.r.t. lower indices only. We see that in order to evaluate (4.42) one would also have to expand the coordinate functions x^μ themselves.

The solution to this problem is to promote x_B to be the origin of a new coordinate system ξ^μ in which geodesics emanating from x_B are straight lines. In these Riemann coordinates by definition one has $\Delta x^\mu = \xi^\mu$ or, equivalently,

$$x^\mu(x_B, \xi) = x^\mu_B + \xi^\mu, \tag{4.44}$$

making them the natural choice of coordinate system to evaluate (4.42). Comparison of (4.43) and (4.44) shows that the symmetrised covariant derivatives of the Christoffel symbols vanish in Riemann coordinates, $\Gamma^\mu_{(\rho_1 \cdots \rho_n)} = 0$. From this relation one can iteratively express the partially symmetrised derivatives of the Christoffel symbols to arbitrary order in terms of the Riemann tensor, and then use these expressions to manifestly covariantise the expansion (4.42), leading to

$$g_{\mu\nu}(x_B+\xi) = g_{\mu\nu}(x_B) - \frac{1}{3}R_{\mu\rho_1\nu\rho_2}\xi^{\rho_1}\xi^{\rho_2} - \frac{1}{3!}\nabla_{\rho_1}R_{\mu\rho_1\nu\rho_2}\xi^{\rho_1}\xi^{\rho_2}\xi^{\rho_3} + \mathcal{O}((\xi)^4). \tag{4.45}$$

As a tensorial equation, this is now valid in any coordinate system.

We also need to evaluate the derivative of the embedding variables X^μ, i.e. of the expansion (4.43). Here it is important to note that, while the symmetrised derivatives of the Christoffel symbols vanish in Riemann coordinates, this is not true for their ordinary derivatives. Therefore the derivative of (4.43) w.r.t. some parameter τ, e.g. along a curve in space-time, leads to an infinite series in Riemann coordinates,

$$\partial_\tau X^\mu(X_B,\xi) = \partial_\tau(X^\mu_B + \Delta X^\mu(\xi)) = \partial_\tau X^\mu_B + \partial_\tau\xi^\mu - \sum_{n=2}^{\infty}\frac{1}{n!}\left(\partial_\nu\Gamma^\mu_{\rho_1\cdots\rho_n}\right)\xi^{\rho_1}\cdots\xi^{\rho_n}\partial_\tau X^\nu_B. \tag{4.46}$$

In manifestly covariant form this reads

$$\begin{aligned}\partial_\tau X^\mu(X_B,\xi) &= \partial_\tau(X^\mu_B+\xi^\mu) = \partial_\tau X^\mu_B + \nabla_\tau\xi^\mu \\ &+ \left[-\frac{1}{3}R^\mu{}_{\rho_1\nu\rho_2}\xi^{\rho_1}\xi^{\rho_2} + \frac{1}{12}\nabla_{\rho_1}R^\mu{}_{\rho_2\rho_3\nu}\xi^{\rho_1}\xi^{\rho_2}\xi^{\rho_3}\right]\partial_\tau X^\nu_B + \mathcal{O}((\xi)^4).\end{aligned} \tag{4.47}$$

Putting everything together, we can now write down the expansion of the string e.o.m. (4.3),

$$\begin{aligned}&\nabla_i\nabla^i\xi^\lambda + R^\lambda{}_{\mu\rho_1\nu}\partial_i X^\mu_B\partial^i X^\nu_B\xi^\rho_1 + 2R^\lambda{}_{\rho_1\rho_2\mu}\partial_i X^\mu_B\nabla^i\xi^{\rho_1}\xi^{\rho_2} \\ &+\frac{1}{2}\left[\nabla_{\rho_1}R^\lambda{}_{\mu\rho_2\nu} + \nabla_\mu R^\lambda{}_{\rho_1\rho_2\nu}\right]\partial_i X^\mu_B\partial^i X^\nu_B\xi^{\rho_1}\xi^{\rho_2} + \mathcal{O}((\xi)^3) = 0.\end{aligned} \tag{4.48}$$

and of the constraints (4.4),

$$\begin{aligned}&g_{\mu\nu}(2\nabla_\tau\xi^\mu\partial_\tau X^\nu_B + 2\nabla_\sigma\xi^\mu\partial_\sigma X^\nu_B + \nabla_\tau\xi^\mu\nabla_\tau\xi^\nu + \nabla_\sigma\xi^\mu\nabla_\sigma\xi^\nu) \\ &\qquad -R_{\mu\rho_1\nu\rho_2}\xi^{\rho_1}\xi^{\rho_2}(\partial_\tau X^\mu_B\partial_\tau X^\nu_B + \partial_\sigma X^\mu_B\partial_\sigma X^\nu_B) + \mathcal{O}((\xi)^3)) = 0 \\ &g_{\mu\nu}(\nabla_\tau\xi^\mu\partial_\sigma X^\nu_B + \nabla_\sigma\xi^\mu\partial_\tau X^\nu_B + \nabla_\tau\xi^\mu\nabla_\sigma\xi^\nu) - R_{\mu\rho_1\nu\rho_2}\xi^{\rho_1}\xi^{\rho_2}\partial_\tau X^\mu_B\partial_\sigma X^\nu_B + \mathcal{O}((\xi)^3)) = 0.\end{aligned} \tag{4.49}$$

7.2. Fermi Expansion. Riemann coordinates are most suitable to evaluate covariant Taylor expansions around a point in space-time. However, if one wishes to expand only transversally to a given geodesic γ, i.e. a one-dimensional object, Fermi coordinates are the most adequate tool. In the following we will restrict the discussion to the case of null Fermi coordinates, i.e. with γ a null geodesic, considered in [**26**] and constructed as follows. First one introduces a quasi-orthonormal frame E^A_μ,

$$ds^2|_\gamma = \eta_{AB}E^AE^B = 2E^+E^- + \delta_{ab}E^aE^b \tag{4.50}$$

parallel transported along γ, with $E^\mu_+ = \dot\gamma^\mu$. The tranversality condition is then implemented by $\xi^\mu_{\gamma(\tau)}E^+_\mu(\gamma(\tau)) = \xi^+_{\gamma(\tau)} = 0$, where $\xi^\mu_{\gamma(\tau)}$ is the vector defining the Riemann coordinate system around the point $\gamma(\tau)$. The rôle of ξ^+ is now played by the affine parameter of the geodesic τ, promoted to be the Fermi coordinate $x^+ = \tau$. The remaining Fermi coordinates are identical to the Riemann coordinates restricted to the transverse hypersurface, i.e.

$$x^{\bar a} = E^{\bar a}_\mu\xi^\mu_{\gamma(\tau)}\Big|_{\xi^+=0} = \xi^{\bar a}_{\gamma(\tau)}. \tag{4.51}$$

In Fermi coordinates, the Christoffel symbols as well as the symmetrised transverse components of their covariant or partial derivatives vanish all along γ,

$$\Gamma^C_{AB}\big|_\gamma = \partial_{(\bar{a}_1} \dots \partial_{\bar{a}_{n-2}} \Gamma^A_{\bar{a}_{n-1}\bar{a}_n)}\big|_\gamma = 0, \tag{4.52}$$

and not only at a certain point, as for Riemann coordinates. The price we have to pay for this is that this is no longer true for the symmetrised higher derivatives including the geodesic direction (a lower +-index). For example, while one obviously has $\Gamma^A_{BC,+} = 0$ by (4.52), one calculates e.g.

$$\Gamma^A_{(+B,C)} = R^A{}_{(BC)+}. \tag{4.53}$$

Similarly to the Riemann case, the derivatives of the Christoffel symbols can be used to determine the explicit expansion of the metric in terms of the components of the Riemann tensor restricted to the geodesic γ. To cubic order (for the quartic terms see [**26**]) one finds

(4.54)
$$\begin{aligned} ds^2 = \;& 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ & - R_{+\bar{a}+\bar{b}}\, x^{\bar{a}} x^{\bar{b}} (dx^+)^2 - \frac{4}{3} R_{+\bar{b}\bar{a}\bar{c}} x^{\bar{b}} x^{\bar{c}} (dx^+ dx^{\bar{a}}) - \frac{1}{3} R_{\bar{a}\bar{c}\bar{b}\bar{d}} x^{\bar{c}} x^{\bar{d}} (dx^{\bar{a}} dx^{\bar{b}}) \\ & - \frac{1}{3} R_{+\bar{a}+\bar{b};\bar{c}} x^{\bar{a}} x^{\bar{b}} x^{\bar{c}} (dx^+)^2 - \frac{1}{4} R_{+\bar{b}\bar{a}\bar{c};\bar{d}}\, x^{\bar{b}} x^{\bar{c}} x^{\bar{d}} (dx^+ dx^{\bar{a}}) - \frac{1}{6} R_{\bar{a}\bar{c}\bar{b}\bar{d};\bar{e}} x^{\bar{c}} x^{\bar{d}} x^{\bar{e}} (dx^{\bar{a}} dx^{\bar{b}}) \\ & + \mathcal{O}(x^{\bar{a}} x^{\bar{b}} x^{\bar{c}} x^{\bar{d}}) \end{aligned}$$

Turning now to the expansion of the coordinates and embedding variables, direct insertion of (4.53) into the expansion (4.43) leads to

$$x^A(\gamma,\xi) = \gamma^A + \Delta x^A(\xi) = \delta^A_+ \tau + \xi^A - R^A{}_{+\bar{c}+} \xi^+ \xi^+ \xi^{\bar{c}} - 2R^A{}_{\bar{b}\bar{c}+} \xi^+ \xi^{\bar{b}} \xi^{\bar{c}} + \mathcal{O}((\xi)^3). \tag{4.55}$$

In constrast to the Riemann expansion it contains terms of arbitrary high order in ξ^A (as long as $\xi^+ \neq 0$). However this expression is valid along γ. Accordingly we find, using (4.52), that no new terms appear after differentiation of the embedding variables,

$$\begin{aligned} \partial_\tau X^A(\gamma,\xi) &= \partial_\tau(\gamma^A + \Delta X^A(\xi)) \\ &= \delta^A_+ + \partial_\tau \xi^A - \partial_\tau(R^A{}_{+\bar{c}+} \xi^+ \xi^+ \xi^{\bar{c}}) - 2\partial_\tau(R^A{}_{\bar{b}\bar{c}+} \xi^+ \xi^{\bar{b}} \xi^{\bar{c}}) + \mathcal{O}((\xi)^3) \end{aligned} \tag{4.56}$$

7.3. Penrose-Fermi Expansion. In [**26**] the Fermi expansion of the metric around a null geodesic was used to define a covariant extension of the Penrose limit to higher orders, i.e. a Penrose-Fermi expansion. In a nutshell the prescription is to rescale the Fermi coordinates together with a conformal transformation of the metric

$$(x^+_\lambda, x^-_\lambda, x^a_\lambda) = (x^+, \lambda^2 x^-, \lambda x^a), \quad ds^2_\lambda = \frac{1}{\lambda^2} ds^2 \tag{4.57}$$

This leads to a reshuffling of the terms in the Fermi expansion whose zero'th order term in λ is the Penrose limit plane wave associated with the metric $g_{\mu\nu}$ and the null geodesic γ,

(4.58)
$$\begin{aligned} ds^2 = \;& 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+} x^a x^b (dx^+)^2 \\ & + \lambda \left[-2R_{+a+-}\, x^a x^- (dx^+)^2 - \frac{4}{3} R_{+bac}\, x^b x^c (dx^+ dx^a) - \frac{1}{3} R_{+a+b;c}\, x^a x^b x^c (dx^+)^2 \right] + \mathcal{O}(\lambda^2) \end{aligned}$$

8. Fermi Coordinates Compatible with the Brinkmann Form

Here we want to show that there always exist Fermi coordinates (x^+, x^-, x^a) which are compatible with the general (Brinkmann) form

$$ds^2 = 2du(dv + A(u, y^k)du + A_i(u, y^k)dy^i) + g_{ij}(u, y^k)dy^i dy^j \tag{4.59}$$

of a metric admitting a null parallel (and hence in particular Killing) vector ∂_v. This means that in this new coordinate system the metric has the same general form as above, and moreover has the features that (a) $x^+ = \tau$, $x^- = 0$, $x^a = 0$ is the basic null geodesic γ, (b) $\partial_+|_\gamma, \partial_-|_\gamma, \partial_a|_\gamma$ is a quasi-orthonormal parallel frame along γ, and (c) all the curves $x^+ = c^+$, $x^- = c^- t$, $x^a = c^a t$ with c^+, c^-, $c^a = const.$ are also geodesics.

In order to identify a suitable null geodesic γ (actually, as we will see, a whole congruence of null geodesics), we first cast the Brinkmann metric (4.59) into the Rosen coordinate form

$$ds^2 = 2dudv + g_{ij}(u, y^k)dy^i dy^j, \tag{4.60}$$

which is always possible [**29**]. It is now readily checked that any curve $u = p^v \tau$, $p^v \neq 0$, with $v, y^i = const.$ is a null geodesic. Pick one of this congruence, set $p^v = 1$, call it γ, shift v so that γ sits at $(v = 0, y^i = y^i_0)$, and introduce the corresponding Fermi coordinate $x^+ = u = \tau$.

Moreover, $x^+ = c^+$ $(p^v = 0)$ is also a solution to the geodesic e.o.m. and thus the hypersurfaces $x^+ = c^+$ can be generated by transverse geodesics emanating from the intersection point with γ. ∂_v is parallel and hence, in particular, parallel transported along γ. Choose $E_+ = \dot\gamma$ and $E_- = \partial_v$ and complete it by $E_a = E^i_a \partial_i$ to a quasi-orthonomal parallel frame along γ. In any one of the spacelike codimension 2 surfaces $v, x^+ = const.$ spanned by the y^i, with induced metric $g_{ij}(x^+, y^k)$, we introduce Riemann normal coordinates x^a around the point (y^i_0) w.r.t. the frame $E_a(x^+)$, i.e. such that $\partial_a|_\gamma = E_a$. Since $g_{ij}(x^+, y^k)$ is independent of v, this can be achieved by a v-independent, but generically x^+-dependent, coordinate transformation of the form $x^a = x^a(x^+, y^i)$. Then the metric takes the form

$$ds^2 = 2dx^+(dv + A(x^+, x^c)dx^+ + A_a(x^+, x^c)dx^a) + g_{ab}(x^+, x^c)dx^a dx^b. \tag{4.61}$$

Note that, while this has the same general form as (4.59), the coordinates are now such that (a) $x^+ = \tau$, $v = 0$, $x^a = 0$ is the Fermi null geodesic γ, and (b) $\partial_+, \partial_v, \partial_a$ is parallel quasi-orthonormal along γ. Furthermore, the geodesic e.o.m. for the x^a are satisfied by $x^a = c^a t$ with $x^+ = c^+$, since A and A_a do not contribute for $\dot x^+ = 0$ and the x^a are spatial Riemann coordinates for g_{ab}.

To completely satisfy criterion (c), we still need to replace v by a coordinate x^- whose geodesic e.o.m. are fulfilled by $x^- = c^- t$, $x^a = c^a t$ and $x^+ = c^+$ for all c^+, c^-, c^a, and such that ∂_- is quasi-orthonormal parallel along γ. The only coordinate transformation left to us is a shift $x^- = v + P(x^+, x^a)$. Note that this shift changes only A and A_a in (4.61) and therefore does not effect the e.o.m. for x^a if $\dot x^+ = 0$. Futhermore, if P is at least quadratic in the x^a, the Jacobian of the coordinate transformation is trivial on γ, and therefore $\partial_+, \partial_-, \partial_a$ is parallel along γ because there it is identical to the above parallel frame $\partial_+, \partial_v, \partial_a$.

After the shift, the x^- e.o.m. is

$$(4.62)\quad \begin{aligned} 2\ddot{x}^- &= -\partial_a\partial_b P(c^+, c^d t)c^a c^b - \frac{d}{dt}(A_a(c^+, c^d t)c^a + \partial_{x^+} g_{ab}(c^+, c^d t)c^a c^b \\ &= -\partial_a\partial_b P(c^+, c^d t)c^a c^b - \partial_a(A_b(c^+, c^d t)c^a c^b + \partial_{x^+} g_{ab}(c^+, c^d t)c^a c^b \end{aligned}$$

where we used $\dot{x}^+ = 0$. We want the right side to vanish. Rescaling c^a by t we get

$$(4.63)\quad \partial_a\partial_b P(c^+, c^d)c^a c^b = -\partial_a A_b(c^+, c^d)c^a c^b + \partial_{x^+} g_{ab}(c^+, c^d)c^a c^b \equiv D_{ab}(c^d)c^a c^b.$$

Expanding both sides in a Taylor series in the c^a, comparison of coefficients gives

$$(4.64)\quad \partial_{(a_1} \cdots \partial_{a_n)} P(c^+, 0) = \partial_{(a_1} \cdots \partial_{a_{n-2}} D_{a_{n-1} a_n)}(c^+, 0).$$

This can always uniquely be solved for given D_{ab}. Finally, as A_a is at least linear in the x^a (the metric restricted to γ is flat) and $\partial_{x^+} g_{ab}$ is at least quadratic (Riemann coordinate metric), P is also at least quadratic in the x^a, as required.

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Subsequent Alterations

The original symbols for the background meric $G_{\mu\nu}$, G_{ij}, G_{ab} and the induced metric g_{ij} have been changed to $g_{\mu\nu}$, g_{ij}, g_{ab} and $\bar{h}_{ij}$ respectively.

CHAPTER 5

Probing space-time singularities

After some introductory remarks about different probes of space-time singularities in section 1, we review the universality of Penrose limits of powerlaw-type singularities [**25**] in section 2 focussing on the interpretation in terms of null geodesic congruences. Finally, in section 3 we motivate the transition from null congruences to massless scalar fields and discuss shortly the technical similarities and differences between both probes.

1. Introduction

The true nature of space-time singularities is, among others [**72**], a very old question of general relativity. A necessary prerequisite one has to address is a physically convincing and mathematically tractable definition of a singularity [**73**]:

> *Intuitively, a space-time singularity is a "place" where the curvature "blows up" or other "pathological behaviour" of the metric takes place. The difficulty in making this notion into a satisfactory, precise definition of a singularity stems from the above terms placed in quotes.*

Certainly, there are many cases where the appearance of a singular behaviour is signalled by the unboundedness of some curvature invariant. However, this might occur at infinite geodesic distance and in this case one would not necessarily want to identify it with a true space-time singularity. Furthermore, there exist many space-times where all polynomial curvature scalars vanish whereas certain frame components of the Riemann curvature tensor still blow up, the most well-known example thereof being the plane waves themselves (cf. chapter 2 section 1.2 and Appendix A section 6).

An approach which proved to be very successful in tackling these problems is to test the curvature by different kinds of probes, i.e. to characterise a singularity by the breakdown or *non-uniqueness of the time-evolution* in question. Obviously, this is a physically appealing approach as it intrinsically encodes a sort of "measurement". The most prominent and successful example is the criterion of *geodesic incompleteness* where the locus of the singularity is defined to be the endpoint (finite range of the affine paremeter) of an *inextendible geodesic.*

Technically much harder to use are stringy probes, the main advantage lying in the additional "internal degrees of freedom" which can become infinitely excited by tidal forces while approaching a singularity[**37**].

As a compromise one can use tensorial probes of a given rank, the simplest and most often employed representative thereof being the scalar field. As we will see in the next chapter in this case the notion of time evolution breakdown is primarily related to the need to impose additional, non-physically determined boundary conditions at the singularity.

After identifying the singularity's "place" (as the endpoint of a geodesic say) one generally continues to classify it according to "curvature" behaviour, e.g. (a) into scalar curvature singularities (polynomial curvature scalars blow up), (b) parallelly propagated curvature singularities (curvature scalars stay finite but the components of the curvature tensor and its derivatives w.r.t. a parallelly propagated orthonormal frame blow up) or (c) non-curvature singularities, seemingly forgetting about the probe and measurability. This however is not true. Note that the operator describing geodesic deviation, i.e. tidal forces, is given precisely by the curvature tensor in a parallel propagated orthonormal frame and it is the singular behaviour of geodesic congruences (clouds of test particles) encoded in (the simplest example of) (b) above which lead to the celebrated *singularity theorems* [**42, 73**].

After the (verbal) transition from curvature components to time-evolution operators of congruences it is tempting to compare the results to the time evolution of other probes. The obvious candidate to compare to a space-time filling congruence is a scalar field. Indeed, as we will see the operators governing the geodesic deviation around a null geodesic, i.e. the Penrose limit wave profile A_{ab} (cf. chapter 2 section 3.2) and the operators describing a massless scalar field share a rather unexpected universal behaviour in the vicinity of physically reasonable space-time singularities.

2. The Universality of Penrose Limits of Power-Law Type Singularities

In [**20**] the Penrose limit of different singularities, i.e. the time evolution operator of the deviation equation along a null geodesic ending in the singularity, was calculated with the surprising result that every such null geodesic in any space-time considered leads to a Penrose limit wave profile with a leading order $(x^+)^{-2}$-behaviour. In a following publication [**25**] the authors showed that a large class of physical singular space-times of spherically symmetric type obeys this *universal powerlaw behaviour*, in particular

(1) Penrose limits of metrics with singularities of power-law type show a universal $(x^+)^{-2}$-behaviour near the singularity,

$$A_{ab}(x^+) \to c_a \delta_{ab} (x^+)^{-2}. \tag{5.1}$$

provided that the strict Dominant Energy Condition (DEC) is satisfied.

We will see below that the behaviour $\sim (x^+)^{-\alpha}$ with $\alpha = 2$ is actually generic and thus always dominates possible but subleading $\alpha < 2$ terms. However, and that is the interesting point, the strict DEC is needed to exclude profiles with $\alpha > 2$.

It was stressed that the corresponding plane wave

$$2dx^+dx^- + (x^+)^{-2} c_a \delta_{ab} x^a x^b dx^+ dx^+ + \delta_{ab} dx^a dx^b \tag{5.2}$$

has a quite remarkable property, namely

(2) such plane waves are singled out by their scale invariance, reflected e.g. in the isometry $(x^+, x^-) \to (\lambda x^+, \lambda^{-1} x^-)$ of the metric (5.1,5.2)

described by the Killing vector

$$x^+\partial_+ - x^-\partial_-, \tag{5.3}$$

which in combination with the usual Heisenberg algebra of Killing vectors of plane waves (cf. Appendix A section 5) renders the space-time homogeneous, i.e. into a singular *homogeneous plane wave* (HPW)[**21**]. Furthermore, it was shown that

(3) the coefficients c_a (related to the harmonic oscillator frequency squares in [**25**] by $c_a = -\omega_a^2$) are bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.

As pointed out in [**25**] the universal $(x^+)^{-2}$-behaviour in the Penrose limit is of some relevance for the study of string theory in singular and/or time-dependent backgrounds. In general, because of the simplifications resulting from the combined conformal and lightcone gauge [**37**], plane waves and more general Brinkmann metrics (cf. chapter 2 section 2) provide an interesting playground to investigate generic problems arising in this context. From (1) one infers that "weakly singular" plane waves with profile $\sim (x^+)^{-\alpha}$, $\alpha < 2$, while perhaps interesting as mere toy-models of time-dependent backgrounds in string theory [**74, 75, 76**], do not arise as Penrose limits of standard cosmological or other singularities, whereas a "strongly singular" behaviour with $\alpha > 2$ can only arise for metrics violating the strict DEC. Together with the interpretation of the Penrose limit as a lowest order metric and a lowest order string expansion obtained in chapter 3 and 4 respectively this singles out the singular HPWs with profile $\sim (x^+)^{-2}$ as the backgrounds to consider in order to obtain insight into the (approximate) properties of string theory near physically reasonable space-time singularities.

Most important to us in the present context is the null geodesic deviation equation (2.73) corresponding to (5.1)

$$\left(-\frac{d^2}{(dx^+)^2}\delta_{ab} + \frac{c_a \delta_{ab}}{(x^+)^2}\right)\xi^b = 0, \tag{5.4}$$

enjoying a *conformal invariance* under rescaling of x^+ inherited from (2). Moreover, in the language of ordinary differential equations (ODE) it lies on the borderline of being *regular singular* (as opposed to *irregular singular*), i.e. the degree of the singularity in the potential does not exceed the degree of the differential operator. Recall that while for ODEs of the regular singular form one can always find a power-series solution using the method of Frobenius, this method mostly fails for irregular singular ODEs. It is intriguing that although the regular variety plays a preferred rôle in mathematical physics, space-time singularities generically allow for the irregular (strongly singular) variety while at the end of the day the latter gets excluded (1) by the (physically motivated) strict DEC.

A similar "regularisation" takes place (3) concerning the allowed interval for the coefficient c_a which is restricted by the strict DEC to be larger or equal to $-\frac{1}{4}$. Note that for an ODE of the form (5.4) $c_a > -\frac{1}{4}$ leads to powerlaw solutions, the borderline case $c_a = -\frac{1}{4}$ to a powerlaw and a logarithmic solution, whereas the excluded range $c_a < -\frac{1}{4}$ leads to a strange oscillatory behaviour of the form $\cos(\sqrt{-4c_a - 1}\log x^+ + \varphi)$.

Naturally, in singular HPWs this type of ODE also shows up in the Killing equations (cf. Appendix A section 5.1), the e.o.m. of geodesics and to leading order in the separated e.o.m. of strings (2.48. In the latter context they have been discussed with exactly the same bound on c_a [**23**] as in this range the string modes can be extended across the singularity at $x^+ = 0$ [**74, 75**].

Rather surprising is the fact to be established in chapter 6 that the same conformal structure as in (5.4) arises generically, i.e. for singular metrics of the power-law type fulfilling the DEC, and without making any reference to the Penrose limit, in the (separated) partial differential equation (PDE) describing the time-evolution of a scalar field.

In this section we want to set the stage for this result and following closely the argument of [**25**] re-derive (1-3). To this end we procede in section 2.1 with a short discussion of

Szekeres-Iyer Metrics. Then in section 2.2 we calculate their null geodesics and in section 2.3 their Penrose limits/null geodesic deviation. Finally, in section 2.4 we describe the rôle of the DEC in this context.

2.1. Szekeres-Iyer Metrics. In the context of the *Cosmic Censorship Hypothesis*, Szekeres and Iyer [**77**] (see also [**78**]) studied a large class of four-dimensional spherically symmetric metrics they called "metrics with power-law type singularities". This class comprises virtually all explicitly known singular spherically symmetric solutions of the Einstein equations, in particular the Friedmann-Robertson-Walker (FRW) metrics, Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, as well as other types of metrics with null singularities[1].

In *double-null form*, these metrics (again $D = d + 2$ dimensions) take the form

$$ds^2 = -\mathrm{e}^{A(u,v)}dudv + \mathrm{e}^{B(u,v)}d\Omega_d^2, \tag{5.5}$$

where $A(u, v)$ and $B(u, v)$ can be expanded into

$$\begin{aligned} A(u,v) =& p \ln x(u,v) + \text{regular terms} \\ B(u,v) =& q \ln x(u,v) + \text{regular terms} \end{aligned} \tag{5.6}$$

in the proximity of the singularity surface $x(u, v) = 0$.

Using the residual coordinate transformations $u \to \tilde{u}(u)$, $v \to \tilde{v}(v)$ preserving the form of the metric (5.5) one can generically make $x(u, v)$ linear in u and v,

$$x(u,v) = ku + lv, \quad k, l = \pm 1, 0, \tag{5.7}$$

where $\eta = kl = 1, 0, -1$ corresponds to spacelike, null and timelike singularities respectively. This gauge choice essentially fixes the coordinates uniquely, and therefore the *critical exponents* p and q contain diffeomorphism invariant information.

The crucial point for the discussion in [**25**] and the following chapter is that generically to analyse the physics close to the singularity it suffices to consider the leading behaviour of such geometries at $x = 0$, namely

$$ds^2 = -x^p dudv + x^q d\Omega_d^2. \tag{5.8}$$

However as one might expect there are special cases, for specific values of the parameters p, q or for null singularities $\eta = 0$, where this leading behaviour cancels in certain components of the Einstein tensor and thus the subleading (in the above sense) terms become important for a full analysis of the singularities [**77, 78**]. Obviously, in this case the analysis becomes much more subtle and consequently we restrict ourselves to the metric (5.8) which, for $\eta \neq 0$ and generic values of p and q, captures the dominant behaviour of the physics near the singularity and refer for a discussion of $\eta = 0$ in this context to [**25**].

For $\eta \neq 0$ one defines $y = ku - lv$ and chooses $k = \eta l = 1$. This brings the metric to the form

$$ds^2 = \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega_d^2. \tag{5.9}$$

For $q \neq 0$ one can change variables to

$$t = y, \quad r = x^{q/2}, \tag{5.10}$$

[1]However, on the other hand the Belinskii-Khalatnikov-Lifshitz (BKL) metrics [**79, 80**] describing the chaotic oscillatory approach to a spacelike singularity are not included.

in terms of which the metric (5.9) becomes

$$\begin{aligned} ds^2 =& \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega_d^2 \\ =& \eta r^{2p/q} dt^2 - \frac{4\eta}{q^2} r^{2(p-q+2)/q} dr^2 + r^2 d\Omega_d^2 \end{aligned} \tag{5.11}$$

i.e. the standard form of a spherically symmetric metric

$$ds^2 = -\eta f(r) dt^2 + \eta g(r) dr^2 + r^2 d\Omega_d^2. \tag{5.12}$$

with the identification

$$\begin{aligned} f(r) =& -\eta r^{2p/q} \\ g(r) =& -\frac{4\eta}{q^2} r^{2(p-q+2)/q}. \end{aligned} \tag{5.13}$$

Note that the notation of t and r is adapted to the case of $\eta = -1$ where the singularity is timelike and t is time. As this is also the case primarily considered in the next chapter we will continue to use this notation even for spacelike singularities where t is actually spacelike.

The special case $q = 0$ corresponds to a so called *shell crossing singularity* [**77**] which is usually not considered to be a true singularity as the transverse sphere is of constant radius $x^q = 1$. Such singularities form for instance in certain collisions of spherical shells of dust. We will omit the case $q \neq 0$ in the following.

2.2. Null Geodesics of Szekeres-Iyer Metrics. Obviously, because of the rotational symmetry one can restrict the discussion to null geodesics in the (y, x, θ) or equivalently the (t, r, θ) plane where θ denotes the co-latitude of the d-sphere,

$$d\Omega_d^2 = d\theta^2 + \sin^2\theta d\Omega_{d-1}^2. \tag{5.14}$$

In terms of the conserved momenta P and L associated with y and θ

$$x^q \dot{\theta} = L, \tag{5.15}$$

and the condition of the geodesic being null is equivalent to

$$\dot{x}^2 = P^2 x^{-2p} + \eta L^2 x^{-p-q}. \tag{5.16}$$

where the dot refers to the derivative w.r.t. the affine parameter x^+.

Due to the expansion around $x = 0$ one can only trust the leading behaviour of this equation as $x \to 0$. On the other hand, it already contains all the (kinematic) information needed to characterise the behaviour of the null geodesics in this limit.

To see this first note that unless $p = q$, one of the two terms on the right-hand-side of (5.16) will obviously dominate as $x \to 0$, and therefore the generic behaviour of a null geodesic near $x = 0$ is identical to that of a geodesic with either $L = 0$ or $P = 0$. In the former case, one finds

$$\dot{x}^2 \sim x^{-2p} \quad \Rightarrow \quad x(x^+) \sim \begin{cases} (x^+)^{1/(p+1)} & p \neq -1 \quad \text{Behaviour 1} \\ \exp x^+ & p = -1 \end{cases} \tag{5.17}$$

The geodesics of interest in this context are those hitting the singularity in $x = 0$ at finite x^+. This happens only for $p > -1$, i.e. behaviour 1. In the latter case, corresponding to null geodesics which asymptotically, as $x \to 0$, behave like geodesics with $P = 0$, one also needs $\eta = +1$, i.e. a spacelike singularity, leading to

$$\dot{x}^2 \sim x^{-p-q} \quad \Rightarrow \quad x(x^+) \sim \begin{cases} (x^+)^{2/(p+q+2)} & p \neq -1 \quad \text{Behaviour 2} \\ \exp x^+ & p + q = -2 \end{cases} \tag{5.18}$$

Here the geodesics run into the singularity at finite x^+ for $p+q > -2$, i.e. behaviour 2. For $\eta = -1$, the situation is largely analogous, the main difference being that now the second term in (5.16) acts as an angular momentum barrier preventing e.g. geodesics with $L \neq 0$ for $q > p$ from reaching the singularity at $x = 0$. For the same reason, for $p = q$ (where behaviour 1 and 2 agree), one finds the additional constraint $|P| > |L|$.

2.3. Penrose Limits of Power-Law Type Singularities. Our next task is to determine the Penrose limits, i.e. the null geodesic deviation matrix, of the Szekeres-Iyer metrics along the null geodesics reaching the singularity $x = 0$ at finite x^+.

To this end we need to construct a parallel quasi-orthonormal frame $E_A = (E_+, E_-, E_a)$ adapted to the null geodesic in question , i.e. $E_+^\mu = \dot{x}^\mu$ (cf. chapter 2 section 3.2). It is convenient to choose E_-, E_1 to be tangential to the (t, r, θ) plane and to supplement this triad by a fixed (x^+-independent) orthonormal frame $E_{\hat{a}}$, $\hat{a} = 2, ..., d$ tangent to the transverse $(d-1)$-sphere.

As shown in Appendix A section 2 for a general metric of the form (5.12) it is unnecessary to specify E_-, E_1 any further. Moreover, one can derive purely algebraically the following expressions for the corresponding Penrose limit wave profile A_{ab}, $a, b = 1, ..., d$

$$A_{11}(x^+) = (r\dot{r}\sqrt{fg})^{-1}\partial^2_{x^+}(r\dot{r}\sqrt{fg}) \tag{5.19}$$

$$A_{\hat{a}\hat{b}}(x^+) = \delta_{\hat{a}\hat{b}}\left(\frac{\ddot{r}(x^+)}{r(x^+)} - \frac{L^2}{r(x^+)^4}\right). \tag{5.20}$$

It follows from the analysis of the previous section that the only possibility of interest for $r(x^+) = x(x^+)^{q/2}$ is the power-law behaviour

$$r(x^+) = (x^+)^m, \tag{5.21}$$

with

$$\begin{array}{lll} \text{Behaviour 1:} & p > -1 & m = q/2(p+1) \\ \text{Behaviour 2:} & p+q > -2 & m = q/(p+q+2). \end{array} \tag{5.22}$$

For the tangential component $A_{11}(u)$ this leads directly to

$$\begin{array}{ll} \text{Behaviour 1:} & A_{11}(x^+) = m(m-1)(x^+)^{-2} \\ \text{Behaviour 2:} & A_{11}(x^+) = pm/q(pm/q-1)(x^+)^{-2}. \end{array} \tag{5.23}$$

and the Penrose limit behaves as a singular homogeneous plane wave in this direction. Since $s(s-1)$ has a minimum $-1/4$ at $s = 1/2$, this leads to the bound

$$A_{11} \to c_1(x^+)^{-2}, \quad c_1 \geq -\frac{1}{4}. \tag{5.24}$$

For the transverse components $A_{\hat{a}\hat{b}}$ the situation is somewhat more complicated. With the power-law behaviour $r(x^+) = (x^+)^m$, the first term in (5.20) is always proportional to $(x^+)^{-2}$. This term is dominant as $x^+ \to 0$ when $m < 1/2$ while it is the angular momentum term that dominates for $m > 1/2$ and in this case leads to a strongly singular plane wave with profile $\sim (x^+)^{-4m}$. In the special case $m = 1/2$, both terms are proportional to

$(x^+)^{-2}$. Thus one has, for $L \neq 0$,

$$m < \frac{1}{2}: \quad A_{\hat{a}\hat{b}}(x^+) \to c_{\hat{a}}\delta_{\hat{a}\hat{b}}(x^+)^{-2}, \qquad c_{\hat{a}} = m(m-1) > -\frac{1}{4} \tag{5.25}$$

$$m = \frac{1}{2}: \quad A_{\hat{a}\hat{b}}(x^+) \to c_{\hat{a}}\delta_{\hat{a}\hat{b}}(x^+)^{-2}, \qquad c_{\hat{a}} = -\frac{1}{4} - c^2L^2 \leq -\frac{1}{4} \tag{5.26}$$

$$m > \frac{1}{2}: \quad A_{\hat{a}\hat{b}}(x^+) \to -L^2(x^+)^{-4m}. \tag{5.27}$$

Here, the additional constant c in the second line arises because the second term in (5.20) depends on the overall scale of $r(x^+)$ whereas the first one obviously does not.

For $p \geq q$, $P \neq 0$ leads to behaviour 1, implying $m = q/2(p+1)$, i.e. always $m < 1/2$. Similarly, $P = 0$ leads to behaviour 2, implying $m = q/(p+q+2)$, i.e. again $m < 1/2$.

However, if $p < q$ this always implies behaviour 2 and for $\eta = 1$, i.e. a spacelike singularity, one can see that $m = q/(p+q+2)$ can take on any value, with $m = 1/2$ along the line $q = p+2$ and $m > 1/2$ for $q > p+2$. On the other hand, if $p < q$ and $\eta = -1$, i.e. for a timelike singularity, as discussed at the very end of the last section geodesics are prevented from reaching the singularity by the angular momentum barrier. For $L = 0$ the analysis is again rather simple as only the first term in (5.20) is present, and one thus finds (5.25) for all values of m. Since $L = 0$ implies Behaviour 1, this means $m = q/2(p+1)$. Along the special line $q = 2(p+1)$ one has $m = 1$ and consequently finds the "flat" Penrose limit $A_{11}(x^+) = A_{\hat{a}\hat{b}}(x^+) = 0$.

2.4. The Significance of the (Strict) Dominant Energy Condition. From the previous discussion one directly infers that Penrose Limits of timelike spherically symmetric singularities of power-law type are singular HPWs with coefficients c_a bounded from below by $-1/4$.

On the other hand it also follows that for spacelike singularities a different (strongly singular) behaviour can in principle occur. However, as was realised in [**25**] this can be ruled out by demanding that the *dominant energy condition* (DEC) be satisfied but not saturated.

Similar to its weak and strong cousins the DEC is a physically reasonable (pointwise) algebraic restriction on the stress-energy tensor $T^\mu{}_\nu$ and via the Einstein field equations on the Einstein tensor $G^\mu{}_\nu$, namely [**42**] for every timelike vector v^μ, $T_{\mu\nu}v^\mu v^\nu \geq 0$, and $T^\mu{}_\nu v^\nu$ is a non-spacelike vector. This can be recast into the statement that for any observer the local energy density is non-negative and the energy flux causal. The stress-energy tensor is said to be of type I [**42**] if $T^\mu{}_\nu$ has one timelike and three, i.e. in the present context $d+1$ spacelike eigenvectors. The corresponding eigenvalues are (minus) the energy density $-\rho$ and the principal pressures P_α, $\alpha = 1, \ldots, d+1$. Thus, for a stress-energy tensor of type I, the DEC is equivalent to

$$\rho \geq |P_\alpha|. \tag{5.28}$$

Following [**25**] we say that the *strict* DEC is satisfied if these are strict inequalities. As we will see below the "extremal" matter content (equation of state) for which at least one of these inequalities is saturated will play a distinguished role.

The Einstein tensor of the metric (5.9) is diagonal (cf. Appendix A section 8),

$$\begin{aligned}
G^x_x &= -\tfrac{1}{2}d(d-1)x^{-q} - \tfrac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)}\\
G^y_y &= -\tfrac{1}{2}d(d-1)x^{-q} + \tfrac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)}\\
G^i_j &= -\tfrac{1}{2}(d-1)(d-2)\delta^i_j x^{-q} + \tfrac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta^i_j x^{-(p+2)}
\end{aligned}$$

and thus of type I. For spacelike singularities, $\eta = +1$, the energy density, radial and transverse pressures are $\rho = -G^x_x$, $P_r = G^y_y$ and $P_i = G^i_i$ respectively, while for $\eta = -1$ the roles of G^x_x and G^y_y are interchanged.

Since for $q > p+2$ the first term in G^x_x and G^y_y dominates over the second term for $x \to 0$, it is obvious that in this case the relation between ρ and P_r becomes extremal in the near-singularity limit,

$$G^x_x - G^y_y \to 0 \quad \Leftrightarrow \quad \rho + P_r \to 0. \tag{5.29}$$

Thus the upshot of this discussion is that $q \leq p+2$ is a necessary condition for the strict DEC to hold. As strongly singular plane waves arise only for $q > p+2$ this establishes that also the Penrose Limits of spacelike spherically symmetric singularities of power-law type satisfying the strict Dominant Energy Condition are singular HPWs. Furthermore, it follows from the discussion in the last section that $c_{\hat{a}} < -1/4$ can only occur for the extremal behaviour $q = p+2$. Together with the analogous results for timelike singularities above this establishes the claims (1), (2) and (3) of section 2.

Again two short comments are in order. First of all, the attribute “strict” of the DEC is neither fully motivated by physical arguments, nor do we lack counter-examples. On the contrary, as has already been seen in **[77]**, space-times with a saturated bound arise rather easily in this context, and even if they turn out to be an artefact of spherical symmetry in the end, the general exclusion of the extremal equations of state has to be further motivated.

Second, the term “strict inequality” as it is used in the present argument should be supplemented by “up to subleading but possibly infinite terms”, and consequently equation (5.29) should rather be interpreted as

$$G^x_x / G^y_y - 1 \to 0. \tag{5.30}$$

3. From null geodesic to massless scalar field probes

The fact that Penrose limits of timelike singularities always behave as $(x^+)^{-2}$, while in the spacelike case generically strongly singular Penrose limits can arise, although only for metrics violating the strict DEC, should not give the false impression that timelike (naked) singularities are “better behaved” than spacelike (censored) singularities. As already mentioned in **[25]** this should rather be interpreted as an indication that massless particles are somewhat inadequate for probing the geometry of timelike singularities as the angular momentum barrier prevents non-radial null geodesics from reaching and thus probing the singularity for large regions in the (p, q)-parameter space.

In this spirit, it is much more intriguing that for spacelike singularities lightlike particle probes with arbitrary angular momentum all detect singular HPWs provided that the strict DEC is satisfied. In particular, this shows that the $(x^+)^{-2}$-behaviour is no simple consequence of say the geodesic deviation equation being a differential equation of second order or similarly the Riemann tensor a second derivative of the metric.

These conclusions are supported by the results of the next chapter where the null geodesic probes are replaced by (massless) scalar fields. Recall from section 2.3 that the Penrose limit wave profile generically includes a term $\sim (x^+)^{-2}$ and the possibly strongly singular behaviour for spacelike singularities arises in the term related to the angular momentum. As we will see a completely analogous behaviour arises in the operator governing the radial motion of the scalar field. Moreover, similar to what we said about null geodesics, the effect which renders a certain class of timelike singularities "well-behaved" w.r.t. the scalar field is directly related to a repulsive potential in this operator, preventing the scalar field from "seeing" the singularity. The only difference to the former case is that for the scalar field the strongly singular behaviour can arise for timelike as well as spacelike singularities and the DEC has to be invoked in both cases to restore universality.

From this point of view, a scalar field seems to be a rather equivalent but slightly more refined and versatile probe of space-time singularities than a geodesic congruence. This interpretation is usually supported **[81]** by the *geometric optics approximation* where congruences are interpreted as *infinite frequency limits* of scalar (or other tensorial) fields. In situations where the space-time scale of variation of the field ϕ is much smaller than that of the curvature it is sensible to make the ansatz

$$\phi(x) = C(x)e^{iS(x)} \tag{5.31}$$

where the derivatives of $C(x)$ are small. Then the massless scalar wave-equation becomes

$$0 = \nabla^\mu \partial_\mu \phi \approx -C\partial^\mu S \partial_\mu S e^{iS} \tag{5.32}$$

i.e. formally equivalent to the Hamilton-Jacobi equation $\partial^\mu S \partial_\mu S = 0$ (cf. Appendix A section 3) describing a null geodesic congruence. In the present context it is even more intriguing that such congruences are also automatically twistfree and S is identical to the action functional used to construct the Penrose limit as discussed in chapter 2 section 3.1.

All this indicates that the results of the next chapter are largely to be expected. On the other hand one might still be puzzled about how the observed "conincidences" can and do arise technically. Being unable to answer this question in the present situation we simply caution the reader about what we are trying to match in the next chapter.

Recall from chapter 2 section 3.2 that on the congruence side the dynamical variables are the transverse deviation vectors ξ^a encoding proper distance to nearby null geodesics. The latter are only subject to the curvature and do obviously not interact. The ξ^a depend only on the affine parameter $\tau = u = x^+$ and it is this single variable playing the role of time which gives rise to the universal singular power-law behaviour $\sim (x^+)^{-2}$. Moreover, as we have seen, no global, in the sense of co-dimension zero, information whatsoever of a space-time-filling null congruence is needed to calculate the Penrose limit or equivalently the null geodesic deviation operator (see also Appendix A section 1). Instead, we are rather dealing with a co-dimension $D-1$ object[2].

On the other side the dynamical variable is the (unitarily transformed) scalar field, i.e. a co-dimension zero object, feeling a "tension" induced by the spatial derivatives acting on it. Moreover, in the timelike case ($\eta = -1$) primarily considered in this context it is a separated spatial direction, namely the Szekers-Iyer coordinate x giving rise to a universal power-law behaviour $\sim x^{-2}$. Obviously, all this contrasts sharply with what we said about the congruence.

[2]Even if we consider that the spherical symmetry of Szekeres-Iyer metrics reduces the global information to an effective geodesic movement in three space-time dimensions as discussed in section 2.2, we are still dealing with co-dimension two.

CHAPTER 6

Scalar Field Probes of Power-Law Space-Time Singularities

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We analyse the effective potential of the scalar wave equation near generic space-time singularities of power-law type (Szekeres-Iyer metrics) and show that the effective potential exhibits a universal and scale invariant leading inverse square behaviour $\sim x^{-2}$ in the "tortoise coordinate" x provided that the metrics satisfy the strict Dominant Energy Condition (DEC). This result parallels that obtained in **[25]** for probes consisting of families of massless particles (null geodesic deviation, a.k.a. the Penrose Limit). The detailed properties of the scalar wave operator depend sensitively on the numerical coefficient of the x^{-2}-term, and as one application we show that timelike singularities satisfying the DEC are quantum mechanically singular in the sense of the Horowitz-Marolf (essential self-adjointness) criterion. We also comment on some related issues like the near-singularity behaviour of the scalar fields permitted by the Friedrichs extension.

1. Introduction

The study of scalar field propagation in non-trivial curved (and possibly singular) backgrounds is of fundamental importance in a variety of contexts including quantum field theory in curved backgrounds, cosmology, the stability and quasi-normal mode analysis of black hole metrics etc.

Typically, this is studied within the context of a particular metric or class of metrics. For certain purposes, however, only the knowledge of the leading behaviour of the metric near a horizon or the singularity is required. In that case, one can attempt to work with a general parametrisation of the metric near that locus and, in this way, ascertain which features of the results that have been obtained previously for particular metrics are special features of those metrics or valid more generally.

In particular, practically all explicitly known metrics with singularities are of "power-law type" **[77]** in a neighbourhood of the singularity (instead of showing, say, some non-analytic behaviour). In the spherically symmetric case, the leading behaviour of generic metrics with such singularities of power-law type is captured by the 2-parameter family

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \tag{6.1}$$

of Szekeres-Iyer metrics [**77, 78, 82**]. The singularity, located in these coordinates at $x = 0$, is timelike for $\eta = -1$ and spacelike for $\eta = +1$. This class of metrics thus provides an ideal laboratory for investigating the behaviour of particles, fields, strings, ... in the vicinity of a generic singularity of this type.

A first investigation along these lines was performed in [**20, 25**] in the context of the Penrose Limit, i.e. of probing a space-time via the geodesic deviation of families of massless particles. There it was shown that the plane wave Penrose limits,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to 2dx^+dx^- + A_{ab}(x^+)x^a x^b(dx^+)^2 + d\vec{x}^2 \quad , \tag{6.2}$$

of metrics with singularities of power-law type have a universal $(x^+)^{-2}$-behaviour near the singularity, $A_{ab}(x^+) \sim (x^+)^{-2}$, provided that the near-singularity stress-energy (Einstein) tensor satisfies the strict dominant energy condition (DEC). This behaviour, which is precisely such that it renders the plane wave metric scale invariant [**21**], had previously been observed in various particular examples and is thus now understood to be a general feature of this class of singularities.

It is then natural to wonder whether a similar universality result can be established in other circumstances or for other kinds of probes and if, analogously, some energy condition plays a role in establishing this. If one considers e.g. the Klein-Gordon equation $\Box\phi = 0$ for scalar fields, it is not difficult to see [**83, 84**] that under certain conditions the scalar effective potential V_{eff} for general metrics with singularities of power-law type displays an inverse square behaviour, $V_{\text{eff}}(x) \sim x^{-2}$, near the singularity. This observation was then used in [**84**] to study the quasi-normal modes for black holes with generic singularities of this type.

The purpose of this note is to study other aspects and consequences of this universality. In particular, we will first show that the results obtained in [**25**], namely the scale invariant inverse square behaviour of the wave profile $A_{ab}(x^+)$, as well as a crucial [**23, 21**] lower bound on the coefficients, have a precise and rather striking analogue in the case of a scalar field. Schematically, this analogy can be expressed as

$$\text{strict DEC} \quad \Rightarrow \quad \begin{cases} A_{ab}(x^+) \to c_a\delta_{ab}(x^+)^{-2} & \text{scale invariant} \quad (c_a \geq -1/4) \\ V_{\text{eff}}(x) \to cx^{-2} & \text{scale invariant} \quad (c \geq -1/4) \end{cases} \tag{6.3}$$

Once again this shows that this inverse square behaviour, that had been observed before in various specific examples in a variety of contexts, is a general feature of a large class of space-time singularities. The precise statements are derived in sections 2.2 and 2.3 and discussed in section 2.4, while sections 2.5 and 2.6 deal with minor variations of this theme.

We hasten to add that if such an inverse square behaviour were universally true without any further qualifications then it would probably have to be true on rather trivial (dimensional) grounds alone. What makes the results obtained here and in [**25**] more interesting is that a priori in either case a more singular behaviour can and does occur and is only excluded provided that some further (e.g. positive energy) condition is imposed.

The significance of the x^{-2}-behaviour is that (as anticipated in (6.3)), the corresponding Schrödinger operator $-\partial_x^2 + cx^{-2}$, to which we will have reduced the Klein-Gordon operator, defines a scale invariant (c is dimensionless) "conformal quantum mechanics" [**85**] problem. Thus, here and in [**25**] we find a rather surprising emergence of scale invariance in the near-singularity limit. One implication of this scale invariance in the plane wave case, discussed in [**86**], is that it leads to a Hagedorn-like behaviour of string theory in this class of backgrounds that is quite distinct from that in plane wave backgrounds with, say, a constant profile and more akin to that in Minkowski space. It would be interesting to explore other consequences of this near-singularity scale invariance.

This class of scale invariant models has recently also appeared and been discussed in various other related settings, most notably in the analysis of the near-horizon (rather than the near-singularity) properties of black holes, see e.g. [**87, 88, 89, 90, 91**], where the emergence of scale invariance can largely be attributed to the near-horizon AdS geometry, as well as in quantum cosmology [**92**].

Having reduced the Klein-Gordon operator to the Schrödinger operator $-\partial_x^2 + cx^{-2}$ (after a separation of variables and a unitary transformation), one can then turn to a more detailed spectroscopy of the Szekeres-Iyer metrics by analysing the properties of this operator. Indeed, as is well known, the inverse square potential is a critical borderline case in the sense that the functional analytic properties of this operator depend in a delicate way on the numerical value of the coefficient c. This value, in turn, depends on the dimension d (number of transverse dimensions) and the Szekeres-Iyer parameter q (it turns out to be independent of p, while the corresponding coefficients c_a in the Penrose limit case typically depend on (p, q) and d).

As one application, we will analyse the Horowitz-Marolf criterion [**81**] for general singularities of power-law type. Horowitz and Marolf defined a static space-time to be quantum mechanically non-singular (with respect to a certain class of test fields) if the evolution of a probe wave packet is uniquely determined by the initial wave packet (as would be the case in a globally hyperbolic space-time) without having to specify boundary conditions at the classical singularity. This criterion can be rephrased as the condition that the (spatial part of the) Klein-Gordon operator be essentially self-adjoint (and thus have a unique self-adjoint extension).

While such a necessarily only semi-classical analysis is certainly not a substitute for a full quantum gravitational analysis, it nevertheless has its virtues since one can learn what kind of problems persist, can arise or can be resolved when passing from test particles to test fields.

Intuitively one might expect a classical singularity with a sufficiently "positive" (in an appropriate sense) matter content to remain singular even when probed by non-stringy test objects other than classical point particles. This line of thought was one of the motivations for analysing the Horowitz-Marolf criterion in this framework, and we will indeed be able to show (section 3.4) that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

A second issue we will briefly address is that of the allowed near-singularity behaviour of the scalar fields for a given self-adjoint extension (section 3.5). A priori, one might perhaps expect a sufficiently repulsive singularity to be regular in the Horowitz-Marolf sense simply because the corresponding unique self-adjoint extension forces the scalar field to be zero at the singularity, thus in a sense again excluding the singularity from the space-time. It is also possible, however, and potentially more interesting, to have a self-adjoint extension with scalar fields that actually probe the singularity in the sense that they are allowed to take on non-zero values there. We propose to call such singularities "hospitable", establish once again a relation, albeit not a strict correlation, with the DEC, and show among other things that, in a suitable sense, half of the Horowitz-Marolf regular power-law singularities are hospitable whereas the others are not.

2. Universality of the Effective Scalar Potential for Power-Law Singularities

2.1. Geometric Set-Up. Even though we will ultimately be interested in the properties of the scalar wave (Klein-Gordon) equation $(\Box - m^2)\phi = 0$ in the Szekeres-Iyer metrics (6.1), to set the stage it will be convenient to begin the discussion in the more general setting of metrics with a hypersurface orthogonal Killing vector. The general set-up here and in section 3.1 is modelled on the approach of [**93**] (with minor adaptations to allow for both timelike and spacelike singularities).

We begin with the n-dimensional metric

$$ds^2 = \eta a^2 dy^2 + h_{ij} dx^i dx^j \tag{6.4}$$

where a and h_{ij} are independent of y, $\xi = \partial_y$ is a hypersurface orthogonal Killing vector with norm $\xi^\mu \xi_\mu = \eta a^2$, and thus timelike (spacelike) for $\eta = -1$ ($\eta = +1$). Correspondingly we assume that the metric h_{ij} induced on the hypersurfaces $\Sigma_y \cong \Sigma$ of constant y is Riemannian (Lorentzian) for $\eta = -1$ ($\eta = +1$).

Denoting the covariant derivatives with respect to the metric h_{ij} by D_i, the wave operator

$$\Box \equiv \frac{1}{\sqrt{-\det g}} \partial_\mu \sqrt{-\det g} g^{\mu\nu} \partial_\nu \tag{6.5}$$

is easily seen to take the form

$$\Box = a^{-2}(\eta \partial_y^2 + a D^i a D_i) \quad . \tag{6.6}$$

Thus the massive wave equation $(\Box - m^2)\phi = 0$ can be written as

$$\partial_y^2 \phi = -A\phi \quad , \tag{6.7}$$

where A is the operator

$$A = \eta a D^i a D_i - \eta a^2 m^2 \quad . \tag{6.8}$$

Assuming now spherical symmetry, the metric takes the warped product form

$$ds^2 = \eta a(x)^2 dy^2 - \eta b(x)^2 dx^2 + c(x)^2 d\Omega_d^2 \tag{6.9}$$

where $d\Omega_d^2$, $d = n - 2$, denotes the standard metric on the d-sphere S^d. It will be apparent from the following that the assumption of spherical symmetry could be relaxed - we will only use the warped product form of the metric in an essential way.

We could fix the residual x-reparametrisation invariance by introducing the "area radius" $r = c(x)$ as a new coordinate. However, for the following it will be more convenient to choose the gauge $a(x) = b(x)$ (i.e. x is a "tortoise coordinate" for $\eta = -1$ respectively "conformal time" for $\eta = +1$),

$$ds^2 = \eta a(x)^2(-dx^2 + dy^2) + c(x)^2 d\Omega_d^2 \quad . \tag{6.10}$$

Then the operator A is

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2} \Delta_d - \eta a^2 m^2 \quad , \tag{6.11}$$

where $\sigma(x) = c(x)^d$ and Δ_d denotes the Laplacian on S^d.

To put A into standard Schrödinger form, we transform from the functions $\phi(x)$ to the half-densities (cf. (6.52)) $\tilde{\phi}(x) = \sigma^{1/2}\phi(x)$. The corresponding unitarily transformed operator $\tilde{A}$ is

$$\begin{aligned} \tilde{A} = \sigma^{1/2} A \sigma^{-1/2} &= -\partial_x^2 + V + \eta a^2 c^{-2} \Delta_d - \eta a^2 m^2 \\ V(x) &= \sigma(x)^{-1/2}(\partial_x^2 \sigma(x)^{1/2}) \quad . \end{aligned} \tag{6.12}$$

After the usual separation of variables in the y-direction,

$$\tilde{\phi}(y,x,\theta^a) = \mathrm{e}^{-iEy}\,\tilde{\phi}(x,\theta^a) \quad , \tag{6.13}$$

and the decomposition into angular spherical harmonics $Y_{\ell\vec{m}}(\theta^a)$, with

$$\begin{aligned} -\Delta_d Y_{\ell\vec{m}}(\theta^a) &= \ell_d^2 Y_{\ell\vec{m}}(\theta^a) \\ \ell_d^2 &= \ell(\ell+d-1) \quad , \end{aligned} \tag{6.14}$$

the Klein-Gordon equation for the metric (6.10) reduces to a standard one-dimensional time-independent Schrödinger equation

$$\left[-\partial_x^2 + V_{\text{eff},\ell}(x)\right]\tilde{\phi}(x) = E^2\tilde{\phi}(x) \tag{6.15}$$

($\tilde{\phi}(x) = \tilde{\phi}_{E,\ell,\vec{m}}(x)$) with effective scalar potential

$$V_{\text{eff},\ell}(x) = V(x) - \eta a(x)^2(\ell_d^2 c(x)^{-2} + m^2) \quad . \tag{6.16}$$

2.2. The Effective Scalar Potential for Power-Law Singularities. The leading behaviour of generic (spherically symmetric) metrics with singularities of power-law type[1], i.e. metrics of the general form **[77]**

$$ds^2 = -dt^2 + [t-\tau(r)]^{2a} f(r,t)^2 dr^2 + [t-\tau(r)]^{2b} g(r,t)^2 d\Omega_d^2 \quad , \tag{6.17}$$

with f and g functions of r and t that are regular and non-vanishing at the location $t = \tau(r)$ of the singularity, is captured by the 2-parameter family of Szekeres-Iyer metrics **[77, 78]** (see also **[25]** and the generalisation to string theory backgrounds discussed in **[82]**)

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \quad . \tag{6.18}$$

The Kasner-like exponents $p, q \in \mathbb{R}$ characterise the behaviour of the geometry near the singularity at $x = 0$. This singularity is timelike for $\eta = -1$ (x is a radial coordinate) and spacelike for $\eta = +1$ (with x a time coordinate). In particular, these metrics possess the hypersurface orthogonal Killing vector ∂_y, and are already in the "tortoise" form (6.10), with $a(x)^2 = x^p$ and $c(x)^2 = x^q$. Thus we can directly read off the effective scalar potential from the results of the previous section.

From (6.12), we deduce, with $\sigma(x) = x^{dq/2}$, that

$$V(x) = s(s-1)x^{-2} \qquad s = \frac{dq}{4} \quad . \tag{6.19}$$

Thus, from (6.16) we find (see also **[84]**)

$$V_{\text{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{6.20}$$

We are interested in the leading behaviour of this potential as $x \to 0$ (subdominant terms can in any case not be trusted as we have only kept the leading terms in the metric (6.18)). For the time being we will consider the massless case $m^2 = 0$ (see section 2.5 for $m^2 \neq 0$).

Provided that $s(s-1) \neq 0$, which term in (6.20) dominates depends on p and q. When $q < p+2$, one finds

$$q < p+2: \qquad V_{\text{eff},\ell}(x) \to s(s-1)x^{-2} \quad . \tag{6.21}$$

[1]Such metrics encompass practically all explicitly known singular spherically symmetric solutions of the Einstein equations like the Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, etc. On the other hand, this class of metrics does prominently *not* include the BKL metrics **[79, 80]** describing the chaotic oscillatory approach to a spacelike singularity. Whether or not such a behaviour occurs depends in a delicate way on the matter content, see e.g. **[94]** and references therein.

The two salient features of this potential are the inverse square behaviour and a coefficient c that is bounded from below by $-1/4$,

$$c = s(s-1) \geq -\frac{1}{4} \quad , \tag{6.22}$$

with equality for $s = 1/2$, i.e. $q = 2/d$.

As mentioned in the introduction, the significance of the x^{-2}-behaviour is that it defines a scale invariant "conformal quantum mechanics" [**85**] problem, discussed more recently in related contexts e.g. in [**87, 88, 89, 90, 91, 92**]. Moreover, for practical purposes [**84, 95**] the virtue of the x^{-2} (as opposed to a more singular) behaviour is that it leads to a standard regular-singular differential operator.

The significance of the bound on c is that in this range the operator $-\partial_x^2 + c/x^2$ is positive, as can be seen by writing

$$-\partial_x^2 + s(s-1)x^{-2} = (\partial_x + sx^{-1})(-\partial_x + sx^{-1}) = (-\partial_x + sx^{-1})^\dagger(-\partial_x + sx^{-1}) \quad . \tag{6.23}$$

When $q = p+2$, the metric is conformally flat, both terms in (6.20) contribute equally, and one again finds the x^{-2}-behaviour

$$q = p+2: \qquad V_{\text{eff},\ell}(x) \to cx^{-2} \quad , \tag{6.24}$$

where now

$$c = s(s-1) - \eta\ell_d^2 \quad . \tag{6.25}$$

Thus in this case c is still bounded by $-1/4$ for timelike singularities, while c can become arbitrarily negative for sufficiently large values of ℓ_d^2 for $\eta = +1$.

Once $q > p+2$, the second term in (6.20) dominates (for $\ell_d^2 \neq 0$), and one finds the more singular leading behaviour

$$q > p+2: \qquad V_{\text{eff},\ell}(x) \to -\eta\ell_d^2\, x^{-2-a} \quad a > 0 \quad . \tag{6.26}$$

Examples of metrics with $q \leq p+2$ are the Schwarzschild and Friedmann-Robertson-Walker (FRW) metrics and indeed, as we will recall below, all metrics satisfying the strict Dominant Energy Condition.

In particular, for the $(d+2)$-dimensional (positive or negative mass) Schwarzschild metric, one has

$$\text{Schwarzschild}: \qquad p = \frac{1-d}{d} \qquad q = \frac{2}{d} \quad , \tag{6.27}$$

as is readily seen by expanding the metric near the singularity and going to tortoise coordinates. Thus the Schwarzschild metric has $s = 1/2$ and c takes on the d-independent extremal value $c = -1/4$, as observed before e.g. in [**95, 84**] in related contexts.

For decelerating cosmological FRW metrics, with cosmological scale factor (in comoving time) $\sim t^h$, $0 < h < 1$,

$$h = \frac{2}{(d+1)(1+w)} \quad , \tag{6.28}$$

with w the equation of state parameter, $P = w\rho$, one finds [**20, 25**]

$$\text{FRW}: \qquad p = q = \frac{2h}{1-h} \quad , \tag{6.29}$$

as can be seen by going to conformal time. A routine calculation shows that the above result (6.21) for the purely x-dependent part of the effective potential (with x conformal time) is actually an exact result, and not an artefact of the near-singularity Szekeres-Iyer approximation.

It remains to discuss the case when $q < p+2$, so that the first term in (6.20) would be dominant, but the coefficient $s(s-1) = 0$. When $s = 0$, then one has $q = 0$ and this is generally interpreted **[77]** as corresponding not to a true central singularity (as the radius of the transverse sphere remains constant as $x \to 0$) but as a shell crossing singularity.

The other possibility is $s = 1$, i.e. $q = 4/d$. This is a case in which (because of the cancellation of the leading terms) subleading corrections to the metric (6.18) can become relevant and should be retained. An example of metrics with $s = 1$ is provided by FRW metrics with a radiative equation of state. Using (6.29), one has

(6.30) $$q = \frac{4}{d} \Leftrightarrow h = \frac{2}{d+2} \Leftrightarrow w = \frac{1}{d+1} \ ,$$

which is precisely the equation of state parameter for radiation. However, as follows from the remark above, in this special case the vanishing of the effective potential for $p = q$ is actually an exact result.

2.3. The Significance of the (Strict) Dominant Energy Condition. We have seen that generically the leading behaviour of the scalar effective potential near a singularity of power-law type is either $\sim x^{-2}$ or $\sim x^{p-q}$. We will now recall from **[77, 25]** that the latter behaviour can arise only for metrics violating the strict Dominant Energy Condition (DEC). While there is nothing particularly sacrosanct about the DEC, and other energy conditions could be considered, the DEC appears to play a privileged role in exploring and understanding the (p, q)-plane of Szekeres-Iyer metrics.

The *Dominant Energy Condition* on the stress-energy tensor T^μ_ν (or Einstein tensor G^μ_ν) **[42]** requires that for every timelike vector v^μ, $T_{\mu\nu}v^\mu v^\nu \geq 0$, and $T^\mu_\nu v^\nu$ be a non-spacelike vector. This may be interpreted as saying that for any observer the local energy density is non-negative and the energy flux causal.

The Einstein tensor of Szekeres-Iyer metrics is diagonal, hence so is the corresponding stress-energy tensor. In this case, the DEC reduces to

(6.31) $$\rho \geq |P_i| \ ,$$

where $-\rho$ and P_i, $i = 1, \ldots, d+1$ are the timelike and spacelike eigenvalues of T^μ_ν respectively. We say that the *strict* DEC is satisfied if these are strict inequalities and we will say that the matter content (or equation of state) is "extremal" if at least one of the inequalities is saturated.

Now it follows from the explicit expression for the components

(6.32) $$\begin{aligned} G^x_x =& -\tfrac{1}{2}d(d-1)x^{-q} - \tfrac{1}{8}\eta dq((d-1)q + 2p)x^{-(p+2)} \\ G^y_y =& -\tfrac{1}{2}d(d-1)x^{-q} + \tfrac{1}{8}\eta dq(2p + 4 - (d+1)q)x^{-(p+2)} \end{aligned}$$

of the Einstein tensor that for $q > p+2$ the relation between $-\rho$ and the radial pressure P_r (identified with G^x_x and G^y_y - which is which depends on the sign of η) becomes extremal as $x \to 0$ **[77, 25]**,

(6.33) $$q > p+2 : \qquad G^x_x - G^y_y \to 0 \quad \Leftrightarrow \quad \rho + P_r \to 0 \ .$$

Put differently, $q \leq p+2$ is a necessary condition for the strict DEC to hold, and thus for metrics satisfying the strict DEC the leading behaviour of the effective potential is always $V_{\text{eff},\ell}(x) \to cx^{-2}$.

As an aside, we note that it follows from (6.32) that precisely those metrics that satisfy the physically more reasonable (non-negative pressure) and more common extremal near-singularity equation of state $\rho = +P_r$ have $q = 2/d$, i.e. $s = 1/2$, leading to the critical value $c = -1/4$ frequently found in applications (to e.g. Schwarzschild-like geometries).

2.4. Comparison with Massless Point Particle Probes (the Penrose Limit). In the previous section we have established that

(1) for metrics with singularities of power-law type satisfying the strict DEC the leading behaviour of the scalar effective potential near the singularity is

$$V_{\text{eff},\ell}(x) \to cx^{-2} \tag{6.34}$$

(2) this class of potentials is singled out by its scale invariance;
(3) the corresponding coefficient c of the effective potential is bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.

These observations bear a striking resemblance to the results obtained recently in [**25**] in the study of plane wave Penrose limits

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to 2dx^+dx^- + A_{ab}(x^+)x^a x^b (dx^+)^2 + d\vec{x}^2 \quad , \tag{6.35}$$

of space-time singularities. Namely, it was shown in [**25**] that

(1) Penrose limits of metrics with singularities of power-law type show a universal $(x^+)^{-2}$-behaviour near the singularity,

$$A_{ab}(x^+) \to c_a \delta_{ab} (x^+)^{-2} \quad , \tag{6.36}$$

provided that the strict DEC is satisfied;
(2) such plane waves are singled out [**21**] by their scale invariance, reflected e.g. in the isometry $(x^+, x^-) \to (\lambda x^+, \lambda^{-1} x^-)$ of the metric (6.35, 6.36);
(3) the coefficients c_a (related to the harmonic oscillator frequency squares by $c_a = -\omega_a^2$) are bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.[2]

The similarity of these two sets of statements is quite remarkable because the objects these statements are made about are rather different. For example, the potential is that of a one-dimensional motion on the half line in one case, and that of a d-dimensional harmonic oscillator (with time-dependent frequencies) in the other.

The analogy with the above statements about scalar effective potentials is brought out even more clearly if one reinterprets [**20, 25**] the Penrose limit in terms of null geodesic deviation in the original space-time. Then this result can be rephrased as the statement that the leading behaviour of the geometry as probed by a family of massless point particles near a singularity is that of a plane wave with a $(x^+)^{-2}$ geodesic effective potential. The analogy with the results of the previous section should now be apparent.

One minor difference between the results obtained here and those of [**25**] is that in the case of Penrose limits the strict DEC needed to be invoked only in the case of spacelike singularities, $\eta = +1$, timelike singularities always giving rise to plane waves with a $(x^+)^{-2}$-behaviour. This should be regarded as an indication (cf. the discussion in [**25**, Section 4.4]) that scalar waves are better probes of timelike singularities than massless point particles.

[2]One significance of this bound on the c_a is that in this range one can consider the possibility to extend the string modes across the singularity at $x^+ = 0$ [**23**].

2.5. Massive Scalar Fields and Geodesic Incompleteness. The simple above analysis can evidently be generalised in various ways, e.g. by considering other kinds of probes. We will briefly comment on the two most immediate generalisations, namely massive and non-minimally coupled scalar fields.

We begin with a massive scalar for which the effective potential is

$$V_{\text{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{6.37}$$

For the mass term to be relevant (dominant) as $x \to 0$ it is clearly necessary that $p < -2$ and $q < 0$. Intuitively one might expect a mass term to be irrelevant at short distances near a singularity. This expectation is indeed borne out: as we will now show, for metrics satisfying the above inequalities the would-be singularity at $x = 0$ is at infinite affine distance for causal geodesics so that such space-times are actually causally geodesically complete.

Null geodesics were analysed in [**25**]. Here we generalise this to causal geodesics. In terms of the conserved angular and y-momentum L and P, the geodesic equation for the metric (6.18) reduces to

$$\dot{x}^2 = P^2 x^{-2p} + \eta L^2 x^{-p-q} + \eta\epsilon x^{-p} \quad , \tag{6.38}$$

where $\epsilon = 0$ ($\epsilon = 1$) for null (timelike) geodesics respectively.

For $\eta = -1$, if the first term in (6.38) is sub-dominant the geodesic effective potential is repulsive (e.g. via the angular momentum barrier) and the geodesics will not reach $x = 0$. Thus generic timelike geodesics will reach $x = 0$ only if (p, q) lie in the positive wedge bounded by the lines $p = 0$ and $p = q$. Radial null geodesics do not feel any repulsive force, and solving

$$\dot{x}^2 \sim x^{-2p} \quad \Rightarrow \quad x(x^+) \sim \begin{cases} (x^+)^{1/(p+1)} & p \neq -1 \\ \exp x^+ & p = -1 \end{cases} \tag{6.39}$$

shows that $x = 0$ is reached at a finite value of the affine parameter only for $p > -1$. We thus conclude that Szekeres-Iyer metrics with $\eta = -1$ and $p \leq -1$ are causally geodesically complete. In particular, therefore, the mass term in the scalar effective potential is sub-dominant for metrics with honest timelike power-law singularities, and for all such metrics the scalar effective potential has the same leading behaviour as in the massless case.

For $\eta = +1$, the situation is more complex as all three terms in (6.38) are positive. If the first term dominates, either because of suitable inequalities satisfied by (p, q) or, for any (p, q), because one is considering radial null geodesics, the analysis and conclusions are identical to the above. Namely, $x = 0$ is at finite affine distance for $p > -1$. Analogously, if the second term dominates (e.g. for angular null geodesics) one finds the condition $p + q > -2$, and if the third term dominates one has $p > -2$. Since one needs $p < -2$ for the mass term to dominate in the scalar effective potential, only the second case is possible. But then the condition $p + q > -2$, with $p < -2$, implies $q > 0$, so that the angular momentum term in the effective potential dominates the mass term.

We thus conclude that, for both $\eta = +1$ and $\eta = -1$, the mass term is always subdominant for metrics that are causally geodesically incomplete at $x = 0$.

As an aside we note that the Szekeres-Iyer metrics for which the mass term does dominate ($p < -2$ and $q < 0$), in addition to being non-singular, also necessarily violate the strict DEC.

2.6. Non-Minimally Coupled Scalar Fields. We will now very briefly also consider a non-minimally coupled scalar field

$$(\Box - \xi R)\phi = 0 \ . \tag{6.40}$$

The Ricci scalar of the Szekeres-Iyer metric (6.18) is

$$R = d(d-1)x^{-q} - \tfrac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \ , \tag{6.41}$$

where once again only the leading order term should be trusted and retained. Thus the new effective potential

$$V^{\xi}_{\text{eff},\ell}(x) = V_{\text{eff},\ell}(x) - \eta\xi x^p R \tag{6.42}$$

is again a sum of two terms, proportional to x^{-2} and x^{p-q} respectively, so that the dominant behaviour is still $\sim x^{-2}$ provided that the metric does not violate the strict DEC. For $q < p+2$ and the conformally invariant coupling

$$\xi = \xi_* = \frac{d}{4(d+1)} \ , \tag{6.43}$$

one finds

$$V^{\xi_*}_{\text{eff},\ell}(x) = \frac{(p-q)d}{4(d+1)}x^{-2} = (p-q)\xi_* x^{-2} \ . \tag{6.44}$$

Note that with this conformally invariant coupling the coefficient c now depends on $p-q$ rather than on q. The appearance of $(p-q)$ could have been anticipated since for $p = q$ the Szekeres-Iyer metric is conformal to an x-independent metric, and hence a conformal coupling cannot generate an x-dependent effective potential. Note also that for the conformal coupling (and, indeed, generic values of ξ) the coefficient c is no longer bounded by $-1/4$ so that the Schrödinger operator is no longer necessarily bounded from below.

3. Self-Adjoint Physics of Power-Law Singularities

In the previous section we have determined the leading behaviour of the scalar wave operators near a power-law singularity. In this section we will now study various aspects of these operators.

3.1. Functional Analysis Set-Up. In order to analyse the properties of the wave operator, we will need to equip the space of scalar fields with a Hilbert space structure. We will be pragmatic about this and introduce the minimum amount of structure necessary to be able to say anything of substance.

We thus return to the discussion of section 2.1, now being more specific about the spaces of functions the various operators appearing there act on [**93**], beginning with the operator A introduced in (6.8),

$$A = \eta a D^i a D_i - \eta a^2 m^2 \ . \tag{6.45}$$

Since $D^i D_i$ is symmetric (formally self-adjoint) with respect to the natural spatial density $\sqrt{-\eta \det h}$ induced on the slices Σ of constant y by the metric (6.4), the operator A is symmetric with respect to the scalar product

$$\begin{aligned} (\phi_1, \phi_2) &= \int d^{n-1}x \, \sigma \phi_1^* \phi_2 \\ \sigma &= a^{-1}\sqrt{-\eta \det h} = \eta\sqrt{-\det g} g^{yy} \ , \end{aligned} \tag{6.46}$$

on $D(A) = C_0^\infty(\Sigma)$,

$$(A\phi_1, \phi_2) = (\phi_1, A\phi_2) \quad . \tag{6.47}$$

Moreover, for $\eta = -1$ the operator A is positive,

$$\eta = -1 \quad \Rightarrow \quad (\phi, A\phi) \geq 0 \quad . \tag{6.48}$$

We are thus led to introduce the Hilbert space $L^2(\Sigma, \sigma d^{n-1}x)$ of functions on Σ square integrable with respect to the above scalar product.

Passing to spherically symmetric metrics (6.9) in the tortoise gauge (6.10), A takes the form (6.11)

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2} \Delta_d - \eta a^2 m^2 \quad , \tag{6.49}$$

where $\sigma(x) = c(x)^d$. Since A is symmetric with respect to the scalar product (6.46), the unitarily transformed operator

$$\tilde{A} = \sigma^{1/2} A \sigma^{-1/2} \quad , \tag{6.50}$$

acting on the half-densities

$$\tilde{\phi}(x) = \sigma(x)^{1/2}\phi(x) \quad , \tag{6.51}$$

is symmetric with respect to the corresponding "flat" ($\sigma(x) \to 1$) scalar product

$$< \tilde{\phi}_1, \tilde{\phi}_2 > := \int dx \, d\Omega \; \tilde{\phi}_1^* \tilde{\phi}_2 = (\phi_1, \phi_2) \quad . \tag{6.52}$$

We now assume that the metric develops a singularity at some value of x, where e.g. the area radius goes to zero, $r \equiv c(x) \to 0$, which we may as well choose to happen at $x = 0$. Thus we consider $x \in (0, \infty)$ and take $\Sigma = \mathbb{R}^{n-1} \backslash \{0\}$, parametrised by x and the angular coordinates.

Then the initial domain of $\tilde{A}$ is $D(\tilde{A}) = C_0^\infty(\mathbb{R}^{n-1} \backslash \{0\})$ or $\tilde{D}(\tilde{A}) = C_0^\infty(\mathbb{R}_+) \otimes C^\infty(S^d)$, which are dense in the unitarily transformed Hilbert space

$$L^2(\mathbb{R}^{n-1} \backslash \{0\}, dx \, d\Omega) \cong L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) \quad . \tag{6.53}$$

Decomposing the second factor into eigenspaces of the Laplacian Δ_d on S^d,

$$L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) = \bigoplus_{\ell=0}^{\infty} L_\ell \quad , \tag{6.54}$$

and defining $\tilde{D}_\ell = \tilde{D} \cap L_\ell$, one has

$$\tilde{A}|_{\tilde{D}_\ell} = \tilde{A}_\ell \otimes \mathbb{I} \quad , \tag{6.55}$$

where

$$\tilde{A}_\ell = -\partial_x^2 + V_{\text{eff},\ell}(x) \tag{6.56}$$

with $V_{\text{eff},\ell}(x)$ given in (6.16).

Questions about the original operator A can thus be reduced to questions about the family $\{\tilde{A}_\ell\}$ of standard Schrödinger-type operators. For example, to show that A is essentially self-adjoint on $D(A)$ it is sufficient to prove that, for each ℓ, $\tilde{A}_\ell$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$.

While one can analyse this question of self-adjointness just as readily for $\eta = +1$ as for $\eta = -1$, the physical significance of this condition in the case of spacelike singularities is not clear to us. Thus we will focus on static space-times with timelike singularities in the following and set $\eta = -1$. An extension of the general formalism to stationary non-static space-times is developed in [**96**].

We conclude this section with a comment on the choice of Hilbert space structure. The L^2 Hilbert space introduced above is certainly a natural choice, but not the only one possible. Based on physical requirements such as the finiteness of the energy of scalar field probes, other (Sobolev) Hilbert space structures have been proposed in the literature - see e.g. [**97, 98**]. The energy is, by definition,

$$E[\phi] = \int_\Sigma \sqrt{h} d^{n-1}x \, T_{\mu\nu}(\phi) \xi^\mu n^\nu \quad , \tag{6.57}$$

where $T_{\mu\nu}(\phi)$ is the stress energy tensor of the scalar field, $\xi = \partial_y$ is the timelike Killing vector, and n the unit normal to Σ. In the present case this reduces to

$$E[\phi] = \int_\Sigma \sigma d^{n-1}x \, T_{yy} \quad , \tag{6.58}$$

which identifies T_{yy} as the energy density with respect to the measure $\sigma d^{n-1}x$ employed above [**98**]. For a minimally coupled complex scalar field one has

$$T_{yy} = \frac{1}{2}\left[\partial_y\phi^*\partial_y\phi + a^2 h^{ij}\partial_i\phi^*\partial_j\phi\right] \quad . \tag{6.59}$$

Thus, with an integration by parts (certainly allowed for $\phi \in D(A)$) the energy can be written as

$$\begin{aligned} E[\phi] &= \int_\Sigma \sigma d^{n-1}x \; (\partial_y\phi^*\partial_y\phi + \phi^* A\phi) \\ &= (\partial_y\phi, \partial_y\phi) + (\phi, A\phi) \quad . \end{aligned} \tag{6.60}$$

For a comparison of the two definitions (6.58) and (6.60) of the energy and the role of boundary terms, see e.g. the discussion in [**99**] and the comment in section 3.5 below. Adopting the expression (6.60) as the definition of the energy suggests introducing a Sobolev structure on the space of scalar fields using the quadratic form

$$Q_A(\phi) = (\phi, A\phi) \tag{6.61}$$

associated to A, via [**97, 98**]

$$||\phi||^2_{H^1} = (\phi,\phi) + Q_A(\phi) \quad , \tag{6.62}$$

thus enforcing the condition that the energy be finite. For present purposes we simply note that at least for the Friedrichs extension A_F of A, based on the closure of the quadratic form $Q_A(\phi)$ with respect to the L^2 norm, the resulting potential energy $Q_{A_F}(\phi)$ is finite (and positive) by definition without having to invoke Sobolev spaces (see also the discussion in [**100, 101**]).[3] We will use specifically this extension in the discussion of section 3.5 below.

[3]Working with such a Sobolev space structure is certainly possible but also complicates the determination of self-adjoint extensions of A, since e.g. studying the closure of A now involves studying the sixth order operator A^3, arising from the term $||A\phi||^2_{H^1} = (A\phi, A\phi) + (A\phi, A^2\phi)$ in the operator norm.

3.2. Essential Self-Adjointness and the Horowitz-Marolf Criterion. The spatial part A of the wave operator is real and symmetric (with respect to an appropriate scalar product on a C_0^∞ domain of A), and as such has self-adjoint extensions, each leading to a well defined (and reasonable [**100**]) time-evolution. If the self-adjoint extension is not unique, however, i.e. if the operator is not essentially self-adjoint, then also the corresponding time-evolution is not uniquely determined. Thus the Horowitz-Marolf criterion [**81**] (unique time-evolution without having to impose boundary conditions at the singularity) amounts to the condition that the operator A be essentially self-adjoint.

To test for essential self-adjointness [**102, 103**], one can e.g. use [**81**] the standard method of Neumann deficiency indices or the Weyl limit point – limit circle criterion (employed in this context in [**104**]). Roughly speaking, in order for A to be essentially self-adjoint the (effective) potential $V_{\text{eff},\ell}$ appearing in the operator $\tilde{A}_\ell$ has to be sufficiently repulsive near $x = 0$ to prevent the waves $\tilde{\phi}$ from leaking into the singularity.

Concretely, in the present case, where we only have control over the operator A near the singularity at $x = 0$, the criteria for the operator $\tilde{A}_\ell$ to be essentially self adjoint on $C_0^\infty(\mathbb{R}_+)$ at $x = 0$ boil down to the following elementary conditions on the effective potential $W \equiv V_{\text{eff},\ell}$ [**102, 103**]:

- If

$$W(x) \geq \frac{3}{4}x^{-2} \tag{6.63}$$

  near zero, then $-\partial_x^2 + W(x)$ is essentially self-adjoint at $x = 0$.
- If for some $\epsilon > 0$

$$W(x) \leq \left(\frac{3}{4} - \epsilon\right) x^{-2} \tag{6.64}$$

  (in particular also if $W(x)$ is decreasing) near $x = 0$, then $-\partial_x^2 + W(x)$ is not essentially self-adjoint at $x = 0$.

The significance of the factor 3/4 can be appreciated by looking at the critical (and relevant for us) case of an inverse square potential

$$W(x) = s(s-1)x^{-2} \ . \tag{6.65}$$

In this case the leading behaviour of the two linearly independent solutions of the equation

$$\left(-\partial_x^2 + W(x)\right) \tilde{\phi}_\lambda(x) = \lambda \tilde{\phi}_\lambda(x) \tag{6.66}$$

near $x = 0$ is given by the two linearly independent solutions of the equation

$$\left(-\partial_x^2 + W(x)\right) \tilde{\phi}_0(x) = 0 \ , \tag{6.67}$$

namely

$$\tilde{\phi}_0 \sim x^s \qquad \text{or} \qquad \tilde{\phi}_0 \sim x^{1-s} \tag{6.68}$$

Thus both solutions are square integrable near $x = 0$ when $2s > -1$ and $2(1-s) > -1$, or

$$-\frac{1}{2} < s < \frac{3}{2} \quad \Leftrightarrow \quad s(s-1) < \frac{3}{4} \ , \tag{6.69}$$

and in this range of $c = s(s-1)$ the potential is limit circle and the self-adjoint extension is not unique. Conversely, it follows that for $c \geq 3/4$ the solutions of equation (6.66) for $\lambda = \pm i$ (which are necessarily complex linear combinations of the two linearly independent real solutions) are not square-integrable near $x = 0$. Thus the deficiency indices are zero and the operator is essentially self-adjoint for $c \geq 3/4$.

Even when there are two normalisable solutions, all is not lost however, as it may be indicative of the possibility (or even necessity) to continue the fields and/or the metric through the singularity [**95**]. Evidently, such an analytic continuation requires some thought (to say the least) in the case of Szekeres-Iyer metrics with generic (non-rational) values of p and q.

3.3. The Horowitz-Marolf Criterion for Power-Law Singularities. In the case at hand, timelike singularities of power-law type, the effective potential is given by (6.20) with $\eta = -1$ and $s = qd/4$. We had already seen in section 2.5 that the mass term is never dominant at $x = 0$ and we can therefore also set $m^2 = 0$. Thus the operator of interest is

$$\begin{aligned} \tilde{A}_\ell &= -\partial_x^2 + V_{\text{eff},\ell}(x) \\ V_{\text{eff},\ell}(x) &= s(s-1)x^{-2} + \ell_d^2 x^{p-q} \quad , \end{aligned} \tag{6.70}$$

It is now straightforward to determine for which values of (p, q) the classical singularities at $x = 0$ become regular or remain singular when probed by scalar waves. First of all, we will show that we can reduce the analysis to the case $\ell = 0$:

- For $q < p + 2$, the first term in the potential is dominant and independent of ℓ. Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is essentially self-adjoint. As we know from (6.63), this condition is satisfied iff $s(s-1) \geq 3/4$.
- For $q > p + 2$, the operators $\tilde{A}_\ell$ for $\ell \neq 0$ are essentially self-adjoint by the criterion (6.63). Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is.
- In the borderline case $q = p + 2$, for $\ell \neq 0$ we have

$$\ell \neq 0 \quad \Rightarrow \quad s(s-1) + \ell_d^2 \geq 3/4 \tag{6.71}$$

  (with equality only for $s = 1/2$ and $\ell = d = 1$). Even in this case, therefore, all the $\tilde{A}_\ell$ with $\ell \neq 0$ are essentially self-adjoint and only $\tilde{A}_{\ell=0}$ needs to be examined.

We can thus conclude that the operator A is essentially self-adjoint iff $s(s-1) \geq 3/4$ and that, in view of (6.69), it fails to be essentially self-adjoint for

$$A \;\text{ not e.s.a.} \quad \Leftrightarrow \quad -\frac{1}{2} < s < \frac{3}{2} \quad \Leftrightarrow \quad -\frac{2}{d} < q < \frac{6}{d} \quad . \tag{6.72}$$

3.4. The Significance of the (Strict) Dominant Energy Condition. While this has been rather straightforward, one of the virtues of the present approach, based on using a class of metrics appropriate for a generic singularity of power-law type, is that it allows us to draw a general conclusion regarding the relation between the Horowitz-Marolf criterion and properties of the matter (stress-energy) content of the space-time near the singularity.

Indeed, as we will now show, whenever the matter content of the near-singularity space-time is sufficiently "positive" (in the sense of the strict DEC, as it turns out), the space-time remains singular according to the Horowitz-Marolf criterion, i.e. when probed with scalar waves.

We can deduce from (6.32) that metrics with timelike power-law singularities satisfying the strict DEC lie in a bounded region inside the strip $0 < q < 2/d$ [**25**]. Indeed, for

$q < p + 2$ only the second terms in (6.32) are relevant, and one finds

$$\rho - P_r = \tfrac{1}{4}dq(2 - dq)\, x^{-(p+2)} \quad . \tag{6.73}$$

Thus one has

$$\rho - P_r > 0 \quad \Leftrightarrow \quad 0 < q < \frac{2}{d} \quad . \tag{6.74}$$

In particular, therefore, it follows from (6.72) that for such metrics the operator A is not essentially self-adjoint and we can draw the general conclusion that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

Even though metrics with $q = 2/d$, say, like negative mass Schwarzschild, still satisfy the bound (6.72), thus remain singular while obeying an extremal equation of state, we cannot strengthen the above statement to include general metrics with extremal equations of state. This can be seen e.g. from examples in [**81**] and is due to the fact that extremal metrics can also be found elsewhere in the (p, q)-plane, in particular in the region $q > p+2$, while violating the bound (6.72).

3.5. The Friedrichs Extension and "Hospitable" Singularities. In the previous section we have discussed self-adjoint extensions of (the spatial part A of) the Klein-Gordon operator. We have not discussed, however, what these self-adjoint extensions imply about the behaviour of the allowed scalar fields ϕ (those in the domain of the self-adjoint extension of A) near the singularity at $x = 0$.

It is certainly possible that self-adjointness can be achieved by allowing only scalar fields that vanish at the singularity. In some sense, then, the singularity remains excluded from the space-time and is not probed directly by the scalar field ϕ. We will see that this is indeed what happens in (in a precise sense) one half of the cases in which there is a unique self-adjoint extension.

However, it is a priori also possible (and perhaps more interesting) to have a well-defined time-evolution (which we take to mean "defined by some self-adjoint extension" [**100**]) with scalar fields that are permitted to be non-zero at the singularity. In that case, the singularity would be probed more directly by the scalar field, and one might then perhaps like to define a classical singularity to be "hospitable" (for a scalar field), if there is a self-adjoint extension which allows the scalar fields to take non-zero values at the locus of the singularity. We will see that this possibility is indeed realised as well, not only for the other half of the essentially self-adjoint cases, but also for e.g. the Friedrichs extension A_F of the operator A in a certain range of parameters for which A is not essentially self-adjoint.

To address these issues, we need to determine the domain of definition of the relevant self-adjoint extension of $\tilde{A}_0 = -\partial_x^2 + cx^{-2}$ for $c = s(s-1) \in [-1/4, \infty)$. For $\tilde{A}_0$ essentially self-adjoint, i.e. $c \geq 3/4$, this can be done by explicitly determining the domain of the closure $\bar{A}_0$ of the operator $\tilde{A}_0$. While we have done this (see also [**105**]), alternatively, for all $c \geq -1/4$, one can determine the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$, constructed from the closure of the associated quadratic form. For $c \geq 3/4$, such that $\tilde{A}_0$ is essentially self-adjoint, its unique self-adjoint extension of course agrees with the Friedrichs extension. Precisely this problem has been addressed and solved in [**106**], and instead of reinventing the wheel here we can draw on the results of that reference to analyse the issue at hand.

The main result of [**106**] of interest to us is their Theorem 6.4. Applied to the operator $\tilde{A}_0$, this theorem[4] states that the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$ is

$$\begin{aligned} D(\tilde{A}_F) = \{f \in L^2(0,\infty) : f(0) = 0, f \in A(0,\infty), \partial_x f \in L^2(0,\infty), \\ x^{-1} f \in L^2(0,\infty), (-\partial_x^2 + cx^{-2}) f \in L^2(0,\infty)\} \end{aligned} \tag{6.75}$$

where $A(0,\infty)$ denotes the space of absolutely continuous functions. In [**106**], this result was established for $c > 0$. As far as we can see, this result is correct, as it stands, also for $-1/4 < c < 0$. We will comment on the special case $c = -1/4$ below.

We will now extract from this result some restrictions on the behaviour of f near $x = 0$ (assuming that we can model the leading behaviour of f as $x \to 0$ by some power of x):

(1) From the condition $x^{-1} f \in L^2$ we learn that $f(x) \sim x^{\frac{1}{2}+\epsilon}$ for some $\epsilon > 0$. Then the conditions $f(0) = 0$ and $\partial_x f \in L^2$ are also satisfied.
(2) The remaining condition $(-\partial_x^2 + cx^{-2}) f \in L^2$ can be satisfied in one of two ways. Either both terms separately are in L^2 or f lies in the kernel of the operator (as $x \to 0$). In the former case, we find the condition $f(x) \sim x^{\frac{3}{2}+\epsilon}$ with $\epsilon > 0$. In the latter case, since the two functions in the kernel are x^s and x^{1-s}, with (as usual) $c = s(s-1)$, we now need to distinguish several cases:
   (a) $c > 3/4$: this means that $s > 3/2$ or $s < -1/2$. The solution x^s with $s > 3/2$, i.e. $f(x) \sim x^{\frac{3}{2}+\epsilon}$, yields nothing new. The solution x^{1-s} with $s > 3/2$ (or, equivalently, the solution x^s with $s < -1/2$) is ruled out by condition 1.
   (b) $c = 3/4$: this means that $s = 3/2$ or $s = -1/2$. In this case, we can allow $x^{3/2}$ and thus relax the domain to include functions $f(x) \sim x^{\frac{3}{2}+\epsilon}$, now with $\epsilon \geq 0$.
   (c) $-1/4 < c < 3/4$: thus $-1/2 < s < 3/2$ and $s \neq 1/2$. Thus the solution x^s is adjoined to the functions $\{x^{\frac{3}{2}+\epsilon}\}$ for $s > 1/2$, and the solution x^{1-s} for $s < 1/2$.

It remains to discuss the special value $c = -1/4$ or $s = 1/2$ which is not covered by the formulation of the domain in (6.75). This is the minimal allowed value of interest to us ($c = s(s-1)$ with s real), and also the minimal value for which the operator remains positive (and thus has a Friedrichs extension). In this case, the two solutions are $x^s = x^{\frac{1}{2}}$ and $x^{\frac{1}{2}} \log x$, and we checked that, as expected, the domain of the Friedrichs extension includes $x^{1/2}$. This can also be deduced e.g. from [**107**], which moreover illustrates nicely some of the weirdness of non-Friedrichs extensions.

The above discussion shows that the two definitions (6.58) and (6.60) of the energy, a priori differing by boundary terms due to the integration by parts, agree for the Friedrichs extension for $c > -1/4$ and differ only by a finite term for $c = -1/4$. The issue of boundary terms for more general domains is discussed in [**99**].

Returning to the original question of determining the behaviour of the allowed scalar fields in the domain of the self-adjoint extension of the spatial part A of the Klein-Gordon operator, we need to now undo the transformation $\phi \to \tilde{\phi}$ from the initial scalar fields

[4] Actually, in [**106**] a more general operator, including in particular a non-zero harmonic oscillator term Bx^2, was studied. However, this term serves only to regularise the wave functions at infinity. Since we are concerned with the behaviour at $x = 0$, this term is of no consequence for the present considerations.

ϕ to the half-densities $\tilde{\phi}$ that we performed in section 2.1 to put A into the form of a standard Schrödinger operator.

This transformation back from $\tilde{\phi}$ to ϕ is accomplished by multiplication by x^{-s}. Now the upshot of the above discussion is that the lowest power of x appearing in the domain of $\tilde{A}_F$ is

$$\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon} & \text{for } s > 3/2 \text{ or } s < -1/2 \\ x^s & \text{for } 1/2 \leq s \leq 3/2 \\ x^{1-s} & \text{for } -1/2 \leq s \leq 1/2. \end{cases} \tag{6.76}$$

Evidently these functions are, in particular, positive powers of x. Thus they, and therefore all functions in the domain, tend to zero for $x \to 0$, consistent with the condition $f(0) = 0$ in (6.75). However this is not necessarily true for the transformed functions, for which one has ($\delta = \delta(s) > 0$ is a positive real number depending on s)

$$\phi_{\min} = x^{-s}\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon-s} = x^{-\delta} & \text{for } s > 3/2 \\ x^0 = 1 & \text{for } 1/2 \leq s \leq 3/2 \\ x^{1-2s} = x^{\delta} & \text{for } -1/2 \leq s < 1/2 \\ x^{\frac{3}{2}+\epsilon-s} = x^{2+\delta} & \text{for } s < -1/2 \end{cases} \tag{6.77}$$

The final result is the simple statement that a ϕ in the domain of the Friedrichs extension A_F of A necessarily goes to zero for $s < 1/2$, ϕ can be non-zero (but remains bounded) for $1/2 < s \leq 3/2$, and can become increasingly singular for large $s > 3/2$.

Note that this statement is not invariant under $s \to 1 - s$. Indeed, while the operator $-\partial_x^2 + s(s-1)x^{-2}$ has this invariance, and therefore also statements about its essential self-adjointness are symmetric under $s \to 1 - s$ (as we have seen), the unitary transformation between ϕ and $\tilde{\phi}$ depends linearly on s and thus leads to a behaviour of the original scalar fields ϕ that does not have this symmetry.

Once again we find a pleasing relation with the DEC, since the watershed happens exactly at $s = 1/2 \Leftrightarrow q = 2/d$ which, as we have seen, corresponds to $\rho = P_r$. Timelike singularities satisfying the strict DEC have $0 < q < 2/d$ (6.74), thus $0 < s < 1/2$. Moreover, metrics with $s \leq -1/2$ have a unique self-adjoint extension ($c \geq 3/4$), thus are regular in the Horowitz-Marolf sense, but are not "hospitable" in the sense described above, while those with $s \geq 3/2$ are.

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Subsequent Alterations

The coordinates (u, v) originally introduced in (6.35) have been changed to (x^+, x^-)

CHAPTER 7

Conclusion

Null Fermi coordinates are an obvious tool to investigate various physical processes involving lightlike particles in a geometrically transparent setup. Moreover, in combination with the standard timelike Fermi coordinates it might be interesting to implement the historical interpretation of the Penrose limit directly in terms of a continuous transformation (if possible) interpolating between both coordinate systems.

At the moment there is not much I can add to Penrose-Fermi metric or string expansions except maybe one technicality. As we have seen in order to establish the equivalence between both expansions the lightcone gauge is crucial, whereas the conformal gauge is not only dispensable but also not necessarily (additionally) available in the first place. Hence it might be worthwile to explicitly get rid of the latter in all the steps of our formalism.

There is a bit more to say about universality instead. Despite its many implications for the Penrose limit in general and the singular homogeneous plane waves in particular at the moment it seems to me that universality primarily tells us something about probes in the vicinity of spacetime singularities. This point of view is underlined by the striking "transferability" from null geodesic congruences to scalar fields and consequently one should check if universality arises also for other probes, e.g. timelike congruences, and in different backgrounds, e.g. stationary (non-static) spacetimes.

Generically, what is most intriguing about universality is how the strict Dominant Energy Condition, i.e. a physically sensible restriction on the metric, gets itself communicated to the kinematics of probes rendering them "well-behaved" in the sense that the linear operator describing their motion is restricted to be regular singular. Although this is very charming from the physical point of view it also raises a disturbing

QUESTION: Why should one actually assume the strict DEC to know anything about probes in the first place?

A first step to settle this question should be to check how much of the linear operator structure can be fixed by purely "dimensional" arguments. For example, as the separated angular momentum is the only dimensional parameter (except the affine parameter or tortoise coordinate) in the operator it is not surprising that the corresponding term is the only one generically allowing for a non-universal behaviour.

Next, one might concentrate on the old relationship between congruences and energy conditions in the context of singularity theorems in the hope to extract some information about the importance of the congruence being mathematically well-behaved. It would also be of some interest to reconsider the relation of the DEC to the strong and weak energy conditions usually employed in this context. Here, special attention should be given to the attribute "strict" so dearly needed for universality while on the other hand not really enforced by physical arguments. On the contrary, as already pointed out by Szekeres and Iyer **[77]** solutions with an extremal matter content show up quite generically and the question is still open if this behaviour is a mere artefact of spherical symmetry.

Once, universality is understood in terms of geodesic congruences it would be interesting to make the connection to scalar fields more “direct”, i.e. implement it on a technical level, maybe using the geometric optics approximation or something similar. For example it would be interesting to relate the affine parametrisation of the geodesic to the unitary transformation of the scalar field.

APPENDIX A

Basics

1. Null Geodesic Deviation

In this section we want to derive the null geodesic deviation equation following [**42**] and [**25**] respectively, providing the relation between (2.72) and (2.73). To this end we embed the given null geodesic γ into an arbitrary but fixed null geodesic congruence, described by the vector field k, i.e.

(A.1) $$\dot{x}^\mu = k^\mu, \quad g_{\mu\nu}k^\mu k^\mu, \quad k^\mu \nabla_\mu k^\nu = 0.$$

with $k|_\gamma = \dot{\gamma}$. Then we construct an adapted parallel pseudo-orthonormal frame E^A, $A = +, -, a$, i.e.

(A.2) $$ds^2 = 2E^+E^- + \delta_{ab}E^aE^b, \quad \nabla_\tau E^A = 0$$

such that $E_+ = k$. This automatically implies

(A.3) $$\nabla_{E_+}E_+ = 0, \quad E_+|_\gamma = \dot{\gamma}.$$

Next we select a 1-parameter family $X(\tau,\sigma)$ of geodesics from the congruence, s.t. for a given σ_0, $X(\tau,\sigma_0)$ is a null geodesic and $X(\tau,0) = \gamma(\tau)$. Infinitesimally close to γ this family is characterised by a *deviation vector field* or *Jacobi field*

(A.4) $$\xi(\tau) = \partial_\sigma|_{\gamma(\tau)} = \partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0}\partial_\mu$$

along γ representing the separation of corresponding points on neighbouring curves[1]. It obviously commutes with $E_+ = \partial_\tau$, i.e. is Lie transported along γ

(A.5) $$L_{E_+}\xi = [E_+,\xi] = \nabla_{E_+}\xi - \nabla_\xi E_+ = [\partial_\tau, \partial_\sigma] = 0.$$

We now make use of the E_A being parallel to replace the covariant derivatives along the congruence/family by partial ones,

(A.6) $$E_+^\mu \nabla_\mu(\xi^A E_A) = \nabla_\tau(\xi^A E_A) = (\partial_\tau \xi^A)E_A.$$

Then, as E_+ is null we find

(A.7) $$(\nabla_A E_+)^- = g(E_+, \nabla_A E_+) = \partial_A g(E_+,E_+) = 0,$$

and similarly for the frame component ξ^- using the geodesic equation (A.3) and (A.5)

(A.8) $$\frac{d}{d\tau}\xi^- = \nabla_{E_+}g(E_+,\xi) = g(E_+,\nabla_{E_+}\xi) = g(E_+,\nabla_\xi E_+) = \partial_\sigma g(E_+,E_+) = 0.$$

The latter equation can be interpreted in terms of light rays emitted from the same source but at different times maintaining a constant separation in time [**42**]. As one is usually only interested in spatial separations one sets

(A.9) $$\xi^- = 0.$$

Using (A.3), (A.5), (A.7) and (A.9) we find that

(A.10) $$\nabla_{E_+}\xi = \nabla_\xi E_+ = \xi^b(\nabla_b E_+)^a E_a + \xi^b(\nabla_b E_+)^+ E_+$$

---

[1] For a geometrical construction of such a family see [**108**, Section 2.1]

i.e.

$$\frac{d}{d\tau}\xi^a = B^a{}_b\xi^b \tag{A.11}$$

with

$$B^a{}_b = (\nabla_b E_+)^a \equiv E^a_\nu E^\mu_b \nabla_\mu E^\nu_+, \tag{A.12}$$

and ξ^+ is determined by the ξ^a

$$\frac{d}{d\tau}\xi^+ = \xi^b(\nabla_b E_+)^+. \tag{A.13}$$

The frame component ξ^+ describes infinitesimal displacements tangential to γ (reparametrisations) and is usually discarded together with ξ^-.

We infer from (A.11) that the transverse components ξ^a satisfy the null geodesic deviation equation

$$\frac{d^2}{d\tau^2}\xi^a = A_{ab}(\tau)\xi^b. \tag{A.14}$$

where

$$A^a{}_b = \frac{d}{d\tau}B^a{}_b + B^a{}_c B^c{}_b. \tag{A.15}$$

Please note that (A.14) is nothing else but a time-dependent harmonic oscillator equation with $(-A_{ab}(\tau))$ the matrix of frequency squares.

A straight forward calculation shows that

$$A^a{}_b = E^a_\nu E^\mu_b R^\nu{}_{\lambda\rho\mu}\dot\gamma^\lambda\dot\gamma^\rho = -R^a{}_{+b+}, \tag{A.16}$$

with R the Riemann curvature tensor of the metric g, establishing the equivalence of (2.72) and (2.73).

Note that we can reinterpret the results above in terms of the expansion, shear and twist of a null geodesic congruence (cf. [**42**, Section 4.2] or [**73**, Section 9.2]), which are identical to the trace, trace-free symmetric and anti-symmetric part of

$$B_{ab} = E^\nu_a E^\mu_b \nabla_\mu k_\nu, \tag{A.17}$$

respectively, i.e.

$$\begin{aligned} \theta &= \operatorname{tr} B \equiv B^a{}_a \\ \sigma_{ab} &= B_{(ab)} - \frac{1}{d}\theta\delta_{ab} \\ \omega_{ab} &= B_{[ab]}, \end{aligned} \tag{A.18}$$

implying the decomposition

$$B_{ab} = \frac{1}{d}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab}. \tag{A.19}$$

The combination of (A.15) and (A.16) leads to

$$\frac{d}{d\tau}B^a{}_b + B^a{}_c B^c{}_b + R^a{}_{+b+} = 0. \tag{A.20}$$

and after substitution of B_{ab} by (A.19) we get for the trace of (A.20)

$$\dot\theta = -\frac{1}{d}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{++} \tag{A.21}$$

describing the time evolution for θ. This is nothing else than the null version of the *Raychaudhuri equation* in $D = d + 2$ space-time dimensions. Similarly, we can calculate the symmetric traceless and antisymmetric part of (A.20), leading to the time-evolutions of shear and twist.

Nevertheless, as we can see from (A.16) although B_{ab} obviously depends on the specific properties of the null geodesic congruence, the combination of expansion, shear and twist and their derivatives appearing in A_{ab} depends only on the components of the curvature tensor and the parallel frame along the original null geodesic. In particular, the geodesic deviation matrix $A_{ab}(\tau)$ is independent of how the null geodesic γ is embedded into some null congruence.

Finally, we note that (A.3) and (A.7) imply that the trace of B is

$$\operatorname{tr} B \equiv B^a{}_a = \nabla_\mu E_+^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}k^\mu), \tag{A.22}$$

a result we will use in the following section.

2. Null Geodesic Deviation in a Static Spherically Symmetric Metric

Here following **[25]** we calculate the transverse null geodesic deviation matrix, i.e. the Penrose limit plane wave profile, for a general static spherically symmetric metric written in Schwarzschild-like coordinates[2]

$$\begin{aligned} ds^2 =& - f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_d^2 \\ d\Omega_d^2 =& d\theta^2 + \sin^2\theta d\Omega_{d-1}^2. \end{aligned} \tag{A.23}$$

Because of the rotational symmetry in the transverse directions, one can choose w.l.o.g. the null geodesic to lie entirely in the (t, r, θ)-plane, reducing the geodesic equations to the first integrals

$$\dot{t} = E/f(r), \quad \dot{\theta} = L/r^2, \quad \dot{r}^2 = E^2/f(r)g(r) - L^2/g(r)r^2, \tag{A.24}$$

with E and L denoting the conserved energy and angular momentum respectively. This defines a geodesic congruence with Hamilton-Jacobi function **[108]**

$$S = -Et + L\theta + R(r) \tag{A.25}$$

where

$$\left(\frac{d}{dr}R\right)^2 = gf^{-1}E^2 - r^{-2}gL^2. \tag{A.26}$$

In order to calculate the tensor field B_{ab} of the previous section 2 one also needs an adapted parallel frame. Obviously

$$E_+ = \dot{r}\partial_r + \dot{t}\partial_t + \dot{\theta}\partial_\theta, \quad E_+|_\gamma = \partial_\tau, \tag{A.27}$$

and there is no need to specify E_- any further. The transverse orthonormal frame is $E_a = (E_1, E_{\hat{a}})$, where $\hat{a} = 2, \ldots, d$ refer to the transverse $(d-1)$-sphere. As no time-evolution takes place in these directions, the $E_{\hat{a}}$ are trivially given by

$$E_{\hat{a}} = \frac{1}{r\sin\theta}e_{\hat{a}} \tag{A.28}$$

with $e_{\hat{a}}$ an orthonormal coframe for $d\Omega_{d-1}^2$. The transverse $SO(d)$-symmetry implies in the notation of section 2

$$B_{1\hat{a}} = A_{1\hat{a}} = 0, \quad B_{\hat{a}\hat{b}}(\tau) = B(\tau)\delta_{\hat{a}\hat{b}}, \quad A_{\hat{a}\hat{b}}(\tau) = A(\tau)\delta_{\hat{a}\hat{b}}. \tag{A.29}$$

[2] For examples as well as the extension to isotropic coordinates, brane-like metrics with extended world volumes or null metrics see **[25]**

Furthermore, using (A.22) one finds

(A.30) $$B_{11}(\tau) = \nabla_\mu \dot{x}^\mu(\tau) - (d-1)\,\mathrm{tr}\,B(\tau).$$

and for $B_{22}(\tau) = B(\tau)$ (assuming $e_2 = \partial_\phi$)

(A.31) $$B_{22} = \Gamma^\phi_{\phi r}\dot{r} + \Gamma^\phi_{\phi\theta}\dot{\theta} = \partial_\tau \log(r(\tau)\sin\theta(\tau)),$$

or

(A.32) $$B_{\hat{a}\hat{b}}(\tau) = \delta_{\hat{a}\hat{b}}\partial_\tau \log(r(\tau)\sin\theta(\tau)).$$

As

(A.33) $$\mathrm{tr}\,B = \partial_\tau \log\left(\dot{r}r^d \sin^{d-1}\theta\sqrt{f(r)g(r)}\right)$$

one has

(A.34) $$B_{11}(\tau) = \partial_\tau \log\left(r(\tau)\dot{r}(\tau)\sqrt{f(r(\tau))g(r(\tau))}\right).$$

If $B_{ab}(\tau)$ is of the logarithmic derivative form

(A.35) $$B_{ab}(\tau) = \delta_{ab}\partial_\tau \log K_a(\tau)$$

one finds using (A.15)

(A.36) $$A_{ab}(\tau) = \delta_{ab}K_a(\tau)^{-1}\partial_\tau^2 K_a(\tau)$$

and thus

(A.37) $$\begin{aligned} A_{11} =& (r\dot{r}\sqrt{fg})^{-1}\partial_\tau^2(r\dot{r}\sqrt{fg}) \\ A_{\hat{a}\hat{b}} =& \delta_{\hat{a}\hat{b}}(r\sin\theta)^{-1}\partial_\tau^2(r\sin\theta). \end{aligned}$$

For the transverse components this leads to

(A.38) $$A_{\hat{a}\hat{b}}(\tau) = \delta_{\hat{a}\hat{b}}\left(\frac{\ddot{r}(\tau)}{r(\tau)} - \frac{L^2}{r(\tau)^4}\right).$$

3. Adapted Penrose coordinates and Hamilton-Jacobi theory

Here we repeat the argument given in **[25]** that locally a general metric $g_{\mu\nu}$ can always be brought to the Penrose form (2.52) using the Hamilton-Jacobi formalism.

The main observation is that the momenta

(A.39) $$p_\mu = g_{\mu\nu}\frac{dx^\nu}{d\tau} = k_\mu$$

associated with the null congruence $k^\mu = (\dot{\tilde{u}} = 1, \dot{\tilde{v}} = \dot{\tilde{y}}^k = 0)$ are

(A.40) $$p_{\tilde{v}} = 1, \quad p_{\tilde{u}} = p_k = 0.$$

Translated to arbitrary coordinates x^μ this means

(A.41) $$p_\mu = \partial_\mu \tilde{v}.$$

and as the geodesic congruence is null by assumption, $g^{\mu\nu}\partial_\mu\tilde{v}\partial_\nu\tilde{v} = 0$, one can identify

(A.42) $$\tilde{v}(x^\mu) = S(x^\mu),$$

with the solution of the (separated) Hamilton-Jacobi equation **[108]**

(A.43) $$g^{\mu\nu}\partial_\mu S\partial_\nu S = 0,$$

associated with the null congruence (A.40),

(A.44) $$\dot{x}^\mu = g^{\mu\nu}\partial_\nu S.$$

Vice versa, any solution S of the equations (A.43,A.44) corresponds to a congruence which is null by equation (A.43), geodesic

$$\dot{x}^\rho \nabla_\rho \dot{x}^\mu = g^{\rho\sigma} g^{\mu\nu} \nabla_\rho \partial_\nu S \partial_\sigma S = \frac{1}{2} g^{\mu\nu} \partial_\nu (g^{\rho\sigma} \partial_\rho S \partial_\sigma S) = 0, \tag{A.45}$$

and (away from singularities of S) twist-free[3]. (cf. (A.17 and A.18)

$$\nabla_{[\mu} \dot{x}_{\nu]} = \nabla_{[\mu} \partial_{\nu]} S = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) S = 0. \tag{A.46}$$

In order to construct the transverse coordinates $\tilde{y}^k$ one has to go a little bit deeper into the Hamilton-Jacobi formalism. Finding explicitly the most general solution to the Hamilton-Jacobi equation (A.43) can be quite complicated in practice but one usually relies on the existence of a *complete solution* labelled by $D = d + 2$ integration constants α_μ **[109, 110]**. The corresponding geodesic congruence is given by $x^\mu = x^\mu(\tau, \alpha_\mu, x_0^\mu)$, with x_0^μ denoting the positions of the geodesics at instant $\tau = 0$, $x_0^\mu = x^\mu(0, \alpha_\mu, x_0^\mu)$. The set of the x_0^μ is assumed to form a (properly submersed) Cauchy hypersurface for the Hamilton-Jacobi equation described by $F(x_0^\mu) = 0$ with everywhere timelike normal vector field $g^{\mu\nu} \partial_\mu F \partial_\nu F < 0$.

Two integration constants are trivial. To see this first note that one of the α_μ corresponds to a constant shift of S. Second, the Hamilton-Jacobi equation is homogeneous of degree two and hence, if S is a solution, then κS, with $\kappa = \text{const} \neq 0$, is another one. This scale invariance is absorbed in the first order geodesic equations, (A.44) by rescaling the affine parameter τ.

The remaining integration constants α_k, $k \in \{1, \ldots, d\}$ are completely fixed by choosing a null geodesic γ with initial momentum $p_\mu^0 = g_{\mu\nu} \dot{x}^\nu|_{\tau=0}$. Note that the mass-shell condition $g^{\mu\nu} p_\mu^0 p_\nu^0 = 0$ is scale invariant and consequently there are only d independent components which can be used to determine the integration constants of the Hamilton-Jacobi function S via the equation

$$p_\mu^0 = \partial_\mu S|_{\tau=0}. \tag{A.47}$$

Given a null geodesic γ, the coordinate transformation from the original coordinates x^μ to Penrose coordinates can be defined using the Hamilton-Jacobi function S and the coordinates x_0^μ of the Cauchy hypersurface in the following way. First, one parametrises the null geodesic congruence as described above (the α_k are given by the momentum of the corresponding geodesic)

$$x^\mu = x^\mu(\tau, x_0^\nu), \quad F(x_0^\mu) = 0, \tag{A.48}$$

and sets

$$\tilde{u} = \tau, \quad \tilde{v} = S(x_0^\mu). \tag{A.49}$$

Note that for this definition to be consistent one needs $S(x^\mu) = S(x_0^\mu)$. This is obviously the case as $\frac{d}{dt} S(x^\mu) = \frac{\partial S}{\partial x^\mu} \dot{x}^\mu = p_\mu \dot{x}^\mu = 0$.

The level sets of S have a null normal vector according to (A.43), whereas for the hypersurface $F = 0$ it is timelike. Therefore $g^{\mu\nu} \partial_\mu S \partial_\nu F < 0$, i.e. the level sets of S transversely intersect the hypersurface $F = 0$. The coordinates $\tilde{y}^k$ are determined by solving the equations $F(x_0^\mu) = 0$ and $S(x_0^\mu) = \tilde{v}$ respectively, i.e. $\tilde{y}^k$ are the coordinates of the intersection

[3]For completeness we also note that the shear of the congruence are directly related to covariant *Hessian* of the Hamilton-Jacobi function via $\nabla_{(\mu} \dot{x}_{\nu)} = \nabla_{(\mu} \partial_{\nu)} S$ and the expansion to its *Laplacian* $\nabla_\mu \dot{x}^\mu = \nabla_\mu \partial^\mu S$.

submanifold and one can rewrite the first equation in (A.48) as the desired transformation relating an arbitrary coordinate system to Penrose coordinates[4]

(A.50) $$x^\mu = x^\mu(\tilde{u}, x_0^\nu(\tilde{v}, \tilde{y}^k)) = x^\mu(\tilde{u}, \tilde{v}, \tilde{y}^k).$$

To see explicitly that in these coordinates the metric takes the form (2.52) note that

(A.51) $$g_{\tilde{u}\tilde{u}} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} = 0$$

because the geodesics $x^\mu(\tau, x_0^\nu)$ are null. Moreover,

(A.52) $$g_{\tilde{u}\tilde{v}} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tilde{v}} = g_{\mu\nu}g^{\mu\rho}\partial_\rho S\frac{\partial x^\nu}{\partial \tilde{v}} = \frac{\partial x^\mu}{\partial \tilde{v}}\partial_\mu S = \frac{\partial \tilde{v}}{\partial \tilde{v}} = 1$$

(A.53) $$g_{\tilde{u}i} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tilde{y}^i} = g_{\mu\nu}g^{\mu\rho}\frac{\partial S}{\partial x^\rho}\frac{\partial x^\nu}{\partial \tilde{y}^i} = \frac{\partial \tilde{v}}{\partial \tilde{y}^i} = 0.$$

4. Plane Gravitational Waves in $D = 4$

In order to understand the physical interpretation of gravitational waves one considers a congruence of null geodesics (gravitons) described by the vector field k. Following the procedure in section 2 we can introduce an adapted quasi-orthonormal frame $E_+ = k$, E_- and E_a to describe the congruence in terms of the tensor field $B_{ab} = E_a^\nu E_b^\mu \nabla_\mu k_\nu$ (A.17) or equivalently in terms of its shear θ, expansion σ_{ab} and twist ω_{ab} (A.18) whereas in the present context it suffices to consider the contracted versions of the latter, i.e. using $D = 4 = d + 2$

(A.54) $$\theta = B^a{}_a, \quad |\sigma| = \sqrt{B_{(ab)}B^{(ab)} - \frac{1}{2}\theta^2}, \quad \omega = \sqrt{B_{[ab]}B^{[ab]}}.$$

In [**5**] a *plane-fronted gravitational wave* in $D = 4$ is defined to be a vacuum space-time which contains a shear-free geodesic null congruence admitting plane wave surfaces, i.e. spacelike 2 surfaces orthogonal to k^μ.

This definition was inspired by electrodynamics, cf. [**111**]. Recall that in a plane electromagnetic wave there exists a vector k^μ tangent to the light rays (photons) and transverse to the electromagnetic field strength, i.e. $F_{\mu\nu}k^\nu = F^*_{\mu\nu}k^\nu = 0$. Furthermore the quadratic invariants vanish identically $F_{\mu\nu}F^{\mu\nu} = F^*_{\mu\nu}F^{\mu\nu} = 0$. As already mentioned in the introduction the gravitational analogues are precisely the Petrov type N regions of space-time [**112**] with k^μ being the *quadruple* (Debever-Penrose) null vector and $R_{\mu\nu\rho\sigma}k^\sigma = 0$ as well as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 0$ and $R^{\mu\nu\rho\sigma}R^*_{\mu\nu\rho\sigma} = 0$. One then shows using the Kundt-Thompson theorem for type N vacuum solutions and the Bianchi identities that the shear of k^μ vanishes. The existence of plane wave surfaces then implies that expansion and twist vanish as well,

(A.55) $$\theta = |\sigma| = \omega = 0,$$

which is the defining property for the Kundt class of gravitational waves. A well-known subclass are the so called *plane fronted waves with parallel rays* with the property of k^μ being covariantly constant, or parallel for short

(A.56) $$\nabla_\mu k_\nu = 0,$$

enforcing obviously (A.55) by (A.17) and (A.54).

[4]As mentioned in [**25**] generically, there is no natural choice for the hypersurface $F = 0$, i.e. for the function F and the latter should rather be interpreted as a conveniently chosen gauge fixing condition in the sense that the Penrose limit metric is independent of it. Different gauges simply correspond to different ways of labelling the geodesics of the congruence on which the adapted coordinates are based.

5. Symmetries of Plane Waves

5.1. The Heisenberg Isometry Algebra of Generic Plane Waves. The following discussion on the isometries of a generic plane wave metric as well as the remaining sections of this chapter are largely taken from [**113**].

In Brinkmann coordinates only one isometry is manifest, namely that generated by the parallel null vector $k = \partial_-$. In Rosen coordinates however, the metric depends neither on v nor on the transverse coordinates y^k, thus in addition to $k = \partial_v$ there are at least d more Killing vectors, namely the ∂_k. Together these form an Abelian translation algebra acting transitively on the null hypersurfaces of constant u. However, this is only a part of a much larger *solvable* isometry algebra, namely a *Heisenberg algebra* which we will discuss in the following.

Systematically, one can find all Killing vectors m by solving the Killing equations

$$L_m g_{\mu\nu} = \nabla_\mu m_\nu + \nabla_\nu m_\mu = 0. \tag{A.57}$$

Here we merely intent to present the results of this analysis in Brinkmann coordinates, for a more detailed discussion we refer to [**21**].

A generic $D = d+2$-dimensional plane wave metric has a $(2d+1)$-dimensional isometry algebra generated by the Killing vector $k = \partial_-$ and the $2d$ Killing vectors

$$l_{(K)} := l(\xi_{(K)}) = \xi_{(K)a}\partial_a - \dot{\xi}_{(K)a}x^a\partial_-, \tag{A.58}$$

where $\xi_{(K)a}$, $K \in \{1, \dots, 2d\}$ are the $2d$ linearly independent solutions of the ubiquitous *transverse null geodesic deviation equation* (cf. equation (2.73) or directly the discussion in section 1)

$$\ddot{\xi}^a(x^+) = A_{ab}(x^+)\xi^b(x^+). \tag{A.59}$$

This intimate relationship between the (macroscopic) Killing vectors l and the (infinitesimal) deviation vectors ξ for plane waves should not surprise us to much. First of all, the restriction of an any Killing vector m to an arbitrary geodesic is a deviation vector [**114**], simply because the flow of m acting on the geodesic generates a 1-parameter family of curves which because of isometry have to be geodesics as well (cf. section 1). Second, as a generic plane wave can be interpreted as the Penrose limit of a suitable space-time all its global information must be encoded in terms of the null geodesic deviation matrix $A_{ab}(x^+)$ along the central null geodesic $\gamma = (x^+, 0, 0)$. Recall that in chapter 2 at the end of section 3.2 we have already seen that for the same reason the geodesic equations are formally identical to the geodesic deviation equations.

The Killing vectors satisfy the algebra

$$\begin{aligned} [l_{(J)}, l_{(K)}] &= W(\xi_{(J)}, \xi_{(K)})k \\ [l_{(J)}, k] &= 0, \end{aligned} \tag{A.60}$$

where $W(\xi_{(J)}, \xi_{(K)})$, the *Wronskian* of the two solutions, is defined by

$$W(\xi_{(J)}, \xi_{(K)}) = \delta_{ab}(\dot{\xi}^a_{(J)}\xi^b_{(K)} - \dot{\xi}^a_{(K)}\xi^b_{(J)}). \tag{A.61}$$

It is constant, i.e. independent of x^+ as a consequence of (A.59). Therefore $W(\xi_{(J)}, \xi_{(K)})$ is a constant, even-dimensional, antisymmetric and non-degenerate, matrix, where the latter property is implied by the linear independence of the solutions $\xi_{(J)}$. Thus we can bring it into the standard Darboux form. Explicitly, a convenient choice of basis for the solutions $\xi_{(J)}$ is obtained by splitting the $\xi_{(J)}$ into two sets of solutions, namely

$$\{\xi_{(J)}\} \to \{\xi^q_{(a)}, \xi^p_{(a)}\} \tag{A.62}$$

characterised by the initial conditions

$$\begin{aligned} \xi^q_{(a)b}(x_0^+) &= 0 \qquad \dot{\xi}^q_{(a)b}(x_0^+) = \delta_{ab} \\ \xi^p_{(a)b}(x_0^+) &= \delta_{ab} \quad \dot{\xi}^p_{(a)b}(x_0^+) = 0. \end{aligned} \tag{A.63}$$

As the Wronskian of these functions is independent of x^+, it can be determined by evaluating it at any $x^+ = x_0^+$. Then one immediately reads off that

$$\begin{aligned} W(\xi^q_{(a)}, \xi^q_{(b)}) &= W(\xi^p_{(a)}, \xi^p_{(b)}) = 0 \\ W(\xi^q_{(a)}, \xi^p_{(b)}) &= \delta_{ab}. \end{aligned} \tag{A.64}$$

Thus the corresponding Killing vectors

$$q_{(a)} = l(\xi^q_{(a)}) \qquad p_{(a)} = l(\xi^p_{(a)}) \tag{A.65}$$

and k satisfy the canonically normalised Heisenberg algebra

$$\begin{aligned} [q_{(a)}, k] &= [p_{(a)}, k] = 0 \\ [q_{(a)}, q_{(b)}] &= [p_{(a)}, p_{(b)}] = 0 \\ [q_{(a)}, p_{(b)}] &= \delta_{ab} k. \end{aligned} \tag{A.66}$$

5.2. Symmetric Plane Waves. A generic plane wave admits just the Heisenberg algebra of isometries (A.66) acting transitively on the null hyperplanes $x^+ = const.$, with a simply transitive Abelian subalgebra. Additional Killing vectors in the transverse directions are related to internal symmetries of $A_{ab}(x^+)$. As an example, note that the conformally flat plane waves (2.21) obviously have an additional $SO(d)$ symmetry and conversely $SO(d)$-invariance implies conformal flatness.

More interesting however are Killing vectors with a ∂_+-component arising from a specific dependence of $A_{ab}(x^+)$ on x^+. Away from the fixed points of such a extra Killing vector its existence renders the plane wave homogeneous. The trivial examples are plane waves with a x^+-independent profile A_{ab},

$$ds^2 = 2dx^+dx^- + A_{ab}x^a x^b (dx^+)^2 + \delta_{ab} x^a dx^b, \tag{A.67}$$

with the extra Killing vector $n = \partial_+$. As A_{ab} is x^+-independent and symmetric, it can be diagonalised by an x^+-independent orthogonal transformation acting on the x^a. In addition, we can change the overall scale of A_{ab}, $A_{ab} \to \mu^2 A_{ab}$, by the boost

$$(x^+, x^-, x^a) \to (\mu x^+, \mu^{-1} x^-, x^a). \tag{A.68}$$

Therefore these metrics can be classified by the eigenvalues of A_{ab} up to permutations and an overall scale.

Constance of A_{ab} implies that the Riemann curvature tensor is covariantly constant,

$$\bar{\nabla}_\mu \bar{R}_{\lambda\nu\rho\sigma} = 0 \Leftrightarrow \partial_+ A_{ab} = 0, \tag{A.69}$$

i.e. in that case the plane wave is locally symmetric.

The existence of the additional Killing vector $n = p_+$ extends the Heisenberg algebra to a *harmonic oscillator algebra*, with n playing the role of the number operator or harmonic oscillator Hamiltonian. Indeed, n and $k = \partial_-$ obviously commute, and the commutator of n with one of the Killing vectors $l(\xi)$ is

$$[n, l(\xi)] = l(\dot{\xi}). \tag{A.70}$$

This is consistent, i.e. the r.h.s. is again a Killing vector, because when A_{ab} is constant and ξ satisfies the null geodesic deviation or harmonic oscillator equation (A.68) then the same is true for its +-derivative $\dot{\xi}$. In terms of the basis (A.65), we have the harmonic oscillator algebra

$$[n, q_{(a)}] = p_{(a)}$$
$$[n, p_{(a)}] = A_{ab}q_{(b)}. \tag{A.71}$$

From this we infer that locally symmetric plane waves are also symmetric in the group theory sense, as they can be realised as a coset (homogeneous) space G/H, where G is the group corresponding to the extended Heisenberg algebra and H the Abelian subgroup generated by, say, the p_a. From (A.66) and (A.71) we see that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$
$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \tag{A.72}$$
$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

which are the conditions for the coset to be symmetric. First discussed as Lorentzian symmetric spaces by Cahen and Wallach [**115**] these plane waves are now referred to as Cahen-Wallach spaces. For a modern treatment in the string theory context see [**116**].

We finally note that the conserved (Noether) charge associated with n and a particle moving along a geodesic $X(\tau)$

$$Q_n = 2n_\mu \dot{X}^\mu = P_+ \tag{A.73}$$

is nothing else but the goedesic lightcone (or up to a factor the harmonic oscillator) Hamiltonian

$$Q_n = P_+ = -H_{\text{LC}} = -\frac{\delta_{ab}\dot{X}^a\dot{X}^b - (P_-)^2 A_{ab} X^a X^b}{2P_-}. \tag{A.74}$$

associated to (A.67) and the lightcone gauge $X^+ = P_-\tau$.

5.3. Singular Scale-Invariant Homogeneous Plane Waves. In the previous section we have seen that plane waves with a constant profile A_{ab} are not only homogeneous but actually symmetric. Hence, it is tempting to ask if there exist plane waves with an x^+-dependent A_{ab} which are not symmetric but still homogeneous. A simple example thereof are plane waves with

$$A_{ab}(x^+) = (x^+)^{-2} C_{ab} \tag{A.75}$$

where C_{ab} is a constant matrix and we can w.l.o.g. assume C_{ab} and A_{ab} to be diagonal, with eigenvalues the oscillator frequency squares $c_a^2 = -\omega_a^2$ (cf. chapter 5 section 2),

$$A_{ab} = -\omega_a^2 \delta_{ab} (x^+)^{-2}. \tag{A.76}$$

The corresponding plane wave metric

$$d\bar{s}^2 = 2dx^+dx^- + C_{ab}x^a x^b \frac{(dx^+)^2}{(x^+)^2} + \delta_{ab}dx^a dx^b \tag{A.77}$$

is invariant under the boost (A.68), corresponding to the extra Killing vector

$$n = x^+\partial_+ - x^-\partial_-. \tag{A.78}$$

Note that in this case $k = \partial_-$ has a non-vanishing commutator with n, namely

$$[n, k] = k \tag{A.79}$$

and consequently is no longer a central element of the isometry algebra. Furthermore, it is easy to see that the commutator of n with an arbitrary Heisenberg algebra Killing vector

$l(\xi)$, ξ^a being a solution to the harmonic oscillator equation (A.59), is again a Heisenberg algebra Killing vector

$$[n, l(\xi)] = l(x^+\dot{\xi}), \tag{A.80}$$

corresponding to the solution $x^+\dot{\xi}_a = x^+\partial_+\xi_a$ of the harmonic oscillator equation.

Precisely these plane waves have been shown to arise universally as the Penrose limits of space-time singularities [**20, 25**] (cf. chapter 5). Moreover, as string theory backgrounds they enjoy many interesting features. Note for example that as a consequence of the scale invariance the lightcone momentum P_- drops out of the transverse string e.o.m (2.46). For more information on these metrics *per se* and in the string theory context we refer to [**21**] and [**23**] respectively.

The conserved (Noether) charge associated with n is now

$$Q_n = 2n_\mu \dot{X}^\mu = X^+P_+ - X^-P_- = -P_-\tau H_{\text{LC}} + \frac{1}{2}\delta_{ab}X^a\dot{X}^b, \tag{A.81}$$

where we used the geodesic e.o.m. and constraint.

6. Curvature Scalars of Plane Waves Vanish Identically

Here we give a condensed version of the argument due to Schmidt [**31, 32**] showing that all curvature scalars of plane waves vanish identically. It can be divided into three major steps:

(1) First, one has to establish that any non-trivial curvature scalar cannot be invariant under constant rescalings of the metric.
(2) One proceeds showing that if there is a coordinate transformation, a *motion* inducing a non-trivial constant rescaling of the metric, i.e. a *homothety*, then it follows from the first step that all elementary curvature invariants vanish at the motion's fixed points.
(3) Finally, one provides the proof that for any point x in a plane wave space-time there exists a homothety with fixed-point x.

An elementary curvature scalars is a product of covariant derivatives

$$\nabla_{\mu_1} \dots \nabla_{\mu_p} R^\mu{}_{\nu\lambda\rho} \tag{A.82}$$

with the necessary number of factors of the inverse metric for a complete contraction. Following our plan above we consider the behaviour of such an elementary curvature scalar under constant rescalings of the metric. It is easy to see that the Christoffel symbols are actually invariant under such a scaling. Consequently, the same is true for the Riemann tensor with index structure $R^\mu{}_{\nu\lambda\rho}$ as well as all its covariant derivatives. However, since positive number of factors of the inverse metric is required to construct a scalar, we infer that non-trivial elementary curvature scalars transform non-trivially under constant rescalings of the metric. This establishes (1).

It follows directly that if there exists a homothety of a metric which is not an isometry, then all curvature invariants have to vanish at the fixed points of this coordinate transformation. The reason for this is that, on the one hand, the curvature scalar must obviously be invariant under coordinate transformations, whereas on the other hand, since this coordinate transformation induces a constant scaling of the metric, the curvature scalar cannot be invariant under this transformation unless it is zero. This establishes (2).

Therefore, in order to prove the vanishing of the curvature invariants of a plane wave, we are left to show that for every point x there exists a non-trivial homothety of the plane wave space-time with fixed point x. This is quite easy to see in Rosen coordinates (2.9) where we have an obvious translation symmetry in v and y^k (cf. section 5.1), s.t. we can assume w.l.o.g. x to be the point $(u, 0, 0)$. However, under the scaling

$$(u, v, y^k) \to (u, \lambda^2 v, \lambda y^k) \tag{A.83}$$

$(u, 0, 0)$ is a fixed point whereas the metric transforms like

$$ds^2 \to \lambda^2 ds^2. \tag{A.84}$$

Consequently, all curvature scalars of a plane wave metric vanish at the points $(u, 0, 0)$ and hence, because of translation invariance, everywhere.

7. Elementary Hereditary Properties of Penrose Limits

Here we give a short discussion concerning the properties of the original metric which are preserved by the Penrose limit. As was realised in **[15]** the proper framework for addressing this issue has been introduced by Geroch in 1969 **[117]** in a much broader context where one considers a one-parameter family of space-times (M_λ, g_λ) for $\lambda > 0$ and discusses the limit space-time for $\lambda \to 0$. In particular, one calls a property of space-times *hereditary* if, whenever a family of space-times has that property, the same is true for all the limits of this family.

In the specific case of Penrose limits, it is advantageous to slightly relax this definition, calling a property of a space-time *hereditary* if, whenever a space-time has this property, the same is true for all its Penrose limits.

Therefore, the first step to investigate if a property of a space-time is hereditary is to verify if it is preserved under the coordinate transformation (2.55) and the conformal scaling of the metric (2.59). Note that we are merely interested in generally covariant, i.e. coordinate independent properties of a metric. Thus we only have to investigate if the property of interest is invariant under a finite rescaling of the metric and then study what happens for $\lambda \to 0$.

One of the most elementary hereditary properties which is valid for any family of space-times was formulated by Geroch **[117]**: If there is a tensor field constructed from the Riemann tensor and its derivatives which vanishes for all $\lambda > 0$, then it also vanishes for $\lambda = 0$. Therefore, we find for the Penrose limit

(1) the Penrose limit of a Ricci-flat metric is Ricci-flat;
(2) the Penrose limit of a conformally flat metric, i.e. vanishing Weyl tensor, is conformally flat. In particular, according to (2.21), it is characterised by the spherically symmetric wave profile $A_{ab}(x^+) = \delta_{ab} A(x^+)$;
(3) the Penrose limit of a locally symmetric metric, i.e. vanishing covariant derivative of the Riemann tensor, is locally symmetric.

Note however that the Penrose limit of an Einstein metric with fixed non-vanishing cosmological constant or scalar curvature is not of the same type, simply because the Ricci scalar, unlike the Ricci tensor, is not scale-invariant, i.e.

$$R_{\mu\nu}(g) = \Lambda g_{\mu\nu} \tag{A.85}$$

is only invariant under a simultaneous scaling of the metric g and Λ,

$$R_{\mu\nu}(\lambda^{-2} g) = R_{\mu\nu}(g) = (\lambda^2 \Lambda)(\lambda^{-2} g_{\mu\nu}). \tag{A.86}$$

Hence we find that the Penrose limit of an Einstein metric is Ricci-flat. Similarly, we can show that all the curvature scalars of a Penrose limit metric vanish identically. This is the "kinematic" variant of the result established in section 6 that all curvature scalars of a plane wave vanish (cf. the discussion of chapter 2 at the end of section 3.1).

Not all hereditary properties (of Penrose limits) are that obvious. Considering for example isometries, one might erroneously assume that for a family of space-times, all possessing a certain number K of Killing vectors, one finds less Killing vectors in the (Penrose) limit because some of them being linearly independent for all $\lambda > 0$ cease to be linearly independent at $\lambda = 0$. However, as was shown by Geroch [**117**]

(4) The number of linearly independent Killing vectors does not decrease in the (Penrose) limit.

As the same is true for Killing spinors and supersymmetries this establishes that the number of supersymmetries preserved by a supergravity configuration can not decrease in the Penrose limit [**15**].

Another quite subtle example concerns the failure of homogeneity to be hereditary w.r.t. the Penrose limit. In a nutshell what happens is that a Killing vector being sum of a translational and a rotational Killing vector, the translational part being responsible for homogeneity, becomes purely rotational in the Penrose Limit. For an explicit counterexample and a detailed discussion of this issue we refer to [**40**] and [**118**] respectively.

8. Curvature of Szekeres-Iyer Metrics

For reference purposes we give here the non-vanishing components of the Ricci and Einstein tensors of the metric,

$$ds^2 = \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega_d^2 \tag{A.87}$$

Indices i, j refer to the metric $\hat{g}_{ij}$ of the transverse sphere (or some other transverse space), with $\hat{R}_{ij}$ and $\hat{R}$ the corresponding Ricci tensor and Ricci scalar.

Christoffel Symbols

$$\begin{aligned} &\Gamma^x{}_{xx} = \Gamma^x{}_{yy} = \Gamma^y{}_{yx} = \frac{p}{2}x^{-1} \\ &\Gamma^x{}_{ij} = \eta\frac{q}{2}\hat{g}_{ij}x^{q-p-1} \\ &\Gamma^i{}_{jx} = \frac{q}{2}\delta^i{}_j x^{-1} \\ &\Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk} \end{aligned} \tag{A.88}$$

Ricci Tensor

$$\begin{aligned} R_{xx} &= \tfrac{1}{4}(2p + 2qd + pqd - q^2 d)x^{-2} \\ R_{yy} &= \tfrac{1}{4}p(qd-2)x^{-2} \\ R_{ij} &= \hat{R}_{ij} + \tfrac{1}{4}\eta q(qd-2)\hat{g}_{ij}x^{q-p-2} \\ &= (d-1)\hat{g}_{ij} + \tfrac{1}{4}\eta q(qd-2)\hat{g}_{ij}x^{q-p-2} \end{aligned} \tag{A.89}$$

Ricci Scalar

$$\begin{aligned} R &= \hat{R}x^{-q} - \tfrac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \\ &= d(d-1)x^{-q} - \tfrac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \end{aligned} \tag{A.90}$$

EINSTEIN TENSOR

(A.91)
$$
\begin{aligned}
G^x_x =& -\tfrac{1}{2}\hat{R}x^{-q} - \tfrac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)}\\
=& -\tfrac{1}{2}d(d-1)x^{-q} - \tfrac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)}\\
G^y_y =& -\tfrac{1}{2}\hat{R}x^{-q} + \tfrac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)}\\
=& -\tfrac{1}{2}d(d-1)x^{-q} + \tfrac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)}\\
G^i_j =& \hat{G}^i_j x^{-q} + \tfrac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta^i_j x^{-(p+2)}\\
=& -\tfrac{1}{2}(d-1)(d-2)\delta^i_j x^{-q} + \tfrac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta^i_j x^{-(p+2)}
\end{aligned}
$$

APPENDIX B

Examples

1. A String Expansion Around a Non-Degenerate Background

In this section we perform a string expansion to first order in the e.o.m. around a non-degenerate background string X_B as was discussed in section 2. We intend to show for a rather simple background how the contribution of the extrinsic curvature alters the e.o.m. and how tedious the explicit calculations, although always feasible, can become in contrast to the expansion around a null geodesic.

Exact solutions in time-dependent space-times which can serve as a simple non-degenerate string background are rare. Here we consider the ringlike solution, discussed in [**119**], propagating in the one-parameter class of Friedmann-Robertson-Walker space-times given by

$$ds^2 = -(dx^0)^2 + R(x^0)^2 \sum_{a=1}^{D-1}(dx^a)^2 = R(\eta)^2 \left(-(d\eta)^2 + \sum_{a=1}^{D-1}(dx^a)^2 \right) \tag{B.1}$$

where x^0 is the cosmic and η the conformal time respectively. The radius $R(\eta)^2$ is of power-law type

$$R(\eta)^2 = B^2 \eta^k \tag{B.2}$$

with B a constant.

From the Polyakov action in conformal gauge for this background

$$S = \int d\sigma^2 \left(-\partial^i X^0 \partial_i X^0 + R(X^0)^2 \partial^i X^a \partial_i X^a \right) \tag{B.3}$$

one derives the e.o.m.

$$\begin{aligned} -\partial^2 X^0 + R(X^0)\frac{\partial R}{\partial X^0} \sum_{a=1}^{D-1}(\partial^i X^a \partial_i X^a) &= 0 \\ \partial_i \left(R^2 \partial^i X^a \right) = 0 \quad 1 \leq a \leq D-1 \end{aligned} \tag{B.4}$$

which are supplemented by the constraints

$$-(\partial_\pm X^0)^2 + R(X^0)^2 (\partial_\pm X^a)^2 = 0. \tag{B.5}$$

The separation of variables ansatz of a ring configuration with τ-dependent radius

$$\begin{aligned} X^0 &= X^0(\tau) \\ X^1 &= f(\tau)\cos n\sigma \\ X^2 &= f(\tau)\sin n\sigma \\ X^a &= c^a = \text{const}, \quad a \geq 3 \end{aligned} \tag{B.6}$$

leads to ordinary differential equations for $X^0(\tau)$ and $f(\tau)$

(B.7)
$$\begin{aligned}\partial_\tau(R^2\partial_\tau f) + R^2 f = 0\\ \partial_\tau^2 X^0 + R(\partial_{X^0}R)(\dot{f}^2 - f^2) = 0\\ (\partial_\tau X^0)^2 - R^2(\dot{f}^2 + f^2) = 0\end{aligned}$$

where $\dot{f}$ means $\partial_\tau f$. For the power-law type metrics above these equations simplify if one uses the conformal time $\eta = \eta(\tau)$ instead

(B.8)
$$\begin{aligned}\eta\ddot{\eta} + k\dot{f}^2 = 0\\ \eta\ddot{f} + k\dot{\eta}\dot{f} + \eta f = 0\\ \dot{\eta}^2 - \dot{f}^2 - f^2 = 0.\end{aligned}$$

The last equation is due to the string constraints and thus its l.h.s. is a constant of motion by the e.o.m. in the first and last equation. The system (B.8) admits a simple set of solutions

(B.9)
$$\eta(\tau) = Ae^{\pm\frac{n\tau}{\sqrt{-k-1}}}, \quad f(\tau) = \frac{A}{\sqrt{-k}}e^{\pm\frac{n\tau}{\sqrt{-k-1}}}$$

parametrised by the winding n and a constant amplitude $A > 0$.

A convenient step in order to evaluate the first order string expansion equations is to adapt the space-time coordinates to the solution, i.e. replacing η by τ and introducing cylindrical coordinates

(B.10)
$$\begin{aligned}\eta &= Ae^{\pm\frac{n\tau}{\sqrt{-k-1}}}\\ x^1 &= \rho\cos n\sigma\\ x^2 &= \rho\sin n\sigma\end{aligned}$$

leading to the metric

(B.11)
$$ds^2 = B^2A^k e^{\frac{knt}{\sqrt{-1-k}}}\left(-\frac{A^2n^2}{|1+k|}e^{\frac{2nt}{\sqrt{-1-k}}}d\tau^2 + dr^2 + n^2r^2d\sigma^2 + \sum_{a=3}^{D-1}(dx^a)^2\right).$$

The solution is now described by

(B.12)
$$X^\mu = (\tau, \rho(\tau) = f(\tau), \sigma, c^a).$$

The tangential part of the frame adapted to the solution (4.7) we get directly by differentiating (B.12) and is easily completed by an orthonormal part to

(B.13)
$$\begin{aligned}E_\tau &= \partial_\tau + \frac{An}{\sqrt{k(k+1)}}e^{\frac{n\tau}{\sqrt{-1-k}}},\\ E_\sigma &= \partial_\sigma,\\ E_1 &= \frac{1}{A^{\frac{k}{2}+1}B}e^{-\frac{(2k+1)n\tau}{2\sqrt{-1-k}}}\partial_\tau + \frac{1}{A^{\frac{k}{2}}B}\sqrt{\frac{k}{k+1}}e^{-\frac{kn\tau}{2\sqrt{-1-k}}}\partial_\rho,\\ E_a &= \frac{1}{A^{\frac{k}{2}}B}e^{-\frac{kn\tau}{2\sqrt{-1-k}}}\partial_{a+1}, \quad a = 2, ..., D-2.\end{aligned}$$

It is now straightforward to calculate the induced metric (4.8)

(B.14)
$$-g_{\tau\tau} = g_{\sigma\sigma} = \frac{A^{k+2}B^2n^2}{k}e^{\frac{(2+k)n\tau}{\sqrt{-1-k}}}, \quad g_{\tau\sigma} = 0$$

the gauge covariant derivative w.r.t. transverse frame rotations (4.15)

$$D_i = \partial_i, \tag{B.15}$$

the extrinsic curvature (4.15)

$$K^1_{\tau\tau} = K^1_{\sigma\sigma} = \frac{A^{k+1}Bn^2}{2\sqrt{-1-k}} e^{\frac{(k+2)n\tau}{2\sqrt{-1-k}}}, \quad \text{else} \quad 0 \tag{B.16}$$

and the frame components of the Riemann tensor

$$\begin{aligned} R_{\tau 1 \tau 1} &= -\frac{n^2}{2}, & R_{\sigma 1 \sigma 1} &= -\frac{(-2+k^2)n^2}{4(k+1)} \\ R_{\tau a \tau a} &= -\frac{kn^2}{4(k+1)}, & R_{\sigma a \sigma a} &= -\frac{kn^2}{4} \quad \text{with} \quad 2 \le a \le D-2, \quad \text{else} \quad 0. \end{aligned} \tag{B.17}$$

Inserting (B.14)-(B.17) into (4.16) we finally get the first order string expansion equations

$$\begin{aligned} \left(-\partial_\tau^2 + \partial_\sigma^2 - \left(\frac{k^2}{4(k+1)} - 1\right) n^2\right) \xi^1 &= 0, \\ \left(-\partial_\tau^2 + \partial_\sigma^2 - \frac{k^2}{4(k+1)} n^2\right) \xi^a &= 0, \quad 2 \le a \le D-2. \end{aligned} \tag{B.18}$$

which are surprisingly regular, containing only constant, however negative, mass terms. The expected poles are completely hidden in the singular behaviour of the background string solution.

2. Another String Expansion Around a Non-Degenerate Background

Here we consider a plane wave background

$$ds^2 = -2dudv + h(u)dx^2 + g(u)dy^2 \tag{B.19}$$

together with the string solution

$$u_c = a\tau, \quad v_c = \frac{b^2}{a^2} H(a\tau), \quad x_c = b\tau, \quad y_c = 0 \tag{B.20}$$

where $H(u) = \int du' h(u')$. This is slightly generalised but dimensionally reduced setup w.r.t. the Big Bang Matrix Model discussed in [**120**]. The normal frame along and adapted to the solution reads

$$\begin{aligned} E_\tau &= \left(a, \frac{b^2}{2a^2} h(a\tau) a, 0, 0\right) \\ E_\sigma &= (0, 0, b, 0) \\ E_1 &= \left(\frac{a}{b\sqrt{h}}, -\frac{b}{2a}\sqrt{h}, 0, 0\right) \\ E_2 &= \left(0, 0, 0, \frac{1}{\sqrt{g}}\right). \end{aligned} \tag{B.21}$$

Again we calculate the induced metric (4.8)

$$-g_{\tau\tau} = g_{\sigma\sigma} = b^2 h(a\tau), \quad g_{\tau\sigma} = 0. \tag{B.22}$$

the gauge-covariant derivative (4.15)

$$D_i = \partial_i, \tag{B.23}$$

the extrinsic curvature (4.15)

$$K^x_{\tau\tau} = K^x_{\sigma\sigma} = \frac{ab}{2}\frac{h'(a\tau)}{\sqrt{h(a\tau)}}, \quad \text{else} \quad 0 \tag{B.24}$$

and the frame components of the Riemann tensor

$$R_{\sigma 1\sigma 1} = \frac{a^2}{h}\frac{h'^2 - 2hh''}{4h}, \quad R_{\tau 2\tau 2} = \frac{a^2}{g}\frac{g'^2 - 2gg''}{4g}, \quad \text{else} \quad 0. \tag{B.25}$$

Insertion of (B.22)-(B.24) into (4.16) leads to

$$\left(-\partial_\tau^2 + \partial_\sigma^2 + \frac{a^2h'^2}{2h^2} + \frac{a^2}{4h^2}(h'^2 - 2hh''))\right)\xi^1 = 0 \tag{B.26}$$

$$\left(-\partial_\tau^2 + \partial_\sigma^2 - \frac{a^2}{4g^2}(g'^2 - 2gg''))\right)\xi^2 = 0. \tag{B.27}$$

The fluctuations ξ^τ and ξ^σ can be chosen arbitrarily on this level (gauge choice). Fixing the lightcone and the static gauge, i.e. $U(\tau,\sigma) = \tau$ and $X(\tau,\sigma) = \sigma$ respectively, we get

$$\begin{aligned} 0 &= \Delta u = \xi^\tau E^u_\tau + \xi^1 E^u_1 \\ 0 &= \Delta x = \xi^\sigma E^1_\sigma \end{aligned} \tag{B.28}$$

or equivalently $\xi^\tau = -\frac{1}{a}E^u_1\xi^1$ as well as $\xi^\sigma = 0$. Using the first relation to eliminate ξ^τ in the equation for Δv we find

$$\Delta v = \xi^\tau E^v_\tau + \xi^1 E^v_1 = -\frac{b\sqrt{h}}{a}\xi^1, \tag{B.29}$$

i.e. the coordinate fluctuation Δv is proportional to the covariant fluctuation ξ^1. Using the relation

$$\partial_\tau^2\left(\frac{\Delta v}{\sqrt{h}}\right) = \frac{3a^2vh'^2}{4h^{5/2}} - \frac{ah'v'}{h^{3/2}} - \frac{a^2vh''}{2h^{3/2}} + \frac{v''}{\sqrt{h}} \tag{B.30}$$

in (B.26) we can extract the e.o.m. for Δv

$$(-\partial_\tau^2 + \partial_\sigma^2)\Delta v + \frac{ah'}{h}\Delta v' = 0 \tag{B.31}$$

which can be integrated to the simple action

$$S_{\Delta v} = \int d\tau d\sigma \left(\frac{1}{h}(-\partial_\tau \Delta v)^2 + \frac{1}{h}(\partial_\sigma \Delta v)^2\right). \tag{B.32}$$

Finally we look at the Δy fluctuation which is given by

$$\Delta y = \xi^2 E^y_2 = \frac{1}{\sqrt{g}}\xi^2. \tag{B.33}$$

Similarly to the Δv fluctuation we find

$$\partial_\tau^2(\sqrt{g}\Delta y) = -\frac{a^2\Delta y g'^2}{4g^{3/2}} + \frac{ag'\Delta y'}{\sqrt{g}} + \frac{a^2\Delta y g''}{2\sqrt{g}} + \sqrt{g}\Delta y'' \tag{B.34}$$

which after insertion into (B.27) leads to

$$(-\partial_\tau^2 + \partial_\sigma^2)\Delta y - \frac{ag'}{g}\Delta y' = 0 \tag{B.35}$$

stemming from the action

$$S_{\Delta y} = \int d\tau d\sigma \left(g(-\partial_\tau \Delta y)^2 + g(\partial_\sigma \Delta y)^2\right). \tag{B.36}$$

This agrees completely with the results of the coordinate calculation performed in [**120**]. Moreover, the covariant approach sheds some light on the fact that the Δv fluctuation takes the place of x which itself has been gauge fixed. In the covariant context this seems to be a mere coordinate coincidence triggered by the lightcone and static gauges. More remarkable is the flip $h \to 1/h$ which after one performs a T-duality gives back the original h in the end.

We finally remark the peculiar similarities of this solution with the one discussed in the previous section w.r.t. extrinsic curvature, i.e. (B.16) vs. (B.24) and gauge covariant derivative, i.e. (B.15) vs. (B.23).

3. Penrose-Fermi Expansion Around the Schwarzschild Event Horizon

In this section we want to calculate the Penrose-Fermi expansion around a null generator of the Schwarzschild horizon in four space-time dimensions. This task is a little bit awkward because the explicit Schwarzschild coordinates degenerate precisely at the horizon whereas the global Kruskal-Szekeres coordinates are not explicit. Optioning for the latter

$$ds^2_{\mathrm{Kr-Sz}} = \frac{16m^2}{r(u,v)} e^{\left(1-\frac{r(u,v)}{2m}\right)} du dv + r(u,v)^2 d\Omega^2 \tag{B.37}$$

with the usual relation

$$uv = (r(u,v) - 2m) e^{\frac{r(u,v)}{2m} - 1}. \tag{B.38}$$

We pick a generator $\gamma = (0, v, \theta_0, \phi_0)$ and complete it to a quasi-orthonormal parallel frame

$$E_+ = \partial_v, \quad E_- = \frac{1}{4m^2}\partial_u, \quad E_1 = \frac{1}{2m}\partial_\theta, \quad E_2 = \frac{1}{2m\sin\theta}\partial_\phi. \tag{B.39}$$

Plugging these equations into the second order Penrose-Fermi expansion (3.97) we get after some straightforward calculations using Mathematica

$$\begin{aligned} ds^2_{\mathrm{Kr-Sz}} &= 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ &+ \lambda^2 \left(\frac{\delta_{ab}x^a x^b}{6m^2} dx^+dx^- + (x^-)^2 dx^+dx^+ - \frac{x^- x^a}{6m^2}\delta_{ab}dx^b dx^+ - \frac{(\epsilon_{ab}x^a dx^b)^2}{12m^2} \right) + \mathcal{O}(\lambda^3). \end{aligned} \tag{B.40}$$

Obviously in the $x^+ - x^-$-plane this reduces to $\mathrm{Ad}S_2$

$$ds^2_{\mathrm{Kr-Sz},\pm} = 2dx^+dx^- + \lambda^2 (x^-)^2 dx^+dx^+ + \mathcal{O}(\lambda^3). \tag{B.41}$$

We finally note, that actually there are global and explicit Schwarzschild coordinates discovered by Klösch and Strobl [**121**]

$$ds^2_{\mathrm{Kl-St}} = 8m \left(dx dy + \frac{y^2}{xy + 2m} dx^2 \right) + (xy + 2m)^2 d\Omega^2. \tag{B.42}$$

Moreover inspection of the Christoffel symbols shows that $\gamma = (x, 0, \theta_0, \phi_0)$ is a null geodesic generator and ∂_y parallel transported along γ as well as geodesic. Thus, in the $x^+ - x^-$-plane x, y coincide up to a boost with the Fermi coordinates x^+, x^- above and the first two terms in (B.42) are nothing else but a (fully integrated) Fermi metric for this plane w.r.t the null generator above.

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